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Koszul-type determinantal formulas for families of mixed multilinear systems

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Abstract

Effective computation of resultants is a central problem in elimination theory and polynomial system solving. Commonly, we compute the resultant as a quotient of determinants of matrices and we say that there exists a *determinantal formula* when we can express it as a determinant of a matrix whose elements are the coefficients of the input polynomials. We study the resultant in the context of *mixed multilinear polynomial systems*, that is multilinear systems with polynomials having different supports, on which determinantal formulas were not known. We construct determinantal formulas for two kind of multilinear systems related to the *Multiparameter Eigenvalue Problem* (MEP): first, when the polynomials agree in all but one block of variables; second, when the polynomials are bilinear with different supports, related to a bipartite graph. We use the *Weyman complex* to construct *Koszul-type determinantal formulas* that generalize Sylvester-type formulas. We can use the matrices associated to these formulas to solve square systems without computing the resultant. The combination of the resultant matrices with the eigenvalue and eigenvector criterion for polynomial systems leads to a new approach for solving MEP.

Key words. Resultant matrix, Multilinear polynomial, Weyman complex, Determinantal formula, Koszul-type matrix, Multiparameter eigenvalue problem

AMS subject classifications. 13P15 14Q20 15A18

1 Introduction

One of the main questions in (computational) algebraic geometry is to decide efficiently when an overdetermined polynomial system has a solution over a projective variety. The resultant answers this question. The resultant is a multihomogeneous polynomial in the coefficients of the polynomials of the system that vanishes if and only if the system has a solution. We can also use it to solve square systems. When we restrict the supports of the input polynomials to make them sparse, we have an analogous concept called the sparse resultant [28]. The sparse resultant is one of the few tools we can use to solve systems taking into account the sparsity of the support of the polynomials. Hence, its efficient computation is fundamental in computational algebraic geometry.

We are interested in the computation of the multiprojective resultant, as it is defined in [17, 20, 43], of sparse systems given by a particular kind of multilinear polynomials. To define the multiprojective resultant as a single polynomial, we restrict ourselves to systems where the number of equations is one greater than

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the dimension of the ambient multiprojective space. In what follows, we refer to this specific situation as an overdetermined system¹. In general, we compute the resultant of a polynomial system (f_0, \dots, f_n) as a quotient of determinants of two matrices whose elements are polynomials in the coefficients of the input polynomials [11, 12, 15, 16, 33, 36, 37]; thus the best we can hope for are linear polynomials. A classical example of such a matrix is the Macaulay matrix, which represents a map $(g_0, \dots, g_n) \mapsto \sum_i g_i f_i$, where each g_i is a polynomial in a finite dimensional vector space. In this case, we say that we have a *Sylvester-type* formula. Other classical formulas include *Bézout-* and *Dixon-type*; nevertheless, the elements of the corresponding matrices are not linear anymore. We refer to [25] and references therein for details.

When we can compute the resultant as the determinant of a matrix we say that we have a *determinantal formula*. Besides general constructions that express any multivariate polynomial as a determinant of a matrix, see for example [35, 49], we are interested in formulas such that the row/column dimension of the corresponding matrix depends linearly on the degree of the resultant. The existence of such formulas is not known in general. When we consider *unmixed* multihomogeneous systems, that is when every polynomial has the same support, these formulas are well studied, e.g., [11, 18, 47, 54]. However, when the supports are not the same, that is in the case of *mixed* multihomogeneous systems, there are very few results. We know determinantal formulas for scaled multihomogeneous systems [22], in which case the supports are scaled copies of one of them, for bivariate tensor-product polynomial systems [9], and for bilinear systems with two different supports [5]. One tool to obtain such formulas is using the Weyman complex [52]. For an introduction to this complex we refer to [53, Sec. 9.2] and [28, Sec. 2.5.C, Sec. 3.4.E].

Resultant computations are also useful in solving 0-dimensional square polynomial systems, say (f_1, \dots, f_N) , taking into account the sparsity; here "square" refers to systems having N polynomials in N variables. For example we can use the *u*-resultant or hide a variable; we refer to [14, Ch. 3] for a general introduction. Whenever a *Sylvester-type formula* is available, through the resultant matrix, we obtain a matrix representing the multiplication map by a polynomial f_0 in $\mathbb{K}[\mathbf{x}]/(f_1, \dots, f_N)$. Then, we solve the system (f_1, \dots, f_N) by computing the eigenvalues and eigenvectors of this matrix, e.g., [4, 21]. The eigenvalues correspond to the evaluations of f_0 at the solutions of the system. From the eigenvectors, at least when there are no multiplicities, we can recover the coordinates of the solutions. For a generalization of this approach to a broader class of resultant matrices, that encapsulates Sylvester-type matrices as a special case, we refer to [5].

1.1 Multilinear polynomial systems

We focus on computing determinantal formulas for mixed multilinear polynomial systems. Besides their mathematical importance, as they are the first non-trivial case of polynomial systems beyond the linear ones, multilinear systems are also ubiquitous in applications, e.g., cryptography [26, 34] and game theory [39].

For $A, B \in \mathbb{N}$ let $\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_1, \dots, \mathbf{Y}_B$ be blocks of variables. We present various *Koszul-type determinantal formulas* (related to the maps in the Koszul complex, see Def. 2.14) for the following two kinds of *mixed multilinear polynomials systems* $(f_0, f_1 \dots f_N)$:

- *star multilinear systems*: these are polynomial systems (f_1, \dots, f_N) , where for each f_k , there is a $j_k \in [B]$ such that

$$f_k \in \mathbb{K}[\mathbf{X}_1]_1 \otimes \dots \otimes \mathbb{K}[\mathbf{X}_A]_1 \otimes \mathbb{K}[\mathbf{Y}_{j_k}]_1,$$

- *bipartite bilinear systems*: these are polynomial systems (f_1, \dots, f_N) , where for each f_k , there are $i_k \in [A]$ and $j_k \in [B]$ such that

$$f_k \in \mathbb{K}[\mathbf{X}_{i_k}]_1 \otimes \mathbb{K}[\mathbf{Y}_{j_k}]_1.$$

To make the system overdetermined and so, to consider its resultant, we complement it with several types of multilinear polynomials f_0 (see the beginning of sections 3 and 4).

Our first main contribution is the theorem below which is an extract of Thm. 3.7 and 4.4.

Theorem 1.1. *Let $\mathbf{f} := (f_1, \dots, f_N)$ be a star multilinear system (Def. 3.1) or a bipartite bilinear system (Def. 4.1). Then, for certain choices of multilinear polynomials f_0 (we present them at the beginning of Sec. 3 and 4), there is a square matrix M such that,*

- *The resultant of \mathbf{f} agrees with the determinant of the matrix, $\mathbf{res}(\mathbf{f}) = \pm \det(M)$.*
- *The number of columns/rows of M is $\text{degree}(\mathbf{res}(\mathbf{f}))$ and its elements are coefficients of \mathbf{f} , possibly with a sign change.*

¹For general overdetermined systems, there exists the concept of *resultant system*, see [51, Sec. 16,5].

The matrix M corresponds to a Koszul-type determinantal formula (Def. 2.14) for \mathbf{f} .

The size of the resultant matrix and the degree of the resultant depend on the multidegree of f_0 . We relate the (expected) number of solutions of (f_1, \dots, f_N) to the degree of the resultant of (f_0, f_1, \dots, f_N) . For star multilinear systems, we present closed formulas for the expected number of solutions of the system (f_1, \dots, f_N) and the size of the matrices; we also express the size of the matrix in terms of the number of solutions. Our techniques to obtain determinantal formulas exploit the properties and the parametrization, through a carefully chosen degree vector, of the Weyman complex and are of independent interest. These results generalize the ones in [5] that correspond to the case $(A = 1, B = 2)$.

1.2 Multiparameter Eigenvalue Problem

A motivating application for the systems and the determinantal formulas that we study comes from the *multiparameter eigenvalue problem* (MEP). We can model MEP using *star multilinear systems* or *bipartite bilinear systems*. The resultant matrices that we construct together with the eigenvalue and eigenvector criterion for polynomial systems, e.g., [13], lead to a new approach for solving MEP.

MEP generalizes the classical eigenvalue problem. It arises in mathematical physics as a way of solving ordinary and partial differential equations when we can use separation of variables (Fourier method) to solve boundary eigenvalue problems. Its applications, among others, include the Spectral and the Sturm-Liouville theory [2, 3, 29, 32, 50]. MEP allows us to solve different eigenvalue problems, e.g., the polynomial and the quadratic two-parameter eigenvalue problems [30, 40]. It is an old problem; its origins date from the 1920' in the works of R. D. Carmichael [10] and A. J. Pell [42].

The precise definition of the problem is as follows. Assume $\alpha \in \mathbb{N}$, $\beta_1, \dots, \beta_\alpha \in \mathbb{N}$, and consider matrices $\{M^{(i,j)}\}_{0 \leq j \leq \alpha} \in \mathbb{K}^{(\beta_i+1) \times (\beta_i+1)}$, where $0 \leq i \leq \alpha$. The MEP consists in finding $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_\alpha) \in \mathbb{P}^\alpha(\mathbb{K})$ and $\mathbf{v}_1 \in \mathbb{P}^{\beta_1}(\mathbb{K}), \dots, \mathbf{v}_\alpha \in \mathbb{P}^{\beta_\alpha}(\mathbb{K})$ such that

$$\left(\sum_{j=0}^{\alpha} \lambda_j M^{(1,j)} \right) \mathbf{v}_1 = 0, \dots, \left(\sum_{j=0}^{\alpha} \lambda_j M^{(\alpha,j)} \right) \mathbf{v}_\alpha = 0, \quad (1)$$

where \mathbb{K} is an algebraically closed field and $\mathbb{P}^n(\mathbb{K})$ is the (corresponding) projective space of dimension $n \in \mathbb{N}$. We refer to $\boldsymbol{\lambda}$ as an eigenvalue, $(\mathbf{v}_1, \dots, \mathbf{v}_\alpha)$ as an eigenvector, and to $(\boldsymbol{\lambda}, \mathbf{v}_1, \dots, \mathbf{v}_\alpha)$ as an eigenpair. For $\alpha = 1$, MEP is the *generalized eigenvalue problem*.

To exploit our tools we need to write MEP as a mixed square bilinear system. For this we introduce the variables $\mathbf{X}_1 = (x_0, \dots, x_\alpha)$ to represent the multiparameter eigenvalues and, for each $1 \leq i \leq \alpha$, the vectors $\mathbf{Y}_t = (y_{t,0}, \dots, y_{t,\beta_t})$ to represent the eigenvectors. This way, we obtain a bilinear system $\mathbf{F} = (f_{1,0}, \dots, f_{\alpha,\beta_\alpha})$, where for each $1 \leq t \leq \alpha$,

$$\left(\sum_{j=0}^{\alpha} x_j M^{(t,j)} \right) \cdot \begin{pmatrix} y_{t,0} \\ y_{t,1} \\ \vdots \\ y_{t,\beta_t} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{\beta_t} \sum_{j=0}^{\alpha} M_{i,0}^{(t,j)} x_j y_{t,i} \\ \vdots \\ \sum_{i=0}^{\beta_t} \sum_{j=0}^{\alpha} M_{i,\beta_t}^{(t,j)} x_j y_{t,i} \end{pmatrix} = \begin{pmatrix} f_{t,0} \\ \vdots \\ f_{t,\beta_t} \end{pmatrix} \quad (2)$$

and, for each $1 \leq t \leq \alpha$, $f_{t,0}, \dots, f_{t,\beta_t} \in \mathbb{K}[\mathbf{X}_1]_1 \otimes \mathbb{K}[\mathbf{Y}_t]_1$. In this formulation, the system in (2) is a particular case of a *star multilinear system* (Def. 3.1) with $A = 1$ and $B = \alpha$, or a particular case of *bipartite bilinear system* (Def. 4.1) with $A = 1$ and $B = \alpha$. There is a one to one correspondence between the eigenpairs of MEP and solutions of \mathbf{F} , that is

$$\begin{aligned} (\boldsymbol{\lambda}, \mathbf{v}_1, \dots, \mathbf{v}_\alpha) \text{ is an} \\ \text{eigenpair of } \{M^{(i,j)}\}. \end{aligned} \iff \begin{aligned} (\boldsymbol{\lambda}, \mathbf{v}_1, \dots, \mathbf{v}_\alpha) \in \mathbb{P}^\alpha(\mathbb{K}) \times \mathbb{P}^{\beta_1}(\mathbb{K}) \times \dots \times \mathbb{P}^{\beta_\alpha}(\mathbb{K}) \\ \text{and } \mathbf{F}(\boldsymbol{\lambda}, \mathbf{v}_1, \dots, \mathbf{v}_\alpha) = 0. \end{aligned}$$

The standard method to solve MEP is Atkinson's *Delta method* [3, Ch. 6, 8]. For each $0 \leq k \leq \alpha$, it considers the overdetermined system \mathbf{F}_k resulting from \mathbf{F} (2) by setting $x_k = 0$. Then, it constructs a matrix Δ_k which is nonsingular if and only if \mathbf{F}_k has no solutions [3, Eq. 6.4.4]. Subsequently, it applies linear algebra operations to these matrices to solve the MEP \mathbf{F} [3, Thm. 6.8.1]. It turns out that the matrices Δ_k are determinantal formulas for the resultants of the corresponding overdetermined systems \mathbf{F}_k . The elements of the matrices of the determinantal formulas Δ_k are polynomials of degree α in the elements of the matrices

$M^{(i,j)}$ [3, Thm. 8.2.1]. The Delta method can only solve *nonsingular MEPs*; these are MEPs where there exists a finite number of eigenvalues [3, Ch. 8]. The main computational disadvantage of Atkinson’s Delta method is the cost of computing the matrices Δ_k . To compute these matrices one needs to consider multiple Kronecker products corresponding to the Laplace expansion of a determinant of size $\alpha \times \alpha$. The interested reader can find more details in [3, Sec. 6.2].

Besides Atkinson’s Delta method, there are recent algorithms such as the *diagonal coefficient homotopy method* [19] and the *fiber product homotopy method* [44] which exploit homotopy continuation methods. These methods seem to be slower than the Delta method, but they can tackle MEPs of bigger size as they avoid the construction of the matrices Δ_k . The Delta method and the homotopy approaches can only compute the *regular* eigenpairs of the MEP, that is, those where the eigenpair is an isolated solution of (2). In contrast to the Delta method, experimentally and in some cases, the *fiber product homotopy method* can also solve *singular MEPs*, see [44, Sec. 10]. We can also use general purpose polynomial system solving algorithms, that exploit sparsity, to tackle MEP. We refer reader to [27], see also [46], for an algorithm to solve unmixed multilinear systems using Gröbner bases, and to [23, 24] using resultants. We also refer to [6, 7] for an algorithm based on Gröbner bases to solve square mixed multihomogeneous systems and to [8, 48] for a numerical algorithm to solve these systems using eigenvalue computations. However, these generic approaches do not fully exploit the structure of the problem.

Our second main contribution is a new approach to solve MEP based on the determinantal formulas that we develop (Alg. 1). We present a novel way to compute the Δ matrices by avoiding the expansion to minors. Moreover, the resultant matrix from which this construction emanates has elements that are linear in the elements of $M^{(i,j)}$, is highly structured, and with few non-zero elements (Rem. 5.2). As the Delta method and the homotopy algorithms, it can only solve *nonsingular MEPs* and recover the *regular eigenvalues*. Contrary to the general purpose approaches, it fully exploits the structure of the system and its complexity relates to the number of eigenpairs. Our approach involves the computation of (standard) eigenvalues and eigenvectors to recover the solutions of the MEP, as the classical symbolic-numeric methods, e.g., [13]. In particular, our method works with exact coefficients as well as with approximations. The code for solving MEP using these resultant matrices is freely available at <https://mbender.github.io/multLinDetForm/sylvesterMEP.m>.

Organization of the paper In Sec. 2.1 we define the multihomogeneous systems and introduce some notation. Later, in Sec. 2.2, we introduce the multihomogeneous resultant. In Sec. 2.3, we introduce Weyman complexes and then, in Sec. 2.4, we explain the Koszul-type formulas. In Sec. 3, we define the *star multilinear systems* and we construct *Koszul-type determinantal formulas* for multihomogeneous systems involving them, and in Sec. 3.2 we study the number of solutions of the systems and we compare them with the sizes of the determinant. We also present an example in Sec. 3.3. In Sec. 4, we define the *bipartite bilinear systems* and construct *Koszul-type determinantal formulas* for multihomogeneous systems involving them. Finally, in Sec. 5, we present an algorithm and an example for solving MEP using our *determinantal formulas*.

2 Preliminaries

For a number $N \in \mathbb{N}$ we use the abbreviation $[N] = \{1, \dots, N\}$.

2.1 Multihomogeneous systems

Let \mathbb{K} be an algebraically closed field, \mathbb{K}^m a vector space, and $(\mathbb{K}^m)^*$ its dual space, where $m \in \mathbb{N}$. Let $q \in \mathbb{N}$ and consider q positive natural numbers, $n_1, \dots, n_q \in \mathbb{N}$. For each $i \in [q]$, we consider the following sets of $n_i + 1$ variables

$$\mathbf{x}_i := \{x_{i,0}, \dots, x_{i,n_i}\} \quad \text{and} \quad \boldsymbol{\partial}\mathbf{x}_i := \{\partial x_{i,0}, \dots, \partial x_{i,n_i}\}.$$

We identify the polynomial algebra $\mathbb{K}[\mathbf{x}_i]$ with the symmetric algebra of the vector space \mathbb{K}^{n_i+1} and the algebra $\mathbb{K}[\boldsymbol{\partial}\mathbf{x}_i]$ with the symmetric algebra of $(\mathbb{K}^{n_i+1})^*$. That is

$$\mathbb{K}[\mathbf{x}_i] \cong S(\mathbb{K}^{n_i+1}) = \bigoplus_{d \in \mathbb{Z}} S_i(d) \quad \text{and} \quad \mathbb{K}[\boldsymbol{\partial}\mathbf{x}_i] \cong S((\mathbb{K}^{n_i+1})^*) = \bigoplus_{d \in \mathbb{Z}} S_i^*(-d).$$

Therefore, for each i , $S_i(d)$ corresponds to the \mathbb{K} -vector space of polynomials in $\mathbb{K}[\mathbf{x}_i]$ of degree d and $S_i^*(-d)$ to the \mathbb{K} -vector space of polynomials in $\mathbb{K}[\boldsymbol{\partial}\mathbf{x}_i]$ of degree d . Note that if $d < 0$, then $S_i(d) = S_i^*(-d) = 0$.

We identify the monomials in $\mathbb{K}[\mathbf{x}_i]$ and $\mathbb{K}[\partial\mathbf{x}_i]$ with vectors in \mathbb{Z}^{n_i+1} . For each $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{n_i}) \in \mathbb{Z}^{n_i+1}$ we set $\mathbf{x}_i^\alpha := \prod_{j=0}^{n_i} x_{i,j}^{\alpha_j}$, and $\partial\mathbf{x}_i^\alpha := \prod_{j=0}^{n_i} \partial x_{i,j}^{\alpha_j}$. We consider the \mathbb{Z}^q -graded polynomial algebra

$$\mathbb{K}[\bar{\mathbf{x}}] := \mathbb{K}[\mathbf{x}_1] \otimes \cdots \otimes \mathbb{K}[\mathbf{x}_q] \cong \bigoplus_{(d_1, \dots, d_q) \in \mathbb{Z}^q} S_1(d_1) \otimes \cdots \otimes S_q(d_q).$$

Notice that for each $\mathbf{d} = (d_1, \dots, d_q) \in \mathbb{Z}^q$, $\mathbb{K}[\bar{\mathbf{x}}]_{\mathbf{d}}$ is the \mathbb{K} -vector space of the multihomogeneous polynomials of multidegree \mathbf{d} , that is polynomials in $\mathbb{K}[\bar{\mathbf{x}}]$ having degree d_i with respect to the set of variables \mathbf{x}_i , for each $i \in [q]$. We say that a polynomial $f \in \mathbb{K}[\bar{\mathbf{x}}]$ is multihomogeneous of multidegree $\mathbf{d} \in \mathbb{Z}^q$, if $f \in \mathbb{K}[\bar{\mathbf{x}}]_{\mathbf{d}} = S_1(d_1) \otimes \cdots \otimes S_q(d_q)$.

Example 2.1. Consider the two blocks of variables $\mathbf{x}_1 = \{x_{1,0}, x_{1,1}\}$ and $\mathbf{x}_2 = \{x_{2,0}, x_{2,1}, x_{2,2}\}$ and the polynomial $x_{1,0}^2 x_{1,1} \otimes x_{2,0} x_{2,1} + x_{1,0} x_{1,1}^2 \otimes x_{2,0}^2 \in \mathbb{K}[\mathbf{x}_1, \mathbf{x}_2]$. It is multihomogeneous of multidegree $(3, 2)$ and we write it simply as $x_{1,0}^2 x_{1,1} x_{2,0} x_{2,1} + x_{1,0} x_{1,1}^2 x_{2,0}^2$.

Following standard notation, we write the monomials of $\mathbb{K}[\bar{\mathbf{x}}]$ as $\prod_{i=1}^q \mathbf{x}_i^{\alpha_i}$ instead of $\bigotimes_{i=1}^q \mathbf{x}_i^{\alpha_i}$. We identify these monomials with vectors in $\mathbb{Z}^{n_1+1} \times \cdots \times \mathbb{Z}^{n_q+1}$. For each $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_q) \in \mathbb{Z}^{n_1+1} \times \cdots \times \mathbb{Z}^{n_q+1}$ we set $\mathbf{x}^\alpha := \prod_{i=1}^q \mathbf{x}_i^{\alpha_i}$. For each multidegree $\mathbf{d} \in \mathbb{Z}^q$, we denote by $\mathcal{A}(\mathbf{d})$ the set of exponents of the monomials of multidegree \mathbf{d} , that is $\mathcal{A}(\mathbf{d}) = \{\boldsymbol{\alpha} \in \mathbb{Z}^{n_1+1} \times \cdots \times \mathbb{Z}^{n_q+1} : \mathbf{x}^\alpha \in \mathbb{K}[\bar{\mathbf{x}}]_{\mathbf{d}}\}$. The cardinality of $\mathcal{A}(\mathbf{d})$ is $\#\mathcal{A}(\mathbf{d}) = \prod_{i=1}^q \binom{d_i+n_i}{n_i}$.

We fix $N = n_1 + \cdots + n_q$. Let $\mathcal{P} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a multiprojective space over \mathbb{K} , where $\mathbb{P}^{n_i} := \mathbb{P}^{n_i}(\mathbb{K})$. A system of multihomogeneous polynomials is a set of multihomogeneous polynomials in $\mathbb{K}[\bar{\mathbf{x}}]$. We say that a system of multihomogeneous polynomials $\{f_1, \dots, f_r\}$ has a solution in \mathcal{P} , if there is $\gamma \in \mathcal{P}$ such that $f_i(\gamma) = 0$, for every $1 \leq i \leq r$. We call a system of multihomogeneous polynomials *square* if $r = N$, and *overdetermined* if $r = N + 1$. Generically, square multihomogeneous systems have a finite number of solutions over \mathcal{P} , while the overdetermined ones do not have solutions. The following proposition bounds the number of solutions of square multihomogeneous polynomial systems.

Proposition 2.2 (Multihomogeneous Bézout bound, [45, Example 4.9]). *Consider a square multihomogeneous system $\mathbf{f} := \{f_1, \dots, f_N\}$ of multidegrees $\mathbf{d}_1, \dots, \mathbf{d}_N \in \mathbb{N}^q$ with $\mathbf{d}_i = (d_{i,1}, \dots, d_{i,q})$, for each $1 \leq i \leq N$. If \mathbf{f} has a finite number of solutions over \mathcal{P} , then their number, counted with multiplicities, see [14, Sec. 4.2], is the coefficient of the monomial $\prod_i Z_i^{n_i}$ in the polynomial $\prod_{k=1}^N (\sum_{i=1}^q d_{k,i} Z_i)$. We refer to this coefficient as the multihomogeneous Bézout bound and we will write it as $\text{MHB}(\mathbf{d}_1, \dots, \mathbf{d}_N)$.*

The multihomogeneous Bézout bound is generically tight, see [17, Thm. 1.11].

2.2 Multihomogeneous resultant

We fix $N + 1$ multidegrees $\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_N \in \mathbb{N}^q$. To characterize the overdetermined multihomogeneous systems of such multidegrees having solutions over \mathcal{P} , we parameterize them and we introduce the resultant. The latter is a polynomial in the coefficients of the polynomials of the system that vanishes if and only if the system has a solution over \mathcal{P} . Our presentation follows [14, Ch. 3.2] adapted to the multihomogeneous case.

Definition 2.3 (Generic multihomogeneous system). Consider the set of variables $\mathbf{u} := \{u_{k,\alpha} : 0 \leq k \leq N \text{ and } \alpha \in \mathcal{A}(\mathbf{d}_k)\}$ and the ring $\mathbb{Z}[\mathbf{u}]$. The generic multihomogeneous polynomial system is the system $\mathbf{F} := \{F_0, \dots, F_N\} \subset \mathbb{Z}[\mathbf{u}][\bar{\mathbf{x}}]$, where

$$F_k := \sum_{\alpha \in \mathcal{A}(\mathbf{d}_k)} u_{k,\alpha} \mathbf{x}^\alpha. \quad (3)$$

The generic multihomogeneous system \mathbf{F} parameterizes every overdetermined multihomogeneous system with polynomials of multidegrees $\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_N$, respectively. For each $\mathbf{c} = (c_{k,\alpha})_{0 \leq k \leq N, \alpha \in \mathcal{A}(\mathbf{d}_k)} \in \mathbb{P}^{\#\mathcal{A}(\mathbf{d}_0)-1} \times \cdots \times \mathbb{P}^{\#\mathcal{A}(\mathbf{d}_N)-1}$, the specialization of \mathbf{F} at \mathbf{c} , that is $\mathbf{F}(\mathbf{c})$, is a multihomogeneous polynomial system in $\mathbb{K}[\bar{\mathbf{x}}]$, say (f_0, \dots, f_N) , where

$$f_k := F_k(\mathbf{c}) = \sum_{\alpha \in \mathcal{A}(\mathbf{d}_k)} c_{k,\alpha} \mathbf{x}^\alpha. \quad (4)$$

Let Ω be its *incidence variety*, that is the algebraic variety containing the overdetermined multihomogeneous systems that have solutions over \mathcal{P} and their solutions,

$$\Omega = \left\{ (\mathbf{p}, \mathbf{c}) \in \mathcal{P} \times (\mathbb{P}^{\#\mathcal{A}(\mathbf{d}_0)-1} \times \cdots \times \mathbb{P}^{\#\mathcal{A}(\mathbf{d}_N)-1}) : (\forall k \in [N]) F_k(\mathbf{c})(\mathbf{p}) = 0 \right\}.$$

Let π be the projection of Ω to $\mathbb{P}^{\#\mathcal{A}(\mathbf{d}_0)-1} \times \dots \times \mathbb{P}^{\#\mathcal{A}(\mathbf{d}_N)-1}$, that is $\pi(\mathbf{p}, \mathbf{c}) = \mathbf{c}$. We can think of $\pi(\Omega)$ as the set of overdetermined multihomogeneous polynomial systems with solutions over \mathcal{P} . This set is an irreducible hypersurface [28, Prop. 3.3.1]. Its defining ideal in $\mathbb{Z}[\mathbf{u}]$ is principal and it is generated by an irreducible polynomial $\mathbf{elim} \in \mathbb{Z}[\mathbf{u}]$ [28, Prop. 8.1.1]. In particular, it holds

$$\text{The system } \mathbf{F}(\mathbf{c}) \text{ has a solution over } \mathcal{P} \iff \mathbf{c} \in \pi(\Omega) \iff \mathbf{elim}(\mathbf{c}) = 0.$$

Following [43, 17, 20], we call $\mathbf{elim} \in \mathbb{Z}[\mathbf{u}]$ the *eliminant*. We warn the reader that the polynomial $\mathbf{elim} \in \mathbb{Z}[\mathbf{u}]$ is called the *resultant* in [28]. In this work we reserve the word resultant for a power of \mathbf{elim} . More precisely, the resultant \mathbf{res} is a polynomial in $\mathbb{Z}[\mathbf{u}]$ such that $\mathbf{res} = \pm \mathbf{elim}^{\mathcal{D}}$, where \mathcal{D} is the degree of the restriction of π to the incidence variety Ω , see [20, Def. 3.1]. Consequently, we have

$$\text{The system } \mathbf{F}(\mathbf{c}) \text{ has a solution over } \mathcal{P} \iff \mathbf{res}(\mathbf{c}) = 0. \quad (5)$$

Proposition 2.4 ([43, Prop. 3.4]). *Let \mathbf{u}_k be the blocks of variables in \mathbf{u} related to the polynomial F_k , that is $\mathbf{u}_k = \{\mathbf{u}_{k,\alpha}\}_{\alpha \in \mathcal{A}(\mathbf{d}_k)}$. The resultant $\mathbf{res} \in \mathbb{Z}[\mathbf{u}]$ is a multihomogeneous polynomial with respect to the blocks of variables $\mathbf{u}_0, \dots, \mathbf{u}_N$. The degree of \mathbf{res} with respect to the variables \mathbf{u}_k is the multihomogeneous Bézout bound (Prop. 2.2) of a square system with multidegrees $\mathbf{d}_0 \dots \mathbf{d}_{k-1}, \mathbf{d}_{k+1} \dots \mathbf{d}_N$,*

$$\text{degree}(\mathbf{res}, \mathbf{u}_k) = \text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{k-1}, \mathbf{d}_{k+1}, \dots, \mathbf{d}_N).$$

The total degree of the resultant is $\text{degree}(\mathbf{res}) = \sum_{k=0}^N \text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{k-1}, \mathbf{d}_{k+1}, \dots, \mathbf{d}_N)$.

Usually we compute the resultant as the quotient of the determinants of two matrices, see [28, Thm. 3.4.2]. When we can compute it as the (exact) determinant of a matrix, then we say that we have a *determinantal formula*.

2.3 Weyman complex

A complex K_\bullet is a sequence of free modules $\{K_v\}_{v \in \mathbb{Z}}$ together with morphisms $\delta_v : K_v \rightarrow K_{v-1}$, such that the image of δ_v belongs to the kernel of δ_{v-1} , that is $(\forall v \in \mathbb{Z}) \text{Im}(\delta_v) \subseteq \text{Ker}(\delta_{v-1})$ or, equivalently, $\delta_{v-1} \circ \delta_v = 0$. We write K_\bullet as

$$K_\bullet : \dots \xrightarrow{\delta_{v+1}} K_v \xrightarrow{\delta_v} K_{v-1} \xrightarrow{\delta_{v-1}} \dots$$

The complex is exact if for all $v \in \mathbb{Z}$ it holds $\text{Im}(\delta_v) = \text{Ker}(\delta_{v-1})$. A complex is bounded when there are two constants a and b such that, for every v such that $v < a$ or $b < v$, it holds $K_v = 0$. If we fix a basis for each K_v , then we can represent the maps δ_v using matrices. For a particular class of bounded complexes, called generically exact (see for example [28, Ap. A]), we can extend the definition of the determinant of matrices to complexes. The non-vanishing of the determinant is related to the exactness of the complex. When there are only two non-zero free modules in the complex (that is all the other modules are zero) we can define the determinant of the complex if and only if both the non-zero free modules have the same rank. In this case, the determinant of the complex reduces to the determinant of the (matrix of the) map between the two non-zero vector spaces. We refer the reader to [1] for an accessible introduction to the determinant of a complex and to [28, Ap. A] for a complete formalization.

The Weyman complex [52, 54, 53] of an overdetermined multihomogeneous system $\mathbf{f} = (f_0, \dots, f_N)$ in $\mathbb{K}[\bar{\mathbf{x}}]$ is a bounded complex that is exact if and only if the system \mathbf{f} has no solutions over \mathcal{P} . More precisely, the determinant of the Weyman complex of the multihomogeneous generic system \mathbf{F} (see Def. 2.3) is well-defined and it is equal to the multihomogeneous resultant [53, Prop. 9.1.3]. If the Weyman complex involves only two non-zero vector spaces, then the resultant of \mathbf{F} is the determinant of the map between these spaces. Thus, in this case, there is a determinantal formula for the resultant.

Theorem 2.5 (Weyman complex, [54, Prop. 2.2]). *Let $\mathbf{F} = (F_0, \dots, F_N)$ in $\mathbb{Z}[\mathbf{u}][\bar{\mathbf{x}}]$ be a generic multihomogeneous system having multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$, respectively (see Def. 2.3). Given a degree vector $\mathbf{m} \in \mathbb{Z}^q$, there exists a complex of free $\mathbb{Z}[\mathbf{u}]$ -modules $K_\bullet(\mathbf{m})$, called the Weyman complex of \mathbf{F} , such that the determinant of the complex $K_\bullet(\mathbf{m})$ agrees with the resultant $\mathbf{res}(F_0, \dots, F_N)$.*

$$K_\bullet(\mathbf{m}) : 0 \rightarrow K_{N+1}(\mathbf{m}) \xrightarrow{\delta_{N+1}(\mathbf{m})} \dots \rightarrow K_1(\mathbf{m}) \xrightarrow{\delta_1(\mathbf{m})} K_0(\mathbf{m}) \xrightarrow{\delta_0(\mathbf{m})} \dots \rightarrow K_{-N}(\mathbf{m}) \rightarrow 0.$$

Moreover, for each $v \in \{-N, \dots, N+1\}$ the $\mathbb{Z}[\mathbf{u}]$ -module $K_v(\mathbf{m})$ is

$$K_v(\mathbf{m}) := \bigoplus_{p=0}^{N+1} K_{v,p}(\mathbf{m}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{u}], \text{ where } K_{v,p}(\mathbf{m}) := \bigoplus_{\substack{I \subset \{0, \dots, N\} \\ \#I=p}} H_{\mathcal{P}}^{p-v}(\mathbf{m} - \sum_{k \in I} \mathbf{d}_k) \otimes \bigwedge_{k \in I} e_k, \quad (6)$$

the term $H_{\mathcal{P}}^{p-v}(\mathbf{m} - \sum_{k \in I} \mathbf{d}_k)$ is the $(p-v)$ -th sheaf cohomology of \mathcal{P} with coefficients in the sheaf $\mathcal{O}_{\mathcal{P}}(\mathbf{m} - \sum_{k \in I} \mathbf{d}_k)$ whose global sections are $\mathbb{K}[\bar{\mathbf{x}}]_{\mathbf{m} - \sum_{k \in I} \mathbf{d}_k}$ (see [31, Sec. II.5]),² and the element $\bigwedge_{k \in I} e_k$ is the singleton $\{e_{I_1} \wedge \dots \wedge e_{I_p}\}$, where $I_1 < \dots < I_p$ are the elements of I , e_0, \dots, e_N is the standard basis of \mathbb{K}^{N+1} , and \wedge is the wedge (exterior) product.

For a multihomogeneous system $\mathbf{f} = (f_0, \dots, f_N)$ in $\mathbb{K}[\bar{\mathbf{x}}]$ that is the specialization of \mathbf{F} at \mathbf{c} , see (4), the Weyman complex $K_{\bullet}(\mathbf{m}; \mathbf{f})$ is the Weyman complex $K_{\bullet}(\mathbf{m})$ where we specialize each variable $u_{k,\alpha}$ at $c_{k,\alpha} \in \mathbb{K}$.

Proposition 2.6 ([54, Prop. 2.1]). *The vector spaces $K_v(\mathbf{m}, \mathbf{f})$ are independent of the specialization of the variables \mathbf{u} , in particular $K_v(\mathbf{m}, \mathbf{f}) = \bigoplus_{p=0}^{N+1} K_{v,p}(\mathbf{m})$. Hence, the rank of $K_v(\mathbf{m})$ as a $\mathbb{Z}[\mathbf{u}]$ -module equals the dimension of $K_v(\mathbf{m}, \mathbf{f})$ as a \mathbb{K} -vector space. The differentials $\delta_v(\mathbf{m}, \mathbf{f})$ depend on the coefficients of \mathbf{f} .*

Following [54], as \mathcal{P} is a product of projective spaces, we use Künneth formula (Prop. 2.7) to write the cohomologies in (6) as a product of cohomologies of projective spaces, that in turn we can identify with polynomial rings.

Proposition 2.7 (Künneth Formula). *The cohomologies of the product of projective spaces in each $K_{v,p}(\mathbf{m})$ of (6) are the direct sum of the tensor product of the cohomologies of each projective space, that is*

$$H_{\mathcal{P}}^{p-v}(\mathbf{m} - \sum_{k \in I} \mathbf{d}_k) \cong \bigoplus_{r_1 + \dots + r_q = p-v} \bigotimes_{i=1}^q H_{\mathbb{P}^{n_i}}^{r_i} \left(m_i - \sum_{k \in I} d_{k,i} \right). \quad (7)$$

By combining Bott formula and Serre's duality, see [41, Sec. 1.1], we can identify the cohomologies of the previous proposition with the rings $\mathbb{K}[\mathbf{x}_i]$ and $\mathbb{K}[\partial \mathbf{x}_i]$. Moreover, for each $p-v$, there is at most one set of values for (r_1, \dots, r_q) such that every cohomology in the tensor product of the right hand side of the previous equation does not vanish. In other words, the right hand side of (7) reduces to the tensor product of certain cohomologies of different projective spaces.

Remark 2.8. For each $1 \leq i \leq q$, $a \in \mathbb{Z}$, it holds

- $H_{\mathbb{P}^{n_i}}^0(a) \cong S_i(a)$, that is the \mathbb{K} -vector space of the polynomials of degree a in the polynomial algebra $\mathbb{K}[\mathbf{x}_i]$.
- $H_{\mathbb{P}^{n_i}}^{n_i}(a) \cong S_i^*(a + n_i + 1)$, that is the \mathbb{K} -vector space of the polynomials of degree $a + n_i + 1$ in the polynomial algebra $\mathbb{K}[\partial \mathbf{x}_i]$.
- If $r_i \notin \{0, n_i\}$, then $H_{\mathbb{P}^{n_i}}^{r_i}(a) \cong 0$.

Remark 2.9. For each $1 \leq i \leq q$, if $H_{\mathbb{P}^{n_i}}^{r_i}(a) \neq 0$, then $r_i \in \{0, n_i\}$. Moreover,

- If $a > -n_i - 1$, then $H_{\mathbb{P}^{n_i}}^{r_i}(a) \neq 0 \iff r_i = 0 \text{ and } a \geq 0$.
- If $a < 0$, then $H_{\mathbb{P}^{n_i}}^{r_i}(a) \neq 0 \iff r_i = n_i \text{ and } a \leq -n_i - 1$.

We define the dual of a complex by dualizing the modules and the maps. The dual of the Weyman complex is isomorphic to another Weyman complex. By exploiting Serre's duality, we can construct the degree vectors of a dual Weyman complex from the degree vector of the primal.

Proposition 2.10 ([53, Thm. 5.1.4]). *Let \mathbf{m} and $\bar{\mathbf{m}}$ be any degree vectors such that $\mathbf{m} + \bar{\mathbf{m}} = \sum_i \mathbf{d}_i - (n_1 + 1, \dots, n_q + 1)$. Then, $K_v(\mathbf{m}) \cong K_{1-v}(\bar{\mathbf{m}})^*$ for all $v \in \mathbb{Z}$ and $K_{\bullet}(\mathbf{m})$ is dual to $K_{\bullet}(\bar{\mathbf{m}})$.*

²The standard notation for the sheaf cohomology $H_{\mathcal{P}}^p(\mathbf{m})$, e.g., [31], is $H^p(\mathcal{P}, \mathcal{L}(\sum_i m_i D_i))$, where each D_i is a Cartier divisor given by the pullback of a hyperplane on \mathbb{P}^{n_i} (via projection) and $\mathcal{L}(\sum_i m_i D_i)$ is the line bundle associated to the Cartier divisor $(\sum_i m_i D_i)$ on \mathcal{P} . We use our notation for simplicity.

2.4 Koszul-type formula

Our goal is to obtain determinantal formulas given by matrices whose elements are linear forms in the coefficients of the input polynomials, that is linear in \mathbf{u} , see (3). Hence, by [53, Prop. 5.2.4], our objective is to choose a degree vector \mathbf{m} so that the Weyman complex reduces to

$$K_{\bullet}(\mathbf{m}) : 0 \rightarrow K_{v,p+v}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \xrightarrow{\delta_v(\mathbf{m})} K_{v-1,p+v-1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \rightarrow 0, \quad (8)$$

where $p = \sum_{k \in I} n_k$, for some set $I \subset \{1, \dots, q\}$. That is, it holds $K_v(\mathbf{m}) = K_{v,p+v}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}]$, $K_{v-1}(\mathbf{m}) = K_{v-1,p+v-1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}]$, and, for all $t \notin \{v-1, v\}$, $K_t(\mathbf{m}) = 0$,

We will describe the map $\delta_v(\mathbf{m})$ through an auxiliary map μ that acts as multiplication. For this, we need to introduce some additional notation. This notation is independent from the rest of the paper and the readers that are familiar with the Weyman complex can safely skip the rest of the section. Let R be a ring; for example $R = \mathbb{Z}[\mathbf{u}]$ or $R = \mathbb{K}$. For each $1 \leq i \leq q$, the polynomial ring $R[\mathbf{x}_i]$, respectively $R[\partial \mathbf{x}_i]$, is a free R -module with basis $\{\mathbf{x}_i^\alpha : \alpha \in \mathcal{A}(d), d \in \mathbb{Z}\}$, respectively $\{\partial \mathbf{x}_i^\alpha : \alpha \in \mathcal{A}(d), d \in \mathbb{Z}\}$. We define the bilinear map

$$\mu_{(i)} : R[\mathbf{x}_i] \times (R[\mathbf{x}_i] \oplus R[\partial \mathbf{x}_i]) \rightarrow R[\mathbf{x}_i] \oplus R[\partial \mathbf{x}_i], \quad (9)$$

which acts follows: for each $d_1, d_2 \in \mathbb{Z}$, $\alpha \in \mathcal{A}(d_1)$ and $\beta, \gamma \in \mathcal{A}(d_2)$, we have

$$\mu_{(i)}(\mathbf{x}_i^\alpha, \mathbf{x}_i^\gamma) = \mathbf{x}_i^{\alpha+\gamma} \quad \text{and} \quad \mu_{(i)}(\mathbf{x}_i^\alpha, \partial \mathbf{x}_i^\beta) = \begin{cases} \partial \mathbf{x}_i^{\beta-\alpha} & \text{if } d_1 \leq d_2 \text{ and } \beta - \alpha \in \mathcal{A}(d_1 - d_2) \\ 0 & \text{otherwise.} \end{cases}$$

The map $\mu_{(i)}$ is graded in the following way, for $f \in S_i(d)$ it holds

$$\mu_{(i)}(f, S_i(D)) \subseteq S_i(D+d) \quad \text{and} \quad \mu_{(i)}(f, S_i^*(D)) \subseteq S_i^*(D+d).$$

We define the bilinear map $\mu := \bigotimes_{i=1}^q \mu_{(i)}$.

Remark 2.11. If we restrict the domain of μ to $\bigotimes_{i=1}^q (R[\mathbf{x}_i] \times R[\mathbf{x}_i]) \simeq (\bigotimes_{i=1}^q R[\mathbf{x}_i]) \times (\bigotimes_{i=1}^q R[\mathbf{x}_i])$, then μ acts as multiplication, ie for $f, g \in (\bigotimes_{i=1}^q R[\mathbf{x}_i])$, it holds $\mu(f, g) = fg$.

Given $f \in (\bigotimes_{i=1}^q R[\mathbf{x}_i])$, we define the linear map

$$\begin{aligned} \mu_f : \bigotimes_{i=1}^q (R[\mathbf{x}_i] \oplus R[\partial \mathbf{x}_i]) &\rightarrow \bigotimes_{i=1}^q (R[\mathbf{x}_i] \oplus R[\partial \mathbf{x}_i]) \\ g &\mapsto \mu_f(g) = \mu(f, g). \end{aligned}$$

Using the isomorphisms of Prop. 2.7 and Rem. 2.8, for $\mathbf{d} \in \mathbb{N}^q$ and $f \in \mathbb{K}[\bar{\mathbf{x}}]_{\mathbf{d}}$, if we restrict the map μ_f to $H_{\mathcal{P}}^r(\mathbf{m})$, for any $r \in \mathbb{N}$, then we obtain the map $\mu_f : H_{\mathcal{P}}^r(\mathbf{m}) \rightarrow H_{\mathcal{P}}^r(\mathbf{m} + \mathbf{d})$.

Definition 2.12 (Inner derivative [54]). Let E be a \mathbb{K} -vector space generated by $\{e_1, \dots, e_N\}$. We define the k -th inner derivative, Δ_k , of the exterior algebra of $\bigwedge E$ as the (-1) -graded map such that, for each i and $1 \leq j_1 < \dots < j_i \leq N$,

$$\begin{aligned} \Delta_k : \bigwedge^i E &\rightarrow \bigwedge^{i-1} E \\ e_{j_1} \wedge \dots \wedge e_{j_i} &\mapsto \Delta_k(e_{j_1} \wedge \dots \wedge e_{j_i}) = \begin{cases} (-1)^{t+1} e_{j_1} \wedge \dots \wedge e_{j_{t-1}} \wedge e_{j_{t+1}} \wedge \dots \wedge e_{j_i} & \text{if } j_t = k \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Given R -modules A_1, A_2, B_1, B_2 and homomorphisms $\mu_1 : A_1 \rightarrow B_1$ and $\mu_2 : A_2 \rightarrow B_2$, their tensor product is the map $\mu_1 \otimes \mu_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ such that, for $a_1 \in A_1$ and $a_2 \in A_2$, $(\mu_1 \otimes \mu_2)(a_1 \otimes a_2) = \mu_1(a_1) \otimes \mu_2(a_2)$.

Proposition 2.13 ([54, Prop. 2.6]). Consider the generic overdetermined multihomogeneous system $\mathbf{F} \in \mathbb{Z}[\mathbf{u}][\bar{\mathbf{x}}]^{N+1}$ with polynomials of multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$, respectively (Def. 2.3). Given a degree vector $\mathbf{m} \in \mathbb{Z}^q$, we consider the Weyman complex $K_{\bullet}(\mathbf{m})$. If there is $v \in \{-N+1, \dots, N+1\}$ and $p \in \{0, \dots, N+1\}$ such that

$$K_v(\mathbf{m}) = K_{v,p}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \quad \text{and} \quad K_{v-1}(\mathbf{m}) = K_{v-1,p-1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}],$$

then the map $\delta_v(\mathbf{m}) : K_v(\mathbf{m}) \rightarrow K_{v-1}(\mathbf{m})$ is $\delta_v(\mathbf{m}) = \sum_{k=0}^N \mu_{F_k} \otimes \Delta_k$, where $\mu_{F_k} \otimes \Delta_k$ denotes the tensor product of the maps μ_{F_k} and Δ_k (Def. 2.12).

Definition 2.14 (Koszul-type determinantal formula). With the notation of Prop. 2.13, when the Weyman complex reduces to

$$K_{\bullet}(\mathbf{m}) : 0 \rightarrow K_{1,p+1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \xrightarrow{\delta_1(\mathbf{m})} K_{0,p}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \rightarrow 0, \quad (10)$$

we say that the map $\delta_1(\mathbf{m})$ is a *Koszul-type determinantal formula*.

Example 2.15. Consider the blocks of variables $\mathbf{x}_1 := \{x_{1,0}, x_{1,1}\}$ and $\mathbf{x}_2 := \{x_{2,0}, x_{2,1}\}$, and the systems $\mathbf{f} := (f_0, f_1, f_2)$ of multidegrees $\mathbf{d}_0 = \mathbf{d}_1 = \mathbf{d}_2 = (1, 1)$. That is,

$$\begin{cases} f_0 = (a_{0,0} x_{1,0} + a_{1,0} x_{1,1}) x_{2,0} + (a_{0,1} x_{1,0} + a_{1,1} x_{1,1}) x_{2,1} \\ f_1 = (b_{0,0} x_{1,0} + b_{1,0} x_{1,1}) x_{2,0} + (b_{0,1} x_{1,0} + b_{1,1} x_{1,1}) x_{2,1} \\ f_2 = (c_{0,0} x_{1,0} + c_{1,0} x_{1,1}) x_{2,0} + (c_{0,1} x_{1,0} + c_{1,1} x_{1,1}) x_{2,1}. \end{cases} \quad (11)$$

As in [23, Lem. 2.2], consider the degree vector $\mathbf{m} = (2, -1)$. So, the Weyman complex is

$$K_\bullet(\mathbf{m}, \mathbf{f}) : 0 \rightarrow K_{1,2}(\mathbf{m}, \mathbf{f}) \xrightarrow{\delta_1(\mathbf{m}, \mathbf{f})} K_{0,1}(\mathbf{m}, \mathbf{f}) \rightarrow 0,$$

where

$$\begin{cases} K_{1,2}(\mathbf{m}, \mathbf{f}) = S_1(0) \otimes S_2^*(-1) \otimes (\{e_0 \wedge e_1\} \oplus \{e_0 \wedge e_2\} \oplus \{e_1 \wedge e_2\}) \\ K_{0,1}(\mathbf{m}, \mathbf{f}) = S_1(1) \otimes S_2^*(0) \otimes (\{e_0\} \oplus \{e_1\} \oplus \{e_2\}). \end{cases}$$

If we consider monomial bases for $K_{1,2}(\mathbf{m})$ and $K_{0,1}(\mathbf{m})$, then we can represent $\delta_1(\mathbf{m})$ with the transpose of the matrix that follows. Note that, the element $\partial 1 \in \mathbb{K}[\partial \mathbf{x}_1, \partial \mathbf{x}_2]$ corresponds to the dual of $1 \in \mathbb{K}[\mathbf{x}_1, \mathbf{x}_2]$.

| | $x_{1,0} \otimes \partial 1 \otimes e_0$ | $x_{1,1} \otimes \partial 1 \otimes e_0$ | $x_{1,0} \otimes \partial 1 \otimes e_1$ | $x_{1,1} \otimes \partial 1 \otimes e_1$ | $x_{1,0} \otimes \partial 1 \otimes e_2$ | $x_{1,1} \otimes \partial 1 \otimes e_2$ |
|---|--|--|--|--|--|--|
| $1 \otimes \partial x_{2,0} \otimes (e_0 \wedge e_1)$ | $-b_{0,0}$ | $-b_{1,0}$ | $a_{0,0}$ | $a_{1,0}$ | 0 | 0 |
| $1 \otimes \partial x_{2,1} \otimes (e_0 \wedge e_1)$ | $-b_{0,1}$ | $-b_{1,1}$ | $a_{0,1}$ | $a_{1,1}$ | 0 | 0 |
| $1 \otimes \partial x_{2,0} \otimes (e_0 \wedge e_2)$ | $-c_{0,0}$ | $-c_{1,0}$ | 0 | 0 | $a_{0,0}$ | $a_{1,0}$ |
| $1 \otimes \partial x_{2,1} \otimes (e_0 \wedge e_2)$ | $-c_{0,1}$ | $-c_{1,1}$ | 0 | 0 | $a_{0,1}$ | $a_{1,1}$ |
| $1 \otimes \partial x_{2,0} \otimes (e_1 \wedge e_2)$ | 0 | 0 | $-c_{0,0}$ | $-c_{1,0}$ | $b_{0,0}$ | $b_{1,0}$ |
| $1 \otimes \partial x_{2,1} \otimes (e_1 \wedge e_2)$ | 0 | 0 | $-c_{0,1}$ | $-c_{1,1}$ | $b_{0,1}$ | $b_{1,1}$ |

The resultant is equal (up to sign) to the determinant of the above matrix, it has total degree 6 (same as the size of this matrix) and 66 terms.

As we saw in the previous example, once we have fixed a basis for the map in Prop. 2.13, we can represent the Koszul-type determinantal formula by the determinant of a matrix. We refer to this matrix as a *Koszul resultant matrix*.

Corollary 2.16 ([53, Prop. 5.2.4]). *Let \mathbf{F} be a generic multihomogeneous system of polynomials with multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$, respectively. Let $\mathbf{m} \in \mathbb{Z}^q$ be a degree vector so that the Weyman complex $K_\bullet(\mathbf{m})$ becomes*

$$K_\bullet(\mathbf{m}) : 0 \rightarrow K_{v,p+v}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \xrightarrow{\delta_v(\mathbf{m})} K_{v-1,p+v-1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \rightarrow 0.$$

Then, the map $\delta_v(\mathbf{m})$ of Prop. 2.13 is linear in the coefficients of \mathbf{F} . Moreover, as the determinant of the complex is the resultant, the rank of both $K_v(\mathbf{m})$ and $K_{v+1}(\mathbf{m})$, as $\mathbb{Z}[\mathbf{u}]$ -modules, equals the degree of the resultant (Prop. 2.4).

We remark that Koszul-type formulas generalizes Sylvester-type formulas.

Proposition 2.17. *Under the assumptions of Prop. 2.13, if $p = 1$ and $v = 0$, the map $\delta_v(\mathbf{m})$ acts as a Sylvester map, that is $(g_0, \dots, g_N) \mapsto \sum_{k=0}^N g_k F_k$. In this case, it holds*

$$\delta_v(\mathbf{m})(g_0 \otimes e_0 + \dots + g_N \otimes e_N) = \left(\sum_{k=0}^N g_k F_k \right) \otimes 1.$$

Determinantal formulas for the multiprojective resultant of unmixed systems, that is systems where the multidegree of each polynomial is the same, were extensively studied by several authors [47, 54, 12, 18]. However, there are very few results about determinantal formulas for *mixed* multihomogeneous systems, that is, when the supports are not the same. We know such formulas for scaled multihomogeneous systems [22], that is when the supports are scaled copies of one of them, and for bivariate tensor-product polynomial systems [38, 9]. In what follows, we use the Weyman complex to derive new formulas for families of mixed multilinear systems.

3 Determinantal formulas for star multilinear systems

We consider four different kinds of overdetermined multihomogeneous systems, related to *star multilinear systems* (Def. 3.1) and we construct determinantal formulas for each of them. These formulas are Koszul- and Sylvester-type determinantal formulas (Def. 2.14). To simplify the presentation, we change somewhat the notation that we used for the polynomial systems in sec. 2.1. We split the blocks of variables in two groups; we replace the blocks of variables \mathbf{x}_k by \mathbf{X}_i or \mathbf{Y}_j and the constants n_k , that correspond to the cardinalities of the blocks, by α_i or β_j . Let $A, B \in \mathbb{N}$ and $q = A + B$. Let $\bar{\mathbf{X}}$ be the set of A blocks of variables $\{\mathbf{X}_1, \dots, \mathbf{X}_A\}$. For each $i \in [A]$, $\mathbf{X}_i := \{x_{i,0}, \dots, x_{i,\alpha_i}\}$; so the number of affine variables in each block \mathbf{X}_i is $\alpha_i \in \mathbb{N}$. We also consider the polynomial algebra $\mathbb{K}[\mathbf{X}_i] = \bigoplus_{d \in \mathbb{Z}} S_{\mathbf{X}_i}(d)$, where $S_{\mathbf{X}_i}(d)$ is the \mathbb{K} -vector space of polynomials of degree d in $\mathbb{K}[\mathbf{X}_i]$. Similarly, $\bar{\mathbf{Y}}$ is the set of B blocks of variables $\{\mathbf{Y}_1, \dots, \mathbf{Y}_B\}$. For each $j \in [B]$, $\mathbf{Y}_j := \{y_{j,0}, \dots, y_{j,\beta_j}\}$; hence the number of variables in each block \mathbf{Y}_j is $\beta_j \in \mathbb{N}$. Moreover, $\mathbb{K}[\mathbf{Y}_j] = \bigoplus_{d \in \mathbb{Z}} S_{\mathbf{Y}_j}(d)$, where $S_{\mathbf{Y}_j}(d)$ is the \mathbb{K} -vector space of polynomials of degree d in $\mathbb{K}[\mathbf{Y}_j]$.

Consider the \mathbb{Z}^q -multigraded algebra $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$, given by

$$\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}] := \bigoplus_{(d_{\mathbf{X}_1}, \dots, d_{\mathbf{X}_A}, d_{\mathbf{Y}_1}, \dots, d_{\mathbf{Y}_B}) \in \mathbb{Z}^q} S_{\mathbf{X}_1}(d_{\mathbf{X}_1}) \otimes \dots \otimes S_{\mathbf{X}_A}(d_{\mathbf{X}_A}) \otimes S_{\mathbf{Y}_1}(d_{\mathbf{Y}_1}) \otimes \dots \otimes S_{\mathbf{Y}_B}(d_{\mathbf{Y}_B}).$$

For a multihomogeneous polynomial $f \in \mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ of multidegree $\mathbf{d} \in \mathbb{Z}^q$, we denote by $d_{\mathbf{X}_i}$, respectively $d_{\mathbf{Y}_j}$, the degree of f with respect to the block of variables \mathbf{X}_i , respectively \mathbf{Y}_j . For each group of indices $1 \leq i_1 < \dots < i_r \leq A$ and $1 \leq j_1 < \dots < j_s \leq B$, we denote by $\mathbb{K}[\mathbf{X}_{i_1}, \dots, \mathbf{X}_{i_r}, \mathbf{Y}_{j_1}, \dots, \mathbf{Y}_{j_s}]_1$ the set of multilinear polynomials in $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ with multidegree $(d_{\mathbf{X}_1}, \dots, d_{\mathbf{X}_A}, d_{\mathbf{Y}_1}, \dots, d_{\mathbf{Y}_B})$, where

$$d_{\mathbf{X}_l} = \begin{cases} 1 & \text{if } l \in \{i_1, \dots, i_r\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad d_{\mathbf{Y}_l} = \begin{cases} 1 & \text{if } l \in \{j_1, \dots, j_s\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $N = \sum_{i=1}^A \alpha_i + \sum_{j=1}^B \beta_j$. We say that a polynomial system is *square* if it has N equations and *overdetermined* if it has $N + 1$. We consider the multiprojective space

$$\mathcal{P} := \mathbb{P}^{\alpha_1} \times \dots \times \mathbb{P}^{\alpha_A} \times \mathbb{P}^{\beta_1} \times \dots \times \mathbb{P}^{\beta_B}.$$

Definition 3.1 (Star multilinear systems). A square multihomogeneous system $\mathbf{f} = (f_1, \dots, f_N)$ in $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ with multidegrees $\mathbf{d}_1, \dots, \mathbf{d}_N \in \mathbb{Z}^q$, respectively, is a *Star multilinear system* if for every $k \in [N]$, there is $j_k \in [B]$ such that

$$f_k \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_{j_k}]_1.$$

For each $j \in [B]$, we denote by \mathcal{E}_j the number of polynomials of \mathbf{f} in $\mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_j]_1$.

We use the term *star* because we can represent such systems using a star graph with weighted edges. The vertices of the graph are the algebras $\mathbb{K}[\mathbf{Y}_1], \dots, \mathbb{K}[\mathbf{Y}_B]$, and $\mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A]$. For each \mathbf{d}_k there is an edge between the vertices $\mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A]$ and $\mathbb{K}[\mathbf{Y}_j]$ whenever $d_{k, \mathbf{Y}_j} = 1$. The weight of the edge between the vertices $\mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A]$ and $\mathbb{K}[\mathbf{Y}_j]$ corresponds to \mathcal{E}_j . That is, when it holds $f_k \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_j]_1$. The graph is a star because every vertex is connected to $\mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A]$ and there is no edge between two vertices $\mathbb{K}[\mathbf{Y}_{j_1}]$ and $\mathbb{K}[\mathbf{Y}_{j_2}]$.

Example 3.2. Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ be blocks of variables. Consider the multihomogeneous system $(f_1, f_2, f_3, f_4) \subset \mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ with the following (pattern of) multidegrees

| | |
|---|--|
| $\begin{array}{l} \mathbf{d}_1 = (d_{k, \mathbf{X}_1}, d_{k, \mathbf{X}_2} \mid d_{k, \mathbf{Y}_1}, d_{k, \mathbf{Y}_2}, d_{k, \mathbf{Y}_3}) \\ \mathbf{d}_2 = (1, 1 \mid 1, 0, 0) \\ \mathbf{d}_3 = (1, 1 \mid 0, 1, 0) \\ \mathbf{d}_4 = (1, 1 \mid 0, 0, 1) \end{array}$ | |
|---|--|

It is a star multilinear system where $\mathcal{E}_1 = 2$, $\mathcal{E}_2 = 1$, and $\mathcal{E}_3 = 1$. The corresponding star graph is the one above.

Remark 3.3. For each square star multilinear system, it holds $N = \sum_{j=1}^B \mathcal{E}_j$. Moreover, if the system has a nonzero finite number of solutions, then for each $j \in \{1, \dots, B\}$ it holds $\mathcal{E}_j \geq \beta_j$, see Prop. 2.2.

3.1 Determinantal formulas

To be able to define the resultant, we study overdetermined polynomial systems (f_0, f_1, \dots, f_N) in $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ where (f_1, \dots, f_N) is a square star multilinear system and f_0 is a multilinear polynomial. The obvious choice for f_0 is to have the same structure as one of the polynomials f_1, \dots, f_N ; still we also choose f_0 to have a different support. This leads to resultants of smaller degrees and so to matrices of smaller size. Aiming at resultant formulas for *any* A and B , we were able to identify the following choices of f_0 , that lead to a determinantal Weyman complex³. In particular, the following f_0 lead to determinantal formulas:

- **Center-Vertex case:** $f_0 \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A]_1$.
- **Outer-Vertex case:** $f_0 \in \mathbb{K}[\mathbf{Y}_j]_1$, for any $j \in [B]$.
- **Edge case:** $f_0 \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_j]_1$, for any $j \in [B]$,
- **Triangle case:** $f_0 \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_{j_1}, \mathbf{Y}_{j_2}]_1$, for any $j_1, j_2 \in [B]$, $j_1 \neq j_2$.

We can view the various multidegrees of f_0 , $\mathbf{d}_0 = (d_{0, \mathbf{X}_1}, \dots, d_{0, \mathbf{X}_A}, d_{0, \mathbf{Y}_1}, \dots, d_{0, \mathbf{Y}_B})$, in the cases above as solutions of the following system of inequalities:

$$\left\{ \begin{array}{ll} (\forall 1 \leq i \leq A) & 0 \leq d_{0, \mathbf{X}_i} \leq 1, \\ (\forall 1 \leq j \leq B) & 0 \leq d_{0, \mathbf{Y}_j} \leq 1, \\ (\forall 1 \leq i_1 < i_2 \leq A) & d_{0, \mathbf{X}_{i_1}} = d_{0, \mathbf{X}_{i_2}}, \text{ and} \\ & \sum_{j=1}^B d_{0, \mathbf{Y}_j} \leq 1 + d_{0, \mathbf{X}_1}. \end{array} \right. \quad (12)$$

Consider the set $\{0, \dots, N\}$ that corresponds to generic polynomials $\mathbf{F} = (F_0, \dots, F_N)$ (Def. 2.3). As many of the polynomials have the same support, we can gather them to simplify the cohomologies of (6). We need the following notation. For each tuple $(s_0, \dots, s_B) \in \mathbb{N}^{B+1}$, let $\mathcal{I}_{s_0, s_1, \dots, s_B}$ be the set of all the subsets of $\{0, \dots, N\}$, such that

- For $1 \leq j \leq B$, the index s_j indicates that we consider exactly s_j polynomials from (F_1, \dots, F_N) that belong to $\mathbb{Z}[\mathbf{u}][\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_j]_1$.
- In addition, if $s_0 = 1$, then 0 belongs to all the sets in $\mathcal{I}_{s_0, s_1, \dots, s_B}$.

That is,

$$\mathcal{I}_{s_0, s_1, \dots, s_B} := \left\{ I : I \subset \{0, \dots, N\}, (0 \in I \Leftrightarrow s_0 = 1) \text{ and} \right. \\ \left. (\forall 1 \leq j \leq B) s_j = \#\{k \in I \setminus \{0\} : F_k \in \mathbb{Z}[\mathbf{u}][\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_j]_1\} \right\}. \quad (13)$$

Example 3.4. If we consider a system (F_1, \dots, F_4) as in Ex. 3.2 and introduce some F_0 , it holds for $\mathbf{F} = (F_0, \dots, F_4)$ that $\mathcal{I}_{1,1,1,0} = \{\{0, 1, 3\}, \{0, 2, 3\}\}$ and $\mathcal{I}_{0,2,0,1} = \{\{1, 2, 4\}\}$.

Notice that if $I, J \in \mathcal{I}_{s_0, s_1, \dots, s_B}$, then I and J have the same cardinality and $\sum_{k \in I} \mathbf{d}_k = \sum_{k \in J} \mathbf{d}_k$, as they correspond to subsets of polynomials of \mathbf{F} with the same multidegrees.

The following lemma uses the sets $\mathcal{I}_{s_0, s_1, \dots, s_B}$ to simplify the cohomologies of (6).

Lemma 3.5. *Consider a generic overdetermined system $\mathbf{F} = (F_0, \dots, F_N)$ in $\mathbb{Z}[\mathbf{u}][\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ of multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$ (Def. 2.3), where (F_1, \dots, F_N) is a square star multilinear system such that, for every $j \in \{1, \dots, B\}$, $\mathcal{E}_j \geq \beta_j$, and \mathbf{d}_0 is the multidegree of F_0 . Following (6), we can rewrite the modules of the Weyman complex $K_v(\mathbf{m}) = \bigoplus_{p=0}^{N+1} K_{v,p} \otimes \mathbb{Z}[\mathbf{u}]$ in the more detailed form*

$$K_{v,p}(\mathbf{m}) \cong \bigoplus_{\substack{0 \leq s_0 \leq 1 \\ 0 \leq s_1 \leq \mathcal{E}_1 \\ \dots \\ 0 \leq s_B \leq \mathcal{E}_B \\ s_0 + s_1 + \dots + s_B = p}} H_p^{p-v} \left(\mathbf{m} - \left(\sum_{j=1}^B s_j, \dots, \sum_{j=1}^B s_j, s_1, \dots, s_B \right) - s_0 \mathbf{d}_0 \right) \otimes \bigoplus_{I \in \mathcal{I}_{s_0, s_1, \dots, s_B}} \bigwedge_{k \in I} e_k. \quad (14)$$

³For other choices of f_0 we found specific values of A and B for which every possible Weyman complex is not determinantal.

Moreover, the following isomorphisms hold for the cohomologies:

$$H_{\mathcal{P}}^{p-v} \left(\mathbf{m} - \left(\sum_{j=1}^B s_j, \dots, \sum_{j=1}^B s_j, s_1, \dots, s_B \right) - s_0 \mathbf{d}_0 \right) \cong \bigoplus_{\substack{r_{\mathbf{X}_1} \dots r_{\mathbf{X}_1}, r_{\mathbf{Y}_1} \dots r_{\mathbf{Y}_B} \in \mathbb{N} \\ \sum_i r_{\mathbf{X}_i} + \sum_j r_{\mathbf{Y}_j} = p-v}} \left(\bigotimes_{i=1}^A H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(m_{\mathbf{X}_i} - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right) \otimes \bigotimes_{j=1}^B H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}} \left(m_{\mathbf{Y}_j} - s_j - s_0 d_{0, \mathbf{Y}_j} \right) \right). \quad (15)$$

Proof. Consider $I, J \subset \mathcal{I}_{s_0, s_1, \dots, s_B}$. Then, by definition, $\#I = \#J$ and $\sum_{k \in I} \mathbf{d}_k = \sum_{k \in J} \mathbf{d}_k = (\sum_{j=1}^B s_j, \dots, \sum_{j=1}^B s_j, s_1, \dots, s_B) + s_0 \mathbf{d}_0$. Hence,

$$\begin{aligned} & \left(H_{\mathcal{P}}^{p-v}(\mathbf{m} - \sum_{k \in I} \mathbf{d}_k) \otimes \bigwedge_{k \in I} e_k \right) \oplus \left(H_{\mathcal{P}}^{p-v}(\mathbf{m} - \sum_{k \in J} \mathbf{d}_k) \otimes \bigwedge_{k \in J} e_k \right) \cong \\ & H_{\mathcal{P}}^{p-v} \left(\mathbf{m} - \left(\sum_{j=1}^B s_j, \dots, \sum_{j=1}^B s_j, s_1, \dots, s_B \right) - s_0 \mathbf{d}_0 \right) \otimes \left(\bigwedge_{k \in I} e_k \oplus \bigwedge_{k \in J} e_k \right). \end{aligned}$$

By definition of $\mathcal{E}_1, \dots, \mathcal{E}_B$ (def. 3.1), the set $\mathcal{I}_{s_0, s_1, \dots, s_B}$ is not empty if and only if $0 \leq s_0 \leq 1$ and for all $i \in \{1, \dots, B\}$ it holds $0 \leq s_i \leq \mathcal{E}_i$. Hence,

$$\{I : I \subset \{0, \dots, N\}, \#I = p\} = \bigcup_{\substack{0 \leq s_0 \leq 1 \\ 0 \leq s_1 \leq \mathcal{E}_1 \\ \dots \\ 0 \leq s_B \leq \mathcal{E}_B \\ s_0 + s_1 + \dots + s_B = p}} \mathcal{I}_{s_0, s_1, \dots, s_B}.$$

Thus, (14) holds. The isomorphism in (15) follows from Prop. 2.7. \square

In what follows, we identify the degree vectors that reduce the Weyman complex to have just two elements and, in this way, they provide us Koszul-type determinantal formulas for star multilinear systems (Def. 2.14). These degree vectors are associated to tuples called *determinantal data*. The determinantal data parameterize the different Koszul-type determinantal formulas that we can obtain using the Weyman complex.

Definition 3.6. Consider a partition of $\{1, \dots, B\}$ consisting of two sets P and D and a constant $c \in \mathbb{N}$. We say that the triplet (P, D, c) is *determinantal data* in the following cases:

- When f_0 corresponds to **Center-Vertex or Edge case**: if holds, $0 \leq c \leq A$.
- When f_0 corresponds to **Outer-Vertex case**: if the following holds,

$$\begin{cases} c = 0 & \text{if } \sum_{j \in P} d_{0, \mathbf{Y}_j} = 0, \text{ or} \\ c = A & \text{if } \sum_{j \in D} d_{0, \mathbf{Y}_j} = 0. \end{cases}$$

- When f_0 corresponds to **Triangle case**: if the following holds,

$$\begin{cases} 0 \leq c \leq A, \\ \sum_{j \in P} d_{0, \mathbf{Y}_j} \leq 1, \text{ and} \\ \sum_{j \in D} d_{0, \mathbf{Y}_j} \leq 1. \end{cases}$$

Equivalently, we say that the triplet (P, D, c) is *determinantal data* for the multidegree \mathbf{d}_0 if the following conditions are satisfied:

$$\left\{ \begin{array}{ll} \sum_{j \in P} d_{0, \mathbf{Y}_j} \leq 1 \\ \sum_{j \in D} d_{0, \mathbf{Y}_j} \leq 1 \\ 0 \leq c \leq A & \text{when } (\forall i \in [A]) \text{ it holds } d_{0, \mathbf{X}_i} = 1, \\ c = 0 & \text{when } (\forall i \in [A]) \text{ it holds } d_{0, \mathbf{X}_i} = 0 \text{ and } \sum_{j \in P} d_{0, \mathbf{Y}_j} = 0, \\ c = A & \text{when } (\forall i \in [A]) \text{ it holds } d_{0, \mathbf{X}_i} = 0 \text{ and } \sum_{j \in D} d_{0, \mathbf{Y}_j} = 0. \end{array} \right. \quad (16)$$

Theorem 3.7. Consider a generic overdetermined system $\mathbf{F} = (F_0, \dots, F_N)$ in $\mathbb{Z}[\mathbf{u}][\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ of multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$ (Def. 2.3), where (F_1, \dots, F_n) is a square star multilinear system. Assume that for every $j \in \{1, \dots, B\}$ it holds $\mathcal{E}_j \geq \beta_j$ (see Rem. 3.3) and that the multidegree of F_0 , \mathbf{d}_0 , is a solution of the system in (12). Then, for each determinantal data (P, D, c) (Def. 3.6) and a permutation $\sigma : \{1, \dots, A\} \rightarrow \{1, \dots, A\}$, the degree vector $\mathbf{m} = (m_{\mathbf{X}_1}, \dots, m_{\mathbf{X}_A}, m_{\mathbf{Y}_1}, \dots, m_{\mathbf{Y}_B})$, defined by

$$\begin{cases} m_{\mathbf{X}_i} = \sum_{j \in D} \beta_j + \sum_{k=1}^{\sigma(i)-1} \alpha_{\sigma^{-1}(k)} + d_{0, \mathbf{X}_i} & \text{for } 1 \leq i \leq A \text{ and } \sigma(i) > c \\ m_{\mathbf{X}_i} = \sum_{j \in D} \beta_j + \sum_{k=1}^{\sigma(i)-1} \alpha_{\sigma^{-1}(k)} - 1 & \text{for } 1 \leq i \leq A \text{ and } \sigma(i) \leq c \\ m_{\mathbf{Y}_j} = \mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} & \text{for } j \in P \\ m_{\mathbf{Y}_j} = -1 & \text{for } j \in D \end{cases}$$

corresponds to the Koszul-type determinantal formula (Def. 2.14)

$$K_{\bullet}(\mathbf{m}) : 0 \rightarrow K_{1, \omega+1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \xrightarrow{\delta_1(\mathbf{m})} K_{0, \omega}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \rightarrow 0,$$

where $\omega = \sum_{k=1}^c \alpha_{\sigma^{-1}(k)} + \sum_{j \in D} \beta_j$.

Proof. To simplify the presentation of the proof, we assume with no loss of generality that σ is the identity map. We rewrite (6) using Lem. 3.5. Hence, we obtain the following isomorphism,

$$H_{\mathbb{P}}^{p-v} \left(\mathbf{m} - \left(\sum_{j=1}^B s_j, \dots, \sum_{j=1}^B s_j, s_1, \dots, s_B \right) - s_0 \mathbf{d}_0 \right) \cong \bigoplus_{r_{\mathbf{X}_i} + \sum_j r_{\mathbf{Y}_j} = p-v} \left(\begin{array}{l} \bigotimes_{j \in P} H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}} (\mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} - s_j - s_0 d_{0, \mathbf{Y}_j}) \otimes \quad \text{[Case Y.1]} \\ \bigotimes_{j \in D} H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}} (-1 - s_j - s_0 d_{0, \mathbf{Y}_j}) \otimes \quad \text{[Case Y.2]} \\ \bigotimes_{i=1}^c H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k - 1 - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right) \otimes \quad \text{[Case X.1]} \\ \bigotimes_{i=c+1}^A H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + d_{0, \mathbf{X}_i} - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right) \quad \text{[Case X.2]} \end{array} \right) \quad (17)$$

We will study the values for $p, v, s_0, \dots, s_B, r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ such that $K_{v,p}(\mathbf{m})$ (14) does not vanish. Clearly, if $0 \leq s_0 \leq 1$ and $(\forall i \in \{1, \dots, B\}) 0 \leq s_i \leq \mathcal{E}_i$, then the module $\bigoplus_{I \in \mathcal{I}_{s_0, s_1, \dots, s_B}} \bigwedge_{k \in I} e_k$ is not zero. Hence, assuming $0 \leq s_0 \leq 1$ and $(\forall i \in \{1, \dots, B\}) 0 \leq s_i \leq \mathcal{E}_i$, we study the vanishing of the modules in (17). By Rem. 2.8, the modules in the right hand side of (17) are not zero only when, for $1 \leq i \leq A$, $r_{\mathbf{X}_i} \in \{0, \alpha_i\}$ and, for $1 \leq j \leq B$, $r_{\mathbf{Y}_j} \in \{0, \beta_j\}$. We can use Rem. 2.9 to show that if (17) does not vanish, then we have the following cases

| | | | |
|------------|-----------------------|---|------|
| [Case Y.1] | For $j \in P$ | $r_{\mathbf{Y}_j} = 0$ and $\mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} \geq s_j + s_0 d_{0, \mathbf{Y}_j}$ | |
| [Case Y.2] | For $j \in D$ | $r_{\mathbf{Y}_j} = \beta_j$ and $s_j + s_0 d_{0, \mathbf{Y}_j} \geq \beta_j$ | |
| [Case X.1] | For $1 \leq i \leq c$ | $r_{\mathbf{X}_i} = \alpha_i$ and $\sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_i} \geq \sum_{j \in D} \beta_j + \sum_{k=1}^i \alpha_{\mathbf{X}_k}$ | (18) |
| [Case X.2] | For $c < i \leq A$ | $r_{\mathbf{X}_i} = 0$ and $\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k \geq \sum_{j=1}^B s_j + (s_0 - 1) d_{0, \mathbf{X}_i}$ | |

From (18), we can deduce the possible values for v such that $K_{v,p}(\mathbf{m})$ does not vanish. From (14), it holds $p = \sum_{j=1}^B s_j + s_0$. By Prop. 2.7, $p - v = \sum_{i=1}^A r_{\mathbf{X}_i} + \sum_{j=1}^B r_{\mathbf{Y}_j}$. Hence, we deduce that

$$v = \sum_{j=1}^B s_j + s_0 - \sum_{j \in D} \beta_j - \sum_{i=1}^c \alpha_i.$$

- First we provide a lower bound for v . Assume that $c > 0$. By [Case X.1], if $i = c$, then

$$\sum_{j=1}^B s_j + s_0 d_{0, \mathbf{x}_c} \geq \sum_{j \in D} \beta_j + \sum_{k=1}^c \alpha_{\mathbf{x}_k}.$$

Hence, as $0 \leq s_0, d_{0, \mathbf{x}_{c+1}} \leq 1$, we conclude that $v \geq 0$ as

$$v = s_0 + \sum_{j=1}^B s_j - \sum_{j \in D} \beta_j - \sum_{k=1}^c \alpha_k \geq s_0 (1 - d_{0, \mathbf{x}_c}) \geq 0.$$

Assume instead that $c = 0$. Then $v = \sum_{j=1}^B s_j + s_0 - \sum_{j \in D} \beta_j$. By [Case Y.2], for each $j \in D$, $\beta_j \leq s_j + s_0 d_{0, \mathbf{Y}_j}$. Moreover, it holds, for each $j \in P$, $0 \leq s_j$. Adding the inequalities we deduce that,

$$\sum_{j=1}^B s_j + s_0 \sum_{j \in D} d_{0, \mathbf{Y}_j} \geq \sum_{j \in D} \beta_j. \quad (19)$$

By definition, $0 \leq \sum_{j \in D} d_{0, \mathbf{Y}_j} \leq 1$. Hence, by (19), $v \geq 0$ as, $v \geq s_0 - s_0 \sum_{j \in D} d_{0, \mathbf{Y}_j} \geq 0$.

- Finally we provide an upper bound for v . Assume that $c < A$. By [Case X.2], if we consider $i = c + 1$, then

$$\sum_{j \in D} \beta_j + \sum_{k=1}^c \alpha_k \geq \sum_{j=1}^B s_j + (s_0 - 1) d_{0, \mathbf{x}_{c+1}}.$$

Hence we conclude that $v \leq 1$, as $0 \leq s_0, d_{0, \mathbf{x}_{c+1}} \leq 1$ and so,

$$v = s_0 + \sum_{j=1}^B s_j - \sum_{j \in D} \beta_j - \sum_{k=1}^c \alpha_k \leq s_0 + (1 - s_0) d_{0, \mathbf{x}_{c+1}} \leq 1.$$

Assume instead that $c = A$. Then $v = \sum_{j=1}^B s_j + s_0 - \sum_{j \in D} \beta_j - \sum_{i=1}^A \alpha_i$. By [Case Y.1], for $j \in P$, $\mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} \geq s_j + s_0 d_{0, \mathbf{Y}_j}$. Moreover, it holds, for each $j \in D$, $\mathcal{E}_j \geq s_j$. As the system (F_1, \dots, F_N) is square, it holds $\sum_{j=1}^B \mathcal{E}_j = \sum_{i=1}^A \alpha_i + \sum_{j \in P} \beta_j + \sum_{j \in D} \beta_j$. Hence, adding the inequalities we obtain

$$\sum_{i=1}^A \alpha_i + \sum_{j \in D} \beta_j = \sum_{j=1}^B \mathcal{E}_j - \sum_{j \in P} \beta_j \geq \sum_{j=1}^B s_j + (s_0 - 1) \sum_{j \in P} d_{0, \mathbf{Y}_j}. \quad (20)$$

By definition, $0 \leq s_0, \sum_{j \in P} d_{0, \mathbf{Y}_j} \leq 1$. Hence, by (20), $v \leq 1$ as,

$$v \leq s_0 - (s_0 - 1) \sum_{j \in P} d_{0, \mathbf{Y}_j} \leq 1.$$

We conclude that the possible values for $v, p, r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ such that (17) is not zero are $v \in \{0, 1\}$, the possible values for $r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ are the ones in (18) and $p = \sum_{k=1}^c \alpha_k + \sum_{j \in D} \beta_j + v$. Let $\omega = \sum_{k=1}^c \alpha_k + \sum_{j \in D} \beta_j$. Hence, our Weyman complex looks like (8), where

$$\delta_1(\mathbf{m}) : K_{1, \omega+1}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}] \rightarrow K_{0, \omega}(\mathbf{m}) \otimes \mathbb{Z}[\mathbf{u}].$$

In what follows, we prove each case in (18). Consider the modules related to the variables \mathbf{Y}_j , for $j \in \{1, \dots, B\}$.

Case (Y.1) We consider the modules that involve the variables in the block \mathbf{Y}_j , for $j \in P$. As $s_j \leq \mathcal{E}_j$ and $s_0, d_{0, \mathbf{Y}_j} \leq 1$, it holds $\mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} - s_j - s_0 d_{0, \mathbf{Y}_j} > -\beta_j - 1$. Hence, by Rem. 2.9,

$$\begin{aligned} H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}}(\mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} - s_j - s_0 d_{0, \mathbf{Y}_j}) &\neq 0 \iff \\ r_{\mathbf{Y}_j} &= 0 \text{ and } \mathcal{E}_j - \beta_j + d_{0, \mathbf{Y}_j} \geq s_j + s_0 d_{0, \mathbf{Y}_j}. \end{aligned} \quad (21)$$

Case (Y.2) We consider the modules that involve the variables in the block \mathbf{Y}_j , for $j \in D$. As $s_j, s_0, d_{0, \mathbf{Y}_j} \geq 0$, then $-1 - s_j - s_0 d_{0, \mathbf{Y}_j} < 0$. Hence, by Rem. 2.9,

$$H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}}(-1 - s_j - s_0 d_{0, \mathbf{Y}_j}) \neq 0 \iff r_{\mathbf{Y}_j} = \beta_j \quad \text{and} \quad s_j + s_0 d_{0, \mathbf{Y}_j} \geq \beta_j. \quad (22)$$

Now we consider the cohomologies related to the blocks of variables \mathbf{X}_i , for $i \in \{1, \dots, A\}$. We assume that the cohomologies related to the blocks of variables \mathbf{Y}_j do not vanish.

Case (X.1) We consider the module related to the blocks $\mathbf{X}_1, \dots, \mathbf{X}_c$. We only need to consider this case if $c > 0$, so we assume $c > 0$. We prove that for each $1 \leq i \leq c$, if the cohomologies related to the variables in the blocks \mathbf{Y}_j , for $1 \leq j \leq B$, and the ones related to $\mathbf{X}_1, \dots, \mathbf{X}_{i-1}$, do not vanish, then

$$\begin{aligned} H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(m_{\mathbf{X}_i} - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right) \neq 0 &\iff \\ r_{\mathbf{X}_i} = \alpha_i \text{ and } \sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_i} &\geq \sum_{j \in D} \beta_j + \sum_{k=1}^i \alpha_{\mathbf{X}_k}. \end{aligned} \quad (23)$$

We proceed by induction on $1 \leq i \leq c$.

- Consider $i = 1$ and the cohomology related to the block \mathbf{X}_1 ,

$$H_{\mathbb{P}^{\alpha_1}}^{r_{\mathbf{X}_1}} \left(\sum_{j \in D} \beta_j - 1 - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_1} \right).$$

As we assumed that $c > 0$ and the triplet (P, D, c) is determinantal data (def. 3.6), by definition either $d_{0, \mathbf{X}_1} = 1$ or both $d_{0, \mathbf{X}_1} = 0$ and $\sum_{j \in D} d_{0, \mathbf{Y}_j} = 0$. Also, it holds $0 \leq s_0, \sum_{j \in D} d_{0, \mathbf{Y}_j} \leq 1$. Hence, from (19), we conclude that,

$$\sum_{j \in D} \beta_j - 1 - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_k} \leq s_0 \sum_{j \in D} d_{0, \mathbf{Y}_j} - 1 - s_0 d_{0, \mathbf{X}_k} < 0$$

Therefore, by Rem. 2.9,

$$\begin{aligned} H_{\mathbb{P}^{\alpha_1}}^{r_{\mathbf{X}_1}} \left(\sum_{j \in D} \beta_j - 1 - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_1} \right) \neq 0 &\iff \\ r_{\mathbf{X}_1} = \alpha_1 \quad \text{and} \quad \sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_1} &\geq \sum_{j \in D} \beta_j + \alpha_{\mathbf{X}_1} \end{aligned}$$

- We proceed by induction, assuming that (23) holds for $i - 1$, we prove the property for i . We consider the cohomology

$$H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j - 1 + \sum_{k=1}^{i-1} \alpha_k - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right).$$

By definition (see (12)), $d_{0, \mathbf{X}_{i-1}} = d_{0, \mathbf{X}_i}$, and by inductive hypothesis, if the previous modules do not vanish, then

$$\sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_i} = \sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_{i-1}} \geq \sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_{\mathbf{X}_k}.$$

Hence, by Rem. 2.9,

$$\begin{aligned} H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j - 1 + \sum_{k=1}^{i-1} \alpha_k - \sum_{j=1}^B s_j - s_0 d_{0, \mathbf{X}_i} \right) \neq 0 &\iff \\ r_{\mathbf{X}_i} = \alpha_i \quad \text{and} \quad \sum_{j=1}^B s_j + s_0 d_{0, \mathbf{X}_i} &\geq \sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + \alpha_i. \end{aligned}$$

Case (X.2) We consider the module related to the blocks $\mathbf{X}_{c+1}, \dots, \mathbf{X}_A$. We only need to consider this case if $c < A$, so we assume $c < A$. We prove that for each $c < i \leq A$, if the cohomologies related to the

variables in the blocks \mathbf{Y}_j , for $1 \leq j \leq B$, and the ones related to $\mathbf{X}_{i+1}, \mathbf{X}_{i+2}, \dots, \mathbf{X}_A$, do not vanish, then

$$\begin{aligned} H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_i} - \sum_{j=1}^B s_j \right) \neq 0 &\iff \\ r_{\mathbf{X}_i} = 0 \quad \text{and} \quad \sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k \geq \sum_{j=1}^B s_j + (s_0 - 1) d_{0, \mathbf{X}_i}. & \end{aligned} \quad (24)$$

We proceed by induction.

- Consider $i = A$ and the cohomology related to the block \mathbf{X}_A ,

$$H_{\mathbb{P}^{\alpha_A}}^{r_{\mathbf{X}_A}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{A-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_A} - \sum_{j=1}^B s_j \right).$$

As we assumed that $c < A$ and the triplet (P, D, c) is determinantal data (Def. 3.6), by definition, either $d_{0, \mathbf{X}_A} = 1$ or both $d_{0, \mathbf{X}_A} = 0$ and $\sum_{j \in P} d_{0, \mathbf{Y}_j} = 0$. Also it holds $0 \leq s_0, \sum_{j \in P} d_{0, \mathbf{Y}_j} \leq 1$. Hence, from (20), we conclude that

$$\sum_{j \in D} \beta_j + \sum_{i=1}^{A-1} \alpha_i + (1-s_0) d_{0, \mathbf{X}_A} - \sum_{j=1}^B s_j \geq -\alpha_A$$

Therefore, by Rem. 2.9,

$$\begin{aligned} H_{\mathbb{P}^{\alpha_A}}^{r_{\mathbf{X}_A}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{A-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_A} - \sum_{j=1}^B s_j \right) \neq 0 &\iff \\ r_{\mathbf{X}_A} = 0 \quad \text{and} \quad \sum_{j \in D} \beta_j + \sum_{k=1}^{A-1} \alpha_k \geq \sum_{j=1}^B s_j + (s_0 - 1) d_{0, \mathbf{X}_A}. & \end{aligned} \quad (25)$$

- We proceed by induction, assuming that (24) holds for $i+1 \leq A$, we prove the property for $i > c$. We consider the cohomology

$$H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_i} - \sum_{j=1}^B s_j \right).$$

By definition (see (12)), $d_{0, \mathbf{X}_{i+1}} = d_{0, \mathbf{X}_i}$. So, if the previous cohomologies do not vanish, by induction hypothesis,

$$\sum_{j \in D} \beta_j + \sum_{k=1}^i \alpha_k \geq (s_0 - 1) d_{0, \mathbf{X}_{i+1}} + \sum_{j=1}^B s_j = (s_0 - 1) d_{0, \mathbf{X}_i} + \sum_{j=1}^B s_j$$

Equivalently,

$$\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_i} - \sum_{j=1}^B s_j \geq -\alpha_i.$$

Hence, by Rem. 2.9,

$$\begin{aligned} H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k + (1-s_0) d_{0, \mathbf{X}_i} - \sum_{j=1}^B s_j \right) \neq 0 &\iff \\ r_{\mathbf{X}_i} = 0 \quad \text{and} \quad \sum_{j \in D} \beta_j + \sum_{k=1}^{i-1} \alpha_k \geq \sum_{j=1}^B s_j + (s_0 - 1) d_{0, \mathbf{X}_i}. & \end{aligned}$$

□

The previous theorem gives us Sylvester-like determinantal formulas in some cases.

Corollary 3.8 (Sylvester-type formulas). *Consider \mathbf{d}_0 corresponding to the Center-Vertex or Edge case. Let $\sigma : \{1, \dots, A\} \rightarrow \{1, \dots, A\}$ be any permutation and consider the determinantal data $(\{1, \dots, B\}, \emptyset, 0)$. Then, by Prop. 2.17, the overdetermined systems from Thm. 3.7 have a Sylvester-like formula coming from the degree vector \mathbf{m} related to the determinantal data $(\{1, \dots, B\}, \emptyset, 0)$ and the permutation σ .*

For each determinantal formula given by Thm. 3.7, we get another one from its dual.

Remark 3.9. Consider a degree vector \mathbf{m} related to the determinantal data (P, D, c) and the permutation σ . Then, the triplet $(D, P, A - c)$ is also determinantal data and the map $i \mapsto (A + 1 - \sigma(i))$ is a permutation of $\{1, \dots, A\}$. Let $\bar{\mathbf{m}}$ be the degree vector associated to $(D, P, A - c)$ and $i \mapsto (A + 1 - \sigma(i))$, then, by Prop. 2.10, $K_\bullet(\mathbf{m})$ is isomorphic to the dual complex of $K_\bullet(\bar{\mathbf{m}})$.

3.2 Size of determinantal formulas

Following general approaches for resultants as in [14, Ch. 3] or specific ideas for Koszul-type formulas as in [5], we can use the matrices associated to the determinantal formulas from Thm. 3.7 to solve the square systems (f_1, \dots, f_N) . To express the complexity of these approaches in terms of the size of the output, that is, the expected number of solutions, we study the size of the determinantal formulas of Thm. 3.7 and we compare them with the number of solutions of the system.

The multihomogeneous Bézout bound (Prop. 2.2) implies the following lemma.

Lemma 3.10. *The expected number of solutions, Υ , of a square star multilinear system is*

$$\Upsilon := \frac{(\sum_{i=1}^A \alpha_i)!}{\prod_{i=1}^A \alpha_i!} \cdot \prod_{j=1}^B \binom{\mathcal{E}_j}{\beta_j}.$$

Lemma 3.11. *The degree of the resultant and the sizes of the matrices corresponding to the determinantal formulas of Thm. 3.7, that is, the rank of the modules $K_0(\mathbf{m})$ and $K_1(\mathbf{m})$, are (See the beginning of Sec. 3.1 for the definition of the four cases and the notation in the bounds) as follows:*

- **Center-Vertex case:** *The rank of the modules is $\Upsilon \cdot (1 + \sum_{i=1}^A \alpha_i)$.*
- **Outer-Vertex case:** *The rank of the modules is $\Upsilon \cdot \frac{\mathcal{E}_j + \beta_j (\sum_{i=1}^A \alpha_i) + 1}{\mathcal{E}_j - \beta_j + 1}$.*
- **Edge case:** *The rank of the modules is $\Upsilon \cdot \frac{(1 + \sum_{i=1}^A \alpha_i)(\mathcal{E}_j + 1)}{\mathcal{E}_j - \beta_j + 1}$.*
- **Triangle case:** *The rank is $\Upsilon \cdot (1 + \sum_{i=1}^A \alpha_i) (1 + \frac{\beta_{j_1}}{\mathcal{E}_{j_1} - \beta_{j_1} + 1} + \frac{\beta_{j_2}}{\mathcal{E}_{j_2} - \beta_{j_2} + 1})$.*

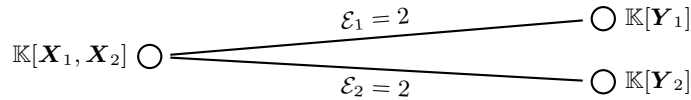
The proof of this lemma can be found in Appendix A.1 and follows from a direct computation, see Prop. 2.4.

3.3 Example

We follow the notation from the beginning of Sec. 3. Consider four blocks of variables $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2$ that we partition to two sets: $\{\mathbf{X}_1, \mathbf{X}_2\}$, of cardinality $A = 2$, and $\{\mathbf{Y}_1, \mathbf{Y}_2\}$, of cardinality $B = 2$. The number of variables in the blocks of the first set are $\alpha = (1, 1)$ and in the second $\beta = (1, 1)$. That is, we consider

$$\mathbf{X}_1 := \{X_{1,0}, X_{1,1}\}, \mathbf{X}_2 := \{X_{2,0}, X_{2,1}\}, \mathbf{Y}_1 := \{Y_{1,0}, Y_{1,1}\}, \mathbf{Y}_2 := \{Y_{2,0}, Y_{2,1}\}.$$

Let (f_1, \dots, f_4) be a square star multilinear system corresponding to the following graph,



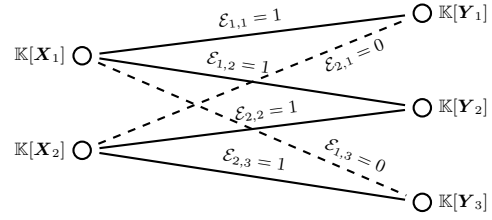
By Lem. 3.10, the expected number of solutions of the system is 8. We introduce a polynomial f_0 and we consider the multiprojective resultant of $\mathbf{f} := (f_0, f_1, \dots, f_N)$. By Lem. 3.11, the degree of the resultant, depending on the choice of f_0 , is as follows:

Definition 4.1 (Bipartite bilinear system). A square multihomogeneous system $\mathbf{f} = (f_1, \dots, f_N)$ in $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ with multidegrees $\mathbf{d}_1, \dots, \mathbf{d}_N \in \mathbb{Z}^q$ is a *bipartite bilinear system* if for every $k \in [N]$, there are $i_k \in [A]$ and $j_k \in [B]$ such that $f_k \in \mathbb{K}[\mathbf{X}_{i_k}, \mathbf{Y}_{j_k}]_1$. For each $i \in [A]$ and $j \in [B]$, let $\mathcal{E}_{i,j}$ be the number of polynomials of \mathbf{f} in $\mathbb{K}[\mathbf{X}_i, \mathbf{Y}_j]_1$.

We use the term *bipartite* because we can represent such systems using a bipartite graph. The vertices of the graph are the algebras $\mathbb{K}[\mathbf{X}_1], \dots, \mathbb{K}[\mathbf{X}_A], \mathbb{K}[\mathbf{Y}_1], \dots, \mathbb{K}[\mathbf{Y}_B]$. For each \mathbf{d}_k there is an edge between the vertices $\mathbb{K}[\mathbf{X}_i]$ and $\mathbb{K}[\mathbf{Y}_j]$ whenever $d_{k, \mathbf{X}_i} = d_{k, \mathbf{Y}_j} = 1$. That is, when it holds $f_k \in \mathbb{K}[\mathbf{X}_i, \mathbf{Y}_j]_1$. The graph is bipartite because we can partition the vertices to two sets, $\{\mathbb{K}[\mathbf{X}_1], \dots, \mathbb{K}[\mathbf{X}_A]\}$ and $\{\mathbb{K}[\mathbf{Y}_1], \dots, \mathbb{K}[\mathbf{Y}_B]\}$ such that there is no edge between vertices belonging to the same set.

Example 4.2. Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ be five blocks of variables. Consider the multihomogeneous system $(f_1, f_2, f_3, f_4) \subset \mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ with multidegrees

$$\begin{array}{l} \mathbf{d}_1 = \left(\begin{array}{cc|ccc} d_{i, \mathbf{X}_1} & d_{i, \mathbf{X}_2} & d_{i, \mathbf{Y}_1} & d_{i, \mathbf{Y}_2} & d_{i, \mathbf{Y}_3} \\ 1 & 0 & 1 & 0 & 0 \end{array} \right) \\ \mathbf{d}_2 = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right) \\ \mathbf{d}_3 = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ \mathbf{d}_4 = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 1 \end{array} \right) \end{array}$$



This system is a bipartite bilinear system where $\mathcal{E}_{1,1} = 1$, $\mathcal{E}_{1,2} = 1$, $\mathcal{E}_{1,3} = 0$, $\mathcal{E}_{2,1} = 0$, $\mathcal{E}_{2,2} = 1$ and $\mathcal{E}_{2,3} = 1$. The corresponding bipartite graph is the one above.

Remark 4.3. For each square bipartite bilinear system, it holds $N = \sum_{i=1}^A \sum_{j=1}^B \mathcal{E}_{i,j}$. Moreover, if the system has a nonzero finite number of solutions, then for $i \in \{1, \dots, A\}$, it holds $\sum_{j=1}^B \mathcal{E}_{i,j} \geq \alpha_i$ and for each $j \in \{1, \dots, B\}$ it holds $\sum_{i=1}^A \mathcal{E}_{i,j} \geq \beta_j$, see Prop. 2.2.

As we did in Sec. 3, we study overdetermined polynomial systems (f_0, f_1, \dots, f_N) in $\mathbb{K}[\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ where (f_1, \dots, f_N) is a square bipartite bilinear system and f_0 is a multilinear polynomial. We consider different types of polynomials f_0 . The obvious choice for f_0 is to have the same structure as one of the polynomials f_1, \dots, f_N ; still we also choose f_0 to have a different support. This leads to resultants of smaller degrees and so to matrices of smaller size. The following f_0 lead to determinantal formulas:

1. $f_0 \in \mathbb{K}[\mathbf{X}_i]_1$, for any $i \in \{1, \dots, A\}$.
2. $f_0 \in \mathbb{K}[\mathbf{Y}_j]_1$, for any $j \in \{1, \dots, B\}$.
3. $f_0 \in \mathbb{K}[\mathbf{X}_i, \mathbf{Y}_j]_1$, for any $i \in \{1, \dots, A\}$ and $j \in \{1, \dots, B\}$.

Theorem 4.4. Consider a generic overdetermined system $\mathbf{F} = (F_0, \dots, F_N)$ in $\mathbb{Z}[\mathbf{u}][\bar{\mathbf{X}}, \bar{\mathbf{Y}}]$ of multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$ (Def. 2.3), where (F_1, \dots, F_N) is a square bipartite bilinear system. Assume that for each $i \in \{1, \dots, A\}$, $\sum_{j=1}^B \mathcal{E}_{i,j} \geq \alpha_i$ and for each $j \in \{1, \dots, B\}$, $\sum_{i=1}^A \mathcal{E}_{i,j} \geq \beta_j$ (see Rem. 4.3), and F_0 is a multilinear polynomial as detailed in the paragraph above of multidegree \mathbf{d}_0 .

The degree vector $\mathbf{m} = (m_{\mathbf{X}_1}, \dots, m_{\mathbf{X}_A}, m_{\mathbf{Y}_1}, \dots, m_{\mathbf{Y}_B})$ define by

$$\begin{cases} m_{\mathbf{X}_i} = \sum_{j=1}^B \mathcal{E}_{i,j} - \alpha_i + d_{0, \mathbf{X}_i} & \text{for } 1 \leq i \leq A \\ m_{\mathbf{Y}_j} = -1 & \text{for } 1 \leq j \leq B \end{cases}$$

corresponds to a Koszul-type determinantal formula (Def. 2.14)

$$K_{\bullet}(\mathbf{m}) : 0 \rightarrow K_{1, \sum_{j=1}^B \beta_j + 1}(\mathbf{m}) \xrightarrow{\delta_1(\mathbf{m})} K_{0, \sum_{j=1}^B \beta_j}(\mathbf{m}) \rightarrow 0.$$

For the sake of brevity, we present the proof of Thm. 4.4 in Appendix A.2 as it is similar to the one of Thm. 3.7. Additionally, we present an example of the determinantal formulas constructed in this section in Appendix A.3.

5 Solving the Multiparameter Eigenvalue Problem

We present an algorithm (and an example) for solving a nonsingular MEP. The polynomial system associated to MEP, see (2), corresponds to a *star multilinear system* (Def. 3.1), where $A = 1$, $B = \alpha$ and $\mathcal{E}_j = \beta_j + 1$, for each $j \in [B]$. In particular, following (2), the system is $\mathbf{f}^{\text{MEP}} := (f_{1,0}, \dots, f_{1,\beta_1}, \dots, f_{\alpha,0}, \dots, f_{\alpha,\beta_\alpha})$, where $f_{i,j} \in \mathbb{K}[\mathbf{Y}_1, \dots, \mathbf{Y}_{i-1}, \mathbf{Y}_{i+1}, \dots, \mathbf{Y}_\alpha]_1$, for $i \in [\alpha]$ and $j + 1 \in [\beta_i + 1]$. The expected number of solutions is $\prod_{j=1}^\alpha (\beta_j + 1)$, see Lem. 3.10.

We introduce a linear form $f_0 \in \mathbb{K}[\mathbf{X}]_1$ and consider the Sylvester-type determinantal formula of Cor. 3.8. The map δ associated to this formula is as follows,

$$\delta : K[\mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1 \times \prod_{i=1}^\alpha \mathbb{K}[\mathbf{Y}_1 \dots \mathbf{Y}_{i-1}, \mathbf{Y}_{i+1} \dots \mathbf{Y}_\alpha]_1 \rightarrow \mathbb{K}[\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1 \\ (g_0, g_{1,0}, \dots, g_{1,\beta_1}, \dots, g_{\alpha,0}, \dots, g_{\alpha,\beta_\alpha}) \mapsto g_0 f_0 + \sum_{i=1}^\alpha \sum_{j=0}^{\beta_i} g_{i,j} f_{i,j}. \quad (26)$$

We fix a monomial basis for the domain and codomain of δ and we construct a matrix C associated to it. We arrange the rows and columns of C so that we can write it as $\begin{bmatrix} C_{2,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}$, such that

- The submatrix $\begin{bmatrix} C_{2,1} & C_{2,2} \end{bmatrix}$ corresponds to the rows $\mathbf{Y}^\theta f_0$, for $\mathbf{Y}^\theta \in \mathbb{K}[\mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1$.
- The submatrix $\begin{bmatrix} C_{1,2} \\ C_{2,2} \end{bmatrix}$ corresponds to the column associated to the monomial $\mathbf{Y}^\theta X_0$.
- If the k -th row of C corresponds to $\mathbf{Y}^\theta f_0$, then the k -th column corresponds to the monomial $\mathbf{Y}^\theta X_0$.

We say that a MEP is *affine* if \mathbf{f}^{MEP} is a zero-dimensional system and for every solution the x_0 -coordinate is not zero. When \mathbf{f}^{MEP} has a finite number of solutions, we can always assume that it is *affine* by performing a structured linear change of coordinates. When the MEP is affine, by [5, Prop. 4.5], the matrix $C_{1,1}$ is invertible. Moreover, by [5, Lem. 4.4], we have a one to one correspondence between the eigenvalues of the MEP and the (classical) eigenvalues of the Schur complement of $C_{2,2}$, $\widetilde{C}_{2,2} := C_{2,2} - C_{2,1} C_{1,1}^{-1} C_{1,2}$: each eigenvalue of $\widetilde{C}_{2,2}$ is the evaluation of $\frac{f_0}{x_0}$ at an eigenvalue of the original MEP. Also, the right eigenspaces of $\widetilde{C}_{2,2}$ correspond to the vector of monomials $x_0 \mathbf{Y}^\alpha$, for $\mathbf{Y}^\theta \in \mathbb{K}[\mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1$, evaluated at each solution of $(f_{1,0}, \dots, f_{1,\beta_1}, \dots, f_{\alpha,0}, \dots, f_{\alpha,\beta_\alpha})$ [21, Lem. 5.1]. Even more, if we multiply these eigenvectors by $\begin{bmatrix} M_{1,1}^{-1} & M_{2,1} \\ I & \end{bmatrix}$, then we recover a vector corresponding to the evaluation of every monomial in $\mathbb{K}[\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1$ evaluated at the original solution. This information suffices to recover all the coordinates of the solution.

Remark 5.1 (Multiplication map). For an affine MEP, the matrix $\widetilde{C}_{2,2}$ corresponds to the multiplication map of the rational function $\frac{f_0}{x_0}$ in the quotient ring $\mathbb{K}[\mathbf{X}, \mathbf{Y}_1, \dots, \mathbf{Y}_\alpha] / \mathbf{f}^{\text{MEP}}$ at multidegree $(0, 1, \dots, 1) \in \mathbb{N}^{\alpha+1}$, with respect to the monomial basis $\{\mathbf{Y}^\theta\}_{\mathbf{Y}^\theta \in \mathbb{K}[\mathbf{Y}_1, \dots, \mathbf{Y}_\alpha]_1}$.

Algorithm 1 SolveMEP($\{\{M^{(i,j)}\}_{j \in [\beta_i+1]}\}_{i \in [\alpha]}\})$

Require: Affine MEP $\{\{M^{(i,j)}\}_{j \in [\beta_i+1]}\}_{i \in [\alpha]}$

- 1: $(f_{1,0}, \dots, f_{\alpha,\beta_\alpha}) \leftarrow$ Multilinear system associated to $\{\{M^{(i,j)}\}_{j \in [\beta_i+1]}\}_{i \in [\alpha]}$ (Eq. 2).
 - 2: $f_0 \leftarrow$ Generic linear polynomial in $\mathbb{K}[\mathbf{X}]_1$.
 - 3: $\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} \leftarrow$ Matrix corresponding to δ (Eq. 26); partitioned in four block
 - 4: $\left\{ \left(\frac{f_0}{x_0}(p), \bar{v}_p \right) \right\}_p \leftarrow$ Set of Eigenvalue-Eigenvector of the Schur compl. of $C_{2,2}$.
 - 5: **for all** $\left(\frac{f_0}{x_0}(p), \bar{v}_p \right) \in \left\{ \left(\frac{f_0}{x_0}(p), \bar{v}_p \right) \right\}_p$ **do**
 - 6: Extract the coordinates of p from $\begin{bmatrix} C_{1,1}^{-1} \cdot C_{2,1} \\ I \end{bmatrix} \cdot \bar{v}_p$.
-

Remark 5.2 (Atkinson's Delta Method). By inspecting the eigenvalues and eigenvectors of the Schur complement of $C_{2,2}$ for $f_0 = \frac{x_i}{x_0}$, we can conclude that $\widetilde{C}_{2,2}$ equals the matrix $\Delta_0^{-1} \Delta_i$ from Atkinson's Delta method, see [3, Chp. 6]. It worth to point out that our construction of $\widetilde{C}_{2,2}$ improves Atkinson's construction of $\Delta_0^{-1} \Delta_i$ as we avoid the symbolic expansion by minors that he considers [3, Eq. 6.4.4]. The dimension of the matrix $C_{2,2}$, and so the dimension of Atkinson's Delta matrices, is $\prod_{i=1}^\alpha (\beta_i + 1) \times \prod_{i=1}^\alpha (\beta_i + 1)$. These matrices are dense. In contrast, the dimension of the matrix C is $(\alpha + 1) \prod_{i=1}^\alpha (\beta_i + 1) \times (\alpha + 1) \prod_{i=1}^\alpha (\beta_i + 1)$ (the degree of the resultant of the system), but the matrix is structured (i.e., multi-Hankel matrix) and sparse; it has at most $(\sum_{i=1}^\alpha \beta_i + \alpha + 1) (\alpha + 1) \prod_{i=1}^\alpha (\beta_i + 1)$ non-zero positions.

In what follows we present an example of our algorithm in the two-parameter eigenvalue problem (2EP). The interested reader can find information about the applications of 2EP in physics in [29]. We consider the 2EP given by the matrices

$$\begin{aligned} M^{(1,0)} &:= \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} & M^{(1,1)} &:= \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} & M^{(1,2)} &:= \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix} \\ M^{(2,0)} &:= \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} & M^{(2,1)} &:= \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} & M^{(2,1)} &:= \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix} \end{aligned} \quad (27)$$

For simplicity, we will name the three blocks of variables as $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, instead of $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Y}_2$. Following (2), we write 2EP as the following bilinear system

$$\begin{aligned} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= \begin{bmatrix} -7x_0 + 12x_1 - 7x_2 & -3x_0 + 2x_1 - x_2 \\ -8x_0 + 13x_1 - 7x_2 & -2x_0 + x_1 - x_2 \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \\ \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} &= \begin{bmatrix} -11x_0 + 7x_1 - 4x_2 & -3x_0 - x_1 \\ 4x_0 + x_1 - x_2 & x_0 + 2x_1 - x_2 \end{bmatrix} \cdot \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \end{aligned}$$

According to Lem. 3.10, the 2EP should have 4 different solutions. To solve this system, we introduce a linear polynomial $f_0 \in \mathbb{K}[\mathbf{X}]$ that separates the eigenvalues of the 2EP, that is, if λ_1 and λ_2 are different eigenvalues, then $\frac{f_0}{x_i}(\lambda_1) \neq \frac{f_0}{x_i}(\lambda_2)$, for some $x_i \in \mathbf{X}$. Then, we consider a Sylvester-like determinantal formula for the resultant of (f_0, \dots, f_4) (Cor. 3.8) and we solve the original system using eigenvalue and eigenvector computations as in [5].

If the MEP problem has a finite number of eigenvalues and all of them are different, then any generic $f_0 \in \mathbb{K}[\mathbf{X}]$ separates the eigenvalues. In our case, we choose $f_0 := -x_0 + 5x_1 - 3x_2$. Following Cor. 3.8, there is a Sylvester-type formula for the resultant of the system $\mathbf{f} := (f_0, \dots, f_4)$ using the degree vector $\mathbf{m} := (1, 1, 1)$. The latter is related to the determinantal data $(\{1, 2\}, \emptyset, 0)$. The Weyman complex reduces to

$$0 \rightarrow \left(\begin{array}{c} S_X(0) \otimes S_Y(1) \otimes S_Z(1) \otimes \{e_0\} \\ \oplus S_X(0) \otimes S_Y(0) \otimes S_Z(1) \otimes \{e_1 \oplus e_2\} \\ \oplus S_X(0) \otimes S_Y(1) \otimes S_Z(0) \otimes \{e_3 \oplus e_4\} \end{array} \right) \xrightarrow{\delta_1(\mathbf{m}, \mathbf{f})} (S_X(1) \otimes S_Y(1) \otimes S_Z(1) \otimes \mathbb{K}) \rightarrow 0,$$

where the map $\delta_1(\mathbf{m}, \mathbf{f})$ is a Sylvester map (Prop. 2.17). Hence, the resultant of \mathbf{f} is the determinant of a matrix C representing this map, which has dimensions 12×12 (Case 1, Lem. 3.11). We split C in $\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}$ according to [5, Def. 4.1].

$$\begin{array}{c|cccccccc|cccc} & x_2y_0z_0 & x_2y_0z_1 & x_2y_1z_0 & x_2y_1z_1 & x_1y_0z_0 & x_1y_0z_1 & x_1y_1z_0 & x_1y_1z_1 & x_0y_0z_0 & x_0y_0z_1 & x_0y_1z_0 & x_0y_1z_1 \\ \hline z_0e_1 & -7 & & -1 & & 12 & & 2 & & -7 & & -3 & \\ z_1e_1 & & -7 & & -1 & & 12 & & 2 & & -7 & & -3 \\ z_0e_2 & -7 & & -1 & & 13 & & 1 & & -8 & & -2 & \\ z_1e_2 & & -7 & & -1 & & 13 & & 1 & & -8 & & -2 \\ y_0e_3 & -4 & & 0 & & 7 & & -1 & & -11 & & -3 & \\ y_1e_3 & & -4 & & 0 & & 7 & & -1 & & -11 & & -3 \\ y_0e_4 & -1 & & -1 & & 1 & & 2 & & 4 & & 1 & \\ y_1e_4 & & -1 & & -1 & & 1 & & 2 & & 4 & & 1 \\ \hline y_0z_0e_0 & -3 & & & & 5 & & & & -1 & & & \\ y_0z_1e_0 & & -3 & & & & 5 & & & & -1 & & \\ y_1z_0e_0 & & & -3 & & & & 5 & & & & -1 & \\ y_1z_1e_0 & & & & -3 & & & & 5 & & & & -1 \end{array}$$

Remark 5.3. If the original MEP has a finite number of eigenvalues, after performing a generic linear change of coordinates in the variables \mathbf{X} , we can assume that there is no solution of (f_1, \dots, f_n) such that $x_0 = 0$.

By [5, Prop. 4.5], as the system (f_1, \dots, f_n) has no solutions such that $x_0 = 0$, the matrix $C_{1,1}$ is nonsingular. Hence, by [5, Lem. 4.4], we have a one to one correspondence between the eigenvalues of the 2EP and the (classical) eigenvalues of the Schur complement of $C_{2,2}$, $\widetilde{C}_{2,2} := C_{2,2} - C_{2,1}C_{1,1}^{-1}C_{1,2}$: each eigenvalue of $\widetilde{C}_{2,2}$ is the evaluation

$$\widetilde{C}_{2,2} = \begin{bmatrix} \frac{7}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{3}{2} & \frac{9}{4} & 2 \\ -\frac{21}{4} & -3 & \frac{27}{4} & \frac{5}{2} \\ \frac{69}{4} & \frac{19}{2} & -\frac{63}{4} & -6 \end{bmatrix}.$$

of $\frac{f_0}{x_0}$ at an eigenvalue of the original 2EP. In our case $\widetilde{C_{2,2}}$ is as it appears at the left.

Let p_1, \dots, p_4 be the four solutions of (f_1, \dots, f_4) . Then, the eigenvalues of $\widetilde{C_{2,2}}$ are $\frac{f_0}{x_0}(p_1) = 1$, $\frac{f_0}{x_0}(p_2) = 2$, $\frac{f_0}{x_0}(p_3) = 3$ and $\frac{f_0}{x_0}(p_4) = -2$. As $\delta_1(\mathbf{m}, \mathbf{f})$ is a Sylvester-like map, the right eigenspaces of $\widetilde{C_{2,2}}$ contain the vector of monomials $\mathbf{v} := \begin{bmatrix} x_0 y_0 z_0 \\ x_0 y_0 z_1 \\ x_0 y_1 z_0 \\ x_0 y_1 z_1 \end{bmatrix}$ evaluated at each solutions of (f_1, \dots, f_4) [21, Lem. 5.1]; this is as it appears in the table below. If each eigenspace has dimension one, then we can recover some coordinates of the solutions by inverting the monomial map given by \mathbf{v} .

| \mathbf{v} | p_1 | p_2 | p_3 | p_4 |
|---------------|-------|-------|-------|-------|
| $x_0 y_0 z_0$ | 1 | 1 | 1 | 1 |
| $x_0 y_0 z_1$ | -3 | -1 | -2 | -3 |
| $x_0 y_1 z_0$ | 1 | 1 | -1 | -3 |
| $x_0 y_1 z_1$ | -3 | -1 | 2 | 9 |

For example, the z_0 -coordinate of p_1 is non-zero as $(x_0 y_0 z_0)(p_1) \neq 0$, and so its z_1 -coordinate equals $\frac{(x_0 y_0 z_1)}{(x_0 y_0 z_0)}(p_1) = -3$. To compute the remaining coordinates, either we substitute the computed coordinates of the solutions in the original system and we solve a linear system, or we extend each eigenvector $\mathbf{v}(p_i)$ to $\mathbf{w}(p_i)$, where $\mathbf{w}(p_i)$ is the solution of the following linear system:

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} \mathbf{w}(p_i) = \frac{f_0}{x_0}(p_i) \begin{bmatrix} 0 \\ \mathbf{v}(p_i) \end{bmatrix}, \text{ and so } \mathbf{w}(p_i) = \begin{bmatrix} -C_{1,1}^{-1} \cdot C_{1,2} \mathbf{v}(p_i) \\ \mathbf{v}(p_i) \end{bmatrix}.$$

Each coordinate of $\mathbf{w}(p_i)$ is a monomial in $\mathbb{K}[\mathbf{X}]_1 \otimes \mathbb{K}[\mathbf{Y}]_1 \otimes \mathbb{K}[\mathbf{Z}]_1$ evaluated at p_i . Hence, we can recover the coordinates of p_i from $\mathbf{w}_i(p_i)$ by inverting a monomial map. In this case, the solutions to (f_1, \dots, f_4) , and so eigenvalues and eigenvectors of 2EP are

| | Ext. Eigenvalues | | | Eigenvectors | |
|-----------|------------------|------------|------------|--------------|--|
| | x_0, x_1, x_2 | y_0, y_1 | z_0, z_1 | | |
| $p_1 = ($ | 1, -1, -3 | 1, 1 | 1, -3 |) | |
| $p_2 = ($ | 1, 3, 4 | 1, 1 | 1, -1 |) | |
| $p_3 = ($ | 1, 1, 1 | 1, -1 | 1, -2 |) | |
| $p_4 = ($ | 1, 1, 2 | 1, -3 | 1, -3 |) | |

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A Additional proofs and examples

A.1 Proof of Lemma 3.11

Proof of Lemma 3.11. Following Cor. 2.16, the size of the Koszul determinantal matrix is the degree of the resultant. By Prop. 2.4, it holds,

$$\text{degree}(\mathbf{res}) = \sum_{k=0}^N \text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{k-1}, \mathbf{d}_{k+1}, \dots, \mathbf{d}_N).$$

As many multidegrees \mathbf{d}_k are identical, we couple the summands in the previous equation. For each $j \in \{1, \dots, B\}$, let $I_j \in \{1, \dots, N\}$ be the index of a polynomial in \mathbf{F} such that $F_j \in \mathbb{K}[\mathbf{X}_1, \dots, \mathbf{X}_A, \mathbf{Y}_{I_j}]$. Recall that \mathcal{E}_j is the number of polynomials with multidegree equal to \mathbf{d}_{I_j} . Hence, we rewrite the degree of the resultant as

$$\text{deg}(\text{Res}_{\mathcal{P}}(\mathbf{d}_0, \dots, \mathbf{d}_n)) = \text{MHB}(\mathbf{d}_1, \dots, \mathbf{d}_n) + \sum_{j=1}^B \mathcal{E}_j \text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{I_j-1}, \mathbf{d}_{I_j+1}, \dots, \mathbf{d}_n).$$

From Lem. 3.10, $\text{MHB}(\mathbf{d}_1, \dots, \mathbf{d}_n) = \frac{(\sum_{i=1}^A \alpha_i)!}{\prod_{i=1}^A \alpha_i!} \cdot \prod_{j=1}^B \binom{\mathcal{E}_j}{\beta_j} = \Upsilon$. By Prop. 2.2, for every $1 \leq j \leq B$, $\text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{I_j-1}, \mathbf{d}_{I_j+1}, \dots, \mathbf{d}_n)$ is the coefficient of $(\prod_{i=1}^A Z_{X_i}^{\alpha_i})(\prod_{t=1}^B Z_{Y_t}^{\beta_t})$ in

$$\left(\sum_{i=1}^A d_{0, \mathbf{X}_i} Z_{X_i} + \sum_{t=1}^B d_{0, \mathbf{Y}_t} Z_{Y_t} \right) \left(\sum_{i=1}^A Z_{X_i} + Z_{Y_j} \right)^{\mathcal{E}_j-1} \prod_{k \in \{1, \dots, B\} \setminus \{j\}} \left(\sum_{i=1}^A Z_{X_i} + Z_{Y_k} \right)^{\mathcal{E}_k}.$$

Consider the last two factors of the previous equation, that is

$$\left(\sum_{i=1}^A Z_{X_i} + Z_{Y_j} \right)^{\mathcal{E}_j-1} \prod_{k \in \{1, \dots, B\} \setminus \{j\}} \left(\sum_{i=1}^A Z_{X_i} + Z_{Y_k} \right)^{\mathcal{E}_k}. \quad (28)$$

Then

$$\text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{I_j-1}, \mathbf{d}_{I_j+1}, \dots, \mathbf{d}_n) = \sum_{s=1}^A d_{0, \mathbf{X}_s} \theta_{j,s}^X + \sum_{l=1}^B d_{0, \mathbf{Y}_l} \theta_{j,l}^Y,$$

where $\theta_{j,s}^X$ is the coefficient of $\frac{(\prod_{i=1}^A Z_{X_i}^{\alpha_i})(\prod_{t=1}^B Z_{Y_t}^{\beta_t})}{Z_{X_s}}$ in (28), and $\theta_{j,t}^Y$ is the coefficient of $\frac{(\prod_{i=1}^A Z_{X_i}^{\alpha_i})(\prod_{t=1}^B Z_{Y_t}^{\beta_t})}{Z_{Y_t}}$ in (28). After some computations, we have

$$\theta_{j,s}^X = \frac{((\sum_{i=1}^A \alpha_i) - 1)!}{(\alpha_s - 1)! \prod_{i \in \{1, \dots, A\} \setminus \{s\}} \alpha_i!} \binom{\mathcal{E}_j - 1}{\beta_j} \prod_{k \in \{1, \dots, B\} \setminus \{j\}} \binom{\mathcal{E}_k}{\beta_k} = \Upsilon \cdot \frac{\alpha_s}{\sum_{i=1}^A \alpha_i} \frac{\mathcal{E}_j - \beta_j}{\mathcal{E}_j},$$

$$\theta_{j,t}^Y = \begin{cases} \frac{(\sum_{i=1}^A \alpha_i)!}{\prod_{i=1}^A \alpha_i!} \binom{\mathcal{E}_j - 1}{\beta_j - 1} \prod_{k \in \{1, \dots, B\} \setminus \{j\}} \binom{\mathcal{E}_k}{\beta_k} = \Upsilon \cdot \frac{\beta_j}{\mathcal{E}_j} & \text{if } t = j, \\ \frac{(\sum_{i=1}^A \alpha_i)!}{\prod_{i=1}^A \alpha_i!} \binom{\mathcal{E}_t}{\beta_t - 1} \binom{\mathcal{E}_j - 1}{\beta_j} \prod_{k \in \{1, \dots, B\} \setminus \{t, j\}} \binom{\mathcal{E}_k}{\beta_k} = \Upsilon \cdot \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} \frac{\mathcal{E}_j - \beta_j}{\mathcal{E}_j} & \text{otherwise.} \end{cases}$$

Using the formulas for $\theta_{j,s}^X$ and $\theta_{j,t}^Y$ we get

$$\begin{aligned} \text{deg}(\text{Res}_{\mathcal{P}}(\mathbf{d}_0, \dots, \mathbf{d}_n)) &= \text{MHB}(\mathbf{d}_1, \dots, \mathbf{d}_n) + \sum_{j=1}^B \mathcal{E}_j \text{MHB}(\mathbf{d}_0, \dots, \mathbf{d}_{I_j-1}, \mathbf{d}_{I_j+1}, \dots, \mathbf{d}_n) = \\ &= \Upsilon + \sum_{j=1}^B \mathcal{E}_j \left(\sum_{s=1}^A d_{0, \mathbf{X}_s} \theta_{j,s}^X + \sum_{l=1}^B d_{0, \mathbf{Y}_l} \theta_{j,l}^Y \right) = \Upsilon + \sum_{j=1}^B \mathcal{E}_j \left(\sum_{s=1}^A d_{0, \mathbf{X}_s} \theta_{j,s}^X \right) + \sum_{j=1}^B \mathcal{E}_j \left(\sum_{l=1}^B d_{0, \mathbf{Y}_l} \theta_{j,l}^Y \right). \end{aligned}$$

Next, we simplify the last two summands of the previous equation. For the first one, as $\sum_{i=1}^A \alpha_i = \sum_{j=1}^B (\mathcal{E}_j - \beta_j)$ and for all s it holds $d_{0, \mathbf{X}_s} = d_{0, \mathbf{X}_1}$, we obtain

$$\sum_{j=1}^B \mathcal{E}_j \left(\sum_{s=1}^A d_{0, \mathbf{X}_s} \theta_{j,s}^X \right) = \sum_{j=1}^B \mathcal{E}_j \left(\sum_{s=1}^A d_{0, \mathbf{X}_s} \Upsilon \frac{\alpha_s}{\sum_{i=1}^A \alpha_i} \frac{\mathcal{E}_j - \beta_j}{\mathcal{E}_j} \right) = \Upsilon d_{0, \mathbf{X}_1} \sum_{i=1}^A \alpha_i.$$

For the second one, we perform the following direct calculations

$$\begin{aligned}
\sum_{j=1}^B \mathcal{E}_j \left(\sum_{t=1}^B d_{0, \mathbf{Y}_t} \theta_{j,t}^{\mathbf{Y}_t} \right) &= \sum_{j=1}^B \mathcal{E}_j \left(\sum_{t \in \{1, \dots, B\} \setminus \{j\}} d_{0, \mathbf{Y}_t} \Upsilon \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} \frac{\mathcal{E}_j - \beta_j}{\mathcal{E}_j} + d_{0, \mathbf{Y}_j} \Upsilon \frac{\beta_j}{\mathcal{E}_j} \right) = \\
&\Upsilon \sum_{j=1}^B \left(\sum_{t=1}^B d_{0, \mathbf{Y}_t} \cdot \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} (\mathcal{E}_j - \beta_j) - d_{0, \mathbf{Y}_j} \cdot \frac{\beta_j (\mathcal{E}_j - \beta_j)}{\mathcal{E}_j - \beta_j + 1} + d_{0, \mathbf{Y}_j} \beta_j \right) = \\
&\Upsilon \sum_{j=1}^B (\mathcal{E}_j - \beta_j) \left(\sum_{t=1}^B d_{0, \mathbf{Y}_t} \cdot \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} \right) + \Upsilon \sum_{j=1}^B d_{0, \mathbf{Y}_j} \cdot \frac{\beta_j}{\mathcal{E}_j - \beta_j + 1} = \\
&\Upsilon \left(1 + \sum_{i=1}^A \alpha_i \right) \left(\sum_{t=1}^B d_{0, \mathbf{Y}_t} \cdot \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} \right).
\end{aligned}$$

At last we have the formula

$$\deg(\text{Res}_{\mathcal{P}}(\mathbf{d}_0, \dots, \mathbf{d}_n)) = \Upsilon \left(1 + d_{0, \mathbf{X}_1} \sum_{i=1}^A \alpha_i + (1 + \sum_{i=1}^A \alpha_i) \left(\sum_{t=1}^B d_{0, \mathbf{Y}_t} \cdot \frac{\beta_t}{\mathcal{E}_t - \beta_t + 1} \right) \right).$$

The proof follows from instantiating the values of $d_{0, \mathbf{X}_1}, d_{0, \mathbf{Y}_1}, \dots, d_{0, \mathbf{Y}_B}$ according to the multidegree of f_0 . \square

A.2 Proof of Theorem 4.4

Proof of Theorem 4.4. In this proof, we follow the same strategy as in the proof of Thm. 3.7. As we did in Sec. 3.1, we can interpret the various multidegrees of f_0 , $\mathbf{d}_0 = (d_{0, \mathbf{X}_1}, \dots, d_{0, \mathbf{X}_A}, d_{0, \mathbf{Y}_1}, \dots, d_{0, \mathbf{Y}_B})$, that we want to prove that lead to determinantal formulas as solutions of the following system of inequalities:

$$\left\{ \begin{array}{ll} (\forall 1 \leq i \leq A) & 0 \leq d_{0, \mathbf{X}_i} \leq 1 \\ (\forall 1 \leq j \leq B) & 0 \leq d_{0, \mathbf{Y}_j} \leq 1 \\ & \sum_{i=1}^A d_{0, \mathbf{X}_i} \leq 1 \\ & \sum_{j=1}^B d_{0, \mathbf{Y}_j} \leq 1. \end{array} \right. \quad (29)$$

Consider the set $\{0, \dots, N\}$ that corresponds to generic polynomials $\mathbf{F} = (F_0, \dots, F_N)$ of multidegrees $\mathbf{d}_0, \dots, \mathbf{d}_N$ (Def. 2.3), where (F_1, \dots, F_n) is a square bipartite bilinear system such that, for each $i \in \{1, \dots, A\}$, $\sum_{j=1}^B \mathcal{E}_{i,j} \geq \alpha_i$ and for each $j \in \{1, \dots, B\}$, $\sum_{i=1}^A \mathcal{E}_{i,j} \geq \beta_j$, and \mathbf{d}_0 is the multidegree of F_0 . As many of these polynomials have the same support, similarly to (13), we can gather them to simplify the cohomologies of (6). For that we introduce the following notation. For each tuple $s_0, s_{1,1}, \dots, s_{A,B} \in \mathbb{N}$, let $\mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}}$ be the set of all the subsets of $\{0, \dots, N\}$, such that

- For $1 \leq i \leq A$ and $1 \leq j \leq B$, the index $s_{i,j}$ indicates that we consider exactly $s_{i,j}$ polynomials from (F_1, \dots, F_N) that belong to $\mathbb{Z}[\mathbf{u}][\mathbf{X}_i, \mathbf{Y}_j]_1$.
- In addition, if $s_0 = 1$, then 0 belongs to all the sets in $\mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}}$

That is,

$$\begin{aligned}
\mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}} &:= \left\{ I : I \subset \{0, \dots, n\}, (0 \in I \Leftrightarrow s_0 = 1) \text{ and} \right. \\
&\quad \left. (\forall 1 \leq i \leq A) (\forall 1 \leq j \leq B) s_{i,j} = \#\{k \in I : f_k \in \mathbb{K}[\mathbf{X}_i, \mathbf{Y}_j]\} \right\}. \quad (30)
\end{aligned}$$

As in Lem. 3.5, we exploit the sets $\mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}}$ to rewrite the cohomologies $K_v(\mathbf{m}) = \bigoplus_{p=0}^{N+1} K_{v,p} \otimes \mathbb{Z}[\mathbf{u}]$ of (6) in the following way,

$$\begin{aligned}
K_{v,p}(\mathbf{m}) &= \bigoplus_{\substack{0 \leq s_0 \leq 1 \\ (\forall 1 \leq i \leq A) (\forall 1 \leq j \leq B) 0 \leq s_{i,j} \leq \mathcal{E}_{i,j} \\ s_0 + \sum_{i=1}^A \sum_{j=1}^B s_{i,j} = p}} \left(H_{\mathcal{P}}^{p-v}(\mathbf{m} - \left(\sum_{j=1}^B s_{1,j}, \dots, \sum_{j=1}^B s_{A,j}, \sum_{i=1}^A s_{i,1}, \dots, \sum_{i=1}^A s_{i,B} \right) - s_0 \mathbf{d}_0) \right. \\
&\quad \left. \otimes \bigoplus_{I \in \mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}}} \bigwedge_{k \in I} e_k \right). \quad (31)
\end{aligned}$$

Hence, using the Künneth Formula (Prop. 2.7) with the degree vector \mathbf{m} defined in Thm. 4.4, we have the following isomorphisms of cohomologies,

$$H_{\mathcal{P}}^{p-v} \left(\mathbf{m} - \left(\sum_{j=1}^B s_{1,j}, \dots, \sum_{j=1}^B s_{A,j}, \sum_{i=1}^A s_{i,1}, \dots, \sum_{i=1}^A s_{i,B} \right) - s_0 \mathbf{d}_0 \right) \cong \bigoplus_{r_{\mathbf{X}_1} + \dots + r_{\mathbf{X}_A} + r_{\mathbf{Y}_1} + \dots + r_{\mathbf{Y}_B} = p-v} \left(\begin{array}{l} \bigotimes_{i=1}^A H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1-s_0) d_{0,\mathbf{X}_i} \right) \quad [\text{Case X}] \\ \bigotimes_{j=1}^B H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}} \left(-1 - \sum_{i=1}^A s_{i,j} - d_{0,\mathbf{Y}_j} s_0 \right) \quad [\text{Case Y}] \end{array} \right) \quad (32)$$

We will study the values for $p, v, s_0, s_{1,1}, \dots, s_{A,B}, r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ such that $K_{v,p}(\mathbf{m})$ does not vanish. Clearly, if $0 \leq s_0 \leq 1$ and $(\forall i \in \{1, \dots, A\}) (\forall j \in \{1, \dots, B\}) 0 \leq s_{i,j} \leq \mathcal{E}_{i,j}$, then the module $\bigoplus_{I \in \mathcal{I}_{s_0, s_{1,1}, \dots, s_{A,B}}} \bigwedge_{k \in I} e_k$ is not zero. Hence, assuming $0 \leq s_0 \leq 1$ and $(\forall i \in \{1, \dots, A\}) (\forall j \in \{1, \dots, B\}) 0 \leq s_{i,j} \leq \mathcal{E}_{i,j}$ we study the vanishing of the modules in the right-side part of (32). We will study the cohomologies independently. By Rem. 2.8, the modules in the right hand side of (32) are not zero only when, for $1 \leq i \leq A$, $r_{\mathbf{X}_i} \in \{0, \alpha_i\}$ and, for $1 \leq j \leq B$, $r_{\mathbf{Y}_j} \in \{0, \beta_j\}$. At the end of the proof we show that if (32) does not vanish then the following conditions hold,

| | | | |
|----------|-----------------------|--|------|
| [Case X] | For $1 \leq i \leq A$ | $r_{\mathbf{X}_i} = 0$ $\sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1-s_0) d_{0,\mathbf{X}_i} \geq 0$ | (33) |
| [Case Y] | For $1 \leq j \leq B$ | $r_{\mathbf{Y}_j} = \beta_j$ $\sum_{i=1}^A s_{i,j} + s_0 d_{0,\mathbf{Y}_j} - \beta_j \geq 0$ | |

Using (33), we study the possible values for v such that $K_{v,p}(\mathbf{m})$ does not vanish. From (31), it holds $p = \sum_{i=1}^A \sum_{j=1}^B s_{i,j} + s_0$. By Prop. 2.7, $p - v = \sum_{i=1}^A r_{\mathbf{X}_i} + \sum_{j=1}^B r_{\mathbf{Y}_j}$. Hence, we deduce that, when $K_{v,p}(\mathbf{m})$ does not vanish, it holds,

$$v = \sum_{i=1}^A \sum_{j=1}^B s_{i,j} + s_0 - \sum_{j=1}^B \beta_j = \sum_{j=1}^B \left(\sum_{i=1}^A s_{i,j} - \beta_j \right) + s_0.$$

We bound the values for v for which $K_{v,p}(\mathbf{m})$ does not vanish.

- First, we lower-bound v . Assume that the cohomologies involving \mathbf{Y}_j are not zero. Hence, if we sum over $j \in \{1, \dots, B\}$ the inequalities of [Case Y], we conclude that

$$0 \leq \sum_{j=1}^B \left(\sum_{i=1}^A s_{i,j} - \beta_j \right) + s_0 \sum_{j=1}^B d_{0,\mathbf{Y}_j} = v + s_0 \left(\sum_{j=1}^B d_{0,\mathbf{Y}_j} - 1 \right).$$

By definition, (29), $0 \leq \sum_{j=1}^B d_{0,\mathbf{Y}_j} \leq 1$, and $0 \leq s_0 \leq 1$, hence $0 \leq v$.

- Finally, we upper-bound v . Assume that the cohomologies involving \mathbf{X}_j are not zero. Hence, if we sum over $i \in \{1, \dots, A\}$ the inequalities of [Case X], we conclude that

$$\begin{aligned} 0 &\leq \sum_{i=1}^A \left(\sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1-s_0) d_{0,\mathbf{X}_i} \right) \\ &= \sum_{i=1}^A \sum_{j=1}^B \mathcal{E}_{i,j} - \sum_{i=1}^A \alpha_i - \sum_{i=1}^A \sum_{j=1}^B s_{i,j} + (1-s_0) \left(\sum_{i=1}^A d_{0,\mathbf{X}_i} \right). \end{aligned}$$

Recall that $N = \sum_{i=1}^A \sum_{j=1}^B \mathcal{E}_{i,j} = \sum_{i=1}^A \alpha_i + \sum_{i=1}^B \beta_i$ and $v = \sum_{i=1}^A \sum_{j=1}^B s_{i,j} + s_0 - \sum_{i=1}^B \beta_i$. Also, as \mathbf{d}_0 is a solution of (29), it holds $0 \leq \sum_{j=1}^B d_{0,\mathbf{Y}_j} \leq 1$, and $0 \leq s_0 \leq 1$. Hence

$$v = \sum_{i=1}^A \sum_{j=1}^B s_{i,j} + s_0 - \sum_{i=1}^B \beta_i \leq s_0 + (1-s_0) \left(\sum_{i=1}^A d_{0,\mathbf{X}_i} \right) \leq 1.$$

We conclude that the possible values for $v, p, r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ such that (32) is not zero are $v \in \{0, 1\}$, the possible values for $r_{\mathbf{X}_1}, \dots, r_{\mathbf{X}_A}, r_{\mathbf{Y}_1}, \dots, r_{\mathbf{Y}_B}$ are the ones in (33) and $p = \sum_{j=1}^B \beta_j + v$. Hence, our Weyman complex looks like (8), where

$$\delta_1(\mathbf{m}) : K_{1, \sum_{j=1}^B \beta_j + 1}(\mathbf{m}) \rightarrow K_{0, \sum_{j=1}^B \beta_j}(\mathbf{m})$$

is a Koszul-type determinantal formula.

In what follows we prove each case in (33).

Case (X) We consider the modules that involve the variables in the block \mathbf{X}_i , for $1 \leq i \leq A$. As $(\forall j) s_{i,j} \leq \mathcal{E}_{i,j}$, $0 \leq s_0 \leq 1$ and $0 \leq d_{0, \mathbf{X}_i} \leq 1$, we have $\sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1 - s_0) d_{0, \mathbf{X}_i} > -1 - \alpha_i$. Hence, by Rem. 2.9,

$$H_{\mathbb{P}^{\alpha_i}}^{r_{\mathbf{X}_i}} \left(\sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1 - s_0) d_{0, \mathbf{X}_i} \right) \neq 0 \iff$$

$$r_{\mathbf{X}_i} = 0 \quad \text{and} \quad \sum_{j=1}^B (\mathcal{E}_{i,j} - s_{i,j}) - \alpha_i + (1 - s_0) d_{0, \mathbf{X}_i} \geq 0.$$

Case (Y) We consider the modules that involve the variables in the block \mathbf{Y}_j , for $1 \leq j \leq B$. As $(\forall j \in \{1, \dots, B\}) s_{i,j} \geq 0$ and $s_0, d_{0, \mathbf{Y}_j} \geq 0$, then $-1 - \sum_{i=1}^A s_{i,j} - s_0 d_{0, \mathbf{Y}_j} < 0$, and so by Rem. 2.9,

$$H_{\mathbb{P}^{\beta_j}}^{r_{\mathbf{Y}_j}} \left(-1 - \sum_{i=1}^A s_{i,j} - s_0 d_{0, \mathbf{Y}_j} \right) \neq 0 \iff$$

$$r_{\mathbf{Y}_j} = \beta_j \quad \text{and} \quad \sum_{i=1}^A s_{i,j} + s_0 d_{0, \mathbf{Y}_j} - \beta_j \geq 0.$$

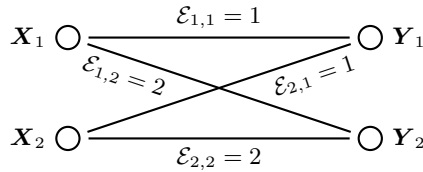
□

A.3 Example of determinantal formula for bipartite bilinear system

Consider four blocks of variables such that $A = 2, B = 2, \alpha = (1, 2), \beta = (1, 2)$, and

$$\begin{cases} X_1 := \{X_{1,0}, X_{1,1}\} \\ X_2 := \{X_{2,0}, X_{2,1}, X_{2,2}\} \\ Y_1 := \{Y_{1,0}, Y_{1,1}\} \\ Y_2 := \{Y_{2,0}, Y_{2,1}, Y_{2,2}\}. \end{cases}$$

Let (f_1, \dots, f_6) be the square bipartite bilinear system represented by the following graph:



We introduce a polynomial $f_0 \in \mathbb{K}[\mathbf{X}_1, \mathbf{Y}_1]_1$ and consider the following overdetermined system \mathbf{f} where,

$$\mathbf{f} := \begin{cases} f_0 := & (a_1 y_{1,0} + a_2 y_{1,1}) x_{1,0} + (a_3 y_{1,0} + a_4 y_{1,1}) x_{1,1} \\ f_1 := & (b_1 y_{1,0} + b_2 y_{1,1}) x_{1,0} + (b_3 y_{1,0} + b_4 y_{1,1}) x_{1,1} \\ f_2 := & (c_1 y_{1,0} + c_2 y_{1,1}) x_{2,0} + (c_5 y_{1,0} + c_6 y_{1,1}) x_{2,1} + (c_3 y_{1,0} + c_4 y_{1,1}) x_{2,2} \\ f_3 := & (d_1 y_{2,0} + d_2 y_{2,2} + d_3 y_{2,1}) x_{1,0} + (d_4 y_{2,0} + d_5 y_{2,2} + d_6 y_{2,1}) x_{1,1} \\ f_4 := & (e_1 y_{2,0} + e_2 y_{2,2} + e_3 y_{2,1}) x_{1,0} + (e_4 y_{2,0} + e_5 y_{2,2} + e_6 y_{2,1}) x_{1,1} \\ f_5 := & (g_1 y_{2,0} + g_2 y_{2,2} + g_3 y_{2,1}) x_{2,0} + (g_7 y_{2,0} + g_8 y_{2,2} + g_9 y_{2,1}) x_{2,1} \\ & + (g_4 y_{2,0} + g_5 y_{2,2} + g_6 y_{2,1}) x_{2,2} \\ f_6 := & (h_1 y_{2,0} + h_2 y_{2,2} + h_3 y_{2,1}) x_{2,0} + (h_7 y_{2,0} + h_8 y_{2,2} + h_9 y_{2,1}) x_{2,1} \\ & + (h_4 y_{2,0} + h_5 y_{2,2} + h_6 y_{2,1}) x_{2,2} \end{cases} \quad (34)$$

