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# ASYMPTOTIC ANALYSIS OF ABSTRACT TWO-SCALE WAVE PROPAGATION PROBLEMS WITH APPLICATIONS \*

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## Abstract.

This work addresses the mathematical analysis, by means of asymptotic analysis, of linear wave propagation problems involving two scales, represented by a single small parameter, and written in an abstract setting. This abstract setting is defined using linear operators in Hilbert spaces. In this setting, we show, under some assumptions on the structure of the wave propagation problems, weak and strong convergence of solutions with respect to the small parameter towards the solution of a well-defined limit problem. Applications in elastodynamics and piezoelectricity are presented.

**Key word.** Wave equation, Asymptotic analysis, Operator theory

**AMS subject classifications.** 35L05, 35B40, 47A50

**1. Introduction.** In the analysis of life science problems or industrial problems by means of partial differential equations (PDE), a class of problem that is frequently encountered is the class of two-scales problems. These problems – after non-dimensionnalisation – involve a unitless small parameter – denoted  $\delta$  in this work. This parameter represents a ratio of two length scales or of two representative speed of propagations for instance. In the most interesting cases, when  $\delta$  is small, the behavior of the underlying PDE is not trivially understood and some kind of asymptotic analysis process is required. More precisely, one study the limit of the solutions to the PDEs when  $\delta$  goes to zero to obtain at the limit simpler PDEs in which the small parameter  $\delta$  is missing. In applied mathematics such problems are very well understood for linear problems, see for instance [3] in the context of periodic structures and homogenization, [8] for the study of the mechanical behavior of plates (i.e. thin domains) or [18, 10] for the study of Stokes equations in nearly incompressible fluids. Many generalizations of these works have been published since then, see for instances [7, 5, 1].

In the semi-group theory the definition of an abstract framework based on linear operator theory [15] is now very standard (see [2, 9] or [19, 4] in the context of control theory). However, very few works in the literature propose a similar abstract framework, based on linear operator theory, (see however [12, 23]) for the analysis of some of the previously mentioned problems. This is precisely the objective of this work. Of course the mentioned class of problems is too large to be meaningfully represented in a single abstract framework, therefore the proposed analysis is restricted to wave propagation problems that possess what is denoted later a two-scale structure. The abstract framework that is presented in this work encompasses many relevant applications in elastodynamics, electromagnetic or piezoelectric problems (see [13, 14, 6]). The focus of this paper is the description of the abstract framework and its rigorous analysis, and present its application to some problems in elastodynamics and piezoelectricity. Note that although the proposed framework can be used to deal with homogenization problems, it is not a specific focus of this work since many dedicated mathematical tools already exists (see [20, 21, 22] to cite a few) to treat this question.

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This article is organized as follows:

- Section 2 is devoted to the abstract definition of two-scale wave propagation problems and their asymptotic limits.
  - Section 2.1 is dedicated to the definition of sequences of two-scale operators in Hilbert spaces. Formally these operators have the following form

$$G_\delta = \overline{\delta^{-1} G_0 + G_1},$$

where  $G_0$  and  $G_1$  are  $\delta$ -independent, densely defined and closed operators in Hilbert spaces. In this section we also introduce the notion of weak scale separation property. This definition corresponds to assumptions on  $G_0$  and  $G_1$  that enable an analysis of the limit process when  $\delta$  goes to zero.

- In Section 2.2 a family of wave propagation problems is introduced. In particular, a simplified formal definition of these problems is

$$s^2 u_\delta + G_\delta^* G_\delta u_\delta = f_\delta,$$

where  $s = i\omega + \eta$  with  $\eta > 0$  and  $\omega \in \mathbb{R}$ . Weak and strong convergence results towards the limit problems are provided. The limit problem reads, formally,

$$s^2 u_0 + \overline{Q_0 G_1^*} \overline{Q_0 G_1} u_0 = f_0,$$

where  $Q_0$  is the orthogonal projection operator on the kernel of  $G_0^*$ . We also introduce in this section the (self-explanatory) notion of propagating and non-propagating solutions, and discussed as well some extensions to a more general class of problems. We conclude by introducing the notion of strong and strict scale separations used in Section to prove convergence estimates.

- Section 3 is devoted to the application of the proposed framework to three problems in physics.
  - Section 3.1 tackles the problem of wave propagation in tissues, in particular, the limit from compressible to incompressible elastodynamics is discussed.
  - Section 3.2 addresses the propagation of elastic waves in thin plates.
  - Section 3.3 concerns the quasi-static approximation in piezo-electricity.
- Section 4 is devoted to establishing convergence estimate of the solutions of the family of problems towards the limit problems when the strong or strict scale separation property holds.

To ease the reading the some proofs of Section 2 are given Section 5.

## 2. Two-scale wave propagation problems.

**2.1. Preliminary definition: Two-scale operators.** We assume given a Hilbert space with values in  $\mathbb{C}$ ,

$$(\mathcal{H}; (\cdot, \cdot)_{\mathcal{H}}) \quad \text{and} \quad (\mathcal{G}; (\cdot, \cdot)_{\mathcal{G}}).$$

In the rest of this work  $\delta$ , is a real parameter and consider that for each  $\delta \in (0, 1)$ , a given densely defined, closed, unbounded operators. We introduce below the definition of a sequence of formal two-scale operators.

**DEFINITION 2.1.** *Let  $G_\delta : D(G_\delta) \subset \mathcal{H} \mapsto \mathcal{G}$  be a sequence of unbounded operators. It is said to be a sequence of formal two-scale operators when, for each  $\delta \in (0, 1)$ ,*

$$(2.1) \quad G_\delta = \delta^{-1} G_0 + G_1,$$

where the operators  $G_0$  and  $G_1$  are  $\delta$ -independent, densely defined and closed operators from  $\mathcal{H}$  to  $\mathcal{G}$ .

The adjective ‘‘formal’’ is used here to mean that, apart from the apparent structure, no interesting property can be deduced for such sequence of operators. In fact, the domain of definition of  $G_\delta$  is given by

$$D(G_\delta) = D(G_0) \cap D(G_1)$$

and may be restricted to  $\{0\}$ . We therefore require a more useful definition of two-scale sequence of operators. For the rest of the work,  $*$  is used to represent the adjoint – in Hilbert spaces – of densely defined operators.

**DEFINITION 2.2.** *Let  $G_\delta : D(G_\delta) \subset \mathcal{H} \mapsto \mathcal{G}$  be a sequence of unbounded operators. It is said to be a sequence of two-scale operators when, for each  $\delta \in (0, 1)$ ,*

$$(2.2) \quad G_\delta = \overline{\delta^{-1} G_0 + G_1},$$

where the operators  $G_0$  and  $G_1$  are  $\delta$ -independent, densely defined and closed operators from  $\mathcal{H}$  to  $\mathcal{G}$ , and

$$(2.3) \quad \begin{cases} D(G_0) \cap D(G_1) \text{ dense in } (D(G_0); \|\cdot\|_{\mathcal{H}} + \|G_0 \cdot\|_{\mathcal{G}}), & (a) \\ D(G_0^*) \cap D(G_1^*) \text{ dense in } (D(G_0^*); \|\cdot\|_{\mathcal{G}} + \|G_0^* \cdot\|_{\mathcal{H}}). & (b) \end{cases}$$

In Definition 2.2 it is implicitly assumed that  $\delta^{-1}G_0 + G_1$  is closable and that  $G_\delta$  is the closure of this operator. Moreover, as a direct consequence of the fact that  $G_0$  is densely defined and (2.3)(a),  $G_\delta$  is densely defined.

**THEOREM 2.3.** *For any sequence of two-scale operators  $G_\delta$ , let  $\{u_\delta\}$  be a sequence in  $D(G_\delta)$ , uniformly bounded in  $\mathcal{H}$  and  $\{g_\delta\}$  be a sequence in  $D(G_\delta^*)$ , uniformly bounded in  $\mathcal{G}$ , then*

$$(2.4) \quad \begin{cases} \forall h \in D(G_0^*) \cap D(G_1^*), (u_\delta, G_0^* h)_{\mathcal{G}} \xrightarrow{\delta \rightarrow 0} 0 \Rightarrow u_\delta \xrightarrow{\mathcal{H}} u_0 \in \text{Ker } G_0, & (a) \\ \forall v \in D(G_0) \cap D(G_1), (g_\delta, G_0 v)_{\mathcal{G}} \xrightarrow{\delta \rightarrow 0} 0 \Rightarrow g_\delta \xrightarrow{\mathcal{G}} g_0 \in \text{Ker } G_0^*, & (b) \end{cases}$$

where the weak convergences hold up to subsequences.

These two properties are proven very similarly. In Section 5 only the proof of (2.4)(a) is provided and it is a direct application of the density results (2.3). Note that Theorem 2.3 has a rather intuitive direct consequence: for a sequence  $u_\delta$  converging towards  $u_0$  in  $\mathcal{H}$ , when there exists  $C > 0$ , independent of  $\delta$  such that

$$(2.5) \quad \|G_\delta u_\delta\|_{\mathcal{G}} \leq C \quad \text{then} \quad u_0 \in \text{Ker } G_0.$$

This property is still not sufficient for our purpose. In what follows three different properties, denoted respectively weak, strong and strict scale separation properties are introduced. These properties describe in more details the relation between the operators  $G_0$  and  $G_1$ . In this section only the notion of weak scale separation is introduced.

DEFINITION 2.4. *A sequence of two-scale operators  $G_\delta$  satisfies the weak scale separation property if*

$$(2.6) \quad \text{Ker } G_0^* \cap D(G_1^*) \text{ dense in } (\text{Ker } G_0^*; \|\cdot\|_{\mathcal{G}}),$$

The fundamental consequence of assumption (2.6) is given below. To state the corresponding theorem the orthogonal projection operator in the kernel of  $G_0^*$  is introduced. It is denoted  $Q_0$  and is defined by,

$$Q_0 \in \mathcal{L}(\mathcal{G}), \quad Q_0^2 = Q_0, \quad (g - Q_0 g, h)_{\mathcal{G}} = 0 \quad \forall (g, h) \in \mathcal{G} \times \text{Ker } G_0^*.$$

Note that  $Q_0$  is a self-adjoint operator.

THEOREM 2.5. *For any sequence of two-scale operators  $G_\delta$  satisfying the weak scale separation property then*

$$(2.7) \quad G_1^* Q_0 \text{ is densely defined in } \mathcal{G} \text{ and } Q_0 G_1 \text{ is closable}$$

and

$$(2.8) \quad Q_0 G_\delta \subset \overline{Q_0 G_1}.$$

Note that the fact that  $Q_0 G_1$  is closable is essential in the analysis of problems involving the corresponding two-scale operator  $G_\delta$ . The reason being that this operator naturally appears in the forthcoming asymptotic analysis. The property of  $Q_0 G_1$  of being closable is essential to obtain a well-posed limit problem in the complete Hilbert space  $D(\overline{Q_0 G_1})$ . Relation (2.8) means that  $\overline{Q_0 G_1}$  is an extension of  $Q_0 G_\delta$ .

**2.2. A family of wave propagation problems and their asymptotic limits.** In this section we assume given a sequence of two-scale operators  $G_\delta$  satisfying the weak scale separation property. To shorten the notation we define the Hilbert space,

$$\mathcal{V}_\delta = D(G_\delta) \subset \mathcal{V}_0 = D(\overline{Q_0 G_1}),$$

equipped respectively with the graph norm of  $G_\delta$  and  $\overline{Q_0 G_1}$ , and we point out that  $\mathcal{V}_\delta \subset \mathcal{V}_0$  is a consequence of (2.8). We consider the following wave propagation problem in Laplace domain: for  $f_\delta \in \mathcal{H}$  given, find  $u_\delta \in \mathcal{V}_\delta$  such that, for all  $v \in \mathcal{V}_\delta$ ,

$$(2.9) \quad s^2(u_\delta, v)_{\mathcal{H}} + (G_\delta u_\delta, G_\delta v)_{\mathcal{G}} = (f_\delta, v)_{\mathcal{H}},$$

where  $s = i\omega + \eta$  with  $\eta > 0$  and  $\omega \in \mathbb{R}$ . By application of the Lax-Milgram theorem one can show that this problem admits a unique solution in  $\mathcal{V}_\delta$ . Moreover, choosing  $v = s u_\delta$  as a test function in (2.9) one can show the following stability estimate

$$(2.10) \quad \|s u_\delta\|_{\mathcal{H}}^2 + \|G_\delta u_\delta\|_{\mathcal{G}}^2 \leq \eta^{-2} \|f_\delta\|_{\mathcal{H}}^2.$$

Note that, such estimate can be used to derive estimates in time domain using the inverse Fourier transform. We refer the reader to [24] or [25] for more details.

### 2.2.1. Weak convergence towards the limit problem.

**THEOREM 2.6.** *Assume that there exists  $f_0 \in \mathcal{H}$  such that  $f_\delta \rightharpoonup_{\mathcal{H}} f_0$ , then, up to a subsequence,*

$$(2.11) \quad u_\delta \rightharpoonup_{\mathcal{V}_0} u_0,$$

where  $u_0 \in \text{Ker } G_0 \cap \mathcal{V}_0$ , satisfies, for all  $v \in \text{Ker } G_0 \cap D(G_1)$ ,

$$(2.12) \quad s^2(u_0, v)_{\mathcal{H}} + (\overline{Q_0 G_1} u_0, \overline{Q_0 G_1} v)_{\mathcal{G}} = (f_0, v)_{\mathcal{H}}.$$

When

$$(2.13) \quad \text{Ker } G_0 \cap D(G_1) \text{ is dense in } (\text{Ker } G_0 \cap \mathcal{V}_0; \|\cdot\|_{\mathcal{V}_0}),$$

then  $u_0$  is uniquely defined by (2.12).

*Proof.* Since  $f_\delta$  weakly converges in  $\mathcal{H}$  it is uniformly bounded in  $\mathcal{H}$  hence, from (2.10), there exists  $C > 0$  independent of  $\delta$  such that

$$(2.14) \quad \|u_\delta\|_{\mathcal{H}}^2 + \|G_\delta u_\delta\|_{\mathcal{G}}^2 \leq C.$$

From (2.5) we deduce that there exists  $u_0 \in \text{Ker } G_0$  such that, up to a subsequence,

$$u_\delta \rightharpoonup_{\mathcal{H}} u_0.$$

For that subsequence, writing  $I = (I - Q_0) + Q_0$  and with the orthogonality property of  $Q_0$  we have

$$\|u_\delta\|_{\mathcal{H}}^2 + \|\overline{Q_0 G_1} u_\delta\|_{\mathcal{G}}^2 \leq \|u_\delta\|_{\mathcal{H}}^2 + \|Q_0 G_\delta u_\delta\|_{\mathcal{G}}^2 + \|(I - Q_0) G_\delta u_\delta\|_{\mathcal{G}}^2 \leq C.$$

Hence,  $u_\delta$  is uniformly bounded in  $\mathcal{V}_0$  and, up to a subsequence,

$$(2.15) \quad u_\delta \rightharpoonup_{\mathcal{V}_0} u_0.$$

For that subsequence, we choose  $v \in \mathcal{D}(G_0) \cap D(G_1)$  in (2.9), we have, setting  $g_\delta = G_\delta u_\delta$ ,

$$(2.16) \quad (g_\delta, G_0 v)_{\mathcal{G}} = \delta(f_\delta, v)_{\mathcal{H}} - \delta s^2(u_\delta, v)_{\mathcal{H}} - \delta(g_\delta, G_1 v)_{\mathcal{G}}.$$

Estimate (2.14) shows that  $g_\delta$  is uniformly bounded in  $\mathcal{G}$ , hence, using (2.16) we can apply Theorem 2.3: there exists  $g_0 \in \text{Ker } G_0^*$  such that, again, up to a subsequence,

$$g_\delta \rightharpoonup_{\mathcal{G}} g_0.$$

Finally, we choose  $v \in \text{Ker } G_0 \cap \mathcal{D}(G_1)$  in (2.9), we have, passing to the limit

$$(2.17) \quad s^2(u_\delta, v)_{\mathcal{H}} + (G_\delta u_\delta, G_1 v)_{\mathcal{G}} = (f_\delta, v)_{\mathcal{H}} \Rightarrow s^2(u_0, v)_{\mathcal{H}} + (g_0, G_1 v)_{\mathcal{G}} = (f_0, v)_{\mathcal{H}}.$$

It remains to characterize  $g_0$ . We have, thanks to (2.15),

$$Q_0 g_0 = Q_0 G_\delta u_\delta = \overline{Q_0 G_1} u_\delta \Rightarrow Q_0 g_0 = \overline{Q_0 G_1} u_0$$

Since  $Q_0 g_0 = g_0$ , we have characterized  $g_0$ . We obtain (2.12) using (2.17) and by writing that, for all  $v \in \text{Ker } G_0 \cap \mathcal{D}(G_1)$ , we have

$$(g_0, G_1 v)_{\mathcal{G}} = (g_0, Q_0 G_1 v)_{\mathcal{G}} = (g_0, \overline{Q_0 G_1} v)_{\mathcal{G}} = (\overline{Q_0 G_1} u_0, \overline{Q_0 G_1} v)_{\mathcal{G}}.$$

When (2.13) holds, by density, one can show that  $u_0$  satisfies (2.12) for every test function in  $\text{Ker } G_0 \cap \mathcal{V}_0$ . One can conclude using the Lax-Milgram theorem. Note that, in that case, each subsequence converges towards the unique solution to problem (2.12).  $\square$

**2.2.2. Quasi-static solutions and propagating solutions.** Assume for this section that (2.13) holds. We introduce the space  $\mathcal{R}_0$  related to the kernel of  $\overline{Q_0 G_1}$ . More precisely, we define

$$(2.18) \quad \mathcal{R}_0 := \text{Ker } \overline{Q_0 G_1} \cap \text{Ker } G_0 \cap \mathcal{V}_0 = \text{Ker } \overline{Q_0 G_1} \cap \text{Ker } G_0.$$

Since by definition  $\overline{Q_0 G_1}$  is closed and densely defined in  $\mathcal{H}$  its kernel is closed in  $\mathcal{H}$ . It is also closed in  $\mathcal{V}_0$  since, for all  $r \in \text{Ker } \overline{Q_0 G_1}$  we have  $\|r\|_{\mathcal{V}_0} = \|r\|_{\mathcal{H}}$ . We deduce then that the space  $\mathcal{R}_0$  is a closed subspace of  $\text{Ker } G_0 \cap \mathcal{V}_0$ . We can therefore define  $\mathcal{S}_0$  as the orthogonal complement of  $\mathcal{R}_0$  in that space. We have

$$(2.19) \quad \text{Ker } G_0 \cap \mathcal{V}_0 = \mathcal{R}_0 \oplus \mathcal{S}_0,$$

where the decomposition is orthogonal with respect to the scalar product in  $\mathcal{V}_0$ , more precisely, we have

$$\forall (r, s) \in \mathcal{R}_0 \times \mathcal{S}_0, \quad 0 = (r, s)_{\mathcal{V}_0} = (r, s)_{\mathcal{H}} + (\overline{Q_0 G_1} r, \overline{Q_0 G_1} s)_{\mathcal{G}} = (r, s)_{\mathcal{H}},$$

hence  $\mathcal{R}_0$  and  $\mathcal{S}_0$  are also orthogonal with respect to the scalar product in  $\mathcal{H}$ . The space  $\mathcal{R}_0$  is called the space of quasi-static solutions whereas the space  $\mathcal{S}_0$  is called the space of propagating solutions. This terminology is justified below. For all  $u_0$  solution to (2.12) we have

$$u_0 = r_0 + s_0$$

where  $r_0 \in \mathcal{R}_0$  and  $s_0 \in \mathcal{S}_0$  are solutions to

$$\begin{cases} \forall r \in \mathcal{R}_0, & s^2(r_0, r)_{\mathcal{H}} = (f_0, r)_{\mathcal{H}}, \\ \forall v \in \mathcal{S}_0, & s^2(s_0, v)_{\mathcal{H}} + (\overline{Q_0 G_1} s_0, \overline{Q_0 G_1} v)_{\mathcal{G}} = (f_0, v)_{\mathcal{H}}. \end{cases}$$

Up to a factor  $s^2$ ,  $r_0$  is the projection of  $f_0$  on the space  $\mathcal{R}_0$  whereas  $s_0$  satisfies an abstract wave propagation problem similar to (2.12).

**2.2.3. Extensions: Lower order terms and more general source term.**

several generalizations of the results given in the previous section can be studied. We focus here on the following wave propagation problem: for  $\ell_\delta \in \mathcal{L}(\mathcal{V}_0)$  given, find  $u_\delta \in \mathcal{V}_\delta$  such that, for all  $v \in \mathcal{V}_\delta$ ,

$$(2.20) \quad s^2(u_\delta, v)_{\mathcal{H}} + (G_\delta u_\delta, G_\delta v)_{\mathcal{G}} + sb(u_\delta, v) = \ell_\delta(v).$$

This problem involves, on the one hand, a sesquilinear form  $b(\cdot, \cdot)$ , that is continuous and positive in  $\mathcal{V}_0$ : for all  $u \in \mathcal{V}_0$  and  $v \in \mathcal{V}_0$ ,

$$(2.21) \quad |b(u, v)| \leq C_b \|u\|_{\mathcal{V}_0} \|v\|_{\mathcal{V}_0}, \quad \Re(b(u, u)) \geq 0,$$

and, on the other hand, a sequence of anti-linear form  $\ell_\delta \in \mathcal{L}(\mathcal{V}_0)$  such that, for all,

$$(2.22) \quad \forall v \in \mathcal{V}_0, \quad |\ell_\delta(v)| \leq c_\ell \|v\|_{\mathcal{H}} + C_\ell \|\overline{Q_0 G_1} v\|_{\mathcal{G}}.$$

Solutions of (2.20) satisfy the following estimate (it is proven using  $v = su$  as a test function),

$$(2.23) \quad \|s u_\delta\|_{\mathcal{H}}^2 + \|G_\delta u_\delta\|_{\mathcal{G}}^2 \leq \frac{1}{\eta^2} (c_\ell + |s|C_\ell)^2.$$

The sesquilinear form  $b$  allows to take into account problems with dissipation, viscosity and/or couplings (see Section 3.3 on piezoelectricity for an application example). The anti-linear form  $\ell_\delta$  is introduced to take into account problems with boundary data for instance. Both  $\ell_\delta$  and  $b$  may depend on  $s$  (as well as the  $c_\ell$ ,  $C_\ell$  and  $C_b$ ) in the case of equations obtained from the Laplace transform of time domain problems. We omit this dependence to simplify the reading. Theorem 2.6 can be extended to this case rather straightforwardly (the proof of the result below is therefore omitted).

**THEOREM 2.7.** *Assume that there exists an anti-linear form  $\ell_0 \in \mathcal{L}(\mathcal{V}_0)$  such that*

$$\forall v \in \text{Ker } G_0 \cap D(G_1), \quad \ell_\delta(v) \xrightarrow{\delta \rightarrow 0} \ell_0(v)$$

*then, up to a subsequence,*

$$u_\delta \xrightarrow{\mathcal{V}_0} u_0.$$

*where  $u_0 \in \text{Ker } G_0 \cap \mathcal{V}_0$ , satisfies, for all  $v \in \text{Ker } G_0 \cap D(G_1)$ ,*

$$(2.24) \quad s^2(u_0, v)_{\mathcal{H}} + (\overline{Q_0 G_1} u_0, \overline{Q_0 G_1} v)_{\mathcal{G}} + sb(u_0, v) = \ell_0(v).$$

*When  $\text{Ker } G_0 \cap D(G_1)$  is dense in  $(\text{Ker } G_0 \cap \mathcal{V}_0; \|\cdot\|_{\mathcal{V}_0})$ , then  $u_0$  is uniquely defined by (2.24).*

**2.2.4. Strong convergence.**

**THEOREM 2.8.** *Assume that there exists an anti-linear form  $\ell_0 \in \mathcal{L}(\mathcal{V}_0)$ , such that*

$$(2.25) \quad \|\ell_\delta - \ell_0\|_{\mathcal{L}(\mathcal{V}_0)} \xrightarrow{\delta \rightarrow 0} 0,$$

*and assume that (2.13) holds, then, the sequence of solutions  $\{u_\delta\} \subset \mathcal{V}_0$  to (2.20), satisfies*

$$(2.26) \quad u_\delta \xrightarrow{\mathcal{V}_0} u_0,$$

*where  $u_0$  is the unique solution to (2.24).*



*Proof.* The following equivalent norm in  $\mathcal{V}_0$  can be defined: for all  $u \in \mathcal{V}_0$ ,

$$|||u|||^2 = \|su\|_{\mathcal{H}}^2 + \|\overline{Q_0 G_1} u\|_{\mathcal{G}}^2 + \frac{|s|^2}{\eta} \Re e(b(u, u)).$$

Using  $su_\delta$  and  $su_0$  as test functions in (2.20) and (2.24) respectively gives

$$\begin{cases} |||u_\delta|||^2 + \|(I - Q_0)G_\delta u_\delta\|_{\mathcal{G}}^2 &= \frac{1}{\eta} \Re e(\ell_\delta(su_\delta)), \\ |||u_0|||^2 &= \frac{1}{\eta} \Re e(\ell_0(su_0)). \end{cases}$$

The first equality above is obtained using,

$$\|G_\delta u_\delta\|_{\mathcal{G}}^2 = \|Q_0 G_\delta u_\delta\|_{\mathcal{G}}^2 + \|(I - Q_0)G_\delta u_\delta\|_{\mathcal{G}}^2 = \|\overline{Q_0 G_1} u_\delta\|_{\mathcal{G}}^2 + \|(I - Q_0)G_\delta u_\delta\|_{\mathcal{G}}^2.$$

Since  $u_\delta$  converges weakly to  $u_0$  in  $\mathcal{V}_0$  (Theorem 2.6) and since, by assumption,  $\Re e(\ell_\delta(su_\delta))$  converges to  $\Re e(\ell_0(su_0))$  we have

$$\lim_{\delta \rightarrow 0} |||u_\delta|||^2 \leq \lim_{\delta \rightarrow 0} \frac{1}{\eta} \Re e(\ell_\delta(su_\delta)) = \frac{1}{\eta} \Re e(\ell_0(su_0)) = |||u_0|||^2.$$

However, thanks to the weak convergence property in  $\mathcal{V}_0$  as well as the equivalence of the norm  $||| \cdot |||$  in this space, we have  $|||u_0|||^2 \leq \liminf_{\delta \rightarrow 0} |||u_\delta|||^2$ . This shows that

$$|||u_\delta||| \xrightarrow{\delta \rightarrow 0} |||u_0|||,$$

which concludes the proof since, weak convergence and convergence of the norm imply strong convergence.  $\square$

**2.3. Scale separations.** In this section two definitions are introduced (strong and strict scale separations). They correspond to simplifying assumptions on  $G_0$  and  $G_1$  in a formal sequence of two-scale operators. Note that both notions are illustrated Section 3.

### 2.3.1. Strong scale separation.

**DEFINITION 2.9.** *A formal sequence of two-scale operators  $G_\delta$  satisfies the strong scale separation property when*

$$(2.27) \quad D(G_0) \cap D(G_1) \text{ dense in } (D(G_0); \|\cdot\|_{\mathcal{H}} + \|G_0 \cdot\|_{\mathcal{G}})$$

and

$$\text{Im } G_0 \perp \text{Im } G_1,$$

where the orthogonality holds for the scalar product in  $\mathcal{G}$ .

**THEOREM 2.10.** *Any formal sequence of two-scale operators  $G_\delta$  that satisfies the strong scale separation property is a sequence of two-scale operators satisfying the weak scale separation property,  $Q_0 G_1 = G_1$  and*

$$\begin{cases} \mathcal{V}_\delta = D(G_0) \cap D(G_1), \\ \mathcal{V}_0 = D(G_1), \\ \mathcal{R}_0 = \text{Ker } G_1 \cap \text{Ker } G_0, \end{cases}$$

and (2.13) holds.

Note that when the strong scale separation holds the problem (2.20) under study reads: find  $u_\delta \in D(G_0) \cap D(G_1)$  such that, for all  $v \in D(G_0) \cap D(G_1)$ ,

$$(2.28) \quad s^2(u_\delta, v)_{\mathcal{H}} + \delta^{-2} (G_0 u_\delta, G_0 v)_{\mathcal{G}} + (G_1 u_\delta, G_1 v)_{\mathcal{G}} + sb(u_\delta, v) = \ell_\delta(v),$$

whereas the limit problem (2.24) simplifies to: find  $u_0 \in \text{Ker } G_0 \cap D(G_1)$ , such that, for all  $v \in \text{Ker } G_0 \cap D(G_1)$ ,

$$(2.29) \quad s^2(u_0, v)_{\mathcal{H}} + (G_1 u_0, G_1 v)_{\mathcal{G}} + sb(u_0, v) = \ell_0(v).$$

Both problems (2.28) and (2.29) have a unique solution by application of the Lax-Milgram theorem.

### 2.3.2. Strict scale separation.

DEFINITION 2.11. *A sequence of formal two-scale operators  $G_\delta$  satisfies the strict scale separation property when  $\text{Im } G_0 \perp \text{Im } G_1$  and  $\text{Im } G_0^* \perp \text{Im } G_1^*$ , where the orthogonality holds for the scalar product in  $\mathcal{G}$  and  $\mathcal{H}$  respectively.*

THEOREM 2.12. *Sequences of formal two-scale operators  $G_\delta$  satisfying the strict scale separation property satisfy the strong scale separation property and  $G_1 P_0 = G_1$ .*

In Theorem 2.12 above,  $P_0$  is the orthogonal projection operator in the kernel of  $G_0$ . It is the self-adjoint operator defined by,

$$P_0 \in \mathcal{L}(\mathcal{H}), \quad P_0^2 = P_0 \quad (u - P_0 u, v)_{\mathcal{H}} = 0 \quad \forall (u, v) \in \mathcal{H} \times \text{Ker } G_0.$$

**3. Applications.** For illustration, two examples in elastic wave propagation are discussed, a problem in piezo-electricity. In this section,  $\Omega \subset \mathbb{R}^3$  is a Lipschitz bounded domain. The space  $L^2(\Omega; \mathbf{C})$  is the space of second order symmetric tensor equipped with the scalar product  $(\cdot, \cdot)_{L^2(\Omega; \mathbf{C})} = (\mathbf{C} \cdot, \cdot)_{L^2(\Omega)^{3 \times 3}}$ , where  $\mathbf{C}$  is a space dependent, uniformly coercive, elasticity tensor. Below,  $u$  is a vector field,  $\varepsilon(u)$  its symmetrized gradient.

**3.1. Nearly-incompressible elastodynamics.** After non-dimensionnalization the problems of wave propagation in nearly incompressible reads: for  $f \in L^2(\Omega)^3$  given, find  $u_\delta \in H^1(\Omega)^3$  such that, for all  $v \in H^1(\Omega)^3$ ,

$$s^2 \int_{\Omega} \rho u_\delta \cdot \bar{v} \, dx + \delta^{-2} \int_{\Omega} \operatorname{div} u_\delta \operatorname{div} \bar{v} \, dx + \int_{\Omega} \mathbf{C} \varepsilon(u_\delta) : \varepsilon(\bar{v}) \, dx = \int_{\Omega} \rho f \cdot \bar{v} \, dx.$$

We define  $\mathcal{H} = L^2(\Omega)^3$  equipped with the scalar product  $(\cdot, \cdot)_{\mathcal{H}} = (\rho \cdot, \cdot)_{L^2(\Omega)^3}$ . We also define  $\mathcal{G} = L^2(\Omega) \times L^2(\Omega; \mathbf{C})$ . We introduce  $G_0$  and  $G_1$ , closed, densely defined operators from  $\mathcal{H}$  to  $\mathcal{G}$ ,

$$D(G_0) = H(\operatorname{div}, \Omega), \quad G_0 u = \begin{pmatrix} \operatorname{div} u \\ 0 \end{pmatrix}, \quad D(G_1) = H^1(\Omega)^3, \quad G_1 u = \begin{pmatrix} 0 \\ \varepsilon(u) \end{pmatrix}.$$

Note that  $H(\operatorname{div}, \Omega)$  is the space of function with square integrable divergence. The operators  $\delta^{-1}G_0 + G_1$  constitute a sequence of formal two-scale operators. It is easy to verify that  $\operatorname{Im} G_0 \perp \operatorname{Im} G_1$ . Moreover, from [10] we have that  $D(G_0) \cap D(G_1) = H^1(\Omega)^3$  is dense in  $D(G_0) = H(\operatorname{div}, \Omega)$ . We have thus verified that the strong scale property holds (hence in particular  $\mathcal{V}_0 = D(G_1)$ ). The limit problem is given by: find  $u_0 \in \mathcal{V}_0^0$  such that, for all  $v \in \mathcal{V}_0^0$ ,

$$s^2(u_0, v) + (G_1 u_0, G_1 v)_{\mathcal{G}} = s^2(u_0, v) + (\varepsilon(u_0), \varepsilon(v))_{L^2(\Omega; \Sigma)} = (f, v)_{\mathcal{H}},$$

where

$$\mathcal{V}_0^0 = \operatorname{Ker} G_0 \cap D(\overline{Q_0 G_1}) = \operatorname{Ker} G_0 \cap D(G_1) = \{u \in H^1(\Omega)^3 \mid \operatorname{div} u = 0\}.$$

We also have

$$\mathcal{R}_0 = \operatorname{Ker} G_1 \cap \operatorname{Ker} G_0 = \{u \in H^1(\Omega)^3 \mid \varepsilon(u) = 0\},$$

namely,  $\mathcal{R}_0$  is the finite dimensional space of rigid body motions.

**3.2. Elastodynamics in plates.** We are now concerned with the elastic wave propagation in plates with thicknesses – in the  $x_3$ –direction – proportional to  $\delta$ . After nondimensionnalisation, the elastodynamics equations are written in a fixed domain  $\Omega = S \times [-1, 1]$  where  $S \subset \mathbb{R}^2$  is a Lipschitz bounded domain. For simplicity, we consider that the problem is isotropic, i.e.,

$$\forall \varepsilon \in \Sigma, \quad \mathbf{C} \varepsilon = \lambda \sum_{j=1}^3 \varepsilon_{jj} I + 2\mu \varepsilon,$$

where  $I$  is the identity tensor in  $\Sigma$  and where  $\lambda$  and  $\mu$  are the space dependent Lamé parameters of the medium. The problem to solve reads: for  $f \in L^2(\Omega)^3$  given, find

$u_\delta \in H^1(\Omega)^3$  such that, for all  $v \in H^1(\Omega)^3$ ,

$$s^2 \int_{\Omega} \rho u_\delta \cdot \bar{v} \, dx + \int_{\Omega} \mathbf{C} (\delta^{-1} \varepsilon_3(u) + \varepsilon_\sigma(u)) : (\delta^{-1} \varepsilon_3(\bar{v}) + \varepsilon_\sigma(\bar{v})) \, dx = \int_{\Omega} \rho f \cdot \bar{v} \, dx.$$

where

$$\varepsilon_\sigma(u) = \frac{1}{2} \begin{pmatrix} 2\partial_1 u_1 & \partial_1 u_2 + \partial_2 u_1 & \partial_1 u_3 \\ \partial_1 u_2 + \partial_2 u_1 & 2\partial_2 u_2 & \partial_2 u_3 \\ \partial_1 u_3 & \partial_2 u_3 & 0 \end{pmatrix},$$

and

$$\varepsilon_3(u) = \frac{1}{2} \begin{pmatrix} 0 & 0 & \partial_3 u_1 \\ 0 & 0 & \partial_3 u_2 \\ \partial_3 u_1 & \partial_3 u_2 & 2\partial_3 u_3 \end{pmatrix}.$$

Note that  $\varepsilon(u) = \varepsilon_3(u) + \varepsilon_\tau(u)$ . As in Section 3.1, in order to write these equations in the abstract setting the space  $\mathcal{H}$  is defined by  $\mathcal{H} = L^2(\Omega)^3$  and  $\mathcal{G}$  by  $\mathcal{G} = L^2(\Omega; \Sigma)$ . Moreover,

$$D(G_0) = \{u \in L^2(\Omega)^3 \mid \varepsilon_3(u) \in L^2(\Omega; \Sigma)\} \quad G_0 u = \varepsilon_3(u),$$

and

$$D(G_1) = \{u \in L^2(\Omega)^3 \mid \varepsilon_\tau(u) \in L^2(\Omega; \Sigma)\} \quad G_1 u = \varepsilon_\tau(u),$$

One can show that  $\delta^{-1}G_0 + G_1$  is closable (it is in fact closed), densely defined, and

$$G_\delta = \overline{\delta^{-1}G_0 + G_1}$$

is defined by

$$D(G_\delta) = \{u \in L^2(\Omega)^3 \mid \varepsilon(u) \in L^2(\Omega; \Sigma)\} = H^1(\Omega)^3,$$

and, for all  $u \in H^1(\Omega)^3$ ,

$$G_\delta u = \delta^{-1} \varepsilon_3(u) + \varepsilon_\sigma(u).$$

Note that the Korn's inequality has been used to deduce that  $D(G_\delta) = H^1(\Omega)^3$ . The problem now reads: find  $u_\delta \in H^1(\Omega)^3$  such that, for all  $v \in H^1(\Omega)^3$ ,

$$s^2(u, v)_\mathcal{H} + (G_\delta u, G_\delta v)_\mathcal{G} = (f, v)_\mathcal{H}.$$

To check that  $G_\delta$  is a sequence of two-scale operators the adjoints  $G_0$  and  $G_1$  have to be computed. In order to define these adjoints, the following divergence operators are introduced

$$\forall \varepsilon \in L^2(\Omega; \mathbf{C}), \quad \mathbf{Div}_3 = \sum_{j=1}^3 \partial_3 \varepsilon_{j3} \mathbf{e}_j, \quad \mathbf{Div}_\sigma = \mathbf{Div} - \mathbf{Div}_3,$$

where  $\mathbf{e}_j$  is the  $j$ th element of the canonical basis and  $\mathbf{Div}$  is the standard divergence operator of a second order tensor. The boundary of  $\Omega$  is decomposed using the following two disjoints subdomains

$$\Gamma_S = S \times \{-1, 1\}, \quad \Gamma_\sigma = \partial S \times (-1, -1).$$

The unitary outward normal on  $\Gamma_\sigma$  is denoted  $\mathbf{n}_\sigma$ . With these notations, the adjoints are given by

$$\begin{cases} D(G_0^*) = \{\varepsilon \in L^2(\Omega; \mathbf{C}) \mid \mathbf{Div}_3(\mathbf{C}\varepsilon) \in L^2(\Omega)^3, (\mathbf{C}\varepsilon)\mathbf{e}_3 = 0 \text{ on } \Gamma_S\}, \\ D(G_1^*) = \{\varepsilon \in L^2(\Omega; \mathbf{C}) \mid \mathbf{Div}_\sigma(\mathbf{C}\varepsilon) \in L^2(\Omega)^3, (\mathbf{C}\varepsilon)\mathbf{n}_\sigma = 0 \text{ on } \Gamma_\sigma\}. \end{cases}$$



The small parameter  $\delta$  represents the ratio of velocities of waves in solids and electromagnetic waves. The space  $H_0(\mathbf{curl}; \Omega)$  is the space of functions with square integrable rotational and vanishing tangential trace. By setting  $\mathcal{H} = L^2(\Omega)^3 \times L^2(\Omega)^3$  and  $\mathcal{G} = L^2(\Omega)^3 \times L^2(\Omega; \Sigma)$  and defining

$$D(G_\delta) = \left\{ \Phi = \begin{pmatrix} \mathbf{E} \\ \mathbf{v} \end{pmatrix} \in H_0(\mathbf{curl}; \Omega_E) \times H^1(\Omega)^3 \right\}, \quad G_\delta \Phi = G_\delta \begin{pmatrix} \mathbf{E} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \delta^{-1} \mathbf{curl} \mathbf{E} \\ \varepsilon(\mathbf{v}) \end{pmatrix},$$

problem (3.2) can be written as: for  $\Psi = (F, J)^t \in \mathcal{H}$  given, find  $\Phi_\delta \in D(G_\delta)$  such that, for all  $\tilde{\Phi} \in D(G_\delta)$ ,

$$(3.3) \quad s^2(\Phi_\delta, \tilde{\Phi})_{\mathcal{H}} + (G_\delta \Phi_\delta, G_\delta \tilde{\Phi})_{\mathcal{G}} + b(s\Phi_\delta, \tilde{\Phi}) = (\Psi, \tilde{\Phi})_{\mathcal{H}},$$

with, for all  $\Phi = \begin{pmatrix} E \\ v \end{pmatrix} \in D(G_\delta)$  and  $\tilde{\Phi} = \begin{pmatrix} \tilde{E} \\ \tilde{v} \end{pmatrix} \in D(G_\delta)$ ,

$$b(\Phi, \tilde{\Phi}) = (\varepsilon(v), \mathbf{d} \tilde{E})_{L^2(\Omega)^3} - (\varepsilon(\tilde{v}), \mathbf{d} E)_{L^2(\Omega)^3}.$$

In order to show that (3.3) is a two scale propagation problem as defined by (2.20) one needs to show first that  $G_\delta$  is a sequence of two-scale operators satisfying the weak scale separation property, second that  $b$  is a continuous sesquilinear form in  $\mathcal{V}_0$ . The operator  $G_0$  and  $G_1$  are defined by

$$D(G_0) = H_0(\mathbf{curl}; \Omega) \times L^2(\Omega)^3, \quad \forall \begin{pmatrix} E \\ v \end{pmatrix} \in D(G_0), \quad G_0 \begin{pmatrix} E \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{curl} E \\ 0 \end{pmatrix}$$

and

$$D(G_1) = L^2(\Omega)^3 \times H_0^1(\Omega)^3, \quad \forall \begin{pmatrix} E \\ v \end{pmatrix} \in D(G_1) \quad G_1 \begin{pmatrix} E \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon(v) \end{pmatrix}.$$

They are closed, densely defined operators. One can verify that the formal sequence of two-scale operators  $G^\delta = \delta^{-1}G_0 + G_1$  satisfies the strict scale separation property hence the weak scale separation property. In particular  $\overline{Q_0 G_1} = G_1$  and

$$\mathcal{V}_0 = D(\overline{Q_0 G_1}) = D(G_1).$$

One can then verify easily that the sesquilinear form  $b$  is indeed continuous in  $\mathcal{V}_0$ . The limit problem reads: for  $\Psi = (F, J)^t \in \mathcal{H}$  given, find  $\Phi_0 = (E_0, v_0)^t \in \text{Ker } G_0 \cap D(G_1)$  such that, for all  $\tilde{\Phi} \in \text{Ker } G_0 \cap D(G_1)$ ,

$$(3.4) \quad s^2(\Phi_0, \tilde{\Phi})_{\mathcal{H}} + (\varepsilon(v_\delta), \varepsilon(\tilde{v}))_{L^2(\Omega; \Sigma)} + b(s\Phi_0, \tilde{\Phi}) = (\Psi, \tilde{\Phi})_{\mathcal{H}},$$

where, thanks to [10] (Chapter 1, Theorem 2.7 and Corrolary 2.9),

$$\text{Ker } G_0 = \left\{ \begin{pmatrix} E \\ v \end{pmatrix} \in \mathcal{H} \mid \mathbf{curl} E = 0 \text{ in } \Omega \right\} = \nabla H_\gamma^1(\Omega) \times L^2(\Omega)^3.$$

with

$$H_\gamma^1(\Omega) = \{\varphi \in H^1(\Omega)/\mathbb{R} \mid \varphi \text{ is constant along } \partial\Omega\}.$$

Therefore introducing  $\varphi_0 \in H_\gamma^1(\Omega)$  such that  $E_0 = \nabla \varphi_0$  and  $u_0 = v_0/s$ , the variational formulation of the limit problem can be simplified. It reads: for  $(F, J)^t \in \mathcal{H}$  given, find  $(\varphi_0, u_0) \in H_\gamma^1(\Omega) \times H_0^1(\Omega)^3$  such that, for all,  $(\tilde{\varphi}, \tilde{u}) \in H_\gamma^1(\Omega) \times H_0^1(\Omega)^3$ ,

$$\begin{cases} (\nabla \varphi_0, \nabla \tilde{\varphi})_{L^2(\Omega)^3} + (\varepsilon(u_0), \mathbf{d} \nabla \tilde{\varphi})_{L^2(\Omega)^3} = (s^{-2}J, \tilde{E})_{L^2(\Omega)^3}, \\ s^2(u_0, \tilde{v})_{L^2(\Omega)^3} + (\varepsilon(u_0), \varepsilon(\tilde{v}))_{L^2(\Omega; \Sigma)} - (\varepsilon(\tilde{v}), \mathbf{d} \nabla \varphi_0)_{L^2(\Omega)^3} = (s^{-1}F, \tilde{v})_{L^2(\Omega)^3}. \end{cases}$$

These last equations are classically known as the quasi-static piezoelectric equations.

**4. Convergence estimates.** As in the proof of Theorem 2.8 the following equivalent norm in  $\mathcal{V}_0$  is introduced: for all  $u \in \mathcal{V}_0$ ,

$$\|u\|^2 = \|su\|_{\mathcal{H}}^2 + \|\overline{Q_0 G_1} u\|_{\mathcal{G}}^2 + \frac{|s|^2}{\eta} \Re(b(u, u)).$$

#### 4.1. Strict scale separation.

**THEOREM 4.1.** *Let the strict scale separation property holds and assume that the range of  $G_0$  is closed, namely there exists  $C_p > 0$  such that*

$$(4.1) \quad \forall u \in \text{Ker } G_0^\perp \cap D(G_0), \quad \|u\|_{\mathcal{H}} \leq C_p \|G_0 u\|_{\mathcal{G}}.$$

Assume, moreover, that there exists an anti-linear form  $\ell_0 \in \mathcal{L}(D(G_1))$  such that

$$\forall v \in \text{Ker } G_0 \cap D(G_1), \quad |\ell_\delta(v) - \ell_0(v)| \leq c_{\ell, \delta} \|v\|_{\mathcal{H}} + C_{\ell, \delta} \|G_1 v\|_{\mathcal{G}}.$$

Then, the sequence of solutions  $\{u_\delta\} \subset D(G_0) \cap D(G_1)$  to (2.28), satisfies

$$(4.2) \quad (\|u_\delta - u_0\|^2 + \delta^{-2} \|G_0 u_\delta\|_{\mathcal{G}}^2)^{\frac{1}{2}} \leq C_\delta$$

where

$$C_\delta = \frac{1}{\eta} \max(c_{\ell, \delta}, |s| C_{\ell, \delta}, \delta C_0) \quad \text{with} \quad C_0 = |s| C_p (c_\ell + C_b \|su_0\|_{D(G_1)}).$$

*Proof.*

**Step 1: Decomposition of the solution.** Since  $\text{Ker } G_0$  is a closed subspace of  $\mathcal{H}$ , the solution  $u_\delta \in D(G_0) \cap D(G_1)$  to (2.28) can be decomposed as

$$(4.3) \quad u_\delta = u_{\delta,0} + z_\delta,$$

where  $u_{\delta,0} \in \text{Ker } G_0$  and  $z_\delta \in (\text{Ker } G_0)^\perp$  and the decomposition is orthogonal with respect to the scalar product in  $\mathcal{H}$ . From the strict scale separation property,

$$(\text{Ker } G_0)^\perp = \overline{\text{Im } G_0^*} \subset \text{Ker } G_1,$$

where the closure is taken for the norm in  $\mathcal{H}$ . This implies that  $z_\delta \in \text{Ker } G_1$ , and therefore

$$u_{\delta,0} \in \text{Ker } G_0 \cap D(G_1) \quad \text{and} \quad z_\delta \in (\text{Ker } G_0)^\perp \cap \text{Ker } G_1.$$

**Step 2: Energy identity for the error equation.** The error term

$$e_\delta = u_{\delta,0} - u_0 \in \text{Ker } G_0 \cap D(G_1),$$

is introduced. From the decomposition (4.3) it can be deduced that, for all  $v \in \text{Ker } G_0 \cap D(G_1)$ ,

$$(4.4) \quad s^2(e_\delta, v)_{\mathcal{H}} + (G_1 e_\delta, G_1 v) + b(se_\delta + sz_\delta, v) = \ell_\delta(v) - \ell_0(v).$$

Choosing  $v = se_\delta$  in (4.4) yields, using the orthogonality of  $e_\delta$  and  $z_\delta$  in  $\mathcal{H}$ ,

$$(4.5) \quad s \|se_\delta\|_{\mathcal{H}}^2 + \bar{s} \|G_1 e_\delta\|_{\mathcal{G}}^2 + b(se_\delta + sz_\delta, se_\delta) = (sl_\delta(e_\delta) - \ell_0(se_\delta)).$$

Moreover, choosing  $v = sz_\delta$  in (2.28) yields

$$(4.6) \quad s \|sz_\delta\|_{\mathcal{H}}^2 + \bar{s} \delta^{-2} \|G_0 z_\delta\|_{\mathcal{G}}^2 + b(su_{\delta,0} + sz_\delta, sz_\delta) = \ell_\delta(sz_\delta).$$

**Step 3: Error estimate.** The next step is to sum (4.5) and (4.6), to get

$$(4.7) \quad \begin{aligned} & \|e_\delta + z_\delta\|^2 + \delta^{-2} \|G_0 z_\delta\|_{\mathcal{G}}^2 \\ &= \frac{1}{\eta} \Re(\ell_\delta(se_\delta) - \ell_0(se_\delta) + \ell_\delta(sz_\delta) - b(su_0, sz_\delta)). \end{aligned}$$

where, due to the properties of  $e_\delta$  and  $z_\delta$ ,

$$\|e_\delta + z_\delta\|^2 = \|se_\delta\|_{\mathcal{H}}^2 + \|sz_\delta\|_{\mathcal{H}}^2 + \|G_1 e_\delta\|_{\mathcal{G}}^2 + \frac{1}{\eta} \Re(b(se_\delta + sz_\delta, se_\delta + sz_\delta)).$$

The three terms in the right hand side of Equation (4.7) are now estimated.

$$(4.8) \quad |(\ell_\delta(se_\delta) - \ell_0(se_\delta))| \leq c_{\ell,\delta} \|se_\delta\|_{\mathcal{H}} + C_{\ell,\delta} |s| \|G_1 e_\delta\|_{\mathcal{G}}$$

moreover, since  $z_\delta \in (\text{Ker } G_0)^\perp \cap \text{Ker } G_1$ , and since the ranged of  $G_0$  is closed,

$$(4.9) \quad |\ell_\delta(sz_\delta)| \leq c_\ell \|sz_\delta\|_{\mathcal{H}} \leq c_\ell C_p |s| \|G_0 z_\delta\|_{\mathcal{G}}.$$

Finally, using the continuity property of  $b$  and again the fact that the ranged of  $G_0$  is closed,

$$(4.10) \quad |b(su_0, sz_\delta)| \leq C_b \|su_0\|_{D(G_1)} \|sz_\delta\|_{D(G_1)} \leq C_b C_p |s| \|su_0\|_{D(G_1)} \|G_0 z_\delta\|_{\mathcal{G}}$$

Summing (4.8), (4.9), (4.10), leads to,

$$(4.11) \quad \begin{aligned} \mathcal{E}_\delta &:= \|e_\delta + z_\delta\|^2 + \delta^{-2} \|G_0 z_\delta\|_{\mathcal{G}}^2 \\ &\leq \frac{c_{\ell,\delta}}{\eta} \|se_\delta\|_{\mathcal{H}} + |s| \frac{C_{\ell,\delta}}{\eta} \|G_1 e_\delta\|_{\mathcal{G}} + \frac{|s|}{\eta} C_p (c_\ell + C_b \|su_0\|_{D(G_1)}) \|G_0 z_\delta\|_{\mathcal{G}}. \quad \square \end{aligned}$$

Hence,  $\mathcal{E}_\delta \leq C_\delta \mathcal{E}_\delta^{\frac{1}{2}}$  which concludes the proof.

**4.2. Strong scale separation.** For conciesness, the following notation is introduced

$$\mathcal{V}_\infty = D(G_0) \cap D(G_1).$$

We introduce the bounded operator  $\mathbf{G}_0 : \mathcal{V}_\infty \mapsto \mathcal{G}$  defined by

$$\forall v \in \mathcal{V}_\infty, \quad \mathbf{G}_0 v = G_0 v.$$

The main theorem of this section relies on the assumption that the range of  $\mathbf{G}_0$  is closed. It is equivalent to an *inf-sup condition* (see [5] for more details).

ASSUMPTION 4.2. *The range of  $\mathbf{G}_0$  is closed, namely there exists  $C_c > 0$  such that*

$$(4.12) \quad \forall u \in (\text{Ker } \mathbf{G}_0)^\perp \cap \mathcal{V}_\infty, \quad \|u\|_{\mathcal{V}_\infty} \leq C_c \|\mathbf{G}_0 u\|_{\mathcal{G}},$$

where the orthogonality holds for the scalar product in  $\mathcal{V}_\infty$ .



**THEOREM 4.3.** *Let the strong scale separation property holds, Assumption 4.2 and let assume that there exists an anti-linear form  $\ell_0 \in \mathcal{L}(D(G_1))$  such that*

$$\forall v \in \text{Ker } G_0 \cap D(G_1), \quad |\ell_\delta(v) - \ell_0(v)| \leq c_{\ell,\delta} \|v\|_{\mathcal{H}} + C_{\ell,\delta} \|G_1 v\|_{\mathcal{G}}.$$

*Then, the sequence of solutions  $\{u_\delta\} \subset D(G_0) \cap D(G_1)$  to (2.28), satisfies*

$$(4.13) \quad \left( \|u_\delta - u_0\|^2 + \delta^{-2} \|G_0 u_\delta\|_{\mathcal{G}}^2 \right)^{\frac{1}{2}} \leq C_\delta$$

where

$$C_\delta = \frac{1}{\eta} \max(c_{\ell,\delta}, |s| C_{\ell,\delta}, \delta \|s g_0\|_{\mathcal{G}}),$$

*and  $g_0$  is the Lagrange multiplier associated to the mixed form of the limit problem: find  $(u_0, g_0) \in \mathcal{V}_\infty \times \text{Im } \mathbf{G}_0$  such that for all,  $v \in \mathcal{V}_\infty$ ,*

$$(4.14) \quad \begin{cases} s^2(u_0, v)_{\mathcal{H}} + (G_1 u_0, G_1 v)_{\mathcal{G}} + b(su_\delta, v) + (g_0, G_0 v)_{\mathcal{G}} = \ell_0(v), \\ G_0 u_0 = 0. \end{cases}$$

*Proof.*

**Step 1: Introduction of a Lagrange multiplier.** For all  $v \in \mathcal{V}_\infty$  a functional  $L \in \mathcal{L}(V_\infty)$  is defined by

$$(4.15) \quad L(v) = \ell_0(v) - s^2(u_0, v)_{\mathcal{H}} - (G_1 u_0, G_1 v)_{\mathcal{G}} - b(su_0, v).$$

From the Riesz representation theorem there exists a unique element  $\ell \in \mathcal{V}_\infty$  such that

$$\forall v \in \mathcal{V}_\infty, \quad (\ell, v)_{\mathcal{V}_\infty} = L(v).$$

Observe now that, since (2.29) holds,  $(\ell, v)_{\mathcal{V}_\infty} = 0$  for all

$$v \in \text{Ker } G_0 \cap D(G_1) = \text{Ker } \mathbf{G}_0 \cap D(G_1).$$

Hence,  $\ell \in (\text{Ker } \mathbf{G}_0)^\perp$  (here the orthogonality relation holds for the scalar product in  $\mathcal{V}_\infty$ ). Now, observe that

$$(\text{Ker } \mathbf{G}_0)^\perp = \overline{\text{Im } \mathbf{G}_0^*}.$$

Since the range of  $\mathbf{G}_0$  is closed by Assumption 4.2, the range of  $\mathbf{G}_0^*$  is closed, hence  $\ell$  belongs to the range of  $\mathbf{G}_0^*$ . More precisely,

$$\ell \in \text{Im } \mathbf{G}_0^* \quad \Rightarrow \quad \exists g \in D(\mathbf{G}_0^*) \text{ such that } \ell = \mathbf{G}_0^* g.$$

The result above can be slightly improved. Since  $\mathbf{G}_0^*$  has closed range, there exists  $C_0 > 0$  such that

$$\forall g \in (\text{Ker } \mathbf{G}_0^*)^\perp, \quad \|g\|_{\mathcal{G}} \leq C_0 \|\mathbf{G}_0^* g\|_{\mathcal{V}_\infty},$$

showing that  $\mathbf{G}_0^*$  is injective on  $(\text{Ker } \mathbf{G}_0^*)^\perp = \overline{\text{Im } \mathbf{G}_0} = \text{Im } \mathbf{G}_0$ . Hence the following unique characterization of  $\ell$  is deduced,

$$\ell \in \text{Im } \mathbf{G}_0^* \quad \Rightarrow \quad \exists! g_0 \in \text{Im } \mathbf{G}_0 \text{ such that } \ell = \mathbf{G}_0^* g_0.$$



**5. Proofs of Section 2.** The proofs below uses several results in operator theory that the reader can find in [11] Chap. 12, or [15] Chap 3. Section 5.

**Proof of Theorem 2.3.** Only (2.4)(a) is proven, the property (2.4)(b) is proven similarly. Since  $u_\delta$  is uniformly bounded in  $\mathcal{H}$  it converges, up to a subsequence, weakly to  $u_0$  in  $\mathcal{H}$ , for all  $h \in D(G_0^*) \cap D(G_1^*)$  their holds,

$$(u_\delta, G_0^* h)_\mathcal{H} \xrightarrow{\delta \rightarrow 0} (u_0, G_0^* h)_\mathcal{H}.$$

When  $(u_\delta, G_0^* h)_\mathcal{G} \xrightarrow{\delta \rightarrow 0} 0$  it holds

$$(u_0, G_0^* h)_\mathcal{H} = 0, \quad \forall h \in D(G_0^*) \cap D(G_1^*).$$

By density of  $D(G_0^*) \cap D(G_1^*)$  into  $D(G_0^*)$  for the graph norm of  $G_0^*$  it can be deduced that the equality above holds for all  $h \in D(G_0^*)$ . Since  $G_0$  is closed and densely defined, this shows that  $u_0 \in D(G_0)$  and  $G_0 u_0 = 0$ .

### Proof of Theorem 2.5.

**The operator  $G_1^* Q_0$  is densely defined.** By definition

$$D(G_1^* Q_0) = \{g \in D(Q_0) = \mathcal{G} \mid Q_0 g \in D(G_1^*)\}$$

For all  $g \in \mathcal{G}$ , an orthogonal decomposition in  $\mathcal{G}$  holds. It is given by  $g = g_0 + g_\perp$  with  $g_0 \in \text{Ker } G_0^*$  and  $Q_0 g_\perp = 0$ . Therefore

$$D(G_1^* Q_0) = \{g = g_0 + g_\perp, g_0 \in \text{Ker } G_0^*, g_\perp \in (\text{Ker } G_0^*)^\perp \mid Q_0 g = g_0 \in D(G_1^*)\},$$

which shows that

$$(5.1) \quad D(G_1^* Q_0) = (\text{Ker } G_0^* \cap D(G_1^*)) \oplus (\text{Ker } G_0^*)^\perp.$$

By assumption  $\text{Ker } G_0^* \cap D(G_1^*)$  is dense in  $\text{Ker } G_0^*$  and since

$$\mathcal{G} = \text{Ker } G_0^* \oplus (\text{Ker } G_0^*)^\perp,$$

it ensues that  $D(G_1^* Q_0)$  is dense in  $\mathcal{G}$ .

**The operator  $Q_0 G_1$  is closable.** Since it has been shown that  $G_1^* Q_0$  is densely defined, it has an adjoint. Since  $Q_0$  is self-adjoint the following relation holds,

$$Q_0 G_1 = Q_0^* G_1^{**} \subset (G_1^* Q_0)^*,$$

meaning that  $(G_1^* Q_0)^*$  is a (closed) extension of  $Q_0 G_1$ . This last property implies that  $Q_0 G_1$  is closable.

**Closure of  $Q_0 G_\delta$ .** Thanks to (2.3), the operator  $\delta^{-1} G_0 + G_1$  is densely defined, its adjoint satisfies

$$G_\delta^* = (\delta^{-1} G_0 + G_1)^* = \overline{\delta^{-1} G_0 + G_1^*} \supset (\delta^{-1} G_0^* + G_1^*)$$

Therefore,

$$(5.2) \quad G_1^* Q_0 = (\delta^{-1} G_0^* + G_1^*) Q_0 \subset G_\delta^* Q_0 = (Q_0 G_\delta)^*.$$

It can be deduced, first, from (5.2) that  $(Q_0 G_\delta)^*$  is also densely defined and therefore  $Q_0 G_\delta$  is closable; second, from (5.2), and the property that  $Q_0$  is bounded and self-adjoint, that

$$\overline{Q_0 G_\delta} = (Q_0 G_\delta)^{**} \subset (G_1^* Q_0)^* = (Q_0 G_1)^{**} = \overline{Q_0 G_1}.$$

**Proof of Theorem 2.10.** Our objective is to prove that

$$\left\{ \begin{array}{l} Q_0 G_1 = G_1 \\ \text{Ker } G_0^* \cap D(G_1^*) \text{ dense in } \text{Ker } G_0^*, \\ D(G_0^*) \cap D(G_1^*) \text{ dense in } D(G_0^*), \\ \delta^{-1} G_0 + G_1 \text{ is closed.} \end{array} \right.$$

The other statements of the theorem are direct consequences of the above properties.

**The equality  $Q_0 G_1 = G_1$ .** By assumption,

$$\forall (u, v) \in D(G_0) \times D(G_1) \quad (G_0 u, G_1 v)_G = 0.$$

By density, this, implies

$$\forall (g, v) \in \overline{\text{Im}(G_0)} \times D(G_1) \quad (g, G_1 v)_G = 0,$$

where  $\overline{\text{Im}(G_0)}$  is the closure of the range of  $G_0$  for the norm in  $\mathcal{G}$ . Now, since  $I - Q_0$  is, by definition, the orthogonal projection operator on the closure of  $\text{Im } G_0$ , there holds

$$\forall (g, v) \in \mathcal{G} \times D(G_1) \quad ((I - Q_0)g, G_1 v)_G = 0,$$

also meaning that, for all  $v \in D(G_1)$ ,  $G_1 v = Q_0 G_1 v$ .

**Density results.** It has been shown that  $Q_0 G_1 = G_1$  hence  $Q_0 G_1$  is closed and densely defined. Therefore  $(Q_0 G_1)^* = G_1^* Q_0^*$  is closed and densely defined in  $\mathcal{G}$ . More precisely,

$$D(G_1^* Q_0) = (\text{Ker } G_0^* \cap D(G_1^*)) \oplus (\text{Ker } G_0^*)^\perp \text{ dense in } \mathcal{G} = \text{Ker } G_0^* \oplus (\text{Ker } G_0^*)^\perp.$$

From the statement it can be deduced that  $\text{Ker } G_0^* \cap D(G_1^*)$  is dense in  $\text{Ker } G_0^*$ . It must be checked now that  $D(G_0^*) \cap D(G_1^*)$  is dense in  $D(G_0^*)$  for the graph norm of  $G_0^*$ . The following vectorial space is defined,

$$\mathcal{Z} = (\text{Ker } G_0^* \cap D(G_1^*)) \oplus ((\text{Ker } G_0^*)^\perp \cap D(G_0^*)),$$

where the decomposition is orthogonal with respect to the scalar product in  $\mathcal{G}$ . Observe that  $\mathcal{Z}$  is a subspace of  $D(G_0^*) \cap D(G_1^*)$  since by assumption

$$(\text{Ker } G_0^*)^\perp = \overline{\text{Im } G_0} \subset \text{Ker } G_1^*.$$

Observe, moreover, that

$$D(G_0^*) = \text{Ker } G_0^* \oplus ((\text{Ker } G_0^*)^\perp \cap D(G_0^*)),$$

where the decomposition is orthogonal with respect to the scalar product in  $\mathcal{G}$ . However, since it has been shown that  $\text{Ker } G_0^* \cap D(G_1^*)$  is dense in  $\text{Ker } G_0^*$  for the norm  $\mathcal{H}$ , it is therefore dense for the graph norm of  $G_0^*$ . This shows that  $\mathcal{Z}$  (hence  $D(G_0^*) \cap D(G_1^*)$ ) is dense in  $D(G_0^*)$  for the same norm.

**The operator  $\delta^{-1}G_0 + G_1$  is closed.** The graph of the operator  $\delta^{-1}G_0 + G_1$  is given by

$$\mathcal{G}_r(\delta^{-1}G_0 + G_1) = \{(u, g) \in \mathcal{H} \times \mathcal{G} \mid u \in D(G_0) \cap D(G_1), g = \delta^{-1}G_0u + G_1u\}.$$

It must be verified that the graph is closed for the composite norm in  $\mathcal{H} \times \mathcal{G}$ . Let  $\{u_n\}$  be a sequence in  $D(G_0) \cap D(G_1)$  such that, for some  $(u, g) \in \mathcal{H} \times \mathcal{G}$ ,

$$u_n \xrightarrow{\mathcal{H}} u, \quad \delta^{-1}G_0u_n + G_1u_n \xrightarrow{\mathcal{G}} g.$$

This implies

$$Q_0G_1u_n = G_1u_n \xrightarrow{\mathcal{G}} Q_0g.$$

However, since  $G_1$  is closed,  $u \in D(G_1)$  and  $Q_0g = G_1u$ . Consequently

$$u_n \xrightarrow{\mathcal{H}} u, \quad \delta^{-1}G_0u_n \xrightarrow{\mathcal{G}} (I - Q_0)g,$$

and since  $G_0$  is closed,  $u \in D(G_0)$  and  $(I - Q_0)g = \delta^{-1}G_0u$  which conclude the proof since

$$u \in D(G_0) \cap D(G_1) \text{ and } g = \delta^{-1}G_0u + G_1u.$$

**Proof of Theorem 2.12.** Our objective is to prove that

$$\begin{cases} G_1P_0 = G_1, \\ D(G_0) \cap D(G_1) \text{ dense in } D(G_0), \end{cases}$$

using that, by assumption,

$$(5.3) \quad \begin{cases} (\text{Ker } G_1)^\perp = \overline{\text{Im}(G_1^*)} \subset \text{Ker } G_0, & (a) \\ (\text{Ker } G_0)^\perp = \overline{\text{Im}(G_0^*)} \subset \text{Ker } G_1. & (b) \end{cases}$$

As a direct consequence of (5.3)(a) it can be deduced that

$$P_0G_1^* = G_1^* \Rightarrow G_1P_0 = G_1.$$

Therefore  $D(G_1P_0) = \text{Ker } G_0 \cap D(G_1) \oplus (\text{Ker } G_0)^\perp$  is dense in  $\mathcal{G}$ , which implies that

$$(5.4) \quad \text{Ker } G_0 \cap D(G_1) \text{ dense in } \text{Ker } G_0.$$

We define the space

$$\mathcal{W} = (\text{Ker } G_0 \cap D(G_1)) \oplus ((\text{Ker } G_0)^\perp \cap D(G_0)),$$

where the decomposition is orthogonal with respect to the scalar product in  $\mathcal{H}$ . Observe that the space  $\mathcal{W}$  is a subspace of  $D(G_0) \cap D(G_1)$  thanks to (5.3)(b). Moreover,

$$D(G_0) = \text{Ker } G_0 \oplus ((\text{Ker } G_0)^\perp \cap D(G_0)),$$

where the decomposition is orthogonal with respect to the scalar product in  $\mathcal{H}$ . It can be concluded that  $\mathcal{W}$  is dense in  $D(G_0)$  for the graph norm of  $G_0$  using (5.4).

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