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ACCESSIBILITY PROPERTIES OF ABNORMAL GEODESICS IN OPTIMAL CONTROL ILLUSTRATED BY TWO CASE STUDIES.

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Abstract. In this article, we use two case studies from geometry and optimal control of chemical network to analyze the relation between abnormal geodesics in time optimal control, accessibility properties and regularity of the time minimal value function.

Introduction. In this article, one considers the time minimal control problem for a smooth system of the form $\frac{\mathrm{d}q}{\mathrm{d}t} = f(q,u)$, where $q \in M$ is an open subset of $\mathbb{R}^n$ and the set of admissible controls is the set $\mathcal{U}$ of bounded measurable mapping $u(\cdot)$ valued in a control domain U, where U is a two-dimensional manifold of $\mathbb{R}^2$ with boundary. According to the Maximum Principle [14], time minimal solutions are extremal curves satisfying the constrained Hamiltonian equation

$$\begin{aligned} \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \\ H(q,p,u) = \boldsymbol{M}(q,p), \end{aligned} \tag{1}$$

where $H(q,p,u) = p \cdot f(q,u)$ is the pseudo (or non maximized) Hamiltonian, while $\boldsymbol{M}(q,p) = \max_{v \in U} H(q,p,v)$ is the true (*maximized*) Hamiltonian. A projection of an extremal curve $z = (q,p)$ on the q-space is called a geodesic.

Moreover since $\boldsymbol{M}$ is constant along an extremal curve and linear with respect to p, the extremal can be either exceptional (*abnormal*) if $\boldsymbol{M} = 0$ or non exceptional if $\boldsymbol{M} \neq 0$. To refine this classification, an extremal subarc can be either regular if the control belongs to the boundary of U or singular if it belongs to the interior and satisfies the condition $\frac{\partial H}{\partial u} = 0$.

Taking $q(0) = q_0$ the accessibility set $A(q_0, t_f)$ in time t_f is the set $\cup_{u(\cdot) \in \mathcal{U}}$ $q(t_f, q_0, u)$, where $t \mapsto q(\cdot, q_0, u)$ denotes the solution of the system, with $q(0) = q_0$ and clearly since the time minimal trajectories belongs to the boundary of the accessibility set, the Maximum Principle is a parameterization of this boundary.

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This set can have some pathologies [10, p. 174], hence the analysis of the extremal dynamics is rather intricate. The same holds for the time minimal value function $T(q_0, q_1) = \{\min\ t;\ q_1 = q(t, q_0, u),\ u(\cdot) \in \mathcal{U}\}$ even in the geodesically complete case.

The analysis of the role of abnormal extremals in the regularity of the boundary of the accessibility set and the value function is a huge task in geometric control. In this article, we concentrate on two case studies whose aim is to illustrate the main feature regarding continuity properties of the value function but also to promote a set of technical tools inherited from geometric invariant theory and singularity theory in the study. It amounts to compute invariants and semi-normal forms to evaluate the small time value function. It leads to classify the extremal curves for specific groups actions related to the underlying optimal control problem. Note that such group takes different forms in our two case studies.

The first case concerns a Zermelo navigation problem which is defined by a pair (F_0, g), where F_0 is a vector field defining the current, while g is a Riemannian metric. The classification problem is related to the action of smooth changes of coordinates on the pair. For the second case, the geometric program is the classification of triples (X, Y, N) where X, Y is a pair of vectors fields representing the controlled dynamics and N is a terminal manifold of codimension one. The group action consists of change of coordinates and feedback transformations preserving the control bound, which takes the form $|u| \leq 1$.

The first problem is one founding example of calculus of variations and was set originally in 1931 by Zermelo [18] and presented in details by Carathéodory [7], in particular in the case of linear wind for which we shall refer as the historical case along this paper. The problem is a ship navigating on a river with a current and aiming to reach the opposite shore.

Introducing the coordinates $q = (x, y)$ on the Euclidean space, where the current is given by $\mu(y)\frac{\partial}{\partial x}$ assuming only dependent upon the distance y to the shore and the set of admissible directions is

$$\dot{q} \in F_0(q) + U, \quad U = S^1.$$

This leads to study the time minimal control problem for the system

$$\dot{q} = F_0(q) + \sum_{i=1}^{2} u_i\, F_i(q),$$

where (F_1, F_2) is an orthonormal frame for the Euclidean metric denoted $g = \mathrm{d}x^2 + \mathrm{d}y^2$. From a pure geometric point of view, one can generalize to a Zermelo navigation problem on a surface of revolution $M \subset \mathbb{R}^3$, endowed with the induced Riemannian metric, with parallel current and represented as a triplet (M, F_0, g). The geodesics can be analyzed up to the action of the pseudo-group G of smooth local change of coordinates.

The domain of navigation can be split into two subdomains:

- strong current domain: $\|F_0\|_g > 1$
- weak current domain: $\|F_0\|_g < 1$

separated by the transitional case called moderate, with $\|F_0\|_g = 1$. In the historical problem with linear current $F_0 = y\frac{\partial}{\partial x}$ it is given by $|y| = 1$.

Introducing the pseudo-Hamiltonian, one gets

$$H(q,p,u) = H_0(q,p) + \sum_{i=1}^{2} u_i H_i(q,p), \tag{2}$$

where $H_i(z) = p \cdot F_i(q),\ i = 1,2$ are the Hamiltonian lifts. In the historical problem the authors introduce the heading angle α of the ship with respect to $\{F_1, F_2\}$, which is the parameterization of the control since F_1, F_2 form a frame. Extremals are regular since $|u| = 1$ belongs to the boundary of U: $|u| \leq 1$ and the Maximum Principle leads to compute the true Hamiltonian:

$$H(z) = H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)}.$$

But an equivalent point of view already introduced in the historical example is to take as accessory control $v = \dot{\alpha}$, derivative of the heading angle.

Such transformation in relation with Goh transformation in optimal control will be called the Carathéodory-Zermelo-Goh transformation. Our study boils down to time minimal control for the single-input control system

$$\dot{\tilde{q}} = X(\tilde{q}) + v\,Y(\tilde{q}), \quad v \in \mathbb{R},$$

$X = F_0(q) + \cos\alpha F_1(q) + \sin\alpha F_2(q)$ and $Y = \frac{\partial}{\partial\alpha}$, $\tilde{q} = (q,\alpha)$. Using this rewriting, the maximization condition gives us: $\frac{\partial H}{\partial \alpha} = 0$ and hence extremals become singular. Moreover $p = (p_x, p_y)$ can be lifted into $\tilde{p} = (p, p_\alpha)$ and the Maximum Principle leads to the constraints:

$$\tilde{p}(t) \cdot Y(\tilde{q}(t)) = \tilde{p}(t) \cdot [Y,X](\tilde{q}(t)) = 0,$$

where $[Y,X](\tilde{q}) = \frac{\partial Y}{\partial \tilde{q}}(\tilde{q})X(\tilde{q}) - \frac{\partial X}{\partial \tilde{q}}(\tilde{q})Y(\tilde{q})$ is the Lie bracket.

Abnormal geodesics are such that $H(q,p,u) = 0$ and they were called limit curves in the historical example. Their geometric interpretation is clear: they exist only in the strong current domain and they are the limit curves of the cone of admissible directions.

One first objective of this article is to make a complete analysis of such abnormal geodesics for the $2d$-Zermelo navigation problem, relating accessibility optimality to regularity of the value function. It completes the series of results presented in [4] describing the relations between singular trajectories in optimal control and feedback invariants. They can be applied to more general problems, where the control is valued in a 2-dimensional manifold with smooth boundary.

The main point is to analyze in this context optimality properties of geodesics which are non immersed curves (i.e. having zero derivatives) using the techniques from singularity theory [2]: computing semi-normal forms and invariants in optimal control, see [13] as a general reference for similar study for singularities of mappings, which goes back to the earliest work of Whitney [17].

The second case of interest consists into the control of chemical reactions networks like the McKeithan network: $\begin{tikzcd} \mathrm{T{+}M} \arrow[r] & \mathrm{A} \arrow[l, bend left] \arrow[r] & \mathrm{B} \arrow[ll, bend left] \end{tikzcd}$, whose aim is to maximize the production of one species, e.g. A. Assuming that the kinetics with respect to the temperature is described by the Arrhenius law the dynamics can be modelled using the graph of reactions. This specific network in the context of the general studies of [6, 12] serves as an example of reversible network with equilibria points, which leads to complicated dynamics.

Taking the first derivative of the temperature as the control, which again consists of a Goh transformation, the problem can be transformed into a time minimal control for a single-input affine control system. The optimization problem can be transformed into a time minimal control problem, where the terminal manifold N is given by fixing at the final time a desired concentration of a chosen species, since both problems share the same geodesics. A lot of preliminary work, see [6], was done to analyze this problem. In this article we shall concentrate to the so-called abnormal (*exceptional*) case where the geodesics are tangent to the terminal manifold N. We complete the results from [3] to analyze this case. Note that this problem can be set in the same frame than the Zermelo navigation problem, where a barrier assimilated to a terminal manifold of codimension one is given by $\|F_0\|_g = 1$. Again the geometric frame related to singularity theory was already in the earliest reference: construction of semi-normal form and concept of unfolding in control related to the codimension of the singularities. Recent progress of formal languages are used to describe algorithms to handle complicated computation, in particular in relation with the (reversible) McKeithan network, to complete previous works justified by the network $A \to B \to C$.

For both case studies, the abnormal case is related to regularity properties of the value function, and discontinuity of the value function is analyzed in relation with classification of accessibility properties.

This article is organized in three sections. In Section 1, we recall general results for time optimality for single-input affine control systems, see [4] as a general reference. Singular trajectories are introduced in relation with feedback invariants and classified with respect to their optimality status. Section 2 analyzes cusp singularities of abnormal geodesics in Zermelo navigation problem, where the historical example is used to compute a semi-normal form. In the final section, we study time minimal syntheses for $3d$-single input system, with terminal manifold of codimension 1, in relation with the McKeithan network. Calculations are intricate and are handled using semi-normal forms. We concentrate again on the abnormal case, in relation with continuity properties of the value function.

1. General concepts and results from optimal control for single-input control system.

We consider a smooth single-input control system of the form

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = X(q(t)) + u(t)\,Y(q(t)),\ \ q \in \mathbb{R}^n,$$

where the control domain U is either $\mathbb{R}$ or the interval $[-1, 1]$ and the set of admissible controls is the set of measurable mappings on $J = [0, t_f(u)]$ valued in U and we denote by $q(\cdot, q_0, u)$ (in short $q(\cdot)$) the response to $u(\cdot)$ defined on a subinterval of J with $q(0) = q_0$.

1.1. Maximum Principle.

Consider the problem of optimizing the time transfer from q_0 to a smooth submanifold N of $\mathbb{R}^n$. Then the Maximum Principle tells us that if a pair $(u(\cdot), q(\cdot))$ is optimal on $[0, t_f]$, then there exists an absolutely continuous vector function $p(\cdot) \in \mathbb{R}^n \setminus \{0\}$ such that if $H(q, p, u) = p \cdot (X(q) + uY(q))$ denotes the Hamiltonian lift, the following conditions are satisfied

1. The triplet (q,p,u) is solution a.e. on $[0,t_f]$ of
$$\begin{aligned} &\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q}, \\ &H(q,p,u) = \max_{v\in U} H(q,p,v). \end{aligned} \tag{3}$$
2. $\boldsymbol{M}(q,p) = \max_{v\in U} H(q,p,v)$ is constant and equal to $-p^0$, where p^0 is non positive.
3. The vector function $p(\cdot)$ satisfies at the final time the transversality condition:
$$p(t_f) \perp T_{q(t_f)}N \tag{4}$$

Definition 1.1. *A triplet (q,p,u) solution of (3) is called an extremal and the q-projection is called a geodesic. If moreover it satisfies the transversality condition (4) it is called a BC-extremal. An extremal is called singular if $H_Y(q(t),p(t)) = p(t)\cdot Y(q(t)) = 0$ a.e. on $[0,t_f]$. Assume $U=[-1,+1]$, a singular control is called strictly feasible if $u(\cdot)\in]-1,+1[$, saturating at time t_s if $|u(t_s)|=1$. An extremal control is called regular if it is given by $u(t) = sign(H_Y(q(t),p(t)))$ a.e. It is called bang-bang if the number of switches is finite. An extremal is called abnormal (or exceptional) if $p^0=0$, so that from the Maximum Principle it is candidate to minimize or maximize the transfer time.*

1.2. **Computation of the singular controls.** If Z_1, Z_2 denote two (smooth) vector fields, the Lie bracket is computed with the convention: $[Z_1,Z_2](q) = \frac{\partial Z_1}{\partial q}(q)Z_2(q) - \frac{\partial Z_2}{\partial q}(q)Z_1(q)$ and if H_1, H_2 are the Hamiltonian lifts of Z_1, Z_2, it is related to the Poisson bracket $\{H_1,H_2\} = \mathrm{d}H_1(\vec H_2)$ by $\{H_1,H_2\}(q,p) = p\cdot[Z_1,Z_2](q)$. In the singular case, deriving twice with respect to time the equation $H_Y(z(t))=0$, where $z(t)=(q(t),p(t))$, one gets:
$$\begin{aligned} &H_Y(z(t)) = \{H_Y,H_X\}(z(t)) = 0, \\ &\{\{H_Y,H_X\},H_X\}(z(t)) + u(t)\,\{\{H_Y,H_X\},H_Y\}(z(t)) = 0. \end{aligned} \tag{5}$$
Hence, if $\{\{H_Y,H_X\},H_Y\}(z(t))$ is not identically zero the singular extremals are given by the Hamiltonian equation:
$$\dot z(t) = H_X(z(t)) + u_s(t)\,H_Y(z(t)), \text{ with } u_s(t) = -\frac{\{\{H_Y,H_X\},H_X\}(z(t))}{\{\{H_Y,H_X\},H_Y\}(z(t))} \tag{6}$$
restricted to $H_Y = H_{[Y,X]} = 0$.

1.3. **Action of the feedback pseudo-group G_F [8].** Take a pair (X,Y). The set of triplets $\{(\varphi,\alpha,\beta)\}$, where φ is a local diffeomorphism and $u=\alpha(x)+\beta(x)v$ with $\beta\neq 0$ is a feedback, acts on the set of pairs (X,Y) and this action defines the pseudo-feedback group G_F. Each local diffeomorphism φ can be lifted into a symplectomorphism using a Matthieu transformation and defines an action of the feedback group on (5) using the symplectomorphism only. One has see [4]:

Theorem 1.2. *The mapping λ, which yields for any pair (X,Y) the constrained differential equation (5), is a covariant i.e. the following diagram is commutative:*
$$\begin{tikzcd} (X,Y) \arrow[r, "\lambda"] \arrow[d, "G_F"'] \arrow[dr, phantom, "\circlearrowleft"] & (5) \arrow[d, "G_F"] \\ (X',Y') \arrow[r, "\lambda"] & (5') \end{tikzcd}$$

where (5') consists of the equation (5) with the substitution $(X,Y) \leftarrow (X',Y')$.

1.4. **The** $3d$**-case.** Assume $n=3$. Introduce the following determinants:

$$D = \det(Y,[Y,X],[[Y,X],Y]),$$
$$D' = \det(Y,[Y,X],[[Y,X],X]),$$
$$D'' = \det(Y,[Y,X],X).$$

Using $H_Y = H_{[Y,X]} = H_{[[Y,X],X]} + u_s H_{[[Y,X],Y]} = 0$, the singular control can be computed eliminating p and depends on q only and this leads to the following:

Proposition 1.3. *1. Assume D be nonzero, the singular controls are defined by the feedback $u_s(q) = -D'(q)/D(q)$ so that the corresponding geodesics are solutions of the vector field: $X_s(q) = X(q) + u_s(q)Y(q)$. Abnormal (exceptional) singular geodesics are contained in the determinantal set $D''(q) = 0$.*

2. The map $\lambda : (X,Y) \mapsto X_s$ is a covariant, restricting the action of the feedback pseudo-group to change of coordinates only.

1.5. **High-order Maximum Principle in the singular case.** From [9], in the singular case the generalized Legendre-Clebsch condition

$$\frac{\partial}{\partial u}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial u}(q(t)) = \{\{H_Y, H_X\}, H_Y\}(q(t)) \geq 0$$

is a necessary optimality condition. This leads to the following.

Proposition 1.4. *In the $3d$-case, candidates to time minimizing are contained in $DD'' \geq 0$ and candidates to time maximizing are contained in the set $DD'' \leq 0$. If the corresponding inequalities are strict, they are respectively called hyperbolic or elliptic.*

2. Abnormal geodesics in planar Zermelo navigation problems.

2.1. **Notations and concepts.** Let (M, g, F_0) be a smooth navigation problem, where (M, g) is a Riemannian metric and F_0 is a vector field defining the current. Our study is local and taking an orthonormal frame (F_1, F_2) for the metric g, the problem can be written as a time minimal problem for the system

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = F_0(q(t)) + \sum_{i=1,2} u_i(t)\, F_i(q(t)), \quad q = (x,y), \tag{7}$$

where the control $u = (u_1, u_2)$ is such that $\|u\| \leq 1$. The domain M can be split into domain of strong current with $\|F_0\|_g > 1$ and weak current with $\|F_0\|_g < 1$ and the case of transition is called moderate with $\|F_0\|_g = 1$. Let q_0 be a point in the strong current domain. Then the tangent model at q_0 is the cone of admissible directions $F_0(q_0) + \sum_{i=1}^2 u_i F_i(q_0)$, $|u_i| \leq 1$ whose boundary is limited by two directions called limit curves by Carathéodory. They are precisely the two abnormal directions given by $H(q,p,u) = H_0(q,p) + \sum_{i=1}^2 u_i H_i(q,p) = 0$, where p is the adjoint vector and $H_i(q,p) = p \cdot F_i(q)$, $i = 0,1,2$.

The pseudo-group of local diffeomorphisms on the plane acts on the pair (F_0, g). The metric can be set locally either in the isothermal form: $g = a(x,y)\,(\mathrm{d}x^2 + \mathrm{d}y^2)$ or the polar form $\mathrm{d}r^2 + m(r,\theta)^2\,\mathrm{d}\theta^2$, where the corresponding coordinates are respectively called isothermal or polar coordinates. In the case of revolution the isothermal and polar form becomes respectively $a(x)\,(\mathrm{d}x^2 + \mathrm{d}y^2)$ and $\mathrm{d}r^2 + m(r)^2\mathrm{d}\theta^2$.

In the historical example, the metric is the Euclidean metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$, while the current is given by $F_0(q) = y\frac{\partial}{\partial x}$ and taking the line to the shore as parallel, it is oriented along the parallel.

2.2. **Maximum Principle.** As explained in the introduction the geodesics curves can be parameterized in two different ways using the Maximum Principle.

2.2.1. *Direct parameterization.* Maximizing the pseudo-Hamiltonian $H(q, u)$ using the constraints $\|u\| \leq 1$ leads to the following:

Proposition 2.1. *Denoting $z = (q, p)$ one has:*

1. *The extremal controls are given by $u_i = \frac{H_i(z)}{\sqrt{H_1(z)^2+H_2(z)^2}}$, $i = 1, 2$ so that $u_1^2 + u_2^2 = 1$.*
2. *The maximized Hamiltonian is given by $\boldsymbol{M}(z) = H_0(z) + \sqrt{H_1(z)^2 + H_2(z)^2}$.*
3. *The maximized Hamiltonian $\boldsymbol{M}$ is constant and can be normalized to $\{\pm 1, 0\}$ and the corresponding geodesics are hyperbolic if $\boldsymbol{M} = 1$, elliptic if $\boldsymbol{M} = -1$ and the abnormal case corresponds to $\boldsymbol{M} = 0$.*

2.2.2. *Parameterization using the Carathéodory-Zermelo-Goh transformation.* Using the heading angle α of the ship amounts to set $u_1 = \cos\alpha$ and $u_2 = \sin\alpha$ so that the pseudo-Hamiltonian takes the form

$$H(z) = H_0(z) + \cos\alpha H_1(z) + \sin\alpha H_2(z).$$

The maximization condition leads to $\frac{\partial H}{\partial u} = 0$. We introduce $X(\tilde{q}) \coloneqq F_0(q) + \cos\alpha F_1(q) + \sin\alpha F_2(q)$ and $Y(\tilde{q}) \coloneqq \frac{\partial}{\partial \alpha}$, where $\tilde{q} = (q, \alpha)$ is the extended state space coordinates. Using section 1, one has.

Proposition 2.2. *Geodesics curves are solutions of the dynamics:*

$$\dot{\tilde{q}} = X_s(\tilde{q}) \coloneqq X(\tilde{q}) + u_s Y(\tilde{q}) \tag{8}$$

with $u_s(\tilde{q}) = -D'(\tilde{q})/D(\tilde{q})$.

The collinear set is a crucial object in our study where singularities of geodesics occur. It is defined by $\{q,\ \exists\alpha;\ F_0(q) = \cos\alpha F_1(q) + \sin\alpha F_2(q)\}$. We have the following properties.

Proposition 2.3.

1. *The geodesics curves solutions of (1) are immersed curves outside the collinear set.*
2. *Only abnormal geodesics can be non immersed curves when meeting the collinear set $\|F_0\|_g = 1$.*

Before going further in our analysis let us analyze the case of revolution with parallel current as a generalization of the historical example.

2.2.3. *The case of revolution with parallel current.* In polar coordinates, one has:

$$F_0(q) = \mu(r)\frac{\partial}{\partial\theta}, \quad g = \mathrm{d}r^2 + m(r)^2\mathrm{d}\theta^2$$

so that

$$F_1 = \frac{\partial}{\partial r}, \quad F_2 = \frac{1}{m(r)}\frac{\partial}{\partial\theta}.$$

Defining:

$$X = \cos\alpha\frac{\partial}{\partial r} + \left(\mu(r) + \frac{\sin\alpha}{m(r)}\right)\frac{\partial}{\partial\theta},$$

we have:

$$[Y,X] = \sin\alpha\frac{\partial}{\partial r} - \frac{\cos\alpha}{m(r)}\frac{\partial}{\partial\theta},$$
$$[[Y,X],X] = \left(-\mu'(r)\sin\alpha + \frac{m'(r)}{m(r)^2}\right)\frac{\partial}{\partial\theta},$$
$$[[Y,X],Y] = \cos\alpha\frac{\partial}{\partial r} + \frac{\sin\alpha}{m(r)}\frac{\partial}{\partial\theta}.$$

Lemma 2.4. *Computing we have:*

- $D = \frac{1}{m(r)}$,
- $D' = -\mu'(r)\sin^2\alpha + \frac{m'(r)}{m(r)^2}\sin\alpha$,
- $D'' = \mu(r)\sin\alpha + \frac{1}{m(r)}$.

Hence D is never vanishing.

This yields the following proposition.

Proposition 2.5. *In the case of revolution, with parallel current one has:*

1. *The pseudo-Hamiltonian in the $\tilde{q}$-representation takes the form:*
$$H = p_r\cos\alpha + p_\theta\left(\mu(r) + \frac{\sin\alpha}{m(t)}\right) + p^0.$$
2. *The Clairaut relation p_θ =constant is satisfied and moreover*
$$p_\theta\left(\mu(r) + \frac{1}{m(r)\sin\alpha}\right) + p^0 = 0. \tag{9}$$
3. *The geodesics equations $\dot{\tilde{q}} = X(\tilde{q}) - \frac{D'(\tilde{q})}{D(\tilde{q})}Y(\tilde{q})$ can be integrated by quadratures, solving the implicit equation (9) to integrate the dynamics of the heading angle α.*
4. *Singular points for the geodesics dynamics occur only restricting to abnormal geodesics in $D'' = 0$ when $D' = 0$.*

In the historical example a cusp singularity was observed in [7] and it will serve as a model to analyze the general case in the frame of singularity theory, since integrability is not a technical requirement. One needs to recall some elementary facts.

2.3. **A brief recap about cusp singularity theory for geodesics** [15]**.** In our problem, one considers a geodesic curve $t \mapsto \sigma(t)$ defined on J and meeting $\|F_0\|_g = 1$ at $t = t_0$. Making a time translation, one can take $J = [-t_0, t_0]$ so that σ touches the boundary at $t = 0$, so that $\dot{\sigma}(0) = 0$.

Definition 2.6. *The point $\sigma(0)$ is a cusp point of order (p,q), $2 \leq p \leq q$ if $\sigma^{(p)}(0)$ and $\sigma^{(q)}(0)$ are independent. The point $\sigma(0)$ is called an ordinary cusp (or a semicubical point) if $p = 2$, $q = 3$, and a ramphoid cusp if $p = 2$, $q = 4$.*

2.3.1. *Semicubical point.* From [16, p. 56], an algebraic model in $\mathbb{R}[x,y]$ at $\sigma(0) = 0$ is given by the equation $x^3 + y^2 = 0$. Moreover it is the transition between a $\mathbb{R}$-node solution of the equation $x^3 - x^2 + y^2 = 0$, where the origin is a double point with two distinct tangents at 0: $x \pm y = 0$ and a $\mathbb{C}-$node solution of $x^3 + x^2 + y^2 = 0$ with two complex tangents at 0 given by $x \pm iy = 0$ and with two distinct components $x = y = 0$ and a smooth real branch.

A neat description from singularity theory suitable in our analysis is given by [1, p. 65] and is associated to a typical perestroika of a plane curve depending on a parameter and having a semicubical cusp point for some value of the parameters, where the curves sweep an umbrella while their inflectional tangents sweep another

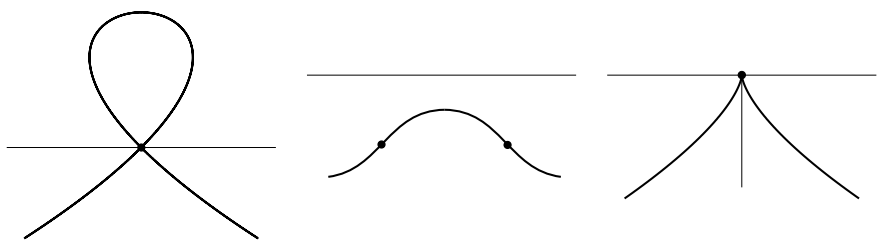


FIGURE 1. Unfolding semicubical cusp.

umbrella surface.

2.3.2. *Semicubical unfolding in the historical example.* In the historical example the geodesics equation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + \cos\alpha, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin\alpha, \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = -1 + \alpha^2. \tag{10}$$

The boundary of moderate current is taken as $y = -1$ and making the translation $Y = y + 1$ and expanding at $\alpha = 0$ up to order 2, the system takes the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Y - \alpha^2/2, \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = \alpha, \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = -1 + \alpha^2. \tag{11}$$

and take a point $q_0 = (x_0, y_0, z_0)$ in a neighbourhood of 0 in the strong current domain $Y < 0$ and let $t \mapsto \sigma(t)$ be a geodesic curve with $\sigma(0) = (x_0, y_0, z_0)$.

Then one has, using direct computations:

Proposition 2.7. *Fixing q_0 and considering the geodesics passing through q_0, we have:*

1. *The abnormal geodesic meets the boundary at a semicubical cusp with vertical tangent.*
2. *Hyperbolic geodesics are self-intersecting curves corresponding to a $\mathbb{R}$-node.*
3. *Elliptic geodesics exist only in the strong current domain $Y < 0$ and correspond to a $\mathbb{C}$-node.*

Hence geodesics curves form an unfolding of the semicubical cusp with one parameter depending upon the initial heading angle α_0, see Fig. 2.

2.4. The analysis of the geodesics curves near the set $\|F_0\|_g = 1$ and regularity of the time minimal value function.

Proposition 2.8. *Let (M, g, F_0) be a two dimensional Zermelo navigation problem and q_1 be a point on the boundary $\{\|F_0\|_g = 1\}$ between the strong current domain and the weak current domain. Assume that the geodesic $\sigma(\cdot)$ is not an immersion at $q_1 = \sigma(0)$. In the extended space (x, y, α) we consider $\tilde{q}_1 = (q_1, \alpha_1)$ a point on*

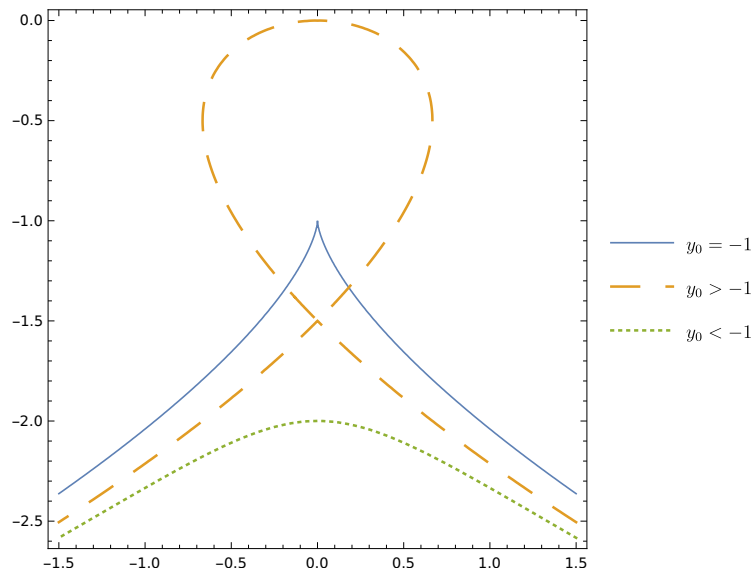


FIGURE 2. Quickest nautical path as a miniversal unfolding of the generic singularity of the abnormal geodesic.

the collinearity set $\mathcal{C}$ *and* $\tilde{\sigma}(\cdot) := (\sigma(\cdot), \alpha(\cdot))$*, the integral curve passing through* $\tilde{q}_1$ *of the differential equation*

$$\dot{\tilde{q}} = X(\tilde{q}) - \frac{D'(\tilde{q})}{D(\tilde{q})} Y(\tilde{q}), \tag{12}$$

where $X = F_0 + \cos\alpha F_1 + \sin\alpha F_2$*,* $Y = \frac{\partial}{\partial\alpha}$*,* $D = \det(Y, [Y, X], [[Y, X], Y])$ *and* $D' = \det(Y, [Y, X], [[Y, X], X])$*.*
Then, we have the two cases:

1. *If* $\dot{\alpha}(0) \neq 0$ *and the curve* $\{\|F_0(q)\|_g = 1\}$ *is regular at* q_1 *then* σ *has a semicubical cusp at* q_1*.*
2. $\dot{\alpha}(0) = 0$*:* $\tilde{q}_1$ *is a singular point with a spectrum of the form* $\{\delta, \delta/2 \pm \sqrt{\delta'}\}$*, where* δ *and* δ' *are real numbers.*

Proof. Normalization. The problem is local in a neighbourhood of q_1 and we can choose a coordinate system (x, y) to normalize at the point q_1 – identified to the origin $(0, 0)$ – the vector field F_0 along the direction $-\frac{\partial}{\partial x}$ i.e. we take two smooth functions b et c such that

$$F_0(x, y) = b(x, y)\,\frac{\partial}{\partial x} + c(x, y)\,\frac{\partial}{\partial y},$$

with $b(0, 0) = -1$ and $c(0, 0) = 0$. Then a frame (F_1, F_2), orthonormal with respect to the metric g set in the isothermal form $g(x, y) = a(x, y)\,\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$, can be taken in such a way that F_0 and F_1 have an opposite direction at q_1:

$$(F_1, F_2) = \left(\frac{1}{\sqrt{a(x, y)}}\,\frac{\partial}{\partial x}, \frac{1}{\sqrt{a(x, y)}}\,\frac{\partial}{\partial y}\right),$$

where a is a smooth positive function. This implies $\alpha_1 = 0$ and $\tilde{q}_1$ is the origin of the extended state space.

In a neighbourhood of q_1 we write $a(x,y) = \sum_{1\le i,j\le k} a_{ij}\, x^i y^j + \varepsilon_1(x,y)$, $b(x,y) = \sum_{1\le i,j\le k} b_{ij}\, x^i y^j + \varepsilon_2(x,y)$, $c(x,y) = \sum_{1\le i,j\le k} c_{ij}\, x^i y^j + \varepsilon_3(x,y)$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are terms of order higher than k.

Computation. Case $\dot\alpha(0) \neq 0$. The boundary between the strong and weak current domains is regular and can be expanded as

$$\{\|F_0\|_g = 1\} = \{a_{00} - 1 + 2\delta_2 x + 2\delta_1 y + \left(c_{01}^2 - \kappa_2\right) y^2 \\ + 2(c_{01}c_{10} - \kappa_1)\, xy + \left(c_{10}^2 - \kappa_2'\right) x^2 + o(|x,y|^2) = 0\},$$

where

$$\begin{aligned} &\delta_1 = a_{01}/2 - b_{01}, \quad \delta_2 = a_{10}/2 - b_{10}, \\ &\kappa_1 = 5/2\, a_{01}a_{10} - a_{10}b_{01} - a_{01}b_{10} + 2\, b_{11} - a_{11}, \\ &\kappa_2 = 5/4\, a_{01}^2 - a_{01}b_{01} + 2\, b_{02} - a_{02}, \\ &\kappa_2' = 5/4\, a_{10}^2 - a_{10}b_{10} + 2\, b_{20} - a_{20}. \end{aligned}$$

Since q_1 belongs to the curve $\{\|F_0\|_g = 1\}$, we have $a_{00} = 1$.

The geodesic $\sigma(\cdot)$ is not an immersion at q_1 since $\dot\sigma(0) = F_0(q_1) + \cos\alpha_1 F_1(q_1) + \sin\alpha_1 F_2(q_1) = 0$ and denoting by $p(\cdot)$ the corresponding adjoint vector, we have $\boldsymbol{M}(q_1, p(0)) = p(0)\cdot\dot\sigma(0) = 0$ and $\sigma(\cdot)$ is an abnormal geodesic.
Integrating (12), $\tilde\sigma(\cdot)$ can be parameterized as

$$\begin{cases} \sigma(t) = \left(-\frac{1}{3}t^3\delta_1^2 + o(t^3), \frac{1}{2}t^2\delta_1 + o(t^2)\right) \\ \alpha(t) = t\delta_1 + o(t) \end{cases}. \tag{13}$$

We have $\delta_1 = \dot\alpha(0) \neq 0$ and from (13) $\sigma(\cdot)$ has a semicubical cusp at q_1 (and this fact is independent of the value of δ_2).

Case $\dot\alpha(0) = 0$. The expansions at $\tilde q_1 = 0$ of the determinants $D(\tilde q)$ and $D'(\tilde q)$ are

$$\begin{aligned} &D(\tilde q) = 1 - a_{10}\, x - a_{01}\, y + o(x,y,\alpha), \\ &D'(\tilde q) = (-a_{01}\delta_2 + a_{10}\,\delta_1 + \kappa_1)/2\, x + \kappa_2\, y + (c_{01} + \delta_2)\,\alpha - \delta_1 + o(x,y,\alpha). \end{aligned} \tag{14}$$

We have $D'(\tilde q_1) = -\delta_1 = -\dot\alpha(0) = 0$, $D(\tilde q_1) \neq 0$ and $\tilde q_1$ is a singular point of the system $\dot{\tilde q} = X_s(\tilde q)$. X_s is Lipschitz with $\tilde\sigma(0) = 0$, therefore $\tilde\sigma(\cdot)$ is reduced to 0. The characteristic polynomial χ of the Jacobian matrix $\frac{\partial X_s}{\partial \tilde q}$ evaluated at $\tilde q_1$ is

$$\chi(s) = -\frac{1}{4}\,(\delta_2 + s)\left((2s + \delta_2)^2 - \lambda\right), \quad \lambda = (2c_{01} + \delta_2)^2 - 4\kappa_2,$$

hence the spectrum is $\{-\delta_2, (-\delta_2 \pm \sqrt{\lambda})/2\}$ if $\lambda \ge 0$ or $\{-\delta_2, (-\delta_2 \pm i\sqrt{-\lambda})/2\}$ otherwise. □

Remark 2.9. *Using the semi-normal form constructed in Proposition 2.8, the expansion at* $\tilde q_1 = 0$ *of the determinant* $D''(\tilde q)$ *up to order* 2 *is*

$$D''(\tilde q) = \frac{\kappa_2'}{2}x^2 + \frac{\kappa_2}{2}y^2 + \frac{1}{2}\alpha^2 + y(c_{01}\alpha - \delta_1) + x\left(c_{10}\alpha + \frac{\kappa_1 y}{2} - \delta_2\right) + o(|x,y,\alpha|^2),$$

which for $\delta_1 := \dot\alpha(0) = 0$ *and* $\delta_2 = 0$ *becomes*

$$D''(\tilde q) = \frac{\kappa_2'}{2}x^2 + \frac{\kappa_2}{2}y^2 + \frac{1}{2}\alpha^2 + c_{01}\, y\alpha + c_{10}\, x\alpha + \frac{\kappa_1}{2}\, xy + o(|x,y,\alpha|^2). \tag{15}$$

Hence when $\{\|F_0(q)\|_g = 1\}$ *is not a regular curve at* q_1*, meaning* $\delta_1 = \delta_2 = 0$*, the surface* $D''(\tilde q) = 0$ *is not regular at* $\tilde q_1$ *and we shall stratify the collinearity set to analyze the singular dynamics on* $D'' = 0$ *(see e.g. Fig. 3).*

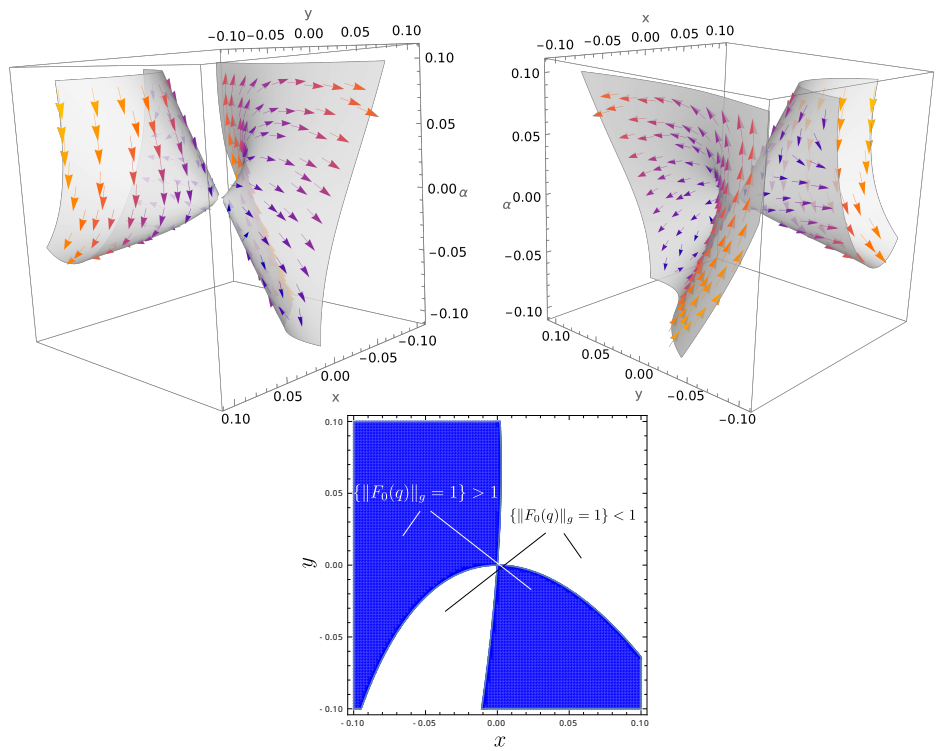


FIGURE 3. *(top)* Singular vector field X_s on the surface $D''(\tilde{q}) = 0$ for the semi-normal form constructed in the proof of Proposition 2.8 for $\delta_1 = \delta_2 = 0$ and random values for the others coefficients a_{ij}, b_{ij}, c_{ij} yielding $\lambda = -4$. *(bottom)* Corresponding strong (blue) and weak (white) current domains. The curve $\{\|F_0(q)\|_g = 1\}$ is not regular.

The following theorem describes the optimality properties of the abnormal and hyperbolic geodesics.

Theorem 2.10. *Let $\tilde{q}_1 = (q_1, \alpha_1)$ such that q_1 is a semicubical cusp at $t = 0$ for the abnormal geodesic $\sigma_a(\cdot)$. There exists a neighbourhood V of q_1, a point q_0 in $V \cap \sigma_a(\cdot)$ in which we have:*

1. *The abnormal arc is time minimizing from q_0 up to the cusp point included.*
2. *Self-intersecting geodesics starting from q_0 in a conic neighbourhood of α_1 are time minimizing up to the intersection point q_1 with the abnormal.*
3. *The value function $T : q_f \mapsto T(q_0, q_f)$ is discontinuous for each $q_f \neq q_1$ on the abnormal geodesic $\sigma_a(\cdot)$.*

Proof. For ξ in the extented state space, we use the notation $\sigma_h(\cdot, \xi)$ (resp. $\sigma_a(\cdot, \xi)$) for the projection on the q-space of the hyperbolic (resp. exceptional) extremal $\tilde{\sigma}(\cdot, \xi)$ passing through ξ at time 0. The point $\tilde{q}_1$ can be identified to 0 and we use the same normalization as in the proof of Proposition 2.8, which leads to consider

the semi-normal form

$$F_0(x,y) = b(x,y)\,\frac{\partial}{\partial x} + c(x,y)\,\frac{\partial}{\partial y},$$
$$F_1 = \frac{1}{\sqrt{a(x,y)}}\,\frac{\partial}{\partial x}, \quad F_2 = \frac{1}{\sqrt{a(x,y)}}\,\frac{\partial}{\partial y},$$

with $a(x,y) = \sum_{1\le i,j\le k} a_{ij}\, x^i y^j + \varepsilon_1(x,y)$, $b(x,y) = \sum_{1\le i,j\le k} b_{ij}\, x^i y^j + \varepsilon_2(x,y)$, $c(x,y) = \sum_{1\le i,j\le k} c_{ij}\, x^i y^j + \varepsilon_3(x,y)$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are terms of order higher than k, $a_{00} = 1$, $b_{00} = -1$, $c_{00} = 0$, $a_{10} = 2\, b_{10}$ and $a_{01} - 2\, b_{01} \neq 0$.

The proof goes as follows. In relation with Fig. 4, we define from $\tilde{q}_1$ the points q_0 and q_2 on the abnormal geodesic as $\sigma_a(t_0, \tilde{q}_1)$ and $\sigma_a(t_2, \tilde{q}_1)$ respectively for some given $t_0 < t_2 < 0$. In the extended space we take $\tilde{q}_2' = (q_2, \alpha_2')$ and a time $t_1 < 0$ such that q_0 is reached from q_2 by a hyperbolic geodesics in time t_1. The following computation aims to express α_2' and t_1 in terms of t_0, t_2.

More precisely, for small nonpositive times t_0, t_1, t_2 and small angle α_2', we expand $q_2 \coloneqq \sigma_a(t_2, \tilde{q}_1)$ and $\sigma_h(t_1, \tilde{q}_2')$ up to order 3 and we obtain

$$\begin{aligned}
\tilde{\sigma}_a(t_2, \tilde{q}_1) = \Big(&- \frac{\delta^2}{3} t_2^3, \frac{\delta}{2} t_2^2, \frac{\delta}{12} t_2 \big(- 5a_{01}\delta t_2^2 + 2c_{01}^2 t_2^2 - 6c_{01} t_2 + 4c_{10}\delta t_2^2 \\
&\qquad + 2\delta^2 t_2^2 - 2\kappa_2 t_2^2 + 12 \big) \Big) + o(t_2^3),
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_h(t_1, \tilde{q}_2') = \Big(&\frac{1}{2}\left(2\alpha_2' t_1 + \delta t_1^2 + \delta t_2^2\right), \frac{1}{6}\big(- 3\alpha_2'^2 t_1 - 6\alpha_2'\delta t_1^2 - 2\delta^2 t_1^3 - 3\delta^2 t_1 t_2^2 - 2\delta^2 t_2^3 \big), \\
&\frac{1}{2}\left(2\alpha_2' t_1 + \delta t_1^2 + \delta t_2^2\right) + \frac{1}{2} c_{01}\delta t_1 t_2^2, \frac{1}{12}\big(3a_{01}\alpha_2'^2 t_1 - 15a_{01}\alpha_2'\delta t_1^2 - 5a_{01}\delta^2 t_1^3 \\
&- 18a_{01}\delta^2 t_1 t_2^2 + 12\alpha_2'^2 c_{10} t_1 - 12\alpha_2'^2\delta t_1 + 6\alpha_2' c_{01}^2 t_1^2 + 12\alpha_2' c_{10}\delta t_1^2 + 6\alpha_2'\delta^2 t_1^2 \\
&- 6\alpha_2'\kappa_2 t_1^2 + 2c_{01}^2\delta t_1^3 + 4c_{10}\delta^2 t_1^3 + 2\delta^3 t_1^3 + 18\delta^3 t_1 t_2^2 - 2\delta\kappa_2 t_1^3 - 6\delta\kappa_2 t_1 t_2^2 \big) \\
&- \alpha_2' c_{01} t_1 + \alpha_2' - \frac{1}{2} c_{01}\delta t_1^2 + \delta t_1 \Big) + o(|t_1, t_2, \alpha_2'|^3).
\end{aligned}$$

Computing, the equation

$$q_0 \coloneqq \sigma_a(t_0, \tilde{q}_1) = \sigma_h(t_1, \tilde{q}_2')$$

is satisfied up to order 2 in t_0, t_1, t_2, α_2', for

$$t_1 = t_2 - t_0 - 2\sqrt{t_0^2 + t_0 t_2 + t_2^2},$$
$$\alpha_2' = \frac{\delta}{2}\,\frac{t_0^2 - t_2^2 - t_1^2}{t_1},$$

since $t_0 < t_2$.

Finally, we compare the cost of the abnormal geodesic from q_0 to q_1, which is $-t_0 > 0$ and the cost $-t_1 - t_2$ of the concatenation of the hyperbolic geodesic from q_0 to q_2 and the abnormal arc from q_2 to q_1 as follows:

$$\begin{aligned}
-t_1 - t_2 &= 2\left(\sqrt{(t_0 - t_2)^2 + 3t_2 t_0} + t_0 - t_2\right) - t_0 \\
&> 2\left(|t_0 - t_2| + t_0 - t_2\right) - t_0 \\
&> -t_0.
\end{aligned}$$

This shows that the value function $q_f \mapsto T(q_0, q_f)$ is discontinuous if q_f is on the abnormal geodesic and $q_f \neq q_1$. □

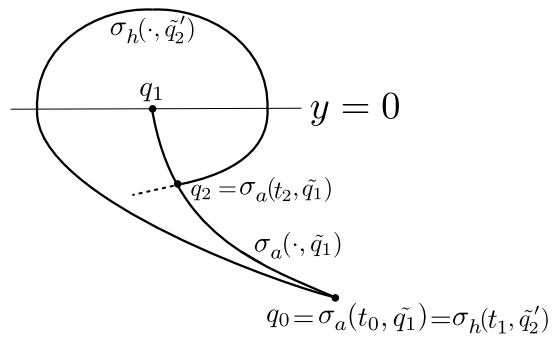
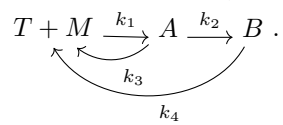


FIGURE 4. Hyperbolic and abnormal geodesics in a neighbourhood of the collinearity set.

3. **Time minimal exceptional geodesics in optimization of chemical networks.**

3.1. **A brief recap about the optimal control of chemical networks.** In this section, we introduce the concepts for the optimization of chemical networks, see [3]. In particular we shall consider the McKeithan network:



The state space is formed by the concentration vector:

$$c = (c_T, c_M, c_A, c_B)$$

of the respective chemical species. We note $\delta_1 = c_T + c_A + c_B$, $\delta_2 = c_M + c_A + c_B$ the first integrals associated to the dynamics and let $x = c_A$, $y = c_B$, then the system is described by the equation:

$$\begin{aligned}
\dot{x} &= -\beta_2 x v^{\alpha_2} - \beta_3 x v^{\alpha_3} - \delta_3 v\,(x+y) + \delta_4 v + v\,(x+y)^2 \\
\dot{y} &= \beta_2 x v^{\alpha_2} - \beta_4 y v^{\alpha_4} \\
\dot{v} &= u, \qquad |u| \leq 1,
\end{aligned}$$

with

$$0 \leq x \leq \delta_1,\ 0 \leq y \leq \delta_2,\ \delta_3 = \delta_1 + \delta_2,\ \delta_4 = \delta_1 \delta_2,$$

the Arrhenius law gives $k_i = A_i \exp(-E_i/RT)$, $i = 1, 2, 3, 4$ (E_i is the activation energy, A_i is the frequency factor, R is the ideal gas constant and T is the temperature) and

$$k_2 = \beta_2 v^{\alpha_2},\ k_3 = \beta_3 v^{\alpha_3}, k_4 = \beta_4 v^{\alpha_4},$$

and $v = k_1$ is the direct control temperature and its derivative is associated to control of the batch reactor by the derivative of the temperature.

Maximizing the production of the A species leads to a time minimal control problem with a terminal manifold $N : x = d$, d being the desired production of A.

We denote by $\dot{q} = X + u\,Y$, $|u| \leq 1$ the control system.

The singular geodesics are solutions of the dynamics

$$\dot{q} = X + u_s\,Y, \quad u_s = -D'/D,$$

where

- $D(q) = ((\alpha_4-1)\alpha_4\beta_4 y v^{\alpha_4-3} - (\alpha_2-1)\alpha_2\beta_2 x v^{\alpha_2-3})(\alpha_2\beta_2 x v^{\alpha_2} + \alpha_3\beta_3 x v^{\alpha_3} + \delta_3 v(x+y) - \delta_4 v - vx^2 - 2vxy - vy^2) + x(\alpha_2^2\beta_2 v^{\alpha_2} - \alpha_2\beta_2 v^{\alpha_2} + (\alpha_3-1)\alpha_3\beta_3 v^{\alpha_3})$ $(\alpha_2\beta_2 x v^{\alpha_2-3} - \alpha_4\beta_4 y v^{\alpha_4-3})$,
- $D'(q) = \beta_2 v^{\alpha_2-2}(\alpha_2\beta_2 x v^{\alpha_2} + \alpha_3\beta_3 x v^{\alpha_3} + \delta_3 v(x+y) - \delta_4 v - vx^2 - 2vxy - vy^2)(\alpha_2\beta_3 x v^{\alpha_3} - \alpha_2\beta_4 x v^{\alpha_4} + (\alpha_2-1)\delta_3 v(x+y) + \delta_4(v-\alpha_2 v) - \alpha_2 vx^2 - 2\alpha_2 vxy - \alpha_2 vy^2 - \alpha_3\beta_3 x v^{\alpha_3} + \alpha_4\beta_4 x v^{\alpha_4} + vx^2 + 2vxy + vy^2) + (\alpha_2\beta_2 x v^{\alpha_2-1} - \alpha_4\beta_4 y v^{\alpha_4-1})((\alpha_2-1)\beta_2 v^{\alpha_2}(\delta_4 - (x+y)(\delta_3 - x - y)) + (\alpha_3-1)\beta_3 v^{\alpha_3}(y(y-\delta_3) + \delta_4 - x^2) + (\alpha_4-1)\beta_4 y v^{\alpha_4}(\delta_3 - 2(x+y)))$,
- $D''(q) = (\beta_2 x v^{\alpha_2-1} - \beta_4 y v^{\alpha_4-1})(\alpha_2\beta_2 x v^{\alpha_2} + \alpha_3\beta_3 x v^{\alpha_3} + \delta_3 v(x+y) - \delta_4 v - vx^2 - 2vxy - vy^2) - (\alpha_2\beta_2 x v^{\alpha_2-1} - \alpha_4\beta_4 y v^{\alpha_4-1})(\beta_2 x v^{\alpha_2} + \beta_3 x v^{\alpha_3} + \delta_3 v(x+y) - \delta_4 v - vx^2 - 2vxy - vy^2)$.

Each optimal solution is a concatenation by arcs σ_+, σ_-, where the control is $u = \pm 1$ and singular arcs σ_s. In this case, the complexity of the determinantal mappings D, D', D'' contrasts with those of the Zermelo navigation problem given in Lemma 2.4 and we handle here this complexity by the use of *different semi-normal forms*, constructed for the action of the pseudo–group $\mathcal{G}$ of local diffeomorphisms φ such that $\varphi(0) = 0$, $\varphi * Y = Y$ and feedback transformation $u \to -u$ (so that σ_+ and σ_- can be exchanged).

3.2. **General concepts and notation.** We consider a local neighborhood U of $q_1 \in N$. For $q \in U$, if the optimal control $u^*(q) \in [-1, 1]$ exists and is unique, it is regular on an open subset of U, union of U_+ where $u^*(q) = 1$ and U_- where $u^*(q) = -1$. The surface S which separates U_+ from U_- is subanalytic and can be stratified into

- a switching locus W: closure of the set of points where u^* is regular and not continuous. We denote by W_+ (resp. W_-) the points of W where the optimal control is $+1$ (resp. -1) on N,
- a cut-locus C: closure of the set of points where a trajectory loses its optimality,
- singular locus Γ_s: union of optimal singular trajectories.

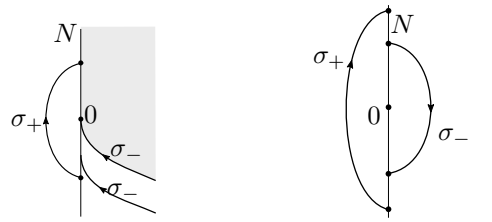
and these strata can be approximated by semialgebraic sets using semi-normal forms.

3.3. **Local syntheses in the exceptional cases.** Take a terminal point q_1 of N, which can be identified to 0. One wants to describe the time minimal syntheses in a small neighborhood U of 0. We denote respectively by $\sigma_\pm^0$, σ_s^0 bang and singular arcs terminating at 0 and we consider only the exceptional case where the arc is tangent to N, which splits into the bang exceptional case or the singular exceptional case. The syntheses are described in details in [6, 12] up to the codimension two situations and we recall the main points to be applied to the McKeithan network.

3.3.1. *The bang exceptional case.* We consider the two cases.
Generic case (codimension one). In this case both arcs σ_+^0 and σ_-^0 arc tangent to N but with a contact of order 2. Using the concept of unfolding, one can define a C^0-foliation of U by invariant planes so that in each plane the system takes the semi-normal form:

$$\begin{aligned} \dot{x} &= by + o(|x|, |y|) \\ \dot{y} &= X_2(x, y) + u, \qquad |u| \le 1, \end{aligned} \tag{16}$$

where $b = n \cdot [Y, X] \neq 0$, which can be normalized to 1, $n = (1, 0)$ being the normal to N identified to $x = 0$. Moreover one can assume that $1 + X_2(0) > 0$ and we have two time minimal synthesis described in Fig. 5.



Case 1: $X_2(0) > 1$ Case 2: $X_2(0) < 1$

FIGURE 5. Local synthesis near $\mathcal{E}$ in the generic case.

The difference between the two cases is related to different accessibility properties of the system. In the case $X_2(0) > 1$, the target N is not accessible from the points in $x > 0$ above the arc σ_-^0 terminating at 0. In the case $X_2(0) < 1$, each point of U can be steered in minimum time to N, the domain U_+ where the optimal control is $+1$ being $x < 0$ and the domain U_- with optimal control -1 being $x > 0$.

Codimension two case. A more complex situation occurs assuming that the arc σ_-^0 has a contact of order three with N while σ_+^0 has a contact of order two. The optimal syntheses cannot be described by foliating N by $2d$-planes as in the previous cases.

One needs to introduce the following assumptions. We assume that $Y = \frac{\partial}{\partial z}$, N is the plane $x = 0$ parameterized as the image of: $(w, s) \mapsto (0, w, s)$. Denoting $n = (1, 0, 0)$, the normal to N at 0, we assume

- bang exceptional case: $n \cdot X(0) = 0$, $n \cdot [Y, X] \neq 0$,
- $\det(X, Y, [Y, X]) \neq 0$ at 0,
- $\{n \cdot X = 0\} \cap N$ is a curve which is neither tangent to X nor to Y at 0.

We introduce the following normalization: along the y-axis, $n \cdot X = 0$ and $[X, Y] = \frac{\partial}{\partial x}$.

Using the concept of semi-normal form the optimal syntheses can be described by the following model:

$$\begin{cases} \dot{x} = z \\ \dot{y} = b, \\ \dot{z} = 1 + u + y \end{cases} \qquad N : (w, s) \mapsto (0, w, s) \ , \tag{17}$$

and we have two types of time minimal syntheses.

Proposition 3.1. *Assume $b < 0$. Then each point of U can be steered to N. Moreover*

1. *$U^+ \setminus N \subset \{x < 0\}$ and $U^- \setminus N \subset \{x > 0\}$*
2. *Optimal trajectories σ_- arrive at any point $(0, w, s < 0)$ or $(0, w \geq 0, s)$ of $N \cap U$.*

The optimal synthesis is described by Fig. 6.

Proposition 3.2. *Assume $b > 0$. In this case the system is not locally controllable at 0. We represent on Fig. 7 the synthesis in this case.*

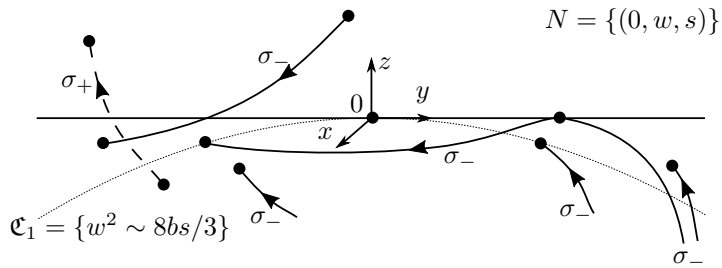


FIGURE 6. Local synthesis near $\mathcal{E}$ in the codimension 2 case for $b < 0$. The dashed curves are in the half-space $x < 0$.

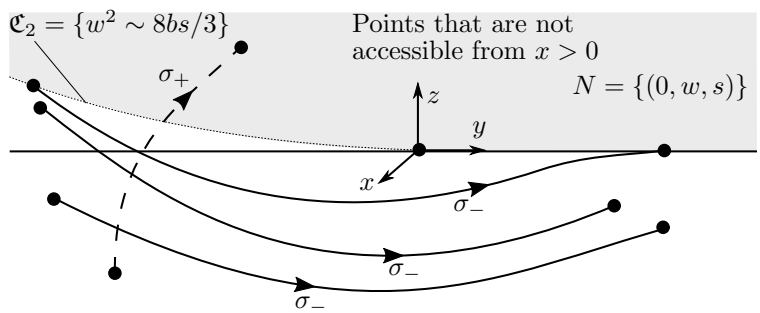


FIGURE 7. Local synthesis near $\mathcal{E}$ in the codimension 2 case for $b > 0$.

We shall refer to [12] for the full details of the computation and description of the syntheses.

3.3.2. *The singular exceptional case.* In this case we can assume $Y = \frac{\partial}{\partial z}$, N is the plane $x = 0$, the normal to N at 0 is $n = (1, 0, 0)$ and moreover:

$$\begin{cases} n \cdot X(0) = 0 \\ n \cdot [Y, X](0) = 0 \end{cases}$$

and we add the following generic conditions

- X and Y are independent at 0,
- $\det(Y, [Y, X], [[Y, X], Y]) \neq 0$ at 0,
- $\{n \cdot X(0) = 0\} \cap N$ is a curve which is not tangent to X at 0.

Using the concept of semi-normal form the optimal syntheses can be described by the following

$$\begin{cases} \dot{x} = y + z^2 \\ \dot{y} = b + b_1\, z \\ \dot{z} = c + u \end{cases} . \tag{18}$$

The different syntheses are described in [12] and we present hereafter a method of computing the time-minimal synthesis for (18) using symbolic computations.

In this model, the exceptional locus $\mathcal{E} \cap N$ is approximated by the parabola: $w + s^2 = 0$ and we denote by $\mathcal{E}_- : \{q \in N,\ n \cdot X(q) < 0\}$ and $\mathcal{E}_+ = \{q \in N,\ n \cdot X(q) > 0\}$.

We have six cases that we can classify using the model.

- *Case 1:* $b > 0,\ u_s(0) > 3$.
- *Case 2:* $b > 0,\ 1 < u_s(0) < 3$.
- *Case 3:* $b > 0,\ 0 < u_s(0) < 1$.
- *Case 4:* $b < 0,\ u_s(0) > 3$.
- *Case 5:* $b < 0,\ 1 < u_s(0) < 3$.
- *Case 6:* $b < 0,\ 0 \leq u_s(0) < 1$.

This describes the complete classification under generic assumptions and the stratification of S can be computed in the original coordinates and applied to the McKeithan network. We illustrate the method in the Case 3 correcting the results obtained in [12].

Illustration of the method based on symbolic computation. We present an algorithm to compute an approximation of the surface S in the codimension 2 case, more specifically we treat the Case 3 described above and given by the model (18) with $b > 0$ and such that the singular trajectory arriving at 0 is not saturating, that is $0 \leq b_1/2 - c < 1$.

Method. The following steps involve symbolic computation to obtain the optimal policy based on [11].

1. Take $q(0) = (0, w, s) \in N \cap U$. We first determine the stratification of the surface N. Since Y is tangent to N, $q(0)$ is a switching point. If $q(0)$ is an ordinary switching point, the optimal control is regular: $u(0) = \mathrm{sign}(p(0) \cdot [Y, X](0))$ using the classification [3], if it is a fold point, the optimal control may be singular and the optimal policy is determined using [5]. Note that since $u_s(0) = b_1/2 - c < 1$, the singular trajectories are small admissible and are either hyperbolic or elliptic, which corresponds respectively to time minimizing or time maximizing trajectories.

2. Then we integrate the system backward in time from $q(0) \in N$ and we expand the flow using Taylor expansion up to order 3 with respect to the time, the weights of the variable t, s, w being respectively $1, 2, 1$.

A trajectory $\sigma_\varepsilon, \varepsilon \in \{-1, 1\}$ can either switch at time $t_1^\varepsilon < 0$ or intersect the surface N at time $t_2^\varepsilon < 0$ or there may exist a time $t_3 < 0$ and a point $q_3 \in V$ such that $\exp(-t_3(X + Y))(q_3) = q(0)$ and $\exp(-t_3(X - Y))(q_3) \in N$.

We deduce the equations characterizing the switching surface, the singular locus Γ_s and the splitting locus C_s.

Switching surface. The switching time, characterized by $p_3(t_1^\varepsilon) = 0$, is

$$t_1^\varepsilon = 2s/(b_1/2 - c - \varepsilon) + \dots$$

and the switching surface $W_- = (x(w, s), y(w, s), z(w, s))$ is

$$\begin{aligned} x(w, s) &= \left(s^2\left(8b_1^2 + \nu_1^2\right) + 3w\nu_1^2 + 6bs\nu_1\right)/(3\nu_1^3/4s) + \dots, \\ y(w, s) &= \left(4s\left(b\nu_1 + b_1^2 s\right)\right)/\nu_1^2 + w + \dots, \\ z(w, s) &= \left(s(b_1 + 2(c + \varepsilon))\right)/\nu_1 + \dots, \end{aligned}$$

where $\nu_1 = b_1 - 2(c + \varepsilon)$.

Singular surface. The singular surface is

$$\Gamma_s(t, w) = \left(bt^2/2 + b_1^2 t^3/6 + tw, bt + b_1^2 t^2/4 + w, b_1 t/2\right) + \dots.$$

Splitting locus. The cut time, characterized by the existence of $q \in U$ such that

$$e^{-t_3(X+Y)}(q) \in N \quad \text{and} \quad e^{-t_3(X-Y)}(q) \in N,$$

is $t_3 = \frac{3s}{b_1/2-c-3} + \dots$ and a parameterization of the splitting locus $C_s = (x(w,s), y(w,s), z(w,s))$ is

$$x(w,s) = \left(2s^2\left(b_1c + b_1(2b_1-9) + 2c^2 + 6\right) + w\nu_2^2 + 3bs\nu_2\right)/(s\nu_2^3/6) + \dots,$$
$$y(w,s) = \left(6s(b\nu_2 + b_1 s(b_1 + c - 3))\right)/\nu_2^2 + w + \dots,$$
$$z(w,s) = s(b_1 + 4c)/\nu_2 + \dots,$$

where $\nu_2 = b_1 - 2c - 6$.

Bang arcs intersecting N twice. The time to reach N from $q(0) \in N$ with a bang control, characterized by $e^{t_2^\varepsilon(X+\varepsilon Y)}(q(0)) \in N$, is

$$t_2^\varepsilon = -2\,(w + s^2)/b + \dots$$

3. The optimal policy is deduced by computing $t^* = \max(t_1^\varepsilon, t_2^\varepsilon, t_3)$ with $\varepsilon = \mathrm{sign}(s\,(n \cdot X(0,w,s)))$ and we represent in Fig. 9 the region of N where $t^*(w,s)$ corresponds to a switching time, a splitting time or a time at which the trajectory intersects N. The surface S separating U_+ and U_- is the union of the switching surface W_-, the singular surface Γ_s foliated by singular arcs, the splitting locus C_s and a subset $\tilde{\mathcal{E}}_- \subset \mathcal{E}_-$ from which $\sigma_\pm$ intersects N in the green region of Fig. 9. The set S is represented in Fig. 8 together with some trajectories emanating from N to the components of S. The green trajectories starting from $\mathcal{E}_+$ intersect N in $\tilde{\mathcal{E}}_-$.

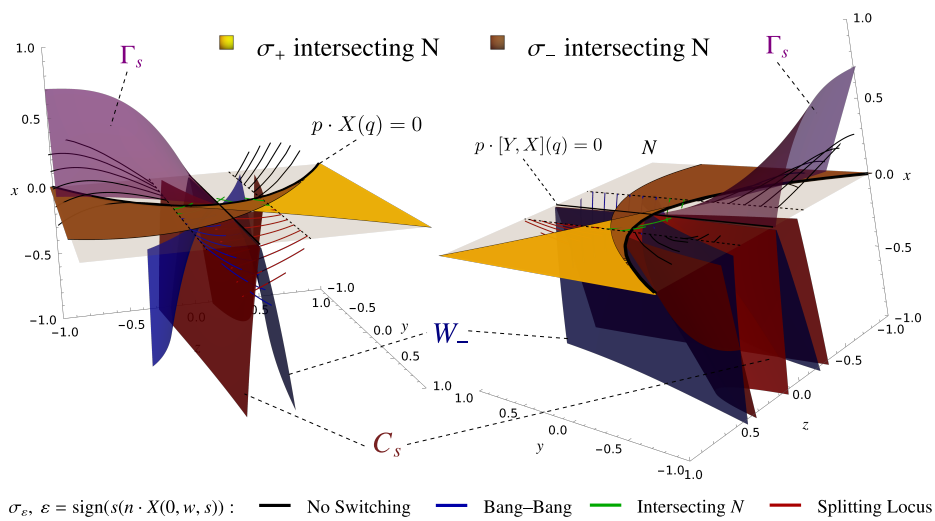


FIGURE 8. Strata of the surface S separating the regions of U where the control is ± 1 for the model (18) with $b = b_1 = 1,\ c = 0$. We also represent the regions where $\sigma_\pm$ intersect N for $t < 0$ and several trajectories which generate a switching locus and a splitting locus.

4. **Conclusion.** In this article, we provide a general framework to analyze accessibility properties of abnormal geodesics using two case studies.

The first case is motivated by the historical Zermelo navigation problem, which is generalized into $2d$-navigation problems. A barrier is formed by decomposing the

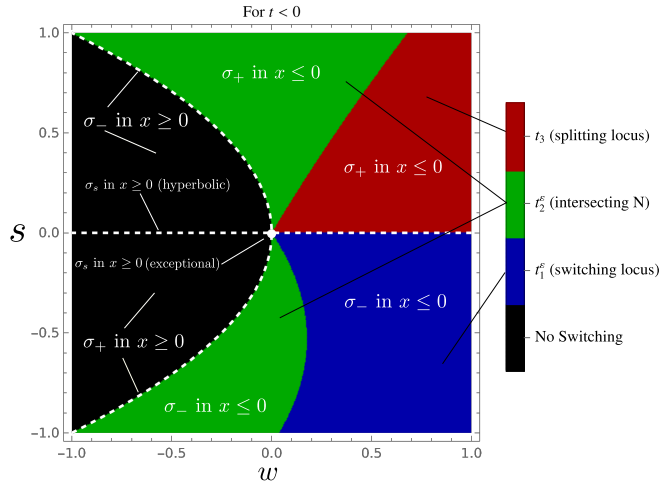


FIGURE 9. Minimum time $t^* = \max(t_1^\varepsilon, t_2^\varepsilon, t_3)$ to reach $(0, w, s) \in N$ from a neighbourhood U of 0 for the model (18) with $b = b_1 = 1, = c = 0$. The exceptional locus is $\mathcal{E} : y = -z^2$ and the singular locus is $\mathcal{S} : n \cdot [Y, X](q) = 0 : z = 0$.

state space into strong and weak current domains. Abnormal geodesics are limit curves of the set of admissible directions. We show that generically they reflect on this barrier with a cusp singularity and we analyze this phenomenon using a semi-normal form to evaluate the value function. In this case one gets precise description of the value function.

This result can be extended to the Zermelo navigation problems (M, F_0, g) in the n–dimensional case. Indeed following [5] one can compute conjugate points in the strict case for normal and abnormal curves, which are immersed curves. One can include situations where semicubical cusps occur along abnormal geodesics.

Our analysis shows that more complicated situations can appear even in the $2d$–case, related to interaction between two abnormal curves. It is described by item 2 in Proposition 2.8. In particular, abnormal curves can be separatrixes of the dynamics and hence $T(q_0, q_1) \to +\infty$ along the abnormal direction.

For the second case study concerning chemical network system, many cases can occur related to abnormal curves and this leads to construct in each case a semi-normal form to handle the problem. Nevertheless the boundary of the small time accessibility set can be stratified in relation with non existence of a Fuller phenomenon [4] near the terminal manifold and the limit of the explicit calculation of this stratification is the complexity of computations, which can be handled by symbolic computations.

Altogether for both case studies, the common thread is the analysis of the singular dynamics for $3d$–single input control systems.

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