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Alain Jean-Marie ^{*}, Tania Jiménez [†], Mabel Tidball [‡]

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Abstract: We introduce the family of Conjectural Learning procedures, in which agents form conjectures about what the opponent will play, as a function of their action or some state variable. We prove general properties of these procedures, comparing them to two dynamic systems generated by nearsighted agents solving repeatedly a static game, either cooperatively, or noncooperatively. We then specify three simple functional forms of conjectures and analyze the five resulting dynamic systems, in terms of steady states and convergence conditions. We next further specify these procedures to the case of a game of groundwater management, using a quadratic functional form for the profit function of agents. We obtain then explicit expressions for steady states and convergence conditions for the dynamic systems. Finally, we conduct numerical experiments in order to assess the “performance” of pairs of agents adopting each of the five behaviors.

Key-words: Game Theory, Dynamic, Learning, Natural Resources Management

* Inria, Alain.Jean-Marie@inria.fr

† LIA, Avignon Université, tania.altman@univ-avignon.fr

‡ CEE-M, Univ Montpellier, INRAE, Institut Agro, mabel.tidball@inrae.fr

**RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Comportements à court terme, à long terme et apprentissage. Application à un problème de gestion de nappe phréatique

Résumé : Nous définissons une famille de procédures d'apprentissage conjectural, dans laquelle les agents formulent des conjectures au sujet de ce que l'adversaire va jouer, en fonction de leur propre action ou d'une variable d'état. Nous prouvons des propriétés générales de ces procédures, en les comparant aux deux systèmes dynamiques générés par des agents myopes qui résolvent à chaque étape un jeu statique, soit de façon coopérative, soit non-coopérative. Nous spécifions ensuite trois formes fonctionnelles simples pour les conjectures et nous analysons les cinq systèmes dynamiques ainsi définis, en termes d'états stationnaires et de conditions de convergence. Puis nous appliquons ces procédures pour un problème de gestion d'une nappe phréatique, en utilisant une forme fonctionnelle quadratique pour la fonction de profit des agents. Nous obtenons alors des formules explicites pour les états stationnaires et des conditions de convergence pour les systèmes dynamiques. Finalement, nous menons des expériences numériques afin d'évaluer les "performances" de paires d'agents qui adoptent chacun des cinq comportements.

Mots-clés : Théorie des Jeux, Dynamique, Apprentissage, Gestion de Ressources Naturelles

1 Introduction

The topic of this paper is the modeling of economic agents, endowed with limited knowledge of their environment, and engaged in a repeated interaction with other agents in a competition for some resource.

Our initial motivation is the question of the management of Common Pool Resources (CPRs) such as reserves of water (lakes, groundwater), some populations of wild plants (forests, pastures), and some populations of wild animals. According to the review of *e.g.* Madani and Dinar [8], agents in practical situations do not act as foreseen by the standard game-theoretical modeling with the well-known Prisoner's Dilemma. What might then be their way of deciding what to do?

When faced with a lack of information, agents are constrained to rely on observation and some form of learning. Many forms indeed exist: for instance Wang et al. [14] review and classify some of them. Among these, agents may form *conjectures* on the opponent's behavior and adapt this conjecture according to observations. We shall follow this line of research.

Endowing agents with a conjecture on what the opponent does is standard in Game theory. The Conjectural Variations model (see *e.g.* Figuières et al. [3]) assumes that each agent believes the opponent reacts to any variation of their actions, and optimizes her utility accordingly. When the observed behaviors match the conjectured ones, the equilibrium obtained is called consistent. In that situation, neither agent has a reason to modify her belief. Otherwise, if the interaction is repeated, it is logical for agents to revise their conjecture to get closer to consistency, thereby learning a consistent conjecture.

This idea was used by Jean-Marie and Tidball [7] in a repeated duopoly game. The conjecture used was an affine function of the control. An interesting feature was that in some cases, the conjecture learned can lead to a Pareto optimal outcome, improving the gains with respect to the Nash equilibrium. Then Quérrou and Tidball [11] applied the same principle and the same conjecture to a dynamic game of resource extraction and Quérrou [10] to a game of pollution control.

In this paper, we continue with this point of view that agents apply a simple reasoning, based on recently observed data, a form of "limited rationality". Indeed, limited rational calculation is expected in a human behavior, see Simon [13]. This is also the idea developed by Madani and Dinar [8], but in their work they essentially assume that agents try to guess their opponents' actions by performing simple statistics on the observations. Our modeling goes one step further by assuming that agents anticipate that the opponents will play strategically, either as a function of their own actions, or as a function of the state. These two behaviors find their inspiration in theoretical results about the use of conjectures in dynamic games. Indeed, Jean-Marie and Tidball have proved in [6] that consistent state-based conjectures and Nash-feedback equilibria coincide in discrete-time dynamic games, Fershtman and Kamien having proved this result in [2] for differential games.

The contribution of the paper is as follows. First, we sketch a general formalism for dynamic games where players have simple conjectures on the behavior of the opponent (for simplification we consider the case of two players only), depending on the last observed action and/or the last observed state. These conjectures are revised if there is a mismatch between predictions and observations. This principle generates a family of dynamic processes called Conjectural Learning procedures. In addition to these incomplete information procedures, we introduce complete-information but nearsighted behaviors in which players repeatedly solve a game using either the Nash equilibrium solution, or the cooperative solution where the sum

of profits is maximized. We call the latter the “nearsighted Pareto” solution in the following. This also generates dynamic processes, characterized by a fixed state-feedback strategy for each agent. We then establish a general link between the steady states of the first and the second family. Specifically: conjectural procedures based on the state only have the same steady states as the nearsighted Nash procedure, whereas conjectural procedures based on the control, modulo some symmetry assumption, have the same steady states as the nearsighted Pareto procedure.

Next, we introduce three specific forms of conjectures, including the usual conjecture affine in the control, the only one used in the literature so far. For each of them, we study existence and uniqueness of steady states, and we provide conditions for local or global convergence of the process to these steady states. We proceed similarly for the two nearsighted procedures already introduced.

We apply this modeling framework to a problem of groundwater exploitation. Water extraction is the only input in the production process of the farmers, and the dynamics is given by the evolution of the level of the water table. In order to mitigate the “tragedy of the commons” typical of such a Common Pool Resource problem when agents are too myopic, we introduce a subjective valuation of the future in the form of a unit price parameter attached to the future level of the resource. We specify profits using a quadratic function, which allows the derivation of explicit formulas and convergence conditions in most of the cases.

Finally, we conduct numerical experiments in order to assess the relative “performance” of the different complete-information and incomplete-information schemes. We evaluate some behavior for a pair of players using two metrics: the total discounted profit and the limit stock of the resource. We also use as benchmark the usual Nash-feedback and cooperative solutions to the corresponding dynamic game, where individual profits and the sum of profits are maximized, respectively. We call respectively these two solutions “farsighted Nash” and “farsighted Pareto”. The experiments reveal that when agents adopt some of the conjectural learning behavior, their total profit can be compared to the farsighted cooperative solution, with even a higher level of the asymptotic stock. This is also the case for the nearsighted schemes. This result however depends on the valuation the agents place on the future stock, which must be adequately chosen. When this valuation is too small, the performance of these schemes is worst than that of the farsighted Nash solution, both for profits and the resource.

The rest of the paper is organized as follows. In Section 2, we present the different dynamic models used in the paper. First, the general conjectural learning model (Section 2.1), then the nearsighted behaviors (Section 2.2), together with results about the links between them and general convergence results. In Section 2.3, we specify the three specific conjectural learning procedures and provide results on their steady states and their convergence. Section 3 is devoted to the analysis of a game of groundwater extraction. The model is specified in general terms in Section 3.1. The specific case of a quadratic profit function is then fully analyzed in Section 3.2: the five dynamic systems previously defined are analyzed individually, then compared. The numerical experiments are reported in Section 4 and perspectives are presented in Section 5.

2 The Models

We first present the general principles of conjectures and learning. We next introduce near-sighted dynamics. Finally, we study some specific functional forms of conjectures.

2.1 A general Learning Model

Consider a dynamic, discrete-time game problem where each of two players $i = 1, 2$ earns in period t :

$$F_i(w_i^t, w_j^t, H^t), \quad (2.1)$$

where the state variable, H^t , evolves according to the dynamics:

$$H^{t+1} = G(w_i^t, w_j^t, H^t). \quad (2.2)$$

Here, at each time period t , w_i^t is Player i 's action. We suppose that functions F_i and G are differentiable with continuous derivatives.

Suppose also that each player at each period makes a *conjecture* about the behavior of the other player of the form

$$w_j^{c,t} = \chi_i^t(w_i^t, H^t), \quad (2.3)$$

for some differentiable functions χ_i^t , $i = 1, 2$, $t = 0, 1, \dots$. Here, $w_j^{c,t}$ is what Player i believes Player j will play, as a function of her own play w_i^t and the current state H^t . The superscript “ c ” stands for “conjectural”. With this assumption on the opponent, choosing the optimal action, which is originally a game problem, becomes an optimization problem.

Assuming then that Player i solves this problem without considering the future of the process, she faces the following “conjectural” optimization problem:

$$\max_{w_i^t \in \mathcal{W}_i(H^t)} \pi_i^{c,t}(w_i^t, H^t), \quad \pi_i^{c,t} := F_i(w_i^t, \chi_i^t(w_i^t, H^t), H^t), \quad (2.4)$$

where $\mathcal{W}_i(H)$ represents the constraints of the problem, which is contingent upon the current state. The solution to the optimization problem (2.4) for $i = 1, 2$ yields the optimal solution for both players, (w_i^{t*}, w_j^{t*}) which is the action they will implement. When this solution is interior, the optimal control w_i^{t*} solves the first order condition, for a given state value H^t :

$$\frac{d\pi_i^{c,t}}{dw_i} = \frac{\partial F_i}{\partial w_i}(w_i^t, \chi_i^t(w_i^t, H^t), H^t) + \frac{\partial F_i}{\partial w_j}(w_i^t, \chi_i^t(w_i^t, H^t), H^t) \frac{\partial \chi_i^t}{\partial w_i}(w_i^t, H^t) = 0. \quad (2.5)$$

However, with respect to conjectures, Player i was expecting that Player j would play $w_j^{c,t*} = \chi_i^t(w_i^{t*}, H^t)$. This conjecture is in general different from w_j^{t*} . Then Player i should revise her conjecture. This update process can take the general form:¹

$$\chi_i^{t+1} = \mathcal{U}_i(\chi_i^t, w_i^{t*}, w_j^{t*}, H^t). \quad (2.6)$$

Note that χ_i^t is a sequence of functions. In practical situations we are going to consider particular functional forms for the conjecture, with a certain numbers of parameters. These

¹An alternative could be $\chi_i^{t+1} = \mathcal{S}_i(w_j^{0*}, H^0, \dots, w_j^{t*}, H^t)$, meaning that the update of the conjecture is based on some “statistic” (hence “ \mathcal{S} ”) performed on all the observations. We shall not develop this alternative here.

parameters are going to be learned in the learning procedure. In (2.6), the parameters are updated (hence “ \mathcal{U} ”) based on the most recent observation of the opponent. Indeed, the *thesis* of this work is that agents will construct their decision based on the most recent data, and not engage in more complex data processing and machine learning.

When it is repeated over time, this learning procedure produces a dynamical system with variables $(H^t, w_i^{t*}, w_j^{t*}, \chi_i^t, \chi_j^t)$, specified by equations (2.2), (2.4) and (2.6). This construction is formalized in the following definition. For the purpose of mathematical statements, it is assumed that the space of conjecture functions χ_i^t is equipped with some topology. In the examples we shall develop, these functions will typically be in a class of parametric functions, and assimilated to some subset of \mathbb{R}^n .

Definition 1 (Conjectural Learning). *A Conjectural Learning procedure (shortened as CLP) is a dynamical system with variables $(H^t, w_i^{t*}, w_j^{t*}, \chi_i^t, \chi_j^t)$, specified by equations (2.2), (2.4) and (2.6). A Conjectural Learning procedure is called*

well-defined if Problem (2.4) has a unique solution for each $i = 1, 2$ and every $t = 0, 1, \dots$. We then denote with $\text{sol}_i^c(\chi_i^t, H^t)$ this solution for Player i .

regular if a) the functions sol_i^c are continuous in their variables, and b) the update functional \mathcal{U}_i is continuous in its variables.

consistent if there is no update of the conjecture when, and only when the observation of the agent matches her conjecture: formally, if

$$\mathcal{U}_i(\chi_i, w_i, w_j, H) = \chi_i \iff w_j = \chi_i(w_i, H).$$

We now state a result that applies to the general learning procedure just exposed.

Proposition 1. *Assume a well-defined, regular and consistent Conjectural Learning procedure converges to a limit $(H^\infty, w_i^\infty, w_j^\infty, \chi_i^\infty, \chi_j^\infty)$ then we have:*

$$w_j^\infty = \chi_i^\infty(w_i^\infty, H^\infty) \quad i \neq j \tag{2.7}$$

$$H^\infty = G(w_i^\infty, w_j^\infty, H^\infty) \tag{2.8}$$

$$w_i^\infty = \text{sol}_i^c(\chi_i^\infty, H^\infty) \quad i = 1, 2. \tag{2.9}$$

In particular, if the solution of (2.4) when $H^t = H^\infty$ is interior, and if functions χ_i^∞ , $i = 1, 2$, are continuous with continuous partial derivatives with respect to w_i , (2.9) reads as:

$$\frac{\partial F_i}{\partial w_i}(w_i^\infty, w_j^\infty, H^\infty) + \frac{\partial F_i}{\partial w_j}(w_i^\infty, w_j^\infty, H^\infty) \frac{\partial \chi_i^\infty}{\partial w_i}(w_i^\infty, H^\infty) = 0. \tag{2.10}$$

Proof. Assume therefore that the sequence $(H^t, w_i^{t*}, w_j^{t*}, \chi_i^t, \chi_j^t)$ converges. Since (2.2) holds and G is continuous, then (2.8) holds in the limit. Also, since $w_i^t = \text{sol}_i^c(\chi_i^t, H^t)$ for every t and sol_i^c is assumed to be continuous (regularity), (2.9) holds in the limit. Likewise, since (2.6) holds for every t and \mathcal{U}_i is continuous (regularity), in the limit we have: $\mathcal{U}_i(\chi_i^\infty, w_i^\infty, w_j^\infty, H^\infty) = \chi_i^\infty$. Then, by consistency, this implies $w_j^\infty = \chi_i^\infty(w_i^\infty, H^\infty)$, which is (2.7). \square

Remark 1. When solutions to (2.4) are unique (well-defined CLP) and interior, the continuity of the partial derivatives appearing in (2.5) implies the continuity of sol_i^c (regularity, part a)).

2.2 Nearsighted behaviors with complete information

We now introduce two models of “almost myopic” agents, which we call *nearsighted*. In these models, agents consider their immediate revenue given by Equation (2.1). With this gain function in mind, assumed common knowledge, the players choose to play non-cooperatively (maximizing their individual profit) or cooperatively (maximizing the joint profit). We study both alternatives in the following sections. Those behaviors will be taken as comparison benchmarks for conjectural learning behaviors.

2.2.1 Definitions

The game or optimization problem solved at each time step t depends only on the state variable H^t . As in Definition 1, we call “well-defined” the situation where this solution exists and is unique for each value of H^t . Denote with $\text{sol}_i(H^t)$, $i = 1, 2$, this nearsighted solution. Taking into account the dynamics (2.2) and repeating the nearsighted solution at each period of time, this generates a dynamic system with generic equations:

$$w_i^t = \text{sol}_i(H^t) \quad i = 1, 2, \quad (2.11)$$

$$H^{t+1} = G(w_i^t, w_j^t, H^t) . \quad (2.12)$$

This construction is formalized with the following definition.

Definition 2 (Nearsighted Scheme). *A Nearsighted Scheme is a dynamical system with variables (w_i^t, w_j^t, H^t) , specified by Equations (2.11) and (2.12).*

A Nearsighted Scheme is called

***well-defined** if the function $\text{sol}_i(H^t)$, solution of the game problem at state t , exists and is unique for each $i = 1, 2$ and every $t = 0, 1, \dots$;*

***regular** if the functions $\text{sol}_i(\cdot)$ are continuous.*

We are interested in the asymptotic behavior of this system and its relationship with the learning procedure of Section 2.1.

2.2.2 Nearsighted vs Myopic

The kind of behavior we are considering here is usually called “myopic”, because it does not take into account the dynamics. We use the term “nearsighted” to refer to the feature that players can take into account the near future in their optimization program. Indeed, assume a gain model of the form (2.1), where the function F_i has the specific form:

$$F_i(w_i^t, w_j^t, H^t) = \pi_i(w_i^t, w_j^t, H^t) + \rho S_i(H^{t+1}), \quad (2.13)$$

where H^{t+1} is given by the dynamics in (2.2), namely $H^{t+1} = G(w_i^t, w_j^t, H^t)$, $S_i(\cdot)$ can be interpreted as a “scrap” value function and ρ as a discount factor.

One interpretation is that Player i optimizes some immediate objective function π_i , plus some gain depending on the future of the resource H^{t+1} . This gain represents the subjective perception that the agent’s future gains somehow depend on the state of the resource. The agents however do not think as far as trying to solve a multiple-stage optimization or game problem. As a particular case, when $S(\cdot) = 0$ we have the usual myopic situation.

Remark 2. When agents optimize a profit function of the form (2.13) with a scrap value different from zero and they maximize step by step as presented in Sections 2.1 and 2.2.1, we say that they have a nearsighted behavior. Section 2.1 deals with nearsighted behavior with *incomplete* information: the agents do not know the profit function of the other player. Section 2.2.1 deals with nearsighted behavior with *complete* information.

2.2.3 Non cooperative nearsighted equilibrium

In this situation, suppose that the players play a non cooperative game, i.e. at each period t both of them solve:

$$\max_{w_i^t \in \mathcal{W}_i(H^t)} F_i(w_i^t, w_j^t, H^t), \quad i = 1, 2. \quad (2.14)$$

We call this behavior the nearsighted Nash equilibrium. Supposing interior solutions for all t , the choice of w_i^t solves the first-order conditions which are:

$$\frac{\partial F_i}{\partial w_i}(w_i^t, w_j^t, H) = 0, \quad i = 1, 2, \quad t = 1, 2, \dots \quad (2.15)$$

This defines the functions $\text{sol}_i(H)$ in (2.11), denoted here sol_i^N .

Remark 3. In the results that follow, we are going to consider only the case where solutions are interior and unique. Some of these results may still hold when players have independent constraints on their strategy: we prefer to avoid the technicalities for this first study. When constraints exist, and especially when these constraints are coupling, it is well-known that multiple Nash equilibria may exist. Some equilibrium selection, such as Rosen's solution [12], may be needed, both for a proper definition of the solution, and for guaranteeing convergence of iterative schemes.

Then we have the first result:

Proposition 2. *If the Nearsighted scheme, where the solutions sol_i^N are defined by (2.14) is well-defined, regular and converges, then the limit $(w_i^\infty, w_j^\infty, H^\infty)$ satisfies:*

$$H^\infty = G(w_i^\infty, w_j^\infty, H^\infty) \quad (2.16)$$

$$w_i^\infty = \text{sol}_i^N(H^\infty), \quad i = 1, 2. \quad (2.17)$$

If the solution is interior, the limit satisfies

$$\frac{\partial F_i}{\partial w_i}(w_i^\infty, w_j^\infty, H^\infty) = 0, \quad i = 1, 2. \quad (2.18)$$

Proof. The proof is obvious taking into account the continuity of the function G and that of sol_i . \square

Remark 4. When the solution of (2.14) is unique (well defined scheme) and interior, the continuity of partial derivative appearing in the first-order conditions (2.15) implies the continuity of sol_i (regularity), and (2.18).

Comparing with Proposition 1, we have the following corollary.

Proposition 3 (Convergence of conjectural learning to a nearsighted Nash steady state). *Consider a Conjectural Learning procedure as in Definition 1, well-defined, regular and consistent. Assume this procedure converges to a limit $(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}, \chi_i^{c\infty}, \chi_j^{c\infty})$, where the solution is interior. Then this limit is a steady state of the Nearsighted Scheme with Nash solution if and only if:*

$$\frac{\partial \chi_i^{c\infty}}{\partial w_i}(w_i^{c\infty}, H^{c\infty}) = 0 \quad \text{or} \quad \frac{\partial F_i}{\partial w_j}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) = 0, \quad i = 1, 2.$$

Proof. The results from the comparison of Propositions 1 and 2. If the Conjectural Learning procedure converges to an interior solution, the limit solves (2.10), according to Proposition 1:

$$\frac{\partial F_i}{\partial w_i}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) + \frac{\partial F_i}{\partial w_j}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) \frac{\partial \chi_i^{c\infty}}{\partial w_i}(w_i^{c\infty}, H^{c\infty}) = 0.$$

If either of the conditions $\partial \chi_i^{c\infty} / \partial w_i = 0$ or $\partial F_i / \partial w_j = 0$ holds, it reduces to (2.15). In addition the limit solves Equation (2.16) since it coincides with (2.8). Therefore, the limit is a fixed point of the Nash nearsighted scheme. Conversely, for such a fixed point to satisfy (2.10), it is required that $\partial \chi_i^{c\infty} / \partial w_i \times \partial F_i / \partial w_j = 0$, hence the condition. \square

This proposition in particular says that, if the conjecture depends only on the evolution of the state variable and if the learning process converges to some interior fixed point, then it must converge to the interior nearsighted Nash steady state. In that case indeed, $\partial \chi_i^t / \partial w_i = 0$ for all t and this is reasonable to expect the same property at the limit.

It also applies to the case where, for each player i , the function F_i does not depend on the opponent's action at all: $\partial F_i / \partial w_j = 0$, $i \neq j$. In that case, the conjecture is of course useless.

2.2.4 Cooperative nearsighted solution

Now suppose the players play a cooperative game at each period, i.e. they maximize the joint profit:

$$\max_{w_i^t, w_j^t} [F_i(w_i^t, w_j^t, H^t) + F_j(w_i^t, w_j^t, H^t)]. \quad (2.19)$$

We call this solution the nearsighted cooperative, or nearsighted Pareto solution.

Assuming an interior solution, the optimal actions satisfy first-order conditions that are here:

$$\frac{\partial F_i}{\partial w_i}(w_i, w_j, H) + \frac{\partial F_j}{\partial w_i}(w_i, w_j, H) = 0, \quad i = 1, 2. \quad (2.20)$$

This defines the functions $\text{sol}_i(H)$ of (2.11), denoted here sol_i^P . As before, we call ‘‘well-defined’’ the situation where this solution exists and is unique for all H .

If the dynamic procedure converges we have

Proposition 4. *If the Nearsighted scheme, where the solutions sol_i^P are defined by (2.20) is well-defined, regular and converges, then the limit $(w_i^\infty, w_j^\infty, H^\infty)$ satisfies:*

$$H^\infty = G(w_i^\infty, w_j^\infty, H^\infty), \quad (2.16)$$

$$w_i^\infty = \text{sol}_i^P(H^\infty), \quad i = 1, 2. \quad (2.21)$$

When the solution is interior, the limit satisfies

$$\frac{\partial F_i}{\partial w_i}(w_i^\infty, w_j^\infty, H^\infty) + \frac{\partial F_j}{\partial w_i}(w_i^\infty, w_j^\infty, H^\infty) = 0, \quad i = 1, 2. \quad (2.22)$$

Proof. The proof is obvious taking into account the continuity of the function G , that of sol_i^P and that of the partial derivatives appearing in the first-order conditions (2.20), and the dynamics at the steady state. \square

Comparing with Proposition 1, we have the following result.

Proposition 5 (Convergence of Conjectural Learning to a Nearsighted cooperative steady state). *Consider a Conjectural Learning procedure as in Proposition 1 which is well-defined, regular and consistent. If the procedure converges to an interior limit $(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}, \chi_i^{c\infty}, \chi_j^{c\infty})$ and if, in addition, for $i = 1, 2$,*

$$\frac{\partial F_i}{\partial w_j}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) = \frac{\partial F_j}{\partial w_i}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty})$$

and:

$$\frac{\partial \chi_i}{\partial w_i}(w_i^{c\infty}, H^{c\infty}) = 1 \quad i = 1, 2,$$

then this limit is an interior steady state of the Nearsighted Scheme with the cooperative solution.

Proof. The result follows from the comparison of Propositions 1 and 4. The interior limit satisfies Equation (2.10):

$$\frac{\partial F_i}{\partial w_i}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) + \frac{\partial F_i}{\partial w_j}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) \frac{\partial \chi_i^{c\infty}}{\partial w_i}(w_i^{c\infty}, H^{c\infty}) = 0.$$

Under the additional conditions, this is equivalent to:

$$\frac{\partial F_i}{\partial w_i}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) + \frac{\partial F_j}{\partial w_i}(w_i^{c\infty}, w_j^{c\infty}, H^{c\infty}) = 0,$$

for each $i = 1, 2$, which is just (2.22). The limit is therefore a steady-state of the cooperative Nearsighted Scheme. \square

This proposition says, in particular, that if both the learning scheme and the nearsighted Pareto scheme converge, if solutions are interior, and if the game is symmetric with a conjecture of the type $\chi_i(w) = w + \beta$, the learning process converges to the nearsighted cooperative steady state. See the forthcoming Propositions 10 and 11.

2.2.5 Convergence

The convergence of a nearsighted scheme reduces to the convergence of the recurrence, obtained by combining (2.11) and (2.12):

$$H^{t+1} = G(\text{sol}_i(H^t), \text{sol}_j(H^t), H^t).$$

Accordingly, a sufficient condition for convergence to a fixed point of this equation, is the following.

Proposition 6 (Convergence of nearsighted schemes). *A sufficient condition for the local convergence of a nearsighted procedure to an interior steady state H^∞ is:*

$$\left| \frac{\partial \text{sol}_i}{\partial H}(H^\infty) \frac{\partial G}{\partial w_i} + \frac{\partial \text{sol}_j}{\partial H}(H^\infty) \frac{\partial G}{\partial w_j} + \frac{\partial G}{\partial H} \right| < 1, \quad (2.23)$$

the partial derivatives of G being evaluated at $(\text{sol}_i(H^\infty), \text{sol}_j(H^\infty), H^\infty)$.

Proof. The recurrence can be described as $H^{t+1} = f(H^t)$ and it is well-known that $|f'(H^\infty)| < 1$ is a sufficient condition for local convergence. See e.g. [4, Theorem 6.5]. \square

2.3 Some affine conjectures and their relationship with nearsightedness

In this section, we study several particular functional forms of conjectures. We provide characterizations of the steady states of the corresponding Conjectural Learning procedures, and discuss when these steady states correspond to those of Nearsighted procedures. We also provide sufficient conditions for convergence of the learning procedures.

Some results hold when players are symmetric, a situation which we now define formally.

Definition 3 (Symmetric players). *The players are called symmetric if $F_i(w_i, w_j, H) = F_j(w_j, w_i, H)^2$ for all values of w_i, w_j and H .*

2.3.1 Affine conjecture on w

We now turn to the most usual functional form in the literature (see [7, 11]) where the function χ_i depends linearly on w_i only:

$$w_j^c = \chi_i(w_i) := \bar{w}_j + \beta_i(w_i - \bar{w}_i), \quad i \neq j. \quad (2.24)$$

Here, $\bar{w} = (\bar{w}_1, \bar{w}_2)$ is some given (common knowledge) reference value and β_i is the supposed sensitivity of Player j to actions of Player i . This parameter is going to evolve with the learning processes with an update as in (2.6). We specify this update now.

Player i plays w_i , assumes that Player j will play as in (2.24) and optimizes her action w_i^* . When she looks at the real action of Player j , w_j^* , she “realizes” that her conjecture should have been

$$w_j^* = \bar{w}_j + \beta'_i(w_i^* - \bar{w}_i).$$

In other words, she observes a β'_i equal to

$$\beta'_i = \frac{w_j^* - \bar{w}_j}{w_i^* - \bar{w}_i},$$

and she revises his learning procedure as follows

$$\beta_i^{t+1} = (1 - \mu_i)\beta_i^t + \mu_i \frac{w_j^{t*} - \bar{w}_j}{w_i^{t*} - \bar{w}_i}, \quad (2.25)$$

for some learning factor $\mu_i \in (0, 1]$. This process is initialized with some given β_i^0 , $i = 1, 2$. The actions played at step t are obtained solving (2.5). This results in a dynamic system

²Note that when the scrap value is non zero this definition implies symmetry in objective gain and also in dynamics.

involving variables $(w_i^{t*}, w_j^{t*}, H^t, \beta_i^t, \beta_j^t)$. Note that this process is consistent (see Definition 1), because here, $\chi_i = \mathcal{U}_i(\chi_i, w_i, w_j, H)$ means

$$\beta_i = (1 - \mu_i)\beta_i + \mu_i \frac{w_j^* - \bar{w}_j}{w_i^* - \bar{w}_i},$$

and since $\mu_i > 0$, this is equivalent to $\beta_i(w_i^* - \bar{w}_i) = w_j^* - \bar{w}_j$, which, in view of (2.24), means $w_j^* = \chi_i(w_i^*)$. This consistency, of course, results from the way the value β_i' was constructed.

Lemma 1. *Assume that the Conjectural Learning procedure specified above is well-defined, regular and converges to some limit $(w_i^{\infty*}, w_j^{\infty*}, H^\infty, \beta_i^\infty, \beta_j^\infty)$. Then $\beta_i^\infty \beta_j^\infty = 1$ and $\chi_i(w_i^\infty) = w_j^\infty$. Assume in addition that $w_i^{\infty*} \neq \bar{w}_i$ for $i = 1, 2$. Then:*

$$\beta_i^\infty = \frac{w_j^{\infty*} - \bar{w}_j}{w_i^{\infty*} - \bar{w}_i}.$$

Proof. The first part of the statement follows from (2.25) and the continuity of the solutions $w_i^* = \text{sol}_i^c(\beta_i, H)$ (assumption on regularity). Indeed, under the convergence assumption, the ratio $(w_j^{t*} - \bar{w}_j)/(w_i^{t*} - \bar{w}_i)$ converges to β_i^∞ , even if $w_i^{\infty*} = \bar{w}_i$. Then the identity $\beta_i^\infty \beta_j^\infty = 1$ is obvious. The second identity also follows from (2.25) after multiplication by $w_i^{t*} - \bar{w}_i$. The third one is equivalent to it when $w_i^{t*} \neq \bar{w}_i$. \square

Applying Proposition 5, we can deduce the following result.

Proposition 7. *Assume that players are symmetric as in Definition 3. Assume in addition that*

$$\frac{\partial F_i}{\partial w_j}(w, w, H) = \frac{\partial F_j}{\partial w_i}(w, w, H) \quad (2.26)$$

for all w and H . Consider the Conjectural Learning procedure specified above with $\bar{w}_i = \bar{w}_j$. Assume it is well-defined, regular and converges to some interior limit $(w_i^{\infty}, w_j^{\infty*}, H^\infty, \beta_i^\infty, \beta_j^\infty)$ with $w_i^{\infty*} = w_j^{\infty*}$ and $\beta_i^\infty = \beta_j^\infty = 1$. Finally, assume that the cooperative nearsighted procedure is well-defined.*

Then, the limit of the CLP is the cooperative nearsighted steady state.

Proof. If the CLP converges with $w_i^\infty = w_j^\infty = w^\infty$ and the solution is interior, then it satisfies (2.10), that is:

$$\begin{aligned} 0 &= \frac{\partial F_i}{\partial w_i}(w^\infty, w^\infty, H^\infty) + \frac{\partial F_i}{\partial w_j}(w^\infty, w^\infty, H^\infty) \frac{\partial \chi_i^\infty}{\partial w_i}(w^\infty, H^\infty) \\ &= \frac{\partial F_i}{\partial w_i}(w^\infty, w^\infty, H^\infty) + \frac{\partial F_i}{\partial w_j}(w^\infty, w^\infty, H^\infty) \beta_i^\infty. \end{aligned}$$

By assumption, we have $\beta_i^\infty = 1$. Since in addition (2.26) is assumed, we obtain:

$$0 = \frac{\partial F_i}{\partial w_i}(w^\infty, w^\infty, H^\infty) + \frac{\partial F_j}{\partial w_i}(w^\infty, w^\infty, H^\infty).$$

In other words, $(w^\infty, w^\infty, H^\infty)$ solves Equation (2.22). Since the procedure is assumed to be well defined, this equation has a unique solution, so the statement is proved. \square

We now state convergence results for this Conjectural Learning procedure.

Proposition 8. Consider a symmetric Conjectural Learning procedure with conjectures as in (2.24). Denote with $r(\beta, H)$ the common reaction function, solution to (2.5), the solution of Problem (2.4) being assumed interior. Assume r is continuous and differentiable. Assume further that the values $\beta_i = 1$, $i = 1, 2$, $w_i = w_j = w^\infty$ and H^∞ solve the fixed-point equations. Denote finally with ω :

$$\omega = \frac{\partial r}{\partial \beta}(1, H^\infty) \frac{1}{w^\infty - \bar{w}} .$$

If the following conditions hold,

$$\frac{\partial G}{\partial H} + \frac{\partial r}{\partial H} \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) \in (-1, 1) \quad (2.27)$$

$$\mu_i(1 + 2 \max(\omega, 0)) < 2 \quad \text{for } i = 1, 2, \quad (2.28)$$

$$1 + 2 \min(\omega, 0) > 0, \quad (2.29)$$

where the functions are evaluated at $(\beta_1^\infty, \beta_2^\infty, H^\infty) = (1, 1, H^\infty)$, then the procedure is locally convergent around this steady state.

The proof of this proposition is presented in Appendix A.2.

Observe that Condition (2.28) can be realized by choosing μ_i small enough. However, Condition (2.29) might not be satisfiable if ω is negative and large. On the contrary, it is trivially satisfied if $\omega \geq 0$.

When symmetry is even greater, then convergence of the subsequence β^t always occurs, as stated in the following result.

Proposition 9. Consider the Conjectural Learning procedure defined in this section, under the assumption of symmetry defined in Definition 3 with in addition $\bar{w}_i = \bar{w}_j$ and $\beta_i^0 = \beta_j^0$. If the property (2.27) holds, then the procedure converges locally and $\beta_i^\infty = \beta_j^\infty = 1$.

Proof. The proof proceeds as in that of Proposition 4.5 in [7]. Under the assumptions, $w_i^{t*} = w_j^{t*}$ for all t , and it follows that the recurrence $\beta_i^{t+1} = (1 - \mu_i)\beta_i^t + \mu_i$ becomes independent of the other variables. It converges to 1 because $\mu_i \in (0, 1]$. Then, in the fixed point $(1, 1, H^\infty)$, the dynamic system behaves as the one-dimensional recurrence $H^{t+1} = G(r(1, H^t), r(1, H^t), H^t)$, and (2.27) is sufficient for its local convergence (see also the proof of Proposition 6). \square

2.3.2 Conjecture “imitate w ”

Assume here that the conjecture takes the form:

$$w_j^c = \chi_i(w, H) := w_i + \beta_i, \quad i \neq j. \quad (2.30)$$

Player i thinks that her opponent will just copy her value, up to some constant: hence the name “imitate w ” we give to this conjecture. Then the adjustment mechanism of the conjecture β_i takes the form:

$$w_j^* = w_i^* + \beta_i' \quad \text{that is,} \quad \beta_i' = w_j^* - w_i^*$$

so that, using the same adjustment method as in Section 2.3.1,

$$\beta_i^{t+1} = (1 - \mu_i)\beta_i^t + \mu_i(w_j^{t*} - w_i^{t*}) . \quad (2.31)$$

for some $\mu_i \in (0, 1]$. This defines a Conjectural Learning procedure as in Section 2.1. This procedure is consistent: by construction, $\beta_i^{t+1} = \beta_i^t$ if and only if $w_j^{*t} = w_i^{*t} + \beta_i^t$. Then it easily follows that:

Lemma 2. *Assume that the Conjectural Learning procedure specified above is well-defined, regular and converges to some limit $(w_i^{\infty*}, w_j^{\infty*}, H^\infty, \beta_i^\infty, \beta_j^\infty)$, not necessarily interior. Then:*

$$\beta_i^\infty = w_j^{\infty*} - w_i^{\infty*}, \quad \chi_i(w_i^{\infty*}, H^\infty) = w_j^{\infty*}, \quad \beta_i^\infty + \beta_j^\infty = 0.$$

Considering Proposition 5, we have the following result.

Proposition 10. *Assume that players are symmetric as in Definition 3. Consider the Conjectural Learning procedure specified above. Assume it is well-defined, regular and converges to some interior limit $(w_i^{\infty*}, w_j^{\infty*}, H^\infty, \beta_i^\infty, \beta_j^\infty)$. Finally, assume that the cooperative nearsighted procedure is well-defined.*

Then, the limit of the CLP is the cooperative nearsighted steady state.

Proof. The proof is immediate from Proposition 5, because the CLP is consistent, the players are symmetric and $\partial\chi_i/\partial w_i(w, H) = 1$ for all values of w and H . \square

Proposition 11. *Consider a symmetric Conjectural Learning procedure with conjectures as in (2.30). Denote with $r(\beta, H)$ the common reaction function, solution to (2.5), the solution of Problem (2.4) being assumed interior. Assume further that the values $\beta_i = \beta_j = 0$, $w_i = w_j = w^\infty$ and H^∞ solve the fixed-point equations. Denote finally with ω :*

$$\omega = \frac{\partial r}{\partial \beta}(0, H^\infty).$$

If the following conditions hold,

$$\frac{\partial G}{\partial H} + \frac{\partial r}{\partial H} \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) \in (-1, 1) \quad (2.27)$$

$$\mu_i(1 + 2 \max(\omega, 0)) < 2 \quad \text{for } i = 1, 2, \quad (2.32)$$

$$1 + 2 \min(\omega, 0) > 0, \quad (2.33)$$

where the functions are evaluated at $(\beta_1^\infty, \beta_2^\infty, H^\infty) = (0, 0, H^\infty)$, then the procedure is locally convergent around this steady state.

The proof of this proposition is presented in Appendix A.3.

Observe that Condition (2.32) can be realized by choosing μ_i small enough, and that if $\omega \leq 0$, it is trivially satisfied for any $\mu_i \leq 1$. However, Condition (2.33) might not be satisfiable if ω is negative and large. On the contrary, it is trivially satisfied if $\omega \geq 0$.

2.3.3 Affine conjecture on H

Another possibility is to assume that the function χ_i depends linearly on H only. We adopt the general form:

$$w_j^c = \chi_i(H) := \bar{w}_j + \beta_i(H - \bar{H}_i) \quad i \neq j. \quad (2.34)$$

In that form, the quantity \bar{H}_i can be interpreted as a reference level proper of Player i , with respect to which deviations are, supposedly, observed by Player j . As in the previous section, the parameter β_i is the subject of learning. The other values \bar{H}_i and \bar{w}_j are parameters assumed to be fixed and of common knowledge.

Here, the update procedure takes the following form. After observing the value w_j^* played by Player j , Player i concludes that the value of β_i should have been:

$$\beta_i' = \frac{w_j^* - \bar{w}_j}{H - \bar{H}_i},$$

so that the update is:

$$\beta_i^{t+1} = (1 - \mu_i)\beta_i^t + \mu_i \frac{w_j^{t*} - \bar{w}_j}{H^t - \bar{H}_i}. \quad (2.35)$$

Again, this procedure is consistent: by construction, $\beta_i^{t+1} = \beta_i^t$ if and only if $w_j^{t*} = \bar{w}_j + \beta_i^t(H^t - \bar{H}_i)$. Accordingly, we have the result:

Lemma 3. *Assume that the Conjectural Learning procedure specified above is well-defined, regular and converges to some limit $(w_i^{\infty*}, w_j^{\infty*}, H^\infty, \beta_i^\infty, \beta_j^\infty)$ such that $H^\infty \neq \bar{H}_i$, $i = 1, 2$. Then:*

$$\beta_i^\infty = \frac{w_j^{\infty*} - \bar{w}_j}{H^\infty - \bar{H}_i}.$$

From Proposition 3 we have the following result:

Proposition 12. *Consider a Conjectural Learning procedure with a conjecture of the form (2.34), which is well-defined and regular. Assume that Equation (2.18) has unique solution. Then if the Conjectural Learning procedure converges, it does to the nearsighted Nash steady state.*

We conclude this section with a sufficient condition for convergence.

Proposition 13. *Consider a symmetric Conjectural Learning procedure with conjectures as in (2.34). Denote with $r(\beta, H)$ the common reaction function, solution to (2.5), the solution of Problem (2.4) being assumed interior. Let $(\beta^\infty, \beta^\infty, w^\infty, w^\infty, H^\infty)$ solve the fixed-point equations, with $H^\infty \neq \bar{H}$. Denote finally with ω_1 and ω_2 :*

$$\omega_1 = \frac{1}{|H^\infty - \bar{H}|} \frac{\partial r}{\partial \beta}(\beta^\infty, H^\infty) \quad \omega_2 = \frac{\partial r}{\partial H}(\beta^\infty, H^\infty).$$

If the following conditions hold, for some $\lambda > 0$:

$$\frac{\partial G}{\partial H} + \omega_2 \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) + \frac{|\omega_1|}{\lambda} \left(\left| \frac{\partial G}{\partial w_1} \right| + \left| \frac{\partial G}{\partial w_2} \right| \right) < 1 \quad (2.36)$$

$$\frac{\partial G}{\partial H} + \omega_2 \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) - \frac{|\omega_1|}{\lambda} \left(\left| \frac{\partial G}{\partial w_1} \right| + \left| \frac{\partial G}{\partial w_2} \right| \right) > -1 \quad (2.37)$$

$$|\omega_1| + \lambda |\beta^\infty - \omega_2| < 1, \quad (2.38)$$

where the functions are evaluated at $(\beta^\infty, \beta^\infty, H^\infty)$, then the procedure is locally convergent around this steady state.

The proof of this proposition is presented in Appendix A.4.

2.4 Farsighted behaviors with complete information

Assuming players have complete information on the profit function of their opponent, they can opt for one of the infinite-horizon solutions, which we call here “farsighted”. We shall consider two standard solutions: the Nash feedback and the cooperative (or “Pareto optimal”) solution.

For the Nash feedback equilibrium each player solves:

$$\max_{\{w_i^t, t=0,1,\dots\} \in \mathcal{FB}_i} \sum_{t=0}^{\infty} \rho^t \pi_i(w_i^t, w_j^t, H^t), \quad H^{t+1} = G(w_i^t, w_j^t, H^t), \quad (2.39)$$

where \mathcal{FB}_i denotes the set of feedback policies of Player i , and for the cooperative solution they solve

$$\max_{\{w_i^t, i=1,2, t=0,1,\dots\}} \sum_{i=1}^2 \sum_{t=0}^{\infty} \rho^t \pi_i(w_i^t, w_j^t, H^t), \quad H^{t+1} = G(w_i^t, w_j^t, H^t). \quad (2.40)$$

Remark 5. Note that in the farsighted solutions, the instantaneous gain function only considers the objective gain $\pi_i(w_i^t, w_j^t, H^t)$. The subjective part of the gain, that is, the term $\rho S_i(H^{t+1})$ in (2.13), is not really earned by the player.

We do not discuss further the computation and the properties of the farsighted solutions in this general case. We only require that they exist and are unique. More details will be given in the application.

3 A groundwater exploitation problem

We consider as application a groundwater exploitation problem, inspired by Provencher and Burt [9]. In the following sections, we introduce the model, then apply the mechanisms described in Section 2.

3.1 The general model

Water extraction is the only input in the production process of two farmers, and the dynamics is given by the evolution of the level of the water table. We adopt the simple reservoir model:

$$H^{t+1} = G(w_i^t, w_j^t, H^t) = H^t + R - \alpha(w_i^t + w_j^t), \quad (3.1)$$

where R is the annual recharge (assumed constant) and $\alpha \in (0, 1]$ is a return flow coefficient.

Farmers face strategic interactions in this model in two possible distinct ways. First, if the pumping cost is made to depend on the height of the water table *during* or *at the end* of the time period, then the whole profit function (revenue minus costs) depends on the action of the opponent. Second, as mentioned in Section 2.2.2, farmers can also take into account the future state of the resource in their (subjective) utility evaluation.

With this in scope, we choose as profit function for the farmers:

$$\begin{aligned} F_i(w_i^t, w_j^t, H^t) &= P_i(w_i^t) - w_i C((1 - \theta)H^t + \theta H^{t+1}) + \rho \gamma_i H^{t+1} \\ &= P_i(w_i^t) - w_i C(H^t + \theta(R - \alpha(w_i^t + w_j^t))) + \rho \gamma_i (H^t + R - \alpha(w_i^t + w_j^t)). \end{aligned} \quad (3.2)$$

The first term $P_i(w_i^t)$ represents the objective revenue from the crop when the quantity of water w_i^t is used. The second term represents the cost of extraction, or pumping cost, during the period. The function $C(\cdot)$, the unit cost of extraction, is assumed to be such that $C'(\cdot) < 0$, $C''(\cdot) \geq 0$. Unit costs increase when the level of the water table is low, and vice versa. Note that extraction is realized at time t and the cost charged is considered at time $t + \theta$, $\theta \in [0, 1]$, somewhere in the interval $[t, t+1]$. This allows to take into account interaction while extraction is done. Setting $\theta = 0$ means that when evaluating the cost, the water level is considered before extraction (optimistic valuation). The opposite case $\theta = 1$ is the worse case where the cost is evaluated after extraction, when the water table is the lowest.

Finally, the third term $\gamma_i H^{t+1}$ is the extra subjective benefit of maintaining the resource. As mentioned previously, ρ is the discount factor, and γ_i his resource preference, that is a “subjective marginal benefit” of having one more unit of resource at the next period. Comparing with (2.13), this amounts to choosing as scrap value $S(H) = \gamma_i H$.

3.2 A linear-quadratic example

In order to investigate further, analytically and numerically, the different behaviors introduced in Section 2, we consider specific functional forms. Choosing the linear-quadratic setting will allow the derivation of explicit formulas. Accordingly, we assume:

$$P_i(w_i) = a_i w_i - \frac{b_i}{2} (w_i)^2, \quad C(H) = c_0 - c_1 H,$$

which results in the perceived profit function: from (3.2),

$$\begin{aligned} F_i(w_i, w_j, H) = & a_i w_i - \frac{b_i}{2} (w_i)^2 - w_i (c_0 - c_1 H - \theta c_1 (R - \alpha(w_i + w_j))) \\ & + \gamma_i \rho (H + R - \alpha(w_i + w_j)). \end{aligned} \quad (3.3)$$

The parameters a_i , c_0 and γ_i have unit $\text{€}/m^3$, parameters b_i and c_1 are in $\text{€}/m^6$, α, ρ, θ are dimensionless, and quantities R , H and w are measured in m^3 .

In the following sections, we develop the calculations for the dynamic schemes defined in Section 2, in the particular case of the functional forms just introduced. We begin with the two Nearsighted schemes (Section 3.2.1), then continue with three Conjectural Learning procedures (Sections 3.2.2, 3.2.3 and 3.2.4). In each case, we discuss the steady-state of the system and its local convergence. For Conjectural Learning procedures, we check that they are well-defined and regular (they are consistent, as mentioned in Section 2) and that the results linking their steady states to those of the Nearsighted schemes (Proposition 7 and Proposition 12) are applicable.

3.2.1 Nearsighted behaviors

We start with writing the solution for the nearsighted behaviors of Section 2.2.3 and 2.2.4.

In the case of nearsighted Nash, the solution of the problem gives the following feedback law:

$$\begin{aligned} w_i^N = \text{sol}_i^N(H) = & \frac{b_j + \alpha \theta c_1}{3\alpha^2 \theta^2 c_1^2 + 2\alpha \theta c_1 (b_1 + b_2) + b_1 b_2} (a_i - c_0 + c_1 H + R c_1 \theta - \alpha \gamma_i \rho) \\ & + \frac{\alpha \theta c_1 (a_i - a_j - \alpha \rho (\gamma_i - \gamma_j))}{3\alpha^2 \theta^2 c_1^2 + 2\alpha \theta c_1 (b_1 + b_2) + b_1 b_2} \end{aligned} \quad (3.4)$$

Then, solving the steady-state stock conservation equation $R = \alpha(w_i + w_j)$, the limit stock and steady-state controls are found to be:

$$c_1 H^{N\infty} = c_0 + \frac{(b_1 + \alpha\theta c_1)(b_2 + \alpha\theta c_1)}{b_1 + b_2 + 2\alpha\theta c_1} \frac{R}{\alpha} - \frac{(a_1 - \alpha\rho\gamma_1)(b_2 + \alpha\theta c_1) + (a_2 - \alpha\rho\gamma_2)(b_1 + \alpha\theta c_1)}{b_1 + b_2 + 2\alpha\theta c_1} \quad (3.5)$$

$$w_i^{N\infty} = \frac{1}{b_1 + b_2 + 2\alpha\theta c_1} \left(a_i - a_j + \frac{R}{\alpha} (b_j + \alpha\theta c_1) - \alpha\rho(\gamma_i - \gamma_j) \right). \quad (3.6)$$

In the symmetric case, the feedback law is:

$$\text{sol}_i^N(H) = \frac{1}{b + 3\alpha\theta c_1} (a - c_0 + c_1 H + R c_1 \theta - \alpha\rho\gamma). \quad (3.7)$$

At the steady state, $w_i^{N\infty} = R/(2\alpha)$ and:

$$H^{N\infty} = \frac{c_0 - a}{c_1} + \frac{R(b + \alpha\theta c_1)}{2\alpha c_1} + \frac{\rho\gamma\alpha}{c_1}. \quad (3.8)$$

In the case of nearsighted Pareto, the feedback law is:

$$\begin{aligned} w_i^P = \text{sol}_i^P(H) &= \frac{b_j}{2\alpha\theta c_1(b_1 + b_2) + b_1 b_2} (a_i - c_0 + c_1 H + R c_1 \theta - 2\alpha\gamma_i \rho) \\ &+ \frac{2\alpha\theta c_1(a_i - a_j) + \alpha\rho b_j(\gamma_i - \gamma_j)}{2\alpha\theta c_1(b_1 + b_2) + b_1 b_2}. \end{aligned} \quad (3.9)$$

Then, solving the steady-state stock conservation equation, the limit stock and the steady state controls are found to be:

$$H^{P\infty} = \frac{c_0}{c_1} + \frac{R b_1 b_2 - \alpha(b_2 a_1 + b_1 a_2)}{\alpha c_1 (b_1 + b_2)} + \frac{\rho(\gamma_1 + \gamma_2)\alpha}{c_1} + R\theta \quad (3.10)$$

$$w_i^{P\infty} = \frac{R b_j + \alpha(a_i - a_j)}{\alpha(b_i + b_j)} = \frac{R}{2\alpha} - \frac{R(b_i - b_j)}{2\alpha(b_i + b_j)} + \frac{a_i - a_j}{b_i + b_j}. \quad (3.11)$$

In the symmetric case, the feedback law is:

$$\text{sol}_i^P(H) = \frac{1}{b + 4\alpha\theta c_1} (a - c_0 + c_1 H + R c_1 \theta - 2\alpha\rho\gamma). \quad (3.12)$$

At the steady state, $w_i^{P\infty} = R/(2\alpha)$ and:

$$H^{P\infty} = \frac{c_0 - a}{c_1} + \frac{R(b + 2\alpha\theta c_1)}{2\alpha c_1} + 2 \frac{\rho\gamma\alpha}{c_1}. \quad (3.13)$$

Convergence. Sufficient conditions for convergence are obtained by applying Proposition 6. The partial derivatives of G are here: $\partial G/\partial w_i = -\alpha$ for $i = 1, 2$, $\partial G/\partial H = 1$. Condition (2.23) writes then in general as:

$$\left| 1 - \alpha \left(\frac{\partial \text{sol}_i}{\partial H}(H^\infty) + \frac{\partial \text{sol}_j}{\partial H}(H^\infty) \right) \right| < 1,$$

equivalently:

$$0 < \alpha \left(\frac{\partial \text{sol}_i}{\partial H}(H^\infty) + \frac{\partial \text{sol}_j}{\partial H}(H^\infty) \right) < 2.$$

In addition, reaction functions in (3.4) and (3.9) are linear in the state, so that $\partial \text{sol}_i / \partial H$ is actually a constant. We obtain, for the nearsighted Nash solution, the condition:

$$0 < \frac{\alpha c_1 (b_1 + b_2 + 2\alpha\theta c_1)}{3\alpha^2\theta^2 c_1^2 + 2\alpha\theta c_1 (b_1 + b_2) + b_1 b_2} < 2,$$

or equivalently:

$$0 < 2\theta(3\theta - 1)\alpha^2 c_1^2 + (4\theta - 1)(b_1 + b_2)\alpha c_1 + b_1 b_2. \quad (3.14)$$

The complete discussion appears to be quite involved, so we focus on the symmetric case: in this case, after factoring out $b + \alpha\theta c_1$, the condition reduces to:

$$0 < (3\theta - 1)\alpha c_1 + b. \quad (3.15)$$

Similarly, for the nearsighted cooperative solution, a sufficient condition for convergence is obtained successively as:

$$\begin{aligned} 0 &< \frac{\alpha c_1 (b_1 + b_2)}{2\alpha\theta c_1 (b_1 + b_2) + b_1 b_2} < 2 \\ 0 &< (4\theta - 1)\alpha c_1 (b_1 + b_2) + 2b_1 b_2 \\ \theta &> \frac{1}{4} \left(1 - \frac{2b_1 b_2}{\alpha c_1 (b_1 + b_2)} \right). \end{aligned} \quad (3.16)$$

In the symmetric case, Condition (3.16) reduces to $\theta > (1 - b/(\alpha c_1))/4$. Solving (3.15) yields the condition that $\theta > (1 - b/(\alpha c_1))/3$. In summary:

Proposition 14. *In the symmetric case, the nearsighted Nash procedure converges locally if $\theta > (1 - b/(\alpha c_1))/3$ and the nearsighted cooperative procedure converges locally if $\theta > (1 - b/(\alpha c_1))/4$.*

3.2.2 Affine conjecture on w

With conjecture (2.24) the maximization problem is

$$\begin{aligned} \max_{w_i^t} & \left[a_i w_i^t - \frac{b_i}{2} (w_i^t)^2 - [c_0 - c_1 (H^t + \theta(R - \alpha(w_i^t + \bar{w}_j + \beta_i^t (w_i^t - \bar{w}_i)))] w_i^t \right. \\ & \left. + \rho \gamma_i [H^t + R - \alpha(w_i^t + \bar{w}_j + \beta_i^t (w_i^t - \bar{w}_i))] \right]. \end{aligned}$$

The function to be maximized is quadratic with leading term $-b_i/2 - (c_1\alpha\theta)(1 + \beta_i^t)$. Solving the first-order condition (2.5) for w_i yields, for an interior extremum:

$$\begin{aligned} w_i^{t*} &= r_i(\beta_i^t, H^t) \\ &= \frac{1}{b_i + 2c_1\alpha\theta(1 + \beta_i^t)} (c_1 H^t + c_1\theta(R - \alpha(\bar{w}_j - \beta_i^t \bar{w}_i)) - \rho \gamma_i \alpha(1 + \beta_i^t) + a_i - c_0). \end{aligned} \quad (3.17)$$

The Conjectural Learning procedure build with the conjecture (2.24) is therefore well-defined if $b_i/2 + (c_1\alpha\theta)(1 + \beta_i^t) > 0$ at each step t . The solution is interior if w_i^{t*} given by (3.17) is nonnegative at each step t .

In the analysis of the steady state and convergence of this dynamic system, we will limit ourselves to the symmetric case of Definition 3. In that case, the fixed-point equations from Proposition 1 are:

$$w_i^\infty = \frac{1}{b + 2c_1\alpha\theta(1 + \beta_i^\infty)} (c_1H^\infty + c_1\theta(R - \alpha\bar{w}(1 - \beta_i^\infty)) - \rho\gamma\alpha(1 + \beta_i^\infty) + a - c_0), \quad i = 1, 2 \quad (3.18)$$

$$w_j^\infty = \bar{w} + \beta_i^\infty(w_i^\infty - \bar{w}), \quad i, j = 1, 2, i \neq j \quad (3.19)$$

$$\frac{R}{\alpha} = w_i^\infty + w_j^\infty. \quad (3.20)$$

The complete solution of this system of equations is provided in Appendix B.1.1. The solution reveals that, depending on conditions on the parameters, one or three solutions may exist: one symmetric and two non-symmetric solutions, but symmetric of each other.

The symmetric solution is:

$$\beta_i^\infty = 1, \quad w_i^\infty = \frac{R}{2\alpha}, \quad H^\infty = \frac{2\alpha(c_0 - a) + R(b + 2c_1\alpha\theta) + 4\alpha^2\rho\gamma}{2c_1\alpha} = H^{P^\infty}. \quad (3.21)$$

It turns out not to depend on \bar{w} .

Remark 6. The equality $H^\infty = H^{P^\infty}$ is a consequence of Proposition 7. We check now that this proposition is applicable here. Indeed, note that from (3.2), we have:

$$\frac{\partial F_i}{\partial w_j}(w_i, w_j, H) = w_i\alpha C'(H + \theta R - \theta\alpha(w_i + w_j)) - \rho\gamma_i\alpha.$$

Then, if $\gamma_i = \gamma_j$,

$$\frac{\partial F_i}{\partial w_j}(w, w, H) = \frac{\partial F_j}{\partial w_i}(w, w, H),$$

for all w and H . This means that the functions F_i and F_j satisfy (2.26).³ In the symmetric solution, $\beta_i^\infty = \beta_j^\infty = 1$ and $w_i^\infty = w_j^\infty$. Therefore, if the CLP converges to this interior solution, the stock converges to H^{P^∞} , the steady state of the cooperative nearsighted scheme.

Convergence. For the convergence of the process to this symmetric solution, results can be obtained by applying Proposition 8 or even Proposition 9. We check the assumptions needed to apply the first one. The critical quantity is $\omega = (\partial r/\partial\beta)(1, H^\infty)(w^\infty - \bar{w})^{-1}$. From the value of $r(\beta, H)$ in (3.17), we have:

$$\frac{\partial r}{\partial\beta}(1, H^\infty) = -\frac{Rc_1\theta + \rho\gamma\alpha}{b + 4c_1\alpha\theta} \quad \text{and therefore} \quad \omega = -\frac{Rc_1\theta + \rho\gamma\alpha}{b + 4c_1\alpha\theta} \frac{1}{R/(2\alpha) - \bar{w}}.$$

³This property holds for a general pumping cost function $C(\cdot)$, even if the production parameters a_i and b_i are not symmetric.

Assume first that $\bar{w} < w^\infty = R/(2\alpha)$. Then $\omega < 0$ and Condition (2.28) is naturally satisfied while Condition (2.29) is equivalent to $\omega > -1/2$. This in turn is equivalent to:

$$2\rho\gamma\alpha < b\left(\frac{R}{2\alpha} - \bar{w}\right) - 4\alpha\theta c_1\bar{w}. \quad (3.22)$$

If on the other hand $\bar{w} > R/(2\alpha)$, then $\omega > 0$. Condition (2.29) is automatically satisfied, and Condition (2.28) is satisfied for all $\mu_i < 1$ whenever $1 + 2\omega < 2$. This sufficient condition is in turn equivalent to $\omega < 1/2$ and successively:

$$\begin{aligned} Rc_1\theta + \rho\gamma\alpha &< \frac{1}{2}(b + 4c_1\alpha\theta)\left(\bar{w} - \frac{R}{2\alpha}\right) \\ 2\alpha\rho\gamma &< b\left(\bar{w} - \frac{R}{2\alpha}\right) + 4\theta c_1(\alpha\bar{w} - R). \end{aligned} \quad (3.23)$$

We also have to check the condition (2.27). With the current form of G , this is:

$$\left|1 - 2\alpha\frac{\partial r}{\partial H}(1, H^\infty)\right| < 1 \quad \iff \quad \left|1 - \frac{2\alpha c_1}{1 + 4c_1\alpha\theta}\right| < 1$$

which is the condition encountered in (3.16). We have proved the following result: sufficient conditions for convergence that are independent of the values of μ_i .

Proposition 15. *When players are symmetric, the Conjectural Learning procedure with conjecture in (2.24) converges locally to the symmetric solution (3.21) under the following conditions: a) $\theta > (1 - (b/\alpha c_1))/4$; b) either $\bar{w} < R/(2\alpha)$ and (3.22) or $\bar{w} > R/(2\alpha)$ and (3.23).*

For the completely symmetric case, Proposition 9 states that Condition (2.27) is sufficient. We then have the simpler result:

Lemma 4. *When players are symmetric with in addition $\bar{w}_i = \bar{w}_j$ and $\beta_i^0 = \beta_j^0$, the Conjectural Learning procedure with conjecture in (2.24) converges locally if $\theta > (1 - (b/\alpha c_1))/4$.*

A final note on the non-symmetric solutions of fixed-point equations (3.18)–(3.20): these solutions are repelling for all values of learning speeds μ_i , at least for some configurations of parameters. See Appendix B.1.2.

3.2.3 Conjecture “imitate w ”

With a conjecture as in (2.30), the maximization problem (2.4) is:

$$\max_{w_i^t} \left[a_i w_i^t - \frac{b_i}{2} (w_i^t)^2 - [c_0 - c_1(H^t + \theta(R - \alpha(2w_i^t + \beta_i^t)))] w_i^t + \rho\gamma_i [H^t + R - \alpha(2w_i^t + \beta_i^t)] \right].$$

It is quadratic in w_i^t with leading coefficient $-b_i/2 - 2\alpha c_1 < 0$. It is therefore strictly concave and the Conjectural Learning procedure is well-defined. In the interior case, solving the first-order condition yields:

$$w_i^{t*} = r_i(\beta_i^t, H^t) = \frac{1}{4\alpha c_1\theta + b_i} (a_i - c_0 + c_1 H^t + Rc_1\theta - \alpha\theta c_1\beta_i^t - 2\alpha\rho\gamma_i). \quad (3.24)$$

This is clearly a continuous function, so that the Conjectural Learning procedure is regular.

In fact, it turns out that the recurrence generated by the Conjectural Learning procedure is linear. Indeed, the equations are:

$$\begin{aligned} w_1^{*t} &= \frac{1}{4\alpha c_1 \theta + b_1} (a_1 - c_0 + Rc_1 \theta - 2\alpha\rho\gamma_1 + c_1 H^t - \alpha c_1 \theta \beta_1^t) \\ w_2^{*t} &= \frac{1}{4\alpha c_1 \theta + b_2} (a_2 - c_0 + Rc_1 \theta - 2\alpha\rho\gamma_2 + c_1 H^t - \alpha c_1 \theta \beta_2^t) \\ \beta_1^{t+1} &= (1 - \mu_1)\beta_1^t + \mu_1 (w_2^{*t} - w_1^{*t}) \\ \beta_2^{t+1} &= (1 - \mu_2)\beta_2^t + \mu_2 (w_1^{*t} - w_2^{*t}) \\ H^{t+1} &= H^t + R - \alpha(w_1^{*t} + w_2^{*t}). \end{aligned}$$

Thus, the recurrence can be written as:

$$\begin{aligned} \begin{pmatrix} \beta_1^{t+1} \\ \beta_2^{t+1} \\ H^{t+1} \end{pmatrix} &= \begin{pmatrix} 1 - \mu_1 + \mu_1 \alpha \theta c_1 \phi_1 & -\mu_1 \alpha \theta c_1 \phi_2 & \mu_1 c_1 (\phi_2 - \phi_1) \\ -\mu_2 \alpha \theta c_1 \phi_1 & 1 - \mu_2 + \mu_2 \alpha \theta c_1 \phi_2 & \mu_2 c_1 (\phi_1 - \phi_2) \\ \alpha^2 c_1 \theta \phi_1 & \alpha^2 c_1 \theta \phi_2 & 1 - \alpha c_1 (\phi_1 + \phi_2) \end{pmatrix} \begin{pmatrix} \beta_1^t \\ \beta_2^t \\ H^t \end{pmatrix} \\ &+ \begin{pmatrix} \mu_1 (\phi_2 A_2 - \phi_1 A_1) \\ \mu_2 (\phi_1 A_1 - \phi_2 A_2) \\ R - \alpha (\phi_1 A_1 + \phi_2 A_2) \end{pmatrix} \end{aligned} \quad (3.25)$$

where

$$\phi_i = \frac{1}{4\alpha c_1 \theta + b_i}, \quad \text{and} \quad A_i := a_i - c_0 + Rc_1 \theta - 2\alpha\rho\gamma_i, \quad i = 1, 2.$$

Consequently, the steady-state equations turn out to be linear and admit a unique solution:

$$\beta_1^\infty = -\beta_2^\infty = \frac{1}{b_1 + b_2 + 4c_1 \alpha \theta} \left((b_1 - b_2) \frac{R}{\alpha} - 2(a_1 - a_2) + 4\rho\alpha(\gamma_1 - \gamma_2) \right) \quad (3.26)$$

$$w_i^\infty = \frac{R}{2\alpha} - \frac{\beta_i^\infty}{2} \quad (3.27)$$

$$c_1 H^\infty = c_0 + c_1 \theta R + (b_1 + b_2) \frac{R}{4\alpha} - \frac{a_1 + a_2}{2} + \rho\alpha(\gamma_1 + \gamma_2) - \frac{b_1 - b_2}{4} \beta_1^\infty. \quad (3.28)$$

Remark 7. Equation (3.28) highlights the effects of asymmetry on the steady-state value of the stock. When players are symmetric, this steady-state stock coincides with the value H^{P^∞} given by (3.13) and $w_i^\infty = R/2\alpha$. This was predicted by Proposition 5, since, because of symmetry, $\partial F_i / \partial w_j = \partial F_j / \partial w_i$ and since the conjecture structurally satisfies $\partial \chi_i / \partial w_i = 1$. Moreover, in the non-symmetric case, it is sufficient that $b_1 = b_2$ to have $H^{P^\infty} = H^\infty$. This condition does not imply that $w_i^{P^\infty} = w_i^\infty$, though.

Convergence. We now apply Proposition 11 to obtain conditions for convergence. Since the recurrence is linear, local convergence conditions are actually global.

Proposition 16. *Consider the Conjectural Learning procedure defined above in the symmetric case. If $\theta > (1 - b/(\alpha c_1))/4$, then the procedure converges.*

Proof. This Conjectural Learning procedure is a linear recurrence of the form $X^{t+1} = JX^t + K$, where the matrix J coincides with the Jacobian matrix as defined in (A.9). The recurrence converges (globally) if Condition (2.27) and Conditions (2.32)–(2.33) of Proposition 11 are satisfied. Condition (2.27) is equivalent to (3.16), as it was in the proof of Proposition 8.

For the two remaining conditions, the critical value is $\omega = (\partial r_i)/(\partial \beta)(\beta^\infty, H^\infty) = -\alpha\theta c_1 / (b + 4\alpha\theta c_1)$. Since this is negative, Condition (2.32) is automatically satisfied. It is readily seen that $\omega \in (-1/4, 0)$, so that Condition (2.33) is satisfied as well. This concludes the proof. \square

3.2.4 Affine conjecture on H

We now turn to conjectures that are a function of H alone. We adopt the following form:

$$w_j^{AH} = \beta_i(H_t + R), \quad i \neq j.$$

This is an instance of the general form (2.34), with $\bar{H}_i = -R$, $\bar{w}_i = 0$, $i = 1, 2$. In this form, the the conjecture of Player i is that her opponent will consume some proportion of the stock available taking into account the recharge R .

With this conjecture, the maximization problem of Player i is:

$$\max_{w_i} \left[a_i w_i^t - \frac{b_i}{2} (w_i^t)^2 - \left[c_0 - c_1(H^t + \theta(R - \alpha(w_i^t + w_j^{t,AH}))) \right] w_i^t + \rho\gamma_i [H^t + R - \alpha(w_i^t + w_j^{t,AH})] \right].$$

This problem is quadratic and strictly concave since $b_i/2 + \alpha\theta c_i > 0$. Its solution is then unique and the Conjectural Learning procedure based on it is well-defined. In the interior case, first order conditions (2.5) give the optimal action:

$$w_i^{t*} = r_i(\beta_i^t, H^t) = \frac{a_i - c_0 + c_1(H^t + \theta R - \alpha\beta_i^t\theta(H^t + R)) - \rho\alpha\gamma_i}{b_i + 2c_1\alpha\theta}. \quad (3.29)$$

This is a continuous (actually, linear) function β_i and H . Therefore, the Conjectural Learning procedure is regular.

The steady-state equations are then:

$$\begin{aligned} w_i^\infty &= \beta_j^\infty(H^\infty + R), \quad i \neq j \\ w_i^\infty + w_j^\infty &= \frac{R}{\alpha} \\ w_i^\infty &= \frac{a_i - c_0 + c_1(H^\infty + \theta R - \alpha\beta_i^\infty\theta(H^\infty + R)) - \rho\alpha\gamma_i}{b_i + 2c_1\alpha\theta}, \quad i = 1, 2. \end{aligned}$$

Solving these, we find as unique steady state:

$$\begin{aligned} w_i^{AH\infty} &= \frac{1}{b_1 + b_2 + 2\alpha\theta c_1} \left(a_i - a_j - \alpha\rho(\gamma_i - \gamma_j) + \frac{R}{\alpha}(b_j + \alpha\theta c_1) \right) \\ c_1 H^{AH\infty} &= c_0 + \frac{(b_1 + \alpha\theta c_1)(b_2 + \alpha\theta c_1)}{b_1 + b_2 + 2\alpha\theta c_1} \frac{R}{\alpha} - \frac{(a_1 - \alpha\rho\gamma_1)(b_2 + \alpha\theta c_1) + (a_2 - \alpha\rho\gamma_2)(b_1 + \alpha\theta c_1)}{b_1 + b_2 + 2\alpha\theta c_1} \\ \beta_i^{AH\infty} &= \frac{w_i^{AH\infty}}{H^{AH\infty} + R}. \end{aligned}$$

Remark 8. These limits coincide with the quantities in (3.6) and (3.5). This is in accordance with Proposition 3 and Proposition 12: when it converges, this CLP does to the nearsighted Nash steady state. No symmetry condition is required.

Remark 9. Note that with this conjecture we do not have the property $\beta_i^\infty \beta_j^\infty = 1$, as in the conjecture of Section 2.3.1. However, we have:

$$\frac{\beta_i^{AH\infty}}{\beta_j^{AH\infty}} = \frac{w_i^{AH\infty}}{w_j^{AH\infty}}.$$

Convergence. In order to state sufficient conditions for the convergence of the current conjectural learning scheme, we apply Proposition 13. We restrict the discussion to the symmetric case. The convergence result can then be stated as:

Proposition 17. *Consider the Conjectural Learning procedure defined above in the symmetric case. Let $(\beta_i^{AH\infty}, w_i^{AH\infty}, H^{AH\infty})$ be its unique steady state, and assume $H^{AH\infty} > 0$. Then if both $\theta \geq (3 - 2b/(\alpha c_1))/4$ and $\theta \leq b/(\alpha c_1)$, the procedure converges locally.*

The proof of this result is in Appendix B.2.

3.2.5 Farsighted solutions

The solutions for farsighted Nash and farsighted cooperative (Pareto) solutions can be obtained using either the maximum principle, or fixed-point formulas based on the fact that the value function is quadratic (see *e.g.* [1, Section 6.2.3]).

The solution yields the following values for the steady-state stock and the state-feedback control law (the label ‘‘FP’’ stands for farsighted Pareto):

$$H^{FP\infty} = \frac{(c_0 - a_1)b_2 + (c_0 - a_2)b_1}{c_1(b_1 + b_2)} + \frac{Rb_1b_2}{\alpha c_1(b_1 + b_2)} + R\theta + \frac{R\rho}{1 - \rho}. \quad (3.30)$$

$$w_i^{FP\infty} = \frac{Rb_j + (a_i - a_j)\alpha}{\alpha(b_1 + b_2)} = \frac{R}{2\alpha} - \frac{R(b_i - b_j)}{2\alpha(b_1 + b_2)} + \frac{a_i - a_j}{b_1 + b_2}. \quad (3.31)$$

In the symmetric situation, the first formula reduces to:

$$H^{FP\infty} = \frac{c_0 - a}{c_1} + \frac{Rb}{2\alpha c_1} + R\theta + \frac{R\rho}{1 - \rho}. \quad (3.32)$$

For the Nash-feedback solution, no such explicit formulas seem to exist. In the experiments of Section 4, we used a numerical solution.

3.2.6 Summary and comparisons

In this section, we compile results obtained for six dynamic procedures analyzed thus far: two nearsighted, three Conjectural Learning and the cooperative farsighted. We leave out the Nash farsighted procedure for lack of explicit enough formulas.

In the general (non-symmetric) case, from Proposition 3 we know that the steady state of the Conjectural Learning procedure affine in H coincides with that of the nearsighted Nash scheme. In the symmetric case, Proposition 5 implies that the steady state of Conjectural Learning procedures depending on w coincide with that of the nearsighted Pareto scheme. Finally, the comparison reduces to the three complete information schemes.

From the comparison of the different formulas, we sketch a parametric discussion about feedback laws, asymptotic consumption and stocks. Comparison of profits will be the topic of the experiments in Section 4.

Comparisons of feedback laws. The feedback laws and best-reply functions are as follows. The superscripts “aw”, “iw” and “aH” refer to “affine in w ”, “imitate w ” and “affine in H ”, respectively.

$$\begin{aligned} \text{sol}_i^N(H) &= \frac{b_j + \alpha\theta c_1}{3\alpha^2\theta^2 c_1^2 + 2\alpha\theta c_1(b_1 + b_2) + b_1 b_2} (a_i - c_0 + c_1 H + R c_1 \theta - \alpha\gamma_i \rho) \\ &+ \frac{\alpha\theta c_1 (a_i - a_j - \alpha\rho(\gamma_i - \gamma_j))}{3\alpha^2\theta^2 c_1^2 + 2\alpha\theta c_1(b_1 + b_2) + b_1 b_2} \end{aligned} \quad (3.4)$$

$$\text{sol}_i^P(H) = \frac{b_j (a_i - c_0 + c_1 H + R c_1 \theta - \alpha\rho(\gamma_1 + \gamma_2)) + 2\alpha\theta c_1 (a_i - a_j)}{2\alpha\theta c_1 (b_1 + b_2) + b_1 b_2} \quad (3.9)$$

$$r_i^{aw}(\beta_i, H) = \frac{1}{b_i + 2\alpha c_1 \theta (1 + \beta_i)} (a_i - c_0 + c_1 H + R c_1 \theta - \alpha c_1 \theta (\bar{w}_j - \beta_i \bar{w}_i)) - \rho \alpha \gamma_i (1 + \beta_i) \quad (3.17)$$

$$r_i^{iw}(\beta_i, H) = \frac{1}{b_i + 4\alpha c_1 \theta} (a_i - c_0 + c_1 H + R c_1 \theta - \alpha c_1 \theta \beta_i - 2\alpha \rho \gamma_i) \quad (3.24)$$

$$r_i^{aH}(\beta_i, H) = \frac{1}{b_i + 2\alpha c_1 \theta} (a_i - c_0 + c_1 H + R c_1 \theta - \alpha c_1 \beta_i \theta (H + R)) - \rho \alpha \gamma_i. \quad (3.29)$$

The feedback law for the farsighted cooperative solution does not appear here for simplicity. In the symmetric case, the nearsighted feedback laws have a simpler form:

$$\text{sol}_i^N(H) = \frac{1}{b + 3\alpha\theta c_1} (a - c_0 + c_1 H + R c_1 \theta - \alpha\rho\gamma) \quad (3.7)$$

$$\text{sol}_i^P(H) = \frac{1}{b + 4\alpha c_1 \theta} (a - c_0 + c_1 H + R c_1 \theta - 2\alpha\rho\gamma). \quad (3.12)$$

Comparing these functions, we see that the sensitivity to the stock (slope) is larger for the Nash solution, and that the intercept (when $H = 0$) is larger as well. This is consistent in the sense that the Nash behavior is expected to be more aggressive on the resource than the Pareto behavior.

Also, comparing function $r_i^{iw}(\cdot)$ in (3.24) (in the symmetric case) and function $\text{sol}_i^P(\cdot)$ in (3.12), we observe a large similarity, the only difference being the term $-\alpha\theta c_1 \beta_i$ in the numerator of (3.24). In this symmetric case, the steady-state of the CPL “imitate w ” is $\beta_i = 0$ (see (3.26)). So, as convergence occurs, β_i^t tends to 0 and the reactions coincide for both schemes. This explains the similarity in the simulation results which we will observe in Section 4.

Comparisons in w^∞ . The steady-state consumption levels are obtained for nearsighted and farsighted schemes, in the general (non-symmetric) case, in (3.6), (3.11) and (3.31):

$$w_i^{N\infty} = \frac{1}{b_1 + b_2 + 2\alpha\theta c_1} \left(a_i - a_j - \alpha\rho(\gamma_i - \gamma_j) + \frac{R}{\alpha} (b_j + \alpha\theta c_1) \right) \quad (3.6)$$

$$w_i^{P\infty} = \frac{R}{2\alpha} - \frac{R(b_i - b_j)}{2\alpha(b_i + b_j)} + \frac{a_i - a_j}{b_i + b_j} \quad (3.11)$$

$$w_i^{FP\infty} = \frac{R}{2\alpha} - \frac{R(b_i - b_j)}{2\alpha(b_1 + b_2)} + \frac{a_i - a_j}{b_1 + b_2}. \quad (3.31)$$

We observe in (3.11) and (3.31) that the extraction at the steady is the same for both cooperative situations: nearsighted and farsighted. This is somewhat surprising, since the first one depends on the subjective parameter γ , whereas the second one does not. The parameters γ_i do appear in $w_i^{N\infty}$.

Comparisons in H^∞ . Collecting results in (3.5), (3.10) and (3.30),

$$H^{N\infty} = \frac{c_0}{c_1} + \frac{(b_1 + \alpha\theta c_1)(b_2 + \alpha\theta c_1)}{c_1(b_1 + b_2 + 2\alpha\theta c_1)} \frac{R}{\alpha} - \frac{(a_1 - \alpha\rho\gamma_1)(b_2 + \alpha\theta c_1) + (a_2 - \alpha\rho\gamma_2)(b_1 + \alpha\theta c_1)}{c_1(b_1 + b_2 + 2\alpha\theta c_1)} \quad (3.5)$$

$$H^{P\infty} = \frac{c_0}{c_1} - \frac{a_1 b_2 + a_2 b_1}{c_1(b_1 + b_2)} + \frac{R b_1 b_2}{\alpha c_1 (b_1 + b_2)} + R\theta + \frac{\rho(\gamma_1 + \gamma_2)\alpha}{c_1} \quad (3.10)$$

$$H^{FP\infty} = \frac{c_0}{c_1} - \frac{a_1 b_2 + a_2 b_1}{c_1(b_1 + b_2)} + \frac{R b_1 b_2}{\alpha c_1 (b_1 + b_2)} + R\theta + \frac{R\rho}{1 - \rho}. \quad (3.30)$$

Observe that the stock $H^{P\infty}$ depends on the parameters γ_i only through their sum, and is thus independent of the way this sum is distributed between farmers. This was also the case for the feedback law $\text{sol}_i^P(H)$ in (3.9). In the symmetric case, we have:

$$H^{N\infty} = \frac{c_0 - a}{c_1} + \frac{Rb}{2\alpha c_1} + \frac{\theta R}{2} + \frac{\rho\gamma\alpha}{c_1} \quad (3.8)$$

$$H^{P\infty} = \frac{c_0 - a}{c_1} + \frac{Rb}{2\alpha c_1} + \theta R + 2 \frac{\rho\gamma\alpha}{c_1} \quad (3.13)$$

$$H^{FP\infty} = \frac{c_0 - a}{c_1} + \frac{Rb}{2\alpha c_1} + \theta R + \frac{\rho R}{1 - \rho}. \quad (3.32)$$

Pareto vs Nash. We start with comparing $H^{N\infty}$ and $H^{P\infty}$ in the general case. We can easily see that

$$a_1 = a_2, \quad \text{or} \quad b_1 = b_2, \quad \text{or} \quad \theta = 0 \quad \text{implies} \quad H^{P\infty} > H^{N\infty}.$$

In a more general context we can prove that when $\rho = 0$, and provided that $a_i > c_0$,

$$R > \frac{\alpha[\alpha\theta(a_1 + a_2 - 2c_0)c_1 + b_1(a_2 - c_0) + b_2(a_1 - c_0)]}{(\alpha\theta c_1 + b_2)(\alpha\theta c_1 + b_1)} \quad \text{implies} \quad H^{P\infty} > H^{N\infty} > 0.$$

Also, it can be proved that:

$$\frac{\partial(H^{P\infty} - H^{N\infty})}{\partial\rho} > 0.$$

From these two facts, we conclude that $H^{P\infty} > H^{N\infty} > 0$ for all $0 \leq \rho < 1$.

Complete vs Incomplete Information. There exist values of the parameters γ_i such that the steady-state stock of nearsighted and conjectural schemes coincides with that of the cooperative solution. In the general case, from the comparison of (3.10) and (3.30) we see that when $\gamma_1 + \gamma_2 = Rc_1/(\alpha(1 - \rho))$, the steady-states H^∞ are the same.

In the symmetric case, we find:

$$H^{FP\infty} - H^{P\infty} = \frac{\rho R}{1 - \rho} - \frac{2\rho\gamma\alpha}{c_1} = 0 \quad \iff \quad \rho = 0 \quad \text{or} \quad \gamma^P = \frac{Rc_1}{2\alpha(1 - \rho)}$$

$$H^{FP\infty} - H^{N\infty} = \frac{\rho R}{1-\rho} + \frac{\theta R}{2} - \frac{\rho\alpha\gamma}{c_1} = 0 \iff \gamma^N = \frac{Rc_1}{\alpha(1-\rho)} + \frac{Rc_1\theta}{2\alpha\rho}$$

$$\gamma^N - \gamma^P = \frac{Rc_1}{2\alpha(1-\rho)} + \frac{Rc_1\theta}{2\alpha\rho} > 0.$$

In order to obtain the same steady state in the state variable, the valuation of the scrap value must be greater in the case of conjectures in H than in the case of conjectures in w . This is again expected since the Nash behavior is more aggressive on the resource, and for each γ , $H^N(\gamma) < H^P(\gamma)$.

4 Numerical experiments

In complement to the theoretical analysis of Section 3.2, this section reports numerical simulations of the seven dynamic schemes introduced in Section 2. The purpose is to compare the “performance” of agents using these schemes. The criteria we use for this are discussed in Section 4.1. In addition to the model parameters, each agent behavior has its own set of parameters. This opens a multitude of possibilities for simulations. Consistently with our theoretical analysis, we choose to focus on *intrinsic* properties of the behaviors. Namely, we restrict our attention to strategic interactions in which agents engage in the same type of behavior. Moreover, we consider mostly symmetric agents. The study of asymmetric situations where agents of different types confront each other is beyond the scope of this paper.

In the rest of this section, we first present our point of view on how to evaluate the different situations (Section 4.1). We briefly discuss interior solutions and valid trajectories (Section 4.2). Then we present the parameters used in the simulations (Section 4.3) and the results of the simulations (Section 4.4).

4.1 Evaluation

The “performance” of a learning scheme can be assessed using many criteria. Taking into account environmental and economic concerns, we select for our comparisons: a) the limit/steady state stock of water H^∞ , representing the environment, and b) the total discounted sum of profits for both players, representing the welfare of the society.⁴ Accordingly, we perform simulations for a certain number T of time steps, and we record: a) the values of H^T as a proxy to H^∞ , and b) the total discounted welfare

$$V^T := \sum_{i=1}^2 \sum_{t=0}^T \rho^t \pi_i(w_i^t, w_j^t, H^t).$$

The parameter T is chosen in a way that convergence of the dynamical systems and the measured quantities is observed. So, for practical purposes, $H^T \simeq H^\infty$ and $V^T \simeq V^\infty$, where V^∞ is the joint profit optimized in (2.40).

The graphs reported below represent the result of a given simulation in the plane (H^∞, V^∞) . We do not claim that one scheme is better than another: in the spirit of multi-criteria evaluation, we associate to each situation a vector of values, and leave it to some decision maker to determine its best compromise or take into account other aspects of the problem.

⁴In the symmetric situations, the graphs will display the welfare of each player, that is, $V^\infty/2$.

For some situations, simulation is actually not necessary since theory provides the metrics we need. This is the case for H^∞ , which is predicted in each situation, see Section 3.2.6. This is also the case when the behavior of agents is a linear feedback of the state: this happens for complete information behaviors, nearsighted and farsighted. In that case, the formulas of Appendix B.3 allow to calculate the total gain. In all situations where an explicit formula is available, we have used it for double checking the simulation.

4.2 Valid trajectories

The analysis of Section 3.2 has largely ignored the constraints on controls and states of the physical problem: it has assumed interior solutions for maximization problems, and there is no statement about the feasibility of (candidate) optimal trajectories, except near steady states.

In the experiments of this section, the trajectories are checked for the following properties:

- $H^t \geq 0$ and $(1 - \theta)H^t + \theta H^{t+1} \leq c_0/c_1$ for all t , implying $C((1 - \theta)H^t + \theta H^{t+1}) \geq 0$ for all t ;
- $w^t \geq 0$ for all t .

The diagrams will not present results corresponding to trajectories that are not valid.

4.3 Parameters and experimental plan

The parameters chosen for the experiments are summarized in Table 1.

Parameter	interpretation	value
R	recharge	1
α	pumping return flow coefficient	1
a	marginal productivity of water	1
b	productivity saturation coefficient	0.04
c_0	baseline unit pumping cost	1
c_1	pumping cost increase per water table unit	0.01
H^0	initial water table level	30
θ	weight of future stock	1/2
ρ	discount factor	0.8 and 0.95
γ	subjective unit value of future stock	0 to 2
μ_i	learning rate	0.1 to 0.9
T	simulation time horizon	100

Table 1: Parameters of the experiments

Note that the form chosen for the unit pumping cost, namely $c_0 - c_1H$, implies that the water table is assumed to have maximum height $H_{max} = c_0/c_1$. With the parameters of Table 1, $H_{max} = 100$, which corresponds to 100 years of recharge since $R = 1$. The weight parameter $\theta = 1/2$ models a pumping uniformly distributed over the time period.

The learning procedures have proper parameters: the learning rates $\mu_i \in (0, 1]$ and the scheme-specific parameter, β_i , at the beginning of the process. We generally choose the initial value β_i^0 to be: a) relatively far from the steady state, and b) meaningful with respect to the learning procedure. For the CLP affine in w , the steady state is 1, and we pick $\beta_i^0 = 0$.

This conjecture corresponds to an agent believing the other one will not react to her choices. For the CLP “imitate w ”, the steady state is 0, and we pick $\beta_i^0 = 2$, a reasonable shift in consumption, given the initial value of the stock. Finally, for the CLP affine in H , the steady state is of the order of $1/H^\infty$, therefore quite close to 0. So we pick $\beta_i^0 = 1/2$, meaning that the agent expects that the other one will consume half of the stock.

Convergence. The parameters of Table 1 satisfy the convergence conditions of Proposition 14, 15, 16 and 17. These results state only a local convergence (except for Proposition 16) but convergence has been observed from the initial values β_i^0 in all situations. Some trajectories were not valid however, as discussed in Section 4.2.

Experiments. In our experiments, we test the sensitivity of the “performance” vector (H, V) with respect to μ_i (learning speed parameter), and γ . The parameter γ is relevant to nearsighted schemes and conjectural learning procedures. When γ varies, the vector $(H(\gamma), V(\gamma))$ draws a parametric curve in the diagrams. On the other hand, the farsighted schemes do not depend on γ , and will serve as comparison benchmarks.

The theoretical analysis of Section 3.2 provides insights allowing to anticipate some results. First of all, the value H^∞ turns out to be *linear and increasing* as a function of γ : see formulas (3.8) and (3.13) in Section 3.2.6. However, the linearity coefficient is not the same: $\rho\alpha/c_1$ for the nearsighted Nash scheme and the conjectural learning procedures with conjectures in H , and $2\rho\alpha/c_1$ for the nearsighted Pareto scheme and the conjectural learning procedures with conjectures in w . This means that a same value of H^∞ will be obtained for: either the same values of γ , or different values, depending on the scheme adopted.

Second, as mentioned in Section 4.1, the behavior of nearsighted agents is an affine state feedback of the form $\nu_0 + \nu_1 H$. Checking again Section 3.2.6, we see that the value of ν_0 is also affine in γ . The value V^{ν_0, ν_1} is quadratic and concave in ν_0 : this can be deduced from (B.18) in Appendix B.3. Consequently, $V(\gamma)$ is quadratic and concave as a function of $H^\infty(\gamma)$.

As it appears, this concave dependency is a general feature of all dynamic schemes. In the simulations reported below, we display the parametric curve $(H^\infty(\gamma), V(\gamma))$ in different situations.

4.4 Results

In the presentation of the results, we will use the following keys for naming the behavior of agents:

NS	nearsighted
FS	farsighted
CLP-AH	conjectural learning procedure, affine in H
CLP-Aw	conjectural learning procedure, affine in w
CLP-Iw	conjectural learning procedure, imitate w

4.4.1 Complete information behaviors

We compare in this section the performance realized by agents adopting one of the behaviors requiring complete knowledge of the opponent’s utility: the nearsighted and farsighted procedures. The nearsighted procedures depend on the parameter γ which weights the importance of the remaining stock. Accordingly, Figure 1 represents the total discounted economic profit V^∞ and H^∞ as a function of γ .

Arrows are used to show how H^∞ depends on γ . The larger γ is, the larger the difference, as predicted in Section 4.3.

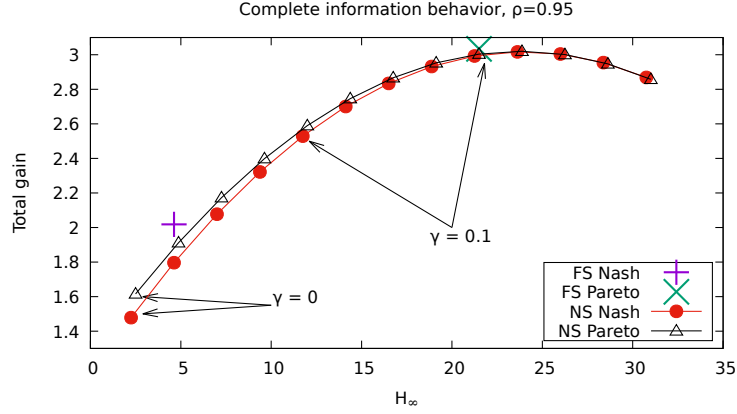


Figure 1: $(H^\infty(\gamma), V(\gamma))$ for complete information schemes

It is observed that when $\gamma = 0$ (the agent does not care about the future at all), the performance is worse than the Nash-feedback farsighted performance, in both the economic and the environmental metrics. However, for certain values of γ , the performance of nearsighted agents can get very close to the cooperative farsighted performance. The values of γ for which the largest value of $V(\gamma)$ is observed are $\gamma^P \simeq 0.112$ and $\gamma^N \simeq 0.229$ for the Pareto and Nash nearsighted schemes, respectively. The respective values are $V^P \simeq 3.0203$ with $H^{P\infty} \simeq 23.775$ and $V^N = 3.0173$ with $H^{N\infty} \simeq 23.996$, to be compared with the farsighted cooperative value $V^{FP} \simeq 3.0346$ and $H^{FP\infty} = 21.5$.

For larger values of γ , corresponding to larger values of $H^\infty(\gamma)$, trajectories cease to be valid. In those cases, $H^0 < H^\infty(\gamma)$ and we observe with the chosen parameters that the optimal feedback prescribes a negative extraction.

4.4.2 Incomplete information behaviors

We now compare the performance of learning agents. These Conjectural Learning procedures have their own parameters in addition to the parameters of the utility functions, namely, learning parameters μ_i and initial values of the learning functions, commonly referred to as: β_i^0 , $i = 1, 2$. We only examine the sensitivity of the performance of agents with respect to parameters μ_i . Figures 2, 3 and 4 display the total discounted economic profit and H^∞ as a function of γ and for different values of μ_i . The three CLPs of Section 2.3 are successively represented: CLP affine in w (Sections 2.3.1 and 3.2.2), “imitate w ” (Sections 2.3.2 and 3.2.3) and affine in H (Sections 2.3.3 and 3.2.4).

For the CLP affine in w , it is observed that smaller values of μ (hence a slow learning) can yield both high values of the profit and the stock, for certain values of γ . In that case, the trajectories of stocks and controls remain valid. The performance is not very sensitive to larger values of μ , but the larger values of γ result in invalid trajectories.

For the CLP “imitate w ”, there is almost no sensitivity to the value of μ . This is explained by the relatively small weight of the term $\alpha c_1 \theta \beta_i$ in front of $c_1 H$ in the best-reply function r_i^{iw} , see (3.24).

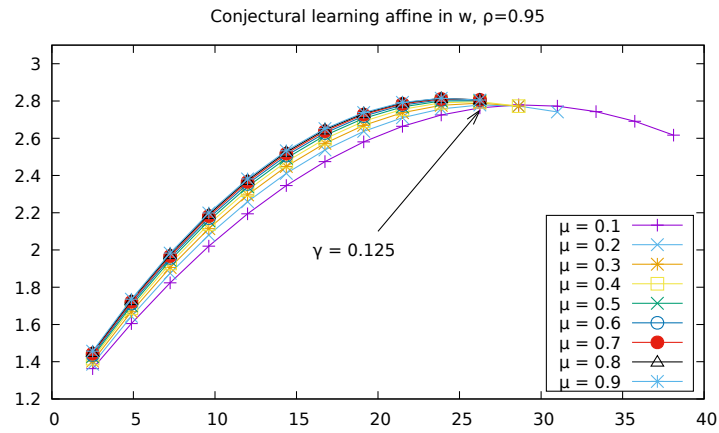


Figure 2: $(H^\infty(\gamma), V(\gamma))$. Learning affine in w .

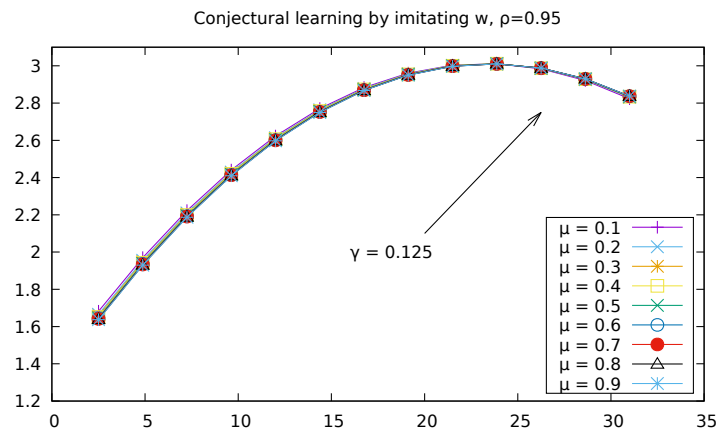


Figure 3: $(H^\infty(\gamma), V(\gamma))$. Learning “imitate w ”.

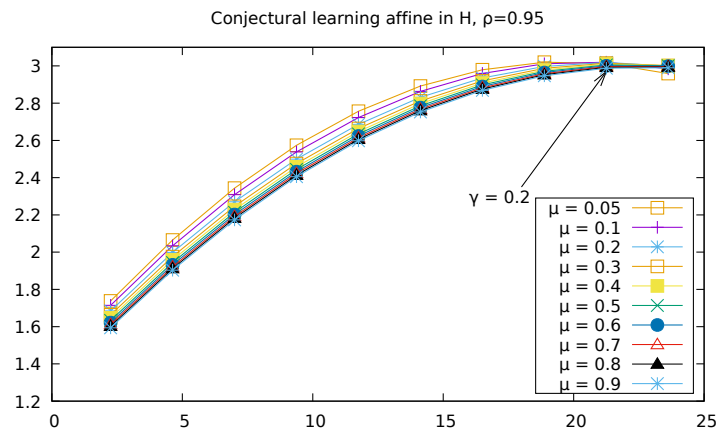


Figure 4: $(H^\infty(\gamma), V(\gamma))$. Learning affine in H .

Finally, the the CLP affine in H has the feature that smaller learning rates result in larger total profits, for almost all values of the parameter γ . The sensitivity to μ however tends to vanish where the profit is the largest.

In the following Figure 5, we superimpose the different learning behaviors for selected values of μ . We conclude that, for a *given* value of the asymptotic environmental metric H^∞ , both the CLP affine in H and the CLP “imitate w ” yield a higher economic performance than the CLP affine in w . As observed above, the performance of the first scheme does not depend much on μ , so we selected the value $\mu = 0.9$ for it. We selected $\mu = 0.1$ which yields the best maximum for the CLP affine in H . The maximum value of the gain V is realized at $\gamma^{aw} = 0.125$ with $V^{aw} \simeq 2.8050$ and $H^\infty = 26.25$, $\gamma^{aH} = 0.2$ with $V^{aH} \simeq 3.0019$ and $H^\infty = 21.25$, $\gamma^{iw} = 0.100$ with $V^{iw} \simeq 2.9975$ and $H^\infty = 21.5$.

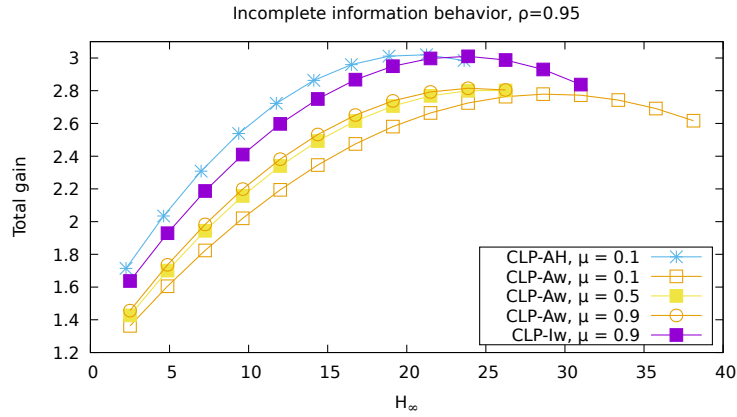


Figure 5: $(H^\infty(\gamma), V(\gamma))$. Comparison of Conjectural Learning procedures

4.4.3 Comparison of all schemes

Next, we compare the performance realized by all schemes discussed previously: farsighted, nearsighted and Conjectural Learning schemes. This appears in Figure 6. The insert is a close-up around the top of the curves.

From this figure, it is concluded that globally, nearsighted schemes, the CLP affine in H and the CLP “imitate w ” have about the same maximal performance with respect to the metric V . In the detail around the maximal performance, nearsighted schemes perform slightly better than the CLP affine in H , itself slightly superior to the CLP “imitate w ”. The value of the limit stock is however larger for the latter scheme.

We now give a look at the trajectories generated by the different schemes. We select for μ and γ the parameters that yields (approximately) the maximum total discounted profit $V(\gamma)$ in the previous experiments. For the CLPs affine in w and “imitate w ”, the value taken is $\gamma = 0.125$ (see Figure 2). For the CLP affine in H , this value is $\gamma = 0.2$ (see Figure 4). The comparison is done with the trajectories resulting from farsighted behaviors. The different trajectories are displayed in Figure 7. The insert shows a zoom of the first time units.

The trajectories obtained for the farsighted Nash behavior and for the Pareto optimum are strikingly different. The farsighted Nash trajectory exhibits a large consumption of the

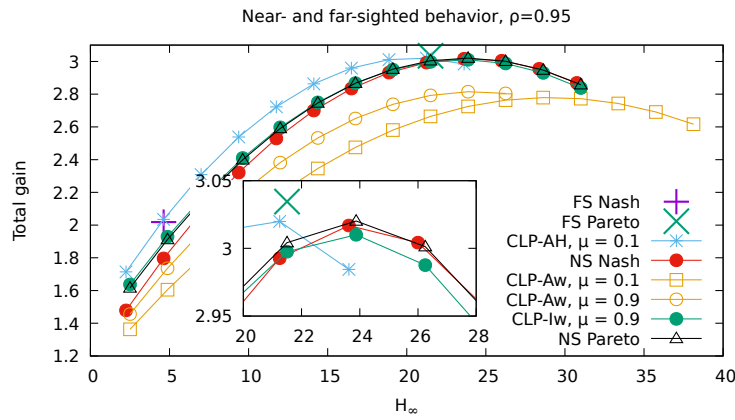


Figure 6: $(H^\infty(\gamma), V(\gamma))$. All behaviors, $\rho = 0.95$

resource and a sharp decrease of the stock, typical of the “tragedy of the commons”. The farsighted Pareto trajectory is more interesting. It starts with a parsimonious consumption, comparable with that of the nearsighted Pareto and “imitate w ” schemes. However, it maintains a relatively high level of consumption longer than these schemes, and finally reaches a lowest asymptotic resource.

The trajectory for the CLP affine in H is similar to the nearsighted Pareto one, in the sense that it has almost the same asymptotic value. The consumption is however smaller in the beginning but then increases so that both trajectories of the stock become superimposed.

The trajectory for the CLP affine in w is also different from the other ones. It starts with a sharp decrease in the stock, which reduces it actually below its asymptotic value. The following steps slowly recover the missing stock until convergence. This initial error in the conjecture is certainly what causes this learning scheme to have a substantially worse performance in terms of gain, see Figure 6.

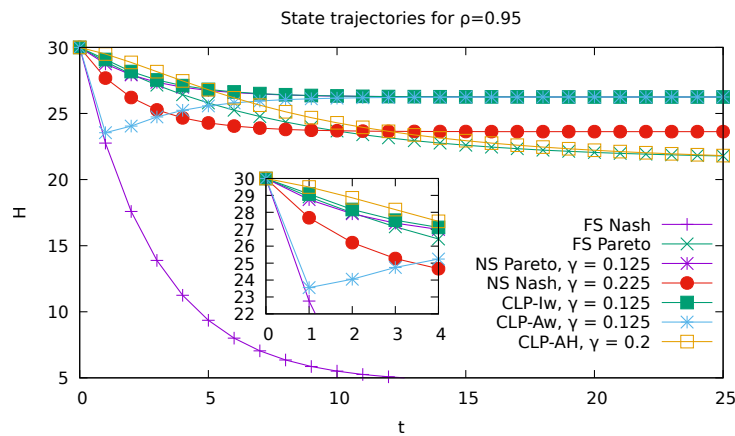


Figure 7: Evolution of the resource for the γ which gives the maximum profit. $\rho = 0.95$.

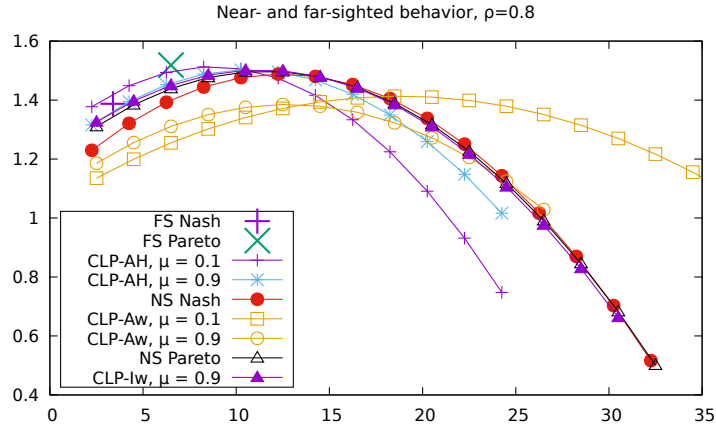


Figure 8: $(H^\infty(\gamma), V(\gamma))$. All behaviors, $\rho = 0.8$

4.4.4 Sensitivity with respect to the discount rate

Finally, we briefly have a look at the influence of ρ on the performance. The discount factor intervenes in the decision of shortsighted and learning agents as a coefficient affecting the valuation of the future stock, see (3.3).

Figure 8 displays the results for $\rho = 0.8$. Compare to Figure 6 for $\rho = 0.95$.

With such a reduced discount rate, the positions of farsighted Nash and Pareto performance points are much closer and have lower values of both asymptotic stock H^∞ and value. This is expected since agents are typically more aggressive when they discount more the future. The performance of nearsighted schemes and conjectural learning procedures in terms of value, can approach that of farsighted Pareto, but with a higher value of the asymptotic stock. This feature already present for $\rho = 0.95$ is magnified with $\rho = 0.8$. It is even more salient for smaller values of ρ (graphs not shown here). The values of γ that correspond to these best performances are smaller than for $\rho = 0.95$: it is consistent that agents should pay less attention to future stocks when the discount rate is smaller.

4.4.5 Asymmetric cases

We complete the experimental section with a brief investigation of the impact of asymmetry on the gain of the players. Specifically, we consider two asymmetric situations.

In a first experiment, we set the parameters of both players to the same values with the exception of a . We analyze the CLP “imitate w ” which seems to achieve the best performance in the symmetric case. We compare with the complete-information, nearsighted Pareto scheme, and the farsighted Pareto as benchmark. As in the previous experiments, we let γ vary. We report $H^\infty(\gamma)$, $V^1(\gamma)$, $V^2(\gamma)$ and the sum $V^1(\gamma) + V^2(\gamma)$. For a given value of γ , the value of $H^\infty(\gamma)$ is the same for the CLP and the nearsighted scheme. The result appears in Figure 9.

One striking feature of Figure 9 is the large asymmetry of gains for both players, given that the asymmetry in parameters is relatively small: $\sim 1\%$ for parameters a_i . Player 1 with a (slight) productivity advantage gets a large proportion of the total profit. The total gain appears to be still comparable with the optimal one (Farsighted Pareto solution), at least for a proper choice of the parameters γ_i .

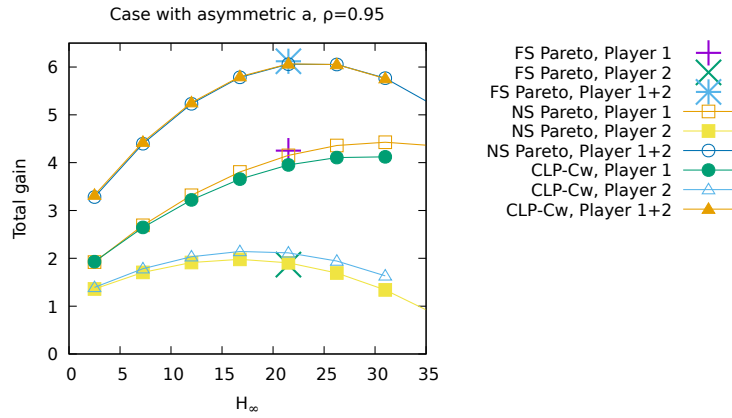


Figure 9: $(H^\infty(\gamma), V(\gamma))$. Asymmetric values of a

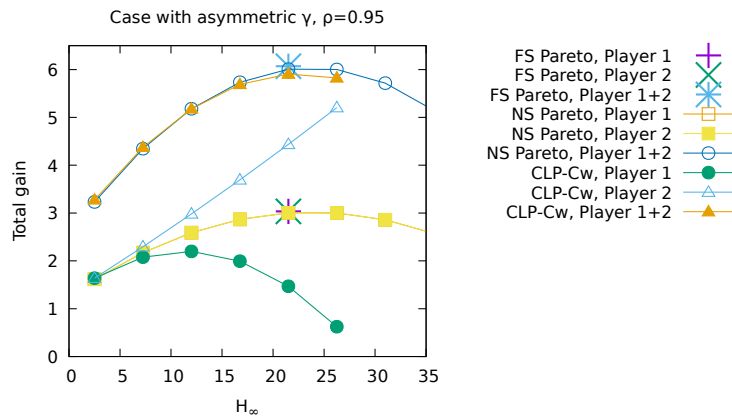


Figure 10: $(H^\infty(\gamma), V(\gamma))$. Asymmetric values of γ

In a second experiment, we set all parameters equal except γ . We assume that $\gamma_1 = (11/9)\gamma_2$. As above, we report results as a function of $\gamma = (\gamma_1 + \gamma_2)/2$. The result is displayed in Figure 10.

We still observe a large asymmetry of gains for both players having the CLP-Iw behavior, although parameters γ_i have only an asymmetry of order $\sim 10\%$. Player 1 with a larger concern for future stock gets a much smaller share of the total gain when her behavior is the CLP-Iw. When the behavior is nearsighted Pareto, players obtain the same share: this can be seen indeed from (3.9) and explained by the fact that the function to be optimized is symmetric in (w_1, w_2) when only γ_i differ. The maximal total gain attainable still appears to be comparable with the optimal one.

5 Discussion and extensions

We have formally defined and analyzed a group of behaviors for agents with limited rationality, engaged in a dynamic game with two players and one state variable. A first class of behaviors,

called 'nearsighted', assumes that agents have a complete knowledge of the opponent's profit function, but have a limited care for the future in their optimization. In the second class, agents do not have this information, and instead form conjectures on the behavior of their opponent. They do however observe this behavior, and revise their conjectures accordingly. We have determined the convergence properties of the dynamical systems generated by these behaviors. In particular, we have proved that the limit state of nearsighted schemes coincides with that of conjectural schemes under simple conditions on the nature of the conjecture.

We have applied these models of behavior to the problem of managing the extraction of groundwater between two producers, and proposed to evaluate the "performance" of agents in along the two dimensions of asymptotic stock of the resource, and total discounted profit. Since nearsighted agents typically over-exploit the resource (both in situations of complete and incomplete information), we have introduced in their utility function a valuation of the future resource, associated to a parameter γ .

The numerical simulations we have provided indicate that the simple agent behaviors we have introduced can perform almost as well as the optimal farsighted Pareto solution, in terms of profit, for certain values of this parameter γ . This performance is even associated to a *larger* asymptotic stock, which can be seen as an additional benefit. Among the incomplete-information learning schemes, the one affine in the stock H and the one consisting in copying the opponent's consumption w seem to perform best. The performance of the third learning scheme, the one affine in w , is hindered by potentially bad initial choices.

On the other hand, this parameter γ is subjective in nature, and real players have *a priori* no reason or means to choose it close to its "optimal" value. This may be an opportunity for developing public policies, either by suggesting some valuation γ or imposing it through taxes. In this situation, determining which behavior is followed by agents is essential, since the value of H^∞ depends both on γ and on this behavior.

Future investigations should investigate more elaborate agent behavior, for instance including elementary statistics on past observation. In the domain of conjectural learning procedures, conjectures with more than one parameter similarly pose the question of using several observations to adjust the agent's model of the opponent.

Finally, in the context of the application to resource management, it should be interesting to mix agents with different behaviors and study the impact of beliefs and shortsightedness on profits and the state of the resource.

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A Proofs of convergence results

We state in this appendix the technical theorems that support the convergence results of Section 2. We start with general matrix stability results. Next, we specialize these results to the different dynamical systems we have introduced.

A.1 Stability results

We propose in this section sufficient conditions for stability that will be used for the analysis of the different dynamical systems constructed in the paper. Those are based on the Geršgorin circle theorem (see e.g. [5, Section 6.1]) which we recall here for the sake of completeness:

Theorem 18 (Geršgorin). *Consider a $n \times n$ matrix A with complex entries. The spectrum of A lies in the union of the discs of centers A_{ii} and radiuses $\sum_{j \neq i} |A_{ij}|$, for $i = 1, \dots, n$.*

In the analysis of stability of discrete-time dynamic systems, we are interested in locating the spectrum of matrices inside the unit disk. We then have the following result as an obvious consequence of Geršgorin's theorem.

Theorem 19. *Consider a real, $n \times n$ matrix A . The spectrum of A lies in the unit disc if the following conditions are satisfied for all i :*

$$\sum_{j \neq i} |A_{ij}| < \min\{1 - A_{ii}, 1 + A_{ii}\}. \quad (\text{A.1})$$

It turns out that the dynamic systems we consider have variables of different nature: different units and different ranges. In that situation, the following modification of Theorem 19 is relevant. It results from the facts that: A and $A_\Sigma := \Sigma^{-1}A\Sigma$ have the same spectrum, for all nonsingular diagonal matrices Σ , and that if $\Sigma = \text{diag}(\sigma_i)$, the entries of A_Σ are $(A_\Sigma)_{ij} = \sigma_i^{-1}A_{ij}\sigma_j$.

Theorem 20. *Consider a real, $n \times n$ matrix A . If there exist numbers $\sigma_i > 0$, $i = 1, \dots, n$, such that for each i ,*

$$\sum_{j \neq i} |A_{ij}|\sigma_j < \sigma_i \min\{1 - A_{ii}, 1 + A_{ii}\}, \quad (\text{A.2})$$

then the spectrum of A lies in the unit disc.

We now give a specific application of these results. Consider now a real, 2×2 matrix with the structure:

$$M = \begin{pmatrix} 1 - \mu_1 - \mu_1\omega & \mu_1\omega \\ \mu_2\omega & 1 - \mu_2 - \mu_2\omega \end{pmatrix}, \quad (\text{A.3})$$

for some parameters $\mu_1, \mu_2 > 0$ and ω . The eigenvalues of M can be located as follows.

Lemma 5. *A sufficient condition for the eigenvalues of M to lie strictly inside the unit circle is:*

$$1 + 2 \min(\omega, 0) > 0, \quad \text{and} \quad \mu_i(1 + 2 \max(\omega, 0)) < 2, \quad i = 1, 2. \quad (\text{A.4})$$

If $\mu_1 = \mu_2 = \mu$, a necessary and sufficient condition for the eigenvalues of M to lie strictly inside the unit circle is

$$\mu < 2, \quad 1 + 2\omega > 0, \quad \text{and} \quad \mu(1 + 2\omega) < 2. \quad (\text{A.5})$$

Proof. Applying Theorem 19, a sufficient condition for the spectrum of M to lie in the unit disc is:

$$\begin{aligned}\mu_i|\omega| < 1 - (1 - \mu_i - \mu_i\omega) &\iff \mu_i(1 + \omega - |\omega|) > 0 \\ \mu_i|\omega| < 1 + (1 - \mu_i - \mu_i\omega) &\iff \mu_i(1 + \omega + |\omega|) < 2.\end{aligned}$$

Given that $\omega + |\omega| = 2 \max(\omega, 0)$ and $\omega - |\omega| = 2 \min(\omega, 0)$, and $\mu_i > 0$, these conditions rewrite as (A.4).

When $\mu_1 = \mu_2 = \mu$, the eigenvalues of M are exactly $1 - \mu$ and $1 - \mu - 2\mu\omega$. Requiring that these lie in the interval $(-1, 1)$ is equivalent to (A.5). \square

A.2 Proof of Proposition 8

Consider for the moment a general (not necessarily symmetric) Conjectural Learning procedure with a fixed point $(\beta_1^\infty, \beta_2^\infty, H^\infty)$. Let $w_1^\infty = r_1(\beta_1^\infty, H^\infty)$ and $w_2^\infty = r_2(\beta_2^\infty, H^\infty)$. Introduce the following notation:

$$r_{11} = \frac{\partial r_1}{\partial \beta_1}(\beta_1^\infty, H^\infty), \quad r_{22} = \frac{\partial r_2}{\partial \beta_2}(\beta_2^\infty, H^\infty), \quad r_{1H} = \frac{\partial r_1}{\partial H}(\beta_1^\infty, H^\infty), \quad r_{2H} = \frac{\partial r_2}{\partial H}(\beta_2^\infty, H^\infty).$$

The Jacobian matrix at the fixed point then writes as:

$$J = \begin{pmatrix} 1 - \mu_1 - \frac{\mu_1 r_{11} \beta_1^\infty}{w_1^\infty - \bar{w}_1} & \frac{\mu_1 r_{22}}{w_1^\infty - \bar{w}_1} & \frac{\mu_1 (r_{2H} - \beta_1^\infty r_{1H})}{w_1^\infty - \bar{w}_1} \\ \frac{\mu_2 r_{11}}{w_2^\infty - \bar{w}_2} & 1 - \mu_2 - \frac{\mu_2 r_{22} \beta_2^\infty}{w_2^\infty - \bar{w}_2} & \frac{\mu_2 (r_{1H} - \beta_2^\infty r_{2H})}{w_2^\infty - \bar{w}_2} \\ r_{11} \frac{\partial G}{\partial w_1} & r_{22} \frac{\partial G}{\partial w_2} & \frac{\partial G}{\partial H} + r_{1H} \frac{\partial G}{\partial w_1} + r_{2H} \frac{\partial G}{\partial w_2} \end{pmatrix}. \quad (\text{A.6})$$

The partial derivatives of G are evaluated at $(w_1^\infty, w_2^\infty, H^\infty)$.

With the assumption on symmetry, both players have the same reaction function: $r_1(\beta, H) = r_2(\beta, H)$. Since it is also assumed that $\beta_1^\infty = \beta_2^\infty = 1$, then $r_{11} = r_{22}$ and $r_{jH} = \beta_i^\infty r_{iH}$, $i \neq j$. After making substitutions in (A.6), the Jacobian matrix becomes:

$$J_S = \begin{pmatrix} 1 - \mu_1 - \frac{\mu_1 r_{11}}{w_1^\infty - \bar{w}_1} & \frac{\mu_1 r_{11}}{w_1^\infty - \bar{w}_1} & 0 \\ \frac{\mu_2 r_{11}}{w_2^\infty - \bar{w}_2} & 1 - \mu_2 - \frac{\mu_2 r_{11}}{w_2^\infty - \bar{w}_2} & 0 \\ r_{11} \frac{\partial G}{\partial w_1} & r_{11} \frac{\partial G}{\partial w_2} & \frac{\partial G}{\partial H} + r_{1H} \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) \end{pmatrix}. \quad (\text{A.7})$$

The eigenvalues of J_S are

$$\lambda_3 = \frac{\partial G}{\partial H} + r_{1H} \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right)$$

and those of the submatrix:

$$J'_S = \begin{pmatrix} 1 - \mu_1 - \mu_1 \omega & \mu_1 \omega \\ \mu_2 \omega & 1 - \mu_2 - \mu_2 \omega \end{pmatrix}, \quad \omega = \frac{r_{11}}{w_1^\infty - \bar{w}_1}.$$

Local stability occurs when all three eigenvalues have a modulus strictly less than 1. Under the Condition (2.27), this is the case for λ_3 . The application of Lemma 5 allows to conclude that Conditions (2.28) and (2.29) imply this for the eigenvalues of J'_S .

A.3 Proof of Proposition 11

Consider for the moment a general (not necessarily symmetric) CLP with a fixed point $(\beta_1^\infty, \beta_2^\infty, H^\infty)$. Let $w_1^\infty = r_1(\beta_1^\infty, H^\infty)$ and $w_2^\infty = r_2(\beta_2^\infty, H^\infty)$. Introduce the following notation:

$$r_{11} = \frac{\partial r_1}{\partial \beta_1}(\beta_1^\infty, H^\infty), \quad r_{11} = \frac{\partial r_2}{\partial \beta_2}(\beta_2^\infty, H^\infty), \quad r_{1H} = \frac{\partial r_1}{\partial H}(\beta_1^\infty, H^\infty), \quad r_{2H} = \frac{\partial r_2}{\partial H}(\beta_2^\infty, H^\infty).$$

The Jacobian matrix at the fixed point then writes as:

$$J = \begin{pmatrix} 1 - \mu_1 - \mu_1 r_{11} & \mu_1 r_{22} & \mu_1 (r_{2H} - r_{1H}) \\ \mu_2 r_{11} & 1 - \mu_2 - \mu_2 r_{22} & \mu_2 (r_{1H} - r_{2H}) \\ r_{11} \frac{\partial G}{\partial w_1} & r_{22} \frac{\partial G}{\partial w_2} & \frac{\partial G}{\partial H} + r_{1H} \frac{\partial G}{\partial w_1} + r_{2H} \frac{\partial G}{\partial w_2} \end{pmatrix}. \quad (\text{A.8})$$

The partial derivatives of G are evaluated at w_1^∞ and w_2^∞ .

With the assumption on symmetry, both players have the same reaction function: $r_1(\beta, H) = r_2(\beta, H)$.⁵ Then $r_{11} = r_{22} = r_\beta$ and $r_{1H} = r_{2H} = r_H$. After making substitutions in (A.6), the Jacobian matrix becomes:

$$J_S = \begin{pmatrix} 1 - \mu_1 - \mu_1 r_\beta & \mu_1 r_\beta & 0 \\ \mu_2 r_\beta & 1 - \mu_2 - \mu_2 r_\beta & 0 \\ r_\beta \frac{\partial G}{\partial w_1} & r_\beta \frac{\partial G}{\partial w_2} & \frac{\partial G}{\partial H} + r_H \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) \end{pmatrix}. \quad (\text{A.9})$$

The eigenvalues of J_S are

$$\lambda_3 = \frac{\partial G}{\partial H} + r_H \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right)$$

and those of the submatrix:

$$J'_S = \begin{pmatrix} 1 - \mu_1 - \mu_1 r_{11} & \mu_1 r_{11} \\ \mu_2 r_{11} & 1 - \mu_2 - \mu_2 r_{11} \end{pmatrix}.$$

This is the same situation as in Section A.2, with a different expression for ω . The application of Lemma 5 allows to conclude that Conditions (2.32) and (2.33) imply that the eigenvalues of J'_S lie in the unit circle, hence local stability.

A.4 Proof of Proposition 13

Consider for the moment a general (not necessarily symmetric) CLP with a fixed point $(\beta_1^\infty, \beta_2^\infty, H^\infty)$. Let $w_1^\infty = r_1(\beta_1^\infty, H^\infty)$ and $w_2^\infty = r_2(\beta_2^\infty, H^\infty)$. Introduce the following notation:

$$r_{11} = \frac{\partial r_1}{\partial \beta_1}(\beta_1^\infty, H^\infty), \quad r_{11} = \frac{\partial r_2}{\partial \beta_2}(\beta_2^\infty, H^\infty), \quad r_{1H} = \frac{\partial r_1}{\partial H}(\beta_1^\infty, H^\infty), \quad r_{2H} = \frac{\partial r_2}{\partial H}(\beta_2^\infty, H^\infty).$$

⁵Here, it is *not* assumed that $\beta_1^\infty = \beta_2^\infty$.

The Jacobian matrix at the fixed point then writes as:

$$J = \begin{pmatrix} 1 - \mu_1 & \frac{\mu_1 r_{22}}{H^\infty - \bar{H}_1} & \frac{\mu_1}{H^\infty - \bar{H}_1} (r_{2H} - \beta_1^\infty) \\ \frac{\mu_2 r_{11}}{H^\infty - \bar{H}_2} & 1 - \mu_2 & \frac{\mu_2}{H^\infty - \bar{H}_2} (r_{1H} - \beta_2^\infty) \\ r_{11} \frac{\partial G}{\partial w_1} & r_{22} \frac{\partial G}{\partial w_2} & \frac{\partial G}{\partial H} + r_{1H} \frac{\partial G}{\partial w_1} + r_{2H} \frac{\partial G}{\partial w_2} \end{pmatrix}. \quad (\text{A.10})$$

The partial derivatives of G are evaluated at $(w_1^\infty, w_2^\infty, H^\infty)$.

With the assumption on symmetry, both players have the same reaction function: $r_1(\beta, H) = r_2(\beta, H)$. Then $r_{11} = r_{22} = r_\beta$ and $r_{1H} = r_{2H} = r_H$. Also, $\bar{H}_1 = \bar{H}_2 = \bar{H}$. We then apply Theorem 20 with $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = \lambda|H^\infty - \bar{H}|$, for some $\lambda > 0$. We obtain the sufficient conditions:

$$\begin{aligned} & \text{for } i = 1, 2, \quad \frac{\mu_i |r_\beta|}{|H^\infty - \bar{H}|} + \mu_i \lambda |r_H - \beta^\infty| < \min\{\mu_i, 2 - \mu_i\} \\ & \left(\frac{\partial G}{\partial H} + r_H \frac{\partial G}{\partial w_1} + r_H \frac{\partial G}{\partial w_2} \right) \lambda |H^\infty - \bar{H}| + |r_\beta \frac{\partial G}{\partial w_1}| + |r_\beta \frac{\partial G}{\partial w_2}| < \lambda |H^\infty - \bar{H}| \\ & \left(\frac{\partial G}{\partial H} + r_H \frac{\partial G}{\partial w_1} + r_H \frac{\partial G}{\partial w_2} \right) \lambda |H^\infty - \bar{H}| - \left(|r_\beta \frac{\partial G}{\partial w_1}| + |r_\beta \frac{\partial G}{\partial w_2}| \right) > -\lambda |H^\infty - \bar{H}|. \end{aligned}$$

With the notation of the proposition, and since $0 < \mu_i \leq 1$, the first one is equivalent to:

$$|\omega_1| + \lambda |\omega_2 - \beta^\infty| < 1,$$

which is (2.38). The second and the third ones are respectively:

$$\begin{aligned} & \frac{\partial G}{\partial H} + \omega_2 \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) + \frac{|\omega_1|}{\lambda} \left(\left| \frac{\partial G}{\partial w_1} \right| + \left| \frac{\partial G}{\partial w_2} \right| \right) < 1, \\ & \frac{\partial G}{\partial H} + \omega_2 \left(\frac{\partial G}{\partial w_1} + \frac{\partial G}{\partial w_2} \right) - \frac{|\omega_1|}{\lambda} \left(\left| \frac{\partial G}{\partial w_1} \right| + \left| \frac{\partial G}{\partial w_2} \right| \right) > -1, \end{aligned}$$

which are (2.36) and (2.37).

B Analysis of the example

In this section, we expose some of the calculations related to the example specified in Section 3.2.

B.1 Affine conjecture on w

This part is devoted to the analysis of the dynamical system generated by the CLP affine in w , in the symmetric case (which includes $\bar{w}_i = \bar{w}_j$). It will be convenient to perform first a change of variables and rename some parameters.

Consider first the change of variables: $W_i^t = w_i^t - \bar{w}$, $i = 1, 2$, $t = 0, \dots, \infty$. Define then

$$A := \theta R + \frac{a - c_0 - \rho\gamma\alpha}{c_1} - \bar{w} \left(3\alpha\theta + \frac{b}{c_1} \right) \quad B := -\theta\alpha\bar{w} - \frac{\rho\gamma\alpha}{c_1}$$

$$C := \frac{b}{c_1} + 2\alpha\theta \quad D := 2\alpha\theta \quad Z := \frac{R}{\alpha} - 2\bar{w}.$$

With this notation, the dynamical system under study is:

$$W_i^t = \frac{H^t + A + B\beta_i^t}{C + D\beta_i^t} \quad (\text{B.1})$$

$$\beta_i^{t+1} = (1 - \mu_i)\beta_i^t + \mu_i \frac{W_j^t}{W_i^t} \quad (\text{B.2})$$

$$H^{t+1} = H^t + \alpha Z - \alpha(W_1^t + W_2^t). \quad (\text{B.3})$$

B.1.1 Limit points for the affine conjecture on w

We are looking points $(\beta_1, \beta_2, W_1, W_2, H)$ which are possible limits of the recurrence (B.1)–(B.3). According to Lemma 1, we have $\beta_i\beta_j = 1$, and also:

$$W_1 = \beta_2 W_2 \quad (\text{B.4})$$

$$W_2 = \beta_1 W_1, \quad (\text{B.5})$$

even if $W_i^t \rightarrow 0$ (or $W_i = 0$) for some i . Next, using (B.1) and (B.3), the limits satisfy:

$$W_1 = \frac{H + A + B\beta_1}{C + D\beta_1}, \quad (\text{B.6})$$

$$W_2 = \frac{H + A + B\beta_2}{C + D\beta_2}. \quad (\text{B.7})$$

$$W_1 + W_2 = Z. \quad (\text{B.8})$$

We proceed with solving the system (B.6)–(B.5). It is assumed that $C \neq D$, which is equivalent to $b \neq 0$.

We first eliminate $W_2 = \beta_1 W_1$. Next, eliminating the term $H + A$ from (B.6) and (B.7), we obtain:

$$\begin{aligned} (C + D\beta_1)W_1 - B\beta_1 &= (C + D\beta_2)W_2 - B\beta_2 = (C + D\beta_2)\beta_1 W_1 - B\beta_2 \\ W_1(1 - \beta_1)(C - D) &= B(\beta_1 - \beta_2). \end{aligned} \quad (\text{B.9})$$

One first case in the discussion is when $B = 0$. In that case, since $C \neq D$ by assumption, either $\beta_1 = 1$ or $W_1 = 0$. Assume $\beta_1 = 1$: then we find as solution:

$$\beta_1 = \beta_2 = 1, \quad W_1 = W_2 = \frac{Z}{2} \quad H = \frac{Z}{2}(C + D) - (A + B). \quad (\text{B.10})$$

Assume now that $B = 0$ and $W_1 = 0$. Then $W_2 = 0$ and necessarily $Z = 0$. If this is not the case, there is no solution with $W_1 = 0$. If this is the case, $H = -A$ and the solution is as in (B.10) except that β_1 and β_2 are not determined by the equations.

Assuming now $B \neq 0$, we use $\beta_2 = 1/\beta_1$ and in (B.9), to get:

$$\beta_1 W_1 (1 - \beta_1)(C - D) = B(\beta_1^2 - 1).$$

Again, we find that $\beta_1 = 1$ solves this equation, which gives the solution (B.10). Assuming that $\beta_1 \neq 1$, we are left with the system of two equations:

$$\beta_1 W_1 (C - D) = -B(1 + \beta_1)$$

$$W_1(1 + \beta_1) = Z,$$

which implies:

$$\begin{aligned}\beta_1(1 + \beta_1)W_1(C - D) + B(1 + \beta_1)^2 &= 0 \\ \beta_1 Z(C - D) + B(1 + \beta_1)^2 &= 0 \\ B\beta_1^2 + (2B + Z(C - D))\beta_1 + B &= 0.\end{aligned}\tag{B.11}$$

The two roots of the polynomial in (B.11) have their product equal to 1. We pick any of them as β_1 , the other one will be β_2 . Explicitly, they are:

$$\beta_1 = -1 + \frac{Z(D - C) + \sqrt{\Delta}}{2B} \quad \beta_2 = -1 + \frac{Z(D - C) - \sqrt{\Delta}}{2B},\tag{B.12}$$

where

$$\Delta := Z(D - C)(Z(D - C) - 4B).\tag{B.13}$$

Replacing in the other equations and simplifying the rest of the solution is:

$$W_1 = \frac{Z}{2} - \frac{\sqrt{\Delta}}{2(D - C)} \quad W_2 = \frac{Z}{2} + \frac{\sqrt{\Delta}}{2(D - C)} \quad H = CZ + B - A.\tag{B.14}$$

A second solution is deduced by symmetry, exchanging the role of β_1 and β_2 . It is checked that these solutions, when placed in the dynamics (B.1)–(B.3), result in a fixed point.

We conclude with checking whether these solutions are real or complex for the parameters of the harvesting model. Given the values of B, C, D and Z , we have

$$\begin{aligned}D - C &= -\frac{b}{c_1} < 0 \\ Z(D - C) - 4B &= -\frac{b}{c_1} \left(\frac{R}{2\alpha} - 2\bar{w} \right) - 4\alpha(\theta\bar{w} - \rho\gamma).\end{aligned}$$

Depending on the value of \bar{w} , Z may be of either sign. It seems that Δ can be of either sign as well.

B.1.2 Stability of fixed points

Starting from Equation (A.6) of Appendix B.1.1, it can be shown that the Jacobian of the dynamics (B.2)–(B.3) (where W_i^t , $i = 1, 2$, has been eliminated with (B.1)) can be written $J = I + \bar{J}$ with:

$$\bar{J} = \begin{pmatrix} -\mu_1 + \mu_1 \frac{\sigma\beta_1}{W_1\psi_1^2} & -\mu_1 \frac{\sigma}{W_1\psi_2^2} & -\mu_1 \frac{B(1 - \beta_1)}{W_1\sigma} \\ -\mu_2 \frac{\sigma}{W_2\psi_1^2} & -\mu_2 + \mu_2 \frac{\sigma\beta_2}{W_2\psi_2^2} & -\mu_2 \frac{B(1 - \beta_2)}{W_2\sigma} \\ \frac{\alpha\sigma}{\psi_1^2} & \frac{\alpha\sigma}{\psi_2^2} & \alpha \frac{2B - DZ}{\sigma} \end{pmatrix}\tag{B.15}$$

where we have used the shorthand notation:

$$\psi_i := C + D\beta_i, \quad \sigma := D(H + A) - BC.$$

The eigenvalues of J are 1 plus those of \bar{J} . When the fixed point is given by (B.10), the analysis of these eigenvalues is done in Section 3.2.2. When it is given by (B.12)–(B.14), the eigenvalues of \bar{J} are less easy to locate, but it can be proved that one of them is real and positive, at least if $Z > 0$ (that is, $\bar{w} < R/(2\alpha)$) and $D > 0$ (that is, $\theta > 0$). Indeed, the determinant of \bar{J} turns out to be:

$$|\bar{J}| = \frac{\mu_1 \mu_2 \alpha \Delta}{Z \sigma (C - D)},$$

where Δ is defined in (B.13). If the solution (B.12) exists, $\Delta \geq 0$. Also, because of (B.14), $\sigma = D(H + A) - BC = D(CZ + B) - BC = DCZ - B(C - D) \geq 0$ because $C > D > 0$, $B \leq 0$ and $Z > 0$. We conclude that $|\bar{J}| \geq 0$.

On the other hand, the eigenvalues λ of \bar{J} are solution of the polynomial $\chi(\lambda) = |\bar{J} - \lambda I|$. This is a polynomial of degree 3 with leading coefficient -1 , such that $\chi(0) = |\bar{J}| \geq 0$. Therefore, it has a root in the interval $[0, +\infty)$. As a consequence, the Jacobian J has an eigenvalue larger than 1, and the fixed point (B.12)–(B.14) is not attractive if $\Delta = 0$, and repelling if $\Delta > 0$.

B.2 Proof of Proposition 17

The following notation will be convenient for this discussion. Let:

$$\psi := \frac{b + \theta \alpha c_1}{c_1}. \quad (\text{B.16})$$

Applying Proposition 13, the sufficient condition for convergence to a steady state $(\beta^\infty, \beta^\infty, H^\infty)$ depends on the critical values:

$$\begin{aligned} \omega_1 &= \frac{1}{|H^\infty + R|} \frac{\partial r}{\partial \beta} = -\frac{\alpha \theta c_1}{b + 2\alpha \theta c_1} = -\frac{\alpha \theta}{\psi + \alpha \theta} \\ \omega_2 &= \frac{\partial r}{\partial H} = \frac{c_1(1 - \alpha \theta \beta^\infty)}{b + 2\alpha \theta c_1} = \frac{1 - \alpha \theta \beta^\infty}{\psi + \alpha \theta}. \end{aligned}$$

Note that $\omega_1 \leq 0$. Given the specification of function G , the conditions of Proposition 13 are:

$$\begin{aligned} |\omega_1| + \lambda |\omega_2 - \beta^\infty| &< 1 \\ 1 - 2\alpha \omega_2 + \frac{2\alpha}{\lambda} |\omega_1| &< 1 \\ 1 - 2\alpha \omega_2 - \frac{2\alpha}{\lambda} |\omega_1| &> -1. \end{aligned}$$

We first express these conditions as constraints on β^∞ , then convert those to constraints on θ . From the first condition,

$$\begin{aligned} |\omega_1| + \lambda |\omega_2 - \beta^\infty| &< 1 \\ \frac{\alpha \theta}{\psi + \alpha \theta} + \lambda \left| \frac{1 - \alpha \theta \beta^\infty}{\psi + \alpha \theta} - \beta^\infty \right| &< 1 \\ \alpha \theta + \lambda |1 - (\psi + 2\alpha \theta) \beta^\infty| &< \psi + \alpha \theta \\ |1 - (\psi + 2\alpha \theta) \beta^\infty| &< \frac{\psi}{\lambda} \end{aligned}$$

$$\begin{aligned} -\frac{\psi}{\lambda} < 1 - (\psi + 2\alpha\theta)\beta^\infty < \frac{\psi}{\lambda} \\ \frac{1}{\psi + 2\alpha\theta} \left(1 - \frac{\psi}{\lambda}\right) < \beta^\infty < \frac{1}{\psi + 2\alpha\theta} \left(1 + \frac{\psi}{\lambda}\right). \end{aligned}$$

Next, we consider the two other conditions:

$$\begin{aligned} \frac{2\alpha}{\lambda}|\omega_1| - 2 < -2\alpha\omega_2 < -\frac{2\alpha}{\lambda}|\omega_1| \\ \frac{\alpha}{\lambda} \frac{\alpha\theta}{\psi + \alpha\theta} - 1 < -\frac{\alpha(1 - \alpha\theta\beta^\infty)}{\psi + \alpha\theta} < -\frac{\alpha}{\lambda} \frac{\alpha\theta}{\psi + \alpha\theta} \\ \frac{\alpha^2\theta}{\lambda} - (\psi + \alpha\theta) < -\alpha + \alpha^2\theta\beta^\infty < -\frac{\alpha^2\theta}{\lambda} \\ \frac{1}{\lambda} + \frac{1}{\alpha\theta} - \frac{\psi + \alpha\theta}{\alpha^2\theta} < \beta^\infty < \frac{1}{\alpha\theta} - \frac{1}{\lambda}. \end{aligned}$$

Summarizing the two cases, a sufficient condition for local convergence is:

$$\max \left\{ \frac{1}{\psi + 2\alpha\theta} \left(1 - \frac{\psi}{\lambda}\right), \frac{1}{\lambda} + \frac{1}{\alpha\theta} - \frac{\psi + \alpha\theta}{\alpha^2\theta} \right\} < \beta^\infty < \min \left\{ \frac{1}{\psi + 2\alpha\theta} \left(1 + \frac{\psi}{\lambda}\right), \frac{1}{\alpha\theta} - \frac{1}{\lambda} \right\}. \quad (\text{B.17})$$

As a function of λ , the right-hand side of (B.17) is maximal when both functions coincide, which happens when $\lambda = 2\theta\alpha$. Setting λ to this value, we get:

$$\begin{aligned} \max \left\{ \frac{1}{\psi + 2\alpha\theta} \left(1 - \frac{\psi}{2\alpha\theta}\right), \frac{1}{2\alpha\theta} + \frac{1}{\alpha\theta} - \frac{\psi + \alpha\theta}{\alpha^2\theta} \right\} < \beta^\infty < \frac{1}{2\alpha\theta} \\ \frac{1}{2\alpha\theta} \max \left\{ \frac{2\alpha\theta - \psi}{\psi + 2\alpha\theta}, 3 - 2\theta - \frac{2\psi}{\alpha} \right\} < \beta^\infty < \frac{1}{2\alpha\theta}. \end{aligned}$$

Finally, if $\psi \geq 2\alpha\theta$ and $\psi \geq \alpha(3/2 - \theta)$, the left-hand side in this inequality is negative because both terms inside the “min” are negative. Since $\beta^\infty > 0$, this side of the inequality is automatically satisfied. Then, the condition reduces successively to

$$\begin{aligned} 2\alpha\beta^\infty &< \frac{1}{\theta} \\ \frac{R}{H^\infty + R} &< \frac{1}{\theta} \\ \theta R &< H^\infty + R \end{aligned}$$

which is always true since $\theta \leq 1$ and $H^\infty > 0$ by assumption. Therefore, these conditions $\psi \geq 2\alpha\theta$ and $\psi \geq \alpha(3/2 - \theta)$ are sufficient for local stability. They are equivalent respectively to $\theta \leq b/(\alpha c_1)$ and $\theta \geq (3 - 2b/(\alpha c_1))/4$.

B.3 Feedback strategies and profits

Consider the model specified by (3.3), where we only take into account the objective instantaneous gain and we assume symmetry, namely,

$$\tilde{F}_i(w_i, w_j, H) = aw_i - \frac{b}{2}(w_i)^2 - w_i(c_0 - c_1[H - \theta(R - \alpha(w_i + w_j))]).$$

Assume that the dynamics are given by (3.1), and consider that both players play the same state-feedback strategy of the form:

$$w_i^t = \nu_1 H^t + \nu_0 .$$

As observed in Section 4.3, this is the situation for the full-information schemes, and in that case parameters ν_0 depend linearly on γ whereas parameters ν_1 do not depend on γ .

Then the total discounted objective gain, for each player, and the steady state of stock H and consumption w of this strategy are:

$$V^{\nu_0, \nu_1}(H_0) = \frac{\nu_1 (H_0 - H^\infty)^2}{1 - \rho(1 - 2\alpha\nu_1)^2} \left(c_1 - \frac{\nu_1 (4\alpha c_1 \theta + b)}{2} \right) \quad (\text{B.18})$$

$$+ \frac{H_0 - H^\infty}{1 - \rho(1 - 2\alpha\nu_1)} \left[\nu_1 (a - c_0 - bw^\infty + c_1 H^\infty - c_1 R\theta) + c_1 w^\infty \right]$$

$$+ \frac{w^\infty}{1 - \rho} \left(a - c_0 - w^\infty \frac{b}{2} + c_1 H^\infty \right)$$

$$H^\infty = \frac{R - 2\alpha\nu_0}{2\alpha\nu_1} \quad (\text{B.19})$$

$$w^\infty = \frac{R}{2\alpha}. \quad (\text{B.20})$$

This is a quadratic and convex function of H_0 , since it is expected that the leading term $c_1 - \nu_1(4\alpha c_1 \theta + b)/2$ is positive. This is also a quadratic function of H^∞ . The leading term is:

$$A_2 = \frac{\nu_1}{1 - \rho(1 - 2\alpha\nu_1)^2} \left(c_1 - \frac{\nu_1 (4\alpha c_1 \theta + b)}{2} \right) - \frac{\nu_1 c_1}{1 - \rho(1 - 2\alpha\nu_1)}$$

$$= - \frac{2\alpha\rho\nu_1^2(1 - 2\alpha\nu_1)}{(1 - \rho(1 - 2\alpha\nu_1)^2)(1 - \rho(1 - 2\alpha\nu_1))} - \frac{\nu_1^2}{2} \frac{4\alpha c_1 \theta + b}{1 - \rho(1 - 2\alpha\nu_1)^2}.$$

This leading term is therefore negative, and the gain function is concave as a function of H^∞ . Since H^∞ is linear in ν_0 , V is quadratic and concave as well, as a function of ν_0 .



**RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

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