



**HAL**  
open science

## Powers of paths in tournaments

Nemanja Draganić, François Dross, Jacob Fox, António Girão, Frédéric Havet,  
Dániel Korándi, William Lochet, David Munhá Correia, Alex Scott, Benny  
Sudakov

► **To cite this version:**

Nemanja Draganić, François Dross, Jacob Fox, António Girão, Frédéric Havet, et al.. Powers of paths in tournaments. *Combinatorics, Probability and Computing*, 2021, pp.1-5. 10.1017/S0963548321000067 . hal-03269230

**HAL Id: hal-03269230**

**<https://inria.hal.science/hal-03269230v1>**

Submitted on 10 Nov 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Powers of paths in tournaments

Nemanja Draganić\*    François Dross†    Jacob Fox‡    António Girão§  
Frédéric Havet¶    Dániel Korándi||\*\*    William Lochet††    David Munhá Correia\*  
Alex Scott||    Benny Sudakov\*

## Abstract

In this short note we prove that every tournament contains the  $k$ -th power of a directed path of linear length. This improves upon recent results of Yuster and of Girão. We also give a complete solution for this problem when  $k = 2$ , showing that there is always a square of a directed path of length  $\lceil 2n/3 \rceil - 1$ , which is best possible.

## 1 Introduction

One of the main themes in extremal graph theory is the study of embedding long paths and cycles in graphs. Some of the classical examples include the Erdős–Gallai theorem [?] that every  $n$ -vertex graph with average degree  $d$  contains a path of length  $d$ , and Dirac’s theorem [?] that every graph with minimum degree  $n/2$  contains a Hamilton cycle. A famous generalization of this, conjectured by Pósa and Seymour, and proved for large  $n$  by Komlós, Sárközy and Szemerédi [?], asserts that if the minimum degree is at least  $kn/(k + 1)$ , then the graph contains the  $k$ -th power of a Hamilton cycle.

In this note, we are interested in embedding directed graphs in a tournament. A tournament is an oriented complete graph. The  $k$ -th power of the directed path  $\vec{P}_\ell = v_0 \dots v_\ell$  of length  $\ell$  is the graph  $\vec{P}_\ell^k$  on the same vertex set containing a directed edge  $v_i v_j$  if and only if  $i < j \leq i + k$ . The  $k$ -th power of a directed cycle is defined analogously. An old result of Bollobás and Häggkvist [?] says that, for large  $n$ , every  $n$ -vertex tournament with all indegrees and outdegrees at least  $(1/4 + \varepsilon)n$  contains the  $k$ -th power of a Hamilton cycle

---

\*Department of Mathematics, ETH Zurich, Zurich, Switzerland. Emails: {nemanja.draganic, david.munhacanascorreia, benjamin.sudakov}@math.ethz.ch. Research supported in part by SNSF grant 200021\_196965

†Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400, Talence, France. Email: francois.dross@u-bordeaux.fr. Research supported in part by ERC grant No 714704.

‡Department of Mathematics, Stanford University, Stanford, CA, USA. Email: jacobfox@stanford.edu. Research supported by a Packard Fellowship and by NSF award DMS-185563.

§E-mail: tzigirao@gmail.com.

¶CNRS, Université Côte d’Azur, I3S, INRIA, Sophia Antipolis, France. Email: frederic.havet@inria.fr. Research supported in part by Agence Nationale de la Recherche under contract Digraphs ANR-19-CE48-0013-01.

||Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, United Kingdom. Emails: {korandi, scott}@maths.ox.ac.uk.

\*\*Supported by SNSF Postdoc.Mobility Fellowship P400P2\_186686.

††Department of Informatics, University of Bergen, Norway. Email: william.lochet@uib.no. Research supported by ERC grant No 819416.

(the constant  $1/4$  is optimal). However, we cannot expect to find powers of directed cycles in general, as the transitive tournament contains no cycles at all.

What about powers of directed paths? A classical result, which appears in every graph theory book (see, e.g., [?]), says that every tournament contains a directed Hamilton path. On the other hand, Yuster [?] recently observed that some tournaments are quite far from containing the square of a Hamilton path. In particular, there is an  $n$ -vertex tournament that does not even contain the square of  $\vec{P}_{2n/3}$ , and more generally, for every  $k \geq 2$ , there are tournaments with  $n$  vertices and no  $k$ -th power of a path with more than  $nk/2^{k/2}$  vertices. In the other direction, Yuster proved that every tournament with  $n$  vertices contains the square of a path of length  $n^{0.295}$ . This was improved very recently by Girão [?], who showed that for fixed  $k$ , every tournament on  $n$  vertices contains the  $k$ -th power of a path of length  $n^{1-o(1)}$ . Both papers noted that no sublinear upper bound is known. Our main result shows that the maximum length is in fact linear in  $n$ .

**Theorem 1.** *For  $n \geq 2$ , every  $n$ -vertex tournament contains the  $k$ -th power of a directed path of length  $n/2^{4k+6k}$ .*

The proof of this theorem combines Kővári–Sós–Turán style arguments, used for the bipartite Turán problem, and median orderings of tournaments. A median ordering is a vertex ordering that maximizes the number of forward edges. ?? and Yuster’s construction show that an optimal bound on the length has the form  $n/2^{\Theta(k)}$ . It would be interesting to find the exact value of the constant factor in the exponent. Optimizing our proof can yield a lower bound of  $n/2^{ck+o(k)}$  with  $c \approx 3.9$ , but is unlikely to give the correct bound.

We also improve the exponential constant in the upper bound from  $1/2$  to  $1$ .

**Theorem 2.** *Let  $k \geq 5$  and  $n \geq k(k+1)2^k$ . There is an  $n$ -vertex tournament that does not contain the  $k$ -th power of a directed path of length  $k(k+1)n/2^k$ .*

Note that this theorem also holds trivially for  $k \leq 4$ , when  $k(k+1)n/2^k > n$ .

Finally, we can solve the problem completely in the special case of  $k = 2$ . Once again, the proof uses certain properties of median orderings.

**Theorem 3.** *For  $n \geq 1$ , every  $n$ -vertex tournament contains the square of a directed path of length  $\ell = \lceil 2n/3 \rceil - 1$ , but not necessarily of length  $\ell + 1$ .*

Theorems ??, ?? and ?? are proved in Sections ??, ?? and ??, respectively.

## 2 Lower bound

We will need the following Kővári–Sós–Turán style lemma.

**Lemma 4.** *Let  $G$  be a directed graph with disjoint vertex subsets  $A$  and  $B$  with  $|A| = 2k + 1$ ,  $|B| \geq 2^{4k+4}k$ , and every vertex in  $A$  has at least  $(1 - \frac{1}{2k+1})|B|/2$  outneighbours in  $B$ . Then  $A$  contains a subset  $A'$  of size  $k$  that has at least  $(2k + 1)2^{2k}$  common outneighbours in  $B$ .*

*Proof.* Suppose there is no such set  $A'$ . Then every  $k$ -subset of  $A$  appears in the inneighbourhood of less than  $(2k+1)2^{2k}$  vertices in  $B$ . So if  $d^-(v)$  denotes the number of inneighbours a vertex  $v \in B$  has in  $A$ , then we have

$$\binom{2k+1}{k} \cdot (2k+1)2^{2k} = \binom{|A|}{k} \cdot (2k+1)2^{2k} > \sum_{v \in B} \binom{d^-(v)}{k}. \quad (1)$$

On the other hand,  $\sum_{v \in B} d^-(v) \geq |A|(1 - \frac{1}{2k+1})|B|/2 = k|B|$ . By Jensen's inequality,  $\sum_{v \in B} \binom{d^-(v)}{k} \geq |B| \cdot \left(\sum_{v \in B} \frac{d^-(v)}{|B|}\right)^k = |B| \geq 2^{4k+4}k$ . This contradicts (??).  $\square$

One more ingredient we need for the proof of ?? is the folklore fact that every tournament on  $2^m$  vertices contains a transitive subtournament of size  $m+1$ . This is easily seen by taking a vertex of outdegree at least  $2^{m-1}$  as the first vertex of the subtournament, and then recursing on the outneighbourhood.

*Proof of ??.* Order the vertices as  $0, 1, \dots, n-1$  to maximize the number of forward edges, i.e., the number of edges  $ij$  such that  $i < j$ . As was mentioned in the introduction, we will refer to such a sequence as a *median ordering* of the vertices. We denote an ‘‘interval’’ of vertices with respect to this ordering by  $[i, j) = \{i, \dots, j-1\}$ , where  $0 \leq i < j \leq n$ .

We will embed  $\vec{P}_\ell^k$  inductively using the following claim.

**Claim.** Let  $t = 2^{4k+4}k$  and  $t \leq i \leq n - (2k+1)t$ . For every subset  $A^* \subseteq [i-t, i)$  of size  $2^{2k}$ , there is an index  $i+t \leq j \leq i+(2k+1)t$  and a set  $A' \subseteq A^*$  of size  $k$  such that  $A'$  induces a transitive tournament and its vertices have at least  $2^{2k}$  common outneighbours in  $[j-t, j)$ .

*Proof.* There is a subset  $A \subseteq A^*$  of size  $2k+1$  that induces a transitive tournament. Let  $B = [i, i+(2k+1)t)$ . Then every vertex  $v \in A$  has at least  $kt = \left(1 - \frac{1}{2k+1}\right)|B|/2$  outneighbours in  $B$ . Indeed, otherwise  $v$  would have more than  $(k+1)t$  inneighbours in the interval  $B$ , so moving  $v$  to the end of this interval would increase the number of forward edges in the ordering, contradicting our choice of the vertex ordering.

We can thus apply ?? to find a  $k$ -subset  $A' \subseteq A$  with least  $(2k+1)2^{2k}$  common outneighbours in  $B$ . Partition  $B$  into  $2k+1$  intervals of size  $t$ , and we can choose  $j$  accordingly so that  $A'$  has at least  $2^{2k}$  common outneighbours in the interval  $[j-t, j)$ .  $\square$

The theorem trivially holds for  $n < 2^{2k}$ , so assume  $n \geq 2^{2k}$ . Let  $i_0 = 2^{2k}$  and  $A_0 = [0, 2^{2k})$ , and apply the Claim with  $i = i_0$  and  $A^* = A_0$ . We get a set  $A' \subset A_0$  of size  $k$  that induces a transitive tournament, i.e., the  $k$ -th power of some path  $v_0 \dots v_{k-1}$ . Moreover, this  $A'$  has at least  $2^{2k}$  common outneighbours in some interval  $[j-t, j)$  with  $i_0 + t \leq j \leq i_0 + (2k+1)t$ . Let us define  $i_1 = j$ , and choose  $A_1$  to be any  $2^{2k}$  of the common outneighbours.

At step  $s$ , we apply the Claim again with  $i = i_s$  and  $A^* = A_s$  to find the  $k$ -th power of some path  $v_{sk} \dots v_{(s+1)k-1}$  in  $A_s$  with  $2^{2k}$  common outneighbours in some  $[i_{s+1}-t, i_{s+1})$  with  $i_s + t \leq i_{s+1} \leq i_s + (2k+1)t$ , and repeat this process until some step  $\ell$  with  $i_\ell > n - (2k+1)t$ . Note that intervals  $[i_s - t, i_s)$  and  $[i_{s+1} - t, i_{s+1})$  are always disjoint. Finally,  $A_\ell$  must also contain a transitive tournament of size  $2k+1$ . Call these vertices  $v_{\ell k}, \dots, v_{(\ell+2)k}$ . Observe that  $n - (2k+1)t < i_\ell \leq 2^{2k} + \ell(2k+1)t$ , so  $n < (\ell+2)(2k+1)t$ .

Then  $v_0 \dots v_{(\ell+2)k}$  is a directed path of length  $(\ell+2)k \geq kn/(2k+1)t \geq n/(2^{4k+6}k)$  whose  $k$ -th power is contained in the tournament. In fact, we proved a bit more: the tournament contains all edges of the form  $v_a v_b$  with  $a < b$  and  $\lfloor a/k \rfloor + 1 \geq \lfloor b/k \rfloor$ .  $\square$

### 3 Upper bound

Let  $\ell_k(n)$  denote the smallest integer  $\ell$  such that there is an  $n$ -vertex tournament that does not contain  $\vec{P}_\ell^k$ , or in other words, the largest integer such that every  $n$ -vertex tournament contains the  $k$ -th power of a directed path on  $\ell$  vertices.

To prove ??, we first note that  $\ell_k(n)$  is subadditive.

**Lemma 5.** *For any  $k, n, m \geq 1$ , we have  $\ell_k(n+m) \leq \ell_k(n) + \ell_k(m)$ .*

*Proof.* Let  $T_1$  and  $T_2$  be extremal tournaments on  $n$  and  $m$  vertices, respectively, not containing the  $k$ -th power of any directed path of length  $\ell_k(n)$  and  $\ell_k(m)$ . Let  $T$  be the tournament on  $n+m$  vertices, obtained from the disjoint union of  $T_1$  and  $T_2$  by adding all remaining edges directed from  $T_1$  to  $T_2$ . Then any  $k$ -th power of a path in  $T$  must be the concatenation of the  $k$ -th power of a path in  $T_1$  and the  $k$ -th power of a path in  $T_2$ , and hence it must have length at most  $(\ell_k(n) - 1) + (\ell_k(m) - 1) + 1 < \ell_k(n) + \ell_k(m)$ .  $\square$

Our improved upper bound is based on the following construction.

**Lemma 6.** *For every  $k \geq 5$ , we have  $\ell_k(2^{k-1}) < \frac{k(k+1)}{2}$ .*

*Proof.* Let  $n = 2^{k-1}$  and  $\ell = \frac{k(k+1)}{2}$ , and note that  $\vec{P}_{\ell-1}^k$  has  $k\ell - \ell$  edges.

Let  $T$  be a random  $n$ -vertex tournament obtained by orienting the edges of  $K_n$  independently and uniformly at random. The probability that a fixed sequence of  $\ell$  vertices  $v_0 \dots v_{\ell-1}$  forms a copy of  $\vec{P}_{\ell-1}^k$  is  $2^{-(k-1)\ell}$ . There are  $\binom{n}{\ell} \cdot \ell!$  such sequences, so the probability that  $T$  contains the  $k$ -th power of a path of length  $\ell - 1$  is at most  $\binom{n}{\ell} \cdot \ell! \cdot 2^{-(k-1)\ell} < n^\ell \cdot 2^{-(k-1)\ell} = 1$ . So with positive probability  $T$  does not contain  $\vec{P}_{\ell-1}^k$ , therefore  $\ell_k(2^{k-1}) \leq \ell - 1$ .  $\square$

Combining ???? and using the monotonicity of  $\ell_k(n)$ , we get

$$\ell_k(n) \leq \left\lceil \frac{n}{2^{k-1}} \right\rceil \cdot \ell_k(2^{k-1}) \leq \left( \frac{n}{2^{k-1}} + 1 \right) \left( \frac{k(k+1)}{2} - 1 \right) \leq \frac{k(k+1)n}{2^k}$$

for  $n \geq k(k+1)2^k$ , establishing ??.

### 4 The square of a path

*Proof of ??.* Recall that  $\ell_2(n)$  is the largest integer such that every  $n$ -vertex tournament contains the square of a path on  $\ell$  vertices. Proving ?? is therefore equivalent to showing  $\ell_2(n) = \lceil 2n/3 \rceil$  for every  $n \geq 1$ .

It is easy to check that  $\ell_2(1) = 1$  and  $\ell_2(2) = \ell_2(3) = 2$ , so  $\ell_2(n) \leq \lceil 2n/3 \rceil$  follows from ?? by induction, as  $\ell_2(n) \leq \ell_2(n-3) + \ell_2(3) = \ell_2(n-3) + 2$  holds for every  $n > 3$ . For the lower bound we need to take a closer look at median orderings.

**Claim.** Every median ordering  $x_1, \dots, x_n$  of a tournament has the following properties:

- (a) All edges of the form  $x_i x_{i+1}$  are in the tournament.
- (b) If  $x_i x_{i-2}$  is an edge of the tournament, then “rotating”  $x_{i-2} x_{i-1} x_i$  gives two other median orderings  $x_1, \dots, x_{i-3}, x_{i-1}, x_i, x_{i-2}, x_{i+1}, \dots, x_n$  and  $x_1, \dots, x_{i-3}, x_i, x_{i-2}, x_{i-1}, x_{i+1}, \dots, x_n$ .
- (c) If  $x_i x_{i-2}$  is an edge of the tournament, then each of  $x_{i-2}, x_{i-1}, x_i$  is an inneighbour of  $x_{i+1}$ , and at most one of them is an outneighbour of  $x_{i+2}$ .

*Proof.* Property (??) holds, as otherwise we could swap  $x_i$  and  $x_{i+1}$  to get an ordering with more forward edges, contradicting our assumption. Property (??) holds because rotating  $x_{i-2} x_{i-1} x_i$  has no effect on the number of forward edges.

These two properties together imply that each of  $x_{i-2}, x_{i-1}, x_i$  is an inneighbour of  $x_{i+1}$ . Suppose, to the contrary of (??), that two of them are outneighbours of  $x_{i+2}$ . By rotating  $x_{i-2} x_{i-1} x_i$  if needed, we may assume that these are  $x_{i-1}$  and  $x_i$ . But then we can also rotate  $x_i x_{i+1} x_{i+2}$  so that  $x_{i+2}$  comes right after  $x_{i-1}$  in a median ordering. This contradicts (??).  $\square$

Let us now say that  $i$  is a *bad index* in a median ordering  $x_1, \dots, x_n$  if  $x_i x_{i-2}$  is an edge, and at least one of  $x_{i+2} x_i$  and  $x_{i+2} x_{i-1}$  is also an edge.

**Lemma 7.** *Every tournament has a median ordering without any bad indices.*

*Proof.* Suppose this fails to hold for some tournament, and take a median ordering  $x_1, \dots, x_n$  that minimizes the largest bad index  $i$ . As  $i$  is a bad index,  $x_i x_{i-2}$  is an edge, and  $x_i$  or  $x_{i-1}$  is an outneighbour of  $x_{i+2}$ . By (??),  $x_{i-2} x_{i-1} x_i$  can be rotated so that  $x_{i+2} x'_{i-2}$  is an edge in the new median ordering  $x_1, \dots, x_{i-3}, x'_{i-2}, x'_{i-1}, x'_i, x_{i+1}, \dots, x_n$ . Then neither  $x_{i+2} x'_i$  nor  $x_{i+2} x'_{i-1}$  is an edge, since by (??), only one of  $x'_{i-2}, x'_{i-1}, x'_i$  is an outneighbour of  $x_{i+2}$ . Also by (??),  $x'_{i-1} x_{i+1}$  and  $x'_i x_{i+1}$  are edges, so both of  $x_{i+1}$  and  $x_{i+2}$  are outneighbours of  $x'_{i-1}$  and  $x'_i$ . This means that none of  $i, i+1, i+2$  is a bad index in this new ordering, and hence the largest bad index is smaller than  $i$ . This is a contradiction.  $\square$

Now we are ready to prove  $\ell_2(n) \geq \lceil 2n/3 \rceil$ . Take an  $n$ -vertex tournament with median ordering  $x_1, \dots, x_n$  as in ??, and let  $I = \{i_1 < i_2 < \dots < i_k\}$  be the set of indices  $i$  such that  $x_i x_{i-2}$  is not an edge (in particular,  $i_1 = 1$  and  $i_2 = 2$ ). We claim that  $x_{i_1} \dots x_{i_k}$  is a directed path on  $k \geq \lceil 2n/3 \rceil$  vertices whose square is contained in the tournament.

To see this, first observe that if the index  $i+2$  is not in  $I$ , then both  $i$  and  $i+1$  are in  $I$ . Indeed, if  $x_{i+2} x_i$  is an edge, then  $x_{i+1} x_{i-1}$  cannot be one because of (??), and  $x_i x_{i-2}$  cannot be one because  $i$  is not a bad index. This immediately implies  $k \geq \lceil 2n/3 \rceil$ .

It remains to check that  $x_{i_j-2} x_{i_j}$  and  $x_{i_j-1} x_{i_j}$  are all edges in the tournament. By the above observation, we know that  $i_j - 3 \leq i_{j-2} < i_{j-1} < i_j$ . Here  $x_{i_j-1} x_{i_j}$  is an edge by (??), and  $x_{i_j-2} x_{i_j}$  is an edge by the definition of  $I$ . So the only case left is to show that  $x_{i_j-2} x_{i_j}$  is an edge when  $i_{j-2} = i_j - 3$ .

In this case there is an index  $i_j - 3 < i < i_j$  that is not in  $I$ , i.e.,  $x_i x_{i-2}$  is an edge in the tournament. But then if  $i = i_j - 1$ , then  $x_{i_j-2} x_{i_j}$  is an edge because of (??), while otherwise  $i = i_j - 2$ , and  $x_{i_j-2} x_{i_j}$  is an edge because  $i$  is not a bad index. This concludes our proof.  $\square$

## References

- [1] B. Bollobás and R. Häggkvist, Powers of Hamilton cycles in tournaments, *J. Combin. Theory Ser. B* 50 (1990), 309–318.
- [2] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* 2 (1952), 69–81.
- [3] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Hungar.*, 10 (1959), 337–356.
- [4] A. Girão, A note on long powers of paths in tournaments, *arXiv:2010.02875* preprint.
- [5] J. Komlós, G.N. Sárközy and E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Comb.* 2 (1998), 43–60.
- [6] R. Yuster, Paths with many shortcuts in tournaments, *Discrete Math.* 334 (2021), 112168.
- [7] D. West, **Introduction to Graph Theory**, 2nd ed., Prentice Hall, 2001.