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Robust output feedback model predictive control for constrained linear systems via interval observers ⋆

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Abstract

This work addresses the problem of robust output feedback model predictive control for discrete-time, constrained linear systems corrupted by (bounded) state and measurement disturbances. Using the available information on measurements and uncertainty bounds, the objective is to stabilize such a system while robustly respecting the imposed constraints on state and control. To this end, interval observer and predictor with guaranteed performance are incorporated into the classic MPC scheme. This new approach offers advantages such as enlarged feasible regions for the optimal control problem, low computational burden, and ease of design.

Key words: predictive control, output feedback, robust control

1 Introduction

The control of constrained dynamical systems is known to be a very complex (or even impossible) task to be tackled using classic feedback solutions [1]. Since this scenario is recurrent (for instance, actuators are physically limited and many industrial plants have strict safety and reliability requirements), Model Predictive Control (MPC) has been exhaustively studied over the last decades, since it offers a way of handling constrained optimal control problems for multivariate systems [2].

The basic idea behind MPC is to use current information on the state, at each decision instant, to predict its future behaviour through a dynamical model. Coupling this prediction to an optimization problem, the best control sequence that achieves stabilization (while respecting constraints on both states and control signals) is determined, and the first element of this sequence is applied to the system [2].

Due to these features, MPC has grown to be very popular in both industry [3], where thousands of successful applications have been already reported [4], and academic research. Despite its intuitive algorithm, some challenges arise when dealing with uncertain systems [5]: *(i)* the prediction relies on models that are often discrepant regarding the real system, thanks to uncertainties, noise and perturbations, *(ii)* MPC requires full state measurement, which is not always available and leads to the need of estimation tools – adding even more uncertainty to the problem due to estimation errors [6]. Then, an MPC scheme is said to be *robust* if it achieves the control task, with guaranteed satisfaction of the state and control constraints for all possible realizations of a certain range of uncertainties [7].

The latter case characterizes the *output feedback MPC* (OF-MPC) problem and is the framework considered in this paper. Many approaches have been studied to tackle this problem: in [8,9] min-max optimization, which considers the minimization of the cost function and the worst-case of the disturbances, have been proposed. Also, early solutions to such a problem require the repeated solution of Linear Matrix Inequalities (LMIs) [10,11]. These two methods are known to be computationally demanding.

Set-membership estimation (such as Moving Horizon Estimation (MHE)) has also been adapted to MPC [12–14]. Using such estimation methods, the available mea-

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surement is used to compute sets of admissible states (regarding the available dynamics, bounds on the disturbance and initial uncertainty), which are then used in the MPC algorithm to guarantee constraint satisfaction. Although a considerable computational effort is required to update these uncertainty sets, many approximations (such as parallelotopes, ellipsoids or limited-complexity polytopes [12]) are available.

Tube-based approaches [15,16] have also grown popular for their reduced computational complexity. By predicting the trajectory of the nominal system, the controller is designed to enforce that any deviation of the perturbed one remains inside a tube (whose center is the nominal trajectory). These tubes are pre-computed invariant sets [17], leading to an optimal control problem with tightened constraints.

In this work, we propose a solution to the OF-MPC problem by introducing *interval observers* (IO) [18] in the design. These observers – while being a special class of set-membership estimators – use input-output information to compute a guaranteed estimation of the admissible values (*i.e.*, intervals) of the states, at each instant of time. The design of an interval observer is based on the concept of *cooperative* (or *positive*) systems (see [19]). Indeed, the design of IOs has been extensively studied and extended over the last years [20].

Contributions: we propose new open-loop interval estimators with stability guarantees, called interval predictors (IP), as well as a suitable stabilizing feedback control law. Combining these new features with the existing theory of IOs, we propose an MPC algorithm that offers a computationally inexpensive way to cope with uncertainty, avoiding the use of complex set-algebraic or estimation schemes, as usually performed by classical solutions. Furthermore, if compared to these aforementioned solutions, our approach shows *(i)* larger feasible regions since it does not tighten the constraint set to ensure constraint satisfaction, *(ii)* lower conservativeness when propagating the uncertainty over the prediction horizon, due to the obtained intervals are not over-approximated, *(iii)* the resulting algorithm resembles the classical MPC, and its implementation is easier with all parameters being determined by solving (offline) LMIs.

This paper is organized as follows: Section 2 states the problem statement, Section 3 discusses the design and features of the interval estimators, Section 4 proposes the MPC scheme together with its stability and robust constraint satisfaction guarantees, Section 5 illustrates the approach with a numerical example and Section 6 concludes this study, giving also future directions of research.

This work is an extension of a previous conference paper [21], containing new results and improvements such as proofs, the design of different gains for the observer and predictor aiming to obtain better accuracy, the selection of a less conservative envelope of interval estimates, improved terminal set estimate, and revamped comparative examples.

Notation:

- The sets of real and integer numbers are defined by $\mathbb{R}$ and $\mathbb{Z}$, respectively, then $|\cdot|$ represents the absolute value for an element of these sets; $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$.
- A matrix M is said to be non-negative if all of its elements are non-negative. A matrix M is said to be Schur stable if all of its eigenvalues have absolute value less than one. The identity matrix is denoted by $\mathcal{I}_n \in \mathbb{R}^{n\times n}$.
- For a function $x : \mathbb{Z}_+ \to \mathbb{R}^n$, we use the convention $x_k = x(k)$ and denote $|x|_\infty = \sup_{k\in\mathbb{Z}_+} \|x_k\|$. Furthermore, we define as ℓ_∞^n the set of all functions such that $|x|_\infty < \infty$;
- Let $x_1, x_2 \in \mathbb{R}^n$ be two vectors and $A_1, A_2 \in \mathbb{R}^{n\times n}$ be two matrices, then the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are to be understood component-wise. For a matrix A we define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors), and also denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$ the relation $A \prec 0$ (resp. $A \succeq 0$) means that $A \in \mathbb{R}^{n\times n}$ is negative (resp. positive semi-) definite. If A is diagonal, then $A > 0$ is equivalent to $A \succ 0$.

2 Problem statement

Consider the following uncertain, linear, discrete-time system given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + v_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector and $y_k \in \mathbb{R}^p$ is the available measurement vector; the signals $w \in \mathcal{L}_\infty^n$ and $v \in \mathcal{L}_\infty^p$ are, respectively, process and measurement noise; the constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known. The following assumption is imposed:

Assumption 1 *Initial conditions of* (1) *are bounded such as* $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, *for some known* $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$. *Furthermore, the additive perturbations* $w_k \in [\underline{w}_k, \overline{w}_k]$ *and* $v_k \in [\underline{v}_k, \overline{v}_k]$ *for all* $k \in \mathbb{Z}_+$, *where* $\underline{w}, \overline{w} \in \ell_\infty^n$ *and* $\underline{v}, \overline{v} \in \ell_\infty^p$ *are known signals.*

Thus, the three sources of uncertainties in (1), *i.e.*, x_0, w_k and v_k, are enclosed in given intervals $[\underline{x}_0, \overline{x}_0]$, $[\underline{w}_k, \overline{w}_k]$ and $[\underline{v}_k, \overline{v}_k]$, which is the classical hypothesis for the design of IOs.

Problem 1 (OF-MPC) *Let* $[\underline{x}_0, \overline{x}_0] \in \mathbb{X}$ *and Assumption 1 be satisfied. The objective is to design an output feedback controller stabilizing system* (1) *in a vicinity of the origin while robustly satisfying state and control constraints*

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U}, \quad \forall k \in \mathbb{Z}_+$$

for any admissible realization of disturbances w_k *and* v_k*, where* $\mathbb{X} \subset \mathbb{R}^n$ *and* $\mathbb{U} \subset \mathbb{R}^m$ *are given bounded convex sets.*

The main interest of using interval observers and/or predictors resides on the fact that, thanks to cooperativity and under Assumption 1, they generate estimates $\underline{x}_k, \overline{x}_k \in \mathbb{R}^n$ such that the relation

$$\underline{x}_k \le x_k \le \overline{x}_k, \quad \forall k \in \mathbb{Z}_+ \tag{2}$$

is satisfied. Hence, this information can be easily used to check fulfillment of state constraints, since

$$[\underline{x}_k, \overline{x}_k] \subset \mathbb{X} \Rightarrow x_k \subset \mathbb{X}.$$

Motivated by these features, we will follow the paradigm of MPC and divide the solution of Problem 1 into three steps: *(i)* design an observer to obtain information on the states through the available measurement, *(ii)* design a predictor to implement an optimization problem over a time window of fixed length, and *(iii)* design the controller. For points *(i)* and *(ii)* we will design an IO and an IP, respectively, while for point *(iii)* both these estimators will be incorporated into an adapted formulation of the dual-mode MPC (see [1]). We will show that the main advantage of this formulation is its constructiveness and its weaker computational complexity if compared to classical solutions of the same problem, provided that some additional restrictions on positivity of the estimation errors can be satisfied by the design of proper gains.

The roles of IO and IP are well distinguished in two scenarios: the IO, using output injection/correction, will be responsible for estimating the set-membership of the states at each instant of time $k \in \mathbb{Z}_+$, while the IP (which is designed using solely known information, such as bounds on the disturbances) will be responsible for estimating the state inclusion intervals for $k+1, \dots, k+N$, where $N > 0$ is the window length for prediction in the MPC routine.

3 Design of interval observer and predictor

In the sequel, we will use the following results:

Lemma 1 *[22] Let* $x \in \mathbb{R}^n$ *be a vector variable,* $\underline{x} \le x \le \overline{x}$ *for some* $\underline{x}, \overline{x} \in \mathbb{R}^n$*. If* $A \in \mathbb{R}^{m \times n}$ *is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \tag{3}$$

Lemma 2 *[20] For* $A \in \mathbb{R}_+^{n \times n}$*, the system*

$$x_{k+1} = Ax_k + \omega_k,\ \omega : \mathbb{Z}_+ \to \mathbb{R}_+^n,\ \omega \in \ell_\infty^n,\ k \in \mathbb{Z}_+$$

has a non-negative solution $x_k \in \mathbb{R}_+^n$ *for all* $k \in \mathbb{Z}_+$ *provided that* $x_0 \ge 0$*.*

Lemma 3 *[19] A matrix* $A \in \mathbb{R}_+^{n \times n}$ *is Schur stable iff there exists a diagonal matrix* $P \in \mathbb{R}^{n \times n}$*,* $P \succ 0$*, such that* $A^\top PA - P \prec 0$*.*

3.1 Interval observer

Let us first investigate the design of an interval observer for (1) by using output injection of the available measurement. To this end, rewritting (1) yields

$$x_{k+1} = (A - LC)x_k + Bu_k + Ly_k - Lv_k + w_k \tag{4}$$

for any $L \in \mathbb{R}^{n \times p}$. Following [23], replacing the uncertain terms (*i.e.*, $w_k - Lv_k$) in (4) by their interval bounds applying (3), the following IO can be proposed:

$$\begin{aligned} \overline{x}_{k+1} &= D_o\overline{x}_k + Bu_k + L_oy_k - L_o^+\underline{v}_k + L_o^-\overline{v}_k + \overline{w}_k \\ \underline{x}_{k+1} &= D_o\underline{x}_k + Bu_k + L_oy_k - L_o^+\overline{v}_k + L_o^-\underline{v}_k + \underline{w}_k \end{aligned} \tag{5}$$

where $D_o = A - L_oC$, $L_o \in \mathbb{R}^{n \times p}$ being a gain matrix to be determined and $\underline{x}_0, \overline{x}_0$ as specified in Assumption 1. Note that the precision of (5) can be evaluated by the width of its interval, *i.e.,* $\delta x_k = \overline{x}_k - \underline{x}_k$, whose dynamics are given by:

$$\delta x_{k+1} = D_o\delta x_k + \delta w_k + |L_o|\delta v_k \tag{6}$$

where $\delta w_k = \overline{w}_k - \underline{w}_k$ and $\delta v_k = \overline{v}_k - \underline{v}_k$ determine the uncertainty size of the state and the output disturbances, respectively. Since $\delta x_0 = \overline{x}_0 - \underline{x}_0$, we obtain:

$$\delta x_k = D_o^k(\overline{x}_0 - \underline{x}_0) + \sum_{i=0}^{k-1} D_o^{k-1-i}(\delta w_i + |L_o|\delta v_i) \tag{7}$$

for all $k \in \mathbb{Z}_+$. Hence, the values of δx_k are completely determined by the choice of L_o and the uncertainty levels given in Assumption 1. Our first result shows a procedure to compute this gain.

Theorem 1 *Let Assumption 1 be satisfied. If the following inequalities are verified for a scalar* $\rho > 0$*, a diagonal* *matrix* $P \in \mathbb{R}^{n\times n}$ *and matrices* $W^+, W^- \in \mathbb{R}_+^{n\times p}$*:*

$$\min_{\rho,P,W^+,W^-} \rho$$
$$P > 0,\ PA - (W^+ - W^-)C \geq 0,$$
$$\begin{bmatrix} P-\mathcal{I}_n & 0 & A^\top P - C^\top (W^+ - W^-)^\top \\ \star & \rho \mathcal{I}_n & E^\top \\ \star & \star & P \end{bmatrix} \succ 0,$$
$$E = \begin{bmatrix} P & W^+ + W^- \end{bmatrix},$$

then system (5) *with gains* $L_o^+ = P^{-1}W^+$ *and* $L_o^- = P^{-1}W^-$ *is an IO for system* (1)*, i.e., relation* (2) *holds and* $\delta x_k \in \ell_\infty^n$*. Furthermore, the transfer* $(\delta w_k, \delta v_k) \longmapsto \delta x$ *has a gain less than* $\sqrt{\rho}$*.*

PROOF. Let the estimation errors be given by $\overline{e}_k = \overline{x}_k - x_k$ and $\underline{e}_k = x_k - \underline{x}_k$, whose increments have the following form:

$$\begin{aligned} \overline{e}_{k+1} &= D_o\overline{e}_k + \overline{w}_k - w_k + L_o v_k - L_o^+\underline{v}_k + L_o^-\overline{v}_k, \\ \underline{e}_{k+1} &= D_o\underline{e}_k + w_k - \underline{w}_k - L_o v_k - L_o^+\overline{v}_k + L_o^-\underline{v}_k. \end{aligned} \tag{8}$$

Then, under Assumption 1 and Lemma 1, all exogenous inputs (*i.e.*, the independent right-most terms) in (8) are non-negative. Consequently, if D_o is also non-negative, we have $\underline{e}_k, \overline{e}_k \geq 0$ for all $k \in \mathbb{Z}_+$ under Lemma 2. This requirement is imposed by the first inequality on this theorem and implies relation (2).

Note that, under Assumption 1, the asymptotic stability of (6) and (8) sums up to the Schur stability of D_o. In this light, let $\nu_k = [\delta w_k^\top, \delta v_k^\top]^\top$ and consider a candidate Lyapunov function $V_k = \delta x_k^\top P \delta x_k$, whose increments for (6) are given by

$$V_{k+1} - V_k = \begin{bmatrix} \delta x_k \\ \nu_k \end{bmatrix}^\top \underbrace{\begin{bmatrix} D_o^\top P D_o - P & D_o^\top E \\ E^\top D_o & E^\top P^{-1} E \end{bmatrix}}_{\Pi} \begin{bmatrix} \delta x_k \\ \nu_k \end{bmatrix}. \tag{9}$$

Thus, a sufficient condition for Schur stability of D_o is $\Pi \prec 0$. Now, by introducing a performance index given by $J = \sum_{k=0}^{\infty} \delta x_k^\top \delta x_k - \gamma^2 \nu_k^\top \nu_k + (V_{k+1} - V_k)$, the desired ℓ_∞ performance [24] is achieved by rendering $J < 0$, while minimizing γ. Taking (9) into account, this condition is equivalent to

$$\begin{bmatrix} D_o^\top P D_o - P + \mathcal{I}_n & D_o^\top E \\ E^\top D_o & E^\top P^{-1} E - \gamma^2 \mathcal{I}_n \end{bmatrix} \prec 0.$$

Since P is non-singular, we can decompose the relation above as follows:

$$\begin{bmatrix} D_o^\top P \\ E^\top \end{bmatrix} P^{-1} \begin{bmatrix} P D_o & E \end{bmatrix} - \begin{bmatrix} P - \mathcal{I}_n & 0 \\ 0 & \gamma^2 \mathcal{I}_n \end{bmatrix} \prec 0$$

and hence, by applying the Schur complement in the inequality above, and by introducing the variables $\rho = \gamma^2$, $W^+ = PL_o^+$ and $W^- = PL_o^-$, the inequalities given in this theorem are obtained, proving the claim. □

Depending on the pair (A, C), there might be no L_o such that the LMIs of Theorem 1 are satisfied, where the first one ensures non-negativity of D_o, while the second provides boundedness of δx_k. However, the first requirement can be relaxed [22,25] by introducing a cooperative change of coordinates (see Appendix A).

Let us denote

$$\Delta_k = \left[-\frac{\delta x_k}{2}, \frac{\delta x_k}{2}\right], \quad \bar{\mathbb{X}}_f = \bigcup_{k\in\mathbb{Z}_+} \Delta_k$$

as an admissible interval around the origin at each instant $k \in \mathbb{Z}_+$ and the maximal terminal set, respectively. Roughly speaking, Δ_k provides the estimated interval for $x_k \simeq \frac{\underline{x}_k + \overline{x}_k}{2} \simeq 0$ (*i.e.*, when the control goal is reached), and $\bar{\mathbb{X}}_f$ covers all possible cases for $k \in \mathbb{Z}_+$. The set Δ_k can be estimated, for instance, through the solution of (6) given by (7) (note that all terms in this dynamics are known under Assumption 1).

3.2 Interval Predictor

As readily seen, IO (5) computes an interval $[\underline{x}_k, \overline{x}_k]$ in which the admissible values of x_k are guaranteed to be confined for all $k \in \mathbb{Z}_+$. But, in counterpart, (5) requires information on y_k, which is obviously unknown on future time steps. Thus, the IO is unsuitable for prediction of the system behavior (as needed for the MPC algorithm).

In this spirit, an option is to design an IP which satisfies condition (2) but requires solely available information, such as the system dynamics and the bounds of the disturbance w_k, under the following mild hypothesis (which can be always achieved by a proper change of coordinates):

Assumption 2 *Let* $C \geq 0$*.*

By definition, the predictor gain (it replaces L in (4)) $L_p = L_p^+ - L_p^-$, where $L_p^+, L_p^- \in \mathbb{R}_+^{n\times p}$. Then, under (2), Assumption 2 and Lemma 1, we can state that

$$L_p^+ C\underline{x}_k - L_p^- C\overline{x}_k \leq L_p C x_k \leq L_p^+ C\overline{x}_k - L_p^- C\underline{x}_k$$

which allows us to replace in (4) the terms which are unavailable for prediction, *i.e.*, $Ly_k - Lv_k + w_k = LCx_k + w_k$ with their bounds in order to obtain the following IP:

$$\begin{aligned} \overline{z}_{k+1} &= D_p\overline{z}_k + Bu_k + L_p^+ C\overline{z}_k - L_p^- C\underline{z}_k + \overline{w}_k \\ \underline{z}_{k+1} &= D_p\underline{z}_k + Bu_k + L_p^+ C\underline{z}_k - L_p^- C\overline{z}_k + \underline{w}_k \end{aligned} \tag{10}$$

where $\overline{z}_k$ and $\underline{z}_k$ are, respectively, the predicted upper and lower bounds for x_k, $k \in \mathbb{Z}_+$ and $D_p = A - L_pC$. As readily seen, (10) is composed only by known terms (under Assumptions 1) and also u_k, which is to be computed in the control algorithm.

Introducing a change of coordinates to new variables $z_k^\star = \frac{\overline{z}_k + \underline{z}_k}{2}$ (the center of the interval) and $\delta z_k = \overline{z}_k - \underline{z}_k$ (the amplitude of the interval), one gets the following equivalent representation of dynamics of (10):

$$z_{k+1}^\star = Az_k^\star + Bu_k + w_k^\star \quad (11)$$

$$\delta z_{k+1} = (A + 2L_p^- C)\delta z_k + \delta w_k \quad (12)$$

where, similarly, $w_k^\star = \frac{\overline{w}_k + \underline{w}_k}{2}$ and $\delta w_k = \overline{w}_k - \underline{w}_k$. The dynamics of the center of the predicted interval $z_k^\star$ is independent of L_p, but it is controlled by u_k, hence, (11) can be used in MPC algorithm. The dynamics of the interval width δz_k is governed by the interval width of the state disturbance δw_k, and it has to be optimized by a proper choice of L_p. This brings us to the following result:

Theorem 2 *Let assumptions 1–2 be satisfied, and there exist a diagonal matrix $P \in \mathbb{R}^{n\times n}$, matrices $Q, \Gamma \in \mathbb{R}^{n\times n}$ and $U^-, U^+ \in \mathbb{R}_+^{n\times p}$ such that the following linear matrix inequalities are verified:*

$$\max_{P,Q,\Gamma,U^+,U^-} \; Q - \Gamma$$

$$PA - U^+C + U^-C \geq 0,$$

$$\begin{bmatrix} P - Q & 0 & A^\top P + 2C^\top U^{-\top} \\ 0 & \Gamma & P \\ PA + 2U^-C & P & P \end{bmatrix} \succeq 0$$

$$P \succ 0,\; Q \succ 0,\; \Gamma \succ 0.$$

Then for the system (1), the IP (10) with $L_p^- = P^{-1}U^-$, $L_p^+ = P^{-1}U^+$ and $\underline{z}_0 = \underline{x}_0$, $\overline{z}_0 = \overline{x}_0$ satisfies the relation:

$$\underline{z}_k \leq x_k \leq \overline{z}_k, \quad \forall k \in \mathbb{Z}_+, \quad (13)$$

and $\delta z \in \ell_\infty^n$.

PROOF. Assume that the gain L_p is selected such that the matrix D_p is non-negative. Then, the realization of relation (13) for system (1), (10) can be proven following the same arguments as in Theorem 1. Recalling that $L_p = L_p^+ - L_p^-$ with $L_p^+, L_p^- \in \mathbb{R}_+^{n\times p}$, then

$$A - L_pC = A - L_p^+C + L_p^-C \geq 0$$

is the condition to verify under Assumption 2.

Let us check the stability of (12). Note that

$$0 \leq A - L_p^+C + L_p^-C \leq A + 2L_p^-C$$

since $L_p^-C \geq 0$ and $L_p^+C \geq 0$ under Assumption 2. Then $A + 2L_p^-C \in \mathbb{R}_+^{n\times n}$ and according to Lemma 3, we can select a diagonal matrix $P \in \mathbb{R}^{n\times n}$ considering $V_k = \delta z_k^\top P \delta z_k$ as a Lyapunov function candidate (with $P \succ 0$), whose increment takes, for any $Q \in \mathbb{R}^{n\times n}$ and $\Gamma \in \mathbb{R}^{n\times n}$, the following form:

$$V_{k+1} - V_k = \begin{bmatrix} \delta z_k \\ \delta w_k \end{bmatrix}^\top \Sigma \begin{bmatrix} \delta z_k \\ \delta w_k \end{bmatrix} - \delta z_k^\top Q \delta z_k + \delta w_k^\top \Gamma \delta w_k$$

where

$$\Sigma = \begin{bmatrix} D^\top PD - P + Q & D^\top P \\ PD & P - \Gamma \end{bmatrix}$$

and $D = A + 2L_p^-C$. If $Q \succ 0$, $\Gamma \succ 0$ and $\Sigma \preceq 0$, then the stability conditions are fulfilled and the system (12) is input-to-state stable (ISS) [26] with respect to the input δw_k:

$$V_{k+1} - V_k \leq -\delta z_k^\top Q \delta z_k + \delta w_k^\top \Gamma \delta w_k.$$

Now, let us show that all imposed conditions can be represented in the form of LMIs. First, since $P > 0$ and

$$\Sigma = \begin{pmatrix} D^\top P \\ P \end{pmatrix} P^{-1} \begin{pmatrix} PD & P \end{pmatrix} - \begin{pmatrix} P - Q & 0 \\ 0 & \Gamma \end{pmatrix},$$

then by applying the Schur complement, the condition $\Sigma \preceq 0$ can be equivalently rewritten as

$$\begin{bmatrix} P - Q & 0 & D^\top P \\ 0 & \Gamma & P \\ PD & P & P \end{bmatrix} \succeq 0.$$

Recalling that $PD = PA + 2PL_p^-C$ and defining $U^- = PL_p^- \in \mathbb{R}_+^{n\times p}$ and $U^+ = PL_p^+ \in \mathbb{R}_+^{n\times p}$ as two new non-negative matrix variables, the inequality above becomes linear in the decision variables P, Q, Γ and U^-:

$$\begin{bmatrix} P - Q & 0 & A^\top P + 2C^\top U^{-\top} \\ 0 & \Gamma & P \\ PA + 2U^-C & P & P \end{bmatrix} \succeq 0.$$

Again, since $P \succ 0$ and diagonal, the condition $A - L_p^+C + L_p^-C \geq 0$ follows from

$$PA - U^+C + U^-C \geq 0,$$

which is also linear in P, U^- and U^+. The theorem is proven. $\square$

3.3 Guaranteeing a smaller envelope of estimates

In the previous work [21], it has been discussed that, if the measurement noise is sufficiently big, the inclusion $[\underline{x}_k, \overline{x}_k] \subseteq [\underline{z}_k, \overline{z}_k]$ might not hold. This can be readily seen by considering the relative error between the IO and the IP, $\delta\xi_k = \delta z_k - \delta x_k$, whose increments (assuming the same gain $L = L_o = L_p$ for both observer and predictor) take the following form

$$\delta\xi_{k+1} = (A - LC)\delta\xi_k + |L|(C\delta z_k - \delta v_k),$$

thus, if $C\delta z_k < \delta v_k$, the inclusion is obviously transgressed.

In this work, since we aim to obtain better accuracy by designing two different gains L_o and L_p for the observer and the predictor, such a conclusion is not as straightforward. However, under the conditions imposed on theorems 1 and 2 we have

$$x_k \in [\underline{x}_k, \overline{x}_k] \quad \text{and} \quad x_k \in [\underline{z}_k, \overline{z}_k]$$

for all $k \in \mathbb{Z}_+$. Therefore, the intersection $[\underline{x}_k, \overline{x}_k] \cap [\underline{z}_k, \overline{z}_k]$ is not empty and the following combined interval

$$x_k \in [\hat{\underline{x}}_k, \hat{\overline{x}}_k] \tag{14}$$

where $\hat{\underline{x}}_k = \max\{\underline{x}_k, \underline{z}_k\}$ and $\hat{\overline{x}}_k = \min\{\overline{x}_k, \overline{z}_k\}$ can be used at each decision instant $k \in \mathbb{Z}_+$, providing an envelope of estimates that is less conservative regarding the uncertainties considered on Assumption 1.

4 IO-MPC Design

This section presents the output feedback MPC scheme with guaranteed robust constraint satisfaction based on IO (5) and IP (10) given in the previous section.

In order to guarantee a proper asymptotic behavior of the closed-loop system under the MPC, we will use the *dual mode formulation* [27], in which the MPC is designed to steer the system to a terminal set $\mathbb{X}_f \subset \mathbb{X}$, where a local (static) controller can be applied.

4.1 Dual-mode realization

Before discussing the MPC algorithm to be proposed, we stress the stabilization of the IP through a (static) state feedback. Note that IP (10) can be stabilized by designing a feedback given by

$$u_k = S_f w_k^\star + K_f z_k^\star \tag{15}$$

for its completely known center dynamics (11) with an additional perturbation b_k (robustness with respect to such an additional input will be needed later in the analysis of the closed-loop system), leading to the following closed-loop system:

$$z_{k+1}^\star = (A + BK_f)z_k^\star + (\mathcal{I}_n + BS_f)w_k^\star + b_k \tag{16}$$

with the gains $K_f \in \mathbb{R}^{m\times n}$ and $S_f \in \mathbb{R}^{m\times n}$ guaranteeing input-to-state stability of the closed-loop system and minimizing the influence of $w_k^\star$, respectively. These gains can be selected as follows:

Proposition 1 *Let $S_f = \Sigma P^{-1}$ and $K_f = \Upsilon P^{-1}$, where $P \in \mathbb{R}^{n\times n}$ and $\Sigma, \Upsilon \in \mathbb{R}^{m\times n}$ are solutions of a linear optimization problem:*

$$\max_{Q,\Gamma_1,\Gamma_2,P,\Sigma,\Upsilon} Q - \Gamma_1 - \Gamma_2$$

$$Q = Q^\top \succ 0,\ \Gamma_1 = \Gamma_1^\top \succ 0,\ \Gamma_2 = \Gamma_2^\top \succ 0,\ P = P^\top \succ 0,\ \Pi \succeq 0$$

$$\Pi = \begin{bmatrix} P - Q & 0 & 0 & PA^\top + \Upsilon^\top B^\top \\ 0 & \Gamma_1 & 0 & P + \Sigma^\top B^\top \\ 0 & 0 & \Gamma_2 & P \\ AP + B\Upsilon & P + B\Sigma & P & P \end{bmatrix}.$$

Then the center dynamics (11) *with the control* (15) *is ISS (from the input $(w_k^\star, b_k)$ to the state $z_k^\star$) with the optimal attenuation of the disturbances and with an ISS-Lyapunov function $V(z_k^\star) = z_k^{\star\top} P^{-1} z_k^\star$.*

PROOF. Consider the closed-loop system (16) and the increment of the candidate Lyapunov function $V(z_k^\star)$:

$$V(z_{k+1}^\star) - V(z_k^\star) = \begin{bmatrix} z_k^\star \\ w_k^\star \\ b_k \end{bmatrix}^\top \tilde{\Pi} \begin{bmatrix} z_k^\star \\ w_k^\star \\ b_k \end{bmatrix} - z_k^{\star\top}\tilde{Q}z_k^\star + w_k^{\star\top}\tilde{\Gamma}_1 w_k^\star + b_k^\top \tilde{\Gamma}_2 b_k,$$

$$\tilde{\Pi} = \begin{bmatrix} \tilde{A}^\top P^{-1}\tilde{A} - P^{-1} + \tilde{Q} & \tilde{A}^\top P^{-1}\tilde{D} & \tilde{A}^\top P^{-1} \\ \tilde{D}^\top P^{-1}\tilde{A} & \tilde{D}^\top P^{-1}\tilde{D} - \tilde{\Gamma} & \tilde{D}^\top P^{-1} \\ P^{-1}\tilde{A} & P^{-1}\tilde{D} & P^{-1} - \tilde{\Gamma}_2 \end{bmatrix},$$

$$\tilde{A} = A + BK_f, \quad \tilde{D} = \mathcal{I}_n + BS_f$$

for any $\tilde{Q}, \tilde{\Gamma}_1, \tilde{\Gamma}_2 \in \mathbb{R}^{n\times n}$. If $\tilde{\Pi} \preceq 0$, then

$$V(z_{k+1}^\star) - V(z_k^\star) \le -\alpha V(z_k^\star) + w_k^{\star\top}\tilde{\Gamma}_1 w_k^\star + b_k^\top \tilde{\Gamma}_2 b_k \tag{17}$$

meaning that the system is ISS, provided that $\tilde{Q} \succeq \alpha P^{-1}$, $\tilde{\Gamma}_1 \succ 0$ and $\tilde{\Gamma}_2 \succ 0$, where $\alpha > 0$ always exists if $\tilde{Q} \succ 0$. Note that

$$\tilde{\Pi} = -\begin{bmatrix} P^{-1} - \tilde{Q} & 0 & 0 \\ 0 & \tilde{\Gamma}_1 & 0 \\ 0 & 0 & \tilde{\Gamma}_2 \end{bmatrix} + \begin{bmatrix} \tilde{A}^\top P^{-1} \\ \tilde{D}^\top P^{-1} \\ P^{-1} \end{bmatrix} P \begin{bmatrix} P^{-1}\tilde{A} & P^{-1}\tilde{D} & P^{-1} \end{bmatrix}$$

is such that, by applying the Schur complement, the LMIs given in the formulation of this proposition are verified for $\Sigma = S_f P$, $\Upsilon = K_f P$, $\tilde{Q} = P^{-1}QP^{-1}$, $\tilde{\Gamma}_1 = P^{-1}\Gamma_1 P^{-1}$, $\tilde{\Gamma}_2 = P^{-1}\Gamma_2 P^{-1}$. □

Remark 1 *We can also impose an additional constraint on P, in order to guarantee some optimal performance for the terminal cost under the control* (15)*.*

Remark 2 *If $w_k^\star = 0$ for $k \in \mathbb{Z}_+$ then an obvious choice is $S_f = 0$, which can be imposed as a constraint on the optimization problem given by Proposition 1.*

Using the property (17) we are in position to define the terminal set:

$$\mathbb{X}_f = \Big\{x \in \mathbb{R}^n : x^\top P^{-1} x \le \alpha^{-1}[\sup_{k\ge 0} w_k^{\star\top}\tilde{\Gamma}_1 w_k^\star + \sup_{k\ge 0} \sup_{\substack{\delta\in\Delta_k \\ v\in[\underline{v}_k,\overline{v}_k]}} (\delta + v - v_k^\star)^\top L_o^\top \tilde{\Gamma}_2 L_o(\delta + v - v_k^\star)]\Big\}.$$

The terminal set $\mathbb{X}_f$ represents the domain that must be reached by the MPC in a finite number of steps, where the static (and possibly sub-optimal) control (15) takes over to guarantee asymptotic stability and robustness to the closed-loop system. As we can conclude this set is robustly positively invariant [28] for (11) under the control (15) with certain class of perturbations.

We need the following property, which is conventionally imposed in MPC [1]:

Assumption 3 *Let the terminal set $\mathbb{X}_f \subseteq \bar{\mathbb{X}}_f \cap \mathbb{X}$ and $u_k \in \mathbb{U}$ under control* (15)*, provided that $z_k^\star \in \mathbb{X}_f$.*

Also, Assumption 3 states a necessary condition for the existence of a solution to Problem 1 (at least locally, in the vicinity of the origin), that is based on the estimates provided by the IO (5). Roughly speaking, it means that if we start in a neighborhood of the origin and Assumption 1 is satisfied, then the obtained interval estimates $[\underline{x}_k, \overline{x}_k]$ belong to $\mathbb{X}_f \subseteq \mathbb{X}$ provided that the control u_k is designed in a way keeping the interval center $x_k^\star = \frac{\underline{x}_k + \overline{x}_k}{2}$ (and consequently x_k) close to the origin. Furthermore, the static feedback control (15) must satisfy the constraints on the control input.

If $w_k^\star = 0$ and the constraint set $\mathbb{U}$ is ellipsoidal, the second part of Assumption 3 can be alleviated by additionally constraining the conditions in Proposition 1 [10].

Finally, similarly to [27], the control law for (1) will be selected as

$$u_k = \begin{cases} s_0^k & x_k^\star \notin \mathbb{X}_f \\ S_f w_k^\star + K_f x_k^\star & x_k^\star \in \mathbb{X}_f \end{cases} \tag{18}$$

where s_0^k is the feedforward control computed by the MPC algorithm, as we will discuss in the next subsection.

4.2 Design of the predictive controller

Since framer (10) depends solely on known variables, the prediction can be performed in a receding horizon fashion with a window of length $N > 1$. Indeed, the central trajectory of the predicted interval (which contains x_k) is modeled by (11) and is controlled by u_k, having a stable width (given by (12)), provided that the conditions of Theorem 2 are satisfied.

At each $k \in \mathbb{Z}_+$ we will initialize the IP (10) as $\underline{z}_{k,0} = \hat{\underline{x}}_k$ and $\overline{z}_{k,0} = \hat{\overline{x}}_k$, and having a sequence of inputs $\mathcal{S}_N = \{s_0, \dots, s_{N-1}\}$ with $s_i \in \mathbb{U}$ for all $i = 0, \dots, N-1$, we will calculate the values $\underline{z}_{k,i+1}, \overline{z}_{k,i+1}$ for $i = 0, \dots, N-1$ under substitution $u_{k+i} = s_i$ (it is supposed in Assumption 1 that $\underline{w}_k$ and $\overline{w}_k$ are given for all $k \in \mathbb{Z}_+$). In order to utilize (14) we will define $\underline{z}_{k+1} = \underline{z}_{k,1}$ and $\overline{z}_{k+1} = \overline{z}_{k,1}$ for $k \in \mathbb{Z}_+$. Then, the optimal control problem (OCP) to be solved by the MPC is as follows:

$$\mathcal{S}_N^k \coloneqq \arg\min_{\mathcal{S}_N} V_N(z_{k,0}^\star, \dots, z_{k,N}^\star, \mathcal{S}_N) \tag{19}$$

under the following constraints:

$$\begin{aligned} &\underline{z}_{k,i+1}, \overline{z}_{k,i+1} \text{ are computed by (10)}, \\ &\underline{z}_{k,0} = \hat{\underline{x}}_k,\ \overline{z}_{k,0} = \hat{\overline{x}}_k, \\ &[\underline{z}_{k,i+1}, \overline{z}_{k,i+1}] \subset \mathbb{X}, \quad s_i \in \mathbb{U}, \\ &z_{k,N}^\star \in \mathbb{X}_f, \end{aligned}$$

The cost function V_N is defined by

$$V_N(z_{k,0}^\star, \dots, z_{k,N}^\star, \mathcal{S}_N) = V_f(z_{k,N}^\star) + \sum_{i=0}^{N-1} \ell(z_{k,i}^\star, s_i)$$

with

$$V_f(z) = z^\top W z,\ \ell(z, s) = z^\top H z + s^\top R s$$

and the weighting matrices $W, H \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are positive definite and symmetric. The MPC routine is given by following algorithm:

Algorithm 1. IO-MPC
Input: Initial conditions $\underline{x}_0$,$\overline{x}_0$, matrices W,$H \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, terminal set $\mathbb{X}_f$ and static gains K_f and S_f.

for each decision instant $k \in \mathbb{Z}_+$ **do**
  Measure y_k and update IO (5).
  Initialize IP such as $[\overline{z}_{k,0}, \underline{z}_{k,0}] = [\hat{\underline{x}}_k, \hat{\overline{x}}_k]$.
  Compute $z_k^\star$ and $x_k^\star$.
  if $x_k^\star \in \mathbb{X}_f$ **do**
    Assign u_k as the static feedback (15).
  else
    Solve optimal control problem (19).
    Assign $u_k = s_0^k$.
  end if
  Apply u_k to system (1).
end for

Since the combined interval (14) is computed at each instant $k \in \mathbb{Z}_+$, under Assumption 3 there exists $k_0 \in \mathbb{Z}_+$ such that $[\hat{\underline{x}}_k, \hat{\overline{x}}_k] \cap \mathbb{X}_f \neq \emptyset$ for all $k \geq k_0$, provided that the interval center $z_k^\star \simeq 0$, which has to be ensured by a proper design of the control u_k.

Now we are in position to formulate the main result of this section:

Theorem 3 *Let* $[\underline{x}_0, \overline{x}_0] \subset \mathbb{X}$, *and assumptions 1–3 hold with* $[\underline{w}_{k+1}, \overline{w}_{k+1}] \subseteq [\underline{w}_k, \overline{w}_k]$ *for all* $k \in \mathbb{Z}_+$. *Then the closed-loop system given by* (1), (5) *and* (18) *has the following properties:*

(1) Recursive feasibility with reaching $\mathbb{X}_f$ *in* N *steps;*
(2) ISS for the dynamics of $x_k^\star$ *and practical ISS for* x_k *in the terminal set* $\mathbb{X}_f$*;*
(3) Constraint satisfaction.

PROOF. The proof relies on conventional arguments for MPC schemes (see [1]).

Suppose that for any $[\underline{x}_0, \overline{x}_0] \subset \mathbb{X}$ there exists a sequence of inputs $\mathcal{S}_N^0$ solving (19). Hence, selecting the input $u_i = s_i^0$ we ensure that $[\underline{z}_{0,i+1}, \overline{z}_{0,i+1}] \subset \mathbb{X}$ for all $i = 0, \dots, N-1$ and $z_{0,N}^\star \in \mathbb{X}_f$. Under the combined interval (14), by applying to (1) the control $u_0 = s_0^0$ we obtain that $[\hat{\underline{x}}_1, \hat{\overline{x}}_1] \subset [\underline{z}_{0,1}, \overline{z}_{0,1}] \subset \mathbb{X}$, and the procedure can be iteratively repeated since $[\underline{w}_1, \overline{w}_1] \subseteq [\underline{w}_0, \overline{w}_0]$. Moreover, on each iteration $k \in \mathbb{Z}_+$, $[\underline{z}_{k,i+1}, \overline{z}_{k,i+1}] \subset \mathbb{X}$ for all $i = 0, \dots, N-1$ and $z_{k,N}^\star \in \mathbb{X}_f$. In addition, since under (14) we have that $[\hat{\underline{x}}_{k+1}, \hat{\overline{x}}_{k+1}] \subset [\underline{z}_{k,1}, \overline{z}_{k,1}]$, the control sequence $\mathcal{S}_N^k$ that steers $z_{k,N}^\star$ to the set $\mathbb{X}_f$ with $u_k = s_0^k$ can also be applied at the next step with $u_{k+1} = s_1^k$ (without resolution of (19)) ensuring iterative achievement of $\mathbb{X}_f$ in N steps. This implies the point *(1)*.

Now, from (5), we have that the dynamics of $x_k^\star = \frac{\underline{x}_k + \overline{x}_k}{2}$ has the form:

$$x_{k+1}^\star = (A - L_oC)x_k^\star + Bu_k + L_o(y_k - v_k^\star) + w_k^\star.$$

If $x_k^\star \in \mathbb{X}_f$, the control (15) is applied (see (18)). Then,

$$\begin{aligned} x_{k+1}^\star = &(A + BK_f - L_oC)x_k^\star + (\mathcal{I}_n + BS_f)w_k^\star \\ &+ L_o(y_k - v_k^\star). \end{aligned}$$

Since $L_o(y_k - v_k^\star) = L_o(Cx_k + v_k - v_k^\star)$ and $x_k = x_k^\star + \sigma_k$ with $\sigma_k \in \Delta_k$, we obtain the dynamics

$$\begin{aligned} x_{k+1}^\star = &(A + BK_f)x_k^\star + (\mathcal{I}_n + BS_f)w_k^\star \\ &+ L_o(\sigma_k + v_k - v_k^\star). \end{aligned}$$

By contrasting the equation above with (16), we have that $b_k = L_o(\sigma_k + v_k - v_k^\star)$. Therefore, the control law (15) also renders $x_k^\star$ to be input-to-state stable. Since $\mathbb{X}_f$ is an invariant set by construction, the interval midpoint $x_k^\star$ will be kept in the terminal set $\mathbb{X}_f$ for all future steps. Such a property by recursion leads to forward invariance of $\mathbb{X}_f$.

Note that, by definition, $\|x_k\| \leq \|x_k^\star\| + \|\delta x_k\|$ and $\|x_0^\star\| \leq \|x_0\| + \|\delta x_0\|$. Thanks to the ISS property of $x_k^\star$ discussed above, and the fact that δx_k is bounded by the implications of Theorem 1 and under Assumption 1, the practical ISS of x_k follows. This implies the point *(2)*.

Finally, the solution of (19) guarantees that $x_k \in [\hat{\underline{x}}_{k+1}, \hat{\overline{x}}_{k+1}] \subset \mathbb{X}$ as a direct consequence of relations (2) and (13), the combined interval (14), and $u_k = s_0^k \in \mathbb{U}$, that together with Assumptions 3 and the discussed features of set $\mathbb{X}_f$ imply the point *(3)*. □

4.3 Complexity and performance

The OCP (19) is a quadratic programming (QP) problem that, assuming a proper implementation, is very similar to the nominal MPC. The complexity for solving (19) is $\mathcal{O}(Nn)$, *i.e.*, is linear with respect to the dimension of the system.

If compared to solutions based on Tubes [17,29] or MHE [14], the proposed method offers many advantages: *(i)* it does not require any steady-state assumption for the observer nor any further development for compensating the initial uncertainty, since this is automatically handled by the convergence of the pair IO/IP, *(ii)* the scheme is constructive, since all gains are obtained by the solution of LMIs (it should be noted that a naive selection of observer/controller gains in approaches that uses set approximations for propagating the uncertainties might influence the performance of the MPC, see the discussion in section 5.2 of [14] and the comparison below), *(iii)* thanks to the guaranteed enclosing $x_k \in [\overline{x}_k, \underline{x}_k]$, if $[\underline{x}_0, \overline{x}_0] \in \mathbb{X}$ and Assumption 3 is satisfied, the feasible region (w.r.t. $z_k^\star$) is similar to the one of the nominal MPC.

5 Numerical Example

Let us now demonstrate the performance of the proposed IO-MPC in numerical experiments. We considered the (linearized) continuous stirred tank reactor (CSTR), where an exothermic and irreversible reaction $S \to P$ occurs [30]. The model is composed of two states, the concentration of reactant C_S and the temperature of the reactor T, and one controlled input, the coolant stream T_C.

The resulting linear system is composed by the following matrices:

$$A = \begin{bmatrix} 0.745 & -0.002 \\ 5.610 & 0.780 \end{bmatrix},\ B = \begin{bmatrix} 5.6 \times 10^{-6} \\ 0.464 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (20)$$

The constraints on state and control are given by $\mathbb{X} = [-2, 2] \times [-10, 5]$ and $\mathbb{U} = [-4.5, 4.5]$, respectively. The disturbance sets are assumed to be given by $\mathbb{W} = [-0.02, 0.02] \times [-0.2, 0.2]$ and $\mathbb{V} = [-0.3, 0.3]$.

The prediction horizon is selected as $N = 10$, and weighting matrices $H = 1000\mathcal{I}_2$ and $R = 0.001$. For the IO-MPC, the terminal set and the controller $K_f = [-6.99, -0.50]$ are obtained by solving the conditions given in Proposition 1, whereas the terminal cost W is defined as its correspondent Lyapunov function.

Comparison with Tube-MPC

For comparison purposes, we have implemented the Tube-based MPC from [29]. For this implementation, a Luenberger observer was designed by pole placement (for the pair $(A^\top, C^\top)$). The terminal set and the static controller $K_{LQ} = [5.58, 0.45]$ are obtained by computing the associated LQR with weighting matrices $Q_{LQ} = 0.1\mathcal{I}_n$ and $R_{LQ} = 0.1$.

Figure 1 shows the comparison of the feasible sets for both techniques. As expected, due to the constraint tightening used to guarantee robust constraint satisfaction in the Tube-MPC, the obtained feasible region for the IO-MPC is much wider (and similar to the nominal MPC). Furthermore, Figure 1 also illustrates the impact of a naive design of the gains for the Tube-MPC. Clearly, the obtained regions are much different depending where the closed-loop poles of the terminal controller/observer are placed, whereas our method has all gains readily obtained by conditions in the form of LMIs, which reduces the number of parameters to be tuned.

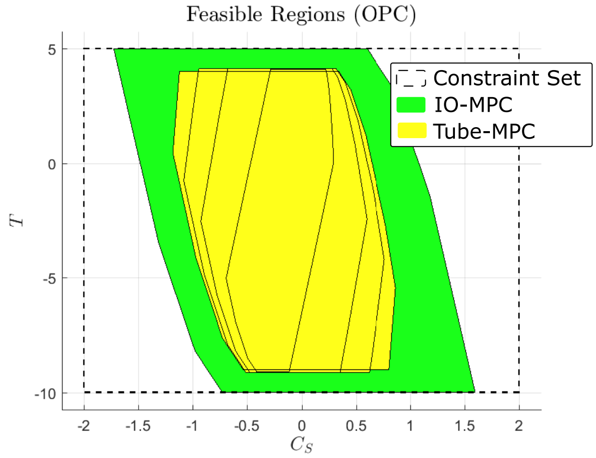

Fig. 1. Comparison of the feasible regions (approximations obtained by ray-shooting using YALMIP [31]). For the Tube-MPC, these regions were obtained by placing different poles when designing the Luenberger observer.

For the Tube-MPC, we selected the Luenberger observer which offered the biggest feasible region (by trial-and-error), which is initialized with $\hat{x}_0 = (-0.65, -7)^\top$, while the IO is initialized with $\underline{x}_0 = (-0.8, -7.8)^\top$ and $\overline{x}_0 = (-0.5, -6)^\top$. The reactor is then simulated over a time window $T_{simu} = 60$ time steps, considering several initial conditions satisfying $x_0 \in [\underline{x}_0, \overline{x}_0]$ (note that this setup is feasible for both techniques) and several realizations of w_k and v_k. For comparison purposes, let us define a performance index $J_p = \sum_{k=0}^{T_{simu}} x_k^\top x_k + u_k^\top u_k$. Table 1 illustrates the average results for 25 simulation runs:

Table 1
Comparison between IO-MPC and Tube-MPC (time simulations)

	IO-MPC	**Tube-MPC**
Perf. index J_p	232.650	253.560
OCP sol. time	0.0265	0.0524
Final values (around origin)	$[-0.070, 0.074] \times [-0.993, 0.993]$	$[-0.047, 0.057] \times [-1.792, 1.743]$

These results show that, even for a region where both techniques are feasible, the IO-MPC shows a faster solution of the optimization problem, as well as a better performance index. This latter fact was expected, since the control constraint was not tightened, meaning that the MPC algorithm can use its full range to achieve faster stabilization. However, since the Tube-MPC is designed to control the trajectories of the observer, it stabilizes the system closer to the origin, whereas the IO-MPC can only drive the envelope of trajectories close to it.

Simulation results

Due to space limitations, we will not present time simulations for the Tube-MPC. Let us initialize the IO with $\overline{x}_0 = (1.2, -8)^\top$ and $\underline{x}_0 = (1, -9)^\top$. Notice that this region is not feasible at all for the Tube-MPC presented previously. For simplicity, let $[\underline{z}_0, \overline{z}_0] = [\underline{x}_0, \overline{x}_0]$ and, again, let the reactor be simulated considering several $x_0 \in [\underline{x}_0, \overline{x}_0]$.

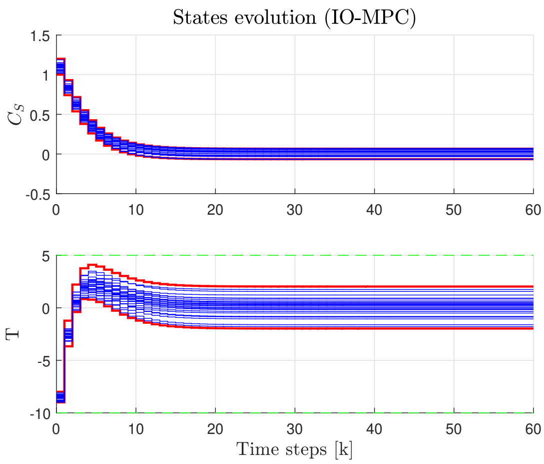

Fig. 2. Evolution of the states under the IO-MPC.

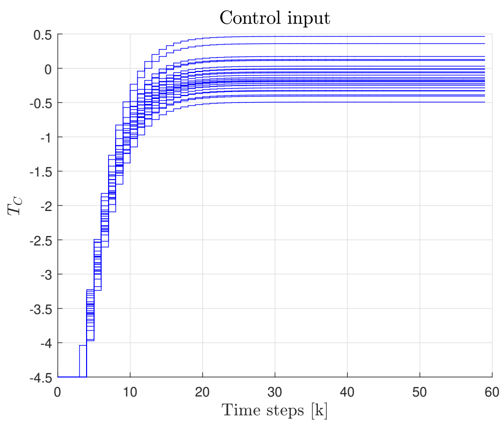

Fig. 3. Control input under the IO-MPC

This simulation scenario is shown in Figure 2, for several realizations of the perturbations v_k and w_k. As it can be seen, all trajectories of the perturbed system satisfied the constraints and were stabilized close to the origin. Furthermore, it is worth noticing that the system trajectories are able to get very close to the constraint boundaries, indicating very low conservativeness. Finally, Figure 3 shows the input applied to the system.

All simulations have been performed using YALMIP [31] in MATLAB 2017a, using an Intel i7-8565U processor (1.8GHz) and 16GB RAM. Also, the toolboxes MPT3 [32] and PnPMPC [33] were used for computing invariant sets.

6 Conclusion

In this paper, we presented a novel robust output feedback MPC using interval observers and predictors. Profiting from the simple form of such estimators, robust constraint satisfaction is achieved with reduced computational complexity and low conservativeness. The efficiency of presented results is illustrated by numerical simulation of a chemical reactor in comparison with classical solutions. This new approach offers interesting directions for future research on output feedback MPC schemes, such as considering non-dual MPC tools or providing extensions to linear parameter-varying (LPV), time-delayed and some classes of nonlinear systems.

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A Cooperative change of coordinates

Theorem 4 *[22] Let Assumption 1 be satisfied, the matrix D_o be Schur stable, and there exist a matrix $R \in \mathbb{R}_+^{n\times n}$ such that $\lambda(D_o) = \lambda(R)$ and the pairs (D_o, e_1) and (R, e_2) are observable for some $e_1 \in \mathbb{R}^{1\times n}$, $e_2 \in \mathbb{R}^{1\times n}$. Then, the relation (2) is satisfied for*

$$\begin{aligned}
&\underline{\xi}_0 = (S^{-1})\underline{x}_0 - (S^{-1})^-\overline{x}_0, \quad \overline{\xi}_0 = (S^{-1})\overline{x}_0 - (S^{-1})^-\underline{x}_0\\
&\overline{\xi}_{k+1} = R\overline{\xi}_k + Fy_k - F^+\underline{v}_k + F^-\overline{v}_k + (S^{-1})^+\underline{w}_k - (S^{-1})^-\overline{w}_k\\
&\underline{\xi}_{k+1} = R\underline{\xi}_k + Fy_k - F^-\overline{v}_k + F^+\underline{v}_k + (S^{-1})^+\overline{w}_k - (S^{-1})^-\underline{w}_k\\
&\underline{x}_k = S^+\underline{\xi}_k - S^-\overline{\xi}_k, \quad \overline{x}_k = S^+\overline{\xi}_k - S^-\underline{\xi}_k
\end{aligned}$$

where $S = O_R O_{D_o}^{-1}$ (O_{D_o} and O_R are the observability matrices of the pairs (D_o, e_1) and (R, e_2), respectively), and $F = S^{-1}L_o$.

Remark 3 *For the sake of compactness, a procedure for determining vectors e_1, e_2 and matrices R, F in Theorem 4 will not be discussed in this work. However, once this new observer is designed, its use is straightforward since all terms for computing $\underline{\xi}_k$ and $\overline{\xi}_k$ are known. Then, $\overline{x}_k, \underline{x}_k$ are readily determined and can be used for the MPC algorithm described in the paper. Thus, for brevity of formulation we use the IO (5) and the result of Theorem 1.*