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The Central Valuations Monad

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Abstract

We give a commutative valuations monad \mathcal{Z} on the category **DCPO** of dcpo's and Scott-continuous functions. Compared to the commutative valuations monads given in [2], our new monad \mathcal{Z} is larger and it contains all push-forward images of valuations on the unit interval $[0, 1]$ along lower semi-continuous maps. We believe that this new monad will be useful in giving domain-theoretic denotational semantics for statistical programming languages with continuous probabilistic choice.

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1 Introduction

The valuations monad \mathcal{V} on the category **DCPO** of dcpo's and Scott-continuous functions is a staple of the domain-theoretic approach for denotational semantics of programming languages with probabilistic choice and recursion [3, 4]. For a dcpo D , $\mathcal{V}D$ consists of *subprobability valuations* on D , which are the Scott-continuous functions ν from the set σD of Scott open subsets of D to $[0, 1]$ satisfying *strictness* ($\nu(\emptyset) = 0$) and *modularity* ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). The set $\mathcal{V}D$ is a dcpo in the *stochastic order*: $\nu_1 \leq \nu_2$ if and only if $\nu_1(U) \leq \nu_2(U)$ for all $U \in \sigma D$. The *unit* of \mathcal{V} at dcpo D is the map $\eta_D: D \rightarrow \mathcal{V}D$ by $\eta_D(x) = \delta_x$, the *Dirac valuation* at x , defined by $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ otherwise. For a Scott-continuous map $f: D \rightarrow \mathcal{V}E$, the *Kleisli extension* f^\dagger of f is defined by $f^\dagger(\nu)(U) = \int_{x \in X} f(x)(U) d\nu$ for $\nu \in \mathcal{V}D$ and $U \in \sigma E$. The integral in this definition is a Choquet type integral: for a general Scott-continuous function $h: D \rightarrow [0, 1]$, the value of $\int_{x \in X} h d\nu$ is defined to be the Riemann integral $\int_0^1 \nu(h^{-1}(t, 1]) dt$. Following this, the action of \mathcal{V} on a Scott-continuous function $g: D \rightarrow E$ between dcpo's D and E is $\mathcal{V}(g) \stackrel{\text{def}}{=} (\eta_E \circ g)^\dagger$; concretely, for $\nu \in \mathcal{V}D$ and $U \in \sigma E$, $\mathcal{V}(g)(\nu)(U) = \nu(g^{-1}(U))$.

While it is well-known that \mathcal{V} can be restricted to a commutative monad on the category **DOM** of domains and Scott-continuous functions, it is unknown whether \mathcal{V} can be restricted to any Cartesian closed full subcategory of **DOM**. This is known as the *Jung-Tix problem* [5].

One may note that the category **DCPO** itself is Cartesian closed and \mathcal{V} is a monad on it. What does one lose if we use the category **DCPO** and monad \mathcal{V} for semantics? A short answer is that compared to **DOM**, \mathcal{V} is not known to be *commutative* over **DCPO**, which is an important property for the denotational semantics of programming languages. Commutativity of \mathcal{V} over **DCPO** is equivalent to showing the following Fubini-style equation

$$\int_{x \in D} \int_{y \in E} \chi_U d\mu d\nu = \int_{y \in E} \int_{x \in D} \chi_U d\nu d\mu \quad (1)$$

holds for all dcpo's D and E , all Scott open subsets U of $D \times E$ and all $\nu \in \mathcal{V}D, \mu \in \mathcal{V}E$, where χ_U is the characteristic function of U . As pointed out in [2], the main difficulty in

establishing (1) over **DCPO** is that the Scott topology on the product $\text{dcpo } D \times E$ may be different from the product topology $\sigma D \times \sigma E$.

Instead of directly proving (1), we showed (together with Lindenhovius) how to construct three submonads of \mathcal{V} that are commutative on **DCPO**, and used each one to give a sound and (strongly) adequate semantics to **PFPC** (Probabilistic FixPoint Calculus) [2]. The simplest of those three monads is the monad \mathcal{M} . For each $\text{dcpo } D$, $\mathcal{M}D$ is defined to be the smallest sub- dcpo of $\mathcal{V}D$ that contains $\mathcal{S}D$, the family of *simple valuations* on D , where a simple valuation is a finite convex sum of Dirac valuations. The other two commutative monads are denoted \mathcal{W} and \mathcal{P} and the following inclusions hold for each $\text{dcpo } D$: $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{V}D$.

Each of our three monads is large enough to interpret *discrete* probabilistic choice in **PFPC**. However, it is unclear if any of these monads is large enough to interpret *continuous* probabilistic choice. In this note, we define a new commutative valuations monad \mathcal{Z} on the category **DCPO** which is larger than \mathcal{M} , \mathcal{W} and \mathcal{P} with the hope of addressing this problem.

2 Central valuations

Our idea for defining \mathcal{Z} is inspired by the notion of centre in group theory (which always forms an abelian subgroup) and the notion of centre of a premonoidal category (which always forms a monoidal subcategory) [6].

► **Definition 1.** A subprobability valuation ν on a $\text{dcpo } D$ is called a *central valuation* if for any $\text{dcpo } E$, any valuation μ on E , and any Scott-continuous function $h: D \times E \rightarrow [0, 1]$, we have

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x, y) d\nu d\mu.$$

We shall write $\mathcal{Z}D$ for the set of all central valuations on a $\text{dcpo } D$.

It is easy to see that simple valuations are central, and that the central valuations are closed under directed suprema under the stochastic order. Thus, for each $\text{dcpo } D$, $\mathcal{Z}D$ is a sub- dcpo of $\mathcal{V}D$ containing $\mathcal{S}D$. Moreover, we have the following theorem, which can be proved using the *disintegration formula* in [1].

► **Theorem 2.** The assignment $\mathcal{Z}(-)$ extends to a commutative monad over the category **DCPO** when equipped with the (co)restricted monad operations of \mathcal{V} . In other words, \mathcal{Z} is a commutative submonad of \mathcal{V} .

In fact, it is proved in [2] that all *point-continuous valuations* are central and therefore $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{Z}D \subseteq \mathcal{V}D$ for each $\text{dcpo } D$. Therefore \mathcal{Z} is the largest commutative submonad of \mathcal{V} known so far. Furthermore, observe that $\mathcal{Z} = \mathcal{V}$ iff \mathcal{V} is a commutative monad on **DCPO**. The latter has been an open problem since 1989, and our simple observation leads us to believe \mathcal{Z} is a very large commutative submonad of \mathcal{V} .

It is not difficult to see that, in order to model sampling against continuous probability distributions on the interval $[0, 1]$, the monad used for the semantics should at least contain the push-forward images of the Lebesgue valuation on $[0, 1]$ (equipped with the metric topology) along lower semi-continuous maps. We can demonstrate even more is true of our new monad \mathcal{Z} .

► **Theorem 3.** Let $f: [0, 1] \rightarrow D$ be a lower semi-continuous map into a $\text{dcpo } D$. If ν is any valuation on $[0, 1]$, then $f_*(\nu) \stackrel{\text{def}}{=} \lambda O \in \sigma D. \nu(f^{-1}(O))$ is a central valuation on D .

We have not been able to establish the above theorem for any of the monads \mathcal{M} , \mathcal{W} or \mathcal{P} , so we believe that \mathcal{Z} is a promising candidate for modeling *continuous* probabilistic choice. We plan to address this in future work.

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