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# Maximal cable tensions of a N-1 cable-driven parallel robot with elastic or ideal cables

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**Abstract.** Determining what will be the maximal cable tensions of a cable-driven parallel robot (CDPR) when it moves over a given workspace is an important step in the design phase as it will allow to choose the cable diameter and to provide a requested information for tuning the CDPR actuation. In this paper we consider a suspended N-1 CDPR with  $n$  cables where all cables are attached at the same point, which leads to a 3-dof robot. We assume a quasi-static behavior of the robot and assume that the cable are either ideal or elastic so that we neglect the sagging effect. Under these assumption we show that the maximum of the cable tensions may be determined in a very fast way by solving a set of second-order polynomials which will lead to the poses at which the maximum of each cable tension will occur. For example for a four-cables CDPR determining the maximal cable tension requires to solve at most 149 second order polynomials.

**Keywords:** cable-driven parallel robot, static, cable tension

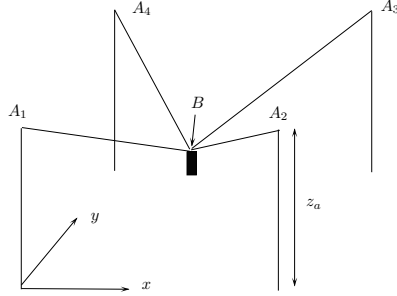
## 1 Introduction

Determining the largest cable tensions over a given workspace is an essential part of the design of a CDPR. This problem has been addressed for classical parallel robots [1, 2] and has been used for their design [3]. However this problem is somewhat different for CDPR because of their redundancy, configuration changes, cable slackness and influence of the discrete time control [4] that may lead to different tensions for a given pose of the CDPR or on a given trajectory. Most works on static analysis address only indirectly this issue either by computing different types of workspace under the constraint that the tensions lie within given bounds [5–8] or by trying to impose by control an optimal force distribution [9–13]. In the later case the force distribution is difficult to ensure for fully constrained CDPRs because of the difficulty to reliably measure the cable tensions, while for suspended CDPRs this scheme can only be adopted if the cable are elastic (for non-elastic cables executing a given tension control scheme at a given pose requires an **exact** control of the cable lengths, an assumption that is clearly unrealistic). However for safety reasons it is of interest to determine the largest cable tensions in the worst case (i.e. with no assumption on the control) for a given CDPR when the platform moves in a given workspace. Clearly dynamics will play a role on the tensions [14–16] but however a first step is to consider only quasi-static motion.

To the best of the author knowledge the only work having addressed the determination of the largest cable tensions has been presented by Riehl [17] for a 3-1 CDPR with

sagging cables. For a given pose the cable tensions are determined assuming ideal cables and then a numerical procedure is used to derive the tensions with sagging cables. Then the given workspace is sampled (i.e. the workspace is approximated by a finite set of poses, called *node poses*) and the maximum cable tensions is derived from the calculation at the node poses. The first limitation of this work is that the inverse kinematics (IK) of the 3-1 CDPR with sagging cable has most probably a single solution while for a  $N-1$  CDPR with  $N > 3$  the IK may have multiple solutions so that all solutions of the IK have to be determined in order to determine the largest cable tensions in the worst case. The second limitation is related to the sampling of the workspace which leads to a large computation time for determining an approximation of the largest tensions as the real maximum may possibly not occur at a node pose. For example for a  $5 \times 5 \times 5$  meters workspace that is sampled with a 5cm step size with a computation time for calculating the cable tensions at a node of 5 ms we will get a computation time of 10mn 25s for determining the maximal cable tensions, which is problematic if this calculation is used in a design process that require a large number of such a calculation. The objective of this paper is to show that the pose at which the maximal tension will occur can be calculated exactly in a very short time.

In this paper we will consider a suspended 3dof  $N-1$  CDPR with  $N \geq 4$  cables attached to the same point on the platform so that this robot allows only for translational motions (figure 1). Furthermore we will assume a geometry that is respected for all the existing prototypes: the exit point  $A_i$  have the same height  $z_a$  while the convex hull of the projection of these points on the ground is a rectangle  $\mathcal{R}$ . A consequence is that the available workspace for this CDPR in terms of horizontal motions is  $\mathcal{R}$ . Regarding cable model we will assume ideal cables(no mass, no elasticity). Note that the results we will obtain also cover the case of elastic cables with no mass, whatever the elasticity model is, as elasticity just play a role for determining the cable lengths to reach a given pose but does not influence the cable tensions at this pose.



**Fig. 1.** A example of a  $N - 1$  CDPR

As mentioned previously the exit point of the cables will be denoted  $A_i$  while the common point of the cables will be  $B$  with coordinates  $(x_b, y_b, z_b)$ . We number the  $A_i$  point in such a way that the points  $A_1, A_2, A_3$  are three corners of the rectangle  $\mathcal{R}$  and

we define a reference frame whose origin will be  $A_1$  (whose coordinates are therefore  $(0,0,z_a)$ ) and  $A_2, A_3$  will have as coordinates  $(x_{a_2} > 0, 0, z_a)$ ,  $(x_{a_3} \geq 0, y_{a_3} > 0, 0)$ , while the others  $A_i$  coordinates are  $(0 \leq x_{a_i} \leq x_{a_2}, 0 \leq y_{a_i} \leq y_{a_3}, z_a)$ . We will assume that  $x_{a_i} = x_{a_2}$  or  $y_{a_i} = y_{a_3}$  so that any  $A_i$  is located on one edge of the rectangle  $\mathcal{R}$ . The point on the ground that results from a vertical projection of a point  $M$  will be denoted  $M^p$ . We will also assume that the CDPR platform is only submitted to gravity and that its mass is  $m$ . In a quasi-static case the center of mass of the platform is located below  $B$  so that no torque is exerted at  $B$ . We will also assume that the set of poses of the CDPR for which we will calculate the maximal cable tensions (the *desired workspace*) is a rectangle included in  $\mathcal{R}$  that is defined by the constraints  $0 \leq x_r^1 \leq x \leq x_r^2 \leq x_{a_2}$ ,  $0 \leq y_r^1 \leq y \leq y_r^2 \leq y_{a_3}$ . Clearly the maximal height of the desired workspace should be lower than  $z_a$ .

## 2 Calculation of the maximal cable tensions

For ideal cable a CDPR property is that most of the time only 3 of the cables will be under tension. Indeed when 3 cables are under tension their lengths leads to a unique possible pose  $B_a$  for  $B$ : hence to have any other cable  $j$  under tension at  $B_a$  may occurs only when its length  $\rho_j$  is exactly  $\|\mathbf{A}_j \mathbf{B}_a\|$  as if  $\rho_j > \|\mathbf{A}_j \mathbf{B}_a\|$ , then cable  $j$  is slack and if  $\rho_j < \|\mathbf{A}_j \mathbf{B}_a\|$ , then  $B$  will move to a pose different from  $B_a$ . A consequence is that trying to create by control a configuration in which 4 cables are under tension will require an infinite accuracy on the measurement of the cable lengths together with a exact knowledge of the location of the  $A_i$  points. However having more than 3 cables under tension may occur occasionally for example when we initially set the lengths of cable 1,2,3 in such a way that  $B_a^p$  lies in the triangle  $A_1^p A_2^p A_3^p$  while cable 4 is slack and then coil cable 4 until its lengths is largely lower than  $\|\mathbf{A}_j \mathbf{B}_a\|$ . Hence for computing exactly the maximum cable tensions we have also to consider that more than 3 cables are under tension as this temporary configuration may occur on any trajectory whatever the control law is and although we are not able to determine when this configuration will occur.

In the next section we will first start to investigate the case where only 3 cables are under tension, the others being slack.

### 2.1 Maximal tension with only 3 cables under tension

For determining the maximal tension in all the  $n$  cables of the CDPR under the assumption that only 3 of them are under tension we will have to consider all the combinations of 3 cables among the  $n$  cable i.e.  $n(n-1)(n-2)/6$  cases. However the treatment of all these cases is basically the same so that we may consider the example case of cables 1,2,3 being under tension.

If we assume that cable 1,2 and 3 are under tension, while the other cables are slack and the CDPR is in mechanical equilibrium, then the point  $B_a^p$  should lie in the triangle  $A_1^p A_2^p A_3^p$ . If  $\tau_i$  denotes the tension of cable  $i$  and  $\rho_i$  its length, then the mechanical equilibrium condition is a linear system of 3 equations in the  $\tau_1, \tau_2, \tau_3$  unknowns so that we

get

$$\begin{aligned}\tau_1 &= \frac{mg\rho_1(xy_{a_3} + x_{a_2}y - x_{a_2}y_{a_3} - x_{a_3}y)}{x_{a_2}y_{a_3}(z - z_a)} \\ \tau_2 &= -\frac{mg\rho_2(xy_{a_3} - x_{a_3}y)}{x_{a_2}y_{a_3}(z - z_a)} \quad \tau_3 = -\frac{mg\rho_3y}{y_{a_3}(z - z_a)}\end{aligned}$$

where  $z - z_a$  is negative. We notice that  $\tau_i$  may be written as  $\tau_i = mg\rho_i U_i(x, y)/V_i(z)$  where  $U_i$  is linear in  $x, y$ , while  $V_i$  is always negative.

Regarding  $\tau_1$ , the term  $U_1 = xy_{a_3} + x_{a_2}y - x_{a_2}y_{a_3} - x_{a_3}y$  is the signed distance of  $B$  to the line 23 which is negative as soon as  $B^p$  lies in the triangle  $A_1^p A_2^p A_3^p$ , so that  $\tau_1 > 0$ . Regarding  $\tau_2$ , the term  $xy_{a_3} - x_{a_3}y$  is positive as soon as  $B^p$  is inside the triangle  $A_1^p A_2^p A_3^p$  so that  $\tau_2$  is positive in that case, while  $\tau_3$  is positive as soon as  $y > 0$ .

We may now look at the influence of the altitude  $z$  on the value of  $\tau_i$ : we notice that this influence is related to the variation of the term  $Z = \rho_i/(z - z_a)$  as each  $\tau_i$  is established as  $K_i \rho_i/(z - z_a)$  where the  $K_i < 0$  are not dependent upon  $z$ . As  $\rho_i$  may be written as  $\sqrt{(x - x_{a_i})^2 + (y - y_{a_i})^2 + (z - z_a)^2}$  the derivative of  $Z$  with respect to  $z$  is always negative so that we may state a rather trivial and expected result:

**Theorem 1:** if only 3 cables are under tension the largest cable tension will be obtained for the highest altitude of the platform

Note that if the  $A_i$  are not lying in a horizontal plane the above theorem will still hold although the tension will increase when the distance between  $B$  and the plane  $A_1 A_2 A_3$  decreases (implying that it will also hold if pulleys are used at the  $A_i$  points).

Now let us look at the derivatives of the  $\tau_i$  with respect to  $x, y$ . Their derivatives may be written as

$$\frac{\partial \tau_i}{\partial x} = \frac{\rho_i^2 \partial U_i / \partial x + U_i(x - x_{a_i})}{W_i} \quad \frac{\partial \tau_i}{\partial y} = \frac{\rho_i^2 \partial U_i / \partial y + U_i(y - y_{a_i})}{W_i}$$

where  $W_i$  has a constant sign. Looking at the numerator of these derivatives we note that because of the linearity of  $U_i$  the term  $\partial U_i / \partial x, y$  is a constant so that  $\rho_i^2 \partial U_i / \partial x, y$  is a second order function in  $x, y$  and so are also the terms  $U_i(x - x_{a_i}), U_i(y - y_{a_i})$ . Hence the numerators of  $\partial \tau_i / \partial x, \partial \tau_i / \partial y$  are two quadratic equations  $\mathcal{C}_i^x, \mathcal{C}_i^y$ . Note that we get 2 such equations for  $\tau_1, \tau_2$  but only one for  $\tau_3$  as  $\partial \tau_3 / \partial x$  cancel only for  $y = 0$  or  $x = x_{a_3}$ : hence for the whole set of cables we get a total of 5 quadratic equations. The conics  $\mathcal{C}_i^x = 0, \mathcal{C}_i^y = 0$  split the desired workspace in different components in which the derivatives of  $\tau_i$  with respect to  $x, y$  have a constant sign. A consequence is that the extremum of a cable tension may be obtained at:

- the set of poses  $\mathcal{S}_C$  which satisfy  $\mathcal{C}_i^x = \mathcal{C}_i^y = 0$ : the resultant with respect to  $x$  of these two equations always factor out in either one quadratic equation in  $y$  or  $x$  (for  $\tau_3$ ) or in two quadratic equations (for  $\tau_1, \tau_2$ ). Determining the intersection of the conics leads to up to 4 poses for each  $\tau_i$  that are obtained by solving two quadratic polynomials so that we have to solve a total of 6 quadratic polynomials

- the set  $S_E$  consisting of the three corners of the triangle  $A_1^p A_2^p A_3^p$  that are projected at the maximum altitude
- the set  $S_B$  for of the poses for which either  $x$  or  $y$  is extremal (i.e being equal to  $x_r^1, x_r^2, y_r^1, y_r^2$  and  $\mathcal{C}_i^x = 0$  or  $\mathcal{C}_i^y = 0$ . As we have 5 conics and four extremal coordinates determining this set require to solve 20 quadratic polynomials
- the set  $S_U$  of poses such that  $U_j = 0, j \in [2, 3]$  (which defines a line in the  $x - y$  plane) and  $\mathcal{C}_i^x = 0$  or  $\mathcal{C}_i^y = 0$  with  $i \neq j$ . In that case we have either  $\tau_2 = 0$  or  $\tau_3 = 0$  (only 2 cables are under tension) and  $S_U$  is obtained by solving 7 quadratic polynomials

Note that we retain as member of the sets only the one that are inside the desired workspace. Obtaining the sets  $S_B, S_C, S_E, S_U$  therefore require to solve a total of 33 second order polynomials. As we have to repeat the process for each triple of cables we have to solve  $33n(n-1)(n-2)/6$  quadratic polynomials. For example if we have a 4-cables CDPR we will have to solve 132 second order polynomials.

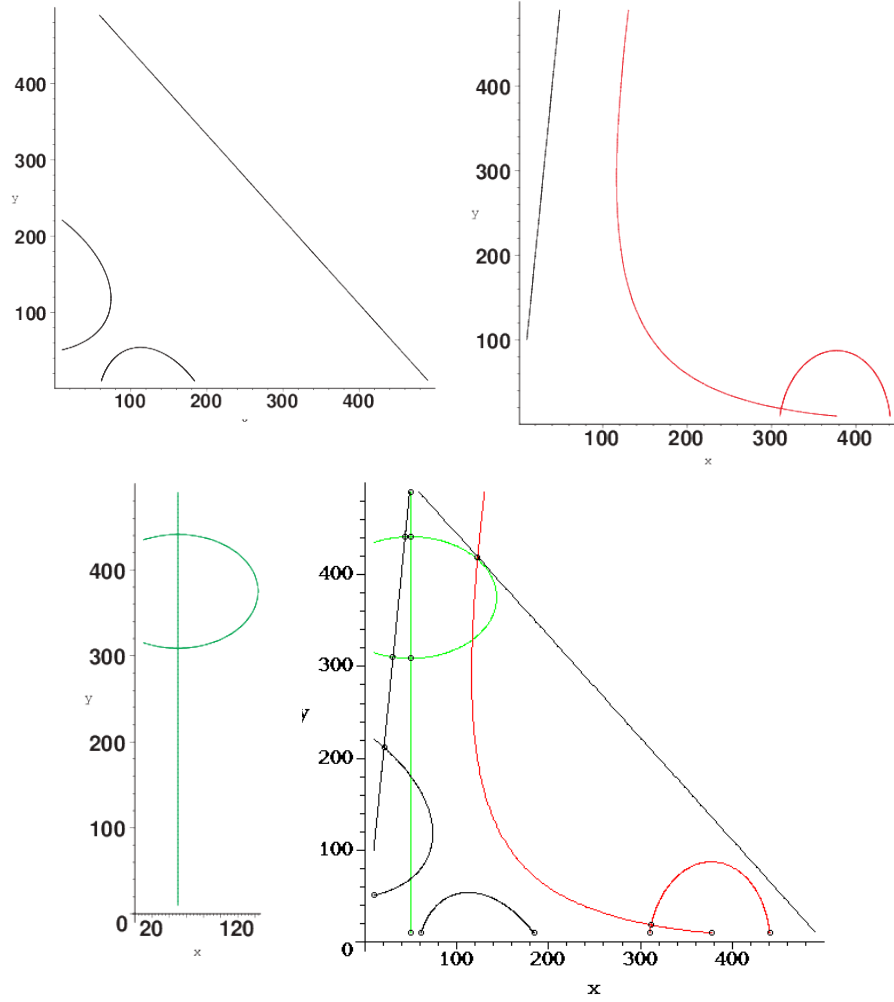
**Examples:** The length unit in this paper is the centimeter. For the first example we choose  $x_{a_2} = 500, x_{a_3} = 50, y_{a_3} = 500$ , the workspace being restricted to  $xi^1 = 10, xr^2 = 490, yr^1 = 10, yr^2 = 490$  and we consider the cable 1, 2 and a cable attached at (50,500). The value of  $z_a$  is set to 450, the load weight is 100N, while the height of the workspace is limited to 300. A second example just differs from the first one by having  $x_{a_3} = 0$ . In figure 2 we present the various geometrical elements that are considered for the calculation are shown, the points that are used to compute the maximal tensions (in solid circles) while table 1 presents the maximal tensions.

	x	y	$\tau$ max
cable 1 (1st example)	21.184	211.844	100.06
cable 2 (1st example)	309.962	10	99.82
cable 3 (1st example)	31.02	310.23	100.36
cable 1 (2nd example)	183.379	10	96.94
cable 2 (2nd example)	309.23	10	100.143
cable 3 (2nd example)	10	309.23	100.143

**Table 1.** Maximal cable tension for the 2 examples

## 2.2 Maximal tensions with 4 cables under tension

As mentioned previously we cannot by control put a CDPR in a configuration where 4 cables (or more) are under tension and stop the CDPR in that configuration. But during the execution of a trajectory such a configuration may temporarily occur after which the CDPR will move to a configuration with three or two cables only are under tension. Intuitively it may be thought that at a given pose when having four cables under tension instead of three will lead to a decrease of the cable tensions but unfortunately intuition is wrong. Indeed as seen in the 3-cables case the maximum of  $\tau_1$  is obtained for a pose that is usually on the line joining  $A_1$  to another of the  $A_i$  points. Now if we introduce a fourth cable we have to consider the  $\tau_1$  that will be obtained for a pose lying on the line  $A_1 A_4$  and it may perfectly occurs that there is pose on that lien such that  $\tau_1$  is larger than the maximum obtained in the 3-cables case.



**Fig. 2.** The elements considered for the calculation of the maximal tensions. Top left we have the 2 conics and line  $U_1 = 0$  for cable 1, top right the same elements for cable 2, bottom left for cable 3 (which has a single conic). At the bottom right we have the same elements in a single drawing and In solid circle we have the poses that are considered for the calculation.

Consider for example a four cables CDPR with  $A_i$  coordinates  $(0,0,z_a), (x_{a2} > 0,0,z_a), (0,y_{a3} > 0,z_a)$  while the  $A$  coordinates of a fourth cable are  $(0 < x_{a4} \leq x_{a2}, y_{a3}, z_a)$ . Using the static equilibrium linear equations we may establish for a given pose  $\mathbf{X}$  the tensions in the cables 1, 2, 3 as a function of the tension  $\tau_4$  in the fourth cable. These tensions may be written as

$$\tau_i = \tau_i^0 + w_i \tau_4, i \in [1, 3] \quad (1)$$

where  $\tau_i^0$  is the tension when only the 3 cables 1,2,3 are under tension (which is positive if  $\mathbf{X}^p$  lies in the triangle  $A_1^p A_2^p A_3^p$ ) and the  $w_i$  coefficients are obtained as:

$$w_1 = \frac{x_{a4} \rho_1}{\rho_4 x_{a2}} \quad w_2 = -\frac{x_{a4} \rho_2}{\rho_4 x_{a2}} \quad w_3 = -\frac{\rho_3}{\rho_4}$$

so that  $w_1 > 0, w_2 < 0, w_3 < 0$ . Hence if indeed  $\tau_2, \tau_3$  decrease when a positive tension is applied on cable 4, at the opposite  $\tau_1$  will increase. As we cannot master the tension  $\tau_4$  it is therefore necessary to investigate its increasing effect on  $\tau_1$ . However there is an upper limit for the value of  $\tau_4$  which is obtained when either  $\tau_2$  or  $\tau_3$  is equal to 0 as this implies a change in the CDPR configuration. Note also that the maximum for  $\tau_1$  may be obtained at a pose that is different from the one we have obtained when calculating the maximum for the 3-cables case. Indeed the increase of  $\tau_1$  due to the presence of  $\tau_4$  is dependent upon the pose so that there is no reason that  $\tau_1^0 + w_i \tau_4$  may not be larger than the same quantity obtained at the pose where  $\tau_1$  has been the maximum for the 3-cables case.

The maximum of  $\tau_1$  is obtained at a pose such that either  $\tau_2$  or  $\tau_3$  (or both) cancels or the derivatives of  $\tau_1$  with respect to  $x, y$  cancel. The later case is not possible as the numerator of both derivatives is equal to  $\rho_1 mg$ . We will thus determine  $\tau_1$  if  $\tau_2 = 0, \tau_3 = 0, \tau_2 = \tau_3 = 0$  and we will retain the one leading to the largest  $\tau_1$ .

**Case with  $\tau_2 = 0$**  First we determine the values of  $\tau_4$  that cancel  $\tau_2$  and report these values in  $\tau_1$  so that  $\tau_1$  is now a function of only  $x, y, z$ . We then consider the derivatives of  $\tau_1$  with respect to  $x, y$  and determine the constraints that cancel these terms. The derivative of  $\tau_1$  with respect to  $x$  cancel only for  $x = 0$  or  $y = y_{a3}$  while the derivative of  $\tau_1$  with respect to  $y$  leads to a conic  $C_{y2} = 0$ .

**Case with  $\tau_3 = 0$**  First we determine the values of  $\tau_4$  that cancel  $\tau_3$  and report these values in  $\tau_1$  so that  $\tau_1$  is now a function of only  $x, y, z$ . The derivatives of  $\tau_1$  with respect to  $x, y$  leads to two conics  $C_{x3} = 0, C_{y3} = 0$ .

**Case with  $\tau_2 = \tau_3 = 0$**  We have also to consider the case where  $\tau_4$  cancel both  $\tau_2$  and  $\tau_3$ : this will occur if  $x = x_{a4} y / y_{a3}$ . We report  $x$  and the corresponding value of  $\tau_4$  into  $\tau_1$  and then calculate the derivative of  $\tau_1$  with respect to  $y$  leads to a second order polynomial  $P_y$  in  $y$ .

Hence the maximum of  $\tau_1$  will be obtained at a pose such that:

- $x$  or  $y$  have one of the extremal value  $x_r^1, x_r^2, y_r^1, y_r^2$  and  $C_{x3}$  or  $C_{y3}$  or  $C_{y2}$  cancel. As  $x$  or  $y$  are fixed the conic leads to a quadratic polynomial in the remaining variable. As we have 3 conics and four extremal values we have to solve 12 quadratic polynomials.



- $C_{x_3} = C_{y_3} = 0$ : the resultant of  $C_{x_3}$  and  $C_{y_3}$  in  $x$  factors out in 2 quadratic polynomials. Therefore to deal with this case we have to solve 4 quadratic polynomials
- the roots of  $P_y$

Overall getting the maximum for the 4-cables cases amount to solve 17 quadratic polynomials. In all this cases we get a set of possible poses from which we eliminate the one that are outside the desired workspace. For each of the remaining poses we calculate the value of  $\tau_1$  and retain the maximal value. We have to repeat this process by considering all  $A = \prod_{j=0}^{n-2} (n-j)$  quadruples of cables among the  $n$  cables, thereby having to solve  $17A$  quadratics.

Note that in this section we have considered the case where both  $\tau_2$  and  $\tau_3$  cancel, so that we also consider the case where only  $n-2$  cables are under tension.

In summary for a 4-cables CDPR determining the maximal tension for all cables requires at most to solve  $17+132=149$  second order polynomials, thereby leading to a very fast algorithm.

**Example:** using the same data than for the second example of the previous section with  $x_{a_4} = 250, y_{a_4} = 500$  leads to a maximum of  $\tau_1$  of 107.743 at the pose  $x = 103.197, y = 206.394$ . We note that this value is approximately 10% higher than the value we have obtained for three cables. The pose at which this maximum is obtained is also different from the pose obtained in the three-cables case.

### 3 Extension to $N > 4$ cables

We may generalize the result obtained in the previous section to a CDPR with an arbitrary number of cables. Again we consider the pose  $\mathbf{X}$  whose  $\mathbf{X}^p$  is included in the triangle  $A_1^p A_2^p A_3^p$  where the coordinates of  $A_1, A_2, A_3$  are  $(0, 0, z_a), (x_{a_2} > 0, 0, z_a), (0, y_{a_3} > 0, z_a)$ . We will assume that the other cables have as coordinates either  $(x_{a_2}, y_{a_j}, z_a)$  with  $0 \leq y_{a_j} \leq y_{a_3}$  or  $(x_{a_j}, y_{a_3}, z_a)$  with  $0 \leq x_{a_j} \leq x_{a_2}$  i.e.  $A_j$  lies on an edge of the rectangle  $\mathcal{R}$ . The static equilibrium condition may be written as

$$\mathbf{J}_3(\tau_1, \tau_2, \tau_3)^T = \left( \sum_{k=4}^{k=n} -\frac{(x-x_{a_k})\tau_k}{\rho_k}, \sum_{k=4}^{k=n} -\frac{(y-y_{a_k})\tau_k}{\rho_k}, -mg - \sum_{k=4}^{k=n} \frac{(z-z_a)\tau_k}{\rho_k} \right)^T \quad (2)$$

where  $\mathbf{J}_3$  is the transpose of the inverse kinematic jacobian restricted to cable 1,2,3. Let us define  $\mathbf{J}_2$  as the matrix derived from  $\mathbf{J}_3$  by substituting the second column of  $\mathbf{J}_3$  by  $(u_1, u_2, u_3)$ . The determinant of  $\mathbf{J}_2, \mathbf{J}_3$  are obtained as

$$|\mathbf{J}_2| = -\frac{y_{a_3}(u_1(z-z_a) - u_3x)}{\rho_1\rho_3} \quad |\mathbf{J}_3| = \frac{(z-z_a)x_{a_2}y_{a_3}}{\rho_1\rho_2\rho_3} \quad (3)$$

so that  $-|\mathbf{J}_2|/|\mathbf{J}_3|$  is  $(u_1(z-z_a) - u_3x)\rho_2/((z-z_a)x_{a_2})$ . If we set  $u_1, u_2, u_3$  to the components of the right-hand side of equation (2) this ratio will be equal to  $\tau_2$ . Consequently  $\tau_2$  may be written as

$$\tau_2 = a_0 + \sum_{k=4}^{k=n} a_k \tau_k$$

where the  $a_k$  are constants that depend only upon the CDPR pose and geometry. The coefficient  $a_k$  of  $\tau_k$  in  $\tau_2$  is

$$a_k = \frac{((z - z_a) \frac{(x - x_{a_k})}{\rho_k} - x \frac{(z - z_a)}{\rho_k}) \rho_2}{(z - z_a) x_{a_2}} = - \frac{x_{a_k} \rho_2}{x_{a_2} \rho_k}$$

which is always negative. Thus any additional cable having a positive tension will reduce the value of  $\tau_2$ . The same procedure may be used to determine the coefficient of  $\tau_k$  in  $\tau_3$  whose value is  $-y_{a_k} \rho_3 / (y_{a_3} \rho_k)$  whose value is also always negative so that  $\tau_3$  decreases when additional cables are used. On the other hand the coefficient  $c_{k1}$  of  $\tau_k$  in  $\tau_1$  is

$$c_{k1} = - \frac{\rho_1 (y_{a_3} (x_{a_2} - x_{a_k}) - x_{a_2} y_{a_k})}{\rho_k y_{a_3} x_{a_2}}$$

as we have either  $x_{a_k} = x_{a_2}$  or  $y_{a_k} = y_{a_3}$ , then we get

$$c_{k1} = \frac{\rho_1 y_{a_k}}{y_{a_3} \rho_k} \quad \text{or} \quad c_{k1} = \frac{\rho_1 x_{a_k}}{x_{a_2} \rho_k}$$

so that  $c_{k1}$  is always positive. Hence  $\tau_1$  increases whenever a cable with a positive tension is added.

**Theorem 2:** if we consider a triplet of cables that represent three successive corners  $i, j, k$  and two edges of the desired workspace, then for any number of additional cables  $l$  whose  $A$  lies on one of the two other edges of the workspace a positive tension leads to an increase of one element in the set  $\{\tau_i, \tau_j, \tau_k\}$  and to a decrease for the other elements of the set.

Let us assume that  $\tau_i$  is the cable tension that increases with  $\tau_l$ . The derivatives of  $c_{k1}$  with respect to  $x, y$  is always a quadratic equation and consequently determining the value of  $\tau_l$  that leads to the largest  $\tau_i$  under the constraint  $\tau_j \geq 0, \tau_k \geq 0$  follows the same process than for the 4-cables case.

## 4 Conclusion

We have proposed in this paper a fast algorithm for computing the maximal cable tensions of a N-1 CDPR over a given workspace that provides exactly the result by solving only a limited set of 2nd order equations, without having to sample the workspace. Furthermore as all these equations are independent the algorithm may be implemented on a multi-core processor so that the computation time will be extremely low. A first possible extension will be to deal with an arbitrary disposition of the winch points together with a more general way to describe the workspace: we believe that as soon as this description involves only linear elements or is parametric this extension will not be difficult. A second possible extension will be to consider an arbitrary number of cables: here again although we will have to manage the combinatorial aspect the principle of the proposed algorithm will remain the same. Finally we may consider uncertainties in the design: as the calculation relies only on simple, low-degree, algebraic equations

it will be relatively easy to determine the worst case maximal tension for example by using interval analysis.

Regarding open issues we may consider taking into account the cables sagging: here the open question for the N-1 case is if the maximal cable tensions with sagging will be obtained at the same pose than when considering ideal cables (in which case taking sagging into account will not drastically change the algorithm, including taking into account the uncertainties on the robot geometry) ? We may have also to consider the case where the platform exhibits limited oscillations so that the force exerted at  $B$  lie in a cone.

Finally extending this work to 6 dof CDPRs will be quite complex as cable/cable and platform/cable interference and limitation on the cable lengths will have to be taken into account.

Extension to a dynamics analysis for a given trajectory appears to be possible as the platform dynamics may be solved to determine the forces exerted at  $B$  but will be more complex as time will be an additional variable so that a temporal-geometrical analysis will have to be performed.

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