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# Yaglom limit for critical neutron transport

S. C. Harris\*, E. Horton<sup>†</sup>, A. E. Kyprianou<sup>‡</sup> and M. Wang<sup>§</sup>

April 30, 2021

## Abstract

We consider the classical Yaglom limit theorem for the Neutron Branching Process (NBP) in the setting that the mean semigroup is critical, i.e. its leading eigenvalue is zero. We show that the law of the process conditioned on survival is asymptotically equivalent to an exponential distribution. As part of the proof, we also show that the probability of survival decays inversely proportionally to time. Although Yaglom limit theorems have recently been handled in the setting of spatial branching processes and superprocesses, [22, 23], as well as in the setting of isotropic homogeneous Neutron Branching Processes, [21], our approach and the main novelty of this work is based around a precise result for the scaled asymptotics for the  $k$ -th martingale moments of the NBP (rather than the Yaglom limit itself). Our proof of the asymptotic martingale moments turns out to offer a general approach to asymptotic martingale moments of critical branching Markov processes with a non-local branching mechanism. Indeed this is the context in which we give both our moment proofs and the Yaglom limit.

**Key words:** Neutron Transport Equation, branching Markov process, semigroup theory, Perron-Frobenius decomposition, Yaglom limit, quasi-stationary limit.

**Mathematics Subject Classification:** Primary 82D75, 60J80, 60J75. Secondary 60J99

## 1 Introduction

The neutron transport equation (NTE) describes the flux of neutrons with a given velocity across a planar cross-section centred at a given point in an inhomogeneous fissile medium. The backwards form of the NTE can be written as follows,

$$\begin{aligned} \frac{\partial}{\partial t} \psi_t(r, v) &= v \cdot \nabla \psi_t(r, v) - \sigma(r, v) \psi_t(r, v) \\ &+ \sigma_{\mathbf{s}}(r, v) \int_V \psi_t(r, v') \pi_{\mathbf{s}}(r, v, v') dv' + \sigma_{\mathbf{f}}(r, v) \int_V \psi_t(r, v') \pi_{\mathbf{f}}(r, v, v') dv', \end{aligned} \quad (1.1)$$

where the flux  $\psi_t(r, v)$  is a function of time,  $t \geq 0$ , and the configuration variables  $(r, v) \in D \times V$  where the spatial domain  $D \subseteq \mathbb{R}^3$  is non-empty, smooth, bounded and convex such that  $\partial D$  has

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zero Lebesgue measure, and  $V = \{v \in \mathbb{R}^3 : v_{\min} \leq |v| \leq v_{\max}\}$  is the velocity space, where  $0 < v_{\min} < v_{\max} < \infty$ . Furthermore, the other components of (1.1) have the following interpretation:

- $\sigma_{\mathbf{s}}(r, v)$ : the rate at which scattering occurs from incoming velocity  $v$ ,
- $\sigma_{\mathbf{f}}(r, v)$ : the rate at which fission occurs from incoming velocity  $v$ ,
- $\sigma(r, v)$ : the sum of the rates  $\sigma_{\mathbf{f}} + \sigma_{\mathbf{s}}$  and is known as the total cross section,
- $\pi_{\mathbf{s}}(r, v, v')dv'$ : the scattering yield at velocity  $v'$  from incoming velocity  $v$ ,  
satisfying  $\int_V \pi_{\mathbf{s}}(r, v, v')dv' = 1$ , and
- $\pi_{\mathbf{f}}(r, v, v')dv'$ : the neutron yield at velocity  $v'$  from fission with incoming velocity  $v$ ,  
satisfying  $\int_V \pi_{\mathbf{f}}(r, v, v')dv' < \infty$ .

We also enforce the following initial and boundary conditions

$$\begin{cases} \psi_0(r, v) = g(r, v) & \text{for } r \in D, v \in V, \\ \psi_t(r, v) = 0 & \text{for } t \geq 0 \text{ and } r \in \partial D \text{ if } v \cdot \mathbf{n}_r > 0, \end{cases} \quad (1.2)$$

where  $\mathbf{n}_r$  is the outward unit normal at  $r \in \partial D$  and  $g : D \times V \rightarrow [0, \infty)$  is a bounded, measurable function. Throughout we will rely on the following assumptions in some (but not all) of our results:

- (H1): The cross-sections  $\sigma_{\mathbf{s}}$ ,  $\sigma_{\mathbf{f}}$ ,  $\pi_{\mathbf{s}}$  and  $\pi_{\mathbf{f}}$  are uniformly bounded away from infinity.**
- (H2): We have  $\sigma_{\mathbf{s}}\pi_{\mathbf{s}} + \sigma_{\mathbf{f}}\pi_{\mathbf{f}} > 0$  on  $D \times V \times V$ .**
- (H3): There is an open ball  $B$  compactly embedded in  $D$  such that  $\sigma_{\mathbf{f}}\pi_{\mathbf{f}} > 0$  on  $B \times V \times V$ .**
- (H4): the fission offspring are bounded in number by the constant  $N_{\max} > 1$ .**

Note, the assumption (H1) ensures that all activity occurs at a maximum rate. Assumption (H2) ensures that at least some activity occurs, whether it be scattering or fission; together with (H3), it ensures that there is at least some fission as well as scattering. Finally (H4) is a physical constraint that is natural to nuclear fission, typically no more than 3 neutrons are produced during an average fission event. Figure 1 illustrates the complex nature of the inhomogeneity in the domain one typically considers.

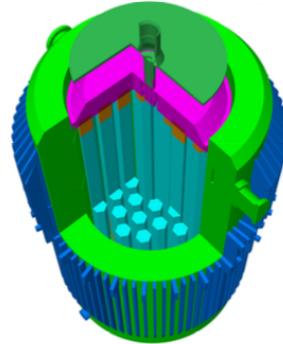


Figure 1: *The geometry of a nuclear reactor core representing a physical domain  $D$ , onto which the different cross-sections  $\sigma_{\mathbf{s}}$ ,  $\sigma_{\mathbf{f}}$ ,  $\pi_{\mathbf{s}}$ ,  $\pi_{\mathbf{f}}$  are mapped, also as a function of neutron velocity.*

Recent literature has explored the relationship between the NTE and a class of stochastic processes; cf. [6, 7, 14, 13, 8]. Roughly speaking, the solution to the NTE can be seen as a semi-group of the underlying physical process of neutrons, described as a branching Markov process. As such, let us spend a little time giving sense to what this means. We recall the neutron branching process (NBP) defined in [14, 13, 6], which at

time  $t \geq 0$  is represented by a configuration of particles which are specified via their physical location and velocity in  $D \times V$ , say  $\{(r_i(t), v_i(t)) : i = 1, \dots, N_t\}$ , where  $N_t$  is the number of particles alive at time  $t$ . The NBP is then given by the empirical distribution of these configurations,

$$X_t(A) = \sum_{i=1}^{N_t} \delta_{(r_i(t), v_i(t))}(A), \quad A \in \mathcal{B}(D \times V), \quad t \geq 0, \quad (1.3)$$

where  $\delta$  is the Dirac measure, defined on  $\mathcal{B}(D \times V)$ , the Borel subsets of  $D \times V$ . In addition, we denote by  $\mathbb{P}_{\delta_{(r,v)}}$  the law of  $X$  initiated from a single particle with configuration  $r \in D, v \in V$ , and by  $\mathbb{E}_{\delta_{(r,v)}}$  the corresponding expectation operator.

The evolution of  $(X_t, t \geq 0)$  is a stochastic process valued in the space of atomic measures  $\mathcal{M}(D \times V) := \{\sum_{i=1}^n \delta_{(r_i, v_i)} : n \in \mathbb{N}, (r_i, v_i) \in D \times V, i = 1, \dots, n\}$  which evolves randomly as follows.

A particle positioned at  $r$  with velocity  $v$  will continue to move along the trajectory  $r + vt$ , until one of the following things happens.

(i) The particle leaves the physical domain  $D$ , in which case it is instantaneously killed.

(ii) Independently of all other neutrons, a scattering event occurs when a neutron comes in close proximity to an atomic nucleus and, accordingly, makes an instantaneous change of velocity. For a neutron in the system with position and velocity  $(r, v)$ , if we write  $T_s$  for the random time that scattering may occur then, independently of any other physical event that may affect the neutron,  $\Pr(T_s > t) = \exp\{-\int_0^t \sigma_s(r + vs, v) ds\}$ , for  $t \geq 0$ .

When scattering occurs at space-velocity  $(r, v)$ , the new velocity  $v' \in V$  is selected with probability  $\pi_s(r, v, v') dv'$ .

(iii) Independently of all other neutrons, a fission event occurs when a neutron smashes into an atomic nucleus. For a neutron in the system with initial position and velocity  $(r, v)$ , if we write  $T_f$  for the random time that fission may occur, then, independently of any other physical event that may affect the neutron,  $\Pr(T_f > t) = \exp\{-\int_0^t \sigma_f(r + vs, v) ds\}$ , for  $t \geq 0$ .

When fission occurs, the smashing of the atomic nucleus produces lower mass isotopes and releases a random number of neutrons, say  $N \geq 0$ , which are ejected from the point of impact with randomly distributed, and possibly correlated, velocities, say  $\{v_i : i = 1, \dots, N\}$ . See Figure 2. The outgoing velocities are described by the atomic random measure

$$\mathcal{Z}(A) := \sum_{i=1}^N \delta_{v_i}(A), \quad A \in \mathcal{B}(V). \quad (1.4)$$

If such an event occurs at location  $r \in D$  from a particle with incoming velocity  $v \in V$ , we denote by  $\mathcal{P}_{(r,v)}$  the law of  $\mathcal{Z}$ , and by  $\mathcal{E}_{(r,v)}$  the corresponding expectation operator. The probabilities  $\mathcal{P}_{(r,v)}$  are such that, for bounded and measurable  $g : V \rightarrow [0, \infty)$ ,

$$\int_V g(v') \pi_f(r, v, v') dv' = \mathcal{E}_{(r,v)} \left[ \int_V g(v') \mathcal{Z}(dv') \right] \quad (1.5)$$

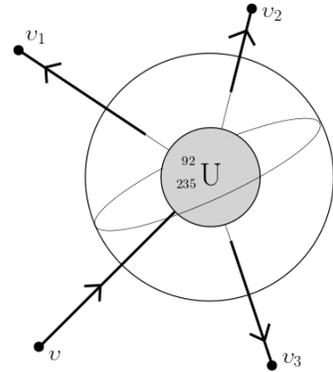


Figure 2: A fission event between e.g. a Uranium-235 atom and a neutron with incoming velocity  $v$  produces three outgoing neutrons with velocity  $v_1, v_2, v_3$ .

Note, the possibility that  $\Pr(N = 0) > 0$ , which is tantamount to neutron capture (that is, where a neutron slams into a nucleus but no fission results and the neutron is absorbed into the nucleus), is included in the notion of fission.

We will tend to work with the process  $X$  by describing the behaviour of  $\langle f, X_t \rangle$ ,  $t \geq 0$ , for  $f \in L_\infty^+(D \times V)$ , the space of non-negative, uniformly bounded functions on  $D \times V$ , where  $\langle f, X_t \rangle := \int_{D \times V} f(r, v) X(dr, dv) = \sum_{i=1}^{N_t} f(r_i(t), v_i(t))$ . Having introduced this inner product notation, we note that we will immediately abuse it by writing e.g.  $\langle f, g \rangle$  to mean  $\int_{D \times V} f(r, v) g(r, v) dr dv$ , as well as  $\langle g, \mathcal{Z} \rangle$  to mean  $\int_V g(v) \mathcal{Z}(dv) = \sum_{i=1}^N g(v_i)$ . In the latter case, we can otherwise think of the point process  $\mathcal{Z}$  in (1.5) as placing offspring at  $\{(r_i, v_i), i = 1, \dots, N\}$  with  $r_i = r$ ,  $i = 1, \dots, N$  under  $\mathcal{P}_{(r, v)}$ .

The NBP is thus parameterised by the quantities  $\sigma_s(r, v)$ ,  $\pi_s(r, v)$ ,  $\sigma_f(r, v)$  and the measures  $\mathcal{P}_{(r, v)}$  for  $r \in D, v \in V$ . Accordingly, we refer to it as a  $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$ -NBP. Although a  $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$ -NBP is uniquely defined, a NBP specified by  $(\sigma_s, \pi_s, \sigma_f, \pi_f)$  alone is not. Nonetheless, it is easy to show that for a given  $\pi_f$ , a  $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$ -NBP satisfying (1.5) always exists. See the discussion in Section 2 of [14].

Although it is not yet clear what ‘critical’ means for the NBP, our objective in this article is to show that, like all other branching processes in a critical regime, the NBP respects a so-called Yaglom limit. The latter can be thought of as consisting of two components. The first is that for all  $r \in D, v \in V$ , there exists a constant  $c_1(r, v) \in (0, \infty)$  such that

$$\lim_{t \rightarrow \infty} t \mathbb{P}_{\delta_{(r, v)}}(\zeta > t) = c_1(r, v), \quad (1.6)$$

where  $\zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}$ . The second component is that, for each  $f \in L_\infty^+(D \times V)$ , there exists a constant  $c_2(f)$ , depending on  $f$ , such that for each  $r \in D$  and  $v \in V$ ,

$$\text{Law} \left( \frac{\langle f, X_t \rangle}{t} \middle| \zeta > t \right) \text{ converges weakly to that of } \mathbf{e}_{c_2(f)}, \quad (1.7)$$

as  $t \rightarrow \infty$  where  $\mathbf{e}_{c_2(f)}$  is an exponentially distributed random variable with rate  $c_2(f)$ . In essence, the Yaglom limit theorem demonstrates that, up to exponentially distributed noise, mass decays linearly to extinction.

Yaglom’s limit theorem is very classical for Galton-Watson processes, cf. [1], and a cornerstone of the theory of branching processes. Remarkably, it is only recently that Yaglom limit theorems have found their way to the literature for spatial branching processes, for example [22] in the setting of branching Brownian motion and [23, 24, 25] in the setting of superprocesses. Unfortunately none of the approaches taken there can be transferred to the current setting on account of the fact that there is non-local branching, which complicates calculations significantly. Interestingly, there has been previous work on the Yaglom limit in the setting of NBPs dating back to the 1970s, cf. [21]. However, this paper has the advantage of an isotropic assumption for scattering and the point process  $\mathcal{Z}$ , as well as homogeneous cross-section, which allows the system to be treated as if it has local branching. More precisely,  $\mathcal{Z}$  has an independent number of points, each of which are uniformly distributed on  $V$ , which is otherwise taken to be  $\mathbb{S}^2$ . In the sequel we will refer to this as the *isotropic case*.

The situation we consider here is significantly more complicated than the isotropic case. The anisotropic behaviour of the fission means that we cannot treat branching as a local phenomenon as in [21]. A large component of our proof of (1.6) and (1.7) (stated more precisely in Theorems

3.1 and 3.2 below) appeals to a new combinatorial methodology which is necessary on account of the spatial inhomogeneity and non-local nature of the branching mechanism. Specifically, our new approach and the main novelty of this work is based around an inductive argument, which describe the convergence of all of the associated martingale moments of the NBP. For the proof of the latter, we make use of the recently demonstrated spine decomposition, cf. [14], in order to extract the leading order terms using a combinatorial decomposition. In addition to this, we provide a new asymptotic analysis of a non-linear form of the NTE which does not currently exist in the literature. The new analysis we present here offers a degree of robustness in that it can be developed further, offering applicability to general branching Markov processes. Indeed, we justify this claim by proving our results for a general class of branching Markov processes with non-local branching mechanisms, which we define in section 5.

In order to understand the relationship of the Yaglom limit for the NBP with the NTE, we need to introduce a more appropriate representation of the the NTE (1.1).

## 2 NTE in mild form

To make a clear link between the NTE and the NBP, a more natural way to formulate the NTE (1.1) in the probabilistic setting is via its *Duhamel* representation since the irregular nature of gradient operator, (1.1) makes it hard to work with in the pointwise sense. The following integral equation was introduced in [14] as a mild form of the NTE,

$$\psi_t[g] = \mathbf{U}_t[g] + \int_0^t \mathbf{U}_s[(\mathbf{S} + \mathbf{F})\psi_{t-s}[g]]ds, \quad t \geq 0, \quad (2.1)$$

where

$$\mathbf{U}_t[g](r, v) = g(r + vt, v)\mathbf{1}_{(t < \kappa_{r,v}^D)}, \quad \kappa_{r,v}^D = \inf\{t > 0 : r + vt \notin D\},$$

the operator  $\mathbf{S}$  has action on  $L_\infty^+(D \times V)$ , the space of non-negative and uniformly bounded measurable functions on  $D \times V$ , given by

$$\mathbf{S}g(r, v) = \sigma_{\mathbf{s}}(r, v) \int_{D \times V} [g(r, v') - g(r, v)]\pi_{\mathbf{s}}(r, v, v')dv', \quad g \in L_\infty^+(D \times V),$$

and, similarly,  $\mathbf{F}$  has action

$$\mathbf{F}g(r, v) = \sigma_{\mathbf{f}}(r, v) \int_{D \times V} g(r, v')\pi_{\mathbf{f}}(r, v, v')dv' - \sigma_{\mathbf{f}}(r, v)g(r, v), \quad g \in L_\infty^+(D \times V).$$

In [6, 14, 10, 9] it was shown that under (H1) and (H2), for  $g \in L_\infty^+(D \times V)$  there is a solution to (2.1) which is bounded in the sense that there exist constants  $C_1, C_2 > 0$  such that  $\psi_t[g](r, v) \leq C_1 \exp(C_2 t)$ , uniformly for all  $t \geq 0$ .

We can associate the solution to the NTE with the following linear semigroup of the NBP,

$$\psi_t[g](r, v) := \mathbb{E}_{\delta_{(r,v)}}[\langle g, X_t \rangle], \quad t \geq 0, r \in \bar{D}, v \in V, \quad (2.2)$$

where, differently to the notation  $\langle \cdot, \cdot \rangle$  in (1.5), we are using the definition

$$\langle g, X_t \rangle = \int_{D \times V} g(r, v)X_t(dr, dv) = \sum_{i=1}^{N_t} g(r_i(t), v_i(t)).$$

It is worth noting that we will occasionally abuse this notation and, for  $f, g \in L_\infty^+(D \times V)$  we will write  $\langle f, g \rangle = \int_{D \times V} f(r, v)g(r, v)drdv$ .

As shown in [6, 14], there is a second stochastic process whose average behaviour also provides a solution to the NTE, called the neutron random walk (NRW).

Define, for  $r \in D$ ,  $v, v' \in V$ ,

$$\alpha(r, v)\pi(r, v, v') = \sigma_{\mathbf{s}}(r, v)\pi_{\mathbf{s}}(r, v, v') + \sigma_{\mathbf{f}}(r, v)\pi_{\mathbf{f}}(r, v, v'). \quad (2.3)$$

Then the  $\alpha\pi$ -NRW can be defined as the process whose pathwise evolution is such that, if its current position and velocity are  $r \in D$  and  $v \in V$ , respectively, then the process will move forward in time in  $D$  with linear motion  $r + vt$ . Unless the NRW first exits the domain  $D$ , at rate  $\alpha(r + vt, v)$ , a scattering event occurs. Suppose  $r' \in D$  is the NRW's position at the scattering event. Then a new velocity  $v'$  is chosen with probability  $\pi(r', v, v')dv'$ . Thus the process evolves by moving in straight lines between scatter times, until it exits the domain  $D$ .

Suppose we denote the probabilities of the  $\alpha\pi$ -NRW by  $\mathbf{P} := (\mathbf{P}_{(r,v)}, (r, v) \in D \times V)$ . Moreover, for  $r \in D$  and  $v \in V$ , let

$$\beta(r, v) = \sigma_{\mathbf{f}}(r, v) \left( \int_V \pi_{\mathbf{f}}(r, v, v')dv' - 1 \right) \geq - \sup_{r \in D, v \in V} \sigma_{\mathbf{f}}(r, v) > -\infty, \quad (2.4)$$

where the lower bound is due to assumption (H1). The following lemma, taken from [14], identifies the law  $\mathbf{P}$  in terms of the solution to the NTE (2.1), the so called *many-to-one* representation.

**Lemma 2.1** (Many-to-one). *Under assumptions (H1) and (H2), the law  $\mathbf{P}$  is related to NTE (2.1) via a semigroup representation, different to (2.2), given by*

$$\psi_t[g](r, v) = \mathbf{E}_{(r,v)} \left[ e^{\int_0^t \beta(R_s, \Upsilon_s) ds} g(R_t, \Upsilon_t) \mathbf{1}_{\{t < \tau^D\}} \right], \quad t \geq 0, r \in D, v \in V, \quad (2.5)$$

where  $\tau^D = \inf\{t > 0 : R_t \notin D\}$ .

Like many branching processes, understanding the long-term growth of the NBP boils down to an understanding of the lead eigenvalue associated to the semigroup (2.2) in the spirit of Perron-Frobenius theory. Classical neutron transport theory has otherwise seen this as an issue of criticality of the underlying NTE (1.1), i.e. the sign of the leading eigenvalue associated to the latter.

**Theorem 2.1.** *Suppose that (H1) holds, in addition to the condition*

$$(\mathbf{H2})^*: \inf_{r \in D, v, v' \in V} \alpha(r, v)\pi(r, v, v') > 0.$$

*Then, for the semigroup  $(\psi_t, t \geq 0)$  identified by (2.1), there exists a  $\lambda_* \in \mathbb{R}$ , a positive<sup>1</sup> right eigenfunction  $\varphi \in L_\infty^+(D \times V)$  and a left eigenmeasure which is absolutely continuous with respect to Lebesgue measure on  $D \times V$  with density  $\tilde{\varphi} \in L_\infty^+(D \times V)$ , both having associated eigenvalue  $e^{\lambda_* t}$ , and such that  $\varphi$  (resp.  $\tilde{\varphi}$ ) is uniformly (resp. a.e. uniformly) bounded away from zero on each compactly embedded subset of  $D \times V$ . In particular, for all  $g \in L_\infty^+(D \times V)$ ,*

$$\langle \tilde{\varphi}, \psi_t[g] \rangle = e^{\lambda_* t} \langle \tilde{\varphi}, g \rangle \quad (\text{resp. } \psi_t[\varphi] = e^{\lambda_* t} \varphi) \quad t \geq 0. \quad (2.6)$$

Moreover, there exists  $\varepsilon > 0$  such that

$$\sup_{g \in L_\infty^+(D \times V): \|g\|_\infty \leq 1} \left\| e^{-\lambda_* t} \varphi^{-1} \psi_t[g] - \langle \tilde{\varphi}, g \rangle \right\|_\infty = O(e^{-\varepsilon t}) \text{ for all } t \geq 0. \quad (2.7)$$

<sup>1</sup>To be precise, by a positive eigenfunction, we mean a mapping from  $D \times V \rightarrow (0, \infty)$ . This does not prevent it being valued zero on  $\partial D$ , as  $D$  is an open bounded, convex domain.

**Remark 2.1.** It is usual to normalise the eigenfunctions  $\tilde{\varphi}$  and  $\varphi$  such that  $\langle \tilde{\varphi}, \varphi \rangle = 1$ .

The criticality of the NBP (equiv. NTE) is determined by the sign of  $\lambda_*$ . When  $\lambda_* > 0$  (resp.  $\lambda_* < 0$ ), the NBP is supercritical (resp. subcritical), situations which are undesirable for nuclear reactors. The case of criticality,  $\lambda_* = 0$ , has the property that there is a mean steady-state for the NBP in the sense that  $\psi_t[g](r, v) \rightarrow \langle \tilde{\varphi}, g \rangle \varphi(r, v)$  as  $t \rightarrow \infty$ .

Despite the latter, in Theorem 5.2 of [14], under the additional assumption

**(H3)\*: There exists an open ball  $B$ , compactly embedded in  $D$ , such that**

$$\inf_{r \in B, v, v' \in V} \sigma_{\mathbf{f}}(r, v) \pi_{\mathbf{f}}(r, v, v') > 0,$$

it was shown for the critical NBP that, for all  $r \in D, v \in V$ ,

$$\mathbb{P}_{\delta_{(r,v)}}(\zeta < \infty) = 1 \text{ where } \zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}.$$

### 3 Main results

In the rest of the paper, we restrict ourselves to the case where  $\lambda_* = 0$ . Before we state our main results, we will need to introduce additional notation and one more assumption. For  $g \in L_{\infty}^+(V)$  define the spatial variance of fission velocities by

$$\mathbb{W}[g](r, v) = \mathcal{E}_{(r,v)} [\langle g, \mathcal{Z} \rangle^2 - \langle g^2, \mathcal{Z} \rangle] = \mathcal{E}_{(r,v)} \left[ \sum_{\substack{1 \leq i, j \leq N \\ i \neq j}} g(v_i) g(v_j) \right]. \quad (3.1)$$

We will often abuse our notation and, for  $g \in L_{\infty}^+(D \times V)$ , write  $\mathbb{W}[g]$  in place of  $\mathbb{W}[g(r, \cdot)]$ . As such, which we now introduce the key constant

$$\Sigma = \langle \tilde{\varphi}, \sigma_{\mathbf{f}} \mathbb{W}[\varphi] \rangle, \quad (3.2)$$

which is finite thanks to (H1), (H4) and the boundedness of  $\tilde{\varphi}$  (cf. Theorem 2.1).

The following assumption will be key to the proof of Theorem 3.1 stated below.

**(H5): There exists a constant  $C > 0$  such that for all  $g \in L_{\infty}^+(D \times V)$ ,**

$$\langle \tilde{\varphi}, \sigma_{\mathbf{f}} \mathbb{W}[g] \rangle \geq C \langle \tilde{\varphi}, \hat{g}^2 \rangle,$$

**where**  $\hat{g} : D \rightarrow [0, \infty) : r \mapsto \int_V g(r, v') dv'$ .

Assumption (H5) can be thought of an irreducibility type condition on the fission operator, arguably playing an analogous role to e.g. the one uniform ellipticity plays for diffusion operators. It ensures that the NBP has a better chance to explore the state space. Indeed, we will later apply this condition to  $g = u_t$ , the survival probability. In this case, the righthand side can be seen as the survival probability of two independent particles started in the stationary distribution  $\tilde{\varphi}$ , but whose velocities are randomised uniformly. Hence, (H5) implies that if there is a fission event (in the stationary distribution), the process should have a better chance of survival than producing two particles with isotropic velocities. We note that this condition is satisfied in the setting of [21] and

in the isotropic case if we assume that, for all  $r \in D, v \in V$ ,  $\mathcal{E}_{(r,v)}[N(N-1)] \geq c > 0$  for some constant  $c$ , as well as in the analogous setting of [22] for branching Brownian motion on a compact domain.

We now state the main results.

**Theorem 3.1.** *Suppose that assumptions (H1), (H2)\*, (H3)\*, (H4) and (H5) hold and  $\lambda_* = 0$ . Then, for all  $r \in D$  and  $v \in V$ ,*

$$\lim_{t \rightarrow \infty} t \mathbb{P}_{\delta_{(r,v)}}(\zeta > t) = \frac{2\varphi(r, v)}{\Sigma}.$$

**Theorem 3.2** (Yaglom limit for critical neutron transport). *Suppose that assumptions (H1), (H2)\*, (H3)\*, (H4) and (H5) hold with  $\lambda_* = 0$ . For  $f \in L^+(D \times V)$  and for all  $r \in D$  and  $v \in V$ ,*

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\delta_{(r,v)}} \left[ \exp \left( -\theta \frac{\langle f, X_t \rangle}{t} \right) \middle| \zeta > t \right] = \frac{1}{1 + \langle \tilde{\varphi}, f \rangle \Sigma \theta / 2}. \quad (3.3)$$

As alluded to above, another way of reading Theorem 3.2 is to define  $\mathbb{P}_{\delta_{(r,v)}}^t(\cdot) = \mathbb{P}_{\delta_{(r,v)}}(\cdot | \zeta > t)$ , in which case  $(t^{-1} \langle f, X_t \rangle, \mathbb{P}_{\delta_{(r,v)}}^t)$  as  $t \rightarrow \infty$  converges weakly to an exponential random variable with rate  $2 / \langle \tilde{\varphi}, f \rangle \Sigma$ .

## 4 Three step approach to the Yaglom limit

The way we will approach the proof of Theorem 3.2 is through three fundamental steps.

### 4.1 Step 1

The first step, which is detailed in section 8, deals with the limiting behaviour of the martingale moments of the NBP. To this end, let us consider the following martingale change of measure,

$$\frac{d\mathbb{P}_\mu^\varphi}{d\mathbb{P}_\mu} \Big|_{\mathcal{F}_t} = \frac{\langle \varphi, X_t \rangle}{\langle \varphi, \mu \rangle} \quad t \geq 0, \mu \in \mathcal{M}(D \times V). \quad (4.1)$$

Note that this change of measure is well-defined due to the fact that  $\varphi$  is a harmonic function, i.e.

$$\mathbb{E}_\mu[\langle \varphi, X_t \rangle] = \langle \varphi, \mu \rangle, \quad t \geq 0, \mu \in \mathcal{M}(D \times V), \quad (4.2)$$

together with the branching Markov property.

Via a combination of the spine decomposition induced by (4.1), along with general results for Poisson processes and some combinatorial results, we will prove the following asymptotic moment result.

**Theorem 4.1.** *Under the assumptions of Theorem 3.2, for all  $k \geq 1$ , we have*

$$\lim_{t \rightarrow \infty} \sup_{r \in D, v \in V} \left| \frac{1}{t^k} \mathbb{E}_{\delta_{(r,v)}}^\varphi \left[ \langle \varphi, X_t \rangle^k \right] - (k+1)! \left( \frac{\Sigma}{2} \right)^k \right| = 0. \quad (4.3)$$

Equivalently, for all  $k \geq 1$ , we have the limiting moment behaviour under  $\mathbb{P}_{\delta_{(r,v)}}$ ,

$$\lim_{t \rightarrow \infty} \sup_{r \in D, v \in V} \left| \frac{1}{\varphi(r, v) t^k} \mathbb{E}_{\delta_{(r,v)}} \left[ \langle \varphi, X_t \rangle^{k+1} \right] - (k+1)! \left( \frac{\Sigma}{2} \right)^k \right| = 0. \quad (4.4)$$

This result is new in its own right, independently of the Yaglom limit and little seems to be known in the literature for (asymptotic) limits of spatial branching processes.

## 4.2 Step 2

For the next step, in section 9 we prove Theorem 3.1. Our approach will take inspiration from the original approach to the Yaglom limit theorem for the NBP in §5 of [21]. That is, we develop a non-linear integral equation for the probability  $\mathbb{P}_{\delta_{(r,v)}}(\zeta > t)$  and show, using an analytical approach of the operators involved, that its asymptotic behaviour is linear. However, one major difference with the setting of §5 of [21] is that the anisotropic inhomogeneous nature of the branching mechanism forces a significantly more complex analysis of the aforementioned non-linear equation. Hence, we first obtain coarse upper and lower bounds of order  $1/t$ , which are then used to obtain the precise limit.

We note that this also concludes the proof of Theorem 3.2, but only in the setting that  $f = \varphi$ . Indeed,

$$\begin{aligned} \mathbb{E}_{\delta_{(r,v)}}[\langle \varphi, X_t \rangle^k | \zeta > t] &= \frac{1}{t^k} \frac{\mathbb{E}_{\delta_{(r,v)}}[\langle \varphi, X_t \rangle^k \mathbf{1}_{(\zeta > t)}]}{\mathbb{P}_{\delta_{(r,v)}}(\zeta > t)} \\ &= \varphi(r, v) \frac{\mathbb{E}_{\delta_{(r,v)}}^\varphi[\langle \varphi, X_t \rangle^{k-1} / t^{k-1}]}{t \mathbb{P}_{\delta_{(r,v)}}(\zeta > t)}, \end{aligned} \quad (4.5)$$

which gives the desired result on account of the fact that the exponential distribution is uniquely identifiable by its moments.

## 4.3 Step 3

At this point in the argument, we have established the desired convergence of martingale moments, and hence the Yaglom limit (3.3), for the case that  $f = \varphi$ . In the final step, detailed in section 10, we remove this restriction by considering the decomposition

$$f = \tilde{f} + \langle \tilde{\varphi}, f \rangle \varphi \quad \text{where} \quad \tilde{f} = f - \langle \tilde{\varphi}, f \rangle \varphi.$$

The reasoning is a relatively standard way to wrap things up, e.g. as in [22, 23, 25]. We use a second moment estimate to show that, for all  $r \in D$  and  $v \in V$ ,  $\langle f, X_t \rangle / t$  converges weakly to zero under  $\mathbb{P}_{\delta_{(r,v)}}(\cdot | \zeta > t)$ , as  $t \rightarrow \infty$ . As such, the principal contribution in the convergence of  $\langle f, X_t \rangle / t$  as  $t \rightarrow \infty$  is its projection on to  $\varphi$ , i.e. the limit of  $\langle \tilde{\varphi}, f \rangle \langle \varphi, X_t \rangle / t$ .

In fact, the techniques outlined in the above three steps are robust enough to apply to a more general setting. Indeed, we will demonstrate the robustness of our methods by showing that Theorems 3.1 and 3.2 actually hold for a general class of non-local Markov branching processes, which we will specify in the following section.

## 5 General non-local Markov branching processes

As previously hinted, our methods are generic enough that they can be applied to the setting of a general branching Markov process with non-local branching. To be more precise about what we mean by the latter, let  $E$  be a Lusin space. We will write  $L_\infty^+(E)$  for the space of non-negative uniformly bounded functions on  $E$ . We will slightly abuse our existing notation (albeit in a way

that will cause no confusion later) and let  $X = (X_t, t \geq 0)$  be a  $(\mathbf{P}, \Phi)$ -Markov branching process (MBP), where  $\mathbf{P} = (\mathbf{P}_t, t \geq 0)$  is a Markov semigroup on  $E$  and

$$\Phi(x, f) = \gamma(x) \mathcal{E}_x \left( \prod_{k=0}^N f(x_k) - f(x) \right), \quad f \in L_\infty^+(E), x \in E,$$

is the branching mechanism such that  $\gamma(x) \geq 0$  is the instantaneous branching rate and  $\mathcal{P}_x$  is the law of the non-local offspring  $\{x_1, \dots, x_N\}$  produced at a branching event positioned at  $x \in E$ . More precisely, in the spirit of (1.3),  $X$  is an atomic measure-valued stochastic process in which particles move independently according to a copy of the Markov process associated to  $\mathbf{P}$  such that, when a particle is positioned at  $x \in E$ , at the instantaneous rate  $\gamma(x)$ , the process will branch and a random number of offspring, say  $N$ , are thrown out in positions, say  $x_1, \dots, x_N$  in  $E$ , according to  $\mathcal{P}_x$ . For convenience, and again, abusing our earlier notation, we will write

$$\mathcal{Z}(A) = \sum_{i=1}^N \delta_{x_i}(A)$$

for any Borel  $A$  in  $E$ .

We do not need  $\mathbf{P}$  to have the Feller property, and it is not necessary that  $\mathbf{P}$  is conservative. That said, if so desired, we can append a cemetery state  $\{\dagger\}$  to  $E$ , which is to be treated as an absorbing state, and regard  $\mathbf{P}$  as conservative on the extended space  $E \cup \{\dagger\}$  (which can also be treated as a Lusin space) and  $\Phi(\{\dagger\}, \cdot) \equiv 0$  (i.e. no branching activity on the cemetery state).

To emphasise the generality of our model, we provide some examples of non-local branching processes in the literature that are encompassed by our setting. The reader is also referred to [15, 16, 17] for an introduction to general branching Markov processes.

**Neutron branching process:** The NBP is, of course, included in this setting. In this case the branching rate is given by  $\gamma = \sigma_{\mathbf{f}}$ , the motion semigroup  $(\mathbf{P}_t, t \geq 0)$  corresponds to that of a NRW that scatters at rate  $\sigma_{\mathbf{s}}$  with new velocity being chosen by  $\pi_{\mathbf{s}}$ , and the non-locality appears in the velocity variable at branching events. The non-linear branching mechanism and associated equation was given in [13].

**Branching Lévy processes:** Our second example is a one-dimensional branching Lévy processes as defined in [19]. This process is defined via a triple,  $(\Lambda, L, \Sigma)$ , where a particle moves according to a Lévy process,  $L$ , and at the random time  $\Lambda$ , which is exponentially distributed with parameter  $\gamma$ , the particle dies and instantaneously scatters a random number of independent copies of  $X$  according to the point process  $\Sigma(dy) = \sum_i \delta_{x_i}(dy)$  relative to its point of death.

**Uchiyama process:** A relative of the previous model is the one given in [26]. Particles live in  $\mathbb{R}^d$  and do not move, but after an independent and exponentially distributed random time they give birth to offspring according to an independent copy of a point process in  $\mathbb{R}^d$  centred at the parent's position. In essence this is a continuous-time  $d$ -dimensional branching random walk.

**Biggins process:** For interest, we mention the very general branching process given in [2], also called the Biggins process in [18]. This process is similarly described by a triple  $(Z, M, \chi)$ , where  $Z$  describes the reproduction,  $M$  describes the movement, and  $\chi$  is a measure of the importance of an individual. More precisely, a particle moves according to a stochastic process,  $(M(a), a \geq 0)$ , where  $a$  is the age of the particle. At a random time, the parent particle gives birth to a random number of individuals according to the point process  $Z$  on  $\mathbb{R} \times \mathbb{R}_+$ , where the first coordinate describes

the spatial displacement of the child from the parent's birth position, and the second coordinate gives the parent's age at the time of that child's birth. Finally,  $\chi$ , is another random process that can be thought of as the importance of the particle in the population as it ages. This model takes the best of several worlds and incorporates features of branching Markov processes, continuous-time branching random walks and Crump-Mode-Jagers processes. Although this process is even more general than the ones we consider in this paper, it is worth remarking that it is likely that the techniques employed here are transferable to the setting of Biggins processes, modulo some technical innovation.

Returning to the  $(\mathbf{P}, \Phi)$ -branching process, suppose that  $(N_t, t \geq 0)$  is the process giving the number of particles alive at each time  $t \geq 0$  and  $\{x_i(t) : i = 1, \dots, N_t\}$  gives their configurations. Then, the linear semigroup associated with  $X$  is given by

$$\psi_t[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle] := \mathbb{E}_{\delta_x} \left[ \sum_{i=1}^{N_t} f(x_i(t)) \right], \quad t \geq 0, f \in L_{\infty}^+(E). \quad (5.1)$$

Further, define the (linear) branching operator as follows:

$$\mathcal{A}[f](x) := \gamma(x) \mathcal{E}_x \left[ \sum_{i=1}^N f(x_i) - f(x) \right], \quad x \in E, f \in L_{\infty}^+(E). \quad (5.2)$$

We introduce the following assumption.

**(G1): The branching rate,  $\gamma(x)$ , and the mean number of offspring,  $m(x) = \mathcal{E}_x[N]$ , are uniformly bounded above.**

As in the setting of the neutron branching process, cf. [6], by breaking the expectation in (5.1) on the first branching event and then appealing to Lemma 1.2, Chapter 4 in [11] to tidy up the resulting integral equation, it is straightforward to prove the following proposition.

**Proposition 5.1.** *Under (G1), the semigroup  $(\psi_t, t \geq 0)$  is the unique solution to the evolution equation*

$$\psi_t[f](x) = \mathbf{P}_t[f](x) + \int_0^t \mathbf{P}_s[\mathcal{A}[\psi_{t-s}[f]]](x) ds, \quad t \geq 0, f \in L_{\infty}^+(E). \quad (5.3)$$

We now consider an alternative representation of the semigroup  $(\psi_t, t \geq 0)$  via a so-called many-to-one formula, which will be convenient for later. Consider the process  $Y = (Y_t, t \geq 0)$  which evolves according to the dynamics of  $(\mathbf{P}_t, t \geq 0)$  albeit that, with instantaneous rate  $\gamma(x)m(x)$ , it jumps from its current position,  $x \in E$ , into a Borel set  $A \subseteq E$  with probability  $m(x)^{-1} \mathcal{E}_x(\mathcal{Z}(A))$ . Letting  $\mathbf{P} = (\mathbf{P}_x, x \in E)$  denote the law of this process, we can again follow standard reasoning highlighted above and condition on the first first jump of  $Y$  to deduce that

$$\eta_t[f](x) := \mathbf{E}_x[f(Y_t)] = \mathbf{P}_t[f](x) + \int_0^t \mathbf{P}_s[\gamma m \hat{\mathcal{A}}[\eta_{t-s}[f]]](x) ds, \quad (5.4)$$

where

$$\hat{\mathcal{A}}[f](x) = \frac{1}{m(x)} \mathcal{E}_x \left[ \sum_{i=1}^N f(x_i) \right] - f(x).$$

Again in the spirit of e.g. [6], it is not difficult to show that one can relate the linear semigroup  $(\psi_t, t \geq 0)$  to the process  $Y$  via the following many-to-one formula.

**Lemma 5.1** (Many-to-one formula). *Suppose (G1) is satisfied. Then*

$$\psi_t[f](x) = \mathbf{E}_x \left[ e^{\int_0^t B(Y_s) ds} f(Y_t) \right], \quad t \geq 0, f \in L_\infty^+(E), \quad (5.5)$$

where  $B(x) = \gamma(x)(m(x) - 1)$ .

In a similar manner, we can consider a many-to-two formula. In order to do so, we define the following operator

$$\mathcal{V}[f, g](x) = \mathcal{E}_x \left[ \sum_{\substack{i, j=1 \\ i \neq j}}^N f(x_i) g(x_j) \right], \quad f, g \in L_\infty^+(E), x \in E. \quad (5.6)$$

We will often abuse notation and write  $\mathcal{V}[g](x)$  instead of  $\mathcal{V}[g, g](x)$  for  $g \in L_\infty^+(E)$  and  $x \in E$ .

**Lemma 5.2.** *Let  $g \in L_\infty^+(E)$ ,  $x \in E$  and  $t \geq 0$ . Suppose (G1) holds and that*

$$\sup_{x \in E} \mathcal{E}_x[N^2] < \infty.$$

*Then we have the following many-to-two formula*

$$\mathbb{E}_{\delta_x} \left[ \langle f, X_t \rangle \langle g, X_t \rangle \right] = \psi_t[fg](x) + \int_0^t \psi_s \left[ \gamma \mathcal{V}[\psi_{t-s}[f], \psi_{t-s}[g]] \right](x) ds. \quad (5.7)$$

As with the proof of Lemma 5.1 and Lemma 5.2, we have excluded the proof of Proposition 5.1 as it is fairly standard. In addition to [6], we refer the reader to [14, 7] for the proofs in the case of the NTE, with the general case using similar techniques.

We now consider the long-term average behaviour of the process  $X$ . In the special case of the NTE, Theorem 2.1 provides us with the existence of the leading eigenvalue  $\lambda_* \in \mathbb{R}$  and corresponding eigenfunctions,  $\varphi$  and  $\tilde{\varphi}$ , which describe the leading order behaviour of the linear semigroup. We will soon see that this type of result is fundamental to analysing the martingale moments of the MBP, thus we introduce the following assumption.

**(G2):** **There exist  $\varphi \in L_\infty^+(E)$  and a finite measure  $\tilde{\varphi}$  on  $E$  such that the analogue of Theorem 2.1 holds true with  $\lambda_* = 0$ ,  $D \times V$  replaced by  $E$  and  $\tilde{\varphi}(x)dx$  replaced by the finite measure  $\tilde{\varphi}$  for the  $(\mathbb{P}, \Phi)$ -MBP.**

Although (G2) may seem like a strong assumption, it can be shown to hold for a large class of MBPs. We refer the reader to [3, 4, 14] and the references therein for various assumptions that imply (G2), as well as several examples where one can prove such results.

As in the previous section we still have that

$$W_t := \frac{\langle \varphi, X_t \rangle}{\varphi(x)}, \quad x \in E, t \geq 0,$$

is a mean one martingale, and can thus be used to define the following change of measure,

$$\left. \frac{d\mathbb{P}_\mu^\varphi}{d\mathbb{P}_\mu} \right|_{\mathcal{F}_t} = \frac{\langle \varphi, X_t \rangle}{\langle \varphi, \mu \rangle} \quad t \geq 0, \mu \in \mathcal{M}(E), \quad (5.8)$$

for the  $(\mathbb{P}, \Phi)$ -MBP.

We also abuse notation and emulate the previous definition of  $\Sigma$  in (3.2), by setting

$$\Sigma = \langle \tilde{\varphi}, \gamma \mathcal{V}[\varphi] \rangle.$$

With this notation in mind, our general moment limit theorem which drives the first step of the Yaglom limit reads as follows.

**Theorem 5.1** (Asymptotic martingale moments). *Suppose that (G1) and (G2) hold. Further, suppose that for  $k \geq 1$ ,*

$$\sup_{x \in E} \mathcal{E}_x \left[ N^{k+1} \right] < \infty. \quad (5.9)$$

Then for all  $j \leq k$ ,

$$\sup_{x \in E} \mathbb{E}_{\delta_x}^{\varphi} [\langle \varphi, X_t \rangle^j] < \infty, \quad t \geq 0, \quad (5.10)$$

and

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \left| \frac{1}{t^j} \mathbb{E}_{\delta_x}^{\varphi} [\langle \varphi, X_t \rangle^j] - (j+1)! \left( \frac{\Sigma}{2} \right)^j \right| = 0. \quad (5.11)$$

Equivalently, for  $j \leq k$ , under  $\mathbb{P}$  we have

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \left| \frac{1}{\varphi(x) t^j} \mathbb{E}_{\delta_x} [\langle \varphi, X_t \rangle^{j+1}] - (j+1)! \left( \frac{\Sigma}{2} \right)^j \right| = 0.$$

For the remaining steps of the proof of the Yaglom limit, we remain in the setting of the general MBP, however, we further impose the following assumptions.

**(G3): The number of offspring produced at a branching event is bounded above by a constant,  $N_{\max}$ .**

**(G4): There exists a constant  $C > 0$  such that for all  $g \in L_{\infty}^+(E)$ ,**

$$\langle \tilde{\varphi}, \gamma \mathcal{V}[g] \rangle \geq C \langle \tilde{\varphi}, g \rangle^2.$$

Again, we abuse notation and define the extinction time  $\zeta := \inf\{t \geq 0 : \langle 1, X_t \rangle = 0\}$ . Then, in the setting of the  $(\mathbb{P}, \Phi)$ -branching process, Theorems 3.1 and 3.2 take the following forms.

**Theorem 5.2.** *Suppose that assumptions (G1), (G2), (G3) and (G4) hold. Then, for all  $x \in E$ ,*

$$\lim_{t \rightarrow \infty} t \mathbb{P}_{\delta_x}(\zeta > t) = \frac{2\varphi(x)}{\Sigma}.$$

**Theorem 5.3** (Yaglom limit for MBPs). *Suppose that assumptions (G1), (G2), (G3) and (G4) hold. For  $f \in L_{\infty}^+(E)$  and for all  $x \in E$ ,*

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x} \left[ \exp \left( -\theta \frac{\langle f, X_t \rangle}{t} \right) \middle| \zeta > t \right] = \frac{1}{1 + \langle \tilde{\varphi}, f \rangle \Sigma \theta / 2}. \quad (5.12)$$

There does not appear to be a comparable result to Theorem 5.1 for general branching processes in the literature, least of all for processes with non-local branching mechanisms. However, we will make a few remarks below concerning the only branching Markov process models for which we are aware Theorem 5.2 and 5.3 have been proved.

**Isotropic neutron transport:** First, let us consider [21]. Here the authors proved Theorem 5.2 under the assumption of isotropic fission and scattering, and where the fission rate and yield are independent of the position and velocity of the colliding neutron. In particular, we have  $\sigma_{\mathbf{f}}(r, v) = \sigma$ , for some constant  $\sigma > 0$ , and further, the law of the offspring  $\mathcal{P}$  does not depend on  $(r, v)$ . In this case, the constant  $\Sigma$  can be written as

$$\Sigma = \sigma \mathcal{E}[N(N-1)] \langle \tilde{\varphi}, \hat{\varphi}^2 \rangle,$$

where  $\hat{\varphi} : D \rightarrow [0, \infty) : r \mapsto (4\pi)^{-1} \int_V \varphi(r, v) dv$ , which is exactly the constant that appears in [21, Theorem 5.1]. Further, comparing this form of  $\Sigma$  with assumption (H5) motivates the appearance of this assumption in Theorem 3.1.

**Branching Brownian motion:** Secondly, we consider the model given in [22]. Here, the author considers a branching Brownian motion on a compact domain  $D$ . More precisely, particles move according to a Brownian motion in  $D$  and at rate  $\gamma$ , a branching event occurs, producing some number of offspring at the site of the branching event. If a particle hits the boundary of  $D$ , it is killed. Again, in this case, the branching rate  $\gamma$  and the offspring distribution are spatially independent. Further, in this setting, the left and right eigenfunctions are equal. Hence, the constant  $\Sigma$  reads

$$\Sigma = \gamma \mathcal{E}[N(N-1)] \int_D \varphi(x)^3 dx,$$

as given in [22, Theorem 1.4].

In the spirit of the branching Brownian motion model described above, Branching Lévy processes, Uchiyama processes and Biggins processes should, in principal, be candidates from which one can similarly develop a Yaglom limit result by killing particles when they exit a compact domain in such a way that the system is critical. In general, without killing in such a manner, one will not have a chance to satisfy the required condition (G2).

**Multi-type continuous-time Galton Watson process:** Technically speaking, a multi-type continuous-time Galton Watson process is a spatial branching particle system in which  $E$  is countable. Parents do not move during their lifetime, which is independent and exponentially distributed depending on the particle's state say  $q_i$ ,  $i \in E$ , and reproduce on their death, sending their offspring both to their current site as well as other sites in  $E$  in such a way that the distribution of the offspring depends on the state of the parent. Although there is greater dependency on the parent's state but a less general state space, this model is similar in spirit to Uchiyama's model. In the setting that  $E$  is finite, the Yaglom limit theorem is a classical result, see e.g. Chapter V.5 of [1]. In our setting, the conditions of Theorem 5.3 are automatically satisfied, e.g. (G2) is essentially supported by the Perron-Frobenius Theorem for the mean semigroup matrix.

For the case that  $E$  is countable but infinite, Theorem 5.3, let  $M(t)$  denote the matrix whose  $(i, j)$ -th entry represents the mean number of type  $j$  individuals alive at time  $t \geq 0$  when the process is initiated from a single type  $i$  individual and, for  $\lambda \in \mathbb{R}$ , set  $H(\lambda) = \int_0^\infty e^{\lambda t} M(t) dt$ . It was shown in [20] that if  $M(t)$  is irreducible and  $\lambda$  is such that  $H_{i,j}(\lambda) < \infty$  for a given pair  $i, j \in E$ , then  $H_{i,j}(\lambda) < \infty$  for all  $i, j \in E$ . We can then define the spectral radius  $\lambda_* = \sup\{\lambda \geq -\infty : H(\lambda) < \infty\}$ .

Thus, the first part of assumption (G2) holds, however, it is still unclear as to whether the uniform convergence in (2.7) holds in this case. Theorems 5.2 and 5.3 also require that the spectral radius satisfies  $\lambda_* = 0$ , but nonetheless give new Yaglom limit results, which we do not believe is currently in the literature.

We should also note that there are recent analogues of Theorem 5.2 and 5.3 for superprocesses, cf. [23], however this case does not fall under our setting of branching particle processes. Moreover, the known results for superprocess are focused on local branching mechanisms. In forthcoming work we will address some of the results presented here for superprocesses with non-local branching mechanisms.

In the setting of the  $(\mathbb{P}, \Phi)$ -branching process, the proof of Theorem 5.1, and as a consequence Theorems 5.2 and 5.3, is based around the use of the spine decomposition, together with some complex combinatorial calculations. For this reason, we look next to the aforesaid spine decomposition and its ergodic properties.

## 6 Spine decomposition

It is well known that for many branching processes, the change of measure (5.8) induces the so-called spine decomposition. Since this will be key to the proof of Theorem 5.1, we first spend some time discussing it.

To this end, let us introduce another  $\mathcal{M}(E)$ -valued stochastic process

$$X^\varphi := (X_t^\varphi, t \geq 0) \text{ with probabilities } \tilde{\mathbb{P}}^\varphi := (\tilde{\mathbb{P}}_\mu^\varphi, \mu \in \mathcal{M}(E)), \quad (6.1)$$

which is defined through the following pathwise construction.

1. From the initial configuration  $\mu \in \mathcal{M}(E)$  with an arbitrary enumeration of particles, the  $i^*$ -th individual is selected and marked ‘*spine*’ so that, given  $\mu$ ,

$$\text{the probability } i^* = j \text{ is given by } \frac{\varphi(x_j)}{\langle \varphi, \mu \rangle} \text{ for } j = 1, \dots, N.$$

2. The individuals  $j \neq i^*$  in the initial configuration that are not marked ‘*spine*’, each issue independent copies of  $(X, \mathbb{P}_{\delta_{x_j}})$  respectively.
3. For the marked individual, issue a single particle whose motion is determined by the semigroup

$$\mathbb{P}_t^\varphi[f](x) := \frac{1}{\varphi(x)} \mathbb{P}_t[f\varphi](x), \quad x \in E.$$

4. The marked individual undergoes branching at the accelerated rate

$$\rho(x) := \gamma(x) \frac{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]}{\varphi(x)} \quad (6.2)$$

when in physical configuration  $x \in E$ , at which point, it scatters a random number of particles according to the random measure on  $E$  given by  $(\mathcal{Z}, \mathcal{P}_x^\varphi)$  where

$$\frac{d\mathcal{P}_x^\varphi}{d\mathcal{P}_x} = \frac{\langle \varphi, \mathcal{Z} \rangle}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]}. \quad (6.3)$$

5. When the marked individual branches in physical configuration  $x \in E$ , set

$$\mu = \sum_{i=1}^n \delta_{x_i}, \text{ where, in the previous step, } \mathcal{Z} = \sum_{i=1}^n \delta_{x_i},$$

and repeat step 1.

The process  $X_t^\varphi$  describes the physical configuration of all the particles in the system at time  $t \geq 0$ , i.e. ignoring the marked genealogy. We will also be interested in the evolution of the single genealogical line of descent that has been marked ‘spine’. This process, referred to simply as *the spine*, will be denoted by  $Y^\varphi := (Y_t^\varphi)_{t \geq 0}$ . Together, the processes  $(X^\varphi, Y^\varphi)$  make a Markov pair, whose probabilities we will denote by  $(\mathbb{P}_{\mu, x}^\varphi, \mu \in \mathcal{M}(E), x \in E)$ . Note in particular that

$$\tilde{\mathbb{P}}_\mu^\varphi = \sum_{i=1}^n \frac{\varphi(x_i)}{\langle \varphi, \mu \rangle} \tilde{\mathbb{P}}_{\mu, x_i}^\varphi$$

where  $\mu = \sum_{i=1}^n \delta_{x_i}$ .

The spine decomposition induced by (4.1) is thus summarised by the following theorem.

**Theorem 6.1.** *Under assumptions (G1) and (G2), the process  $(X^\varphi, \tilde{\mathbb{P}}^\varphi)$  is Markovian and equal in law to  $(X, \mathbb{P}^\varphi)$ , where  $\mathbb{P}^\varphi = (\mathbb{P}_\mu^\varphi, \mu \in \mathcal{M}(E))$  was defined in (5.8).*

Let  $\tilde{\mathbf{P}}^\varphi$  denote the law of the spine,  $Y^\varphi$ . We are also interested the behaviour of  $Y^\varphi$  as an autonomous process. The following lemma identifies the spine as a change of measure applied to the process  $Y$  defined in Lemma 5.1.

**Lemma 6.1.** *Under assumptions (G1) and (G2), the process  $(Y^\varphi, \tilde{\mathbf{P}}^\varphi)$  is equal in law to  $(Y, \mathbf{P}^\varphi)$  where*

$$\left. \frac{d\mathbf{P}_x^\varphi}{d\mathbf{P}_x} \right|_{\mathcal{F}_t} = e^{\int_0^t B(Y_s) ds} \frac{\varphi(Y_t)}{\varphi(x)} \mathbf{1}_{\{t < \mathbf{k}\}}, \quad t \geq 0, x \in E, \quad (6.4)$$

where  $\mathbf{k} := \inf\{t \geq 0 : Y_t \notin E\}$  and we recall that  $B(x) = \gamma(x)(m(x) - 1)$ . From this we see that the semigroup  $(\mathbf{P}_t^\varphi, t \geq 0)$  associated to  $(Y, \mathbf{P}^\varphi)$  is conservative and satisfies, for  $f \in L_\infty^+(E)$ ,  $\mathbf{P}_t^\varphi[f] = \varphi^{-1} \psi_t[\varphi f]$ ,  $t \geq 0$ , with stationary limit distribution

$$\varphi(x) \tilde{\varphi}(dx), \quad x \in E.$$

In particular, the above two results mean that under  $\mathbb{E}^\varphi$ , we can write

$$\frac{\langle \varphi, X_t \rangle}{t} = \frac{\varphi(Y_t)}{t} + \frac{1}{t} \sum_{j=1}^{n_t} \Xi_j(Y_{T_j}, t - T_j), \quad (6.5)$$

where  $n_t$  is the number of fission events on the spine at times  $(T_j, j \geq 1)$  and, given  $x \in E$  and  $0 \leq u < \infty$ , the random variables  $\Xi_j(x, u)$  are independent and equal in law to

$$\sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_u^i \rangle \text{ under } \eta_x^\varphi := \mathcal{P}_{(r, v)}^\varphi \bigotimes_{i=1}^N \mathbb{P}_{\delta_{(r, v_i)}}. \quad (6.6)$$

Note that, as  $\|\varphi\|_\infty < \infty$ , the spine term in (6.5) converges uniformly to zero.

For  $x \in E$  and  $t > 0$ , suppose we define the random variable  $\mathbf{u}_t$  with law  $\mathfrak{P}_Y^t$ , which depends on the path of  $Y$  such that

$$\mathfrak{P}_Y^t(\mathbf{u}_t \in ds) = \frac{\rho(Y_s)}{\int_0^t \rho(Y_u) du} ds, \quad (6.7)$$

where  $\rho$  was defined in (6.2). Recalling that the ordering of the arrivals in the sum of the Poisson arrival process  $(T_i, i = 1, \dots, n_t)$  is not important, given  $n_t$  and  $Y$ , we can appeal to standard descriptions of Poisson point processes to deduce that (6.5) can equivalently be written

$$\frac{\langle \varphi, X_t \rangle}{t} = \frac{\varphi(Y_t)}{t} + \frac{1}{t} \sum_{i=1}^{n_t} \Xi_i(Y_{\mathbf{u}_t^i}, t - \mathbf{u}_t^i), \quad (6.8)$$

where, conditional on  $n_t$  and  $Y$ ,  $\mathbf{u}_t^i$  are iid and their law is given by (6.7).

## 7 Ergodic properties of the spine

Next, we state and prove a theorem relating to the long-term behaviour of the spine, which will be of general use throughout our computations. It is loosely based on Lemma 5.1 of [25]. In order to state it, let us introduce a class of functions  $\mathcal{C}$  on  $E \times [0, \infty) \times [0, \infty)$  such that  $F$  belongs to class  $\mathcal{C}$  if

$$F(x, s) := \lim_{t \rightarrow \infty} F(x, s, t), \quad x \in E, s \geq 0,$$

exists,

$$\sup_{x \in E, 0 < s < 1} |\varphi F(x, s)| < \infty, \quad (7.1)$$

and

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \varphi(x) |F(x, s) - F(x, s, t)| = 0. \quad (7.2)$$

**Theorem 7.1.** *Suppose that  $F_1, \dots, F_k \in \mathcal{C}$ . Then, for all  $j \leq k$ ,*

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j \int_0^1 F_i(Y_{ut}, u, t) du \right] - \prod_{i=1}^j \int_0^1 \langle \varphi \tilde{\varphi}, F_i(\cdot, u) \rangle du \right| = 0,$$

where  $F_i(\cdot, u) = \lim_{t \rightarrow \infty} F_i(\cdot, u, t)$ .

*Proof.* We will show that, for  $0 \leq u_1, \dots, u_k \leq 1$ ,  $F_1, \dots, F_k \in \mathcal{C}$  and  $j \leq k$ ,

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j F_i(Y_{u_i t}, u_i, t) \right] - \prod_{i=1}^j \langle \varphi \tilde{\varphi}, F_i(\cdot, u_i) \rangle \right| = 0. \quad (7.3)$$

Once that has been established, it is a simple matter of dominated convergence to deduce that, for

all  $j \leq k$ ,

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j \int_0^1 F_i(Y_{ut}, u, t) du \right] - \prod_{i=1}^j \int_0^1 \langle \varphi \tilde{\varphi}, F_i(\cdot, u) \rangle du \right| \\
&= \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \int_0^1 \cdots \int_0^1 \prod_{i=1}^j F_i(Y_{u_i t}, u_i, t) du_1 \cdots du_j \right] \right. \\
&\quad \left. - \int_0^1 \cdots \int_0^1 \prod_{i=1}^j \langle \varphi \tilde{\varphi}, F_i(\cdot, u_i) \rangle du_1 \cdots du_j \right| \\
&\leq \int_0^1 \cdots \int_0^1 \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j F_i(Y_{u_i t}, u_i, t) \right] - \prod_{i=1}^j \langle \varphi \tilde{\varphi}, F_i(\cdot, u_i) \rangle \right| du_1 \cdots du_j \\
&= 0,
\end{aligned}$$

as required.

The proof of (7.3) works by induction. We start by proving (7.3) in the case that  $k = 1$ . Suppose  $F \in \mathcal{C}$ . As  $\mathbf{E}_x^\varphi[F(Y_{ut}, u, t)] = \mathbf{P}_{ut}^\varphi[F(\cdot, u, t)](x)$ , we can use Lemma 6.1 and write

$$\begin{aligned}
& \sup_{x \in E} |\mathbf{P}_{ut}^\varphi[F(\cdot, u, t)](x) - \langle \varphi \tilde{\varphi}, F(\cdot, u) \rangle| \\
&\leq \sup_{x \in E} \frac{1}{\varphi(x)} |\psi_{ut}[\varphi F(\cdot, u, t)](x) - \psi_{ut}[\varphi F(\cdot, u)](x)| \\
&\quad + \sup_{x \in E} \left| \frac{1}{\varphi(x)} \psi_{ut}[\varphi F(\cdot, u)](x) - \langle \tilde{\varphi}, \varphi F(\cdot, u) \rangle \right|. \tag{7.4}
\end{aligned}$$

Since  $\psi_t$  is a linear operator, from (7.2), for  $t$  sufficiently large, the first term on the right-hand side of (7.4) can be estimated by  $\varphi^{-1} \psi_{ut}[\varepsilon]$  for each  $\varepsilon > 0$ . Hence, by the triangle inequality, the fact that  $\langle \tilde{\varphi}, \varepsilon \rangle \leq c\varepsilon \langle \tilde{\varphi}, 1 \rangle$  for some constant  $c > 0$  and Theorem 2.1, for  $t$  sufficiently large,

$$\begin{aligned}
& \sup_{x \in E} |\mathbf{P}_{ut}^\varphi[F(\cdot, u, t)](x) - \langle \varphi \tilde{\varphi}, F(\cdot, u) \rangle| \\
&\leq \sup_{x \in E} \left| \frac{1}{\varphi(x)} \psi_{ut}[\varepsilon](x) - \langle \tilde{\varphi}, \varepsilon \rangle \right| + \varepsilon \|\tilde{\varphi}\|_1 \\
&\quad + \sup_{x \in E} \left| \frac{1}{\varphi(x)} \psi_{ut}[\varphi F(\cdot, u)](x) - \langle \tilde{\varphi}, \varphi F(\cdot, u) \rangle \right| \\
&\leq c(e^{-kt} + \varepsilon) \tag{7.5}
\end{aligned}$$

for some strictly positive constants  $c$  and  $k$  as in (2.7). This both ensures that the left-hand side of (7.4) tends to zero as  $t \rightarrow \infty$  as required.

Now suppose that (7.3) is true for  $k = n - 1$  and, without loss of generality, that  $0 \leq u_1 \leq \cdots \leq u_n \leq 1$ . Then, by the Markov property,

$$\mathbf{E}_x^\varphi \left[ \prod_{i=1}^n F_i(Y_{u_i t}, u_i, t) \right] = \int_E \mathbf{P}_x^\varphi(Y_{u_1 t} \in dy') F_1(y', u_1, t) \mathbf{E}_{y'}^\varphi \left[ \prod_{i=2}^n F_i(Y_{(u_i - u_1)t}, u_i, t) \right].$$

Now define

$$\tilde{F}(y', u_1, t) = F_1(y', u_1, t) \mathbf{E}_{y'}^\varphi \left[ \prod_{i=2}^n F_i(Y_{(u_i - u_1)t}, u_i, t) \right].$$

Note, moreover, that, using the induction hypothesis,

$$\tilde{F}(y', u_1) := \lim_{t \rightarrow \infty} \tilde{F}(y', u_1, t) = F_1(y', u_1) \prod_{i=2}^n \langle \varphi \tilde{\varphi}, F_i(\cdot, u_i) \rangle, \quad y' \in E,$$

which further respects the required boundedness conditions in the statement of the proposition. Hence, appealing again to the induction hypothesis for  $n = 1$ , we get the required result.  $\square$

## 8 Step 1: Proof of Theorem 5.1

We now prove Theorem 5.1. We emphasise that we will work in the setting of a general  $(\mathbf{P}, \Phi)$ -MBP as introduced in sections 5 and 6. The strategy of the proof will be to use induction. For the inductive step, we use an interim calculation that looks at the asymptotic behaviour of contributions to the  $k$ -th moment that comes from the subtrees along the spine.

*Proof.* As previously mentioned, the spine decomposition will play an important role in the proof and we will rely heavily on the decomposition (6.8). In addition to this, a number of combinatorial results for moment formulae will come into play, for which we have placed two fundamental results in the Appendix that we will call upon. The proof is long and we break it into a number of blocks of calculations which fall under informal headings.

**Block 1:** *The theorem holds for  $k = 1$ .*

We start by noting that, the moment assumption (5.9) implies that both  $\sup_{x \in E} \mathcal{E}_x[N] < \infty$  and  $\sup_{x \in E} \mathcal{V}[\varphi](x) < \infty$ .

Recalling  $\eta^\varphi$  introduced in (6.6), we also note that, for  $x \in E$  and  $0 \leq u \leq t$ , we have

$$\begin{aligned} \eta_x^\varphi \left[ \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right] &= \mathcal{E}_x \left[ \frac{\langle \varphi, \mathcal{Z} \rangle}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]} \sum_{j=1}^N \frac{\varphi(x_j)}{\langle \varphi, \mathcal{Z} \rangle} \sum_{\substack{i=1 \\ i \neq j}}^N \mathbb{E}_{\delta_{x_i}}[\langle \varphi, X_{t-u}^i \rangle] \right] \\ &= \frac{1}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{i=1 \\ i \neq j}}^N \varphi(x_i) \right] \\ &= \frac{\mathcal{V}[\varphi](x)}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]}. \end{aligned} \tag{8.1}$$

Using the spine decomposition (6.8), we may thus write

$$\begin{aligned} \mathbb{E}_{\delta_x}^\varphi \left[ \frac{\langle \varphi, X_t \rangle}{t} \right] - \mathbb{E}_{\delta_x}^\varphi \left[ \frac{\varphi(Y_t)}{t} \right] &= \frac{1}{t} \mathbb{E}_{\delta_x}^\varphi \left[ \sum_{i=1}^{n_t} \frac{\mathcal{V}[\varphi](Y_{\mathbf{u}_i^i})}{\mathcal{E}_{Y_{\mathbf{u}_i^i}}[\langle \varphi, \mathcal{Z} \rangle]} \right] \\ &= \mathbb{E}_{\delta_x}^\varphi \left[ \frac{n_t \int_0^t \rho(Y_s) \frac{\mathcal{V}[\varphi](Y_s)}{\mathcal{E}_{Y_s}[\langle \varphi, \mathcal{Z} \rangle]} ds}{t \int_0^t \rho(Y_u) du} \right] \\ &= \mathbb{E}_{\delta_x}^\varphi \left[ \frac{n_t \int_0^t [\gamma(Y_s) \mathcal{V}[\varphi](Y_s) / \varphi(Y_s)] ds}{t \int_0^t \rho(Y_u) du} \right] \\ &= \frac{1}{t} \mathbf{E}_x^\varphi \left[ \int_0^t \frac{\gamma(Y_s) \mathcal{V}[\varphi](Y_s)}{\varphi(Y_s)} ds \right], \end{aligned} \tag{8.2}$$

where we have used (6.2) in the penultimate line and that, given  $Y$ ,  $n_t$  is Poisson distributed with mean  $\int_0^t \rho(Y_u) du$  to obtain the final equality. Changing variables so that  $s = ut$  and noting that the spine term will converge to 0, we can use (8.2) to get

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbb{E}_{\delta_x}^\varphi \left[ \frac{\langle \varphi, X_t \rangle}{t} \right] - \Sigma \right| \\
&= \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \int_0^1 \frac{\gamma(Y_{ut}) \mathcal{V}[\varphi](Y_{ut})}{\varphi(Y_{ut})} ds \right] - \Sigma \right| \\
&= \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \int_0^1 \mathbf{P}_{ut}^\varphi[g](x) du - \langle \varphi \tilde{\varphi}, g \rangle \right| \\
&= 0
\end{aligned} \tag{8.3}$$

where we have used Theorem 7.1 with  $F(x, u, t) = g(x) = \gamma(x) \mathcal{V}[\varphi](x) / \varphi(x)$ .

To show that (5.10) holds, we consider the right-hand side of (8.2). Due to assumption (G1), we have  $\bar{B} := \sup_{x \in E} B(x) \vee 1 < \infty$ . Due to this, Fubini's theorem and the change of measure (6.4), we have

$$\begin{aligned}
\mathbf{E}_x^\varphi \left[ \int_0^t \frac{\gamma(Y_s) \mathcal{V}[\varphi](Y_s)}{\varphi(Y_s)} ds \right] &= \int_0^t \frac{1}{\varphi(x)} \mathbf{E}_x \left[ e^{\int_0^s B(Y_u) du} \varphi(Y_s) \frac{\gamma(Y_s) \mathcal{V}[\varphi](Y_s)}{\varphi(Y_s)} ds \right] \\
&\leq e^{\bar{B}t} \int_0^t \frac{1}{\varphi(x)} \mathbf{E}_x [\gamma(Y_s) \mathcal{V}[\varphi](Y_s)] ds \\
&\leq e^{\bar{B}t} \int_0^t (C e^{-\varepsilon s} + \Sigma) ds,
\end{aligned}$$

where we have used assumption (G2) in the final line. Due to (5.9), the right-hand side above is finite, and hence (5.10) holds for  $k = 1$ .

**Block 2:** *Decomposing the  $k$ -th power of the sum in (6.8) assuming the theorem for  $k - 1$*

We now assume that the theorem holds for all powers less than or equal to  $k - 1$  and proceed to show that the result holds true for  $k$ . We can appeal to the spine decomposition (6.8) and the adaptation of the multinomial formula for expectations of iid random variables in Lemma A.2 in

the Appendix to write

$$\begin{aligned}
& \frac{1}{t^k} \mathbb{E}_{\delta_x}^\varphi \left[ \left( \langle \varphi, X_t \rangle - \frac{\varphi(Y_t)}{t} \right)^k \right] \\
&= \mathbb{E}_{\delta_x}^\varphi \left[ \frac{1}{t^k} \left( \sum_{i=1}^{n_t} \Xi_i(Y_{\mathbf{u}_t^i}, t - \mathbf{u}_t^i) \right)^k \right] \\
&= (\Sigma/2)^k \mathbb{E}_{\delta_x}^\varphi \left[ \sum_{j=1}^k 2^j \binom{n_t}{j} \sum_{[k_1, \dots, k_j]_+} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)! \right. \\
&\quad \left. \times \prod_{i=1}^j \frac{\eta_{Y_{\mathbf{u}_t^i}}^\varphi \left[ \Xi_i(Y_{\mathbf{u}_t^i}, t - \mathbf{u}_t^i)^{k_i} \right]}{(k_i - 1)! (\Sigma/2)^{k_i - 1} \Sigma t^{k_i}} \right] \\
&= (\Sigma/2)^k \mathbb{E}_{\delta_x}^\varphi \left[ \sum_{j=1}^k 2^j \frac{n_t(n_t - 1) \cdots (n_t - j + 1)}{j! \left( \int_0^t \rho(Y_u) du \right)^j} \sum_{[k_1, \dots, k_j]_+} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)! \right. \\
&\quad \left. \times \prod_{i=1}^j \frac{1}{t} \int_0^t \rho(Y_u) \frac{\eta_{Y_u}^\varphi \left[ \Xi_i(Y_u, t - u)^{k_i} \right]}{(k_i - 1)! (\Sigma/2)^{k_i - 1} \Sigma t^{k_i - 1}} du \right]. \tag{8.4}
\end{aligned}$$

The sums in the second and third equalities above are taken over the set  $[k_1, \dots, k_j]_+$ , which arises from the aforementioned result in the Appendix and consists of all combinations of strictly positive  $\{k_1, \dots, k_j\}$  such that  $\sum_{i=1}^j k_i = k$  if  $j \leq n_t$  and otherwise  $[k_1, \dots, k_j]_+ = \emptyset$ .

Suppose now we write  $I_t = \int_0^t \rho(Y_{u_i}) du$  so that, given  $Y$ ,  $n_t$  has a Poisson distribution with rate  $I_t$ . In particular, given  $Y$  the expectation of the functional  $n_t(n_t - 1) \cdots (n_t - j + 1)$  is given by precisely  $I_t^j$ . Hence we can develop the right-hand side of (8.4) and get

$$\begin{aligned}
& \frac{1}{t^k} \mathbb{E}_{\delta_x}^\varphi \left[ \left( \langle \varphi, X_t \rangle - \frac{\varphi(Y_t)}{t} \right)^k \right] \\
&= (\Sigma/2)^k \sum_{j=1}^k 2^j \sum_{[k_1, \dots, k_j]_+} \frac{1}{j!} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)! \\
&\quad \times \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j \frac{\frac{1}{t} \int_0^t \rho(Y_u) \eta_{Y_u}^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^{k_i} \right]}{(k_i - 1)! (\Sigma/2)^{k_i - 1} \Sigma t^{k_i - 1}} du \right], \tag{8.5}
\end{aligned}$$

where we recall the notation for  $\eta_x^\varphi$  was given in (6.6). In the next two blocks, we develop the  $\eta_{Y_u}^\varphi[\dots]$  terms further and obtain asymptotics for them that allow us to complete the proof in the final block.

**Block 3:** *Developing the  $\eta_{Y_u}^\varphi[\dots]$  terms in (8.5) assuming the theorem for  $k - 1$*

We need to control the terms

$$\eta_{Y_u}^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^{k_i} \right]$$

in (8.5), noting in particular that none of the  $k_i$ -s will exceed the value  $k$ .

In a similar spirit to (8.1), we have for any integer  $1 \leq p \leq k$ ,

$$\begin{aligned}
& \eta_x^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^p \right] \\
&= \mathcal{E}_x \left[ \frac{\langle \varphi, \mathcal{Z} \rangle}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]} \sum_{j=1}^N \frac{\varphi(x_j)}{\langle \varphi, \mathcal{Z} \rangle} \bigotimes_{i=1}^N \mathbb{E}_{\delta_{x_i}} \left[ \left( \sum_{\substack{i=1 \\ i \neq j}}^N \langle \varphi, X_{t-u}^i \rangle \right)^p \right] \right] \\
&= \frac{1}{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{p_j=0 \\ [p_1, \dots, p_N]}} \binom{p}{p_1, \dots, p_N} \prod_{i=1}^N \mathbb{E}_{\delta_{x_i}} [\langle \varphi, X_{t-u} \rangle^{p_i}] \right], \tag{8.6}
\end{aligned}$$

where in the final equality we have used the multinomial formula in Lemma A.1 of the Appendix; in particular, the sum is over the set  $[p_1, \dots, p_N]$  of all combinations of  $(p_i, i = 1, \dots, N)$  such that  $\sum_{i=1}^N p_i = p$ .

We start by noting that there are no more than  $p$  strictly positive elements in the product on the right-hand side above since, regardless of the value of  $N$ , the set  $\{1, \dots, p\}$  cannot be partitioned into more than  $p$  non-empty blocks. Then, the induction hypothesis (set as valid for  $k - 1$ ), the fact that  $\sum_{[p_1, \dots, p_N]} \binom{p}{p_1, \dots, p_N} = N^p$ , the boundedness of  $\varphi$  and the assumption (5.9) ensures that, uniformly in  $x$ , the entirety of the right-hand side, and hence the left-hand side, of (8.6) is finite.

To show that this implies (5.10), we first note that using the same techniques as in the proof of Theorem 7.1, one can show that, under the assumptions of this theorem, there exists a constant  $K > 0$  such that

$$\sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{i=1}^j \int_0^1 F_i(Y_{ut}, u, t) du \right] - \prod_{i=1}^j \int_0^1 \langle \varphi \tilde{\varphi}, F_i(\cdot, u) \rangle du \right| \leq K,$$

Then, feeding (8.6) into the right-hand side of (8.5), given the definition of  $\rho$  in (6.2), and the above inequality, we see that (5.10) holds.

**Block 4:** *Uniform asymptotics of  $\eta_{Y_u}^\varphi[\dots]$  terms in (8.5) assuming the theorem for  $k - 1$*

It follows from (8.6) that, for  $t \geq 1$ ,

$$\begin{aligned}
& \sup_{u \leq t} \left| \frac{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]}{t^{k-1}} \eta_x^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^k \right] \right. \\
& \quad \left. - \frac{1}{t^{k-1}} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{i=1 \\ i \neq j}}^N \mathbb{E}_{\delta_{x_i}} [\langle \varphi, X_{t-u} \rangle^k] \right] \right| \\
& \leq \frac{1}{t} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{[k_1, \dots, k_N]_2} \binom{k}{k_1, \dots, k_N} \prod_{i=1}^N \sup_{u \leq t} \mathbb{E}_{\delta_{x_i}} [\langle \varphi, X_s \rangle^{k_i}] / t^{k_i-1} \right] \\
& \leq \frac{1}{t} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{[k_1, \dots, k_N]_2} \binom{k}{k_1, \dots, k_N} \prod_{i=1}^N \binom{k_i}{k_i-1} \varphi(x_i) \frac{\mathbb{E}_{\delta_{x_i}}^\varphi [\langle \varphi, X_t \rangle^{k_i-1}]}{t^{k_i-1}} \right]
\end{aligned}$$

where  $[k_1, \dots, k_N]_2$  are the combinations of non-negative integers  $\{k_1, \dots, k_N\}$  such that  $\sum_{i=1}^N k_i = k$ , with at least two of the  $k_i > 0$ . Using similar arguments to the treatment of the right-hand side of (8.6), the assumption (5.9), the fact that  $\|\varphi\|_\infty < \infty$  and the induction hypothesis at  $k-1$  gives us that, uniformly in  $x \in E$ , the right-hand side tends to zero. That is to say

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sup_{x \in E} \sup_{u \leq t} \left| \frac{\mathcal{E}_x[\langle \varphi, \mathcal{Z} \rangle]}{t^{k-1}} \eta_x^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^k \right] \right. \\
& \quad \left. - \frac{1}{t^{k-1}} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{i=1 \\ i \neq j}}^N \mathbb{E}_{\delta_{x_i}} [\langle \varphi, X_{t-u} \rangle^k] \right] \right| = 0. \quad (8.7)
\end{aligned}$$

This gives the uniform leading order behaviour of the term on the left-hand side of (8.6).

**Block 5:** *Uniform asymptotic martingale moments assuming the theorem for  $k-1$*

If we can show that, for each  $j \leq k$ ,

$$\lim_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{\ell=1}^j \frac{\frac{1}{t} \int_0^t du \rho(Y_u) \eta_{Y_u}^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq i^*}}^N \langle \varphi, X_{t-u}^i \rangle \right)^{k_\ell} \right]}{(k_\ell - 1)! (\Sigma/2)^{k_\ell-1} \Sigma t^{k_\ell-1}} \right] - 1 \right| = 0, \quad (8.8)$$

then this, along with Lemma A.3 in the Appendix, back in (8.5) this yields

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1}{t^k} \mathbb{E}_{\delta_x}^\varphi \left[ \langle \varphi, X_t \rangle^k \right] &= (\Sigma/2)^k \sum_{j=1}^k 2^j \sum_{[k_1, \dots, k_j]_+} \frac{1}{j!} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)! \\
&= (\Sigma/2)^k (k+1)!
\end{aligned}$$

uniformly for  $x \in E$ , which completes the proof of the theorem.

The remainder of the proof is thus focused on showing (8.8). Suppose that  $1 \leq j \leq k$  and  $k_1, \dots, k_j \in \mathbb{N}$  are such that  $\sum_{i=1}^j k_i = k$ . Then, from (8.7) used in conjunction with (8.5) we have

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{\ell=1}^j \frac{\frac{1}{t} \int_0^t du \rho(Y_u) \eta_x^\varphi \left[ \left( \sum_{\substack{i=1 \\ i \neq j}}^N \langle \varphi, X_{t-u}^i \rangle \right)^{k_\ell} \right]}{(k_\ell - 1)! (\Sigma/2)^{k_\ell - 1} \Sigma t^{k_\ell - 1}} \right] - 1 \right| \\
&= \limsup_{t \rightarrow \infty} \sup_{x \in E} \left| \mathbf{E}_x^\varphi \left[ \prod_{\ell=1}^j \frac{\frac{1}{t} \int_0^t du \frac{\gamma(Y_u)}{\varphi(Y_u)} \mathcal{E}_{Y_u} \left[ \sum_{j=1}^N \varphi(Y_j) \sum_{\substack{i=1 \\ i \neq j}}^N \mathbb{E}_{\delta_{Y_i}} [\langle \varphi, X_{t-u} \rangle^{k_\ell}] \right]}{(k_\ell - 1)! (\Sigma/2)^{k_\ell - 1} \Sigma t^{k_\ell - 1}} \right] - 1 \right| \\
&= \limsup_{t \rightarrow \infty} \sup_{x \in E} \frac{1}{\Sigma^j} \left| \mathbf{E}_x^\varphi \left[ \prod_{\ell=1}^j \int_0^1 ds F_\ell(Y_{st}, s, t) \right] - \Sigma^j \right|, \tag{8.9}
\end{aligned}$$

where, for  $\ell = 1, \dots, j$ ,  $x \in E$ ,  $0 \leq u \leq 1$  and  $t \geq 0$ ,

$$\begin{aligned}
F_\ell(x, s, t) &= \frac{\gamma(x)}{\varphi(x)} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{i=1 \\ i \neq j}}^N \varphi(x_i) \frac{\mathbb{E}_{\delta_{x_i}}^\varphi [\langle \varphi, X_{t(1-s)} \rangle^{k_\ell - 1}]}{(k_\ell - 1)! (\Sigma/2)^{k_\ell - 1} t^{k_\ell - 1}} \right] \\
&= k_\ell (1-s)^{k_\ell - 1} \frac{\gamma(x)}{\varphi(x)} \mathcal{E}_x \left[ \sum_{j=1}^N \varphi(x_j) \sum_{\substack{i=1 \\ i \neq j}}^N \varphi(x_i) \frac{\mathbb{E}_{\delta_{x_i}}^\varphi [\langle \varphi, X_{t(1-s)} \rangle^{k_\ell - 1}]}{k_\ell! (\Sigma/2)^{k_\ell - 1} (t(1-s))^{k_\ell - 1}} \right].
\end{aligned}$$

Since we are assuming that the statement of the theorem holds for  $k - 1$ , it is easy to verify with the help of (5.9) that  $F_i \in \mathcal{C}$  (where  $\mathcal{C}$  was defined immediately before Theorem 7.1) and that

$$F_\ell(x, s) := \lim_{t \rightarrow \infty} F_\ell(x, s, t) = k_\ell (1-s)^{k_\ell - 1} \frac{\gamma(x)}{\varphi(x)} \mathcal{V}[\varphi](x).$$

Note also that  $k_\ell (1-s)^{k_\ell - 1} \Sigma = \langle \varphi \tilde{\varphi}, F_\ell(\cdot, s) \rangle$  so that  $\int_0^1 \langle \varphi \tilde{\varphi}, F_\ell(\cdot, s) \rangle ds = \Sigma$ . As a consequence, Theorem 7.1 tells us that the right-hand side of (8.9) converges to zero, as required.  $\square$

## 9 Step 2: Proof of Theorem 5.2

We recall that we are in the setting of the  $(\mathbb{P}, \Phi)$ -branching process under the assumptions of Theorem 5.2. For a bounded function  $g : E \rightarrow [0, 1]$ , let us define the non-linear semigroup

$$u_t[g](x) = \mathbb{E}_{\delta_x} \left[ 1 - \prod_{i=1}^{N_t} g(x_i(t)) \right], \quad x \in E,$$

with the convention that if  $\zeta \leq t$  then the above product is equal to 1. We will also use the notation

$$u_t(x) := u_t[\mathbf{0}](x) = \mathbb{P}_{\delta_x}(\zeta > t) \quad \text{and} \quad a(t) := \langle \tilde{\varphi}, u_t \rangle, \tag{9.1}$$

where  $\mathbf{0}$  is the zero function.

**Proposition 9.1.** For all bounded and measurable functions  $g \in L_\infty^+(E)$  such that  $\|g\|_\infty \leq 1$  and  $x \in E, t \geq 0, u_t[g]$  satisfies

$$u_t[g](x) = \psi_t[1 - g](x) + \int_0^t \psi_s[\mathcal{G}[u_{t-s}[g]]](x)ds, \quad (9.2)$$

where the operator  $\mathcal{G}$  is given by

$$\mathcal{G}[h](x) := \gamma(x)\mathcal{E}_x \left[ 1 - \prod_{i=1}^N (1 - h(x_i)) - \sum_{i=1}^N h(x_i) \right] \quad (9.3)$$

for  $h \in L_\infty^+(E)$  such that  $\|h\|_\infty \leq 1$ . It follows that  $a(t)$ , defined in (9.1), satisfies

$$a(t) = a(0) + \int_0^t \langle \tilde{\varphi}, \mathcal{G}[u_{t-s}] \rangle ds = a(0) + \int_0^t \langle \tilde{\varphi}, \mathcal{G}[u_s] \rangle ds, \quad t \geq 0. \quad (9.4)$$

*Proof.* In some respects, the proof mirrors the one for Proposition 5.1 by splitting on the first branching event, so we will only sketch the arguments here. Recall that  $(\mathbf{P}_t, t \geq 0)$  is the underlying motion and that the branching operator  $\mathcal{A}$  given in (5.2). By conditioning on the first branching event, and then applying Lemma 1.2, Chapter 4 in [12], we find that, for  $x \in E, t \geq 0$  and  $g \in L_\infty^+(E)$  bounded by unity,

$$u_t[g](x) = \mathbf{P}_t[1 - g](x) + \int_0^t \mathbf{P}_s[\mathcal{A}[u_{t-s}[g]]](x)ds + \int_0^t \mathbf{P}_s[\mathcal{G}[u_{t-s}[g]]](x)ds. \quad (9.5)$$

Next, recall the process  $(Y_t, t \geq 0)$  under  $\mathbf{P}_x$  defined just before Lemma 5.1. Define  $(\tilde{u}_t, t \geq 0)$  via

$$\tilde{u}_t[g](x) = \mathbf{E}_x \left[ e^{\int_0^t B(Y_s)ds} (1 - g(Y_t)) \right] + \mathbf{E}_x \left[ \int_0^t e^{\int_0^s B(Y_u)du} \mathcal{G}[\tilde{u}_{t-s}[g]](Y_s) \right], \quad (9.6)$$

for  $x \in E, t \geq 0$  and  $g \in L_\infty^+(E)$  such that  $\|g\|_\infty \leq 1$ . Note that for the moment we don't claim a solution to (9.6) exists. By conditioning the right-hand side of (9.6) on the first jump of  $Y$  and invoking Lemma 1.2, Chapter 4 of [12] (bearing in mind the dynamics of  $Y$  given just before Lemma 5.1) we can check that  $(\tilde{u}_t, t \geq 0)$  is a solution to (9.6) if and only if it solves (9.5). In particular,  $(u_t, t \geq 0)$  satisfies (9.6). Combined with the many-to-one formula, this implies that

$$u_t[g](x) = \psi_t[1 - g](x) + \int_0^t \psi_s[\mathcal{G}[u_{t-s}[g]]](x)ds, \quad (9.7)$$

which is (9.2). As  $\tilde{\varphi}$  is a left eigenfunction for the semigroup  $(\psi_t, t \geq 0)$  and as we are dealing with the critical case, we have, for all  $t \geq 0$ , that  $\langle \tilde{\varphi}, \psi_t[g] \rangle = \langle \tilde{\varphi}, g \rangle$ . Using this fact and taking  $g = \mathbf{0}$  in (9.7), we find that

$$a(t) = \langle \tilde{\varphi}, \psi_t[\mathbf{1}] \rangle + \int_0^t \langle \tilde{\varphi}, \psi_s[\mathcal{G}[u_{t-s}[g]]] \rangle ds = \langle \tilde{\varphi}, \mathbf{1} \rangle + \int_0^t \langle \tilde{\varphi}, \mathcal{G}[u_{t-s}[g]] \rangle ds,$$

where  $\mathbf{1}$  is the function which is identically unity. The above equality gives rise to the first identity in (9.4). The second identity in (9.4) then follows by a change of variable.  $\square$

Our aim is to extract the proof of Theorem 5.2 from (9.2) and (9.4). To this end, we need a series of technical lemmas. The first lemma regards the analytical behaviour of the  $\mathcal{V}[h](x)$  defined in (5.6) and  $\mathcal{G}[h](x)$  defined in (9.3).

**Lemma 9.1.** *The following statements hold.*

(i) *For all  $x \in E$  and  $h : E \rightarrow [0, 1]$ , we have  $0 \geq \mathcal{G}[h](x) \geq -\|\gamma\|_\infty M$ , where  $M = \sup_{x \in E} m(x)$ .*

(ii) *There exists  $C \in (0, \infty)$  such that, for all  $h : E \rightarrow [0, \frac{1}{2}]$ ,*

$$\sup_{x \in E} \left| \mathcal{G}[h](x) + \frac{1}{2} \gamma(x) \mathcal{V}[h](x) \right| \leq C \sup_{x \in E} h(x)^3. \quad (9.8)$$

(iii) *There exists  $C \in (0, \infty)$  such that for all  $h : E \rightarrow [0, \frac{1}{2}]$ ,*

$$-\mathcal{G}[h](x) \geq C \gamma(x) \mathcal{V}[h](x) \geq 0, \quad x \in E.$$

(iv) *There exists  $C \in (0, \infty)$  such that for all  $h_1, h_2 : E \rightarrow [0, 1]$ , we have*

$$\sup_{x \in E} \left| \mathcal{V}[h_1](x) - \mathcal{V}[h_2](x) \right| \leq C \sup_{x \in E} |h_1(x) - h_2(x)|.$$

*Proof.* (i) The non positivity of  $\mathcal{G}[h]$  can be demonstrated using an iterative argument. For  $n \geq 1$  and  $(x_i, 1 \leq i \leq n) \in [0, 1]^n$ ,

$$f_n(x_1, \dots, x_n) := 1 - \prod_{i=1}^n (1 - x_i) - \sum_{i=1}^n x_i \leq 0.$$

To see why this is true, we note that  $f_1(x_1) = 0$  and more generally, we have  $f_{n+1}(x_1, \dots, x_n, 0) = f_n(x_1, \dots, x_n)$  as well as  $\partial_{n+1} f_{n+1}(x_1, \dots, x_{n+1}) \leq 0$ . This yields  $\mathcal{G}[h] \leq 0$ .

For the second inequality in (i), it suffices to observe that since  $0 \leq h \leq 1$

$$\mathcal{E}_x \left[ 1 - \prod_i (1 - h(x_i)) - \langle h, \mathcal{Z} \rangle \right] \geq -\mathcal{E}_x [\langle h, \mathcal{Z} \rangle] \geq -\sup_{x \in E} m(x),$$

which is bounded due to (G1).

(ii) Let us write  $L(h) = -\langle \log(1 - h), \mathcal{Z} \rangle$ , which is non negative. Then, we have

$$\begin{aligned} \left| \mathcal{G}[h](x) + \frac{1}{2} \gamma(x) \mathcal{V}[h](x) \right| &\leq \gamma(x) \mathcal{E}_x [1 - e^{-L(h)} - L(h) + \frac{1}{2} L(h)^2] \\ &\quad + \gamma(x) \mathcal{E}_x [\langle -\log(1 - h) - h - \frac{1}{2} h^2, \mathcal{Z} \rangle] \\ &\quad + \frac{1}{2} \gamma(x) \mathcal{E}_x [|\langle \log(1 - h), \mathcal{Z} \rangle^2 - \langle h, \mathcal{Z} \rangle^2|]. \end{aligned} \quad (9.9)$$

Applying the elementary bounds that  $0 \leq 1 - e^{-x} - x + \frac{1}{2} x^2 \leq \frac{1}{6} x^3$  for all  $x \geq 0$  and that  $0 \leq -\log(1 - x) - x - \frac{1}{2} x^2 \leq x^3$  for  $x \in [0, \frac{1}{2}]$ , since both  $\gamma$  and the number of offspring produced at a branching event are uniformly bounded, we see the first two terms on the right-hand side of (9.9) are bounded by  $C \|h\|_\infty^3$ . For the third term on the right-hand side of (9.9), writing  $\log(1 - h) = -h + R(h)$  with  $|R(h)| \leq C \|h\|_\infty^2$ , we get

$$|\langle \log(1 - h), \mathcal{Z} \rangle^2 - \langle h, \mathcal{Z} \rangle^2| \leq 2 \langle h, \mathcal{Z} \rangle \cdot \langle |R(h)|, \mathcal{Z} \rangle + \langle R(h), \mathcal{Z} \rangle^2 \leq C \|h\|_\infty^3,$$

which leads to (ii).

(iii) First note that for all  $n \geq 2$  and  $(x_i, 1 \leq i \leq n) \in [0, \frac{1}{2}]^n$ ,

$$\prod_{i=1}^n (1 - x_i) - 1 + \sum_{i=1}^n x_i \geq \frac{1}{2^{n-1}} \sum_{1 \leq i, j \leq n: i \neq j} x_i x_j.$$

This can be shown with an induction on  $n$  and its proof is therefore omitted. The result then follows due to (G3).

(iv) The final claim can be checked from the definition of  $\mathcal{V}$ , combined with (G3).  $\square$

We now prove a sequence of results that show that  $a(t)$  and  $u_t(x)$  can be bounded above and below by a constant divided by  $t$ . We can then use this to show that the difference between  $u_t(x)/\varphi$  and  $a(t)$  decays sufficiently quickly as  $t \rightarrow \infty$ , which we then, in turn, use to obtain the precise constant given in Theorem 5.2.

We start with the lower bound.

**Lemma 9.2.** *There exists  $C \in (0, \infty)$  such that*

$$\mathbb{P}_{\delta_x}(\zeta > t) \geq \frac{\varphi(x)}{Ct} \quad \text{and} \quad a(t) \geq \frac{1}{Ct}$$

for all  $t \geq 1$ .

*Proof.* Let us recall the change of measure (5.8) and that, from Theorem 5.1, there exists a  $C > 0$  such that

$$\sup_{r \in D, v \in V} \mathbb{E}_{\delta_x}^\varphi[\langle \varphi, X_t \rangle] \leq Ct$$

for all  $t \geq 1$ . By Jensen's inequality, we then get

$$\mathbb{P}_{\delta_x}(\zeta > t) = \mathbb{E}_{\delta_x}^\varphi \left[ \frac{\varphi(x)}{\langle \varphi, X_t \rangle} \right] \geq \frac{\varphi(x)}{\mathbb{E}_{\delta_x}^\varphi[\langle \varphi, X_t \rangle]} \geq \frac{\varphi(x)}{Ct}, \quad t \geq 1. \quad (9.10)$$

The lower bound for  $a(t)$  then follows from an integration with  $\tilde{\varphi}$ , recalling that we have normalised the left and right eigenfunctions so that  $\langle \tilde{\varphi}, \varphi \rangle = 1$ .  $\square$

Before turning to the upper bound, we first state and prove a lemma regarding the extinction time of the process.

**Lemma 9.3.** *For all  $x \in E$ , we have  $\mathbb{P}_{\delta_x}(\zeta < \infty) = 1$ .*

*Proof.* The ideas for this proof are taken from §11 of [14] but we write the details for the current setting here for convenience.

We start by proving that for all  $x \in E$  and  $t \geq 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = \mathbf{1}_{\{\zeta < \infty\}}, \quad \mathbb{P}_{\delta_x} - a.s.. \quad (9.11)$$

On the event  $\{\zeta < \infty\}$ , it is immediate that, for all  $x \in E$ ,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{X_t}(\zeta \leq t_0) = 1, \quad (9.12)$$

$\mathbb{P}_{\delta_x}$ -almost surely.

Let  $(T_n)_{n \in \mathbb{N}}$  be any increasing sequence of stopping times. Using the strong Markov property and (9.12), we have that, for all  $n \in \mathbb{N}$ ,

$$\mathbb{P}_{\delta_x}(\zeta < \infty) = \mathbb{E}_{\delta_x} [\mathbb{P}_{X_{T_n}}(\zeta < \infty)] \geq \mathbb{E}_{\delta_x} [\mathbb{P}_{X_{T_n}}(\zeta \leq t_0)].$$

Using this inequality and Fatou's Lemma, we deduce that

$$\begin{aligned} \mathbb{P}_{\delta_x}(\zeta < \infty) &\geq \liminf_{n \rightarrow \infty} \mathbb{E}_{\delta_x} [\mathbb{P}_{X_{T_n}}(\zeta \leq t_0)] \\ &\geq \mathbb{E}_{\delta_x} \left[ \liminf_{n \rightarrow \infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) \right] \\ &\geq \mathbb{E}_{\delta_x} [\mathbf{1}_{\{\zeta < \infty\}}] + \delta \mathbb{P}_{\delta_x} \left( \zeta = \infty \text{ and } \liminf_{n \rightarrow \infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) \geq \delta \right). \end{aligned}$$

It follows that, for all  $\delta \in (0, 1]$ , we have  $\mathbb{P}_{\delta_{(r,v)}}(\zeta = \infty \text{ and } \liminf_{n \rightarrow \infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) \geq \delta) = 0$ . This implies that, on  $\{\zeta = \infty\}$ ,  $\liminf_{n \rightarrow \infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) = 0$ . Since this is true for any sequence of increasing stopping times, we deduce that, on  $\{\zeta = \infty\}$ ,  $\limsup_{n \rightarrow \infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = 0$ . Together with (9.12), this gives us

$$\lim_{n \rightarrow \infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = \mathbf{1}_{\{\zeta < \infty\}}$$

$\mathbb{P}_{\delta_x}$ -almost surely, as required.

Next we prove that for all  $x \in E$ , on  $\{\zeta = \infty\}$ , we have  $\mathbb{P}_{\delta_x}$ -almost surely that

$$\lim_{t \rightarrow \infty} \langle \varphi, X_t \rangle = \infty. \quad (9.13)$$

First note that for any  $x \in E$  and  $t \geq 0$ , we have

$$\mathbb{P}_{\delta_x}(t < \zeta) \leq \mathbb{E}_{\delta_x}[N_t] = \psi_t[\mathbf{1}_E](x).$$

Using (2.7), we deduce that there exists a  $t_0 > 0$  such that, for all  $x \in E$ ,

$$\mathbb{P}_{\delta_x}(t_0 < \zeta) \leq \psi_{t_0}[\mathbf{1}_E](x) \leq 2\varphi(x).$$

It is straightforward to show that  $\sup_{x \in E} \mathbb{P}_x(t_0 < \zeta) < 1$ . Hence, there exists a constant  $c_0 \in (0, 1)$  such that, uniformly for all  $x \in E$ ,

$$\mathbb{P}_{\delta_x}(t_0 < \zeta) \leq c_0 \wedge 2\varphi(x).$$

Using the branching property, we deduce that, for all  $\mu = \sum_{i=1}^n \delta_{x_i} \in \mathcal{M}(E)$ ,

$$\mathbb{P}_{\mu}(\zeta \leq t_0) \geq \prod_{i=1}^n (1 - c_0 \wedge 2\varphi(x_i)).$$

Now, using (9.11) we have,

$$\mathbf{1}_{\{\zeta < \infty\}} = \limsup_{n \rightarrow \infty} \mathbb{P}_{X_n}(\zeta \leq t_0) \geq \limsup_{n \rightarrow \infty} \prod_{i=1}^{N_n} (1 - c_0 \wedge 2\varphi(x_i(n))), \quad (9.14)$$

where  $x_i(n)$  denotes the  $i^{\text{th}}$  particle alive at time  $n$ ,  $i = 1, \dots, N_n$ .

Since  $c_0 < 1$ , taking logarithms in (9.14) and using  $\log x \leq x - 1$ , we deduce that, on  $\{\zeta = \infty\}$ ,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} (c_0 \wedge 2\varphi(x_i(n))) = \infty$$

and (9.13) follows.

To conclude the proof of the lemma, we note that as  $\varphi$  is uniformly bounded above,  $W_t = \langle \varphi, X_t \rangle / \varphi(x) \geq C \langle \varphi, X_t \rangle$ . Since,  $W$  is uniformly convergent, (9.13) implies that  $\{\zeta < \infty\}$   $\mathbb{P}_{\delta_x}$ -a.s..  $\square$

**Lemma 9.4.** *Under the assumptions of Theorem 5.2, there exists a constant  $C > 0$  such that for all  $t \geq t_0$ ,*

$$a(t) \leq \frac{C}{t} \quad \text{and} \quad \sup_{x \in E} u_t(x) \leq \frac{C}{t}. \quad (9.15)$$

*Proof.* We first show that

$$a(t) \rightarrow 0 \quad \text{and} \quad \sup_{x \in E} u_t(x) \rightarrow 0$$

as  $t \rightarrow \infty$ .

Recall  $\zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}$  is the extinction time. Due to Lemma 9.3, we have  $\mathbb{P}_{\delta_x}(\zeta < \infty) = 1$ , namely,  $u_t(x) = \mathbb{P}_{\delta_x}(\zeta > t) \rightarrow 0$ , as  $t \rightarrow \infty$ , for all  $x \in E$ . Clearly, for each  $x$ ,  $t \mapsto u_t(x) = \mathbb{P}_{\delta_x}(\zeta > t)$  is non increasing. Let us write  $u_\infty(x) = \lim_{t \rightarrow \infty} u_t(x)$ . Similarly, we denote by  $a(\infty)$  the limit of  $a(t)$ . Letting  $t \rightarrow \infty$  in (9.4), we find by monotone convergence that,

$$0 \leq - \int_0^\infty \langle \tilde{\varphi}, \mathcal{G}[u_s] \rangle ds = a(0) - a(\infty) < \infty.$$

In particular, this implies that  $\limsup_{t \rightarrow \infty} \langle \tilde{\varphi}, -\mathcal{G}[u_t] \rangle = 0$ . On the other hand,  $u_t(x) \rightarrow u_\infty(x)$  for each  $x$ . Due to Lemma 9.1 (i), it is straightforward to check that  $\mathcal{G}[u_t](x) \rightarrow \mathcal{G}[u_\infty](x)$  for each  $x$ , and that  $0 = \limsup_{t \rightarrow \infty} \langle \tilde{\varphi}, -\mathcal{G}[u_t] \rangle = \langle \tilde{\varphi}, -\mathcal{G}[u_\infty] \rangle = 0$ . Hence,  $\mathcal{G}[u_\infty](x) = 0$  for a.e.  $x$ . According to Lemma 9.1 (i), this implies  $u_\infty(x) = 0$  for a.e.  $x$ . Therefore,  $a(t) = \langle \tilde{\varphi}, u_t \rangle \rightarrow 0$ , as  $t \rightarrow \infty$ , by dominated convergence. To get to the uniform convergence of  $u_t$ , let us note that  $u_{t+s}(x) = u_t[1 - u_s](x)$  by the Markov branching property. Appealing to (9.2) and Lemma 9.1 (i), we find that

$$0 \leq u_{t+s}(x) = \psi_t[u_s](x) + \int_0^t \psi_l[\mathcal{G}[u_{t+s-l}]](x) dl \leq \psi_t[u_s](x), \quad (9.16)$$

by Lemma 9.1 (i). Combined with (2.7), this yields

$$\|u_{t+s}\|_\infty \leq \|\psi_t[u_s]\|_\infty \leq a(s)\|\varphi\|_\infty + O(e^{-\varepsilon t}). \quad (9.17)$$

Taking first  $t \rightarrow \infty$  and then  $s \rightarrow \infty$  gives the desired result.

Now we prove the required upper bound on  $a(t)$  and  $\|u_t\|_\infty$ . Due to the above calculations, we may fix  $t_0 > 0$  such that  $\|u_t\|_\infty \leq 1/2$  for all  $t \geq t_0$ . Note that the integrand in (9.4) is bounded due to our assumptions and Lemma 9.1 (i). It follows that  $a(t)$  is differentiable. Differentiating (9.4) for  $t \geq t_0$  and then applying the bound in Lemma 9.1 (iii), we obtain that for  $t \geq t_0$ ,

$$a'(t) = \langle \tilde{\varphi}, \mathcal{G}[u_t] \rangle \leq -C_1 \langle \tilde{\varphi}, \gamma \mathcal{V}[u_t] \rangle \leq -C_2 \langle \tilde{\varphi}, u_t \rangle^2 = -C_2 a(t)^2,$$

where we have used assumption (G4) in the second inequality.

Integrating from  $t_0$  to  $t$  yields

$$a(t) \leq (C_2(t - t_0) + a(t_0)^{-1})^{-1} \leq (C_2 t)^{-1},$$

where the last inequality holds for  $t$  sufficiently large. The upper bound for  $a(t)$  then follows. We may then apply the same techniques as in (9.17) by setting  $s = t$  to obtain the uniform bound for  $u_t(x)$ .  $\square$

The next result shows that the long-term behaviour of  $u_t/\varphi$  and  $a(t)$  are the same, which will be key to obtaining the correct constants in the bounds obtained in the previous lemma.

**Lemma 9.5.** *Suppose that there exist  $\kappa, \eta \in (0, \infty)$  such that  $\|u_t\|_\infty \leq \kappa t^{-\eta}$  for all  $t > 0$ . Then we can find some constant  $C \in (0, \infty)$  which does not depend on  $\kappa$  such that*

$$\sup_{x \in E} \left| \frac{u_t(x)}{\varphi(x)} - a(t) \right| \leq C \kappa^2 t^{-2\eta}, \quad \text{for all } t > 0.$$

*Proof.* Comparing (9.2) with (9.4), we find that

$$\begin{aligned} \left| \frac{u_t(x)}{\varphi(x)} - a(t) \right| &\leq \left| \frac{\psi_t[\mathbf{1}](x)}{\varphi(x)} - \langle \tilde{\varphi}, \mathbf{1} \rangle \right| + \int_0^t \left| \frac{\psi_{t-s}[\mathcal{G}[u_s]](x)}{\varphi(x)} - \langle \tilde{\varphi}, \mathcal{G}[u_s] \rangle \right| ds \\ &\leq C_1 e^{-\varepsilon t} + \int_0^t C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds, \end{aligned}$$

where the constant  $\varepsilon > 0$  comes from (2.7). Thanks to Lemma 9.4, we can find  $t_0 > 0$  such that  $\sup_{x \in E} \sup_{t \geq t_0} u_t(x) \leq 1/2$ . Take  $t \geq 2t_0$ , then the integral above can be bounded as follows:

$$\begin{aligned} \int_0^t C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds &= \int_0^{t/2} C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds + \int_{t/2}^t C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds \\ &\leq \frac{C_1 \|\gamma\|_\infty N_{\max}}{\varepsilon} e^{-\varepsilon t/2} + \int_{t/2}^t C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds, \end{aligned} \quad (9.18)$$

where we have used Lemma 9.1 (i). We note that  $\mathcal{V}[u_s] \leq N_{\max}^2 \|u_s\|_\infty^2$ . Using this together with Lemma 9.1 (ii), we deduce for the second term in (9.18) that  $\|\mathcal{G}[u_s]\|_\infty \leq \|\mathcal{V}[u_s]\|_\infty + \|u_s^3\|_\infty \leq C_2 \|u_s\|_\infty^2$ . The latter is bounded by  $C_2 \kappa^2 s^{-2\eta}$  due to the assumption of the lemma. Therefore,

$$\begin{aligned} \int_{t/2}^t C_1 e^{-\varepsilon(t-s)} \|\mathcal{G}[u_s]\|_\infty ds &\leq C_3 \kappa^2 \int_{t/2}^t e^{-\varepsilon(t-s)} s^{-2\eta} ds \\ &= \frac{C_3 \kappa^2}{\varepsilon} \left( t^{-2\eta} - 4e^{-\varepsilon t/2} t^{-2\eta} \right) + C_4 \int_{t/2}^t e^{-\varepsilon(t-s)} s^{-2\eta-1} ds = O(t^{-2\eta}). \end{aligned}$$

Putting the pieces together, we obtain the claimed bound in the lemma.  $\square$

We are now ready to prove our first main result.

*Proof of Theorem 3.1.* Applying Lemma 9.5 with  $\eta = 1$  and  $\kappa$  being some positive constant (this is permitted thanks to Lemma 9.4), we have

$$\sup_{x \in E} \left| \frac{u_t(x)}{\varphi(x)} - a(t) \right| = O(t^{-2}), \quad t \rightarrow \infty.$$

On the other hand, we've seen in Lemma 9.2 that  $a(t)^{-1} = O(t)$ . It follows that

$$\sup_{x \in E} \left| \frac{u_t(x)}{\varphi(x)a(t)} - 1 \right| = O(t^{-1}), \quad t \rightarrow \infty. \quad (9.19)$$

Applying Lemma 9.1 (iv), we deduce that

$$\begin{aligned} \sup_{x \in E} |\mathcal{V}[u_t](x) - a(t)^2 \mathcal{V}[\varphi](x)| &= \sup_{r \in D, v \in V} a(t)^2 \left| \mathcal{V} \left[ \frac{u_t}{a(t)} \right](x) - \mathcal{V}[\varphi](x) \right| \\ &\leq C a(t)^2 \sup_{x \in E} \left| \frac{u_t(x)}{a(t)} - \varphi(x) \right| = O(t^{-3}). \end{aligned} \quad (9.20)$$

Therefore, for all  $t \geq t_0$ ,

$$\begin{aligned} a(t) - a(t_0) &= \int_{t_0}^t \langle \tilde{\varphi}, \mathcal{G}[u_s] \rangle ds \\ &= -\frac{1}{2} \int_{t_0}^t \left( \langle \tilde{\varphi}, \gamma \mathcal{V}[u_s] \rangle + O(\|u_s\|_\infty^3) \right) ds \\ &= -\frac{1}{2} \int_{t_0}^t \left( \langle \tilde{\varphi}, \gamma \mathcal{V}[u_s] \rangle + O(s^{-3}) \right) ds \\ &= -\frac{1}{2} \int_{t_0}^t \left( a(s)^2 \langle \tilde{\varphi}, \gamma \mathcal{V}[\varphi] \rangle + O(s^{-3}) \right) ds \\ &= -\frac{\Sigma}{2} \int_{t_0}^t a(s)^2 (1 + o(1)) ds, \end{aligned}$$

where we have used (9.8) in the second equality, (9.15) in the third, (9.20) in the fourth and Lemma 9.2 in the final equality. This implies that

$$a(t) \sim \frac{2}{\Sigma t} \text{ as } t \rightarrow \infty.$$

The desired asymptotic for  $u_t$  then follows from (9.19). □

**Remark 9.2.** We end this section with a few comments on Theorem 5.2 in the setting of the NTE, or equivalently, Theorem 3.1. We note that assumption (H5) is weaker than (G4) in the case where  $E$  is bounded. To compensate this, the proof of Theorem 3.1 requires some more work. The idea in this case is to use the same techniques as in Lemma 9.4 to obtain coarse upper and lower bounds but of order  $1/\sqrt{t}$ . Then, Lemma 9.5 allows us to bootstrap these bounds to obtain bounds of order  $1/t$ . From there, we can then conclude as in the general case.

## 10 Step 3: Proof of Theorem 5.3

Returning to (4.5), the conclusions of Theorem 5.1 and Theorem 5.2 yield

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x} [\langle \varphi, X_t \rangle^k | \zeta > t] = \lim_{t \rightarrow \infty} \frac{\mathbb{E}_{\delta_x}^\varphi [\langle \varphi, X_t \rangle^{k-1} / t^{k-1}]}{t \mathbb{E}_{\delta_x}^\varphi [1 / \langle \varphi, X_t \rangle]} = (\Sigma/2)^k k!, \quad k \in \mathbb{N}.$$

The right-hand side above is precisely the  $k^{\text{th}}$  moment of an exponential distribution with parameter  $2/\Sigma$ . Hence, at this stage, again thanks to the *moment problem* for the exponential distribution, noting that  $\langle \tilde{\varphi}, \varphi \rangle = 1$ , we conclude that Theorem 5.3 holds for the special case that  $f = \varphi$ .

We now deal more generally with the case that  $f \in L_\infty^+(E)$ .

*Proof of Theorem 5.3.* As alluded to above, we introduce  $\tilde{f} = f - \langle \tilde{\varphi}, f \rangle \varphi$  and consider the behaviour of  $\langle \tilde{f}, X_t \rangle / t$  as  $t \rightarrow \infty$ . The first thing we should note in this direction is that

$$\langle \tilde{\varphi}, \tilde{f} \rangle = \langle \tilde{\varphi}, f \rangle - \langle \tilde{\varphi}, f \rangle \langle \tilde{\varphi}, \varphi \rangle = 0,$$

where we have used that  $\langle \tilde{\varphi}, \varphi \rangle = 1$ . Hence, Theorem 2.1 tells us that for  $t$  sufficiently large

$$\sup_{x \in E} \left| \frac{\psi_t[\tilde{f}](x)}{\varphi(x)} \right| = O(e^{-\epsilon t}),$$

for some  $\epsilon > 0$ . In particular, we have  $\sup_{x \in E} \psi_t[\tilde{f}](x) \rightarrow 0$  as  $t \rightarrow \infty$ .

We also note that, under the assumption that  $0 \leq f \leq \varphi$  we have the estimate

$$-\langle \tilde{\varphi}, f \rangle \varphi \leq \tilde{f} \leq (1 - \langle \tilde{\varphi}, f \rangle) \varphi. \quad (10.1)$$

Further, since  $f \leq \varphi$ ,  $\langle \tilde{\varphi}, f \rangle \leq \langle \tilde{\varphi}, \varphi \rangle = 1$ , the upper bound in (10.1) is positive. As such,  $|\tilde{f}| \in L_\infty^+(E)$ .

We will use these facts to prove that  $\langle \tilde{f}, X_t \rangle / t$  converges weakly under  $\mathbb{P}_{\delta_x}(\cdot | \langle 1, X_t \rangle > 0)$  to zero as  $t \rightarrow \infty$ . In that case, thanks again to Slutsky's Theorem we can deduce that, for  $x \in E$ ,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x} \left[ \exp \left( -\theta \frac{\langle \tilde{f}, X_t \rangle}{t} \right) \middle| \zeta > t \right] \\ &= \lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x} \left[ \exp \left( -\theta \frac{\langle \tilde{f}, X_t \rangle}{t} - \theta \langle \tilde{\varphi}, f \rangle \frac{\langle \varphi, X_t \rangle}{t} \right) \middle| \zeta > t \right] \\ &= \frac{1}{1 + \Sigma \theta \langle \tilde{\varphi}, f \rangle / 2}, \end{aligned} \quad (10.2)$$

as required.

Let us thus consider the estimate

$$\mathbb{P}_{\delta_x}(|\langle \tilde{f}, X_t \rangle|/t > \epsilon | \zeta > t) \leq \frac{1}{\epsilon^2} \frac{\mathbb{E}_{\delta_x} [\langle \tilde{f}, X_t \rangle^2 / t]}{t \mathbb{P}_{\delta_x}(\langle 1, X_t \rangle > 0)}. \quad (10.3)$$

In order to deal with the numerator on the righthand side above, we will use the *many-to-two* formula given in Lemma 5.2. We restate it here in the required form for convenience.

Let  $g \in L_\infty^+(E)$ . Then the many-to-two formula reads

$$\mathbb{E}_{\delta_x} \left[ \langle g, X_t \rangle^2 \right] = \psi_t[g^2](x) + \int_0^t \psi_s \left[ \gamma \mathcal{V}[\psi_{t-s}[g]] \right] (x) ds, \quad (10.4)$$

for  $x \in E$  and  $t \geq 0$ , where  $\mathcal{V}$  was given in (5.6).

Hence, setting  $g = \tilde{f}$  and dividing by  $t^2$ , we note that the first term on the right-hand side of (10.4) has the behaviour

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sup_{x \in E} \left| \psi_t[\tilde{f}^2](r, v) \right| \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sup_{x \in E} \left| \psi_t[\tilde{f}^2](x) - \langle \tilde{\varphi}, \tilde{f}^2 \rangle \varphi \right| + \frac{1}{t} \langle \tilde{\varphi}, \tilde{f}^2 \rangle \|\varphi\|_\infty = 0,$$

where the final equality uses Theorem 2.1. For the integral term in (10.4), we can handle it using the ergodic convolution Theorem 7.1, the change of measure Lemma 6.1 and the many-to-one formula in Lemma 5.5. In particular, for  $u \in [0, 1]$  and  $t \geq 0$ ,

$$\begin{aligned} \psi_{ut} \left[ \gamma \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]] \right] (x) &= \frac{\psi_{ut} \left[ \varphi \frac{\gamma}{\varphi} \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]] \right] (x)}{\varphi(x)} \varphi(x) \\ &= \mathbf{E}_x^\varphi \left[ \frac{\gamma(Y_{ut})}{\varphi(Y_{ut})} \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]](Y_{ut}) \right] \varphi(x). \end{aligned}$$

Moreover, defining  $F(x, u, t) = \varphi(x)^{-1} \gamma(r, v) \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]](x)$ , Lemma 9.1 and the fact that  $\psi_t[\tilde{f}] \rightarrow 0$  imply that

$$F(x, u) := \lim_{t \rightarrow \infty} F(x, u, t) = \lim_{t \rightarrow \infty} \frac{\gamma(x)}{\varphi(x)} \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]](x) = 0,$$

so that the conditions of Theorem 7.1 are trivially met (i.e.  $F \in \mathcal{C}$ ). Hence, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x} \left[ \frac{\langle \tilde{f}, X_t \rangle^2}{t} \right] = \lim_{t \rightarrow \infty} \int_0^1 \mathbf{E}_x^\varphi \left[ \frac{\gamma(Y_{ut})}{\varphi(Y_{ut})} \mathcal{V}[\psi_{t(1-u)}[\tilde{f}]](Y_{ut}) \right] du = 0.$$

Using this back in (10.3), together with the convergence of  $t\mathbb{P}_{\delta_x}(\langle 1, X_t \rangle > 0)$  as  $t \rightarrow \infty$  we see that

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x}(|\langle \tilde{f}, X_t \rangle|/t > \varepsilon | \zeta > t) \leq \lim_{t \rightarrow \infty} \frac{1}{\varepsilon^2} \frac{\mathbb{E}_{\delta_x} \left[ \langle \tilde{f}, X_t \rangle^2 / t \right]}{t\mathbb{P}_{\delta_x}(\langle 1, X_t \rangle > 0)} = 0.$$

This tells us that  $\langle \tilde{f}, X_t \rangle / t$  converges weakly to zero as  $t \rightarrow \infty$  and hence, thanks to (10.2), this completes the proof of the Yaglom limit for general  $f \in L_\infty^+(E)$ .  $\square$

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## A Combinatorial results

Here we give some combinatorial results that are of importance to us. The first is the multinomial expansion formula, the second is a variant of the multinomial formula for sums of iid random variables. The third is an identity which does not appear to be easily available in the literature, but is nonetheless relatively straightforward to prove.

**Lemma A.1** (Multinomial identity). *Suppose that  $a_1, \dots, a_n$  are real-valued and  $k \in \mathbb{N}$ . Then*

$$\left( \sum_{i=1}^n a_i \right)^k = \sum_{[k_1, \dots, k_n]} \binom{k}{k_1, \dots, k_n} \prod_{i=1}^n a_i^{k_i}, \quad (\text{A.1})$$

where the sum is over the set  $[k_1, \dots, k_n]$  of all combinations of non-negative  $(k_i, i = 1, \dots, n)$  such that  $\sum_{i=1}^n k_i = k$ , which are assigned as power indices to  $(a_1, \dots, a_n)$ .

The set  $[k_1, \dots, k_n]$  is very complex, particularly when one takes into account the fact that, for example, when  $n > k$ , some of the  $k_i$  are necessarily zero. The combinatorics become more complicated when we apply this lemma to the setting that the  $a_i$  are iid random variables and the left-hand side of (A.1) is averaged.

**Lemma A.2.** *Suppose that  $Y_1, \dots, Y_n$  are iid random variables which are equal in distribution to  $(Y, P)$ , then*

$$\mathbb{E} \left[ \left( \sum_{i=1}^n Y_i \right)^k \right] = \sum_{j=1}^k \binom{n}{j} \sum_{[k_1, \dots, k_j]_+} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j \mathbb{E}[Y^{k_i}] \quad (\text{A.2})$$

where the sum is over the set  $[k_1, \dots, k_j]_+$  of all combinations of strictly positive  $\{k_1, \dots, k_j\}$  such that  $\sum_{i=1}^j k_i = k$  if  $j \leq n$  and otherwise  $[k_1, \dots, k_j]_+ = \emptyset$ .

*Proof of Lemma A.2.* We start by re-examining (A.1). To avoid having to deal with the zero values of  $k_i$  in (A.1) we would like to write

$$[k_1, \dots, k_n] = \bigcup_{j=1}^k \bigcup_{\{i_1, \dots, i_j\} \in \mathcal{I}_j} \bigcup_{\{k_1, \dots, k_j\} \in [k_1, \dots, k_j]_+},$$

where the first union in  $j$  identifies the number  $k_i$  which are strictly positive, the second union is over,  $\mathcal{I}_j$ , the set of all possible choices of  $j$  indices  $\{i_1, \dots, i_j\}$  from  $\{1, \dots, n\}$  and, the third union is taken over  $[k_1, \dots, k_j]_+$ , the set of combinations of  $j$  strictly positive powers  $\{k_1, \dots, k_j\}$ , such that  $\sum_{\ell=1}^j k_\ell = k$ , that are assigned to  $\{i_1, \dots, i_j\}$  by matching index. Note that  $[k_1, \dots, k_j]_+ = \emptyset$  for  $j > n$  as one cannot select more than  $n$  different of the  $Y_i$ -s across the  $k$  brackets in the product  $(Y_1 + \dots + Y_n)^k$ .

We can thus re-write (A.1)

$$\left( \sum_{i=1}^n Y_i \right)^k = \sum_{j=1}^k \sum_{\{i_1, \dots, i_j\} \in \mathcal{I}_j} \sum_{[k_1, \dots, k_j]_+} \binom{k}{k_1, \dots, k_j} \prod_{\ell=1}^j Y_{i_\ell}^{k_\ell}. \quad (\text{A.3})$$

In essence, thinking of the left-hand side as  $k$  brackets of the form  $(Y_1 + \dots + Y_n)$  that are multiplied together, we first pick the indices we want to appear in multiplying one term from each bracket,

then look at the number of ways these indices can be selected from each of the  $k$  brackets (this explains the appearance of the multinomial coefficient again).

The fact that  $Y_1, \dots, Y_n$  are iid means that

$$\mathbb{E} \left[ \prod_{\ell=1}^j Y_{i_\ell}^{k_\ell} \right] = \prod_{\ell=1}^j \mathbb{E}[Y^{k_\ell}],$$

which is the same for multiple combinations of  $\{i_1, \dots, i_j\}$  and  $\{k_1, \dots, k_j\}$ . Indeed, for a given  $j \in \{1, \dots, k\}$  and combination  $[k_1, \dots, k_j]_+$  there are  $\binom{n}{j}$  ways of picking  $j$  indices from  $n$ . In conclusion (A.2) holds.  $\square$

We conclude with an unusual combinatorial lemma that we could not find in the literature.

**Lemma A.3.** *We have we have*

$$(k+1)! = \sum_{j=1}^k 2^j \sum_{[k_1, \dots, k_j]_+} \frac{1}{j!} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)!, \quad k \in \mathbb{N}.$$

*Proof.* Recall that, for  $j, k \in \mathbb{N}$  such that  $1 \leq j \leq k$ , the  $(k, j)$ -Bell polynomial is given by

$$B_{k,j}(x_1, \dots, x_{k-j+1}) := \sum_{(\ell_1, \dots, \ell_{k-j+1})} \frac{k!}{\ell_1! \dots \ell_{k-j+1}!} \prod_{p=1}^{k-j+1} \left( \frac{x_p}{p!} \right)^{\ell_p},$$

where the sum is taken over all combinations of  $(\ell_1, \dots, \ell_{k-j+1})$  such that

$$\sum_{i=1}^{k-j+1} \ell_i = j \quad \text{and} \quad \sum_{i=1}^{k-j+1} i \ell_i = k. \quad (\text{A.4})$$

One should think of  $(\ell_1, \dots, \ell_{k-j+1})$  as representing the partition of  $\{1, \dots, k\}$  into  $1 \leq j \leq k$  non-empty sub blocks. The quantity  $\ell_i$  is thus the number of blocks of size  $i$  so that the first sum in (A.4) indicates that the total number of blocks is  $j$  and the total number of items in these blocks is  $k$ . Note that some of the values of  $\ell_i$  may be zero. Note also that the index runs up to  $k-j+1$  as the largest of the  $j$  blocks can be obtained by having  $j-1$  blocks of size 1 and 1 block of size  $k-j+1$ . As such, the  $(k, j)$ -Bell polynomial tells us the number of ways of generating  $j$  non-empty sub blocks from  $\{1, \dots, k\}$  by setting  $x_1 = \dots = x_{k-j+1} = 1$ .

If a configuration  $\{k_1, \dots, k_j\}$  has block count  $\{\ell_1, \dots, \ell_{k-j+1}\}$ , then

$$\prod_{i=1}^j (k_i - 1)! = \prod_{p=1}^{k-j+1} ((p-1)!)^{\ell_p}.$$

For each  $j = 1, \dots, k$ , counting the functionals  $\prod_{i=1}^j (k_i - 1)!$  with its respective multiplicity is equivalent to counting the functionals  $\prod_{p=1}^{k-j+1} ((p-1)!)^{\ell_p}$  with their respective multiplicity.

Considering again the multinomial formula (A.2), we see that for each  $j = 1, \dots, k$  and  $\{k_1, \dots, k_j\} \in [k_1, \dots, k_j]_+$  the functional  $\prod_{i=1}^j (k_i - 1)!$  has a respective multiplicity given by

$$\frac{1}{j!} \binom{k}{k_1, \dots, k_j}.$$

Note, the factor  $1/j!$  is present as we are not concerned about the order of the blocks. Similarly, for each  $\{\ell_1, \dots, \ell_{k-j+1}\} \in (\ell_1, \dots, \ell_{k-j+1})$ , the functional  $\prod_{p=1}^{k-j+1} ((p-1)!)^{\ell_p}$  has multiplicity

$$\frac{1}{\ell_1! \cdots \ell_{k-j+1}!} \times \frac{k!}{(1!)^{\ell_1} \cdots ((k-j+1)!)^{\ell_{k-j+1}}},$$

where the second fraction is the combinatoric for selecting  $\ell_1$  blocks of size 1,  $\ell_2$  blocks of size 2... and  $\ell_{k-j+1}$  blocks of size  $k-j+1$ , and the first fraction factors out the multiplicity of order of blocks within each size category.

It follows that

$$\begin{aligned} & \sum_{j=1}^k 2^j \sum_{[k_1, \dots, k_j]_+} \frac{1}{j!} \binom{k}{k_1, \dots, k_j} \prod_{i=1}^j (k_i - 1)! \\ &= \sum_{j=1}^k 2^j \sum_{(\ell_1, \dots, \ell_{k-j+1})} \frac{1}{\ell_1! \cdots \ell_{k-j+1}!} \frac{k!}{\prod_{p=1}^{k-j+1} (p!)^{\ell_p}} \prod_{p=1}^{k-j+1} ((p-1)!)^{\ell_p} \\ &= \sum_{j=1}^k 2^j \sum_{(\ell_1, \dots, \ell_{k-j+1})} \frac{k!}{\ell_1! \cdots \ell_{k-j+1}!} \prod_{p=1}^{k-j+1} \left( \frac{(p-1)!}{p!} \right)^{\ell_p} \\ &= \sum_{j=1}^k 2^j B_{k,j}(0!, \dots, (k-j)!). \end{aligned} \tag{A.5}$$

Standard texts on combinatorics (see e.g [5]) identify

$$B_{k,j}(0!, \dots, (k-j)!) = \left[ \begin{matrix} k \\ j \end{matrix} \right], \tag{A.6}$$

which is a Stirling number of the first kind. Moreover, for the latter, it is known that

$$\sum_{j=1}^k x^j \left[ \begin{matrix} k \\ j \end{matrix} \right] = x(x+1) \cdots (x+k-1). \tag{A.7}$$

Putting the pieces together from (A.5), (A.6) and (A.7), the desired result holds.  $\square$

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