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The Central Valuations Monad

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Abstract

We give a commutative valuations monad \mathcal{Z} on the category **DCPO** of dcpo's and Scott-continuous functions. Compared to the commutative valuations monads given in [2], our new monad \mathcal{Z} is larger and it contains all push-forward images of valuations on the unit interval $[0, 1]$ along lower semi-continuous maps. We believe that this new monad will be useful in giving domain-theoretic denotational semantics for statistical programming languages with continuous probabilistic choice.

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Category Early Ideas

1 Introduction

The valuations monad \mathcal{V} on the category **DCPO** of dcpo's and Scott-continuous functions is a staple of the domain-theoretic approach for denotational semantics of programming languages with probabilistic choice and recursion [3, 4]. For a dcpo D , $\mathcal{V}D$ consists of *subprobability valuations* on D , which are the Scott-continuous functions ν from the set σD of Scott open subsets of D to $[0, 1]$ satisfying *strictness* ($\nu(\emptyset) = 0$) and *modularity* ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). The set $\mathcal{V}D$ is a dcpo in the *stochastic order*: $\nu_1 \leq \nu_2$ if and only if $\nu_1(U) \leq \nu_2(U)$ for all $U \in \sigma D$. The *unit* of \mathcal{V} at dcpo D is the map $\eta_D: D \rightarrow \mathcal{V}D :: x \mapsto \delta_x$, where δ_x is the *Dirac valuation* at x , defined by $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ otherwise. For a Scott-continuous map $f: D \rightarrow \mathcal{V}E$, the *Kleisli extension* f^\dagger of f is defined by $f^\dagger(\nu)(U) = \int_{x \in X} f(x)(U) d\nu$ for $\nu \in \mathcal{V}D$ and $U \in \sigma E$. The integral in this definition is a Choquet type integral: for a general Scott-continuous function $h: D \rightarrow [0, 1]$, the value of $\int_{x \in X} h d\nu$ is defined to be the Riemann integral $\int_0^1 \nu(h^{-1}(t, 1)) dt$. Following this, the action of \mathcal{V} on a Scott-continuous function $g: D \rightarrow E$ between dcpo's D and E is $\mathcal{V}(g) \stackrel{\text{def}}{=} (\eta_E \circ g)^\dagger$; concretely, for $\nu \in \mathcal{V}D$ and $U \in \sigma E$, $\mathcal{V}(g)(\nu)(U) = \nu(g^{-1}(U))$. Subprobability valuations on general topological spaces and the corresponding integral of lower semi-continuous functions against subprobability valuations can be defined similarly [3].

While it is well-known that \mathcal{V} can be restricted to a commutative monad on the category **DOM** of domains and Scott-continuous functions, it is unknown whether \mathcal{V} can be restricted to any Cartesian closed full subcategory of **DOM**. This is known as the *Jung-Tix problem* [5].

One may note that the category **DCPO** itself is Cartesian closed and \mathcal{V} is a monad on it. What does one lose if we use the category **DCPO** and monad \mathcal{V} for semantics? A short answer is that compared to **DOM**, \mathcal{V} is not known to be *commutative* over **DCPO**, which is an important property for the denotational semantics of programming languages. Commutativity of \mathcal{V} over **DCPO** is equivalent to showing the following Fubini-style equation



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$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x, y) d\nu d\mu \tag{1}$$

holds for all dcpo's D and E , all Scott-continuous functions $h: D \times E \rightarrow [0, 1]$ and all $\nu \in \mathcal{V}D, \mu \in \mathcal{V}E$. As pointed out in [2], the main difficulty in establishing (1) over **DCPO** is that the Scott topology on the product dcpo $D \times E$ may be different from the product topology $\sigma D \times \sigma E$. Actually, we do know that Equation (1) holds for those functions h that are continuous when $D \times E$ is given the product topology $\sigma D \times \sigma E$ ([4, Lemma 2.37]).

Instead of directly proving (1), we showed (together with Lindenhovius) how to construct three submonads of \mathcal{V} that are commutative on **DCPO**, and used each one to give a sound and (strongly) adequate semantics to PFPC (Probabilistic FixPoint Calculus) [2]. The simplest of those three monads is the monad \mathcal{M} . For each dcpo D , $\mathcal{M}D$ is defined to be the smallest sub-dcpo of $\mathcal{V}D$ that contains $\mathcal{S}D$, the family of *simple valuations* on D , where a simple valuation is a finite convex sum of Dirac valuations. The other two commutative monads are denoted \mathcal{W} and \mathcal{P} and the following inclusions hold for each dcpo $D: \mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{V}D$.

Each of our three monads is large enough to interpret *discrete* probabilistic choice in PFPC [2]. However, it is unclear if any of these monads is large enough to interpret *continuous* probabilistic choice. In this note, we define a new commutative valuations monad \mathcal{Z} on the category **DCPO** which is larger than \mathcal{M}, \mathcal{W} and \mathcal{P} with the hope of addressing this problem.

2 Central Valuations

Our idea for defining \mathcal{Z} is inspired by the notion of centre in group theory (which always forms an abelian subgroup) and the notion of centre of a premonoidal category (which always forms a monoidal subcategory) [6].

► **Definition 1.** *A subprobability valuation ν on a dcpo D is called a central valuation if for any dcpo E , any valuation μ on E , and any Scott-continuous function $h: D \times E \rightarrow [0, 1]$, we have*

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x, y) d\nu d\mu.$$

We shall write $\mathcal{Z}D$ for the set of all central valuations on a dcpo D .

It is easy to see that simple valuations are central, and that the central valuations are closed under directed suprema under the stochastic order. Thus, for each dcpo D , $\mathcal{Z}D$ is a sub-dcpo of $\mathcal{V}D$ containing $\mathcal{S}D$. Moreover, we have the following theorem, which can be proved using the *disintegration formula* in [1].

► **Theorem 2.** *The assignment $\mathcal{Z}(-)$ extends to a commutative monad over the category **DCPO** when equipped with the (co)restricted monad operations of \mathcal{V} . In other words, \mathcal{Z} is a commutative submonad of \mathcal{V} .*

Proof. The unit of \mathcal{Z} at dcpo D sends each $x \in D$ to δ_x which is obviously a central valuation.

Let $f: C \rightarrow \mathcal{Z}D$ be a Scott-continuous function. Then f can also be viewed as a Scott-continuous map from C to $\mathcal{V}D$, since $\mathcal{Z}D$ is a sub-dcpo of $\mathcal{V}D$. We prove that $f^\dagger: \mathcal{V}C \rightarrow \mathcal{V}D$ maps central valuations on C to central valuations on D . Towards this end, we pick μ from $\mathcal{Z}C$, and assume that E is a dcpo, ν is an arbitrary subprobability valuation on E

and $h: D \times E \rightarrow [0, 1]$ is a Scott-continuous map. Then by the disintegration formula (see Lemma 3.1(iii) in [1]) we have that

$$\int_{y \in E} \int_{x \in D} h(x, y) d(f^\dagger(\mu)) d\nu = \int_{y \in E} \int_{t \in C} \int_{x \in D} h(x, y) df(t) d\mu d\nu,$$

and the right side of the equation is equal to

$$\int_{t \in C} \int_{x \in D} \int_{y \in E} h(x, y) d\nu df(t) d\mu$$

by the fact that $f(t), t \in D$ and μ are central valuations. Again, by the disintegration formula that is just $\int_{x \in D} \int_{y \in E} h(x, y) d\nu df^\dagger(\mu)$. Hence we have proved that $f^\dagger(\mu)$ is indeed a central valuation provided that μ is. Similar arguments show that the monadic strength also (co)restricts as required. The corresponding (co)restrictions of the monadic operations of \mathcal{V} to \mathcal{Z} validate that \mathcal{Z} is a strong monad on **DCPO**. The commutativity of \mathcal{Z} , which is equivalent to Equation (1) holding for all dcpo's D and E and central valuations μ and ν on them, is then obvious by definition of \mathcal{Z} . ◀

In fact, it is proved in [2] that all *point-continuous valuations* are central and therefore $SD \subseteq MD \subseteq WD \subseteq PD \subseteq ZD \subseteq VD$ for each dcpo D . Therefore \mathcal{Z} is the largest commutative submonad of \mathcal{V} known so far. Furthermore, observe that $\mathcal{Z} = \mathcal{V}$ iff \mathcal{V} is a commutative monad on **DCPO**. The latter has been an open problem since 1989, and our simple observation leads us to believe \mathcal{Z} is a very large commutative submonad of \mathcal{V} .

It is not difficult to see that, in order to model sampling against continuous probability distributions on the interval $[0, 1]$, the monad used for the semantics should at least contain the push-forward images of the Lebesgue valuation on $[0, 1]$ (equipped with the metric topology) along lower semi-continuous maps. We can demonstrate even more is true of our new monad \mathcal{Z} (see Theorem 4 below). For this, let us first recall that a space X is called *core-compact* if the set $\mathcal{O}X$ of all open subsets of X is a continuous lattice in the inclusion order. Equivalently, X is core-compact if and only if for each open subset U of X and $x \in U$, there exists an open subset V such that $x \in V \ll U$, where $V \ll U$ means that V is *way-below* U in the sense of domain theory. Many important spaces are core-compact. For example, each locally compact space is core-compact, and in particular, the unit interval with the usual topology is compact Hausdorff, hence locally compact hence core-compact.

► **Lemma 3.** *Let X be a core-compact topological space. Let D, E be dcpos, and $f: X \rightarrow D$ a lower semi-continuous map, i.e., f is continuous when D is equipped with the Scott topology. Then the map $f \times \text{id}_E: X \times \Sigma E \rightarrow D \times E$ is also lower semi-continuous, where ΣE denotes the topological space $(E, \sigma E)$ and $X \times \Sigma E$ is the topological product of X and ΣE .*

Proof. First, we assume that X is core-compact and prove that $f \times \text{id}_E$ is lower semi-continuous. Towards this end, we pick a Scott open subset O of $D \times E$, and assume that $f \times \text{id}_E((x_0, e_0)) \in O$, that is $(f(x_0), e_0) \in O$. We must find an open neighbourhood U of x_0 in X and a Scott open neighbourhood V of e_0 in E such that $f \times \text{id}_E(U \times V) \subseteq O$. We let $A = \{x \in X \mid (f(x), e_0) \in O\}$. Then A is just $f^{-1}(O_{e_0})$, where $O_{e_0} = \{d \in D \mid (d, e_0) \in O\}$. Since $f: X \rightarrow D$ is lower semi-continuous and O_{e_0} is Scott open in D , we know that A is an open neighbourhood of x_0 in X . Now the core-compactness of X enables us to find, in X , an open subset U and a sequence of open subsets $U_i, i \in \mathbb{N}$ such that $x_0 \in U \ll \dots \ll U_n \dots \ll U_1 \ll A$. For each $U_n, n \in \mathbb{N}$, we define $V_n = \{e \mid f(x, e) \in O \text{ for all } x \in U_n\}$ and let $V = \bigcup_{n \in \mathbb{N}} V_n$. Since for each $n \in \mathbb{N}$, $U_n \subseteq A$, we have for all $x \in U_n$, $(f(x), e_0) \in O$.

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Thus we know that $e_0 \in V_n$ for each $n \in \mathbb{N}$, and hence $e_0 \in V$. Moreover, for any $(x, e) \in U \times V$, there exists a natural number n such that $e \in V_n$, then it follows that $f \times \text{id}_E((x, e)) = (f(x), e) \in f(U) \times V_n \subseteq f(U_n) \times V_n \subseteq O$. The last inclusion is due to the construction of V_n . To sum up, it is true that $f \times \text{id}_E(U \times V) \subseteq O$. Since U is an open subset of X which contains x_0 and $e_0 \in V$, we finish the proof by showing that V is Scott open in E . To this end we let $\{e_i\}_{i \in I}$ be a directed subset of E with $\sup_{i \in I} e_i \in V$. For each $i \in I$, set $W_i = \{x \in X \mid (f(x), e_i) \in O\}$. It is easy to see that $\{W_i \mid i \in I\}$ is a directed family of open subsets of X . Since $\sup_{i \in I} e_i \in V = \bigcup_{n \in \mathbb{N}} V_n$, $\sup_{i \in I} e_i$ is in some V_n . This means that for each $x \in U_n$, $(f(x), \sup_{i \in I} e_i) \in O$. Because O is Scott open, for each $x \in U_n$, there exists $i \in I$ such that $(f(x), e_i) \in O$, i.e., $x \in W_i$. Hence we have that $\sup_{i \in I} e_i \in V_n \subseteq \bigcup_{i \in I} W_i$. Remember that $U_{n+1} \ll U_n$, it follows that $U_{n+1} \subseteq W_j$ for some $j \in I$. By definition of W_j , we know that $f(U_{n+1}) \times \{e_j\} \subseteq O$, which means that $e_j \in V_{n+1}$, this time by definition of V_{n+1} . So we find $j \in I$ with $e_j \in V_{n+1} \subseteq V$, and indeed V is Scott open in E . ◀

► **Theorem 4.** *Let X be a core-compact space and f be a lower semi-continuous map from X to a dcpo D . If ν is a valuation on X , then $f_*(\nu) \stackrel{\text{def}}{=} \lambda O \in \sigma D. \nu(f^{-1}(O))$, the push-forward valuation along f , is a central valuation on D . In particular, for a core-compact dcpo D , all valuations on D are central, i.e., $\mathcal{V}D = \mathcal{Z}D$.*

Proof. By definition, we prove for any dcpo E , continuous valuations μ on E and Scott-continuous map $h: D \times E \rightarrow [0, 1]$ the equation

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu df_*(\nu) = \int_{y \in E} \int_{x \in D} h(x, y) df_*(\nu) d\mu$$

holds.

Note that for each $y \in E$, the map $g \stackrel{\text{def}}{=} (x \mapsto \int_{y \in E} h(x, y) d\mu): D \rightarrow [0, 1]$ is Scott-continuous, and $f: X \rightarrow D$ is lower semi-continuous. Hence for the left side of the above equation we have

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu df_*(\nu) = \int_{x \in X} g(f(x)) d\nu = \int_{x \in X} \int_{y \in E} h(f(x), y) d\mu d\nu.$$

The first equality follows from the so-called *change-of-variable* formula, which can be found in [4]. As a consequence of it, we also have that

$$\int_{y \in E} \int_{x \in D} h(x, y) df_*(\nu) d\mu = \int_{y \in E} \int_{x \in X} h(f(x), y) d\nu d\mu.$$

Since X is core-compact and the function $f: X \rightarrow D$ is lower semi-continuous, by Lemma 3 we know that $f \times \text{id}_E: X \times \Sigma E \rightarrow X \times Y$ is lower semi-continuous. This implies that the map $(x, y) \mapsto h(f(x), y): X \times \Sigma E \rightarrow [0, 1]$ is lower semi-continuous. Hence by Lemma 2.37 in [4] we know that $\int_{x \in X} \int_{y \in E} h(f(x), y) d\mu d\nu = \int_{y \in E} \int_{x \in X} h(f(x), y) d\nu d\mu$, which finishes the proof.

The second claim is a straightforward consequence of the first one. ◀

► **Theorem 5.** *Let $f: [0, 1] \rightarrow D$ be a lower semi-continuous map into a dcpo D . If ν is any continuous valuation on $[0, 1]$, then $f_*(\nu)$ is a central valuation on D .*

Proof. Since $[0, 1]$ is core-compact in the usual topology, the result follows from Theorem 4. ◀

We have not been able to establish the above theorem for any of the monads \mathcal{M}, \mathcal{W} or \mathcal{P} , so we believe that \mathcal{Z} is a promising candidate for modeling *continuous* probabilistic choice. We plan to address this in future work.

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