Parisian times for linear diffusions

Christophe Profeta

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CHRISTOPHE PROFETA

Abstract. We compute the joint distribution of the first times a linear diffusion makes an excursion longer than some given duration above (resp. below) some fixed level. In the literature, such stopping times have been introduced and studied in the framework of *Parisian* barrier options, mainly in the case of Brownian motion with drift. We also exhibit several independence properties, and provide some formulae for the associated ruin probabilities.

1. Statement of the main results

1.1. **Introduction.** Let X be a regular diffusion, without killing, taking values in some interval $I \subseteq \mathbb{R}$. In this note, we are interested in the following *Parisian* stopping times:

$$\kappa_r^{(a,\pm)} = \inf\{t > r,\ t - g_t^{(a)} \geq r \text{ and } \pm (X_t - a) \geq 0\}$$

where $a \in \mathrm{Int}(I)$, the interior of I, $r > 0$ and

$$g_t^{(a)} = \sup\{s \leq t,\ X_s = a\}.$$

These stopping times are the first times the process X makes an excursion above a (resp. below a) longer than $r > 0$. They were introduced by Chesney, Jeanblanc & Yor [3] for the pricing of some barrier options, in which the owner loses the option if the underlying asset X remains long enough below or above some fixed level. Since then, many papers have been devoted to the valuation of such options, see for instance Labart & Lelong [13], but one recurring ingredient is the need to know the distribution of $\kappa_r^{(a,\pm)}$.

For the Brownian motion with drift, the distribution of $\kappa_r^{(a,\pm)}$ was first computed in [3] using Azéma's martingale. This initial approach was extended to the case of two barriers by Anderluh & van der Weide [1]. Another method was developed by Dassio & Wu [7, 8], using a semi-Markov model and a limiting procedure. Finer properties of these stopping times, in particular their densities and their asymptotics, have also been studied, for which we refer the reader to Dassio & Lim [4, 5]. More recently, the Azéma's martingale method was successfully applied to the study of Parisian times at level 0 for the Bessel and the CIR process, see Dassio, Lim & Qu [6].

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The purpose of this paper is thus to study, in full generality, the distribution of the quadruplet $(\kappa_u^{(a,-)}, \kappa_v^{(b,+)}, X_{\kappa_u^{(a,-)}}, X_{\kappa_v^{(b,+)}})$ where $a < b$ and $u, v > 0$. This distribution is indeed at the core of all the aforementioned papers where specific models are studied.

The paper is divided as follows. In the remainder of Section 1, we introduce some notation and state our main result, as well as several corollaries and consequences. Section 2 is devoted to some preliminary lemmas, which will be needed for the proof of the main theorem in Section 3. We then give in Section 4 some theoretical formulae for the probability densities of Parisian times as well as some asymptotics. Finally, a few examples are given in Section 5, including Brownian motion with drift and some Bessel processes.

1.2. **Notations.** We shall work in the framework of Pitman & Yor [15]. Let us denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and expectation of X when starting from $x \in I$, and $(\mathcal{F}_t, t \geq 0)$ its natural filtration. We assume that the infinitesimal generator of X has the form

$$\mathcal{G} = \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

on some appropriate domain, with σ a strictly positive and continuous function and μ a locally bounded function. Recall that X may also be defined by its scale function s and speed measure m which are given by :

$$s'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right) \qquad \text{and} \qquad m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx. \tag{1.1}$$

For $a \in \mathrm{Int}(I)$, we define the first hitting time

$$T_a = \inf\{t > 0,\, X_t = a\}.$$

Its Laplace transform is classically given for $\lambda > 0$ by :

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right] = \begin{cases} \Phi_{\lambda,-}(x)/\Phi_{\lambda,-}(a) & \text{if } x \leq a, \\ \Phi_{\lambda,+}(x)/\Phi_{\lambda,+}(a) & \text{if } x > a, \end{cases}$$

where $\Phi_{\lambda,-}$ and $\Phi_{\lambda,+}$ are two solutions (respectively increasing and decreasing) of the ordinary differential equation $\mathcal{G}u = \lambda u$, subject to some appropriate boundary conditions. Applying the Markov property and using the continuity of the paths of X, we also have for $a \leq x \leq b$ and $\lambda > 0$:

$$\mathbb{E}_x\left[e^{-\lambda T_a}1_{\{T_a<T_b\}}\right] = \frac{W_\lambda^{(b)}(x)}{W_\lambda^{(b)}(a)} \quad \text{and} \quad \mathbb{E}_x\left[e^{-\lambda T_b}1_{\{T_b<T_a\}}\right] = \frac{W_\lambda^{(a)}(x)}{W_\lambda^{(a)}(b)}$$

where, for $c \in I$, the function $W_\lambda^{(c)}$ is defined by :

$$W_\lambda^{(c)}(x) = \Phi_{\lambda,+}(x)\Phi_{\lambda,-}(c) - \Phi_{\lambda,-}(x)\Phi_{\lambda,+}(c).$$

Recall next from Itô & McKean [11, Section 6.2] that T_a admits a continuous probability density function and that there exist two Lévy measures $\nu_+^{(a)}$ and $\nu_-^{(a)}$ such that

$$\nu_\pm^{(a)}(dt) = \pm \lim_{\varepsilon\downarrow 0} \frac{\mathbb{P}_{a\pm\varepsilon}(T_a \in dt)}{s(a\pm\varepsilon) - s(a)} = \pm \frac{1}{s'(a)} \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}_x(T_a \in dt)_{\big|x=a^\pm}.$$

Equivalently, $\nu_+^{(a)}$ and $\nu_-^{(a)}$ may be defined by their (modified) Laplace transforms :

$$\int_{(0,+\infty]} (1-e^{-\lambda t}) \nu_\pm^{(a)}(dt) = \mp \frac{1}{s'(a)} \frac{\Phi'_{\lambda,\pm}(a)}{\Phi_{\lambda,\pm}(a)}.$$

Some care must be taken when dealing with transient diffusions, as in this case, the measure $\nu_\pm^{(a)}$ might have an atom at $t = +\infty$ given by :

$$\nu_\pm^{(a)}(\{+\infty\}) = \mp \frac{1}{s'(a)} \lim_{\lambda\downarrow 0} \frac{\Phi'_{\lambda,\pm}(a)}{\Phi_{\lambda,\pm}(a)}.$$

We also introduce the Green kernel of X which is defined by the formula

$$G_\lambda(x,y) = \int_0^{+\infty} e^{-\lambda t} p(t;x,y) dt \tag{1.2}$$

where $p(t;x,y)$ denotes the transition density of X with respect to the speed measure m, i.e.

$$\mathbb{P}_x(X_t \in dy) = p(t;x,y) m(dy).$$

The Green kernel admits the classic representation

$$G_\lambda(x,y) = \omega_\lambda^{-1} \Phi_{\lambda,-}(x)\Phi_{\lambda,+}(y), \qquad x \leq y,$$

where the Wronskian ω_λ is defined by

$$\omega_\lambda = \frac{\Phi'_{\lambda,-}(x)\Phi_{\lambda,+}(x) - \Phi_{\lambda,-}(x)\Phi'_{\lambda,+}(x)}{s'(x)}$$

and is independent from x.

We finally define the positive and negative meanders of lengths u by :

$$\forall \Lambda_u \in \mathcal{F}_u, \qquad \begin{cases} \mathbb{P}_a^{\uparrow,u}(\Lambda_u) = \lim_{\varepsilon\downarrow 0} \mathbb{P}_{a+\varepsilon}(\Lambda_u | T_a > u) \\ \mathbb{P}_a^{\downarrow,u}(\Lambda_u) = \lim_{\varepsilon\downarrow 0} \mathbb{P}_{a-\varepsilon}(\Lambda_u | T_a > u). \end{cases}$$

The occurrence of meanders in the forthcoming formulae is not surprising as the random variables $X_{\kappa_u^{(a,-)}}$ and $X_{\kappa_v^{(b,+)}}$ are respectively a.s. smaller than a and greater than b.

1.3. **Main results.** Let α and β be two continuous and bounded functions from I to $[0,+\infty)$. For $a \leq x \leq b$ and $\gamma, \lambda, u, v > 0$ let us define the modified Laplace transform of the quadruplet $(\kappa_u^{(a,-)}, \kappa_v^{(b,+)}, X_{\kappa_u^{(a,-)}}, X_{\kappa_v^{(b,+)}})$ by :

$$F_{\alpha,\beta,\gamma,\lambda}(x) = \mathbb{E}_x\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}\right].$$

The main result of the paper is the computation of this function. To simplify the statement, let us set :

$$\Psi_\lambda^{(\pm)}(a,b,r) = \frac{\Phi_{\lambda,\pm}(a)}{\Phi_{\lambda,\pm}(b)} \pm \frac{W_\lambda^{(b)}(a)}{\omega_\lambda}\int_r^{+\infty} e^{-\lambda t}\nu_\pm^{(b)}(dt).$$

Theorem 1. *For $a \leq x \leq b$, the function $F_{\alpha,\beta,\gamma,\lambda}$ admits the expression :*

$$F_{\alpha,\beta,\gamma,\lambda}(x) = \left(\frac{W_{\gamma+\lambda}^{(b)}(x)}{W_{\gamma+\lambda}^{(b)}(a)}\frac{1}{\Psi_{\gamma+\lambda}^{(-)}(b,a,u)} + \frac{W_{\gamma+\lambda}^{(a)}(x)}{W_{\gamma+\lambda}^{(a)}(b)}\right)F_{\alpha,\beta,\gamma,\lambda}(b) \tag{1.3}$$

where

$$F_{\alpha,\beta,\gamma,\lambda}(b)$$
$$= e^{-(\lambda+\gamma)v}\frac{W_{\gamma+\lambda}^{(b)}(a)}{\omega_{\gamma+\lambda}}\frac{\nu_+^{(b)}[v,+\infty]\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}}\right]}{\Psi_{\gamma+\lambda}^{(+)}(a,b,v)-\frac{1}{\Psi_{\gamma+\lambda}^{(-)}(b,a,u)}}\mathbb{E}_a^{\downarrow,u}\left[\alpha\left(X_u\right)\right]\mathbb{E}_b^{\uparrow,v}\left[\frac{\Phi_{\gamma,+}(X_v)}{\Phi_{\gamma,+}(b)}\beta\left(X_v\right)\right]$$

and

$$\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}}\right] = e^{-\gamma u}\frac{\Phi_{\gamma,+}(b)}{\Phi_{\gamma,+}(a)}\frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_\gamma(a,a)}+\int_u^{+\infty}e^{-\gamma t}\nu_-^{(a)}(dt)}.$$

One may observe several independence properties in this formula.

(1) Letting $\gamma \downarrow 0$, we deduce that on the set $\{\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)} < +\infty\}$, the random variables $\kappa_v^{(b,+)}$, $X_{\kappa_u^{(a,-)}}$ and $X_{\kappa_v^{(b,+)}}$ are mutually independent.

(2) Taking $\alpha \equiv 1$, we observe that one may remove the condition $\{\kappa_u^{(a,-)} < +\infty\}$ in the definition of $F_{1,\beta,\gamma,\lambda}$. Letting $\gamma \downarrow 0$ and then $u \to +\infty$, we deduce that on the set $\{\kappa_v^{(b,+)} < +\infty\}$, the random variables $\kappa_v^{(b,+)}$ and $X_{\kappa_v^{(b,+)}}$ are independent. This independence property, when dealing with a single boundary, was already observed in the Brownian case by Chesney, Jeanblanc & Yor [3].

We mention below two situations where Theorem 1 greatly simplifies. First, when $\alpha \equiv \beta \equiv 1$, one obtains the following corollary :

Corollary 2. *For $a < b$ and $\gamma, \lambda > 0$, the Laplace transform of the pair $(\kappa_u^{(a,-)}, \kappa_v^{(b,+)})$ is given by :*

$$\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}\}}\right] = e^{-\lambda v}\frac{W_{\gamma+\lambda}^{(b)}(a)}{\omega_{\gamma+\lambda}}\frac{\int_v^{+\infty}e^{-\gamma t}\nu_+^{(b)}(dt)}{\Psi_{\gamma+\lambda}^{(+)}(a,b,v)-\frac{1}{\Psi_{\gamma+\lambda}^{(-)}(b,a,u)}}\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}}\right].$$

This result follows from the observation that, by definition of the meander,

$$\begin{aligned}\mathbb{E}_b^{\uparrow,v}\left[\frac{\Phi_{\gamma,+}(X_v)}{\Phi_{\gamma,+}(b)}\right] &= \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_{b+\varepsilon}\left[\mathbb{E}_{X_v}\left[e^{-\gamma T_b}\right]1_{\{T_b\geq v\}}\right]}{\mathbb{P}_{b+\varepsilon}(T_b\geq v)}\\ &= \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_{b+\varepsilon}\left[e^{-\gamma(T_b-v)}1_{\{T_b\geq v\}}\right]}{\mathbb{P}_{b+\varepsilon}(T_b\geq v)} = \frac{e^{\gamma v}\int_v^{+\infty}e^{-\gamma t}\nu_+^{(b)}(dt)}{\nu_+^{(b)}[v,+\infty]}\end{aligned}$$

where we have applied the Markov property in the second equality. Another simplification is obtained by letting $a\uparrow b$. Using that $W_\lambda^{(a)}(b)\underset{a\uparrow b}{\longrightarrow}0$ and computing the limit yields the general formula for the one barrier case :

Corollary 3. *For $\gamma, \lambda > 0$, the Laplace transform of the pair $(\kappa_u^{(b,-)}, \kappa_v^{(b,+)})$ is given by :*

$$\begin{aligned}&\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(b,-)}-\lambda\kappa_v^{(b,+)}}1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(b,-)}\}}\right]\\ &\qquad = \frac{e^{-\lambda v}\int_v^{+\infty}e^{-\gamma t}\nu_+^{(b)}(dt)}{\int_v^{+\infty}e^{-(\gamma+\lambda)t}\nu_+^{(b)}(dt)+\int_u^{+\infty}e^{-(\gamma+\lambda)t}\nu_-^{(b)}(dt)+\frac{1}{G_{\gamma+\lambda}(b,b)}}\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(b,-)}}\right].\end{aligned}$$

In particular, when X is a Brownian motion with drift, by letting $\gamma\downarrow 0$, we recover the formula obtained by Dassios & Wu [7].

1.4. **Parisian ruin probabilities.** Letting $\gamma\downarrow 0$ in the Laplace transform of $\kappa_u^{(a,-)}$ given in Theorem 1, we obtain the following ruin probabilities :

(1) if X is recurrent, then $\lim_{\gamma\downarrow 0}G_\gamma(a,a)=+\infty$ and $\nu_-^{(a)}(\{+\infty\})=0$, hence

$$\mathbb{P}_a\left(\kappa_u^{(a,-)}<+\infty\right)=\frac{\nu_-^{(a)}[u,+\infty]}{\nu_-^{(a)}[u,+\infty)}=1.$$

(2) if X is transient, then $G_0(a,a)=\lim_{\gamma\downarrow 0}G_\gamma(a,a)<+\infty$ and

$$\mathbb{P}_a\left(\kappa_u^{(a,-)}<+\infty\right)=\frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_0(a,a)}+\nu_-^{(a)}[u,+\infty)}.$$

In particular, if X goes to $\inf(I)$ a.s. as $t\to+\infty$, then $\frac{1}{G_0(a,a)}=\nu_-^{(a)}(\{+\infty\})$ and this probability also equals 1.

Besides Brownian motion with drift, such Parisian ruin probabilities have also been computed for spectrally negative Lévy processes, see for instance Loeffen, Czarna & Palmowski [14].

Another probability of interest is the probability that $\kappa_v^{(b,+)}$ takes place before $\kappa_u^{(a,-)}$, which may be obtained by letting λ and γ go to 0 in Corollary 2 and Formula (1.3).

Corollary 4. *Assume that either X is recurrent, or X is transient and goes a.s to* $\sup(I)$ *or to* $\inf(I)$ *as $t \to +\infty$. Then, for $a \leq x \leq b$:*

$$\begin{aligned}
&\mathbb{P}_x\left(\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)} < +\infty\right) \\
&= \left(\frac{s(b)-s(x)}{\frac{1}{\mathbb{P}_a(T_b<+\infty)} + (s(b)-s(a))\nu_-^{(a)}[u,+\infty)} + (s(x)-s(a))\right) \frac{\mathbb{P}_b\left(\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)} < +\infty\right)}{s(b)-s(a)}
\end{aligned}$$

with

$$\begin{aligned}
&\mathbb{P}_b\left(\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)} < +\infty\right) \\
&\qquad = \frac{(s(b)-s(a))\nu_+^{(b)}[v,+\infty)\mathbb{P}_b\left(\kappa_u^{(a,-)} < +\infty\right)}{\frac{1}{\mathbb{P}_b(T_a<+\infty)} + (s(b)-s(a))\nu_+^{(b)}[v,+\infty) - \frac{1}{\frac{1}{\mathbb{P}_a(T_b<+\infty)} + (s(b)-s(a))\nu_-^{(a)}[u,+\infty)}}.
\end{aligned}$$

To prove Corollary 4, observe that by definition of the Wronskian ω_λ, we have

$$\frac{W_\lambda^{(b)}(a)}{\omega_\lambda} = \Phi_{\lambda,+}(a)\Phi_{\lambda,+}(b)\int_a^b \frac{s'(z)}{\Phi_{\lambda,+}^2(z)}dz = \int_a^b \frac{\mathbb{E}_b\left[e^{-\lambda T_z}\right]}{\mathbb{E}_z\left[e^{-\lambda T_a}\right]}s'(z)dz.$$

Then, on the one hand, when X is recurrent or X is transient and goes a.s. to $\inf(I)$, we obtain

$$\frac{W_\lambda^{(b)}(a)}{\omega_\lambda} \xrightarrow[\lambda\downarrow 0]{} \int_a^b s'(z)dz = s(b)-s(a).$$

On the other hand, when X is transient and goes a.s. to $\sup(I)$, we may assume without loss of generality that $s(\sup(I)) = 0$ and again

$$\frac{W_\lambda^{(b)}(a)}{\omega_\lambda} \xrightarrow[\lambda\downarrow 0]{} \int_a^b \frac{s(b)/s(z)}{s(z)/s(a)}s'(z)dz = s(b)-s(a).$$

Corollary 4 now follows by letting γ and λ go to 0 in Corollary 2 and Formula (1.3) and using the above limit.

Remark 5. When X is recurrent, since $\mathbb{P}_b(T_a < +\infty) = \mathbb{P}_a(T_b < +\infty) = 1$, Corollary 4 greatly simplifies and one obtains :

$$\mathbb{P}_x\left(\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)}\right) = \frac{\left(1 + (s(x) - s(a))\nu_-^{(a)}[u, +\infty)\right)\nu_+^{(b)}[v, +\infty)}{\nu_-^{(a)}[u, +\infty) + \nu_+^{(b)}[v, +\infty) + (s(b) - s(a))\nu_-^{(a)}[u, +\infty)\nu_+^{(b)}[v, +\infty)}.$$

1.5. **Remarks on the meanders.** We shall give in Section 5 a few examples where all the terms appearing in Theorem 1 may be computed explicitly. To this end, it is worthwhile to note that the terms involving the meanders in Theorem 1 only require the knowledge of the transition density p and of the Lévy measures $\nu_\pm^{(a)}$. Indeed, by applying the Markov property

$$\mathbb{E}_x\left[\alpha(X_u)1_{\{T_a \geq u\}}\right] = \mathbb{E}_x\left[\alpha(X_u)\right] - \int_0^u \mathbb{E}_a\left[\alpha(X_{u-t})\right]\mathbb{P}_x(T_a \in dt).$$

Inverting this transform, we obtain after some simple manipulations :

$$\begin{aligned}&\mathbb{P}_x\left(X_u \in dz | T_a \geq u\right)/m(dz)\\ &\qquad= \frac{p(u; x, z) - p(u; a, z)}{\mathbb{P}_x(T_a \geq u)} + p(u; a, z) + \int_0^u \frac{p(u; a, z) - p(u-t; a, z)}{\mathbb{P}_x(T_a \geq u)}\mathbb{P}_x(T_a \in dt).\end{aligned}$$

Letting then $x \uparrow a$, we deduce that for $z < a$:

$$\begin{aligned}&\mathbb{P}_a^{(\downarrow,u)}\left(X_u \in dz\right)/m(dz)\\ &= -\frac{1}{s'(a)\nu_-^{(a)}[u, +\infty]}\frac{\partial p}{\partial x}(u; x, z)_{|x=a} + p(u; a, z) + \int_0^u \frac{p(u; a, z) - p(u-t; a, z)}{\nu_-^{(a)}[u, +\infty]}\nu_-^{(a)}(dt).\end{aligned}$$

One may also give an alternative definition of the meanders as derivatives. Indeed, for example, by definition of the left derivative :

$$\begin{aligned}\mathbb{P}_a^{(\downarrow,u)}\left(\Lambda_u\right) &= \lim_{\varepsilon \downarrow 0} \frac{-\varepsilon}{\mathbb{P}_{a-\varepsilon}\left(T_a \geq u\right)}\frac{\mathbb{P}_{a-\varepsilon}\left(\Lambda_u 1_{\{T_a \geq u\}}\right)}{-\varepsilon}\\ &= -\frac{1}{s'(a)\nu_-^{(a)}[u, +\infty]}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_x\left(\Lambda_u 1_{\{T_a \geq u\}}\right)\Big|_{x=a^-}.\end{aligned} \tag{1.4}$$

Formula (1.4) will be the one appearing during the proofs.

2. Preliminary computations

Our proof of Theorem 1 will rely on some local estimates for $t \to p(t; a, a)$. Let us introduce the semimartingale local time of X :

$$L_t^a = \sigma^2(a) \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^t 1_{\{|X_s - a| \leq \varepsilon\}} ds, \qquad t \geq 0.$$

From the occupation time formula (see [17, Chapter VI]) and the expression (1.1) of the speed measure m, we have

$$\mathbb{E}_a\left[L_t^a\right] = \frac{2}{s'(a)}\int_0^t p(t;a,a)dt.$$

In particular, using Fatou's lemma, and the fact that $\sigma > 0$, we deduce that

$$\frac{2}{s'(a)}\liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^\varepsilon p(t;a,a)dt = \liminf_{\varepsilon\downarrow 0}\mathbb{E}_a\left[\frac{L_\varepsilon^a - L_0^a}{\varepsilon}\right] \geq \mathbb{E}_a\left[\liminf_{\varepsilon\downarrow 0}\frac{L_\varepsilon^a - L_0^a}{\varepsilon}\right] = +\infty$$

which yields a first estimate :

$$\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\int_0^\varepsilon p(t;a,a)dt} = 0. \tag{2.1}$$

We now prove a lemma which will be repeatedly used in the sequel.

2.1. **Limiting lemma.**

Lemma 6. *Let $a \in Int(I)$ and let φ be a positive and continuous function such that $\mathbb{E}_x\left[\varphi(X_t)\right] < +\infty$ is finite for t small enough and x in the vicinity of a. We assume that $\varphi(a) = 0$ and that φ admits a left and a right derivative at the point a. Then the following asymptotics hold:*

$$\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\geq a\}}\right] \underset{\varepsilon\downarrow 0}{\sim} \frac{1}{s'(a)}\varphi'(a^+)\int_0^\varepsilon p(t;a,a)dt$$

and

$$\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\leq a\}}\right] \underset{\varepsilon\downarrow 0}{\sim} -\frac{1}{s'(a)}\varphi'(a^-)\int_0^\varepsilon p(t;a,a)dt.$$

Proof. We only prove the first asymptotics, as the second one will follow from a similar argument. Take δ small enough and let us decompose the first expectation in Lemma 6 according as whether X_ε is greater or smaller than $a+\delta$:

$$\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\geq a\}}\right] = \mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\geq a+\delta\}}\right] + \mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{a+\delta > X_\varepsilon\geq a\}}\right].$$

Applying the Markov property, the first term on the right-hand side is smaller than

$$\begin{aligned}\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\geq a+\delta\}}\right] &= \mathbb{E}_a\left[\mathbb{E}_{a+\delta}\left[\varphi(X_{\varepsilon-s})1_{\{X_{\varepsilon-s}\geq a+\delta\}}\right]_{\big|s=T_{a+\delta}} 1_{\{T_{a+\delta}\leq\varepsilon\}}\right]\\ &\leq \sup_{s\in[0,\varepsilon]}\mathbb{E}_{a+\delta}\left[\varphi(X_s)\right]\mathbb{P}_a\left(T_{a+\delta}\leq\varepsilon\right)\end{aligned}$$

From Kotani & Watanabe [12], there is the asymptotics for $z \in \text{Int}(I)$:

$$\lim_{\varepsilon\downarrow 0} -2\varepsilon\ln\left(\mathbb{P}_a\left(T_z\leq\varepsilon\right)\right) = \left(\int_a^z \frac{1}{\sigma(u)\sqrt{s'(u)}}du\right)^2. \tag{2.2}$$

As a consequence, we deduce from (2.1) that

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{X_\varepsilon\geq a+\delta\}}\right]}{\int_0^\varepsilon p(t;a,a)dt} = 0$$

which proves that this term is negligible. Then, since $\varphi(a) = 0$, the second term may be bounded by :

$$\inf_{a\le z\le a+\delta} \frac{\varphi(z)}{z-a} \le \frac{\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{a+\delta>X_\varepsilon\ge a\}}\right]}{\mathbb{E}_a\left[(X_\varepsilon - a)1_{\{a+\delta>X_\varepsilon\ge a\}}\right]} \le \sup_{a\le z\le a+\delta} \frac{\varphi(z)}{z-a}. \tag{2.3}$$

Recall now that for any $x \in \mathrm{Int}(I)$, the function $(t, y) \longrightarrow p(t; x, y)$ is a solution of the partial differential equation $\frac{\partial}{\partial t}p = \mathcal{G}p$ on $(0, +\infty) \times \mathrm{Int}(I)$, and that for $z \ne a$, from Kotani & Watanabe [12] :

$$\lim_{t\downarrow 0} p(t; a, z) = 0. \tag{2.4}$$

Since the scale function s and the speed measure m are assumed to be absolutely continuous, an adaptation of their proof also shows that

$$\lim_{t\downarrow 0} \frac{\partial p}{\partial z}(t; a, z) = 0. \tag{2.5}$$

Now, integrating the partial differential equation for p on $[0, \varepsilon]$ and using (2.4), we deduce that for $z \ne a$,

$$p(\varepsilon; a, z) = \int_0^\varepsilon \left(\frac{1}{2}\sigma^2(z)\frac{\partial^2 p}{\partial z^2}(t; a, z) + \mu(z)\frac{\partial p}{\partial z}(t; a, z)\right) dt.$$

Then applying Fubini's theorem, and recalling that $m(dx) = 2/(s'(x)\sigma^2(x))dx$, we may write

$$\begin{aligned}\mathbb{E}_a&\left[(X_\varepsilon - a)1_{\{a+\delta>X_\varepsilon\ge a\}}\right]\\ &= \int_0^\varepsilon \left(\int_a^{a+\delta} \frac{z-a}{s'(z)}\frac{\partial^2 p}{\partial z^2}(t; a, z)dz + \int_a^{a+\delta} (z-a)\mu(z)\frac{\partial p}{\partial z}(t; a, z)m(dz)\right) dt.\end{aligned}$$

Integrating by parts the first term, and using the expression (1.1) of s, we observe that the terms in μ cancel and we obtain :

$$\mathbb{E}_a\left[(X_\varepsilon - a)1_{\{a+\delta>X_\varepsilon\ge a\}}\right] = \int_0^\varepsilon \left(\frac{\delta}{s'(a+\delta)}\frac{\partial p}{\partial z}(t; a, a+\delta) - \int_a^{a+\delta} \frac{1}{s'(z)}\frac{\partial p}{\partial z}(t; a, z)dz\right) dt.$$

Using yet another integration by parts, we finally obtain :

$$\begin{aligned}\mathbb{E}_a\left[(X_\varepsilon - a)1_{\{a+\delta>X_\varepsilon\ge a\}}\right] &= \frac{1}{s'(a)}\int_0^\varepsilon p(t; a, a)dt\\ &+ \int_0^\varepsilon \left(\frac{1}{s'(a+\delta)}\left(\delta\frac{\partial p}{\partial z}(t; a, a+\delta) - p(t; a, a+\delta)\right) + \mathbb{E}_a\left[\mu(X_t)1_{\{a+\delta\ge X_t\ge a\}}\right]\right) dt.\end{aligned}$$

As a consequence, going back to (2.3), we deduce from (2.4), (2.5), and (2.1) that

$$\begin{aligned}\inf_{a\le z\le a+\delta} \frac{\varphi(z)}{z-a} &\le \liminf_{\varepsilon\downarrow 0} \frac{s'(a)\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{a+\delta>X_\varepsilon\ge a\}}\right]}{\int_0^\varepsilon p(t; a, a)dt}\\ &\le \limsup_{\varepsilon\downarrow 0} \frac{s'(a)\mathbb{E}_a\left[\varphi(X_\varepsilon)1_{\{a+\delta>X_\varepsilon\ge a\}}\right]}{\int_0^\varepsilon p(t; a, a)dt} \le \sup_{a\le z\le a+\delta} \frac{\varphi(z)}{z-a}\end{aligned}$$

and the result follows by letting $\delta \downarrow 0$.

$\square$

2.2. **The distribution of** $(\kappa_u^{(a,-)}, X_{\kappa_u^{(a,-)}})$**.** We now prove a simpler version of Theorem 1, involving only one barrier.

Proposition 7. *Conditionally on* $\{\kappa_u^{(a,-)} < +\infty\}$*, the random variables* $\kappa_u^{(a,-)}$ *and* $X_{\kappa_u^{(a,-)}}$ *are independent, and their distributions are respectively given by*

$$\mathbb{E}_a\left[e^{-\gamma \kappa_u^{(a,-)}}\right] = e^{-\gamma u}\frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_\gamma(a,a)} + \int_u^{+\infty} e^{-\gamma t}\nu_-^{(a)}(dt)}$$

and

$$\mathbb{E}_a\left[\alpha\big(X_{\kappa_u^{(a,-)}}\big)|\kappa_u^{(a,-)} < +\infty\right] = \mathbb{E}_a^{\downarrow,u}\left[\alpha\left(X_u\right)\right].$$

Proof. We set

$$d_t^{(a)} = \inf\{s > t,\ X_s = a\}$$

and

$$F_{\alpha,\gamma}(a) = \mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}\right].$$

Let $0 < \varepsilon < u$. We shall decompose the expectation of $F_{\alpha,\gamma}(a)$ according as whether X_ε is greater or smaller than a. When $X_\varepsilon > a$, applying the Markov property at the time $d_\varepsilon^{(a)}$, we obtain

$$\mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon>a\}}\right] = \mathbb{E}_a\left[e^{-\gamma d_\varepsilon^{(a)}}1_{\{X_\varepsilon>a\}}\right]F_{\alpha,\gamma}(a). \tag{2.6}$$

When $X_\varepsilon \leq a$, we need to take care as whether or not the excursion below a straddling ε is longer than u. On the one hand, if this excursion is shorter than u, then

$$\begin{aligned}\mathbb{E}_a&\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}<g_\varepsilon^{(a)}+u\}}\right]\\ &= \mathbb{E}_a\left[e^{-\gamma d_\varepsilon^{(a)}}1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}<g_\varepsilon^{(a)}+u\}}\right]F_{\alpha,\gamma}(a).\end{aligned} \tag{2.7}$$

On the other hand, if this excursion is longer than u, then $\kappa_u^{(a,-)} = g_\varepsilon^{(a)} + u$ a.s. and

$$\begin{aligned}\mathbb{E}_a&\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]\\ &= \mathbb{E}_a\left[e^{-\gamma(g_\varepsilon^{(a)}+u)}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right].\end{aligned} \tag{2.8}$$

Gathering (2.6), (2.7) and (2.8) together and isolating $F_{\alpha,\gamma}(a)$, we deduce that

$$F_{\alpha,\gamma}(a) = \frac{\mathbb{E}_a\left[e^{-\gamma(g_\varepsilon^{(a)}+u)}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]}{1-\mathbb{E}_a\left[e^{-\gamma d_\varepsilon^{(a)}}1_{\{X_\varepsilon>a\}}\right]-\mathbb{E}_a\left[e^{-\gamma d_\varepsilon^{(a)}}1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}<g_\varepsilon^{(a)}+u\}}\right]}$$

and it remains to let $\varepsilon \downarrow 0$. We start with the numerator. Since $g_\varepsilon^{(a)} \leq \varepsilon$, we have

$$\mathbb{E}_a\left[e^{-\gamma g_\varepsilon^{(a)}}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]$$
$$\leq \mathbb{E}_a\left[\max_{s\in[0,\varepsilon]}\alpha\left(X_{s+u}\right)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq u\}}\right]$$
$$= \mathbb{E}_a\left[1_{\{X_\varepsilon\leq a\}}\mathbb{E}_{X_\varepsilon}\left[\max_{s\in[0,\varepsilon]}\alpha\left(X_{s+u-\varepsilon}\right)1_{\{\varepsilon+T_a\geq u\}}\right]\right].$$

Fix $\delta \in (\varepsilon, u)$. Applying again the Markov property, this last expression is then smaller than

$$\mathbb{E}_a\left[1_{\{X_\varepsilon\leq a\}}\mathbb{E}_{X_\varepsilon}\left[\max_{r\in[0,\delta]}\alpha\left(X_{r+u-\delta}\right)1_{\{T_a\geq u-\delta\}}\right]\right]$$
$$= \mathbb{E}_a\left[1_{\{X_\varepsilon\leq a\}}\mathbb{E}_{X_\varepsilon}\left[\mathbb{E}_{X_{u-\delta}}\left[\max_{r\in[0,\delta]}\alpha\left(X_r\right)\right]1_{\{T_a\geq u-\delta\}}\right]\right]$$

From Lemma 6 and Formula (1.4) we deduce that :

$$\mathbb{E}_a\left[1_{\{X_\varepsilon\leq a\}}\mathbb{E}_{X_\varepsilon}\left[\mathbb{E}_{X_{u-\delta}}\left[\max_{r\in[0,\delta]}\alpha\left(X_r\right)\right]1_{\{T_a\geq u-\delta\}}\right]\right]$$
$$\underset{\varepsilon\downarrow 0}{\sim}\ \mathbb{E}_a^{\downarrow,u-\delta}\left[\mathbb{E}_{X_{u-\delta}}\left[\max_{r\in[0,\delta]}\alpha\left(X_r\right)\right]\right]\nu_-^{(a)}[u-\delta,+\infty]\int_0^\varepsilon p(t;a,a)dt.$$

As a consequence, we obtain the upper bound :

$$\limsup_{\varepsilon\downarrow 0}\frac{\mathbb{E}_a\left[e^{-\gamma g_\varepsilon^{(a)}}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]}{\nu_-^{(a)}[u-\delta,+\infty]\int_0^\varepsilon p(t;a,a)dt}\leq \mathbb{E}_a^{\downarrow,u-\delta}\left[\mathbb{E}_{X_{u-\delta}}\left[\max_{r\in[0,\delta]}\alpha\left(X_r\right)\right]\right].$$

Similarly, we have the lower bound :

$$\liminf_{\varepsilon\downarrow 0}\frac{\mathbb{E}_a\left[e^{-\gamma g_\varepsilon^{(a)}}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]}{\nu_-^{(a)}[u,+\infty]\int_0^\varepsilon p(t;a,a)dt}\geq e^{-\lambda\delta}\mathbb{E}_a^{\downarrow,u}\left[\mathbb{E}_{X_u}\left[\min_{r\in[0,\delta]}\alpha\left(X_r\right)\right]\right].$$

Letting $\delta \downarrow 0$, we deduce by continuity that the numerator is equivalent to :

$$\mathbb{E}_a\left[e^{-\gamma g_\varepsilon^{(a)}}\alpha\big(X_{g_\varepsilon^{(a)}+u}\big)1_{\{X_\varepsilon\leq a\}}1_{\{d_\varepsilon^{(a)}\geq g_\varepsilon^{(a)}+u\}}\right]\underset{\varepsilon\downarrow 0}{\sim}\mathbb{E}_a^{\downarrow,u}\left[\alpha\left(X_u\right)\right]\nu_-^{(a)}[u,+\infty]\int_0^\varepsilon p(t;a,a)dt.$$

Looking now at the denominator, we obtain, adding and subtracting indicators functions :

$$\frac{1}{s'(a)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-\gamma T_a}1_{\{T_a\leq u\}}\right]\Big|_{x=a^-}-\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-\gamma T_a}\right]\Big|_{x=a^+}\right)\int_0^\varepsilon p(t;a,a)dt.$$
$$=\left(\frac{1}{G_\gamma(a,a)}+\int_u^{+\infty}e^{-\gamma t}\nu_-^{(a)}(dt)\right)\int_0^\varepsilon p(t;a,a)dt.$$

Gathering the two previous asymptotics, we finally conclude

$$\mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}\right] = e^{-\gamma u}\frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_\gamma(a,a)}+\int_u^{+\infty}e^{-\gamma t}\nu_-^{(a)}(dt)}\mathbb{E}_a^{\downarrow,u}\left[\alpha\left(X_u\right)\right]$$

which ends the proof of Proposition 7. □

Remark 8. We briefly sketch here another proof for the distribution of $\kappa_u^{(a,-)}$ which is inspired from Getoor [10]. Assume that X is recurrent. The right-continuous inverse $(\tau_\ell^a,\, \ell \geq 0)$ of L^a is a subordinator with Laplace transform :

$$\mathbb{E}_a\left[e^{-\gamma\tau_\ell^a}\right] = \exp\left(-\ell\frac{s'(a)}{G_\gamma(a,a)}\right) = \exp\left(-\ell s'(a)\int_0^{+\infty}(1-e^{-\gamma t})\nu^{(a)}(dt)\right).$$

where $\nu^{(a)} = \nu_-^{(a)} + \nu_+^{(a)}$. Furthermore, from Pitman & Yor [15], the two processes

$$\left(\int_0^{\tau_\ell} 1_{\{X_u>a\}}du, \ell \geq 0\right) \quad \text{and} \quad \left(\int_0^{\tau_\ell} 1_{\{X_u<a\}}du, \ell \geq 0\right) \tag{2.9}$$

are independent subordinators with respective Lévy measures given by $s'(a)\nu_+^{(a)}$ and $s'(a)\nu_-^{(a)}$. As a consequence, we may decompose $\tau^{(a)}$ into two independent subordinators

$$\tau^{(a)} = \eta^{(0)} + \eta^{(1)}$$

with respective Lévy measures $s'(a)(\nu_+^{(a)}(dt) + 1_{\{t<u\}}\nu_-^{(a)}(dt))$ and $s'(a)1_{\{t\geq u\}}\nu_-^{(a)}(dt)$. Now $\kappa_u^{(a,-)}$ is the starting point of the first excursion below a of length greater than u. Define the first jump of $\eta^{(1)}$ by

$$R = \inf\{\ell > 0,\ \eta_\ell^{(1)} \neq 0\}.$$

The random variable R is exponentially distributed with parameter $s'(a)\nu^{(a,-)}[u,+\infty]$. Then

$$\begin{aligned}
\mathbb{E}_a\left[e^{-\gamma(\kappa_u^{(a,-)}-u)}\right] &= \mathbb{E}\left[e^{-\gamma\eta_{R-}^{(0)}}\right] \\
&= s'(a)\nu_-^{(a)}[u,+\infty]\int_0^{+\infty}\mathbb{E}\left[e^{-\gamma\eta_{r-}^{(0)}}\right]e^{-s'(a)\nu_-^{(a)}[u,+\infty]r}dr \\
&= \frac{\nu_-^{(a)}[u,+\infty]}{\int_0^{+\infty}(1-e^{-\gamma t})\nu_+^{(a)}(dt)+\int_0^u(1-e^{-\gamma t})\nu_-^{(a)}(dt)+\nu_-^{(a)}[u,+\infty]} \\
&= \frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_\gamma(a,a)}+\int_u^{+\infty}e^{-\gamma t}\nu_-^{(a)}(dt)},
\end{aligned}$$

which yields the Laplace transform of $\kappa_u^{(a,-)}$ when X is recurrent. However, when X is transient, the two subordinators given in Formula (2.9) are no longer independent as explained in [15]. They are only conditionally independent given L_∞^a, since when $\ell = L_\infty^a$, one will be stopped at the time L_∞^a while the other one will jump to $+\infty$. As a consequence, in the transient case, one would need to condition by L_∞^a before mimicking the previous argument.

3. Proof of Theorem 1

The proof of Theorem 1 is similar to the proof of Proposition 7, except that we shall work with two boundaries $a < b$, and thus write down a system of equations involving $F_{\alpha,\beta,\gamma,\lambda}(a)$ and $F_{\alpha,\beta,\gamma,\lambda}(b)$. Let $0 < \varepsilon < \min(u, v)$. To compute $F_{\alpha,\beta,\gamma,\lambda}(a)$ and $F_{\alpha,\beta,\gamma,\lambda}(b)$, we shall decompose the expectation in three terms, according to the position of X_ε with respect to a and b.

3.1. **Study of $F_{\alpha,\beta,\gamma,\lambda}(a)$.** First, if $X_\varepsilon \leq a$, then the excursion below a straddling ε must remain shorter than u. Applying the Markov property at the time $d_\varepsilon^{(a)}$, this yields the formula

$$\begin{aligned}\mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\leq a\}}\right]\\ = \mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(a)}}1_{\{d_\varepsilon^{(a)}<g_\varepsilon^{(a)}+u\}}1_{\{X_\varepsilon\leq a\}}\right]F_{\alpha,\beta,\gamma,\lambda}(a).\end{aligned}\tag{3.1}$$

Next, if $a < X_\varepsilon < b$, we obtain, applying the Markov property at the times $d_\varepsilon^{(a)}$ and $d_\varepsilon^{(b)}$:

$$\begin{aligned}\mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{a<X_\varepsilon<b\}}\right]\\ = \mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(b)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(b)}<d_\varepsilon^{(a)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(b)\\ +\mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(a)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(a)}<d_\varepsilon^{(b)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(a).\end{aligned}\tag{3.2}$$

We denote the last term when $X_\varepsilon \geq b$ by $\Delta_a(\varepsilon)$:

$$\Delta_a(\varepsilon) = \mathbb{E}_a\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\geq b\}}\right].\tag{3.3}$$

We shall prove later that this term has no contribution in the result. As a consequence, gathering (3.1), (3.2) and (3.3), we deduce that :

$$F_{\alpha,\beta,\gamma,\lambda}(a) = \frac{\mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(b)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(b)}<d_\varepsilon^{(a)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(b)+\Delta_a(\varepsilon)}{1-\mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(a)}}1_{\{d_\varepsilon^{(a)}<g_\varepsilon^{(a)}+u\}}1_{\{X_\varepsilon\leq a\}}\right]-\mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(a)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(a)}<d_\varepsilon^{(b)}\}}\right]}$$

and it remains to let $\varepsilon \downarrow 0$. Applying the Markov property and Lemma 6, we deduce that the first term in the numerator is equivalent to :

$$\begin{aligned}\mathbb{E}_a\left[e^{-(\gamma+\lambda)d_\varepsilon^{(b)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(b)}<d_\varepsilon^{(a)}\}}\right] &= \mathbb{E}_a\left[1_{\{a<X_\varepsilon<b\}}\mathbb{E}_{X_\varepsilon}\left[e^{-(\gamma+\lambda)T_b}1_{\{T_b<T_a\}}\right]\right]\\ &\underset{\varepsilon\downarrow 0}{\sim}\frac{1}{s'(a)}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_b}1_{\{T_b<T_a\}}\right]\Big|_{x=a^+}\times\int_0^\varepsilon p(t;a,a)dt.\end{aligned}$$

Similarly, the denominator is equivalent to :

$$\frac{1}{s'(a)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_a}1_{\{T_a<u\}}\right]\Big|_{x=a^-}-\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_a}1_{\{T_a<T_b\}}\right]\Big|_{x=a^+}\right)\times\int_0^\varepsilon p(t;a,a)dt.$$

Finally, from (2.1) and (2.2), the last term $\Delta_a(\varepsilon)$ is negligible since

$$\frac{\Delta_a(\varepsilon)}{\int_0^\varepsilon p(t;a,a)dt} \leq \max_{z\in\mathbb{R}}(\alpha(z)\beta(z))\frac{\mathbb{P}_a(T_b\leq\varepsilon)}{\int_0^\varepsilon p(t;a,a)dt} \xrightarrow[\varepsilon\downarrow 0]{} 0$$

so we obtain, after some simplifications,

$$\begin{aligned}
F_{\alpha,\beta,\gamma,\lambda}(a) &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_b}1_{\{T_b<T_a\}}\right]\Big|_{x=a^+}}{\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_a}1_{\{T_a<u\}}\right]\Big|_{x=a^-} - \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left[e^{-(\gamma+\lambda)T_a}1_{\{T_a<T_b\}}\right]\Big|_{x=a^+}}F_{\alpha,\beta,\gamma,\lambda}(b)\\
&= \frac{\frac{W^{(a)\prime}_{\gamma+\lambda}(a)}{W^{(a)}_{\gamma+\lambda}(b)}}{\frac{\Phi'_{\gamma+\lambda,-}(a)}{\Phi_{\gamma+\lambda,-}(a)} + s'(a)\int_u^{+\infty}e^{-(\gamma+\lambda)t}\nu_-^{(a)}(dt) - \frac{W^{(b)\prime}_{\gamma+\lambda}(a)}{W^{(b)}_{\gamma+\lambda}(a)}}F_{\alpha,\beta,\gamma,\lambda}(b)\\
&= \frac{1}{\Psi^{(-)}_{\gamma+\lambda}(b,a,u)}F_{\alpha,\beta,\gamma,\lambda}(b) \qquad (3.4)
\end{aligned}$$

which gives a first equation relating $F_{\alpha,\beta,\gamma,\lambda}(a)$ and $F_{\alpha,\beta,\gamma,\lambda}(b)$.

Remark 9. By identification, the Laplace transform of T_b on the event $\{T_b < \kappa_u^{(a,-)}\}$ is given by

$$\mathbb{E}_a\left[e^{-\lambda T_b}1_{\{T_b<\kappa_u^{(a,-)}\}}\right] = \frac{1}{\Psi^{(-)}_{\lambda}(b,a,u)}.$$

3.2. **Study of** $F_{\alpha,\beta,\gamma,\lambda}(b)$**.** First, if $X_\varepsilon > b$, we need to take care as whether or not the excursion above b straddling ε is longer than v. On the one hand, if this excursion is longer than v, then $\kappa_v^{(b,+)} = g_\varepsilon^{(b)} + v$, hence, applying as before the Markov property at the time $d_\varepsilon^{(b)}$:

$$\begin{aligned}
&\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\geq b\}}1_{\{d_\varepsilon^{(b)}\geq v+g_\varepsilon^{(b)}\}}\right]\\
&\quad= \mathbb{E}_b\left[e^{-\gamma d_\varepsilon^{(b)}-\lambda(g_\varepsilon^{(b)}+v)}\beta\big(X_{g_\varepsilon^{(b)}+v}\big)1_{\{X_\varepsilon\geq b\}}1_{\{d_\varepsilon^{(b)}\geq v+g_\varepsilon^{(b)}\}}\right]\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}\right].
\end{aligned}$$

On the other hand, if this excursion is shorter than v, then

$$\begin{aligned}
&\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\geq b\}}1_{\{d_\varepsilon^{(b)}<v+g_\varepsilon^{(b)}\}}\right]\\
&\qquad\qquad= \mathbb{E}_b\left[e^{-(\gamma+\lambda)d_\varepsilon^{(b)}}1_{\{X_\varepsilon\geq b\}}1_{\{d_\varepsilon^{(b)}<v+g_\varepsilon^{(b)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(b).
\end{aligned}$$

Next, if $a < X_\varepsilon < b$, we obtain, applying the Markov property at the times $d_\varepsilon^{(a)}$ and $d_\varepsilon^{(b)}$:

$$\begin{aligned}
&\mathbb{E}_b\left[e^{-\lambda\kappa_v^{(b,+)}-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\leq\kappa_u^{(a,-)}<+\infty\}}1_{\{a<X_\varepsilon<b\}}\right]\\
&\qquad= \mathbb{E}_b\left[e^{-(\gamma+\lambda)d_\varepsilon^{(b)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(b)}<d_\varepsilon^{(a)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(b)\\
&\qquad\quad+ \mathbb{E}_b\left[e^{-(\gamma+\lambda)d_\varepsilon^{(a)}}1_{\{a<X_\varepsilon<b\}}1_{\{d_\varepsilon^{(a)}<d_\varepsilon^{(b)}\}}\right]F_{\alpha,\beta,\gamma,\lambda}(a). \qquad (3.5)
\end{aligned}$$

Finally, as in the previous subsection, the last term when $X_\varepsilon \le a$ is again negligible :

$$\Delta_b(\varepsilon) = \mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}-\lambda\kappa_v^{(b,+)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)\beta\big(X_{\kappa_v^{(b,+)}}\big)1_{\{\kappa_v^{(b,+)}\le\kappa_u^{(a,-)}<+\infty\}}1_{\{X_\varepsilon\le a\}}\right].$$

Passing to the limit as $\varepsilon \downarrow 0$ and proceeding as in Proposition 7, we deduce that

$$\begin{aligned}
&\Psi^{(+)}_{\gamma+\lambda}(a,b,v)F_{\alpha,\beta,\gamma,\lambda}(b) - F_{\alpha,\beta,\gamma,\lambda}(a)\\
&= e^{-\lambda v-\gamma v}\frac{W^{(b)}_{\gamma+\lambda}(a)}{\omega_{\gamma+\lambda}}\mathbb{E}_b\left[e^{-\gamma\kappa_u^{(a,-)}}\alpha\big(X_{\kappa_u^{(a,-)}}\big)1_{\{\kappa_u^{(a,-)}<+\infty\}}\right]\mathbb{E}_b^{\uparrow,v}\left[\frac{\Phi_{\gamma,+}(X_v)}{\Phi_{\gamma,+}(b)}\beta\left(X_v\right)\right]\nu_+^{(b)}[v,+\infty]
\end{aligned}$$

and the expression for $F_{\alpha,\beta,\gamma,\lambda}(b)$ follows by plugging (3.4) in this last formula. Finally, to get the general expression of Theorem 1 for $a \le x \le b$, one simply need to apply the Markov property

$$F_{\alpha,\beta,\gamma,\lambda}(x) = \mathbb{E}_x\left[e^{-(\gamma+\lambda)T_a}1_{\{T_a<T_b\}}\right]F_{\alpha,\beta,\gamma,\lambda}(a) + \mathbb{E}_x\left[e^{-(\gamma+\lambda)T_b}1_{\{T_b<T_a\}}\right]F_{\alpha,\beta,\gamma,\lambda}(b)$$

and then use (3.4). □

4. Inversions and Asymptotics

We shall provide in this section a few results regarding the density of Parisian times as well as the asymptotics of their survival function under some specific assumptions.

4.1. **Probability densities.** We start by giving theoretical expressions for the densities of the Parisian times $\kappa_u^{(a,-)}$ and $\kappa_v^{(a,+)}\wedge\kappa_u^{(a,-)}$ when starting from $x=a$. Our approach is similar to that of Dassios & Lim [4, 5] who wrote down the explicit computations in the Brownian case.

Proposition 10. *Let $*$ denote the standard convolution product.*

(1) The probability density of $\kappa_u^{(a,-)}$ is given by

$$\begin{aligned}
&\mathbb{P}_a\left(\kappa_u^{(a,-)} - u \in dt\right)/dt\\
&\qquad= \nu_-^{(a)}[u,+\infty]\sum_{n=0}^{+\infty}(-1)^n\int_0^t p_{t-r}(a,a)\left(p_\cdot(a,a) * \nu_-^{(a)}1_{\{\cdot\ge u\}}\right)^{*(n)}(r)dr.
\end{aligned}$$

(2) For $v \ge u$, the probability density of $\kappa_v^{(a,+)}\wedge\kappa_u^{(a,-)}$ is given by :

$$\begin{aligned}
&\mathbb{P}_a\left(\kappa_v^{(a,+)}\wedge\kappa_u^{(a,-)} - u \in dt\right)/dt\\
&= \nu_-^{(a)}[u,+\infty]\sum_{n=0}^{+\infty}(-1)^n\int_0^t p_{t-r}(a,a)\left(p_\cdot(a,a) * \nu_S^{(a)}\right)^{*(n)}(r)dr\\
&\qquad+ \nu_+^{(a)}[v,+\infty]\sum_{n=0}^{+\infty}(-1)^n\int_{v-u}^{t\wedge(v-u)} p_{t-r-(v-u)}(a,a)\left(p_\cdot(a,a) * \nu_S^{(a)}\right)^{*(n)}(r)dr
\end{aligned}$$

where the measure $\nu_S^{(a)}$ is defined as the sum $\nu_S^{(a)}(dt) = \nu_+^{(a)}(dt)1_{\{t\ge v\}} + \nu_-^{(a)}(dt)1_{\{t\ge u\}}$.

Note that due to the presence of indicator functions, the value of the probability density for a fixed t only depends on a finite number of terms in the sum. Such observation was at the core of the aforementioned papers [4, 5].

Proof. We only prove Point (2), as Point (1) may be obtained by letting $v \uparrow +\infty$. Starting from Corollary 3, we have by symmetry :

$$\mathbb{E}_a\left[e^{-\lambda(\kappa_v^{(a,+)}\wedge\kappa_u^{(a,-)}-u)}\right] = \frac{e^{-\lambda(v-u)}\nu_+^{(a)}[v,+\infty] + \nu_-^{(a)}[u,+\infty]}{\int_v^{+\infty} e^{-\lambda t}\nu_+^{(a)}(dt) + \int_u^{+\infty} e^{-\lambda t}\nu_-^{(a)}(dt) + \frac{1}{G_\lambda(a,a)}}. \tag{4.1}$$

For λ large enough, the right-hand side may be written :

$$\left(e^{-\lambda(v-u)}\nu_+^{(a)}[v,+\infty] + \nu_-^{(a)}[u,+\infty]\right) G_\lambda(a,a) \times \sum_{n=0}^{+\infty}(-1)^n \left(G_\lambda(a,a)\int_0^{+\infty} e^{-\lambda t}\nu_S^{(a)}(dt)\right)^n.$$

The result now follows by inverting this Laplace transform, using the definition of $G_\lambda(a,a)$ given in (1.2). □

4.2. **Asymptotics with one barrier.** We now compute some asymptotics involving Parisian times with one and two barriers. Starting from their Laplace transform, the key formula we shall use is the following relationship between the Green function $G_\lambda(a,a)$ and the Lévy measures $\nu_\pm^{(a)}$, see [15] :

$$\frac{1}{G_\gamma(a,a)} = \gamma\int_0^{+\infty} e^{-\gamma t}\left(\nu_+^{(a)}[t,+\infty] + \nu_-^{(a)}[t,+\infty]\right) dt \tag{4.2}$$

Proposition 11. *Assume that $t \to \nu_+^{(a)}[t,+\infty]$ is regularly varying at $+\infty$:*

$$\nu_+^{(a)}[t,+\infty] \underset{t\to+\infty}{\sim} t^{-\alpha}\ell(t)$$

where $\alpha \in (0,1)$ and ℓ is a slowly varying function. Then,

$$\mathbb{P}_a\left(\kappa_u^{(a,-)} \geq t\right) \underset{t\to+\infty}{\sim} \frac{\nu_+^{(a)}[t,+\infty]}{\frac{1}{G_0(a,a)} + \nu_-^{(a)}[u,+\infty)}.$$

Proof. We start from the Laplace transform of $\kappa_u^{(a,-)}$ which we write :

$$\gamma\int_0^{+\infty} e^{-\gamma t}\mathbb{P}_a(\kappa_u^{(a,-)} \geq t)dt = 1 - e^{-\gamma u}\frac{\nu_-^{(a)}[u,+\infty]}{\frac{1}{G_\gamma(a,a)} + \int_u^{+\infty} e^{-\gamma t}\nu_-^{(a)}(dt)}.$$

Integrating by parts and using Formula (4.2), the denominator is further equal to

$$\gamma\int_0^{+\infty} e^{-\gamma t}\nu_+^{(a)}[t,+\infty]dt + \gamma e^{-\gamma u}\nu_-^{(a)}[u,+\infty] + \gamma\int_0^u e^{-\gamma t}\nu_-^{(a)}[t,+\infty]dt.$$

As a consequence, we obtain

$$\int_0^{+\infty} e^{-\gamma t}\mathbb{P}_a(\kappa_u^{(a,-)} \geq t)dt = \frac{\int_0^{+\infty} e^{-\gamma t}\nu_+^{(a)}[t,+\infty]dt}{\frac{1}{G_\gamma(a,a)} + \int_u^{+\infty} e^{-\gamma t}\nu_-^{(a)}(dt)} + \frac{\int_0^{u} e^{-\gamma t}\nu_-^{(a)}[t,+\infty]dt}{\frac{1}{G_\gamma(a,a)} + \int_u^{+\infty} e^{-\gamma t}\nu_-^{(a)}(dt)}.$$

The result now follows by letting $\gamma \downarrow 0$ and applying the Karamata's Tauberian theorem, while observing that from the monotone convergence theorem and Fubini's theorem

$$\lim_{\gamma\downarrow 0} \int_0^{u} e^{-\gamma t}\nu_-^{(a)}[t,+\infty]dt = \int_0^{u} \nu_-^{(a)}[t,+\infty]dt = \int_0^{+\infty} (t\wedge u)\nu_-^{(a)}(dt) < +\infty$$

since $\nu_-^{(a)}$ is the Lévy measure of a subordinator. $\square$

It is noteworthy to point out that the asymptotics of $\kappa_u^{(a,-)}$ is given by $\nu_+^{(a)}$. In other words, the probability that the diffusion does not make an excursion below a longer that u for a very long time is governed by the distribution of its (large) excursions above a.

4.3. **Asymptotics with two barriers.** In the case of two barriers, as for the exit time from an interval, we shall see that the asymptotics are rather exponential. Let us define the following function f on $\mathbb{R}$:

$$f(\lambda) = \frac{\int_0^{v} e^{-\lambda t}\nu_+^{(b)}[t,+\infty]dt + \int_0^{u} e^{-\lambda t}\nu_-^{(b)}[t,+\infty]dt}{e^{-\lambda v}\nu_+^{(b)}[v,+\infty] + e^{-\lambda u}\nu_-^{(b)}[u,+\infty]}$$

Proposition 12. *Assume that* $\liminf_{\lambda\to-\infty} \lambda f(\lambda) < -1$. *Let* λ^* *be the smallest positive solution of the equation*

$$1 - \lambda^* f(-\lambda^*) = 0$$

and assume that $(\lambda^*)^2 f'(-\lambda^*) < 1$. *Then,*

$$\lim_{x\to+\infty} \frac{1}{x} \ln \mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq x) = -\lambda^*.$$

Proof. We start from the Laplace transform given in (4.1) :

$$1 - \mathbb{E}_b\left[e^{-\lambda\kappa_v^{(b,+)}\wedge\kappa_u^{(b,-)}}\right] = 1 - \frac{e^{-\lambda v}\nu_+^{(b)}[v,+\infty] + e^{-\lambda u}\nu_-^{(b)}[u,+\infty]}{\int_v^{+\infty} e^{-\lambda t}\nu_+^{(b)}(dt) + \int_u^{+\infty} e^{-\lambda t}\nu_-^{(b)}(dt) + \frac{1}{G_\lambda(b,b)}}.$$

Integrating by parts the denominator and using Formula (4.2), we obtain

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{P}_b\left(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq t\right)dt = \frac{f(\lambda)}{1+\lambda f(\lambda)}.$$

As a consequence, applying Taylor's formula at the point λ^* :

$$\int_0^{+\infty} e^{-(\lambda-\lambda^*)t}\mathbb{P}_b\left(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq t\right)dt = \frac{f(\lambda-\lambda^*)}{1+(\lambda-\lambda^*)f(\lambda-\lambda^*)} \underset{\lambda\downarrow 0}{\sim} \frac{1}{(1-(\lambda^*)^2 f'(-\lambda^*))\,\lambda}$$

and thus from the Tauberian theorem

$$\int_0^x e^{\lambda^* t}\mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq t)dt \underset{x\to+\infty}{\sim} \frac{x}{1-(\lambda^*)^2 f'(-\lambda^*)}.$$

It is now a classic matter to obtain a logarithmic asymptotic. One the one hand, by monotony, we first have :

$$\int_0^x e^{\lambda^* t}\mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq t)dt \geq \mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq x)\frac{1}{\lambda^*}\left(e^{\lambda^* x} - 1\right)$$

which implies

$$-\lambda^* \geq \limsup_{x\to+\infty} \frac{1}{x}\ln \mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq x).$$

On the other hand, let $\varepsilon > 0$. We have

$$\int_x^{x(1+\varepsilon)} e^{\lambda^* t}\mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq t)dt \leq \mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq x)\frac{e^{\lambda^* x(1+\varepsilon)} - e^{\lambda^* x}}{\lambda^*}$$

which implies

$$-\lambda^*(1+\varepsilon) \leq \liminf_{x\to+\infty} \frac{1}{x}\ln \mathbb{P}(\kappa_v^{(b,+)} \wedge \kappa_u^{(b,-)} \geq x)$$

and the result follows by letting $\varepsilon \downarrow 0$.

□

5. Examples

We briefly give a few examples where the expressions for the Lévy measures $\nu_-^{(a)}$ and $\nu_+^{(b)}$ may be written down explicitly.

5.1. **Brownian motion with drift μ.** Assume that X is a Brownian motion with drift μ. Its scale function may be chosen such that :

$$s'(x) = e^{-2\mu x}.$$

The two eigenfunctions $\Phi_{\lambda,+}$ and $\Phi_{\lambda,-}$ are given by :

$$\Phi_{\lambda,+}(x) = \exp\left(-x\left(\sqrt{2\lambda+\mu^2}+\mu\right)\right) \quad \text{and} \quad \Phi_{\lambda,-}(x) = \exp\left(x\left(\sqrt{2\lambda+\mu^2}-\mu\right)\right)$$

and

$$\mathbb{P}_x(T_a \in dt) = \frac{|a-x|}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{(a-x-\mu t)^2}{2t}\right)dt.$$

As a consequence, the Lévy measures $\nu_\pm^{(a)}$ only depend on a through the scale function s :

$$\nu_\pm^{(a)}(dt) = e^{2\mu a}\frac{1}{\sqrt{2\pi}t^{3/2}}e^{-\frac{\mu^2}{2}t}dt,$$

and

$$\nu_\pm^{(a)}(\{+\infty\}) = e^{2\mu a}\left(|\mu| \pm \mu\right).$$

When $\mu = 0$, one may choose $s(x) = x$ and applying Remark 5, we recover for instance the formula for a standard Brownian motion (see [8]), i.e. for $a \leq x \leq b$:

$$\mathbb{P}_x\left(\kappa_v^{(b,+)} \leq \kappa_u^{(a,-)}\right) = \frac{\sqrt{u} + (x-a)\sqrt{\frac{2}{\pi}}}{\sqrt{v} + \sqrt{u} + (b-a)\sqrt{\frac{2}{\pi}}}.$$

5.2. **Three-dimensional Bessel process with drift $\mu > 0$ in the wide sense.** Let us consider the solution of the following SDE :

$$X_t = x + B_t + \mu \int_0^t \frac{\cosh(\mu X_s)}{\sinh(\mu X_s)} ds$$

where $x > 0$ and B is a standard Brownian motion. The law of X is that of a Brownian motion with drift $\mu > 0$ conditioned to stay positive, or alternatively that of a three-dimensional Bessel process with drift μ in the wide sense, see Pitman & Yor [16] or Watanabe [18]. Its scale function may be chosen such that

$$s'(x) = \frac{\mu^2}{\sinh^2(\mu x)}$$

and the two eigenfunctions are given by

$$\Phi_{\lambda,+}(x) = \mu \frac{\exp\left(-x\sqrt{2\lambda + \mu^2}\right)}{\sinh(\mu x)} \quad \text{and} \quad \Phi_{\lambda,-}(x) = \mu \frac{\sinh(x\sqrt{2\lambda+\mu^2})}{\sinh(\mu x)}.$$

As a consequence, we deduce from [9, p.258] that

$$\mathbb{P}_x(T_a \in dt)/dt = \begin{cases} \dfrac{\sinh(\mu a)}{\sinh(\mu x)} \dfrac{1}{\sqrt{2\pi}t^{3/2}} \displaystyle\sum_{n=-\infty}^{+\infty} ((2n+1)a - x) e^{-\frac{((2n+1)a-x)^2}{2t} - \frac{\mu^2}{2}t} & \text{if } x < a, \\ \\ \dfrac{\sinh(\mu a)}{\sinh(\mu x)} \dfrac{(x-a)}{\sqrt{2\pi}t^{3/2}} \exp\left(-\dfrac{(x-a)^2}{2t} - \dfrac{\mu^2}{2}t\right) & \text{if } x > a, \end{cases}$$

hence we obtain

$$\nu_-^{(a)}(dt) = \frac{\sinh^2(\mu a)}{\mu^2} \frac{1}{\sqrt{2\pi}t^{3/2}} e^{-\frac{\mu^2}{2}t} \sum_{n=-\infty}^{+\infty} \left(\frac{\mu\cosh(\mu a)}{\sinh(\mu a)} 2na + 1 - \frac{4n^2a^2}{t}\right) e^{-\frac{2n^2a^2}{t}} dt$$

and

$$\nu_+^{(b)}(dt) = \frac{\sinh^2(\mu b)}{\mu^2} \frac{1}{\sqrt{2\pi}t^{3/2}} e^{-\frac{\mu^2}{2}t} dt.$$

Note that the measure $\nu_+^{(b)}$ admits an atom at $+\infty$ which is given by :

$$\nu_+^{(b)}(\{+\infty\}) = \frac{e^{2\mu b} - 1}{2\mu}.$$

As a consequence, we deduce for instance that :

$$\mathbb{P}_a\left(\kappa_u^{(a,-)} < +\infty\right) = \frac{\nu_-^{(a)}[u,+\infty)}{\frac{e^{2\mu a}-1}{2\mu} + \nu_-^{(a)}[u,+\infty)}.$$

5.3. **Reflected Brownian motion.** Assume that $X = |B|$ with B a standard Brownian motion. The two eigenfunctions $\Phi_{\lambda,+}$ and $\Phi_{\lambda,-}$ are given by :

$$\Phi_{\lambda,+}(x) = \exp\left(-x\sqrt{2\lambda}\right) \qquad \text{and} \qquad \Phi_{\lambda,-}(x) = \cosh\left(x\sqrt{2\lambda}\right)$$

and from Borodin & Salminen [2], the distribution of T_a admits the density

$$\mathbb{P}_x(T_a \in dt)/dt = \begin{cases} \dfrac{1}{\sqrt{2\pi}t^{3/2}} \displaystyle\sum_{n=-\infty}^{+\infty} (-1)^n((2n+1)a+x)e^{-\frac{((2n+1)a+x)^2}{2t}} & \text{if } x < a, \\ \\ \dfrac{(x-a)}{\sqrt{2\pi}t^{3/2}} \exp\left(-\dfrac{(x-a)^2}{2t}\right) & \text{if } x > a. \end{cases}$$

Therefore, we deduce that

$$\nu_-^{(a)}(dt) = \frac{1}{\sqrt{2\pi}t^{3/2}} \sum_{n=-\infty}^{+\infty} (-1)^{n+1}\left(\frac{4a^2n^2}{t} - 1\right) e^{-\frac{2a^2n^2}{t}} dt$$

and

$$\nu_+^{(b)}(dt) = \frac{1}{\sqrt{2\pi}t^{3/2}} dt.$$

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Université Paris-Saclay, CNRS, Univ Evry, Laboratoire de Mathématiques et Modélisation d'Evry, 91037, Evry-Courcouronnes, France. *Email* : `christophe.profeta@univ-evry.fr`