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# Full waveform inversion for bore reconstruction of woodwind-like instruments

Augustin Ernoult<sup>1</sup>    Juliette Chabassier<sup>1</sup>    Samuel Rodriguez<sup>2</sup>    Augustin Humeau<sup>3</sup>

1. Project team Makutu, Inria Bordeaux-Sud Ouest, 200 avenue de la vieille Tour, 33405 Talence CEDEX, France
2. University of Bordeaux, CNRS, Arts et Metiers Institute of Technology, Bordeaux INP, INRAE, I2M Bordeaux, F-33400 Talence, France
3. Humeau Factory, 24700 Montpon-Ménéstérol, France

## Abstract

The internal geometry of a wind instrument can be estimated from acoustic measurements. For woodwind instruments, it implies to characterize the inner shape (bore) but also the side holes (dimensions and location). In this study, the geometric parameters are recovered by a gradient-based optimization process, which minimizes the deviation between simulated and measured linear acoustic responses of the resonator for several fingerings through an observable function. The acoustic fields are computed by solving a linear system resulting from the 1D spectral finite elements spatial discretization of the wave propagation equations (including thermo-viscous effects, radiation and side holes). The “full waveform inversion” (FWI) technique exploits the fact that the gradient of the cost function can be computed by solving the same linear system that the one of the direct problem with a different source term. The dependence of the cost function with respect to the choice of the observed quantity, the frequency range and the fingerings used, is first analyzed. Then, the FWI is used to reconstruct, from measured impedances, an elementary instrument with 14 design variables. The results, obtained in about 1 minute on a laptop, are in great agreement with the direct geometric measurements.

## 1 Introduction

A wind instrument can be divided into two parts: its exciter which creates an acoustic oscillation under the musician action, and its resonator in which acoustic waves propagate. The resonator is mainly a tube, characterized by the variation of its internal cross-sectional area along its main axis and eventually by the presence of side holes with specific locations and dimensions. In this study, the bore profile groups all these geometric parameters. In the traditional instrument organology, most woodwind instruments have side holes as opposed to most brass instruments. By extension, in this study, woodwind-like instruments refer to instruments with side holes.

The bore reconstruction of such instruments aims at obtaining their bore geometry from acoustic measurements. This non-destructive characterization can be useful to explore historical instruments, to document museum collections or to build reproductions of them. It can also be used to detect defects on built instruments. This technique can finally be seen as a first step towards automatic design of new musical instruments, for which, to some extent, a similar approach could be used with targeted acoustic specifications instead of acoustic measurements.

In the literature, bore reconstructions from measured data have only been performed on brass instruments (one main tube with no side hole). The first approach was the pulse reflectometry [1, 2] in which the variation of the cross sectional area along the main pipe axis is computed directly from the input impulse response (in the time domain). The reconstructed bore is a step tube, the length of each step being determined by the highest measured frequency. This approach can not be directly applied on instruments with side holes in which the waves propagate in several tubes in parallel. As a second approach, optimization algorithms can be used to find the geometry which minimizes the deviation between simulations and acoustic measurements. Kausel [3] minimizes the residuals between target and simulated impedances of a brass instrument by a zeroth order algorithm (Rosenbrock). This kind of local algorithms, in which no gradient is computed, is robust but not efficient in terms of convergence rate and computational cost. A “virtual trumpet” with up to 100 design variables is reconstructed from simulated targets by imposing the mouthpiece geometry, and an elementary step tube is reconstructed from measurements. In a slightly different approach, Chilekwa [4] uses Rosenbrock algorithm to localize and characterize defects in a cylindrical duct, which can be seen as the reconstruction of a really elementary woodwind instrument.

As mentioned previously, the automatic design of instruments can be seen as a similar issue. To this purpose, Noreland [5] used a gradient based algorithm with analytic gradient to design a brass bore profile composed of up to

400 truncated cones, by optimizing the frequency and magnitude of resonances. This kind of first order algorithms is more efficient but necessitates to estimate the gradient of the cost function. Braden [6] tuned some properties of the impedance of a trombone by using Rosenbrock algorithm, to minimize three cost functions concerning the total impedance, the frequency and the magnitude of the resonances. He parameterized the geometry of the brass bell through a few Bessel horns to reduce the number of design variables. Colinot [7] designed a coaxial saxophone (without side hole) equivalent to an existing conical instrument by minimizing the p-norm of the impedance difference.

Concerning woodwind instruments, Noreland [8] followed by Guilloteau [9] designed a logical clarinet (one hole per note without any fork fingering) by optimizing the resonance frequencies of the two first registers, then the ratio of their magnitudes. They used a gradient-based algorithm in which gradients are estimated by finite-differences. By using a similar approach on more complex instruments with geometric constraints, Ernoult [10] showed that the traditional resonance definition leads to a discontinuous cost-function which can not be easily optimized. A new and more robust definition is proposed to tune the resonance frequency and magnitude. In a slightly different approach, Tournemene [11] proposed to tune some geometric parameters of a brass instrument (up to 10) by minimizing a cost function based on sound characteristics averaged on simulated sounds, obtained with a set of musician parameters. He used a surrogate-assisted derivative-free optimization strategy, particularly suited to this non differentiable problem.

The aim of the present study is to find a robust and efficient methodology to reconstruct instruments with side holes from acoustic measurements. Following the literature, the reconstruction is performed by minimizing the deviation between simulations and measurements of an observable built from the input impedance for several fingerings. To solve this inverse problem, computing the gradient of the cost function without using finite differences leads to a faster and more accurate estimation. This reconstruction process being a first step towards instrument design, a versatile formulation giving possibility to easily change the cost function is developed. The full waveform inversion (FWI) appears perfectly suited to this purpose. This technique, which comes from the seismology community [12], is based on the observation that the gradient of the acoustic fields in each points of the medium with respect to the design variables is solution to the same equations as the forward problem, modeling the wave propagation in a known propagation medium, with only different source terms. It allows the gradient estimation of the inverse problem with a negligible additional computation cost.

The wave propagation model and the finite element method used to compute the impedance (forward problem) through the inversion of a linear system are presented in Sec. 2. In Sec. 3, the general bore reconstruction problem (inverse problem) is presented. The FWI is applied to woodwind instruments, allowing an explicit characterization of the gradient of cost function. Considerations on the observable are carefully given for the case of an elementary instrument in Sec. 4, resulting in a reconstruction strategy applied on measured impedance data. The performance of this techniques are then discussed, and a conclusion is given in Sec. 5. All the tools presented here are implemented and available in the open source software OpenWinD [13] under GPLv3.

## 2 Forward problem

### 2.1 Instrument description

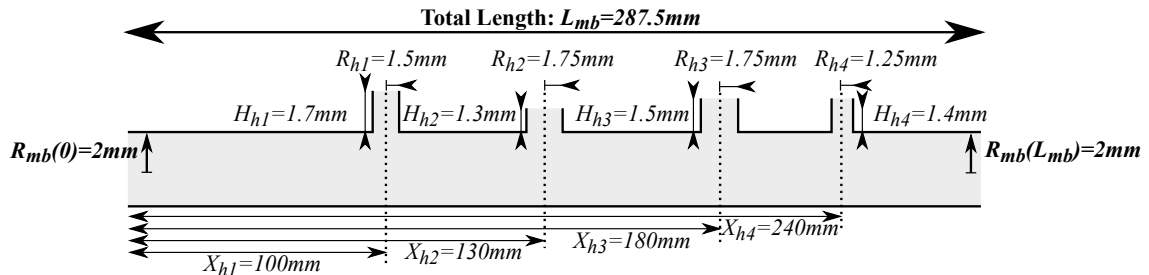


Figure 1: Sketch of the studied cylindrical tube with 4 side holes. It is described by 15 geometric parameters (3 for the main bore profile and 3 for each of the four holes).

The reconstruction methodology (theory and strategy) is presented through a concrete resonator with a simple geometry for which it is possible to measure manually the design parameters (Fig. 1). The main bore (index  $mb$  in

Parameter	Measured Value	Uncertainty
$L_{mb}$	287.5 mm	$\pm 0.5$ mm
$R_{mb}(L_{mb})$	2 mm	$\pm 0.1$ mm
$X_{h,i} \quad i \in [1, 4]$	[100, 130, 180, 240] mm	$\pm 0.5$ mm
$H_{h,i} \quad i \in [1, 4]$	[1.7, 1.3, 1.5, 1.4] mm	$\pm 0.2$ mm
$R_{h,i} \quad i \in [1, 4]$	[1.5, 1.75, 1.75, 1.25] mm	$\pm 0.05$ mm

Table 1: Direct geometric measurements of the design parameters grouped together in the vector  $\xi$ .

the rest of the document) is a cylindrical pipe seen as a particular conical part, described by 3 design parameters: its total length  $L_{mb} = 287.5$  mm and the left and right ends radii, both equal to  $R_{mb}(0) = R_{mb}(L_{mb}) = 2$  mm. It is drilled with four side holes ( $h1, h2, h3, h4$ ) being cylindrical chimneys, each characterized by 3 design parameters: their location on the main pipe  $X_{hi}$ , their chimney height  $H_{hi}$  and their radius  $R_{hi}$ . The values obtained by direct measurement with caliper or graduated scale are indicated in Table 1 with the corresponding estimated uncertainties (the chimney heights being measured indirectly, their uncertainties are higher).

The instrument geometry is therefore described by  $N_\xi = 15$  design parameters grouped together in the vector  $\xi$ . The forward problem presented in the current section, allows the computation of the acoustic response of the instrument in the frequency domain, including its input impedance  $Z$ , from the knowledge of these design parameters.

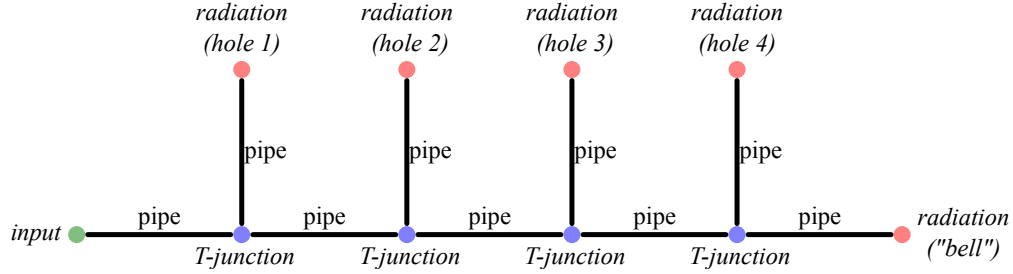


Figure 2: The graph of the studied resonator is composed of 9 pipes and 10 connection conditions.

To model the acoustic propagation, the instrument is described by a graph composed of pipes and connectors which imposes the pipes boundaries conditions (Fig. 2). The studied instrument is composed of 9 pipes and 10 connectors: 5 pipes for the main bore and one for each of the four hole chimneys. In these pipes, the wave propagation is modeled in one dimension by Telegraphists' equations including thermo-viscous effects, detailed in Section 2.2. At the hole locations, the three pipes (upstream, downstream and chimney) are connected by a T-joint junction including acoustics masses detailed in Section 2.3. The open end of each pipes is associated to a radiation condition developed in Section 2.4. The entrance of the instrument is associated to an input condition: a unitary flow is imposed at each frequency.

## 2.2 Acoustic propagation along the pipe

Each pipe  $n \in [1, N]$  is characterized by its radius evolution  $R_n(x, \xi)$  along its main axis  $x$  and its length  $L_n(\xi)$ , both depending on the design parameters grouped in the vector  $\xi$ . The wave propagation is modeled by Telegraphists' equations, following the methodology presented by Tournemene and Chabassier [14]. The profiles of the acoustic pressure  $p_n$  and of the acoustic flow  $u_n$  along the  $n^{\text{th}}$  pipe of length  $L_n$ , at the angular frequency  $\omega$ , follow the equations [15, Chap.5.5, p.239]:

$$\begin{cases} Z_v(x, \omega)u_n + \frac{dp_n}{dx} = 0 \\ Y_t(x, \omega)p_n + \frac{du_n}{dx} = 0 \end{cases} \quad \forall x \in [0, L_n]. \quad (1)$$

It includes the thermo-viscous effects through:

$$\begin{cases} Z_v(x, \omega) = j\omega \frac{\rho}{S_n(x)} (1 - \mathcal{J}(k_v R_n(x)))^{-1} \\ Y_t(x, \omega) = j\omega \frac{S_n(x)}{\rho c^2} (1 + (\gamma - 1)\mathcal{J}(k_t R_n(x))) \end{cases} \quad (2a)$$

$$k_v = \sqrt{j\omega \frac{\rho}{\mu}}, \quad k_t = \sqrt{j\omega \frac{\rho C_p}{\kappa}},$$

where  $S_n(x) = \pi R_n(x)^2$  is the cross sectional area at the position  $x$  and  $\mathcal{J} : \mathbb{C} \mapsto \mathbb{C}$  is a complex function modeling the viscothermal terms [15, Chap.5.5, p.239]

$$\mathcal{J}(z) = \frac{2 J_1(z)}{z J_0(z)} \quad \forall z \in \mathbb{C} \quad (3)$$

with  $J_0(z)$  and  $J_1(z)$  Bessel functions of the first kind. Table 2 describes the physical constants, which are supposed uniform on the entire instrument in this study.

Air density (kg.m <sup>-3</sup> )	$\rho = 1.2929(T_0/(T + T_0))$
Sound celerity (m.s <sup>-1</sup> )	$c = 331.45\sqrt{(T + T_0)/T_0}$
Viscosity (kg.m <sup>-1</sup> .s <sup>-1</sup> )	$\mu = 1.708e^{-5}(1 + 2.9e^{-3}T)$
Therm. conduct. (Cal/(m.s.°C))	$\kappa = 5.77e^{-3}(1 + 3.3e^{-3}T)$
Specific heat (Cal/(kg.°C))	$C_p = 240$
Ratio of spec. heat	$\gamma = 1.402$

Table 2: Physical constant values [15, chap.5.5.2, p.241].  $T$  is the temperature in Celsius and  $T_0 = 273.15$ K. In this study  $T = 20^\circ\text{C}$ .

In equation (1), the pipe length  $L_n$  is taken into account indirectly trough the definition domain of  $x$ . To explicitly write these physical parameters in the equations and give the possibility to modify them without changing the definition domain, the scaling  $x = L_n \tilde{x}$  is used:

$$\begin{cases} L_n Z_v(\tilde{x}, \omega) u_n + \frac{dp_n}{d\tilde{x}} = 0 \\ L_n Y_t(\tilde{x}, \omega) p_n + \frac{du_n}{d\tilde{x}} = 0 \end{cases} \quad \forall \tilde{x} \in [0, 1]. \quad (4)$$

Now, all the physical parameters are coefficients of these equations. With this substitution, the optimization process only modifies the coefficients of the equations by keeping the same definition domain and the same topology. The boundary conditions at  $\tilde{x} = 0$  and  $\tilde{x} = 1$ , necessary to complete this system, depend on the type of connection at each extremity of the pipe.

### 2.3 T-junction

At the location of a side hole, the pressure and acoustic outward flow at the boundary of the chimney pipe ( $p_3, -u_3$ ), at the upstream ( $p_1, u_1$ ) and downstream ( $p_2, -u_2$ ) main bore parts, are related through the linear system [15, Chap.7.7 p.364-370]

$$j\omega \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad (5)$$

$$u_1 - u_2 - u_3 = 0.$$

where  $m_{11}, m_{12}$  and  $m_{22}$  are acoustic masses linked to the junction geometry. In the case of a symmetric junction,  $m_{11} = m_{22}$ , which is assumed within this study. They are generally expressed through the masses  $m_s$  and  $m_a$ , respectively associated to symmetric and antisymmetric modes:  $m_{11} = m_{22} = m_s + m_a/4$  and  $m_{12} = m_s - m_a/4$ , which appear during diagonalization of the mass matrix. This leads to the introduction of two auxiliary variables

$\gamma_{123}, \zeta_{123}$ , such that:

$$\begin{cases} \begin{pmatrix} u_1 \\ -u_2 \\ -u_3 \end{pmatrix} = -T^T \begin{pmatrix} \gamma_{123} \\ \zeta_{123} \end{pmatrix} \\ j\omega M \begin{pmatrix} \gamma_{123} \\ \zeta_{123} \end{pmatrix} + T \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0 \end{cases} \quad (6)$$

with

$$M = \begin{pmatrix} 2m_s & 0 \\ 0 & m_a/2 \end{pmatrix}, \quad \text{and} \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \quad (7)$$

For circular pipes, the masses are linked to the geometric parameters by polynomial expression fitted on data obtained with 3D simulation [16, 17]. Here the following expression [16] is assumed:

$$m_s = \frac{\rho}{\pi R_h} (0.82 - 0.193\delta - 1.09\delta^2 + 1.27\delta^3 - 0.71\delta^4) + m_{mv}, \quad (8)$$

with  $\delta = R_h/R_{mb}$  and  $R_h, R_m$  respectively the inner radii of the chimney pipe and of the main bore pipe at the hole location. The matching volume mass [18]

$$m_{mv} = \frac{\rho}{\pi R_h} \frac{\delta}{8} (1 + 0.207\delta^3) \quad (9)$$

is here included in the symmetric mass and not added to the effective length of the chimney pipe as usual [16, 17]. Indeed, despite the fact that both expressions are similar at the limit  $\omega m_{mv} c S_{hole} / \rho \ll 1$  and that the wave propagation in this volume is debatable, it locates all the coefficients depending on both the main bore and the side hole geometry into the junction. This is consistent with the graph representation (Fig.2). This representation also prompts us to use the long chimney approximation for the acoustic mass  $m_a$  [16]

$$m_a = \frac{\rho R_h}{\pi R_{mb}^2} (-0.37 + 0.087\delta) \delta^2, \quad (10)$$

in which it is independent of the open/close state at the other extremity of the chimney pipe. This approximation is valid when the chimney height is longer than this radius ( $H_h > R_h$ ), however the induced error stays small for the studied geometry (below 2% of deviation from the complete formula for the 4 holes). This approximate model suffers from the major drawback of being non-passive in some practical cases, and could be improved using asymptotic approaches as [19, 20].

## 2.4 Radiation condition

Pipe radiation is modeled by a radiation admittance  $Y_{r,n}$  at each opening (the bell and the 4 holes). For an opening of radius  $R$ , it reads:

$$u_n(1) = Y_{r,n}(\omega, R) p_n(1). \quad (11)$$

It is computed with the ‘‘non-causal’’ formula given by Silva et al. [21]:

$$Y_{r,n}(\omega, R) = \frac{\pi R^2}{\rho c} \frac{1}{j \tan(kR\mathcal{L} + 0.5j \log(|\mathcal{R}|))} \quad (12)$$

with  $k = \omega/c$  the wavenumber and

$$\mathcal{L} = \eta \frac{1 + b_1(kR)^2}{1 + b_2(kR)^2 + b_3(kR)^4 + b_4(kR)^6} \quad (13)$$

$$|\mathcal{R}| = \frac{1 + a_1(kR)^2}{1 + (\beta + a_1)(kR)^2 + a_2(kR)^4 + a_3(kR)^6} \quad (14)$$

with  $\eta, \beta, a_1, a_2, a_3, b_1, \dots, b_4$  real coefficients depending on the radiation condition [21]. The choice of this condition can impact the result of the reconstruction. It is assumed here that the holes radiate with an infinite flanged condition, and the main bore with an unflanged condition. This choice is discussed in more details in Section 4.4. For closed holes, the flow is imposed to zero by assuming  $Y_{r,n} = 0$ .

## 2.5 Numerical resolution

The resolution is done using the high order spectral finite elements method in 1D to approximate Eqs. (4;6;11) along with a unitary flow at the pipe entry (input). Following Tournemene and Chabassier [14], the pressure and flow inside the instrument pipes are sought as piecewise polynomials of variable order on small intervals called elements. The interpolation points and the quadrature points are chosen as the Gauss Lobatto points on each element, which include the end points. The junction and radiation equations relate the acoustical variables as intended. The number of discrete unknowns increases with the polynomial order and with the number of elements. The method converges exponentially with the polynomial order, achieving arbitrarily accurate approximations for low computational costs.

Practically, a linear system must be solved at each frequency:

$$A_h(\omega, \boldsymbol{\xi}) \mathbf{U}_h = L_h \quad (15)$$

where  $\mathbf{U}_h = {}^t (u_{h,1}, p_{h,1}, \dots, u_{h,N}, p_{h,N}, \gamma_{123}, \zeta_{123}, \dots, \gamma_{(N-2)(N-1)N}, \zeta_{(N-2)(N-1)N}) \in \mathbb{C}^{N_h}$  is the vector of the pressure and the flow at each degree of freedom of the mesh, along with the junctions auxiliary variables. The acoustic fields at any point of the instrument can be recovered from the vector of unknowns  $\mathbf{U}_h$ . The vector  $L_h$  imposes the unitary flow at the degree of freedom corresponding to the entrance of the instrument. The matrix  $A_h(\omega, \boldsymbol{\xi})$  has a sparse block structure, related to the unknowns organization inside  $\mathbf{U}_h$ . Eq. (16) illustrates this structure for a pipe with a single side hole.

$$A_h = \begin{pmatrix} A_{h,1} & 0 & 0 & -\frac{0}{\sqrt{2}}C_1 & \frac{0}{\sqrt{2}}C_1 \\ 0 & A_{h,2} & 0 & \frac{1}{\sqrt{2}}E_2 & \frac{1}{\sqrt{2}}E_2 \\ 0 & 0 & A_{h,3} & 0 & -\sqrt{2}E_3 \\ 0 & \frac{1}{\sqrt{2}}C_1^* & 0 & -\frac{1}{\sqrt{2}}E_2^* & 0 \\ 0 & -\frac{1}{\sqrt{2}}C_1^* & 0 & -\frac{1}{\sqrt{2}}E_2^* & 0 \end{pmatrix} \quad (16)$$

with

$$\begin{aligned} A_{h,1} &= \begin{pmatrix} j\omega M_1^u & -B_1 \\ B_1^T & j\omega M_1^p \end{pmatrix} \\ A_{h,n} &= \begin{pmatrix} j\omega M_n^u & -B_n \\ B_n^T & j\omega M_n^p + Y_{r,n}(\omega)C_n \end{pmatrix} \quad \text{for } n \in [2, 3] \end{aligned}$$

The matrices  $M_n^u$ ,  $M_n^p$  and  $B_n$  follow from the finite element discretization of Eq. (4), the matrices  $C_n$  and  $E_n$  relate respectively to the downstream and upstream end of the pipes. The expressions of these matrices are detailed in [14]. They are sparse, and depend on the values of the design parameters  $\boldsymbol{\xi}$ . For any set of design parameters  $\boldsymbol{\xi}$  and angular frequency  $\omega$ , Eq. (15) defines the solution  $\mathbf{U}_h$ , so in a sense,

$$\mathbf{U}_h \equiv \mathbf{U}_h(\omega, \boldsymbol{\xi}) = (A_h(\omega, \boldsymbol{\xi}))^{-1} L_h. \quad (17)$$

In this study, only observables based on the normalized input impedance are analysed. The acoustic flow being imposed to 1 at the entrance (see Sec. 2.1), the input impedance corresponds to the pressure at the entrance of the instrument. The normalized input impedance  $Z \in \mathbb{C}$  is therefore obtained from the simulated acoustic field by using the restriction matrix  $\mathcal{R} \in \mathbb{R}^{1 \times N_h}$ :

$$Z(\omega, \boldsymbol{\xi}) \equiv Z(\mathbf{U}_h(\omega, \boldsymbol{\xi})) = \mathcal{R} \mathbf{U}_h(\omega, \boldsymbol{\xi}). \quad (18)$$

Due to our choice of finite elements, the restriction matrix has only one non-zero value. Here, with  $i_0$  the index of the degree of freedom corresponding to the pressure at the entrance of the instrument:

$$\mathcal{R}_i = \begin{cases} \frac{S_1(0)}{\rho c} & \text{if } i = i_0 \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

with  $S_1(0)$  the cross sectional area at the entrance of the instrument. The input impedance is computed for each of the  $N_\omega$  frequencies and for each of the  $N_{\text{fing}}$  fingerings ( $N_\omega \times N_{\text{fing}}$  matrix inversions) using the python 3 toolbox OpenWInD [13].

## 2.6 Measurement-simulation agreement

For the bore reconstruction, the previously described model is used to estimate the geometry of the resonator from a measured impedance. To ensure the validity of the approach, simulations and measurement must be verified to match on a known geometry in the studied frequency range.

In this study, the impedances are measured with a home-made sensor based on the two microphones, three calibrations method proposed by Gibiat [22]. The bench tube is a 2 mm inner radius and 450 mm long cylinder equipped with 5 microphones (Audix TM1) located at [6, 18, 51, 174.8, 400] mm from the entrance, giving the possibility to measure impedance between 50 Hz and 12 kHz. The data analysis proposed by Dickens [23] is used, which allows to measure with only two calibration steps (blocked: infinite impedance, 100 m tube: unitary impedance).

The agreement between measurements and simulations is validated on a cylinder with an inner radius of 2 mm and a length of 436 mm, for which the theoretical impedance is well known [15, Chap.5.5, p.239-248]. The relative error ( $\|Z_{theo} - Z_{meas}\|_{\ell^2} / \|Z_{meas}\|_{\ell^2}$ ) in the frequency range [100 Hz, 4 kHz] is about 5%. It corresponds to a deviation within  $\pm 2$  cents for the frequency of the peaks and within  $\pm 0.5$  dB for their magnitude. This is consistent with the deviation obtained with other impedance sensors [24].

This deviation mainly comes from measurement uncertainties. The electric and acoustic noise in the acquired data contribute to it, but can be smoothed by averaging. The acoustic loads (blocked and 100 m tube) used for the calibration steps are not perfect due to visco-thermal effects at the wall and bends for the long tube. It induces small bias in the resulting measured impedance. The model used here (Sec.2.2) assumes only one propagative mode corresponding to  $kR \ll 1$ . The same assumption is made for the Zwikker and Kosten model (Eq.2). For the presented geometry, it corresponds to  $f \ll 30$  kHz. The model is therefore theoretically valid on the entire frequency range studied here. However the radiation condition used (infinite flanged or unflanged pipe) does not perfectly correspond to the experimental situation. This approximation also participates to the deviation between measurement and simulation.

## 3 Inverse problem

The bore reconstruction is performed by solving an optimization problem. Its objective is to find the values of the design parameters minimizing the deviation between the simulated impedances (Sec.2.5) and the measured ones (Sec.2.6). This optimization problem is formulated here for the elementary instrument presented in Section 2.1 and can be easily generalized to any other configuration.

### 3.1 Initial geometry

Parameter	Min	Init. Values	Max
$L_{mb}$	0	150 mm	$\infty$
$R_{mb}(L_{mb})$	0	4 mm	$\infty$
$X_{hi} \quad i \in [1, 4]$	0	[10, 20, 30, 40] mm	$L_{mb}$
$H_{hi} \quad i \in [1, 4]$	0	[1, 1, 1, 1] mm	$\infty$
$R_{hi} \quad i \in [1, 4]$	0	[1, 1, 1, 1] mm	$R_{mb}(X_{hi})$

Table 3: Initial values of the different optimized geometric parameters and their bounds.

The method presented in this study is not able to modify the topology of the resonator which must be known and fixed at the initial state: here a conical main pipe with four side holes (Sec.2.1). Only the inner radius at the entrance of the pipe is supposed to be known ( $R_{mb}(0) = 2$  mm). The aim of the inverse problem is to obtain the value of the 14 other geometric parameters from the measured input impedances: the main bore length  $L_{mb}$ , the main bore right end radius  $R_{mb}(L_{mb})$ , the locations of the 4 holes  $X_{hi}$ , their chimney height  $H_{hi}$  and their radius  $R_{hi}$ .

The initial values given to these design parameters are indicated in Table 3 and can be compared to the the values from direct geometric measurements with caliper of Table 1.



## 3.2 Constraints and parameterization

The values of the geometrical parameters of woodwind instruments have strong physical limitations. They all must be strictly positive, the holes must be located on the main bore and their radii must be strictly smaller than the main bore radius at their location. The resulting bounds are summarized in Table 3. These inequalities are ensured by defining a bound-constrained minimization problem. These algorithms do not necessarily guarantee that the bounds are respected for each evaluation of the cost function. A margin of  $10^{-7}$  is therefore included in the bounds but omitted for clarity in the rest of the document. The positiveness of  $R_{mb}(L_{mb})$ ,  $L_{mb}$  and  $H_{hi}$  is secured by fixing their bounds to  $\xi_{min} = 0$  and  $\xi_{max} = \infty$ . To guarantee the inequalities between the holes radii and the main bore radius and between the holes locations and the main bore length, the design parameters modified by the optimization algorithm are chosen as auxiliary parameters  $\xi_p$  and  $\xi_q$  defined as:

$$X_{hi} = \xi_p L_{mb} \quad (20)$$

$$R_{hi} = \xi_q R_{mb}(X_{hi}) \quad (21)$$

and bounded by 0 and 1.

## 3.3 General optimization problem

The reconstruction of the instrument is carried out by treating a least-square optimization problem. The aim of the process is to minimize the difference between the simulated and the measured observable (extracted from the acoustic fields) by adjusting the values of the  $N_\xi$  design parameters. The simulated impedance is compared to the measured one through an observable:

$$\begin{aligned} \phi: \mathbf{C} &\rightarrow \mathbf{C} \\ Z &\rightarrow \phi(Z) \end{aligned} \quad (22)$$

which can be the identity, the modulus, or another non linear function. The choice of this observable is crucial to observe a correct convergence of the algorithm. It is discussed in Section 4.1 for the studied instrument. The only limitation on the choice of  $\phi$  is its differentiability with respect to  $Z$  in the complex sense (See Appendix A). This  $\mathcal{C}^1$  property is necessary for the use of gradient-based optimization algorithms (Sec.3.4).

The problem is treated frequency by frequency and fingering by fingering. For a given fingering, and for each angular frequency  $\omega$ , the real residual  $r(\omega, \xi)$  is defined as

$$r(\omega, \xi) \equiv r(\phi(Z(\omega, \xi)) - \phi(Z^\circ(\omega))) \quad (23)$$

where

$$r(\zeta) = \begin{pmatrix} \Re(\zeta) \\ \Im(\zeta) \end{pmatrix} \quad \forall \zeta \in \mathbf{C}.$$

All the frequencies for all the fingerings are concatenated in the vector  $\mathbf{r}(\omega, \xi)$  of length  $2 \times N_\omega \times N_{\text{fing}}$ . For clarity, the  $\omega$  dependency is omitted in the rest of the document. The choice of using a real residual is motivated by the gradient computation detailed in Sec .3.4. The cost function is defined as :

$$F(\xi) = f(\mathbf{r}(\xi)) = \frac{1}{2} \langle \mathbf{r}(\xi), \mathbf{r}(\xi) \rangle_{\mathbb{R}} \quad (24)$$

where  $f$  is the squared real norm. By taking into account the constraints (Sec. 3.2), the optimization problem finally reads:

$$\begin{aligned} \min_{\xi} \quad & F(\xi) \\ \text{s.t.} \quad & \xi_{min} \leq \xi \leq \xi_{max} \end{aligned} \quad (25)$$

where the inequality between vectors means inequality on each of their components.

## 3.4 Gradient computation

### 3.4.1 General approach

To minimize the cost function  $F(\xi)$ , a gradient-based algorithm is used (e.g. quasi-Newton algorithm), which requires the computation of the gradient  $\nabla_{\xi} F$ . The Full Waveform Inversion method uses the knowledge of the forward problem to compute efficiently the gradient of the cost function. The first step is to write the explicit relation between this gradient and the derivative of the acoustic fields.

As this relation involves complex operators, the complex derivative must be carefully defined. The general principles, presented in details by Barucq *et al.* [25] are recalled in App. A. In particular, as  $\boldsymbol{\xi} \in \mathbb{R}^{N_\epsilon}$  and  $F(\boldsymbol{\xi}) \in \mathbb{R}$ , the following relation can be shown [25] (see App. B):

$$\frac{\partial}{\partial \xi_j} F(\boldsymbol{\xi}) = 2\Re e \left( \frac{df}{d\mathbf{r}}(\mathbf{r}(\boldsymbol{\xi})) \frac{d\mathbf{r}}{dZ}(\boldsymbol{\xi}) \frac{dZ}{d\mathbf{U}_h}(\boldsymbol{\xi}) \frac{\partial \mathbf{U}_h}{\partial \xi_j}(\boldsymbol{\xi}) \right). \quad (26)$$

The Fréchet derivative of the squared norm with respect to the residual  $\frac{df}{d\mathbf{r}}$  is equal to  $\mathbf{r}^T$ , the transpose of  $\mathbf{r}$ , and the Fréchet derivative of the impedance with respect to the acoustic field  $\frac{dZ}{d\mathbf{U}_h}$  is equal to  $\mathcal{R}$ . The expression of the gradient is then:

$$\frac{\partial}{\partial \xi_j} F(\boldsymbol{\xi}) = 2\Re e \left( \mathbf{r}(\boldsymbol{\xi})^T \frac{d\mathbf{r}}{dZ}(\boldsymbol{\xi}) \mathcal{R} \frac{\partial \mathbf{U}_h}{\partial \xi_j}(\boldsymbol{\xi}) \right). \quad (27)$$

The derivative of the residual with respect to the impedance depends on the choice of the observable. This relation is obtained from its definition Eq. (23) and the one of the complex derivative (see App. A):

$$\frac{d\mathbf{r}}{dZ}(\boldsymbol{\xi}) = \frac{1}{2} \begin{pmatrix} \left( \frac{d\phi}{dZ} + \frac{d\bar{\phi}}{dZ} \right) (Z(\boldsymbol{\xi})) \\ -j \left( \frac{d\phi}{dZ} - \frac{d\bar{\phi}}{dZ} \right) (Z(\boldsymbol{\xi})) \end{pmatrix} \quad (28)$$

where  $\bar{\phi}$  is the complex conjugate of  $\phi$ . The necessity for  $\phi$  to be  $\mathcal{C}^1$  with respect to  $Z$  appears here clearly. Finally, the gradient of the cost function is now explicitly linked to the derivative of the acoustic fields  $\frac{d\mathbf{U}_h}{d\xi_j}$  by Eq. (27).

The formal differentiation of the forward problem Eq. (15) with respect to any design parameter  $\xi_j$ ,

$$\frac{d}{d\xi_j} (A_h(\boldsymbol{\xi})U_h(\boldsymbol{\xi}) = L_h) \Rightarrow A_h(\boldsymbol{\xi}) \frac{dU_h}{d\xi_j}(\boldsymbol{\xi}) = -\frac{dA_h}{d\xi_j}(\boldsymbol{\xi})U_h(\boldsymbol{\xi}) \quad (29)$$

makes it explicit that this derivative can be obtained by solving a linear problem similar to Eq. (15) with a different source term, which could also be justified mathematically at the continuous level following a similar approach to [26]. The inverse of the matrix  $A_h$  being already computed (eg, LU factorized) for the forward problem, the additional numerical cost for solving this system is negligible. The differentiation of the matrix  $A_h$  Eq.(16) follows from the differentiation of the coefficients of equations (2), (4), (8), (9), (10) and (12) with respect to the design parameters, along with the assembly procedure entangled with the finite element method.

This approach requires to solve the linear system (29) once for each design parameter. In the FWI, generally used for geophysical problems in which the number of design parameters is huge, the adjoint-state method is preferred [27], which only requires to solve one additional linear system. However the benefit of this method are less important for musical instruments which involve only a few dozen design parameters. Moreover, unlike the adjoint-state method, the one presented here gives the possibility to compute the Jacobian of the residual:

$$\nabla_{\boldsymbol{\xi}} \mathbf{r}(\boldsymbol{\xi}) = \frac{d\mathbf{r}}{dZ}(\boldsymbol{\xi}) \mathcal{R} \frac{\partial \mathbf{U}_h}{\partial \xi_j}(\boldsymbol{\xi}). \quad (30)$$

The knowledge of this Jacobian allows to simply compute an approximation of the Hessian:  $H \approx \nabla_{\boldsymbol{\xi}} \mathbf{r}^T \nabla_{\boldsymbol{\xi}} \mathbf{r}$  such as used in optimization algorithms specifically suited for least-square problems (for ex. Gauss-Newton or Levenberg-Marquardt) [28, Chap.10]. In this study the optimizations are performed by the trust region reflective algorithm implemented in the Scipy python module, which takes advantage of this approximation.

### 3.4.2 Specific cases

This formulation is versatile with respect to the choice of the observable  $\phi(Z)$  which only needs to be  $\mathcal{C}^1$  with respect to  $Z$  (Eq. 28). For example, if the reflection coefficient  $R_{\text{refl}}$  is chosen as observable:

$$\phi(Z) = R_{\text{refl}}(Z) := \frac{Z-1}{Z+1} \quad (31)$$

and defining  $Z = Z_r + jZ_i$  with  $Z_r, Z_i \in \mathbb{R}$  the real and imaginary components of  $Z$ , the derivative of  $\phi$  is (c.f. App. A):

$$\begin{aligned} \frac{d\phi}{dZ} &= \frac{1}{2} \left( \frac{\partial \phi}{\partial Z_r} - j \frac{\partial \phi}{\partial Z_i} \right) = \frac{1}{2} \frac{1}{(Z+1)^2} (2 - j(2j)) \\ &= \frac{2}{(Z+1)^2}. \end{aligned} \quad (32)$$

A similar computation for  $\bar{\phi}$ , the conjugate of the observable, gives:

$$\begin{cases} \frac{d\phi}{dZ} = \frac{2}{(Z+1)^2} \\ \frac{d\bar{\phi}}{dZ} = 0 \end{cases} \quad (33)$$

This computation completes the gradient computation of the cost function with respect to the design parameters. The results for other choices of observables are given in Table 4.

Observable $\phi$	$\frac{d\phi}{dZ}$	$\frac{d\bar{\phi}}{dZ}$
$Z$	1	0
$R_{\text{refl}}(Z) = \frac{Z-1}{Z+1}$	$\frac{2}{(Z+1)^2}$	0
$ Z ^2 = Z \times \bar{Z}$	$\bar{Z}$	
$\arctan\left(\frac{Z-\bar{Z}}{Z+\bar{Z}}\right)$	$-j\frac{\bar{Z}}{2 Z ^2}$	

Table 4: Complex derivative with respect to  $Z$  of different observables. For real observables the two derivatives are equal. The phase and modulus of  $R_{\text{refl}}$  can be obtained by combination.

## 4 Reconstruction Strategy

The formulation of the optimization problem presented here gives the possibility to adjust the construction of the cost function to the targeted application. It implies a choice of the observable function  $\phi$  but also of the considered frequencies and fingerings included in the cost function. In this study, the reconstruction of the elementary instrument is divided in several steps corresponding to the adjustment of different design parameters. The definition of the cost function can possibly be different at each step. These three choices (observable, frequency range and fingerings) are interlaced, each choice impacting the others. For clarity of the presentation, they are here treated separately.

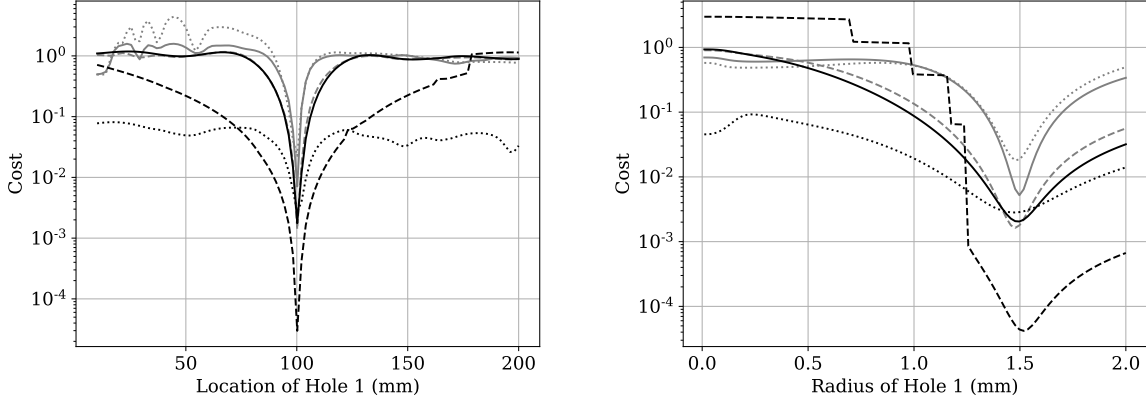
### 4.1 Choosing an observable

Following the gradient computation developed in Sec. 3.4, the observable must be differentiable with respect to the input impedance over the entire design domain. Ideally, an observable giving a convex function over the entire design domain (one minimum and no plateau) ensures a quick convergence of the algorithm towards a unique set of design parameters. In practice, the observable giving the widest basin of attraction and the smoothest evolution with respect to the different design variables is searched.

The observables presented in Tab. 4 are tested, in addition to the modulus and the unwrapped phase of the reflection coefficient. Their evolution along two dimensions of the design space are depicted in Fig. 3, all other design parameters being set to their direct measured value. The chosen frequency axis ([0.1, 4] kHz with a 2 Hz step) is discussed in more details in Sec. 4.2.2.

Both for the location of Hole 1 (Fig. 3a) and to a lesser extent its radius (Fig. 3b), the complex impedance and its modulus show numerous fluctuations leading to many local minima in the design space. They will prevent the algorithm to converge to the correct geometry and must be excluded. This observation already made by [4], can come from the repetition of the impedance shape along the frequency axis which locally favors the concordance of two resonances even if the others do not concur. Fluctuations are also observed for the modulus of the reflection coefficient.

The unwrapped phase seems ideal for assessing the location of Hole 1 (Fig. 3a). The associated cost functions have a smooth evolution with a clear unique basin of attraction. However jumps appear for the radius of Hole 1 (Fig. 3b). It comes from the fact that the reflection coefficient crosses the origin of the complex plane for some specific geometric leading to a locally non differentiable cost function. This aspect has been discussed in more



(a) Costs evolution w.r.t. the the first hole location.

(b) Costs evolution w.r.t. the first hole radius.

Figure 3: Costs evolution for different observables computed for frequencies in  $[0.1, 4]$  kHz with a 2 Hz step, for the fingering "oxxx" with respect to: (a) the location and, (b) the radius of the first hole. The observables are: the complex value (solid lines), the modulus (dotted lines) and the phase (dashed lines) of the impedance (gray) and the reflection coefficient (black).

details in [10] along with a solution which requires derivation and integration with respect to the frequency axis, which is incompatible with the currently proposed formalism.

Finally the complex reflection coefficient and the impedance phase both appear acceptable in terms of convexity with a small benefit for the reflection coefficient (Fig. 3). Similar observations can be made for all design variables taken isolated and for different frequency ranges. The reflection coefficient defined in Eq. (31) is therefore chosen as the observable in this study.

## 4.2 Main bore geometry and holes location

Instead of reconstructing all the design parameters together, a strategy is searched to treat the design parameters independently. As a first step, a strategy aiming at recovering the main bore geometry and the hole locations is searched by using different boundary conditions given by the different fingerings.

### 4.2.1 Fingering choice

A design parameter can be easily adjusted in isolation, when the observable obtained with one fingering is specifically sensitive to this parameter only. To quantify this aspect, the sensitivity of each fingering with respect to the design parameters  $\xi_i$  is computed:

$$\sigma_i(\xi) = \frac{\|\partial_{\xi_i} \phi(Z(\omega, \xi))\|_{L2, \omega}}{\|\phi(Z(\omega, \xi))\|_{L2, \omega}} \quad (34)$$

The sensitivity with respect to the location of the holes and to the main bore length for each fingering is plotted in Fig. 4. The computation is done for the direct measured geometry. This map is poorly sensitive to the absolute value of the design parameters as long as the global geometry is unchanged (holes order, etc). All the fingerings are sensitive to the main bore length and, to a lesser extent, to the location of the open holes, mainly the first ones. The locations of the upstream closed holes have still a small effect on the observable (1 order of magnitude smaller). The downstream closed holes have negligible effect.

Following these observations, the holes locations seem independent, to some extent, if only fingerings with one open hole are used (xxxo, xxox, xxx and oxxx). The location of each hole can be recovered by performing independent optimizations on the corresponding fingering (hole 4 on xxxo, hole 3 on xxox, etc.), after having recovered the main bore length on the all closed fingering.

The result of this process can only be a rough estimation of the geometry because the location of holes cannot be estimated independently from their radius and chimney height, which can not be recovered using a unique fingering (see Sec. 4.3). Concerning the main bore, its length is strongly interlaced with its output radius (conicity). Hopefully, this radius and length can be estimated together with a good enough accuracy on the all closed fingering ('xxxx').

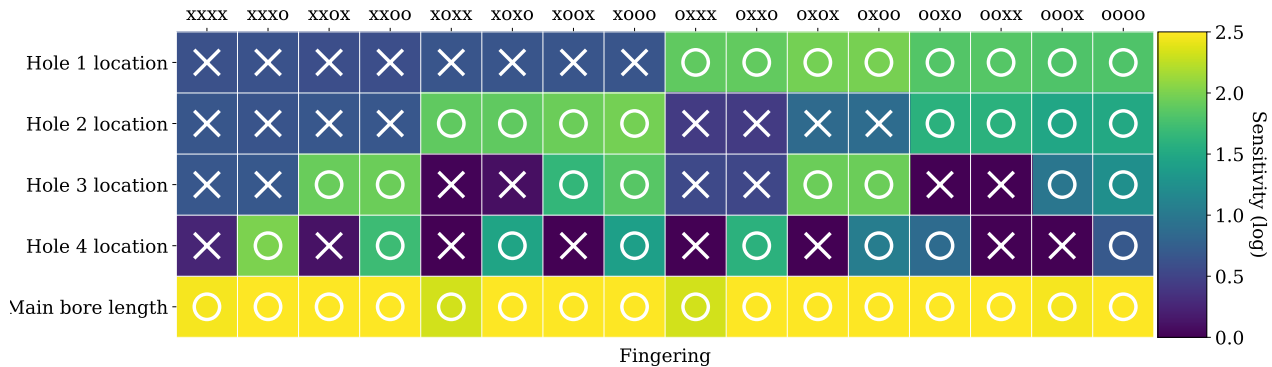


Figure 4: Sensitivity of the reflection coefficient w.r.t the locations design parameters (Eq. (34)) for frequencies in  $[0.1, 4]$  kHz with a 2 Hz step. Open (resp. closed) holes are indicated with circles (resp. crosses).

### 4.2.2 Frequency range

Since the observable and the fingering are selected to recover each hole locations and the main bore geometry, it is interesting to investigate how the choice of the frequency range can facilitate the convergence of the optimization algorithm.

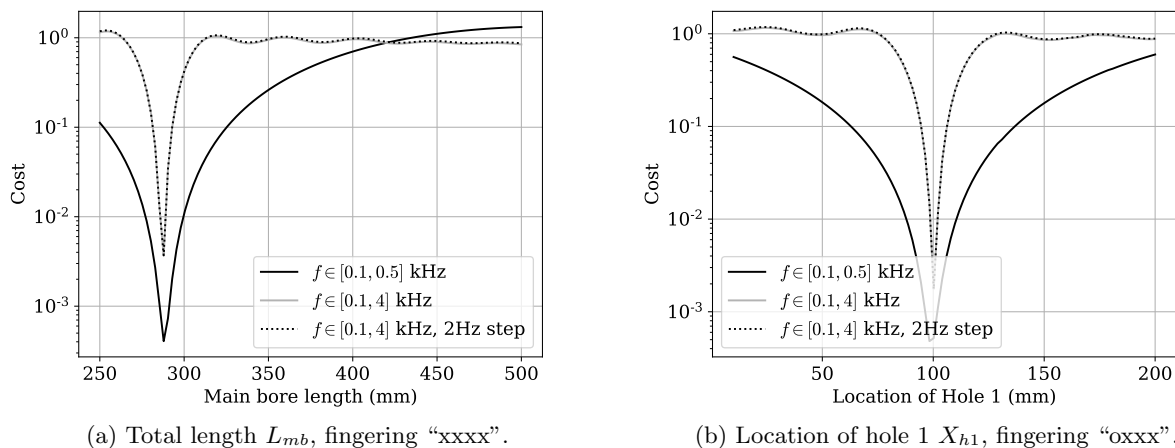


Figure 5: Cost evolution along one dimension of the design space for three frequency ranges:  $[0.1, 0.5]$  kHz and  $[0.1, 4]$  kHz with a 100 Hz step, and  $[0.1, 4]$  kHz with a 2 Hz step.

Fig. 5 compares the evolution of the cost function computed with  $\phi(Z) = R_{\text{ref}}$  on different sets of frequencies with respect to the main bore length (Fig. 5a) and the location of the first hole (Fig. 5b), the other parameters being fixed to their direct measured values.

The first observation is that the frequency step has little influence on the cost value. It can be enlarged from 2 Hz to 100 Hz with negligible effect on the cost function (Fig. 5). Contrary to the usual assumption, the physical information is not concentrated at the resonance but it influences the acoustic response at any frequency. The number and the distribution of the observed frequencies must only be chosen to avoid confusion between geometries which could have similar responses at one given frequency and to reduce the effect of the measurement noise on the reconstructed geometry. The choice of a 100 Hz step seems a good compromise between computation cost and accuracy of the reconstruction.

Secondly, in both cases it appears that reducing the frequency range to  $[100 \text{ Hz}, 500 \text{ Hz}]$  enlarges the basin of attraction. This can come from the fact that, at high frequencies, two resonances can accidentally match for a given geometry leading to a local minimum, which will not be the case at low frequency, especially for frequencies lower

than the first resonance. It is noticeable that finally, only 5 frequency values are necessary to obtain a satisfactory smooth cost-function. The conclusion of the two previous paragraphs on the observable choice (Sec. 4.1) and on the fingering choice (Sec. 4.2.1) are still valid with this frequency range.

### 4.3 Holes dimensions

The conclusions of Sec. 4.2 can not be applied to the radius and the chimney height of the holes. There is no sense in treating separately these two dimensions which have a similar effect on the impedance at low frequencies. Figure 6 shows the influence of the frequency range on the cost function evolution with respect to the first hole

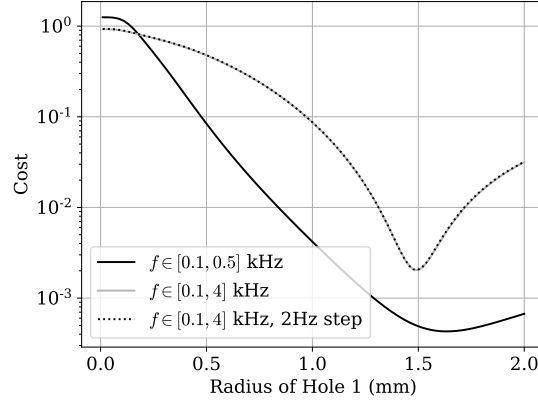
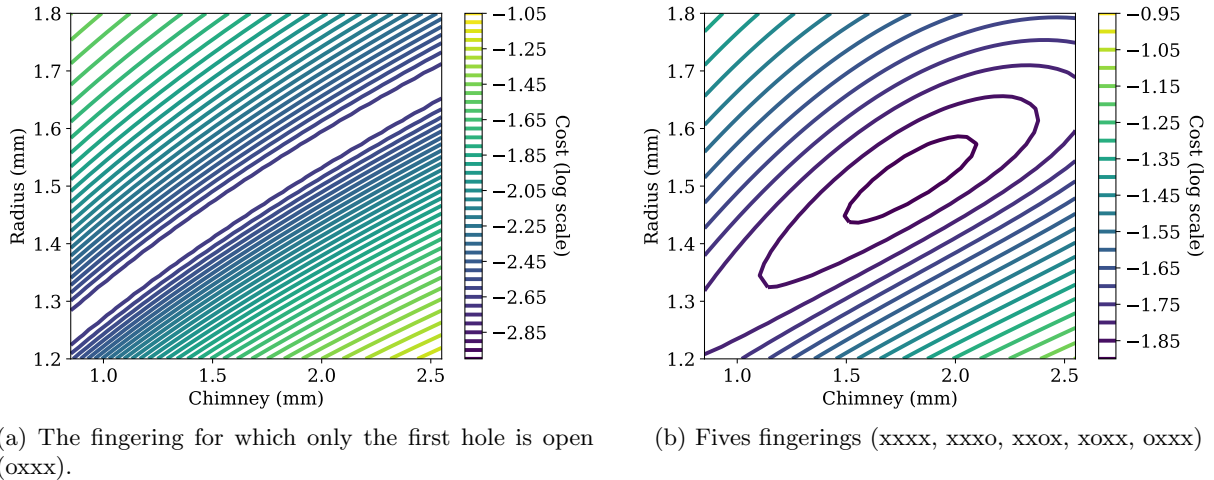


Figure 6: Cost evolution with respect to the radius of Hole 1 on the fingering "xxxx", with three frequency ranges: [0.1, 0.5] kHz and [0.1, 4] kHz with a 100 Hz step, and [0.1, 4] kHz with a 2 Hz step.

radius (a similar effect can be observed with the chimney height). Here again, refining the frequency step does not modify the cost function evolution, a 100 Hz step is sufficient. But this time, the low frequencies are not sufficient: the basin of attraction is so large that the minimum almost disappears. The inclusion of higher range allows to refine the minimum.



(a) The fingering for which only the first hole is open (oxxx). (b) Fives fingerings (xxxx, xxxo, xxox, xoxx, oxxx)

Figure 7: Cost associated to the reflection coefficient, with respect to the first hole radius  $R_{h1}$  and chimney height  $H_{h1}$ , for frequencies in [0.1, 4] kHz with a 0.1 kHz step.

Recovering together both the radius of the hole and its chimney height is a challenge. To clearly distinguish their effects, it is necessary to reach frequencies verifying  $kH_{hi} > 1$ , which corresponds for the presented instrument

to  $f > 27$  kHz for 2 mm. In this range, the modeling is not valid anymore and we are not able to measure the impedance. However, these two parameters differ also by the fact that only the radius appears in the hole radiation. Accounting for several boundary conditions through the different fingerings will help recovering their values. This is illustrated by the observation of the cost function in the design space plane  $(R_{h1}, H_{h1})$  on Fig. 7. When only one fingering (oxxx) is considered, a valley without minimum appears, whereas when fingerings including different open-closed states for Hole 1 are considered, a clear minimum appears. To conclude, recovering holes dimensions necessitates both to consider a wide frequency range and several fingerings.

## 4.4 Results

Following the analysis of Secs. 4.2 and 4.3, a two-step strategy is proposed to reconstruct the instrument.

1. A rough estimation is first performed by recovering only the main bore geometry and the holes locations on low frequencies only ([100, 500] Hz, step 100 Hz).
2. Then, a finer reconstruction is performed by including all design parameters on a wider frequency range ([100, 4000] Hz, step 100 Hz).

The rough estimation is divided in five optimization problems. The main bore geometry ( $L_{mb}$  and  $R_{mb}(L_{mb})$ ) is optimized on the all closed fingering (xxxx). Then each hole location is recovered, from the last one to the first one, by observing the fingering for which only the considered hole is open ("xxxo" for the Hole 4, etc.). Each optimization converges in about 8 iterations and the total rough estimation is computed in about 5 seconds in a personal computer. After this step, the holes locations and total length deviate about 5 mm from their measured values and the radius  $R_{mb}$  deviates about 0.05 mm. The refined reconstruction is then performed by considering the five fingerings ("xxxx", "xxxo", "xxox", "xoxx", "oxxx"), and activating the adjustment of the 14 design parameters. The optimization algorithm converges after about 15 iterations computed in about 1 minute on a personal computer. The total reconstruction is repeated for three sets of measured data with similar computation time.

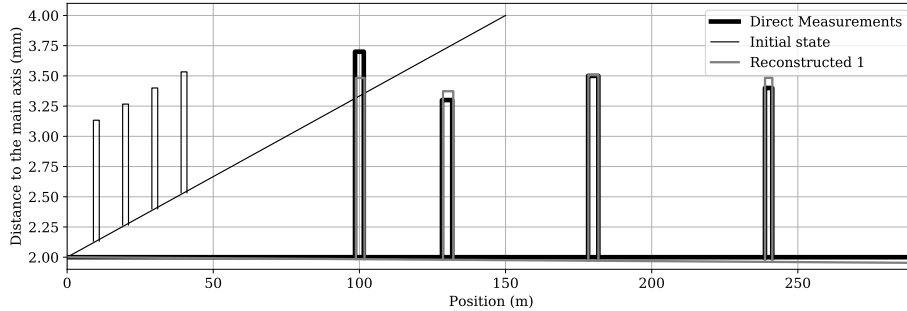


Figure 8: Comparison between the geometry obtained by direct measurement (wide black line), the initial state (thin black line) and the reconstructed geometry from the first set of data (grey). For readability the vertical and horizontal axis have different scales.

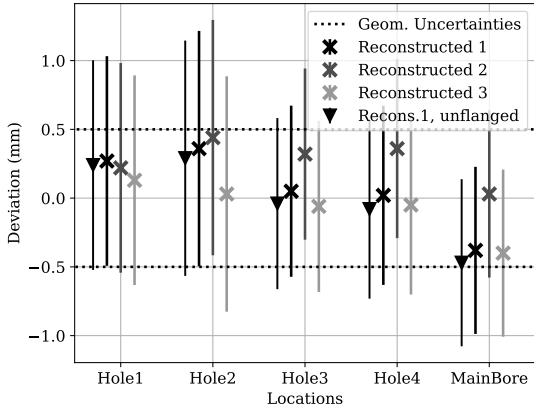
The geometry obtained for the first set of data is compared to the initial state and the direct measurements on Fig. 8. The resulting deviations for the three sets of data are presented in more details on Fig. 9. The direct measurement uncertainties (see Tab. 1) are indicated as dotted lines. The reconstruction uncertainties are estimated by using the sensitivity of the observable  $\sigma_i$  defined in Eq. (34) for the resulting geometry and the impedance measurement accuracy estimated in Sec. 2.6 ( $\Delta\phi(Z) = 5\%$ ):

$$\Delta\xi_i = \frac{\Delta\phi(Z)}{\sigma_i}. \quad (35)$$

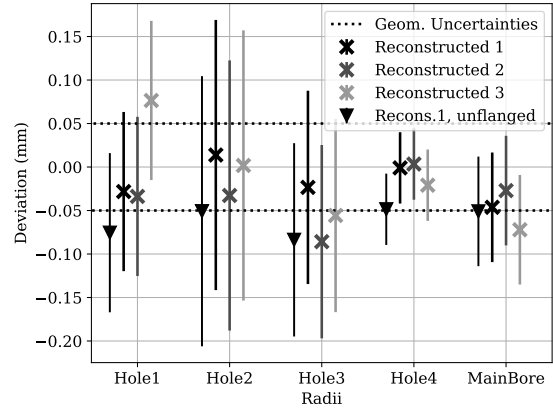
Since the sensitivity depends on the fingering, the uncertainty  $\Delta\xi_i$  is computed for each of them and the smallest value is depicted.

The geometric values obtained by reconstruction are in good agreement with the ones obtained by direct measurements. The deviations on the hole locations and the main bore length are within 0.5 mm (Fig. 9a), within 0.1 mm for the radii (Fig. 9b) and within 0.3 mm for the chimney lengths (Fig. 9c). Regarding the uncertainties and the accuracy needed to rebuild an instrument, these results are satisfactory.

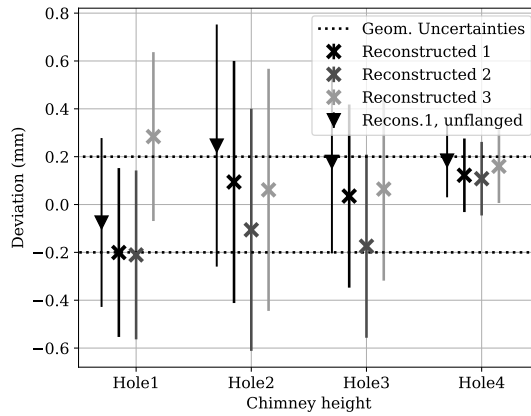




(a) Holes' locations and main bore length.



(b) Hole's radii and main bore radius.



(c) Chimney lengths.

Figure 9: Deviation between manually measured and reconstructed geometries from the three sets of data. The crosses represent the results for the mixed radiation condition (holes flanged, main bore unflanged) and the triangle the results with only unflanged radiation for the first set of data. The direct measurement uncertainties (see Tab. 1) are indicated as dotted lines.

The results variation between the sets of data is more pronounced for the holes dimensions (Fig. 9b, Fig. 9c). The Hole 1 is poorly reconstructed with the third set of data. It can be due to a measurement issue such as a leak. The holes are closed with a piece of Teflon covered with putty. Even put with precaution, the putty can slightly penetrate the chimney, resulting in a different effective length between open and closed states and distorting the reconstruction. As the tube is made of brass, its temperature is very sensitive to handling, which can also alter the measurements.

The holes radii seem mainly slightly underestimated (Fig. 9b). This bias could come from the model assumptions such as the radiation condition (Sec. 2.4). To observe this influence, a reconstruction has been performed on the first set of data with unflanged radiation for all the openings (black triangles on Fig. 9). This mainly reduces the radii and increases the chimney heights. It therefore deteriorates the results. It is possible to use more refined radiation models [29]. Since the main pipe external radius is 4mm, cylindrical flange or finite flange conditions should result here in an intermediary condition in between unflanged and infinite flanged conditions in low frequency. However, these radiation conditions strongly depend on frequency and could eventually improve the reconstruction.

## 5 Conclusion

In this study, the Full Waveform Inversion technique is used for the bore reconstruction of woodwind-like musical instruments. This methodology exploits the fact that the gradient of the cost function can be computed by solving



the same linear system which arises from the finite element spatial discretization of the direct problem of wave propagation. This gradient computation is more accurate and more efficient than finite-difference. It therefore improves the efficiency of gradient-based algorithms to perform the inversion.

The analysis of the cost functions built from different observables reveals that the reflection coefficient is a better candidate than the input impedance to reconstruct the bore profile. Only few frequencies are necessary to correctly estimate the geometric values. To recover the holes locations, the exclusion of high frequencies enlarges the basin of attraction of the cost function. To recover the holes dimensions (radius and chimney height) it is necessary to include these high frequency and to account for several boundary conditions (different fingerings). Following these considerations, the use of local optimization algorithms to perform the reconstruction appears suitable. The angle of the reflection coefficient seems to be a good candidate to improve further the algorithms convergence, in spite of the fact that a loss of regularity occurs due its wrapping. This aspect should be treated properly in future work.

This method is applied successfully to reconstruct an elementary woodwind-like instrument from measured data. For this purpose, a strategy including step by step inversion on specific fingerings is devised. This is a promising step in view of using this method in the context of manufacture. It could allow estimating the internal geometry of historical instruments or to design new ones. The remaining limitations preventing the use of this method on real instruments is the complexity of the main bore shape, limited here to a simple conical part. One interesting lead could be to combine pulse reflectometry on the all closed fingering to have a first estimation of the main bore shape, before performing the FWI strategy proposed in this study. Finally, in seismology, it is well identified that the FWI gives better results if more measurement points are used instead of increasing the number of frequencies. Since our impedance sensor uses five microphones, another interesting lead could be to reconstruct the instrument directly from the fives pressures signal instead of the recovered input impedance.

## 6 Data availability

All tools used in this study have been implemented in the open-source software OpenWInD[13]. The scripts and the data are available under GPLv3 licence at [openwind.gitlabpages.inria.fr/web/](https://openwind.gitlabpages.inria.fr/web/).

## 7 Acknowledgment

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## A Appendix: Complex derivative of complex function

If  $z \in \mathbb{C}$ , the derivative operator  $\frac{d}{dz}$  needs to be precisely defined. This definition follows the work of Barucq *et al.* [25] inspired by the work of Brandwood [30]. Each application  $g : \mathbb{C} \rightarrow \mathbb{C}$  is interpreted as the application  $g : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$  of the complex number  $z$  and its conjugate  $\bar{z}$ , with  $z = z_r + jz_i$  and  $\bar{z} = z_r - jz_i$ . The variables  $z$  and  $\bar{z}$  are then considered as independent variables. The application  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  is defined from  $g$  such as  $f(z_r, z_i) = g(z, \bar{z})$ . The complex derivatives are defined from the real derivative  $\partial/\partial z_r$  and  $\partial/\partial z_i$ :

$$\frac{\partial g}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial z_r} - j \frac{\partial f}{\partial z_i} \right) \quad (36)$$

$$\frac{\partial g}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial z_r} + j \frac{\partial f}{\partial z_i} \right) \quad (37)$$

**Remark:** In the general case where  $g(z, \bar{z}) \in \mathbb{C}$ , it can be noticed that:

$$\overline{\frac{\partial g}{\partial \bar{z}}} = \overline{\frac{1}{2} \left( \frac{\partial f}{\partial z_r} + j \frac{\partial f}{\partial z_i} \right)} = \frac{1}{2} \left( \frac{\partial \bar{f}}{\partial z_r} - j \frac{\partial \bar{f}}{\partial z_i} \right) = \frac{\partial \bar{g}}{\partial z} \quad (38)$$

which is an extension of the corollary A.1 given in the appendix of the paper [25].

## B Appendix: Complex derivative of real function

Let

$$\begin{aligned} h : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow h(x) \equiv g(z(x), \bar{z}(x)) \end{aligned} \quad (39)$$

where  $g : \mathbb{C}^2 \rightarrow \mathbb{R}$  and  $z : \mathbb{R} \rightarrow \mathbb{C}$ . By applying the definition of Eq. (37) we can show that (see [25]):

$$\frac{dh}{dx} = \Re e \left( \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x} \right) = \Re e \left( \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} \right) + \Re e \left( \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x} \right) \quad (40)$$

Applying the result of Eq. (38) twice, we can remark that:

$$\Re e \left( \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x} \right) = \Re e \left( \frac{\overline{\partial g}}{\partial z} \frac{\partial \bar{z}}{\partial x} \right) = \Re e \left( \frac{\partial \bar{g}}{\partial z} \frac{\partial \bar{z}}{\partial x} \right) \quad (41)$$

$$= \Re e \left( \frac{\partial \bar{g}}{\partial z} \frac{\partial z}{\partial \bar{x}} \right) \quad (42)$$

In this specific case, where  $x \in \mathbb{R}$  and  $g(z, \bar{z}) \in \mathbb{R}$  we finally get:

$$\frac{dh}{dx} = 2\Re e \left( \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} \right). \quad (43)$$

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