On the approximation of electromagnetic fields by edge finite elements. Part 4: analysis of the model with one sign-changing coefficient

Patrick Ciarlet

Noname manuscript No.
(will be inserted by the editor)

On the approximation of electromagnetic fields by edge finite elements. Part 4: analysis of the model with one sign-changing coefficient

Patrick Ciarlet Jr.

This version: September 14, 2022

Abstract In electromagnetism, in the presence of a negative material surrounded by a classical material, the electric permittivity, and possibly the magnetic permeability, can exhibit a sign-change at the interface. In this setting, the study of electromagnetic phenomena is a challenging topic. We focus on the time-harmonic Maxwell equations in a bounded set Ω of $\mathbb{R}^3$, and more precisely on the numerical approximation of the electromagnetic fields by edge finite elements. Special attention is paid to low-regularity solutions, in terms of the Sobolev scale $(\boldsymbol{H}^{\mathtt{s}}(\Omega))_{\mathtt{s}>0}$. With the help of T-coercivity, we address the case of one sign-changing coefficient, both for the model itself, and for its discrete version. Optimal a priori error estimates are derived.

Introduction

We study the numerical approximation by finite elements of electromagnetic fields governed by the time-harmonic Maxwell equations, in the presence of a negative material surrounded by a classical material. A negative material can be a metal at optical frequencies, or a metamaterial, see for instance [49, 2]. So, in this setting, the electric permittivity, and possibly the magnetic permeability, can exhibit a sign-change at the interface between the two materials. We consider such a model in a bounded set of $\mathbb{R}^3$, supplemented with a vanishing boundary condition on the tangential trace. To the author's knowledge, the first attempt to address this situation theoretically can be found in [11, 10]; see also [16, 45]. However, little is known regarding the numerical approximation of the model. In the present paper, we provide the numerical analysis for a model with one sign-changing coefficient.
For the numerical approximation, we use (low-order) edge finite elements. We use some recent results [22, 32, 23] to interpolate low-regularity solutions that

POEMS, CNRS, INRIA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France (E-mail: patrick.ciarlet@ensta-paris.fr).

can occur both in a classical setting, that is for a model with fixed-sign, piecewise smooth coefficients [26, 7, 24], and in the presence of an interface between a classical material and a negative material.
In what follows, we shall assume that the electric permittivity ε has a sign-change, while the magnetic permeability μ has a fixed sign (when the roles of ε and μ are reversed, we refer to section 8). Typically, this corresponds to an interface model between a metal surrounded by a classical material (in some ad hoc frequency range). Classically [3, §8], for solving the time-harmonic Maxwell equations, one can choose first-order formulations in both the electric and magnetic fields, or second-order formulations in the electric field only, or in the magnetic field only. Our choice will be a second-order formulation in the electric field.

The outline is as follows. We begin by introducing some notations, together with a precise definition of the mathematical framework considered hereafter. Before investigating the solution of this problem, we propose some comments in section 2 to help identify the difficulties to be addressed. For that, we rely on some well-known facts regarding the classical setting (fixed-sign coefficient), that we shall apply to the new model. We introduce the companion scalar problem and tools, such as the T-coercivity to realize the inf-sup condition. In section 3, we explain how to solve the time-harmonic Maxwell equations. Next, in section 4, we recall the numerical approximation via edge finite elements, and in particular how one can interpolate the electric field, which can (possibly) be of low-regularity. To prove the results regarding convergence of the numerical method, we use some results regarding practical discrete T-coercivity (for the companion scalar problem) which is achieved with the help of T-conform meshes. These are recalled in the appendix A. As a matter of fact, these results allow us to prove the uniform discrete inf-sup condition for the time-harmonic Maxwell equations: this is the object of the next two sections, where we use a result on the div-curl problem established in appendix B. Then, in section 7, we provide a numerical illustration to check that the expected convergence order is achieved, and how the use of T-conform meshes may impact the convergence rate. In section 8, we outline how one can solve theoretically and numerically the case of μ having a sign-change, and ε having a fixed sign. Finally, we give some concluding remarks in the last section.

We refer to [36] for the theoretical and numerical analyses of the two-dimensional time-harmonic Maxwell equations, and to [23] for the analyses of the three-dimensional, div-curl, or div-curlcurl, problem, with one sign-changing coefficient. Let us comment briefly on some alternative finite element methods that have previously been designed to solve numerically scalar problems with sign-changing coefficients (diffusion-like, or time-harmonic). As mentioned above, one uses T-conform meshes when one relies on the T-coercivity theory to prove convergence. On the one hand, the use of plain meshes is tempting. However, to the author's knowledge, convergence theory is incomplete, namely convergence is not guaranteed for all well-posed problems (see section 7) ; and, if one adds dissipation to restore well-posedness, convergence is suboptimal and can only be guaranteed in some special cases (see respectively sections 5.1 and 5.2 in [19]). On the other hand, one may ask whether it is possible to solve subproblems in regions where the sign of the coefficients is constant, coupled by transmission conditions on the interface. It turns out that an iterative solver based on optimal control theory (with a control defined on the interface) has been proposed in [1] to solve diffusion problems. However it requires extra-regularity of the solution. Finally, let us mention a recent work [25], also relying on optimal control theory (with a volume control), that allows one to solve iteratively diffusion problems without any regularity assumption.

1 Setting of the problem

As in [22], we denote constant fields by the symbol *cst*. Vector-valued (respectively tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters). Unless otherwise specified, we consider spaces of real-valued functions. Given a non-empty open set $\mathcal{O}$ of $\mathbb{R}^3$, we use the notation $(\cdot|\cdot)_{0,\mathcal{O}}$ (respectively $\|\cdot\|_{0,\mathcal{O}}$) for the $L^2(\mathcal{O})$ and the $\boldsymbol{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$ inner products (resp. norms). More generally, $(\cdot|\cdot)_{\mathtt{s},\mathcal{O}}$ and $\|\cdot\|_{\mathtt{s},\mathcal{O}}$ (respectively $|\cdot|_{\mathtt{s},\mathcal{O}}$) denote the inner product and the norm (resp. semi-norm) of the Sobolev spaces $H^{\mathtt{s}}(\mathcal{O})$ and $\boldsymbol{H}^{\mathtt{s}}(\mathcal{O}) := (H^{\mathtt{s}}(\mathcal{O}))^3$ for $\mathtt{s} \in \mathbb{R}$ (resp. for $\mathtt{s} > 0$). The index *zmv* indicates zero-mean-value fields. If moreover the boundary $\partial\mathcal{O}$ is Lipschitz, $\boldsymbol{n}$ denotes the unit outward normal vector field to $\partial\mathcal{O}$. It is assumed that the reader is familiar with function spaces related to Maxwell's equations, such as $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}_0(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}(\mathrm{div};\mathcal{O})$, $\boldsymbol{H}_0(\mathrm{div};\mathcal{O})$ etc. A priori, $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$ is endowed with the "natural" norm $\boldsymbol{v} \mapsto (\|\boldsymbol{v}\|^2_{0,\mathcal{O}} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\mathcal{O}})^{1/2}$, etc. We refer to the monographs [43, 41, 3] for details.
The symbol C is used to denote a generic positive constant which is independent of the meshsize, the mesh and the fields of interest; C may depend on the geometry, or on the coefficients defining the model. We use the notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar fields, and C is a generic constant.

Let Ω be a *domain* in $\mathbb{R}^3$, ie. an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. The domain Ω can be *simply connected* (sc) or *not* (nsc) [35]. This means that we assume that one of the two conditions below holds:

- (sc) *'for all curl-free vector field $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega)$, there exists $p \in C^0(\Omega)$ such that $\boldsymbol{v} = \nabla p$ in Ω'*;
- (nsc) *'there exist $I > 0$ non-intersecting, piecewise plane manifolds, $(\Sigma_j)_{j=1,\cdots,I}$, with boundaries $\partial\Sigma_i \subset \partial\Omega$, such that, if we let $\dot{\Omega} = \Omega \setminus \bigcup_{i=1}^{I} \Sigma_i$, for all curl-free vector field $\boldsymbol{v}$, there exists $\dot{p} \in C^0(\dot{\Omega})$ such that $\boldsymbol{v} = \nabla\dot{p}$ in $\dot{\Omega}$'*.

To simplify the computations (without restricting the scope of the study), we assume that the boundary $\partial\Omega$ is *connected.*

We let Ω be surrounded by a perfect conductor. We recall that, for a given pulsation $\omega > 0$, the time-harmonic Maxwell equations set in Ω can be expressed in terms of the complex-valued electric field $\boldsymbol{e}$ only. They write

$$\begin{cases} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) - \omega^2\varepsilon\boldsymbol{e} = \imath\omega\boldsymbol{j} \text{ in } \Omega \\ \operatorname{div}\varepsilon\boldsymbol{e} = \varrho \text{ in } \Omega. \end{cases} \tag{1}$$

Above, the real-valued coefficient ε is the electric permittivity tensor and the real-valued coefficient μ is the magnetic permeability tensor. The complex-valued source terms $\boldsymbol{j}$ and ϱ are respectively the current density and the charge density. They are related by the charge conservation equation

$$-\imath\omega\varrho + \operatorname{div}\boldsymbol{j} = 0 \text{ in } \Omega. \tag{2}$$

Classically, in (1), the equation $\operatorname{div}\varepsilon\boldsymbol{e} = \varrho$ is implied by the second-order equation $\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) - \omega^2\varepsilon\boldsymbol{e} = \imath\omega\boldsymbol{j}$, together with the charge conservation equation (2), so it is omitted from now on. We fix the *a priori* regularity of the current density to $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$, which implies that $\varrho \in H^{-1}(\Omega)$, with dependence $\|\varrho\|_{-1,\Omega} = \omega^{-1}\|\operatorname{div}\boldsymbol{j}\|_{-1,\Omega} \lesssim \|\boldsymbol{j}\|_{0,\Omega}$.

Finally, note that one can split the problem into two parts, where $\Re(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ is related to $-\Im(\boldsymbol{j}) \in \boldsymbol{L}^2(\Omega)$, resp. $\Im(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ is related to $\Re(\boldsymbol{j}) \in \boldsymbol{L}^2(\Omega)$. So, we carry on with $\boldsymbol{e}$ standing either for $\Re(\boldsymbol{e})$ or $\Im(\boldsymbol{e})$, resp. $\boldsymbol{f}$ standing for $-\omega^{-1}\Im(\boldsymbol{j})$ or $\omega^{-1}\Re(\boldsymbol{j})$, that is with *real-valued fields*. One can check that the equivalent variational formulation in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ writes

$$\begin{cases} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ a_\omega(\boldsymbol{e},\boldsymbol{v}) = \omega^2(\boldsymbol{f}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \end{cases} \tag{3}$$

where

$$a_\omega(\boldsymbol{u},\boldsymbol{v}) := (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2(\varepsilon\boldsymbol{u}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).$$

Note that with these notations, one has $\operatorname{div}\varepsilon\boldsymbol{e} = -\operatorname{div}\boldsymbol{f}$ in Ω.

Then, the real-valued coefficient $\xi \in \{\varepsilon,\mu\}$ fulfills one of the two sets of conditions below, which we refer to as the *classical case* and the *interface case* hereafter.

Classical case:

$$\begin{cases} \xi \text{ is a real-valued, symmetric, measurable tensor field on } \Omega, \\ \exists \xi_-,\xi_+ > 0, \forall \boldsymbol{z} \in \mathbb{R}^3,\ \xi_-|\boldsymbol{z}|^2 \le \xi\boldsymbol{z}\cdot\boldsymbol{z} \le \xi_+|\boldsymbol{z}|^2 \text{ a.e. in } \Omega. \end{cases} \tag{4}$$

Interface case: Ω is partitioned into the non-trivial partition $\mathcal{P} := (\Omega_p)_{p=+,-}$, where $\Omega_\pm$ are domains, and $\delta\xi$ fulfills (4), with $\delta_{|\Omega_+} = +1$ and $\delta_{|\Omega_-} = -1$.

For our studies of the time-harmonic Maxwell equations in the electric field, we assume from now on that

$$\boxed{\varepsilon \text{ is as in the } \textit{interface case}\,;\ \mu \text{ is as in the } \textit{classical case.}}$$

2 Some comments

Observe that if the electric field is curl-free, ie. $\mathbf{curl}\,\boldsymbol{e} = 0$, then it may be written as $\boldsymbol{e} = \nabla p_{\boldsymbol{e}}$ for some $p_{\boldsymbol{e}} \in H^1_0(\Omega)$ (cf. Theorem 3.3.9 in [3], as $\partial\Omega$ is connected). Moreover, $p_{\boldsymbol{e}}$ is such that $\operatorname{div}\varepsilon\nabla p_{\boldsymbol{e}} = -\operatorname{div}\boldsymbol{f}$ in $H^{-1}(\Omega)$. So to ensure well-posedness, one must make an assumption on the *companion scalar problem* with Dirichlet boundary condition:

$$\begin{cases} \textit{Find } s \in H^1_0(\Omega) \textit{ such that} \\ (\varepsilon\nabla s|\nabla q)_{0,\Omega} = \langle g, q\rangle_{H^1_0(\Omega)}, \ \forall q \in H^1_0(\Omega), \end{cases} \tag{5}$$

namely, that this scalar problem is well-posed. In other words,

$$\exists C_\star > 0, \ \forall g \in H^{-1}(\Omega), \ \exists! s \text{ solution to (5), with } \|s\|_{H^1_0(\Omega)} \leq C_\star \|g\|_{-1,\Omega}. \tag{6}$$

To measure elements of $H^1_0(\Omega)$, we choose the norm $q \mapsto \|q\|_{H^1_0(\Omega)} := \|\nabla q\|_{0,\Omega}$. If the permittivity ε were to fulfill (4), well-posedness of the scalar problem would *automatically* hold, as an obvious consequence of the fact that $(q, q') \mapsto (\varepsilon\nabla q|\nabla q')_{0,\Omega}$ defines an inner product on $H^1_0(\Omega)$, whose associated norm is equivalent to the $\|\cdot\|_{H^1_0(\Omega)}$-norm.
However, in the present setting, since ε is as in the interface case, this is an *additional assumption*, which is addressed with the help of T-coercivity [13, 9]. We recall the abstract framework below, see [21, 19] for details. Let V be a Hilbert space with norm $\|\cdot\|_V$, and $a(\cdot,\cdot)$ a *symmetric*, continuous bilinear form on $V \times V$. Then, the well-posedness of the problem

$$\textit{Find } u \in V \textit{ such that } a(u, v) = \langle f, v\rangle_V, \ \forall v \in V, \tag{7}$$

which reads

$$\exists C > 0, \ \forall f \in V', \ \exists! u \text{ solution to (7), with } \|u\|_V \leq C \|f\|_{V'}, \tag{8}$$

can be addresssed as follows. One has to prove that the form a is T-coercive, cf. Theorem 1 and Remark 2 of [19]:

$$\exists \alpha > 0, \ \exists T \in \mathcal{L}(V), \ \forall v \in V, \ |a(v, Tv)| \geq \alpha \|v\|^2_V. \tag{9}$$

In other words, the operator T realizes the classical inf-sup condition (see eg. [6]) *explicitly.*

Hence, for the scalar problem (5), and because ε is a symmetric tensor field, well-posedness is equivalent to $(q, q') \mapsto (\varepsilon\nabla q|\nabla q')_{0,\Omega}$ fulfilling an inf-sup condition:

$$\exists \gamma_0 > 0, \ \forall q \in H^1_0(\Omega), \qquad \sup_{q' \in H^1_0(\Omega)\setminus\{0\}} \frac{|(\varepsilon\nabla q|\nabla q')_{0,\Omega}|}{\|q'\|_{H^1_0(\Omega)}} \geq \gamma_0 \|q\|_{H^1_0(\Omega)}. \tag{10}$$

Or, as noted above, this is equivalent to

$$\exists \alpha_0 > 0, \ \exists T_0 \in \mathcal{L}(H^1_0(\Omega)), \forall q \in H^1_0(\Omega), \ |(\varepsilon\nabla q|\nabla(T_0 q))_{0,\Omega}| \geq \alpha_0 \|\nabla q\|^2_{0,\Omega}.$$

Note that the absolute value can be removed. Indeed, the quadratic mapping $q \mapsto (\varepsilon\nabla q|\nabla(T_0 q))_{0,\Omega}$ is continuous in $H^1_0(\Omega)$ and vanishes only for $q=0$, so it takes either positive, or negative, values everywhere in $H^1_0(\Omega)$. Thus, (10) is also equivalent to

$$\begin{aligned}&\exists\alpha_0>0,\ \exists T_0\in\mathcal{L}(H^1_0(\Omega)),\\ &\forall q\in H^1_0(\Omega),\ (\varepsilon\nabla q|\nabla(T_0q))_{0,\Omega}\geq\alpha_0\,\|\nabla q\|^2_{0,\Omega}.\end{aligned}\tag{11}$$

To recapitulate, we assume from now on that

(10)-(11) holds for the companion scalar problem (5).

When we perform the numerical analysis, and in order to obtain explicit convergence rates between the exact and approximate solution to the time-harmonic Maxwell equations, we shall make *two additional assumptions*:

– *the coefficients ε,μ are piecewise smooth*: there exists a partition $\{\Omega_p\}_{p=1,\cdots,P}$ of Ω, made of disjoint domains $(\Omega_p)_{p=1,\cdots,P}$, with $\overline{\Omega}=\cup_{p=1}^P\overline{\Omega_p}$, and such that $\varepsilon_{|\Omega_p},\mu_{|\Omega_p}\in\mathbb{W}^{1,\infty}(\Omega_p)$ for $p=1,\cdots,P$. In relation to the partition and for $\mathtt{s}\geq0$, we define
$$PH^{\mathtt{s}}(\Omega):=\{v\in L^2(\Omega)\ :\ v_{|\Omega_p}\in H^{\mathtt{s}}(\Omega_p),\ 1\leq p\leq P\},\tag{12}$$
endowed with the "natural" norm $\|v\|_{PH^{\mathtt{s}}(\Omega)}:=\Big(\sum_{1\leq p\leq P}\|v_p\|^2_{\mathtt{s},\Omega_p}\Big)^{1/2}$.

– *the data $\boldsymbol{f}$ has extra-regularity*, in the sense that
$$\operatorname{div}\boldsymbol{f}\in H^{-1+\tau_0}(\Omega),\ \text{with } \tau_0\in(0,1] \text{ given.}\tag{13}$$

For further analysis, let us introduce the scalar problem with *modified* right-hand side

$$\begin{cases}\textit{Find } s\in H^1_0(\Omega) \textit{ such that}\\ (\varepsilon\nabla s|\nabla q)_{0,\Omega}=\langle g,q\rangle_{H^1_0(\Omega)}+(\varepsilon\boldsymbol{g}|\nabla q)_{0,\Omega},\ \forall q\in H^1_0(\Omega).\end{cases}\tag{14}$$

If ε were as in the classical case [26,46,31,40,37,7,29], one could prove a shift theorem for the problem (14) when the data $(g,\boldsymbol{g})$ has *extra-regularity* like

$$g\in H^{-1+\tau_0}(\Omega),\ \boldsymbol{g}\in\boldsymbol{H}^1(\Omega),\ \text{with } \tau_0\in(0,1] \text{ given.}$$

In the interface case, there exist similar results in this direction. We refer to [27, 14,18,17,12] for a piecewise constant coefficient ε. So we introduce $\tau_{Dir}\in(0,1]$ depending only on the geometry and on ε such that

$$\begin{aligned}&\forall\mathtt{s}\in[0,\tau_{Dir})\setminus\{1/2\},\ \forall(g,\boldsymbol{g})\in H^{-1+\mathtt{s}}(\Omega)\times\boldsymbol{H}^1(\Omega),\\ &\text{the solution } s \text{ to (14) is such that } s\in PH^{1+\mathtt{s}}(\Omega), \text{ and}\\ &\|s\|_{PH^{1+\mathtt{s}}(\Omega)}\lesssim(\|g\|_{-1+\mathtt{s},\Omega}+\|\boldsymbol{g}\|_{1,\Omega}).\end{aligned}$$

Above, the constant hidden in $\lesssim$ may depend on $\mathtt{s}$, but not on g nor on $\boldsymbol{g}$. By a slight abuse of vocabulary, we call this result the *shift theorem*, respectively τ_{Dir} the *limit regularity exponent.* We assume from now on that

a shift theorem holds with $\tau_{Dir}\in(0,1]$ for the modified scalar problem (14).

Remark 1 We shall also need a shift theorem for the scalar problem involving the magnetic permeability μ with Neumann boundary condition, see (31) below. The result can be found in the above-mentioned references, because μ is as in the classical case. □

So far we focused on curl-free fields. To tackle fields with a non-vanishing curl, we use an ad hoc splitting of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Define

$$\boldsymbol{K}_N(\Omega,\varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ \operatorname{div}\varepsilon\boldsymbol{v} = 0\}.$$

An equivalent (variational) definition is

$$\boldsymbol{K}_N(\Omega,\varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ (\varepsilon\boldsymbol{v}|\nabla q)_{0,\Omega} = 0,\ \forall q \in H^1_0(\Omega)\}.$$

Proposition 1 *One has the continuous, direct sum*

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H^1_0(\Omega)] \oplus \boldsymbol{K}_N(\Omega,\varepsilon). \tag{15}$$

Proof Obviously, $\nabla[H^1_0(\Omega)] + \boldsymbol{K}_N(\Omega,\varepsilon)$ is a subset of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$. According to (6), there exists $p_{\boldsymbol{v}} \in H^1_0(\Omega)$ such that

$$(\varepsilon\nabla p_{\boldsymbol{v}}|\nabla q)_{0,\Omega} = (\varepsilon\boldsymbol{v}|\nabla q)_{0,\Omega},\ \forall q \in H^1_0(\Omega). \tag{16}$$

Now, let $\boldsymbol{k}_{\boldsymbol{v}} = \boldsymbol{v} - \nabla p_{\boldsymbol{v}}$, one has $\boldsymbol{k}_{\boldsymbol{v}} \in \boldsymbol{K}_N(\Omega,\varepsilon)$ by construction. It follows that $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H^1_0(\Omega)] + \boldsymbol{K}_N(\Omega,\varepsilon)$.
Next, let $\boldsymbol{z} \in \nabla[H^1_0(\Omega)] \cap \boldsymbol{K}_N(\Omega,\varepsilon)$ be given. There exists $s \in H^1_0(\Omega)$ such that $\boldsymbol{z} = \nabla s$ and, by definition of $\boldsymbol{K}_N(\Omega,\varepsilon)$, s is governed by (5) with zero right-hand side. By uniqueness of the solution, one has $s = 0$ and so $\boldsymbol{z} = 0$: the sum is direct.
Finally, by definition (16) of $p_{\boldsymbol{v}}$ and according to (11), one has $\alpha_0\,\|\nabla p_{\boldsymbol{v}}\|^2_{0,\Omega} \le (\varepsilon\nabla p_{\boldsymbol{v}}|\nabla(T_0 p_{\boldsymbol{v}}))_{0,\Omega} = (\varepsilon\boldsymbol{v}|\nabla(T_0 p_{\boldsymbol{v}}))_{0,\Omega} \le \|\varepsilon\boldsymbol{v}\|_{0,\Omega}\|\nabla(T_0 p_{\boldsymbol{v}})\|_{0,\Omega}$, so that

$$\|\nabla p_{\boldsymbol{v}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \|\nabla p_{\boldsymbol{v}}\|_{0,\Omega} \le \alpha_0^{-1}\varepsilon_+\|T_0\|_{\mathcal{L}(H^1_0(\Omega))}\|\boldsymbol{v}\|_{0,\Omega},$$
$$\text{and}\ \ \|\boldsymbol{k}_{\boldsymbol{v}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le (1 + \alpha_0^{-1}\varepsilon_+\|T_0\|_{\mathcal{L}(H^1_0(\Omega))})\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

So the sum is continuous. □

In other words, we may introduce the operators of $\mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega), H^1_0(\Omega))$, resp. of $\mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$

$$\pi_1\ :\ \begin{cases}\boldsymbol{H}_0(\mathbf{curl};\Omega) \to H^1_0(\Omega)\\ \boldsymbol{v} \mapsto p_{\boldsymbol{v}}\end{cases},\quad \boldsymbol{\pi}_2\ :\ \begin{cases}\boldsymbol{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{K}_N(\Omega,\varepsilon)\\ \boldsymbol{v} \mapsto \boldsymbol{k}_{\boldsymbol{v}}\end{cases}$$

and write, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{v} = \nabla(\pi_1\boldsymbol{v}) + \boldsymbol{\pi}_2\boldsymbol{v}$. Note that $(\boldsymbol{\pi}_2)^2 = \boldsymbol{\pi}_2$.

We finally recall an important result on the measure of elements of $\boldsymbol{K}_N(\Omega,\varepsilon)$. For its proof, we refer the reader to Corollary 5.2 of [10].

Theorem 1 *Elements of* $\boldsymbol{K}_N(\Omega,\varepsilon)$ *can be measured with the* $\|\,\mathbf{curl}\,\cdot\|_{0,\Omega}$*-norm:*

$$\exists C_W > 0,\ \forall\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon),\quad \|\boldsymbol{k}\|_{0,\Omega} \le C_W\,\|\,\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}, \tag{17}$$
$$\exists C'_W > 1,\ \forall\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon),\quad \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C'_W\,\|\,\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}. \tag{18}$$

3 Solving the exact problem

Recall that μ is as in the classical case (cf. (4)), resp. ε is as in the interface case, and assumption (10)-(11) holds. Using operators π_1 and $\boldsymbol{\pi}_2$, one can provide an equivalent reformulation of the variational formulation (3). Its solution $\boldsymbol{e}$ may be split as

$$\boldsymbol{e} = \boldsymbol{e}_0 + \nabla\phi, \text{ with } \boldsymbol{e}_0 = \boldsymbol{\pi}_2\boldsymbol{e} \text{ and } \phi = \pi_1\boldsymbol{e}. \tag{19}$$

By using the (variational) definition of $\boldsymbol{K}_N(\Omega,\varepsilon)$ (recall that ε is a symmetric tensor field), we notice that $\boldsymbol{e}_0$ and ϕ are respectively governed by

$$\begin{cases} \textit{Find } \boldsymbol{e}_0 \in \boldsymbol{K}_N(\Omega,\varepsilon) \textit{ such that} \\ a_\omega(\boldsymbol{e}_0,\boldsymbol{v}) = \omega^2(\boldsymbol{f}|\boldsymbol{v})_{0,\Omega}, \ \forall \boldsymbol{v} \in \boldsymbol{K}_N(\Omega,\varepsilon). \end{cases} \tag{20}$$

$$\begin{cases} \textit{Find } \phi \in H^1_0(\Omega) \textit{ such that} \\ (\varepsilon\nabla\phi|\nabla q)_{0,\Omega} = \langle \operatorname{div} \boldsymbol{f}, q\rangle_{H^1_0(\Omega)}, \ \forall q \in H^1_0(\Omega). \end{cases} \tag{21}$$

Actually, there is an equivalence result (the proof is left to the reader).

Proposition 2 *A field* $\boldsymbol{e}$ *is a solution to (3) if, and only if,* $\boldsymbol{\pi}_2\boldsymbol{e}$ *is a solution to (20) and* $\pi_1\boldsymbol{e}$ *is a solution to (21).*

According to the assumption on ε, we already know that problem (21) is well-posed. Hence proving the well-posedness of (3) amounts to proving the well-posedness of (20). We recall Theorem 8.15 of [10].

Theorem 2 *The imbedding of* $\boldsymbol{K}_N(\Omega,\varepsilon)$ *in* $\boldsymbol{L}^2(\Omega)$ *is compact.*

As a consequence (cf. Theorem 8.16 of [10]), one has the

Corollary 1 *The variational formulation (20) with unknown* $\boldsymbol{e}_0$ *enters the Fredholm alternative:*

- *either the problem (20) is well-posed, ie. it admits a unique solution* $\boldsymbol{e}_0$ *in* $\boldsymbol{H}_0(\mathbf{curl};\Omega)$*, which depends continuously on the data* $\boldsymbol{f}$*:*

$$\|\boldsymbol{e}_0\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{f}\|_{0,\Omega}\,;$$

- *or, the problem (20) has solutions if, and only if, the data* $\boldsymbol{f}$ *satisfies a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension, and the component of the solution which is orthogonal (in the sense of the* $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ *inner product) to the corresponding linear vector space, depends continuously on* $\boldsymbol{f}$*.*

Finally, each alternative occurs simultaneously for variational formulation (20), and variational formulation (3) with unknown $\boldsymbol{e}$*.*

From now on, we assume that variational formulation (3) is *well-posed*:

$$\forall \boldsymbol{f} \in \boldsymbol{L}^2(\Omega),\ \exists! \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\,\text{sol}^{\text{n}}\,\text{to}\,(3) \text{ and } \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{f}\|_{0,\Omega}. \tag{22}$$

4 Approximation by Nédélec's finite elements

For the ease of exposition[1], we assume that Ω and $\{\Omega_p\}_{p=1,\cdots,P}$ are Lipschitz polyhedra. We consider a family of simplicial meshes of Ω, and we choose the Nédélec's first family of edge finite elements [44, 43] to define finite dimensional subspaces $(\boldsymbol{V}_h)_h$ of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. So $\overline{\Omega}$ is triangulated by a shape regular family of meshes $(\mathcal{T}_h)_h$, made up of (closed) simplices, generically denoted by K. Each mesh is indexed by $h := \max_K h_K$ (the meshsize), where h_K is the diameter of K. And meshes are conforming with respect to the partition $\{\Omega_p\}_{p=1,\cdots,P}$ induced by the coefficients ε, μ: namely, for all h and all $K \in \mathcal{T}_h$, there exists $p \in \{1, \cdots, P\}$ such that $K \subset \overline{\Omega_p}$. Nédélec's $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming (first family, first-order) finite element spaces are then defined by

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \ : \ \boldsymbol{v}_{h|K} \in \mathcal{R}_1(K), \ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{R}_1(K)$ is the vector space of polynomials on K defined by

$$\mathcal{R}_1(K) := \{\boldsymbol{v} \in \boldsymbol{P}_1(K) \ : \ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}, \ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

To approximate the curl-free fields, we need to define a suitable approximation of elements of $H^1_0(\Omega)$. So we introduce finite dimensional subspaces $(M_h)_h$ of $H^1_0(\Omega)$. Lagrange's first-order finite element spaces are defined by

$$M_h := \{q_h \in H^1_0(\Omega) \ : \ q_{h|K} \in P_1(K), \ \forall K \in \mathcal{T}_h\}.$$

The discrete companion scalar problems are

$$\begin{cases} \textit{Find } s_h \in M_h \textit{ such that} \\ (\varepsilon \nabla s_h | \nabla q_h)_{0,\Omega} = \langle g, q_h \rangle_{H^1_0(\Omega)}, \ \forall q_h \in M_h. \end{cases} \tag{23}$$

For approximation purposes, one can use the Lagrange interpolation operator Π_h^L, or the Scott-Zhang interpolation operator Π_h^{SZ}. The latter allows one to interpolate any element of $H^1_0(\Omega)$, with values in M_h, at the expense of local interpolation operators that are not localized to each tetraedron, but are localized to the union of the tetrahedron and its neighbouring tetrahedra. We refer to [33] for details. Unless otherwise specified, we choose $\Pi_h^{grad} = \Pi_h^{SZ}$.

For h given, the discrete variational formulation of the time-harmonic problem (3) is

$$\begin{cases} \textit{Find } \boldsymbol{e}_h \in \boldsymbol{V}_h \textit{ such that} \\ a_\omega(\boldsymbol{e}_h, \boldsymbol{v}_h) = \omega^2 (\boldsymbol{f} | \boldsymbol{v}_h)_{0,\Omega}, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{24}$$

[1] The results obtained in this paper carry over to curved polyhedra, that is domains with piecewise smooth boundaries (see eg. p. 81 in [3] for a precise definition). When dealing with the discretization by first-order edge finite elements in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one may use [28]. Respectively, when dealing with the discretization by Lagrange's first-order finite elements in $H^1_0(\Omega)$, one may use [34]. In particular, it is proven there that optimal interpolation properties hold, ie. one may recover up to $O(h)$ accuracy, provided the field to be interpolated is sufficiently smooth.

To obtain explicit error estimates for the time-harmonic Maxwell equations, a natural idea is to use the interpolation of its solution $\boldsymbol{e}$. This requires some additional definitions and a priori analysis of the regularity of $\boldsymbol{e}$, and of its curl.

Let Π_h^{curl} be the classical global Raviart-Thomas-Nédélec interpolant in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ with values in $\boldsymbol{V}_h$ [44]. We then denote by Π_h^{div} the classical global Raviart-Thomas-Nédélec interpolation operator in $\boldsymbol{H}_0(\mathrm{div};\Omega)$ with values in $\boldsymbol{W}_h$ [48,44], where $(\boldsymbol{W}_h)_h$ are designed with the help of $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming, first-order finite element spaces:

$$\boldsymbol{W}_h := \{\boldsymbol{w}_h \in \boldsymbol{H}_0(\mathrm{div};\Omega)\ :\ \boldsymbol{w}_{h|K} \in \mathcal{D}_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{D}_1(K)$ is the vector space of polynomials on K defined by

$$\mathcal{D}_1(K) := \{\boldsymbol{v} \in \boldsymbol{P}_1(K)\ :\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x},\ \boldsymbol{a} \in \mathbb{R}^3,\ b \in \mathbb{R}\}.$$

Let us recall a few useful properties (see Chapter 5 in [43]). To start with,

Proposition 3 *For all h, it holds that*

$$\nabla[M_h] \subset \boldsymbol{V}_h\,; \tag{25}$$

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\quad \Pi_h^{curl}\boldsymbol{v}_h = \boldsymbol{v}_h\,; \tag{26}$$

$$\mathbf{curl}[\boldsymbol{V}_h] \subset \boldsymbol{W}_h\,; \tag{27}$$

$$\forall \boldsymbol{w}_h \in \boldsymbol{W}_h,\quad \Pi_h^{div}\boldsymbol{w}_h = \boldsymbol{w}_h\,; \tag{28}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ \textit{s.t.}\ \Pi_h^{curl}\boldsymbol{v}\ \textit{exists},\ \Pi_h^{div}(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\Pi_h^{curl}\boldsymbol{v}). \tag{29}$$

There are useful additional properties regarding Π_h^{curl} listed next. Below, when we refer to piecewise-H^s fields, the partition is understood as in (12).

Proposition 4 (discrete exact sequence [44]) *Let h be given, and let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ that can be written as $\boldsymbol{v} = \nabla q$ in Ω, for some $q \in H_0^1(\Omega)$. Then if $\Pi_h^{curl}\boldsymbol{v}$ is well-defined, there exists $q_h \in M_h$ such that $\Pi_h^{curl}\boldsymbol{v} = \nabla q_h$ in Ω.*

Proposition 5 (classical interpolation results) *Assume that $\boldsymbol{v} \in \boldsymbol{PH}^{\mathsf{s}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{PH}^{\mathsf{s}'}(\Omega)$ for some $\mathsf{s} > 1/2$, $\mathsf{s}' > 0$. Then one can define $\Pi_h^{curl}\boldsymbol{v}$ and, in addition, one has the approximation result [5]:*

$$\|\boldsymbol{v} - \Pi_h^{curl}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathsf{s},\mathsf{s}',1)}\{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathsf{s}'}(\Omega)}\}. \tag{30}$$

Furthermore, if $\mathbf{curl}\,\boldsymbol{v}$ is piecewise constant on $\mathcal{T}_h$, one has the improved approximation result (cf. Theorem 5.41 in [43]):

$$\|\boldsymbol{v} - \Pi_h^{curl}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathsf{s},1)}\{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}\}.$$

Remark 2 When Ω_2 is a domain of $\mathbb{R}^2$, note that one can define the Raviart-Thomas-Nédélec interpolant of a field $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}; \Omega_2)$ as soon as $\boldsymbol{v} \in \boldsymbol{PH}^{\mathtt{s}}(\Omega_2)$ for some $\mathtt{s} > 0$ (there is no requirement on the regularity of $\mathrm{curl}\, \boldsymbol{v}$). This result is proven in [4] for fields in $\boldsymbol{H}(\mathrm{div}; \Omega_2)$, and it obviously carries over to fields in $\boldsymbol{H}(\mathrm{curl}; \Omega_2)$ by appropriate coordinates transform. Further, one has the approximation result:

$$\|\boldsymbol{v} - \Pi_h^{curl} \boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{curl};\Omega_2)} \lesssim h^{\min(\mathtt{s},1)} \{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega_2)} + \|\,\mathrm{curl}\, \boldsymbol{v}\|_{0,\Omega_2}\}. \qquad \square$$

Our aim is to apply Π_h^{curl} to the electric field $\boldsymbol{e}$ governed by (3).
First, one must have $\mathbf{curl}\, \boldsymbol{e} \in \boldsymbol{PH}^{\mathtt{s}'}(\Omega)$ for some $\mathtt{s}' > 0$. Since $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we immediately find that $\boldsymbol{c} = \mu^{-1}\, \mathbf{curl}\, \boldsymbol{e}$ belongs to

$$\boldsymbol{X}_T(\Omega, \mu) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega) \ : \ \mu \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}; \Omega)\}.$$

Then, using a shift theorem for the companion scalar problem with Neumann boundary condition

$$\begin{cases} \textit{Find } s \in H^1_{zmv}(\Omega) \textit{ such that} \\ (\mu \nabla s | \nabla q)_{0,\Omega} = \langle g', q \rangle_{H^1_{zmv}(\Omega)}, \ \forall q \in H^1_{zmv}(\Omega), \end{cases} \tag{31}$$

and a regular plus gradient decomposition (see eg. [26, 22]), we introduce $\tau_{Neu} \in (0, 1]$ depending only on the geometry and on μ such that

$$\boldsymbol{X}_T(\Omega, \mu) \subset \cap_{\mathtt{s}' \in [0, \tau_{Neu})} \boldsymbol{PH}^{\mathtt{s}'}(\Omega),$$

with continuous imbedding for all $\mathtt{s}' \in [0, \tau_{Neu})$. Furthermore, using a Weber inequality (cf. Theorem 6.2.5 in [3]), one has that for all $\mathtt{s}' \in [0, \tau_{Neu})$,

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{X}_T(\Omega, \mu), \\ &\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}'}(\Omega)} \lesssim \|\,\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega} + \|\,\mathrm{div}\, \mu \boldsymbol{v}\|_{0,\Omega} + \sum_{1 \le i \le I} |\langle \mu \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{H^{1/2}(\Sigma_i)}|. \end{aligned} \tag{32}$$

As a consequence, we note that since μ is piecewise smooth, it also holds that

$$\mathbf{curl}\, \boldsymbol{e} \in \cap_{\mathtt{s}' \in [0, \tau_{Neu})} \boldsymbol{PH}^{\mathtt{s}'}(\Omega). \tag{33}$$

Then, to guarantee that Π_h^{curl} can be applied to the electric field $\boldsymbol{e}$, one must check whether $\boldsymbol{e} \in \boldsymbol{PH}^{\mathtt{s}}(\Omega)$ for some $\mathtt{s} > 1/2$. To evaluate the exponent $\mathtt{s}$ *a priori*, we use the following regular plus gradient decomposition (see Lemma 2.4 of [38]).

Proposition 6 *There exist operators*

$$\boldsymbol{P} \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl}; \Omega), \boldsymbol{H}^1(\Omega)), \quad Q \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl}; \Omega), H^1_0(\Omega)),$$

such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \quad \boldsymbol{v} = \boldsymbol{P}\boldsymbol{v} + \nabla(Q\boldsymbol{v}). \tag{34}$$

This yields some useful results for elements of $\boldsymbol{K}_N(\Omega, \varepsilon)$.

Corollary 2 *The a priori regularity of elements of* $\boldsymbol{K}_N(\Omega,\varepsilon)$ *is governed by the imbedding:*

$$\boldsymbol{K}_N(\Omega,\varepsilon) \subset \cap_{\mathtt{s}\in[0,\tau_{Dir})}\boldsymbol{PH}^{\mathtt{s}}(\Omega).$$

Moreover, for all $\mathtt{s}\in[0,\tau_{Dir})$*,*

$$\forall \boldsymbol{k}\in\boldsymbol{K}_N(\Omega,\varepsilon),\quad \|\boldsymbol{k}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} \lesssim \|\,\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}. \tag{35}$$

Proof Let $\boldsymbol{k}\in\boldsymbol{K}_N(\Omega,\varepsilon)$. According to proposition 6, one can write $\boldsymbol{k} = \boldsymbol{k}^\star + \nabla s_{\boldsymbol{k}}$ with $\boldsymbol{k}^\star\in\boldsymbol{H}^1(\Omega)$, resp. $s_{\boldsymbol{k}}\in H^1_0(\Omega)$, and it holds that $\|\boldsymbol{k}^\star\|_{1,\Omega} + \|s_{\boldsymbol{k}}\|_{H^1_0(\Omega)} \lesssim \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. In particular, $\mathrm{div}(\varepsilon\nabla s_{\boldsymbol{k}}) = -\,\mathrm{div}\,\varepsilon\boldsymbol{k}^\star$ in Ω, so $s_{\boldsymbol{k}}$ solves the modified scalar problem (14) with data $(0,-\boldsymbol{k}^\star)$. Thanks to the shift theorem, we know that, for all $\mathtt{s}\in[0,\tau_{Dir})$, $s_{\boldsymbol{k}}$ belongs to $PH^{1+\mathtt{s}}(\Omega)$, with the bound $\|s_{\boldsymbol{k}}\|_{PH^{1+\mathtt{s}}(\Omega)} \lesssim \|\boldsymbol{k}^\star\|_{1,\Omega}$. Using the triangle inequality, we conclude that

$$\forall \mathtt{s}\in[0,\tau_{Dir}),\ \boldsymbol{k}\in\boldsymbol{PH}^{\mathtt{s}}(\Omega),\ \text{and}\ \|\boldsymbol{k}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} \lesssim \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

This proves the first part of the corollary. Using finally theorem 1 on the equivalence of norms in $\boldsymbol{K}_N(\Omega,\varepsilon)$, we conclude that (35) holds. ☐

Hence, it may happen that the field to be interpolated, eg. the electric field $\boldsymbol{e}$, does not belong to $\cup_{\mathtt{s}>1/2}\boldsymbol{PH}^{\mathtt{s}}(\Omega)$. In the classical case, the occurence of such a situation is explained in Section 7 of [26]. In the interface case, this can be inferred from the results obtained in [9,12].
On the other hand, to interpolate such a *low regularity field*, one may still choose the quasi-interpolation operator of [32], or the combined interpolation operator of [22,23]. We choose the latter. To get a definition for the combined interpolation operator, denoted by Π_h^{comb}, one needs to be able to split low regularity fields defined on Ω. To that end, we apply proposition 6.

Definition 1 (combined interpolation operator) Let $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$, with $\mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{H}^{\mathtt{s}'}(\Omega)$ for some $\mathtt{s}'>0$. We define

$$\Pi_h^{comb}\boldsymbol{v} := \Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}) + \nabla(\Pi_h^{grad}(Q\boldsymbol{v})).$$

Then, the approximation results for the combined interpolation are a straightforward consequence of the available results for Π_h^{curl} and Π_h^{grad}.

Proposition 7 (combined interpolation results) *Let* $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$*, with* $Q\boldsymbol{v}\in PH^{1+\mathtt{s}}(\Omega)$ *and* $\mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{PH}^{\mathtt{s}'}(\Omega)$ *for some* $\mathtt{s}\geq 0$, $\mathtt{s}'>0$*. One has the approximation result:*

$$\|\boldsymbol{v}-\Pi_h^{comb}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\mathtt{s}',1)}\{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|Q\boldsymbol{v}\|_{PH^{1+\mathtt{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}'}(\Omega)}\}. \tag{36}$$

Furthermore, if $\mathbf{curl}\,\boldsymbol{v}$ *is piecewise constant on* $\mathcal{T}_h$*, one has the improved approximation result:*

$$\|\boldsymbol{v}-\Pi_h^{comb}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},1)}\{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|Q\boldsymbol{v}\|_{PH^{1+\mathtt{s}}(\Omega)}\}.$$

Together with this definition of the combined interpolation operator, we have the results below, to be compared with the well-known results (26) and (29) for the classical interpolation operator.

Proposition 8 *For all h, it holds that*

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \exists q_h \in M_h, \quad \Pi_h^{comb} \boldsymbol{v}_h = \boldsymbol{v}_h + \nabla q_h \,; \tag{37}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \text{ s.t. } \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}^{\mathtt{s}'}(\Omega) \text{ for some } \mathtt{s}' > 0, \tag{38}$$
$$\Pi_h^{div}(\mathbf{curl}\, \boldsymbol{v}) = \mathbf{curl}(\Pi_h^{comb} \boldsymbol{v}).$$

Proof Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$. We note that because $\boldsymbol{v}_h$ is piecewise smooth on $\mathcal{T}_h$, one has $\boldsymbol{v}_h, \mathbf{curl}\, \boldsymbol{v}_h \in \boldsymbol{PH}^{\mathtt{t}}(\Omega)$ for all $\mathtt{t} \in [0, 1/2)$. Hence $\Pi_h^{comb} \boldsymbol{v}_h$ is well-defined according to definition 1. If we write $\boldsymbol{v}_h = (\boldsymbol{v}_h)^\star + \nabla s_{\boldsymbol{v}_h}$, with $(\boldsymbol{v}_h)^\star = \boldsymbol{P}\boldsymbol{v}_h$, resp. $s_{\boldsymbol{v}_h} = Q\boldsymbol{v}_h$, we have $\Pi_h^{comb} \boldsymbol{v}_h := \Pi_h^{curl}(\boldsymbol{v}_h)^\star + \nabla(\Pi_h^{grad} s_{\boldsymbol{v}_h})$.
On the other hand, $\nabla s_{\boldsymbol{v}_h} = \boldsymbol{v}_h - (\boldsymbol{v}_h)^\star$. Since $\Pi_h^{curl}(\boldsymbol{v}_h - (\boldsymbol{v}_h)^\star)$ is well-defined, so is $\Pi_h^{curl}(\nabla s_{\boldsymbol{v}_h})$ and, according to proposition 4, there exists $q_h' \in M_h$ such that $\Pi_h^{curl}(\nabla s_{\boldsymbol{v}_h}) = \nabla q_h'$. Applying now Π_h^{curl} to $(\boldsymbol{v}_h)^\star = \boldsymbol{v}_h - \nabla s_{\boldsymbol{v}_h}$, it follows that

$$\Pi_h^{curl}(\boldsymbol{v}_h)^\star = \Pi_h^{curl} \boldsymbol{v}_h - \nabla q_h' = \boldsymbol{v}_h - \nabla q_h',$$

where the second equality now follows from (26). One concludes that

$$\Pi_h^{comb} \boldsymbol{v}_h := \boldsymbol{v}_h + \nabla(\Pi_h^{grad} s_{\boldsymbol{v}_h} - q_h'),$$

which is precisely (37) with $q_h = \Pi_h^{grad} s_{\boldsymbol{v}_h} - q_h' \in M_h$.
To check (38), let $\boldsymbol{v}$ be split as $\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v} + \nabla(Q\boldsymbol{v})$. Since $\mathbf{curl}(\boldsymbol{P}\boldsymbol{v}) \in \boldsymbol{H}^{\mathtt{s}'}(\Omega)$, according to proposition 5 one may apply (29) to $\boldsymbol{P}\boldsymbol{v}$, leading to $\Pi_h^{div}(\mathbf{curl}(\boldsymbol{P}\boldsymbol{v})) = \mathbf{curl}(\Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}))$. On the other hand, because of the definition 1 of $\Pi_h^{comb} \boldsymbol{v} = \Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}) + \nabla(\Pi_h^{grad}(Q\boldsymbol{v}))$ one has

$$\mathbf{curl}(\Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v})) = \mathbf{curl}(\Pi_h^{comb} \boldsymbol{v} - \nabla(\Pi_h^{grad}(Q\boldsymbol{v}))) = \mathbf{curl}(\Pi_h^{comb} \boldsymbol{v}).$$

Using finally the equality $\mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}(\boldsymbol{P}\boldsymbol{v})$ leads to the claim. □

We now have all the required results to bound the interpolation error of the electric field $\boldsymbol{e}$.

Proposition 9 *Let $\boldsymbol{e}$ be the solution to the time-harmonic Maxwell equations. Let the extra-regularity of the data $\boldsymbol{f}$ be as in (13) with $\tau_0 > 0$ given. One can define $\Pi_h^{comb} \boldsymbol{e}$, and moreover one has the approximation result, for all $\mathtt{s} \in [0, \min(\tau_0, \tau_{Dir}, \tau_{Neu}))$,*

$$\|\boldsymbol{e} - \Pi_h^{comb} \boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\mathtt{s}} \{\| \operatorname{div} \boldsymbol{f}\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\}.$$

Proof Let $\mathtt{s} \in [0, \min(\tau_0, \tau_{Dir}, \tau_{Neu}))$; because $\tau_{Dir} \leq 1$, one has $\mathtt{s} < 1$.
According to proposition 6, we may write $\boldsymbol{e} = \boldsymbol{e}^\star + \nabla s_{\boldsymbol{e}}$ with $\boldsymbol{e}^\star \in \boldsymbol{H}^1(\Omega)$, $s_{\boldsymbol{e}} \in H_0^1(\Omega)$, and $\|\boldsymbol{e}^\star\|_{1,\Omega} + \|s_{\boldsymbol{e}}\|_{H_0^1(\Omega)} \lesssim \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. By construction, $s_{\boldsymbol{e}}$ solves the modified scalar problem (14) with data $(\operatorname{div} \boldsymbol{f}, -\boldsymbol{e}^\star)$. But $\mathtt{s} < \min(\tau_0, \tau_{Dir})$ so,

thanks to the shift theorem, $s_{\boldsymbol{e}} \in PH^{1+\mathrm{s}}(\Omega)$, with the bound $\|s_{\boldsymbol{e}}\|_{PH^{1+\mathrm{s}}(\Omega)} \lesssim \|\operatorname{div} \boldsymbol{f}\|_{-1+\mathrm{s},\Omega} + \|\boldsymbol{e}^\star\|_{1,\Omega}$. Using (22) for the last inequality below, we find

$$\|s_{\boldsymbol{e}}\|_{PH^{1+\mathrm{s}}(\Omega)} \lesssim \|\operatorname{div} \boldsymbol{f}\|_{-1+\mathrm{s},\Omega} + \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\operatorname{div} \boldsymbol{f}\|_{-1+\mathrm{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}.$$

On the other hand, one has $\mathbf{curl}\,\boldsymbol{e} \in \boldsymbol{PH}^{\mathrm{s}}(\Omega)$, cf. (33). Then, with the help of the bound (32) on $\|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\|_{\boldsymbol{PH}^{\mathrm{s}}(\Omega)}$, noting that $\operatorname{div}\mathbf{curl}\,\boldsymbol{e} = 0$ in Ω, and $\langle \mathbf{curl}\,\boldsymbol{e}\cdot\boldsymbol{n}, 1\rangle_{H^{1/2}(\Sigma_i)} = 0$ for $1 \le i \le I$ (see Remark 3.5.2 in [3]), we find

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{e}\|_{\boldsymbol{PH}^{\mathrm{s}}(\Omega)} &\lesssim \|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\|_{\boldsymbol{PH}^{\mathrm{s}}(\Omega)} \lesssim \|\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e})\|_{0,\Omega}\\ &\lesssim \|\boldsymbol{e}\|_{0,\Omega} + \|\boldsymbol{f}\|_{0,\Omega},\end{aligned}$$

where we used the relation $\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) = \omega^2\varepsilon\boldsymbol{e} + \omega^2\boldsymbol{f}$ in Ω. Therefore, one can define $\Pi_h^{comb}\boldsymbol{e}$ and, using (36) and (22) once more, one finds now

$$\begin{aligned}\|\boldsymbol{e} - \Pi_h^{comb}\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &\lesssim h^{\mathrm{s}}\{\|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|s_{\boldsymbol{e}}\|_{PH^{1+\mathrm{s}}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{e}\|_{\boldsymbol{PH}^{\mathrm{s}}(\Omega)}\}\\ &\lesssim h^{\mathrm{s}}\{\|\operatorname{div} \boldsymbol{f}\|_{-1+\mathrm{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\},\end{aligned}$$

which is the desired estimate. □

Remark 3 In particular, we note that even when the electric field $\boldsymbol{e}$ does not belong to $\cup_{\mathrm{s}>1/2}\boldsymbol{PH}^{\mathrm{s}}(\Omega)$, one may use the combined interpolation operator and still obtain "best" interpolation error. On the other hand, it is well-known by using classical interpolation that, when $\tau_{Dir} = \tau_{Neu} = 1$, and for a *regular* data $\boldsymbol{f} \in \boldsymbol{H}(\mathrm{div};\Omega)$, the interpolation error behaves like $O(h)$. □

From this point on, to obtain the well-posedness result for the discretized problems, and finally convergence to the exact solution $\boldsymbol{e}$, one needs to prove a uniform discrete inf-sup condition. For that, we mimic in section 5 the two ingredients that were used to solve the exact variational formulation: uniformly stable discrete decompositions in the spirit of proposition 1 ; uniform equivalence of norms in the spirit of theorem 1. The key ingredient is the study of the approximation (23) of the companion scalar problem (5). And, since it was originally solved with the help of T-coercivity, we consider two situations regarding its approximation. We refer to the appendix A for details. Either we have at hand a "full" T-coercivity involution operator T_0 to solve (5), that can also be used to establish to establish the uniform discrete T-coercivity (60)-(61) of the discrete scalar problems (23). Or, we only have at hand a "weak" explicit T-coercivity involution operator T, cf. (59). The first situation is addressed in section 5.1, whereas the second situation is addressed in section 5.2.

5 Uniform estimates

5.1 Case of a "full" T-coercivity operator

We assume in this section that we have at hand a "full" T-coercivity involution operator T_0 to solve the companion scalar problem (5) (see section A.1), and that the meshes are T-conform, such that (60)-(61) are fulfilled, with consequences listed in section A.2. Define, for any h,

$$\boldsymbol{K}_h(\varepsilon) := \{\boldsymbol{v}_h \in \boldsymbol{V}_h \ : \ (\varepsilon \boldsymbol{v}_h|\nabla q_h)_{0,\Omega} = 0, \ \forall q_h \in M_h\}. \tag{39}$$

Proposition 10 *Assume that (60) holds. For all h, one has the direct sum*

$$\boldsymbol{V}_h = \nabla[M_h] \oplus \boldsymbol{K}_h(\varepsilon). \tag{40}$$

Proof Let h be given. Thanks to (25), we know that $\nabla[M_h] + \boldsymbol{K}_h(\varepsilon)$ is a subset of $\boldsymbol{V}_h$. Then for $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and because the discrete scalar problem (23) is well-posed, there exists one, and only one, $p_{\boldsymbol{v}_h} \in M_h$ such that

$$(\varepsilon \nabla p_{\boldsymbol{v}_h}|\nabla q_h)_{0,\Omega} = (\varepsilon \boldsymbol{v}_h|\nabla q_h)_{0,\Omega}, \ \forall q_h \in M_h. \tag{41}$$

And one has

$$\boldsymbol{k}_{\boldsymbol{v}_h} = \boldsymbol{v}_h - \nabla p_{\boldsymbol{v}_h} \in \boldsymbol{K}_h(\varepsilon), \tag{42}$$

so $\boldsymbol{V}_h = \nabla[M_h] + \boldsymbol{K}_h(\varepsilon)$. Using (60), the fact that the sum is direct is derived exactly as in the continuous case (see the proof of proposition 1). □

For all h, we can use the splitting (40) and the explicit definitions (41)-(42) to introduce the operators

$$\pi_{1h} \ : \ \begin{cases} \boldsymbol{V}_h \to M_h \\ \boldsymbol{v}_h \mapsto p_{\boldsymbol{v}_h} \end{cases}, \quad \boldsymbol{\pi}_{2h} \ : \ \begin{cases} \boldsymbol{V}_h \to \boldsymbol{K}_h(\varepsilon) \\ \boldsymbol{v}_h \mapsto \boldsymbol{k}_{\boldsymbol{v}_h} \end{cases}. \tag{43}$$

In other words, one may write, for all h, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\boldsymbol{v}_h = \nabla(\pi_{1h}\boldsymbol{v}_h) + \boldsymbol{\pi}_{2h}\boldsymbol{v}_h$. Also, one has for all h, $(\boldsymbol{\pi}_{2h})^2 = \boldsymbol{\pi}_{2h}$. Below, we prove the uniform stability of the decomposition (40).

Proposition 11 *Assume that (60) holds. The continuity moduli of the operators $(\pi_{1h})_h$, $(\boldsymbol{\pi}_{2h})_h$ are bounded independently of h.*

Proof Given h and $\boldsymbol{v}_h \in \boldsymbol{V}_h$, one has according to (60) and (41)

$$\begin{aligned} \alpha_0' \, \|\nabla(\pi_{1h}\boldsymbol{v}_h)\|_{0,\Omega}^2 &\leq (\varepsilon\nabla(\pi_{1h}\boldsymbol{v}_h)|\nabla(T_0(\pi_{1h}\boldsymbol{v}_h)))_{0,\Omega} = (\varepsilon\boldsymbol{v}_h|\nabla(T_0(\pi_{1h}\boldsymbol{v}_h)))_{0,\Omega} \\ &\leq \|\varepsilon\boldsymbol{v}_h\|_{0,\Omega}\|\nabla(T_0(\pi_{1h}\boldsymbol{v}_h))\|_{0,\Omega} \\ &\leq \|\varepsilon\boldsymbol{v}_h\|_{0,\Omega}\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\|\nabla(\pi_{1h}\boldsymbol{v}_h)\|_{0,\Omega}, \end{aligned}$$

so that

$$\|\nabla(\pi_{1h}\boldsymbol{v}_h)\|_{0,\Omega} \leq (\alpha_0')^{-1}\varepsilon_+\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\|\boldsymbol{v}_h\|_{0,\Omega}.$$

And then

$$\|\boldsymbol{\pi}_{2h}\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq (1 + (\alpha_0')^{-1}\varepsilon_+\|T_0\|_{\mathcal{L}(H_0^1(\Omega))})\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},$$

so the claim follows. □

Next, one has to check that $\boldsymbol{k}_h \mapsto \|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$ defines a norm on $\boldsymbol{K}_h(\varepsilon)$. And, if the answer is positive, whether this norm of uniformly equivalent in h (ie. with constants that are independent of h) to the $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$-norm on $\boldsymbol{K}_h(\varepsilon)$.

Proposition 12 *Assume that (60) holds. For all h,* $\boldsymbol{k}_h \mapsto \|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega}$ *defines a norm on* $\boldsymbol{K}_h(\varepsilon)$.

Proof Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon)$ be such that $\operatorname{\mathbf{curl}} \boldsymbol{k}_h = 0$ in Ω. Since the boundary $\partial\Omega$ is connected, we get from Theorem 3.3.9 of [3] that there exists $q \in H^1_0(\Omega)$ such that $\boldsymbol{k}_h = \nabla q$ in Ω. Since $\Pi_h^{curl}\boldsymbol{k}_h$ is well-defined (and equal to $\boldsymbol{k}_h$), we know from proposition 4 that there exists $q_h \in M_h$ such that $\Pi_h^{curl}\boldsymbol{k}_h = \nabla q_h$ in Ω. In other words, $\boldsymbol{k}_h = \Pi_h^{curl}\boldsymbol{k}_h = \nabla q_h \in \nabla[M_h]$. So one has $\boldsymbol{k}_h \in \nabla[M_h] \cap \boldsymbol{K}_h(\varepsilon)$ which reduces to $\{0\}$ according to proposition 10: this proves the result. □

Theorem 3 *Assume that (60) holds. Then*

$$\exists C_W^\star > 0,\ \forall h,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon),\quad \|\boldsymbol{k}_h\|_{0,\Omega} \le C_W^\star \|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega}. \tag{44}$$

In addition, let $\mathtt{s} \in (0, \tau_{Dir})$*:*

$$\begin{cases} \exists C_{\mathtt{s}} > 0,\ \forall h,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon), \\ \inf_{\boldsymbol{k}\in\boldsymbol{K}_N(\Omega,\varepsilon)} \|\boldsymbol{k} - \boldsymbol{k}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C_{\mathtt{s}} h^{\mathtt{s}} \|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega}. \end{cases} \tag{45}$$

Proof Let

$$\boldsymbol{H}_0^\Sigma(\operatorname{div}0;\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div};\Omega)\ :\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0,\ 1 \le i \le I\}.$$

Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon)$ be given. According to Theorem 6.1.4 in [3], one has $\operatorname{\mathbf{curl}} \boldsymbol{k}_h \in \boldsymbol{H}_0^\Sigma(\operatorname{div}0;\Omega)$. So, using Corollary 3 in appendix B, we find that there exists one, and only one, solution to the div-curl problem

$$\begin{cases} \textit{Find } \boldsymbol{k} \in \boldsymbol{L}^2(\Omega) \textit{ such that} \\ \operatorname{\mathbf{curl}} \boldsymbol{k} = \operatorname{\mathbf{curl}} \boldsymbol{k}_h \text{ in } \Omega, \\ \operatorname{div}\varepsilon\boldsymbol{k} = 0 \text{ in } \Omega, \\ \boldsymbol{k} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{46}$$

with $\|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega}$. By definition, $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon)$, and it holds

$$\begin{aligned} \|\boldsymbol{k}_h\|_{0,\Omega} &\le \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + \|\boldsymbol{k}\|_{0,\Omega} \\ &\le \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + C_W \|\operatorname{\mathbf{curl}} \boldsymbol{k}\|_{0,\Omega} \\ &= \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + C_W \|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega} \end{aligned} \tag{47}$$

thanks to the triangle inequality, (17) and the definition of $\boldsymbol{k}$. To obtain (44), we bound $\|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega}$ by $\|\operatorname{\mathbf{curl}} \boldsymbol{k}_h\|_{0,\Omega}$, uniformly with respect to h.
By definition of $\boldsymbol{k}$, we know that $\operatorname{\mathbf{curl}}(\boldsymbol{k} - \boldsymbol{k}_h) = 0$ in Ω so, thanks to Theorem 3.3.9. in [3], there exists $q \in H^1_0(\Omega)$ such that $\boldsymbol{k} - \boldsymbol{k}_h = \nabla q$ in Ω. Thus, using (11), we have the bound

$$\alpha_0 \|\boldsymbol{k} - \boldsymbol{k}_h\|^2_{0,\Omega} = \alpha_0 \|\nabla q\|^2_{0,\Omega} \le (\varepsilon\nabla q|\nabla(T_0 q))_{0,\Omega} = (\varepsilon(\boldsymbol{k} - \boldsymbol{k}_h)|\nabla(T_0 q))_{0,\Omega}.$$

Because $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon)$ and $\boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon)$, we note that $(\varepsilon(\boldsymbol{k} - \boldsymbol{k}_h)|\nabla q_h')_{0,\Omega} = 0$, for all $q_h' \in M_h$. Or equivalently, if we recall (60) and its consequence $T_0[M_h] =$

M_h: $(\varepsilon(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla(T_0 q_h))_{0,\Omega}=0$, for all $q_h\in M_h$. Hence, it holds that, for all $q_h\in M_h$:

$$\begin{aligned}\alpha_0\,\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}^2 &\le (\varepsilon(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla(T_0(q-q_h)))_{0,\Omega}\\ &\le \varepsilon_+\,\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\,\|\nabla(T_0(q-q_h))\|_{0,\Omega}\\ &\le \varepsilon_+\,\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\,\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\,\|\nabla(q-q_h)\|_{0,\Omega}.\end{aligned}$$

This implies that

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\le\frac{\varepsilon_+}{\alpha_0}\,\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\,\inf_{q_h\in M_h}\|\nabla(q-q_h)\|_{0,\Omega}.$$

There remains to choose some *ad hoc* $q_h\in M_h$. For that, we prove next that $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h$ belongs to $\nabla[M_h]$.
First, we remark that $\mathbf{curl}(\Pi_h^{comb}\boldsymbol{k})=\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})$ according to (38). Next, we express $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})$ in terms of $\mathbf{curl}\,\boldsymbol{k}_h$. By definition of $\boldsymbol{k}$, it holds that $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})=\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k}_h)$, so using (27)-(28), we get that $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})=\mathbf{curl}\,\boldsymbol{k}_h$. In other words, $\mathbf{curl}(\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h)=0$ in Ω. According to Theorem 3.3.9. in [3], there exists $q\in H_0^1(\Omega)$ such that $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h=\nabla q$ in Ω. Moreover, $\Pi_h^{curl}(\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h)$ is well-defined (and equal to $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h$, cf. (26)). Hence we conclude from proposition 4 that there exists $q_h^0\in M_h$ such that $\nabla q_h^0\big(=\Pi_h^{curl}(\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h)\big)=\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h$.
Now, we find that

$$\nabla(q-q_h^0)=(\boldsymbol{k}-\boldsymbol{k}_h)-(\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h)=\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k},$$

so choosing $q_h=q_h^0$ yields

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\le\frac{\varepsilon_+}{\alpha_0}\,\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\,\|\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega}.$$

Thanks to corollary 2 and proposition 7, for any $\mathsf{s}\in(0,\tau_{Dir})$ it holds that

$$\|\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega}\lesssim h^{\mathsf{s}}\{\|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|Q\boldsymbol{k}\|_{PH^{1+\mathsf{s}}(\Omega)}\}.$$

On the other hand, we know that $\|Q\boldsymbol{k}\|_{PH^{1+\mathsf{s}}(\Omega)}\lesssim\|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ (see the proof of corollary 2), so using (18) and the definition of $\boldsymbol{k}$, for any $\mathsf{s}\in(0,\tau_{Dir})$, it actually holds that

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\lesssim\|\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega}\lesssim h^{\mathsf{s}}\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}.$$

Since by construction $\mathbf{curl}(\boldsymbol{k}-\boldsymbol{k}_h)=0$, we have obtained (45).
Noting finally that $h\lesssim\mathrm{diam}(\Omega)$, using (47) we conclude that (44) holds. □

5.2 Case of a "weak" T-coercivity operator

As usual we assume in this section that the companion scalar problem (5) is well-posed. But that we only have at hand a "weak" explicit T-coercivity involution operator T, cf. (59) in section A.1. At the discrete level, one can build "weak" discrete T-coercivity operators provided the meshes are locally T-conform (see section A.2). This yields uniformly bounded discrete operators $(T_h)_{h\leq \mathtt{h}_0}$, where $\mathtt{h}_0 > 0$ is a threshold value, such that (62)-(63) are fulfilled, with consequences listed in section A.2. Consequently, introducing $\boldsymbol{K}_h(\varepsilon)$ as before (see (39)), one has the...

Proposition 13 *In the "weak" T-coercivity framework, for all $h \leq \mathtt{h}_0$, one has the direct sum*

$$\boldsymbol{V}_h = \nabla[M_h] \oplus \boldsymbol{K}_h(\varepsilon). \tag{48}$$

In addition, $\boldsymbol{k}_h \mapsto \|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$ defines a norm on $\boldsymbol{K}_h(\varepsilon)$.
Finally, the operators $(\pi_{1h})_{h\leq \mathtt{h}_0}$ and $(\boldsymbol{\pi}_{2h})_{h\leq \mathtt{h}_0}$ introduced in (43) are well-defined, and their continuity moduli are bounded independently of $h \leq \mathtt{h}_0$.

To conclude the study, we now prove the result below, whose proof follows closely the proof of theorem 3.

Theorem 4 *In the "weak" T-coercivity framework, $\|\,\mathbf{curl}\cdot\|_{0,\Omega}$ defines a norm that is uniformly equivalent to the $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$-norm on $\boldsymbol{K}_h(\varepsilon)$, for h small enough, ie.*

$$\exists C_W^\star > 0,\ \forall h \leq \mathtt{h}_0,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon),\ \|\boldsymbol{k}_h\|_{0,\Omega} \leq C_W^\star\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}. \tag{49}$$

In addition, let $\mathtt{s} \in (0, \tau_{Dir})$:

$$\begin{cases} \exists C_{\mathtt{s}} > 0,\ \forall h \leq \mathtt{h}_0,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon), \\ \inf_{\boldsymbol{k}\in \boldsymbol{K}_N(\Omega,\varepsilon)} \|\boldsymbol{k} - \boldsymbol{k}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq C_{\mathtt{s}} h^{\mathtt{s}}\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}. \end{cases} \tag{50}$$

Proof Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon)$ be given, and let $\boldsymbol{k}$ be the solution to the div-curl problem (46). Exactly as in the proof of theorem 3, we find that there exists $q \in H^1_0(\Omega)$ such that $\boldsymbol{k} - \boldsymbol{k}_h = \nabla q$ in Ω.
Let $h \leq \mathtt{h}_0$. Then, for any $\bar{q}_h \in M_h$, we write the triangle inequality

$$\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} = \|\nabla q\|_{0,\Omega} \leq \|\nabla(q - \bar{q}_h)\|_{0,\Omega} + \|\nabla \bar{q}_h\|_{0,\Omega}.$$

According to (63), there exists $q'_h \in M_h \setminus \{0\}$ such that

$$\|\nabla \bar{q}_h\|_{0,\Omega} \leq (\underline{\gamma}_0)^{-1} \frac{|(\varepsilon\nabla\bar{q}_h|\nabla q'_h)_{0,\Omega}|}{\|\nabla q'_h\|_{0,\Omega}}.$$

Since $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon)$ and $\boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon)$, one has $(\varepsilon(\boldsymbol{k} - \boldsymbol{k}_h)|\nabla q'_h)_{0,\Omega} = 0$ or, in other words, $(\varepsilon\nabla q|\nabla q'_h)_{0,\Omega} = 0$. Hence,

$$\|\nabla \bar{q}_h\|_{0,\Omega} \leq (\underline{\gamma}_0)^{-1} \frac{|(\varepsilon\nabla(\bar{q}_h - q)|\nabla q'_h)_{0,\Omega}|}{\|\nabla q'_h\|_{0,\Omega}} \leq \frac{\varepsilon_+}{\underline{\gamma}_0}\,\|\nabla(q - \bar{q}_h)\|_{0,\Omega}.$$

We find that $\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega} \leq (1+\varepsilon_+/\underline{\gamma}_0)\|\nabla(q-\bar{q}_h)\|_{0,\Omega}$. Since the result holds for any $\bar{q}_h \in M_h$, we have actually proved that

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega} \lesssim \inf_{\bar{q}_h \in M_h} \|\nabla(q-\bar{q}_h)\|_{0,\Omega}.$$

We conclude the proof as before, by noting that $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h \in \nabla[M_h]$. □

6 Uniform discrete inf-sup condition and convergence

We consider directly the "weak" T-coercivity framework. Assuming (62)-(63) holds, we remark that $(\boldsymbol{k}_h, \boldsymbol{k}'_h) \mapsto (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_h|\,\mathbf{curl}\,\boldsymbol{k}'_h)_{0,\Omega}$ fulfills a uniform discrete inf-sup condition on $\boldsymbol{K}_h(\varepsilon)\times\boldsymbol{K}_h(\varepsilon)$, for h small enough. Indeed, according to theorem 4, we have

$$\begin{cases} \exists\tilde{\gamma}>0,\ \forall h\leq \mathtt{h}_0,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon), \\ \displaystyle\sup_{\boldsymbol{k}'_h\in\boldsymbol{K}_h(\varepsilon)\setminus\{0\}} \frac{|(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_h|\,\mathbf{curl}\,\boldsymbol{k}'_h)_{0,\Omega}|}{\|\boldsymbol{k}'_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq \tilde{\gamma}\,\|\boldsymbol{k}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \end{cases} \tag{51}$$

Next, we introduce $A_\omega \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$ defined by

$$(A_\omega \boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}_0(\mathbf{curl};\Omega)} := a_\omega(\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_0(\mathbf{curl};\Omega),$$

and

$$|||a_\omega||| := \sup_{\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v},\boldsymbol{w})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} < \infty.$$

Theorem 5 *Assume that the variational formulation (3) is well-posed. In the "weak" T-coercivity framework, the form a_ω fulfills a uniform discrete inf-sup condition on $\boldsymbol{V}_h\times\boldsymbol{V}_h$ for h small enough, ie.*

$$\begin{cases} \exists C_\omega, \mathtt{h}_\omega>0,\ \forall h\leq \mathtt{h}_\omega,\ \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \\ \displaystyle\sup_{\boldsymbol{v}'_h\in\boldsymbol{V}_h\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_h,\boldsymbol{v}'_h)|}{\|\boldsymbol{v}'_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq C_\omega\,\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \end{cases} \tag{52}$$

Remark 4 Next, we proceed in the spirit of the proof of Theorem 2.2 in [13].

Proof We argue by contradiction. Namely, we assume that

$$\begin{cases} \forall k\in\mathbb{N}\setminus\{0\},\ \exists h_k\leq k^{-1},\ \exists \boldsymbol{v}_{h_k}\in\boldsymbol{V}_{h_k}, \\ \|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}=1 \quad\text{and}\quad \displaystyle\sup_{\boldsymbol{v}'_{h_k}\in\boldsymbol{V}_{h_k}\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_{h_k},\boldsymbol{v}'_{h_k})|}{\|\boldsymbol{v}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \leq k^{-1}. \end{cases} \tag{53}$$

In particular, $\lim_{k\to\infty} h_k = 0$, so it holds that $h_k < \mathtt{h}_0$ for k large enough. So from now on, we consider that $h_k < \mathtt{h}_0$. We write $\boldsymbol{v}_{h_k} = \nabla q_{h_k} + \boldsymbol{k}_{h_k}$, where $q_{h_k} = \pi_{1h_k}\boldsymbol{v}_{h_k}$ and $\boldsymbol{k}_{h_k} = \boldsymbol{\pi}_{2h_k}\boldsymbol{v}_{h_k}$. Note that $(\nabla q_{h_k})_k$ and $(\boldsymbol{k}_{h_k})_k$ are bounded sequences in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, because the continuity moduli of $(\pi_{1h_k})_k$ and $(\boldsymbol{\pi}_{2h_k})_k$ are bounded uniformly with respect to k (cf. Proposition 13).

Step 1. Let us show that $\lim_{k\to\infty} \|\nabla q_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. This is a simple consequence of (63). According to (53):

$$\sup_{q'_{h_k}\in M_{h_k}\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_{h_k}, \nabla q'_{h_k})|}{\|\nabla q'_{h_k}\|_{0,\Omega}} \le k^{-1}.$$

But $a_\omega(\boldsymbol{v}_{h_k}, \nabla q'_{h_k}) = -\omega^2(\varepsilon \boldsymbol{v}_{h_k}|\nabla q'_{h_k})_{0,\Omega} = -\omega^2(\varepsilon\nabla q_{h_k}|\nabla q'_{h_k})_{0,\Omega}$. From (63), we infer that

$$\underline{\gamma}_0\, \omega^2\, \|\nabla q_{h_k}\|_{0,\Omega} \le k^{-1} \to 0 \text{ as } k\to\infty.$$

Step 2. Let us show that $\lim_{k\to\infty} \|\boldsymbol{\pi}_2 \boldsymbol{v}_{h_k}\|_{0,\Omega} = 0$. Let $\boldsymbol{w}\in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, and $\boldsymbol{w}_{h_k}\in \boldsymbol{V}_{h_k}$:

$$\begin{aligned}|a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w})| &\le |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w}-\boldsymbol{w}_{h_k})| + |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w}_{h_k})|\\ &\le |||a_\omega|||\, \|\boldsymbol{w}-\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + k^{-1}\|\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.\end{aligned}$$

According to the basic approximability property of $(\boldsymbol{V}_{h_k})_k$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one can choose $(\boldsymbol{w}_{h_k})_k$ such that $\lim_{k\to\infty}\|\boldsymbol{w}-\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. In particular, $(\boldsymbol{w}_{h_k})_k$ is a bounded sequence in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, and one finds that

$$\lim_{k\to\infty} |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w})| = 0.$$

This result holds for all $\boldsymbol{w}\in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, so we have proved that $A_\omega \boldsymbol{v}_{h_k} \rightharpoonup 0$ (weakly) in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. On the other hand, the variational formulation (3) is well-posed, so A_ω^{-1} exists and $A_\omega^{-1}\in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$. Hence $\boldsymbol{v}_{h_k}\rightharpoonup 0$ (weakly) in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. This implies that $\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\rightharpoonup 0$ (weakly) in $\boldsymbol{K}_N(\Omega,\varepsilon)$. And because the imbedding of $\boldsymbol{K}_N(\Omega,\varepsilon)$ in $\boldsymbol{L}^2(\Omega)$ is compact, one finds that $\lim_{k\to\infty}\|\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\|_{0,\Omega} = 0$.

Step 3. Let us show that $\lim_{k\to\infty}\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. According to (51),

$$\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le \tilde\gamma^{-1} \sup_{\boldsymbol{k}'_{h_k}\in \boldsymbol{K}_{h_k}(\varepsilon)\setminus\{0\}} \frac{|(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega}|}{\|\boldsymbol{k}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}}.$$

Let $\boldsymbol{k}'_{h_k}\in \boldsymbol{K}_{h_k}(\varepsilon)$. By definition of $\boldsymbol{k}_{h_k}$, one finds that

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega} &= (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega}\\ &= a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{k}'_{h_k}) + \omega^2(\varepsilon\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}.\end{aligned}$$

According to (53), one has $|a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{k}'_{h_k})| \le k^{-1}\,\|\boldsymbol{k}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
On the other hand, $\boldsymbol{v}_{h_k} = \nabla(\pi_1\boldsymbol{v}_{h_k}) + \boldsymbol{\pi}_2\boldsymbol{v}_{h_k}$, so

$$|(\varepsilon\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| \le |(\varepsilon\nabla(\pi_1\boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k})_{0,\Omega}| + |(\varepsilon\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}|.$$

The last term is bounded by the Cauchy-Schwarz inequality

$$|(\varepsilon\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| \le \varepsilon_+\|\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\|_{0,\Omega}\|\boldsymbol{k}'_{h_k}\|_{0,\Omega}.$$

There remains to evaluate the first term. For all $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega, \varepsilon)$, one has

$$|(\varepsilon\nabla(\pi_1 \boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k})_{0,\Omega}| = |(\varepsilon\nabla(\pi_1 \boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k} - \boldsymbol{k})_{0,\Omega}|$$
$$\leq \varepsilon_+ \|\nabla(\pi_1 \boldsymbol{v}_{h_k})\|_{0,\Omega} \|\boldsymbol{k}'_{h_k} - \boldsymbol{k}\|_{0,\Omega}.$$

Now, let $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega, \varepsilon)$ be chosen as in theorem 4 (see (50)). Owing to the fact that $\|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 1$ (cf. (53)), one gets the bound

$$|(\varepsilon\nabla(\pi_1 \boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k})_{0,\Omega}| \lesssim {h_k}^{\mathtt{s}} \, \| \mathbf{curl}\, \boldsymbol{k}'_{h_k}\|_{0,\Omega}.$$

Aggregating the above estimates, one finds that

$$\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \left(k^{-1} + \|\boldsymbol{\pi}_2 \boldsymbol{v}_{h_k}\|_{0,\Omega} + {h_k}^{\mathtt{s}}\right),$$

thus leading to $\lim_{k\to\infty} \|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$ according to Step 2.
Step 4. For all k, one has $\|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq \|\nabla q_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ by the triangle inequality, so one concludes that $\lim_{k\to\infty} \|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$, which contradicts (53). □

One can finally derive the (classical) error estimate.

Theorem 6 *Let the assumptions of theorem 5 be fulfilled, and let* $\mathtt{h}_\omega > 0$ *be the threshold value introduced there. Then, for all* $h \leq \mathtt{h}_\omega$*, the discrete variational formulation (24) is well-posed.*
Without further assumption on the regularity of the data $\boldsymbol{f}$*, one has*

$$\lim_{h\to 0} \|\boldsymbol{e} - \boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0. \tag{54}$$

Let the extra-regularity of the data $\boldsymbol{f}$ *be as in (13) with* $\tau_0 > 0$ *given, then one has the error estimate, for all* $\mathtt{s} \in [0, \min(\tau_0, \tau_{Dir}, \tau_{Neu}))$*,*

$$\forall h \leq \mathtt{h}_\omega, \quad \|\boldsymbol{e} - \boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\mathtt{s}}\{\|\operatorname{div} \boldsymbol{f}\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\}. \tag{55}$$

Proof Because the form a_ω fulfills a uniform discrete inf-sup condition for $h \leq \mathtt{h}_\omega$, classical error analysis yields

$$\forall h \leq \mathtt{h}_\omega, \quad \|\boldsymbol{e} - \boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

In the absence of extra-regularity of the data, according to the basic approximability property of $(\boldsymbol{V}_h)_h$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one finds (54). On the other hand, in the case of extra-regularity of the data $\boldsymbol{f}$, we then recover (55) by choosing $\boldsymbol{v}_h = \Pi_h^{comb}\boldsymbol{e}$ (see proposition 9). □

Fig. 1 Left: a T-conform mesh ; right: a plain mesh.

7 Numerical illustrations

In this section, we study numerically a simple model. The domain Ω is $(0,1)\times(-1,1)\times(0,1)$. It is partitioned into $\Omega_+ = (0,1)\times(0,1)\times(0,1)$ and $\Omega_- = (0,1)\times(-1,0)\times(0,1)$. Note that this partition is symmetric with respect to the interface $\Sigma = (0,1)\times\{0\}\times(0,1)$. The pulsation and coefficients are respectively set to

$$\omega = 1\,;\quad \varepsilon_{|\Omega_+} = 1,\ \mu_{|\Omega_+} = 1\,;\quad \varepsilon_{|\Omega_-} \in \{-1.5, -1.1, -1.01\},\ \mu_{|\Omega_-} = .5.$$

In this symmetric geometry, it is known that the companion scalar problem is well-posed as soon as $\varepsilon_{|\Omega_-} \neq -1$; that "full" T-coercivity is achieved with the help of the symmetry with respect to Σ (cf. [9]) ; and, as a consequence, that T-conform meshes are obtained using meshes that are symmetric with respect to Σ (cf. [19]). Below, we assume that the model set in Ω (cf. (3)) is well-posed for the above values of the pulsation and the coefficients.
We choose a piecewise smooth solution $\boldsymbol{e}$, which is consistent with the fact that, in this symmetric setting, $\tau_{Dir} = \tau_{Neu} = 1$ (cf. again [19]). Namely,

$$\begin{aligned} e_1(x_1,x_2,x_3) &= x_1^2 \sin\Big(\frac{\pi}{2}(x_2-1)\Big)\sin(\pi x_3^2),\\ e_2(x_1,x_2,x_3) &= \varepsilon^{-1}\sin(\pi x_1)\,x_2\sin(5\pi x_3),\\ e_3(x_1,x_2,x_3) &= \sin(2\pi x_1)\sin(\pi x_2^2)\,x_3. \end{aligned}$$

It is easily checked that $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, with $\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) \in \boldsymbol{L}^2(\Omega)$ and $\operatorname{div}\varepsilon\boldsymbol{e} \in L^2(\Omega)$. Consequently, the data $\boldsymbol{f} = \omega^{-2}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) - \varepsilon\boldsymbol{e}$ belongs to $\boldsymbol{H}(\operatorname{div};\Omega)$, so one has $\tau_0 = 1$ in (13).
Computations are carried out on two series of meshes. A *T-conform series*: the meshes are generated by meshing Ω_+ first, and then using the symmetry transform with respect to Σ to build the mesh on Ω_- (see figure 1, left). And a *plain series*, where the meshes can be nonsymmetric with respect to Σ (see figure 1, right). All results have been obtained with the help of the `GetDP` software [30].
In figure 2, the error results in $\boldsymbol{H}(\mathbf{curl};\Omega)$-norm are reported. In abscissa, we choose the number of degrees of freedom $N_h = \dim(\boldsymbol{V}_h)$ to the power $1/3$, to compare simulations with similar computational costs. Also, $N_h^{1/3}$ is known to be equivalent to h for regular families of meshes. Overall, results are similar for both series of meshes. However, for the *plain series*, there are

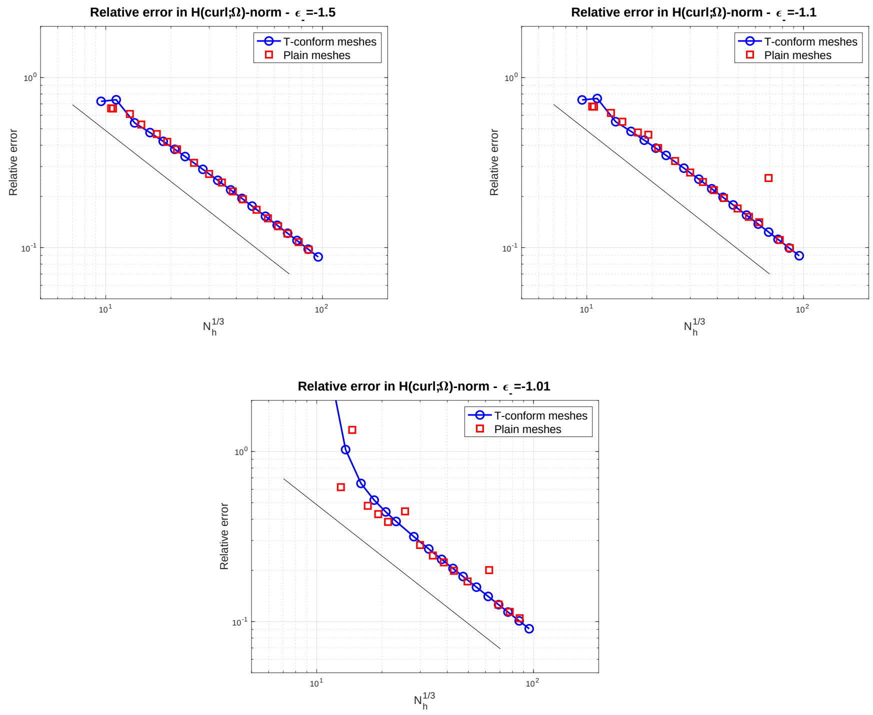

Fig. 2 Relative error in $\boldsymbol{H}(\mathbf{curl};\Omega)$-norm obtained for the three values of $\varepsilon_{|\Omega_-}$, with h varying from 0.1 to 0.01. The line corresponds to the linear scale $O(h) = O(N_h^{1/3})$.

anomalies/glitches for $\varepsilon_{|\Omega_-} \in \{-1.1, -1.01\}$, ie. convergence is not monotonic. On the other hand, for the *T-conform series*, results indicate that the sign-change has little influence on the convergence.
We then report errors in $\boldsymbol{L}^2(\Omega)$-norm (figure 3), and also in $\boldsymbol{L}^2(\Omega)$-norm of the curl of the errors (figure 4). For the errors on the curl, results are more or less nominal (recall that μ does not change sign). While for the $\boldsymbol{L}^2(\Omega)$-norm, results show that convergence is erratic for the *plain series* and, more to the point, it seems that $\|\boldsymbol{e} - \boldsymbol{e}_h\|_{0,\Omega}$ does not decrease when $\varepsilon_{|\Omega_-} = -1.01$. The numerical method is still in a *pre-asymptotic regime* regarding convergence, even though the meshsize is as small as one hundredth of the size (length) of the domain[2]. For the *T-conform series*, convergence is again nominal.
To conclude the analysis of the numerical results, we draw a parallel with some results available in the literature for the companion scalar problem set in a symmetric geometry [19]. Let us isolate the curl-free part of the exact and

[2] This observation is consistent with the fact that only small glitches are seen in figure 2 for the *plain series*. This is due to the fact that the values taken by the curl of the chosen exact solution are orders of magnitude larger than the values of the solution itself.

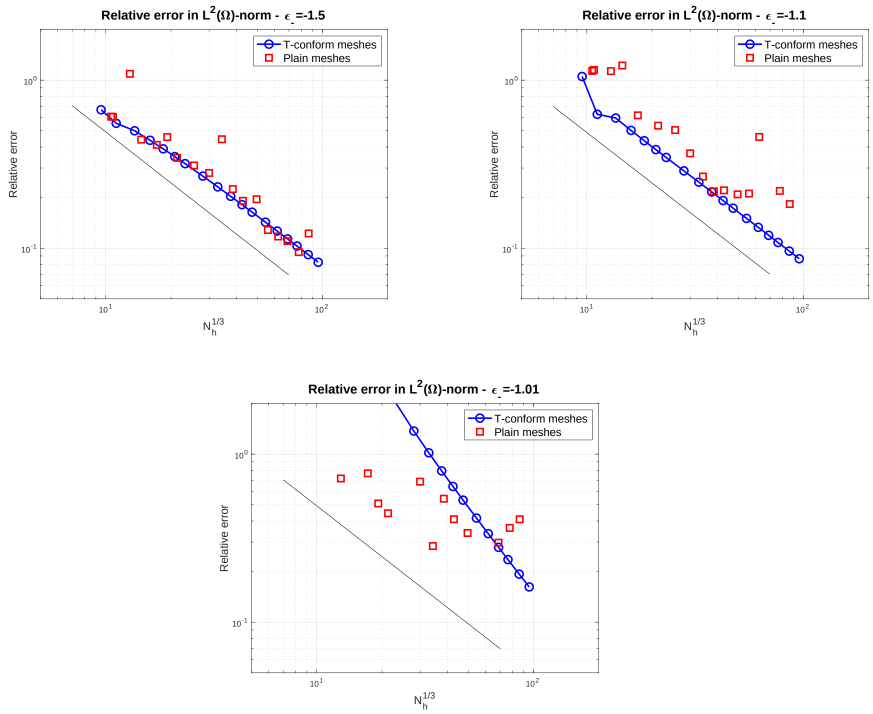

Fig. 3 Relative error in $\boldsymbol{L}^2(\Omega)$-norm obtained for the three values of $\varepsilon_{|\Omega_-}$, with h varying from 0.1 to 0.01. The line corresponds to the linear scale $O(h) = O(N_h^{1/3})$.

discrete solutions, that is $\nabla\phi$ in (19), governed by (21):

$$\begin{cases} \textit{Find } \phi \in H_0^1(\Omega) \textit{ such that} \\ (\varepsilon\nabla\phi|\nabla q)_{0,\Omega} = (\operatorname{div} \boldsymbol{f}|q)_{0,\Omega}, \ \forall q \in H_0^1(\Omega) \end{cases};$$

resp. $\nabla\phi_h$ where $\phi_h = \pi_{1h}\boldsymbol{e}_h$ is governed by

$$\begin{cases} \textit{Find } \phi_h \in M_h \textit{ such that} \\ (\varepsilon\nabla\phi_h|\nabla q_h)_{0,\Omega} = (\operatorname{div} \boldsymbol{f}|q_h)_{0,\Omega}, \ \forall q_h \in M_h. \end{cases}$$

These are respectively the companion scalar problem, and its discretization. The error $\|\nabla\phi - \nabla\phi_h\|_{0,\Omega}$ has been thoroughly investigated in [19]. In particular, numerical examples are provided in a rectangle (a domain of $\mathbb{R}^2$), partitioned into two squares, and it is observed that the use of nonsymmetric meshes leads to serious numerical instabilities: we refer the interested reader precisely to Figure 7, page 23 in [19]. In other words, we get the same behavior, now on the solution of time-harmonic Maxwell equations in a domain of $\mathbb{R}^3$.

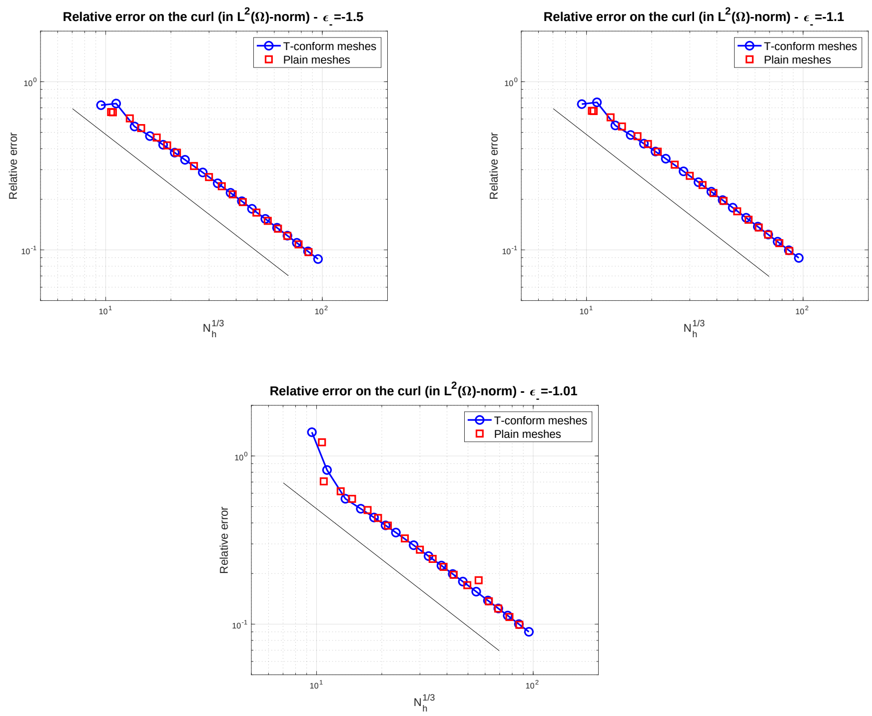

Fig. 4 Relative error in $\boldsymbol{L}^2(\Omega)$-norm of the curl obtained for the three values of $\varepsilon_{|\Omega_-}$, with h varying from 0.1 to 0.01. The line corresponds to the linear scale $O(h) = O(N_h^{1/3})$.

8 Case of sign-changing magnetic permeability

Let us briefly describe how one can proceed if

$$\boxed{\mu \text{ is as in the } \textit{interface case}\,;\ \varepsilon \text{ is as in the } \textit{classical case.}}$$

To address this situation, one expresses the time-harmonic Maxwell equations in terms of the magnetic field only

$$\begin{cases} \textit{Find } \boldsymbol{h} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \textit{ such that:} \\ \mathbf{curl}(\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j})) - \omega^2 \mu \boldsymbol{h} = 0 \text{ in } \Omega, \\ \operatorname{div} \mu \boldsymbol{h} = 0 \text{ in } \Omega\,; \\ \mu \boldsymbol{h} \cdot \boldsymbol{n} = 0 \text{ and } \varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{h} - \boldsymbol{j}) \times \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{56}$$

As before, one can decouple the real and imaginary parts. Eg., if $\boldsymbol{h}$ stands for $\Re(\boldsymbol{h})$ and $\boldsymbol{g}$ stands for $\varepsilon^{-1}\Re(\boldsymbol{j})$, then $\boldsymbol{h}$ solves the equivalent variational formulation

$$\begin{cases} \textit{Find } \boldsymbol{h} \in \boldsymbol{H}(\mathbf{curl}; \Omega) \textit{ such that} \\ a'_\omega(\boldsymbol{h}, \boldsymbol{v}) = (\boldsymbol{g}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega), \end{cases} \tag{57}$$

where

$$a'_\omega(\boldsymbol{u},\boldsymbol{v}) := (\varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2(\mu\boldsymbol{u}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl};\Omega).$$

We observe that one has to study the companion scalar problem with Neumann boundary condition, as introduced in (31). In the present situation however, one has to *assume* that this problem is well-posed, which can again be tackled with the help of T-coercivity: one finds results that are similar to those of appendix A. Then, the study of the well-posedness of (57) proceeds as before. Namely, one introduces

$$\boldsymbol{K}_T(\Omega,\mu) := \{\boldsymbol{v}\in \boldsymbol{X}_T(\Omega,\mu)\ :\ \operatorname{div}\mu\boldsymbol{v} = 0\}.$$

There holds the direct, continuous decomposition

$$\boldsymbol{H}(\mathbf{curl};\Omega) = \nabla[H^1(\Omega)] \oplus \boldsymbol{K}_T(\Omega,\mu),$$

together with equivalence of norms in $\boldsymbol{K}_T(\Omega,\mu)$, and the compact imbedding of $\boldsymbol{K}_T(\Omega,\mu)$ in $\boldsymbol{L}^2(\Omega)$. We refer to the same bibliographical references as in sections 2 and 3.
One then uses

$$\begin{aligned}\boldsymbol{V}_h^+ &:= \{\boldsymbol{v}_h\in \boldsymbol{H}(\mathbf{curl};\Omega)\ :\ \boldsymbol{v}_{h|K}\in \mathcal{R}_1(K),\ \forall K\in\mathcal{T}_h\},\\ M_h^+ &:= \{q_h\in H^1(\Omega)\ :\ q_{h|K}\in P_1(K),\ \forall K\in\mathcal{T}_h\},\end{aligned}$$

to discretize (57), resp. (31). The analysis of the interpolation error on the magnetic field can again be carried out with the combined interpolation operator. To prove the uniform discrete inf-sup condition of the form a'_ω on $\boldsymbol{V}_h^+\times\boldsymbol{V}_h^+$ and error estimates, one has to study the properties of the discrete spaces

$$\boldsymbol{K}_h(\mu) := \{\boldsymbol{v}_h\in\boldsymbol{V}_h^+\ :\ (\mu\boldsymbol{v}_h|\nabla q_h)_{0,\Omega} = 0,\ \forall q_h\in M_h^+\}. \tag{58}$$

Uniform equivalence of norms in $\boldsymbol{K}_h(\mu)$, resp. uniform discrete inf-sup condition, are obtained with techniques that are completely similar to those developed in the proofs of theorems 3 (”full” T-coercivity) and 4 (”weak” T-coercivity), resp. theorem 5.

9 Conclusions and extensions

We have studied the time-harmonic Maxwell equations for a model with one sign-changing coefficient. We have proved optimal convergence rates on the error, when the numerical approximation is computed with the help of the Nédélec’s first family of edge finite elements. For low-regularity solutions, those results are achieved with the help of the combined interpolation operator designed in [22,23]. All those results have been obtained with the help of explicit T-coercivity operators for the derivation of the inf-sup condition.
A possible extension is to have a boundary data, illustrated below for the problem expressed in the electric field. In this case, let us assume for instance

that $\boldsymbol{e}$ has a non-vanishing tangential trace, namely one replaces $\boldsymbol{e} \times \boldsymbol{n} = 0$ on $\partial\Omega$ by $\boldsymbol{e} \times \boldsymbol{n} = \mathsf{e}_\Gamma$ on $\partial\Omega$ in (1), where the data e_Γ defined on $\partial\Omega$ is actually equal to the tangential trace of some field $\boldsymbol{e}^\star \in \boldsymbol{H}(\mathbf{curl};\Omega)$. Introducing $\boldsymbol{e}_0 = \boldsymbol{e} - \boldsymbol{e}^\star \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, one finds that $\boldsymbol{e}_0$ solves the time-harmonic Maxwell equations (1), with modified right-hand sides. Hence one may study these problems as before. In order to determine explicit convergence rates, one needs to have some *ad hoc* extra-regularity assumptions on $\boldsymbol{e}^\star$.
Another interesting extension to consider is to address the time-harmonic Maxwell equations, with *two* sign-changing coefficients.

Acknowledgements The author thanks Théophile Chaumont-Frelet and Lucas Chesnel for many interesting discussions and feedbacks, and Axel Modave for producing the numerical results.

References

1. A. Abdulle, M.E. Huber, and S. Lemaire. An optimization-based numerical method for diffusion problems with sign-changing coefficients. *C. R. Acad. Sci. Paris, Série I*, 355:472–478, 2017.
2. S. Anantha Ramakrishna. Physics of negative refractive index materials. *Rep. Prog. Phys.*, 68:449–521, 2005.
3. F. Assous, P. Ciarlet Jr., and S. Labrunie. *Mathematical foundations of computational electromagnetism*, volume 198 of *Applied Mathematical Sciences*. Springer, 2018.
4. A. Bermudez, P. Gamallo, M. R. Nogueiras, and R. Rodriguez. Approximation of a structural acoustic vibration problem by hexahedral finite elements. *IMA J. Numer. Anal.*, 26:391–421, 2006.
5. A. Bermudez, R. Rodriguez, and P. Salgado. Numerical solution of eddy current problems in bounded domains using realistic boundary conditions. *Comput. Methods Appl. Mech. Engrg.*, 194:411–426, 2005.
6. D. Boffi, F. Brezzi, and M. Fortin. *Mixed and hybrid finite element methods and applications*. Springer-Verlag, 2013.
7. A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *J. Math. Anal. Appl.*, 408:498–512, 2013.
8. A.-S. Bonnet-Ben Dhia, C. Carvalho, and P. Ciarlet, Jr. Mesh requirements for the finite element approximation of problems with sign-changing coefficients. *Numer. Math.*, 138:801–838, 2018.
9. A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet, Jr. T-coercivity for scalar interface problems between dielectrics and metamaterials. *Math. Mod. Num. Anal.*, 46:1363–1387, 2012.
10. A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet, Jr. T-coercivity for the Maxwell problem with sign-changing coefficients. *Communications in Partial Differential Equations*, 39:1007–1031, 2014.
11. A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet, Jr. Two-dimensional Maxwell's equations with sign-changing coefficients. *Appl. Numer. Math.*, 79:29–41, 2014.
12. A.-S. Bonnet-Ben Dhia, L. Chesnel, and X. Claeys. Radiation condition for a non-smooth interface between a dielectric and a metamaterial. *Math. Model. Meth. App. Sci.*, 23:1629–1662, 2013.
13. A.-S. Bonnet-Ben Dhia, P. Ciarlet, Jr., and C.M. Zwölf. Time harmonic wave diffraction problems in materials with sign-shifting coefficients. *J. Comput. Appl. Math.*, 234:1912–1919, corrigendum p. 2616, 2010.
14. A.-S. Bonnet-Ben Dhia, M. Dauge, and K. Ramdani. Spectral analysis and singularities of a non-coercive transmission problem (in French). *C. R. Acad. Sci. Paris, Ser. I*, 328:717–720, 1999.

15. C. Carvalho. *Mathematical and numerical study of plasmonic structures with corners.* PhD thesis, Ecole Polytechnique, 2015.
16. M. Cassier, P. Joly, and M. Kachanovska. Mathematical models for dispersive electromagnetic waves: an overview. *Comput. Math. Appl.*, 74:2792–2830, 2017.
17. L. Chesnel. *Investigation of some transmission problems with sign-changing coefficients. Application to metamaterials (in French).* PhD thesis, Ecole Polytechnique, 2012.
18. L. Chesnel and P. Ciarlet, Jr. Compact imbeddings in electromagnetism with interfaces between classical materials and meta-materials. *SIAM J. Math. Anal.*, 43:2150–2169, 2011.
19. L. Chesnel and P. Ciarlet, Jr. T-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients. *Numer. Math.*, 124:1–29, 2013.
20. P.G. Ciarlet. *The finite element method for elliptic problems.* Classics in Applied Mathematics. SIAM, 2002.
21. P. Ciarlet, Jr. T-coercivity: Application to the discretization of Helmholtz-like problems. *Computers Math. Applic.*, 64:22–34, 2012.
22. P. Ciarlet, Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields. *Computers Math. Applic.*, 71:85–104, 2016.
23. P. Ciarlet, Jr. Mathematical and numerical analyses for the div-curl and div-curlcurl problems with a sign-changing coefficient. *Technical Report HAL*, hal-02567484v1, 2020.
24. P. Ciarlet, Jr. On the approximation of electromagnetic fields by edge finite elements. Part 3: Sensitivity to coefficients. *SIAM J. Math. Anal.*, 52:3004–3038, 2020.
25. P. Ciarlet, Jr., D. Lassounon, and M. Rihani. An optimal control-based numerical method for scalar transmission problems with sign-changing coefficients. *Technical Report HAL*, hal-03666913v1, May 2022.
26. M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *Math. Mod. Num. Anal.*, 33:627–649, 1999.
27. M. Dauge and B. Texier. non-coercive transmission problems in polygons (in French). Technical Report 97-27, IRMAR, Rennes 1 University, Rennes, France, 1997.
28. A. Dello Russo and A. Alonso. Finite element approximation of Maxwell eigenproblems on curved Lipschitz polyhedral domains. *Appl. Numer. Math.*, 59:1796–1822, 2009.
29. K. Disser, H.-C. Kaiser, and J. Rehberg. Optimal Sobolev regularity for linear second-order divergence elliptic operators occuring in real-world problems. *SIAM J. Math. Anal.*, 47:1719–1746, 2015.
30. P. Dular, C. Geuzaine, F. Henrotte, and W. Legros. A general environment for the treatment of discrete problems and its application to the finite element method. *IEEE Transactions on Magnetics*, 34:3395–3398, 1998.
31. J. Elschner, H.-C. Kaiser, J. Rehberg, and G. Schmidt. $W^{1,q}$ regularity results for elliptic transmission problems on heterogeneous polyhedra. *Math. Models Methods Appl. Sci.*, 17:593–615, 2007.
32. A. Ern and J.-L. Guermond. Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions. *Comput. Math. Applic.*, 75:918–932, 2018.
33. A. Ern and J.-L. Guermond. *Finite elements I: approximation and interpolation.* Springer, 2021.
34. M. Feistauer and A. Ženišek. Finite element solution of nonlinear elliptic problems. *Numer. Math.*, 50:451–475, 1987.
35. P.W. Gross and P.R. Kotiuga. *Electromagnetic theory and computation: a topological approach.* MSRI Publications Series, **48**. Cambridge University Press, 2004.
36. M. Halla. On the approximation of dispersive electromagnetic eigenvalue problems in 2D. *Technical Report Arxiv*, arXiv:2104.09318v1, 2021.
37. R. Haller-Dintelmann, H.-C. Kaiser, and J. Rehberg. Direct computation of elliptic singularities across anisotropic, multi-material edges. *J. Math. Sci.*, 172:589–622, 2011.
38. R. Hiptmair. Finite elements in computational electromagnetics. *Acta Numerica*, pages 237–339, 2002.
39. R. Hiptmair, J. Li, and J. Zou. Convergence analysis of finite element methods for $\boldsymbol{H}(\mathbf{curl};\Omega)$-elliptic interface problems. *Numer. Math.*, 122:557–578, 2012.
40. H.-C. Kaiser and J. Rehberg. Optimal elliptic regularity at the crossing of a material interface and a Neumann boundary edge. *J. Math. Sci.*, 169:145–166, 2010.

41. A. Kirsch and F. Hettlich. *Mathematical theory of time-harmonic Maxwell's equations.* Springer, 2016.
42. J. Li, J.M. Melenk, B. Wohlmuth, and J. Zou. Optimal a priori estimates for higher order finite elements for elliptic interface problems. *Appl. Numer. Math.*, 60:19–37, 2010.
43. P. Monk. *Finite element methods for Maxwell's equations.* Oxford University Press, 2003.
44. J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer Math.*, 35:315–341, 1980.
45. H.-M. Nguyen and S. Sil. Limiting absorption principle and well-posedness for the time-harmonic Maxwell equations with anisotropic sign-changing coefficients. *Commun. Math. Phys.*, 379:145–176, 2020.
46. S. Nicaise and A.-M. Sändig. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation. *Math. Model. Methods Appl. Sci.*, 9:855–898, 1999.
47. S. Nicaise and J. Venel. A posteriori error estimates for a finite element approximation of transmission problems with sign changing coefficients. *J. Comput. Appl. Math.*, 235:4272–4282, 2011.
48. P.-A. Raviart and J.-M. Thomas. A mixed finite element method for second order elliptic problems. In *Mathematical aspects of finite element methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer, 1977.
49. D.R. Smith, J.B. Pendry, and M.C.K. Wiltshire. Metamaterials and negative refractive index. *Science*, 305:788–792, 2004.

A Practical T-coercivity for the companion scalar problem

A.1 Explicit T-coercivity operators

In practice, how to realize *explicitly* the T-coercivity for a well-posed companion scalar problem (5) in the interface case? The concept was originally introduced in [13] (see Theorem 2.1).
We provide a list *à la Prévert* to describe a number of situations where explicit T-coercivity operators are available, taking into account the geometry of the domain Ω, and the shape of the interface induced by the partition $\mathcal{P} = (\Omega_p)_{p=+,-}$. In some cases the results are known for domains in $\mathbb{R}^2$ (we use the notations Ω_2, resp. $(\Omega_{2p})_{p=+,-}$). We rely on Refs. [47,9,17, 19,10,15,8] for the precise results:

- the geometry is symmetric with respect to the interface, cf. §5.1 in [47] or §3.1 in [9] ; this implies that the interface is a subset of a hyperplane ;
- the geometry is tubular with respect to the interface, with a smooth interface, cf. §3.4 in [9] ;
- the domain Ω_2 is a disk or an angular sector in $\mathbb{R}^2$, and Ω_{2+} and Ω_{2-} are angular subsectors, cf. §3.2 in [9], or the domain Ω_2 is the union of self-replicating triangles in $\mathbb{R}^2$, and Ω_{2+} and Ω_{2-} are union of contiguous triangles, cf. §3 in [8] ; this implies that the interface has exactly one corner inside Ω_2. This can be generalized to a geometry in $\mathbb{R}^3$, by taking $\Omega := \Omega_2 \times (a, b)$, resp. $\Omega_\pm := \Omega_{2\pm} \times (a, b)$, cf. §7.2 in [10], for some $a < b$; this implies that the interface has exactly one edge, and no vertex, inside Ω.
- Ω is the cube $(-a, a)^3$, Ω_+ or Ω_- is the sub-cube $(0, a)^3$, cf. §7.3 in [10], for some $a > 0$; or §5.2 in [47] for the same setting in a square domain Ω_2 in $\mathbb{R}^2$.

Then one can build explicitly an operator T_0 that fulfills (11). We say that there is a "full" T-coercivity operator T_0 available. In all of the above, the operator T_0 is derived from elementary geometrical transforms, such as symmetries, rotations and angle dilation. Except for the latter, all those transforms can be used after discretization, provided the underlying discrete geometrical structures (in our case, the meshes, see section 4) are *conforming* with respect to the transforms.
One can check that, thanks to the generic definition of the operators T_0 that is used (cf. p.

1915 in [13] or p. 4274 in [47]), in all instances, one has $(T_0)^2 = I_{H^1_0(\Omega)}$.

On the other hand, in many other configurations, and even though the scalar problem (5) is well-posed, only a "weak" T-coercivity operator T, defined in the following sense (see Lemma 2 in [8]), can be built explicitly:

$$\begin{cases} \exists\alpha,\beta>0,\ \exists T\in\mathcal{L}(H^1_0(\Omega)) \text{ bijective,} \\ \forall q\in H^1_0(\Omega),\ (\varepsilon\nabla q|\nabla(Tq))_{0,\Omega}\geq\alpha\,\|\nabla q\|^2_{0,\Omega}-\beta\,\|q\|^2_{0,\Omega}. \end{cases} \tag{59}$$

The main idea (see §4.3 in [9]) to build those operators is to use localization arguments. For that, the mathematical tool of choice is an *ad hoc* partition of unity function. First, one can focus on a neighborhood of the interface. Second, one separates corners and edges (in $\mathbb{R}^2$), or one splits a smooth interface, etc., into elementary blocks that fit locally the situations described above. We provide another list *à la Prévert* in which such a "weak" T-coercivity operator T can be built. The geometry of the domain Ω, and the partition $\mathcal{P}=(\Omega_p)_{p=+,-}$ are such that:

- the geometry is locally symmetric with respect to the interface, cf. §4 in [19] or §7.4 in [10] ;
- the interface is smooth, cf. §2.B.1 in [15].
- the partition of the domain Ω_2 is such that the interface separating Ω_{2+} and Ω_{2-} is polygonal, cf. §4 in [8] ; this can be generalized to a geometry in $\mathbb{R}^3$, by taking $\Omega := \Omega_2\times(a,b)$, resp. $\Omega_\pm := \Omega_{2\pm}\times(a,b)$, for some $a<b$; in principle, in $\mathbb{R}^3$, it could be generalized to a polyhedral interface.

Again in all instances above, one has $T^2 = I_{H^1_0(\Omega)}$, see Lemma 2 in [8].

Remark 5 Notice that (59) also fits the original concept of T-coercivity, cf. §2 in [13].

A.2 Discrete T-coercivity for the companion scalar problem

We assume below that the companion scalar problem (5) is well-posed.
With the help of "full" or "weak" T-coercivity operators for this problem, one may define discrete T-coercivity operators that help prove well-posedness of the discrete scalar problems (23). As a matter of fact, this is made possible thanks to the use, in the definition of the exact operators T_0 ("full" T-coercivity operator) and T ("weak" T-coercivity operator), of elementary geometrical transforms, such as symmetries and rotations. This happens when the interface is part of a hyperplane, polygonal (in $\mathbb{R}^2$) or polyhedral (in $\mathbb{R}^3$). Also, one needs to interpolate the partition of unity function for the "weak" T-coercivity operator. Then, one can implement the discrete operators: this amounts to using (locally for the "weak" T-coercivity operator) T-conform meshes. Namely, the mesh is first built in Ω_-, and then mapped to Ω_+ via the same geometrical transforms as the ones that were chosen to design T_0 or T, in order to define the mesh there. Or the other way around, from Ω_+ to Ω_-. For the "weak" T-coercivity operator, the process is localized to a neighborhood of the interface. We refer to [19,8] for details.
Consequently, when one has at hand a "full" T-coercivity operator T_0, it can also be used to establish the uniform discrete T-coercivity of the discrete scalar problems (23). Namely, T_0 is such that

$$\begin{cases} \forall h,\quad T_0[M_h]\subset M_h, \text{ and} \\ \exists\alpha_0'>0,\ \forall h,\ \forall q_h\in M_h,\ (\varepsilon\nabla q_h|\nabla(T_0q_h))_{0,\Omega}\geq\alpha_0'\,\|\nabla q_h\|^2_{0,\Omega}. \end{cases} \tag{60}$$

As a first consequence of (60), we note that since $(T_0)^2 = I_{H^1_0(\Omega)}$, one has actually $T_0[M_h] = M_h$ for all h. Another by-product of (60) is that $(q_h,q_h')\mapsto(\varepsilon\nabla q_h|\nabla q_h')_{0,\Omega}$ fulfills a *uniform discrete inf-sup condition*, ie.

$$\exists\underline{\gamma}_0>0,\ \forall h,\ \forall q_h\in M_h,\qquad \sup_{q_h'\in M_h\setminus\{0\}}\frac{|(\varepsilon\nabla q_h|\nabla q_h')_{0,\Omega}|}{\|q_h'\|_{H^1_0(\Omega)}}\geq\underline{\gamma}_0\,\|q_h\|_{H^1_0(\Omega)}. \tag{61}$$

So, the discrete scalar problems (23) are well-posed, and the classical error estimate holds: $\|s - s_h\|_{H^1_0(\Omega)} \lesssim \inf_{q_h \in M_h} \|s - q_h\|_{H^1_0(\Omega)}$. See Theorem 2 in [19] for details.
On the other hand, when one has at hand a "weak" T-coercivity operator T, because of the presence of the partition of unity function, one builds "weak" discrete T-coercivity operators (see Lemma 3 in [8]), that is discrete operators $(T_h)_h$ such that

$$\exists C, \mathtt{h}_0 > 0,\ \forall h \le \mathtt{h}_0,\ \exists T_h \in \mathcal{L}(M_h), \qquad \sup_{q \in M_h \setminus \{0\}} \frac{\|\nabla(T - T_h) q_h\|_{0,\Omega}}{\|\nabla q_h\|_{0,\Omega}} \le C\, h.$$

Obviously, $\sup_h \|T_h\|_{\mathcal{L}(M_h)} < \infty$. We call this situation the *"weak" T-coercivity framework*. It follows that one has a *"weak"* discrete T-coercivity property (pp. 820-821 in [8]):

$$\exists \underline{\alpha}, \underline{\beta}, \mathtt{h}_0 > 0,\ \forall h \le \mathtt{h}_0, \quad \forall q_h \in M_h,\ (\varepsilon \nabla q_h | \nabla (T_h q_h))_{0,\Omega} \ge \underline{\alpha}\, \|\nabla q_h\|^2_{0,\Omega} - \underline{\beta}\, \|q_h\|^2_{0,\Omega}. \tag{62}$$

Then, thanks to Proposition 3 in [19] where one argues by contradiction[3], one can prove that $(q_h, q'_h) \mapsto (\varepsilon \nabla q_h | \nabla q'_h)_{0,\Omega}$ fulfills a *uniform discrete inf-sup condition, for h small enough*, ie.

$$\exists \underline{\gamma}_0, \mathtt{h}_0 > 0,\ \forall h \le \mathtt{h}_0,\ \forall q_h \in M_h, \qquad \sup_{q'_h \in M_h \setminus \{0\}} \frac{|(\varepsilon \nabla q_h | \nabla q'_h)_{0,\Omega}|}{\|q'_h\|_{H^1_0(\Omega)}} \ge \underline{\gamma}_0\, \|q_h\|_{H^1_0(\Omega)}. \tag{63}$$

So, one can derive results for the discrete scalar problems (23) that are similar to those that where obtained when a "full" T-coercivity operator was available, *now for h small enough*, that is when $h \le \mathtt{h}_0$.
Finally, when the interface is smooth, the same guidelines apply, see §2.B.1 in [15]. In this case, one needs to have at hand some curvilinear finite elements, such as isoparametric finite elements (cf. §4.3 in [20]), near the interface. It is known that optimal interpolation properties hold, ie. one may recover up to $O(h)$ accuracy using Lagrange's first-order finite elements for a sufficiently smooth scalar field. Or, one can choose the approach of [42] to achieve again optimal convergence rate: for that one needs a family of simplicial meshes which resolve the smooth interface sufficiently well. Observe that for first-order edge finite elements, the latter approach can also be used, to yield $O(h)$ interpolation accuracy for a sufficiently smooth vector field of $\boldsymbol{H}(\mathbf{curl}; \Omega)$ (see [39]).

B The div-curl problem

The general div-curl problem is expressed as

$$\begin{cases} \textit{Find } \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \textit{ such that} \\ \mathbf{curl}\, \boldsymbol{u} = \boldsymbol{f} \text{ and } \operatorname{div} \varepsilon \boldsymbol{u} = g \text{ in } \Omega. \end{cases} \tag{64}$$

In the classical case, according to Theorem 6.1.4 in [3] ($\partial\Omega$ is connected):

$$\boldsymbol{v} \mapsto (\mathbf{curl}\, \boldsymbol{v}, \operatorname{div} \varepsilon \boldsymbol{v})$$

is a bijective mapping from $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$ to $\boldsymbol{H}^\Sigma_0(\operatorname{div} 0; \Omega) \times H^{-1}(\Omega)$.
Hence, to ensure well-posedness of the div-curl model in the classical case, the source terms must be chosen such that

$$\boldsymbol{f} \in \boldsymbol{H}^\Sigma_0(\operatorname{div} 0; \Omega), \quad g \in H^{-1}(\Omega). \tag{65}$$

[3] In theorem 5 in section 6, we proceed similarly to derive a uniform discrete inf-sup condition for the form a_ω. A proof is given there. Note that because we argue by contradiction, bounds are not explicit anymore.

We keep this choice for the div-curl model in the interface case, and use below the operator T_0 introduced in (11). Let

$$\mathbb{V} := \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega) \text{ endowed with } \|(\boldsymbol{v},q)\|_{\mathbb{V}} := (\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|q\|^2_{H^1_0(\Omega)})^{1/2};$$

$$a((\boldsymbol{u},p),(\boldsymbol{v},q)) := (\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\varepsilon\boldsymbol{u}|\nabla q)_{0,\Omega} + (\varepsilon\boldsymbol{v}|\nabla p)_{0,\Omega},\ \forall(\boldsymbol{u},p),(\boldsymbol{v},q) \in \mathbb{V}.$$

We check below that the equivalent variational formulation of problem (64) writes

$$\begin{cases} \textit{Find } (\boldsymbol{u},p) \in \mathbb{V} \textit{ such that} \\ a((\boldsymbol{u},p),(\boldsymbol{v},q)) = (\boldsymbol{f}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \langle g,q\rangle_{H^1_0(\Omega)},\ \forall(\boldsymbol{v},q) \in \mathbb{V}. \end{cases} \tag{66}$$

In (66), the left-hand side defines a continuous bilinear form on $\mathbb{V}$, and the right-hand side defines a continuous linear form on the same function space. The norm of the right-hand side is bounded from above by $\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{-1,\Omega}$.

Lemma 1 *Let $\boldsymbol{f} \in \boldsymbol{H}^\Sigma_0(\mathrm{div}\,0;\Omega)$ and $g \in H^{-1}(\Omega)$ be given. Then if $(\boldsymbol{u},p)$ is a solution to the variational formulation (66), it holds that $p = 0$.*

Proof Choose the test function $(\nabla(T_0p),0)$ in (66). This yields $(\varepsilon\nabla(T_0p)|\nabla p)_{0,\Omega} = 0$. Recall that ε is a symmetric tensor field, so one has $\alpha_0\|\nabla p\|^2_{0,\Omega} = 0$ according to (11), and it follows that $p = 0$. □

Next, one has the classical result, see eg. §6.1.2 in [3].

Proposition 14 *Let $\boldsymbol{f} \in \boldsymbol{H}^\Sigma_0(\mathrm{div}\,0;\Omega)$ and $g \in H^{-1}(\Omega)$ be given. Then it holds that $\boldsymbol{u}$ is a solution to the div-curl problem (64) if, and only if, $(\boldsymbol{u},0)$ is a solution to the variational formulation (66).*

Theorem 7 *The form a is T-coercive.*

Proof Let $(\boldsymbol{u},p) \in \mathbb{V}$ be given. Let us decompose $\boldsymbol{u}$ using (15): $\boldsymbol{u} = \nabla p_{\boldsymbol{u}} + \boldsymbol{k}_{\boldsymbol{u}}$ with $(p_{\boldsymbol{u}},\boldsymbol{k}_{\boldsymbol{u}}) := (\pi_1\boldsymbol{u},\boldsymbol{\pi}_2\boldsymbol{u}) \in H^1_0(\Omega) \times \boldsymbol{K}_N(\Omega,\varepsilon)$.
(i) Assume first that $\boldsymbol{u} = 0$. Choosing $(\boldsymbol{v}^\star,q^\star) = (\nabla(T_0p),0)$ yields

$$a((0,p),(\boldsymbol{v}^\star,q^\star)) = (\varepsilon\nabla(T_0p)|\nabla p)_{0,\Omega} \geq \alpha_0\,\|\nabla p\|^2_{0,\Omega} = \alpha_0\,\|(0,p)\|^2_{\mathbb{V}}.$$

(ii) Consider next that $p = 0$. Because $\boldsymbol{k}_{\boldsymbol{u}} \in \boldsymbol{K}_N(\Omega,\varepsilon)$ with $\mathbf{curl}\,\boldsymbol{k}_{\boldsymbol{u}} = \mathbf{curl}\,\boldsymbol{u}$, one has

$$a((\boldsymbol{u},0),(\boldsymbol{v},q)) = (\mathbf{curl}\,\boldsymbol{k}_{\boldsymbol{u}}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\varepsilon\nabla p_{\boldsymbol{u}}|\nabla q)_{0,\Omega}.$$

One chooses in this case $(\boldsymbol{v}^\star,q^\star) = (\boldsymbol{k}_{\boldsymbol{u}},T_0p_{\boldsymbol{u}})$. Indeed with the help of (11) and (18)

$$\begin{aligned} a((\boldsymbol{u},0),(\boldsymbol{v}^\star,q^\star)) &= \|\,\mathbf{curl}\,\boldsymbol{k}_{\boldsymbol{u}}\|^2_{0,\Omega} + (\varepsilon\nabla p_{\boldsymbol{u}}|\nabla(T_0p_{\boldsymbol{u}}))_{0,\Omega} \\ &\geq (C'_W)^{-2}\|\boldsymbol{k}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \alpha_0\,\|\nabla p_{\boldsymbol{u}}\|^2_{0,\Omega} \\ &\geq \min((C'_W)^{-2},\alpha_0)\left(\|\boldsymbol{k}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\nabla p_{\boldsymbol{u}}\|^2_{0,\Omega}\right) \\ &\geq \gamma\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \gamma\,\|(\boldsymbol{u},0)\|^2_{\mathbb{V}}, \end{aligned}$$

where $\gamma := \frac{1}{2}\min((C'_W)^{-2},\alpha_0) > 0$.
(iii) In the general case, let us consider a "linear combination" of the above, eg. $(\boldsymbol{v}^\star,q^\star) = (\nabla(T_0p) + \boldsymbol{k}_{\boldsymbol{u}},T_0p_{\boldsymbol{u}})$. Then one finds

$$\begin{aligned} a((\boldsymbol{u},p),(\boldsymbol{v}^\star,q^\star)) &= \|\,\mathbf{curl}\,\boldsymbol{k}_{\boldsymbol{u}}\|^2_{0,\Omega} + (\varepsilon\nabla p_{\boldsymbol{u}}|\nabla(T_0p_{\boldsymbol{u}}))_{0,\Omega} + (\varepsilon\nabla(T_0p)|\nabla p)_{0,\Omega} \\ &\geq (C'_W)^{-2}\|\boldsymbol{k}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \alpha_0\,\|\nabla p_{\boldsymbol{u}}\|^2_{0,\Omega} + \alpha_0\,\|\nabla p\|^2_{0,\Omega} \\ &\geq \gamma\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \alpha_0\,\|\nabla p\|^2_{0,\Omega} \\ &\geq \gamma\,\|(\boldsymbol{u},p)\|^2_{\mathbb{V}}, \end{aligned}$$

because $\gamma < \alpha_0$. To conclude the proof, remark that $\mathbb{T} : (\boldsymbol{u},p) \mapsto (\nabla(T_0p) + \boldsymbol{\pi}_2\boldsymbol{u}, T_0(\pi_1\boldsymbol{u}))$ belongs to $\mathcal{L}(\mathbb{V})$. □

Remark 6 In the above proof, $\mathbb{T}$ is an involution, when T_0 is one too: $\mathbb{T}^2 = I_{\mathbb{V}}$.

Corollary 3 *Let* $\boldsymbol{f} \in \boldsymbol{H}_0^{\Sigma}(\mathrm{div}\, 0; \Omega)$, $g \in H^{-1}(\Omega)$ *be given. Then there exists one, and only one, solution to* $(\boldsymbol{u}, p)$ *to (66). In addition,* $p = 0$ *and* $\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{-1,\Omega}$.

On can proceed similarly for the div-curlcurl problem, see [23].