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# Algorithms for fundamental invariants and equivariants (of finite groups)

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*Dedicated to the memory of Karin Gatermann.*

## Abstract

For a finite group, we present three algorithms to compute a generating set of invariant simultaneously to generating sets of basic equivariants, *i.e.*, equivariants for the irreducible representations of the group. The main novelty resides in the exploitation of the orthogonal complement of the ideal generated by invariants; Its symmetry adapted basis delivers the fundamental equivariants.

Fundamental equivariants allow to assemble symmetry adapted bases of polynomial spaces of higher degrees, and these are essential ingredients in exploiting and preserving symmetry in computations. They appear within algebraic computation and beyond, in physics, chemistry and engineering.

Our first construction applies solely to reflection groups and consists in applying symmetry preserving interpolation, as developed by the same authors, along an orbit in general position. The fundamental invariants can be read off the H-basis of the ideal of the orbit while the fundamental equivariants are obtained from a symmetry adapted basis of an invariant direct complement to this ideal in the polynomial ring.

The second algorithm takes as input primary invariants and the output provides not only the secondary invariants but also free bases for the modules of basic equivariants. These are constructed as the components of a symmetry adapted basis of the orthogonal complement, in the polynomial ring, to the ideal generated by primary invariants.

The third algorithm proceeds degree by degree, determining the fundamental invariants as forming a H-basis of the Hilbert ideal, *i.e.*, the polynomial ideal generated by the invariants of positive degree. The fundamental equivariants are simultaneously computed degree by degree as the components of a symmetry adapted basis of the orthogonal complement of the Hilbert ideal.

**Keywords:** finite groups; representation theory; polynomial invariants; equivariants; covariants; symmetry adapted bases; H-basis; computational invariant theory.

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## 1 Introduction

Symmetry is ubiquitous in science and art and underlies a number of mathematical endeavors as for instance invariant theory and representation theory. Preserving and exploiting symmetry in computations has relied on both the use of invariants, and equivariants, as well as symmetry adapted bases of polynomial rings. They are relevant in combinatorics [Ber09, Sta79, TY93], dynamical systems [CL00, Gat00, GSS88, HL13], cryptography [LRS16, Mes91], global optimization [GP04, RTAL13], cubatures and multivariate interpolation [CH15, RH21, RH22], solving polynomial systems [FR09, FS13, Gat90, HL12, HL16], and more widely in physics, chemistry, and engineering [FS92, FK77, MZ01, Mug72, MZF92]. Symmetry adapted bases of polynomial rings are made of *basic equivariants*, *i.e.* equivariants for the irreducible representations of the group. For each irreducible representation, the related set of equivariants forms a module over the invariant ring. The purpose of this article is to offer algorithms to compute relevant sets of generators of these modules, together with generators for the ring of invariants. We shall call these generators the *fundamental invariants and equivariants*.

One can always compute symmetry adapted bases of polynomial rings degree by degree with linear algebra operations thanks to explicit projection operators [Ser77]. Yet, generative approaches are desirable: if fundamental equivariants are known, one can assemble symmetry adapted bases at any degree by multiplying these with polynomials of appropriate degrees in the fundamental invariants. Fundamental equivariants are given combinatorially for some classical families of finite groups [ATY97, Spe35] or are determined explicitly for some representations of relevance in applications, as for instance [CCP08, CCDP15, DPZ15]. We propose algorithms for their computation that applies to any finite group.

The computation of invariants of group actions has been an active topic since the 19th century. Celebrated mathematicians have left their marks on the subject and textbooks reporting major progress still appeared relatively recently [DK15, Stu93]. Semi-invariants and equivariants<sup>1</sup> may appear in the construction of invariants [GY03, Oli17, GHP19] and are sometimes more pertinent or simpler to work with. Despite their relevance, equivariants are scarcely present in the computational invariant theory literature of the last decades, with the exception of [Gat00].

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<sup>1</sup> Starting with the 19th century literature regarding the action of  $SL_2(\mathbb{C})$  on binary forms the term *covariants* has also been used. In other parts of invariant theory *covariant* characterizes the quotient algebra of the polynomial ring by the Hilbert ideal [Ber09, Kan01].

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So far equivariants have been computed by altering the algorithms for invariants. We take a very different approach that stems from the following property: take the ideal generated by sufficiently many invariants and consider an invariant direct complement to this ideal in the polynomial ring. A symmetry adapted basis of this latter essentially provides a set of fundamental equivariants. Any polynomial equivariant can be written in terms of basic equivariants and thus the comprehensive output of our algorithms provide generators for any equivariant modules.

We develop three algorithms. The first one applies to reflection groups. These latter enjoy extensive properties [BG85, Che55, Kan01]. The fundamental invariants are then algebraically independent and the fundamental equivariants are free bases of the modules of basic equivariants, over the ring of invariants. Their computation can be approached either as  $\mathfrak{G}$ -harmonic polynomials or from the cyclic structure of the covariant algebra. Yet it is a rather remarkable observation that the invariants and equivariants can simply be obtained as a direct application of symmetry preserving ideal interpolation as developed in [RH22]. Interpolating along an orbit in general position, the fundamental equivariants are read from a basis of an invariant interpolation space, while the invariants are read from an orthogonal H-basis of the ideal of the orbit. It is this original idea that we expanded on to compute the fundamental invariants and equivariants of any finite groups.

The second construction, in essence, takes as input a set of primary invariants  $h_1, \dots, h_n$  and computes a set of secondary invariants together with basic equivariants that form free bases for their modules over  $\mathbb{C}[h_1, \dots, h_n]$ . This provides us with a Hironaka decomposition of the polynomial ring and its isotypic components as  $\mathbb{C}[h]$ -modules.

The third algorithm computes simultaneously minimal sets of fundamental invariants and equivariants. It constructs, degree by degree, a H-basis of the Hilbert ideal, *i.e.*, the polynomial ideal generated by the invariants of positive degree. This H-basis forms a minimal set of fundamental invariants. The fundamental equivariants form a symmetry adapted basis of the orthogonal complement of the Hilbert ideal and this basis is constructed alongside degree by degree. The degree  $d$  part of the Hilbert ideal and its orthogonal complement are obtained with linear algebra operations. The equivariance of the underlying linear maps, together with the symmetry adapted bases so far computed, allows us to block diagonalize these operations as well as exhibit and avoid redundancies intrinsic to the symmetry under the group action.

All three algorithms rely on the knowledge of symmetry adapted bases of polynomials up to some degree. In return, we can identify fundamental equivariants and thus higher degree symmetry adapted bases can be assembled easily, without resorting to projection operators. This assembly is particularly straightforward with the output of the first and second algorithm when we have a Hironaka decomposition of the basic equivariant modules.

**Previous work and contributions.** For the linear action  $\vartheta$  of an algebraic group computing the invariants of a given degree can be done thanks to the Reynolds operator, *i.e.*, a projection from the polynomial ring to the ring of invariants, or by computing the kernel of a linear map. [Gat00] sorted algorithms for determining a fundamental set of invariants in three categories:

- Using the Hilbert series, in the case of finite groups or compact Lie groups [Stu93, Algorithm 2.2.5] [Gat00, Section 2.1]: adding invariants degree by degree, termination occurs when the algebra they generate has the same Hilbert series as the ring of invariants - this latter being given by Molien's formula.
- Using a homogeneous system of parameters [Stu93, Section 2.5], [Gat00, Section 2.1], [DK15, Section 3.5-7]: a set of primary invariants is first determined. The degree of the secondary invariants can then be deduced from Molien's formula for the Hilbert series of the invariant ring.
- Using the Hilbert ideal. In Derksen's algorithm the Hilbert ideal is obtained by specializing the ideal of the action graph [Der99], [Gat00, Section 2.2], [DK15, Section 4.1]. In the case of finite groups, King's

algorithm constructs a Gröbner basis of the Hilbert ideal degree by degree [Kin13], [DK15, Section 3.8.2]. Hilbert’s original construction [Stu93, Section 4.6] also enters this category.

As mentioned in [PV94], [Gat00, Section 2.2], [DK15, Section 4.2.3], the module of  $(\theta : \vartheta)$ -equivariants can be identified to a submodule of the ring of  $(\theta \oplus \vartheta)$ -invariants. This provides a default option to compute equivariants given an algorithm to compute invariants. One has to note that it is AT the cost of increasing the dimension of the representation space and hence the number of variables in the computations.

[Gat00, Section 2] actually provides more details on how the algorithms for invariants can be extended to compute generators of an equivariant module. In the two first categories one uses Molien’s formula relative to the module of equivariants.

In the case of Derksen’s algorithm, [Gat00, Section 2.2] shows how to modify the algorithm in order to compute only the necessary submodule of the  $(\theta \oplus \vartheta)$ -invariant ring. This is achieved by the use of a specific grading and an adequately truncated Gröbner basis. A comparable modification can be applied to King’s algorithm.

When it comes to computing the invariants our third and main algorithm belongs to the third category: it computes, degree by degree, a H-basis of the Hilbert ideal and the elements of the so obtained H-basis form a minimal generating set of invariants.

When it comes to computing the fundamental equivariants, we exploit the following novel idea in computational invariant theory: The components of a symmetry adapted basis of an invariant direct complement, in the polynomial ring, of the Hilbert ideal provide minimal sets of generators for the basic equivariant modules. We construct such a symmetry adapted basis of the orthogonal complement degree by degree, simultaneously to the H-basis of the Hilbert ideal.

Though the comprehensive output of our algorithm can be obtained by aggregating the outputs of previous algorithms, our algorithm is the first one to provide both the fundamental invariants and the fundamental equivariants with mutualized computations.

Section 2 formalizes the link between symmetry adapted bases and fundamental equivariants. In Section 3 we recall the premises of our algorithms: zero dimensional ideals, interpolation and H-bases. In Section 4, we show that the symmetry preserving ideal interpolation algorithm in [RH22] can be straightforwardly applied to compute the fundamental invariants and equivariants of a reflection group. In Section 5, given a set of primary invariants (or superset of these)  $\{h_1, \dots, h_n\}$ , we apply the algorithms in [RH21, RH22] to compute both a set of secondary invariants and free bases of basic equivariants, then seen as  $\mathbb{C}[h_1, \dots, h_n]$ -modules. The ideas of these two constructions brings us, in Section 6, to an independent algorithm to compute simultaneously minimal sets of fundamental invariants and equivariants for any finite group. The examples we present all along the paper were computed with our implementation in Maple of the algorithms presented.

The base field in this paper is  $\mathbb{C}$  for simplicity of presentation. The algorithms can be extended to work over  $\mathbb{R}$  or any other subfield of  $\mathbb{C}$ . As discussed in the conclusion, the constructions can also be adapted to any field the characteristic which does not divide the order of the group (a.k.a. the non modular case).

## 2 Symmetry adapted bases and basic equivariants

Symmetry adapted bases are an essential tool to take advantage of symmetry in algebraic computations; the main reason for that is the fact that the matrix of a  $\mathfrak{G}$ -morphism is block diagonal in a symmetry adapted basis [FS92]. In polynomial rings, symmetry adapted bases can be understood to consist of *basic* equivariants. Basic equivariants form modules over the invariant ring which allows for finite presentation. We recall these notions, their connections, and preliminary results about them.

We deal with a finite group  $\mathfrak{G}$ . We note  $\mathfrak{r}^{(1)}, \dots, \mathfrak{r}^{(\ell)}$  the inequivalent irreducible matrix representations of  $\mathfrak{G}$  over  $\mathbb{C}$ ;  $\mathfrak{n}_\ell$  the dimension of  $\mathfrak{r}^{(\ell)}$ . All along the article  $\mathfrak{r}^{(1)}$  is understood to be the trivial representation.

## 2.1 Symmetry adapted bases

A representation  $\mathfrak{r} : \mathfrak{G} \rightarrow \text{GL}(V)$ , can be decomposed into irreducible representations,  $\mathfrak{r} = m_1 \mathfrak{r}^{(1)} \oplus \dots \oplus m_\ell \mathfrak{r}^{(\ell)}$ . Hence, the representation space  $V$  admits a basis in which the representation matrices are  $\text{diag} (I_{m_\ell} \otimes \mathfrak{r}^{(\ell)}(g) \mid 1 \leq \ell \leq \ell)$ , for  $g \in \mathfrak{G}$ . Assume that

$$\mathcal{Q} = \bigcup_{\ell=1}^{\ell} \mathcal{Q}^{(\ell)}, \text{ with } \mathcal{Q}^{(\ell)} = \bigcup_{i=1}^{m_\ell} \{q_{i1}^{(\ell)}, \dots, q_{i\mathfrak{n}_\ell}^{(\ell)}\} \text{ where } q_{ik}^{(\ell)} \in V \quad (1)$$

is such a basis. If the components of  $\mathcal{Q}^{(\ell)}$  are reordered as  $\bigcup_{j=1}^{\mathfrak{n}_\ell} \{q_{1j}^{(\ell)}, \dots, q_{m_\ell j}^{(\ell)}\}$  then the matrix of an equivariant endomorphism is block diagonal, with  $\mathfrak{n}_\ell$  identical blocks of size  $m_\ell$  [FS92]. Despite this fact, in this paper it is more practical to present the *symmetry adapted bases* as in (1) though this differs from the convention adopted in [RH21, RH22] that we shall refer to.

The decomposition into irreducible representations is not unique but the decomposition into *isotypic components* is canonical:  $V = \bigoplus_{\ell=1}^{\ell} V^{(\ell)}$  where  $V^{(\ell)}$  is equivalent to  $m_\ell$  times the irreducible representation  $\mathfrak{r}^{(\ell)}$ . The *character* of  $\mathfrak{r}$  is the function  $\chi : \mathfrak{G} \rightarrow \mathbb{C}$  given  $\chi(g) = \text{trace}(\mathfrak{r}(g))$ . If  $\chi^{(\ell)}$  is the character of  $\mathfrak{r}^{(\ell)}$  we have:

$$m_\ell = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \chi^{(\ell)}(g^{-1}) \chi(g)$$

and a projection on  $V^{(\ell)}$  is given by [Ser77, Chapter 2]:

$$\begin{aligned} \pi^{(\ell)} : V &\rightarrow V^{(\ell)} \\ v &\mapsto \frac{\mathfrak{n}_\ell}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \chi^{(\ell)}(g^{-1}) \mathfrak{r}(g)(v). \end{aligned}$$

The construction of a symmetry adapted basis is basically given by [Ser77, Chapter 2, Proposition 8] that we reproduce here for ease of reference.

**Proposition 2.1** *The linear maps  $\pi_{ij}^{(\ell)} : V \rightarrow V$ , for  $1 \leq i, j \leq \mathfrak{n}_\ell$ , defined by*

$$\pi_{ij}^{(\ell)}(v) = \frac{\mathfrak{n}_\ell}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \mathfrak{r}_{ji}^{(\ell)}(g^{-1}) \mathfrak{r}(g)(v)$$

satisfy the following properties:

1. For every  $1 \leq i \leq \mathfrak{n}_\ell$ , the map  $\pi_{ii}^{(\ell)}$  is a projection; it is zero on the isotypic components  $V^{(k)}$ ,  $k \neq \ell$ . Its image  $V^{(\ell, i)}$  is contained in  $V^{(\ell)}$  and

$$V^{(\ell)} = V^{(\ell, 1)} \oplus \dots \oplus V^{(\ell, \mathfrak{n}_\ell)} \quad \text{while} \quad \pi^{(\ell)} = \sum_{i=1}^{\mathfrak{n}_\ell} \pi_{ii}^{(\ell)}. \quad (2)$$

2. For every  $1 \leq i, j \leq \mathfrak{n}_\ell$ , the linear map  $\pi_{ij}^{(\ell)}$  is zero on the isotypic components  $V^{(k)}$ ,  $k \neq \ell$ , as well as on the subspaces  $V^{(\ell, k)}$  for  $k \neq j$ ; it defines an isomorphism from  $V^{(\ell, j)}$  to  $V^{(\ell, i)}$ .
3. For any  $q \in V$  and  $1 \leq k \leq \mathfrak{n}_\ell$  consider  $q_i = \pi_{ik}^{(\ell)}(q) \in V^{(\ell, i)}$  for all  $1 \leq i \leq \mathfrak{n}_\ell$ . If nonzero,  $q_1, \dots, q_{\mathfrak{n}_\ell}$  are linearly independent and generate an invariant subspace of dimension  $\mathfrak{n}_\ell$ . For each  $g \in \mathfrak{G}$ , we have

$$\mathfrak{r}(g)(q_j) = \sum_{i=1}^{\mathfrak{n}_\ell} \mathfrak{r}_{ij}^{(\ell)}(g)(q_i) \quad \forall i, j = 1, \dots, \mathfrak{n}_\ell.$$

The last property can be written in the matricial form

$$[\tau(g)(q_1) \quad \dots \quad \tau(g)(q_{n_\ell})] = [q_1 \quad \dots \quad q_{n_\ell}] \tau^{(\ell)}$$

whether we think of  $q_1, \dots, q_{n_\ell}$  as column vectors, or, for later purpose, polynomials.

If the isotypic component  $V^{(\ell)}$  associated to the  $\ell$ -th irreducible representation of  $\mathfrak{G}$  is canonical, the projections  $\pi^{(\ell,k)}$  and the components  $V^{(\ell,k)} = \pi^{(\ell,k)}(V)$  defined in Proposition 2.1, for  $1 \leq k \leq n_\ell$ , depend on the choice of the representing matrix  $\tau^{(\ell)} : \mathfrak{G} \rightarrow \text{GL}_{n_\ell}(\mathbb{C})$  for this irreducible representation.

The following follows very directly from Proposition 2.1 and describes a way to compute a symmetry adapted basis for  $V$ .

**Corollary 2.2** *If, for  $1 \leq \ell \leq l$ ,  $\{q_1^{(\ell)}, \dots, q_{m_\ell}^{(\ell)}\}$  is a basis for  $V^{(\ell,1)} = \pi_{11}^{(\ell)}(V)$  then*

$$\mathcal{Q}^{(\ell)} = \bigcup_{j=1}^{m_\ell} \{q_j^{(\ell)}, \pi_{21}^{(\ell)}(q_j^{(\ell)}), \dots, \pi_{n_\ell 1}^{(\ell)}(q_j^{(\ell)})\}$$

*is a symmetry adapted basis of  $V^{(\ell)}$  and  $\mathcal{Q} = \bigcup_{\ell=1}^l \mathcal{Q}^{(\ell)}$  is a symmetry adapted basis of  $V$ .*

If the representation  $\tau$  is actually defined over  $\mathbb{R}$ , we can also compute real symmetry adapted bases whose components are in correspondence with the irreducible representations over  $\mathbb{R}$ . This can be obtained by combining conjugate pairs of irreducible representations over  $\mathbb{C}$  and is detailed for instance in [RH21].

## 2.2 Modules of equivariants

Consider a representation  $\varrho : \mathfrak{G} \rightarrow \text{GL}_m(\mathbb{C})$ . A polynomial  $q \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$  is an *invariant polynomial*, or simply an *invariant*, if  $q \circ \varrho(g) = q$  for all  $g \in \mathfrak{G}$ . The set of all invariants for  $\varrho$  form a ring which is denoted  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ .

More generally, if  $\tau : \mathfrak{G} \rightarrow \text{GL}_m(\mathbb{C})$  is a  $m$ -dimensional matrix representation of  $\mathfrak{G}$ , an  $\tau$ -*equivariant* (or more precisely a  $(\varrho : \tau)$ -equivariant) is a row vector  $q \in \mathbb{C}[\mathbf{x}]^m$  such that  $q \circ \varrho(g^{-1}) = q \tau(g)$ , where the right-hand side is a vector-matrix multiplication.

The set of all  $\tau$ -equivariants forms a  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -module that we denote  $\mathbb{C}[\mathbf{x}]_{\tau}^{\mathfrak{G}}$ . A set of  $\tau$ -equivariants  $\mathcal{Q} = \{q_1, \dots, q_k\}$  is generating for  $\mathbb{C}[\mathbf{x}]_{\tau}^{\mathfrak{G}}$  as a  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -module if any other  $\tau$ -equivariant  $q$  can be written as a linear combination of elements of  $\mathcal{Q}$  over  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ :  $q = a_1 q_1 + \dots + a_k q_k$ , with  $a_i \in \mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ .

We call *basic equivariant* an  $\tau^{(\ell)}$ -equivariant, for some  $1 \leq \ell \leq l$ . The  $\tau$ -equivariants, for any matrix representation  $\tau : \mathfrak{G} \rightarrow \text{GL}_m(\mathbb{C})$ , are linear combinations of basic equivariants. Indeed, let  $q \in \mathbb{C}[\mathbf{x}]^m$  be an  $\tau$ -equivariant and  $P \in \mathbb{C}^{m \times m}$  be an invertible matrix such that  $P^{-1} \tau(g) P = \text{diag}(I_{m_\ell} \otimes \tau^{(\ell)}(g) \mid 1 \leq \ell \leq l)$  for all  $g \in \mathfrak{G}$ . Then  $qP$  is a  $(m_1 \tau^{(1)} \oplus \dots \oplus m_l \tau^{(l)})$ -equivariant, *i.e.*, its components are basic equivariants.

Sets  $\mathcal{Q}^{(1)}$  of invariants and  $\mathcal{Q}^{(\ell)}$ ,  $2 \leq \ell \leq l$ , of  $\tau^{(\ell)}$ -equivariants are called *fundamental* if they generate, respectively, the ring  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$  and the  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -modules  $\mathbb{C}[\mathbf{x}]_{\tau^{(\ell)}}^{\mathfrak{G}}$ .

## 2.3 Symmetry adapted bases in polynomial rings

The representation space we are concerned with in this section is the polynomial ring in  $n$  variables  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$  or an invariant subspace of it, as for instance  $\mathbb{C}[\mathbf{x}]_d$ .

We consider the representation  $\rho: \mathfrak{G} \rightarrow \text{GL}(\mathbb{C}[\mathbf{x}])$  induced by a representation  $\varrho: \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{C}^n)$ , i.e.,  $\rho(g)(p) = p \circ \varrho(g^{-1})$ , for  $p \in \mathbb{C}[\mathbf{x}]$ ,  $g \in \mathfrak{G}$ . For a row vector  $\mathbf{q} = [q_1, \dots, q_m] \in \mathbb{C}[\mathbf{x}]^m$  of polynomials we write  $\rho(g)(\mathbf{q})$  for the row vector  $[\rho(g)(q_1), \dots, \rho(g)(q_m)] \in \mathbb{C}[\mathbf{x}]^m$ .

The representation  $\rho: \mathfrak{G} \rightarrow \text{GL}(\mathbb{C}[\mathbf{x}])$  leaves invariant the finite dimensional subspace  $\mathbb{C}[\mathbf{x}]_d$  spanned by the homogeneous polynomials of degree  $d$ . The isotypic components of  $\mathbb{C}[\mathbf{x}]_d$  are denoted  $\mathbb{C}[\mathbf{x}]_d^{(\ell)}$ . We write

$$\mathbb{C}[\mathbf{x}] = \bigoplus_{d \in \mathbb{N}} \mathbb{C}[\mathbf{x}]_d = \bigoplus_{d \in \mathbb{N}} \bigoplus_{\ell=1}^l \mathbb{C}[\mathbf{x}]_d^{(\ell)}, \quad \mathbb{C}[\mathbf{x}]^{(\ell)} = \bigoplus_{d \in \mathbb{N}} \mathbb{C}[\mathbf{x}]_d^{(\ell)}, \quad \mathbb{C}[\mathbf{x}] = \bigoplus_{\ell=1}^l \mathbb{C}[\mathbf{x}]^{(\ell)};$$

and, calling onto the notations introduced in Proposition 2.1,

$$\mathbb{C}[\mathbf{x}]_d^{(\ell)} = \bigoplus_{i=1}^{n_\ell} \mathbb{C}[\mathbf{x}]_d^{(\ell,i)}, \quad \mathbb{C}[\mathbf{x}]^{(\ell,i)} = \bigoplus_{d \in \mathbb{N}} \mathbb{C}[\mathbf{x}]_d^{(\ell,i)}, \quad \mathbb{C}[\mathbf{x}]^{(\ell)} = \bigoplus_{i=1}^{n_\ell} \mathbb{C}[\mathbf{x}]^{(\ell,i)}.$$

$\mathbb{C}[\mathbf{x}]^{(1)}$  is the  $\mathbb{C}$ -vector space of invariants and is equal, setwise, to  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ . The infinite dimensional isotypic components  $\mathbb{C}[\mathbf{x}]^{(\ell)}$ , and their components  $\mathbb{C}[\mathbf{x}]^{(\ell,i)}$ , are  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -modules: if  $p \in \mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ , then  $\pi_{ij}^{(\ell)}(pq) = p \pi_{ij}^{(\ell)}(q)$  for any  $q \in \mathbb{C}[\mathbf{x}]$ .

Now observe that if  $\bigcup_{i=1}^{m_\ell} \{q_{i1}^{(\ell)}, \dots, q_{in_\ell}^{(\ell)}\}$  is a symmetry adapted basis of  $\mathbb{C}[\mathbf{x}]_d^{(\ell)}$  then the row vector  $\mathbf{q}_i^{(\ell)} = [q_{i1}^{(\ell)}, \dots, q_{in_\ell}^{(\ell)}]$  is a  $\tau^{(\ell)}$ -equivariant. The elements of a symmetry adapted basis of an invariant subspace of  $\mathbb{C}[\mathbf{x}]$  combine to form basic equivariants. The components of these equivariants will appear within a bracket when we spell out a symmetry adapted basis in a polynomial ring. Hence a symmetry adapted basis of an invariant subspace of  $\mathbb{C}[\mathbf{x}]$  will subsequently appear as

$$\mathcal{Q} = \bigcup \mathcal{Q}^{(\ell)} \text{ where } \mathcal{Q}^{(\ell)} = \{q_1^{(\ell)}, \dots, q_{m_\ell}^{(\ell)}\} = \left\{ [q_{i1}^{(\ell)}, \dots, q_{in_\ell}^{(\ell)}] \mid 1 \leq i \leq m_\ell \right\} \text{ with } q_{ik}^{(\ell)} \in \mathbb{C}[\mathbf{x}]^{(\ell,k)}.$$

We conclude this section by formalizing the correspondances between the  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -modules  $\mathbb{C}[\mathbf{x}]_{\tau^{(\ell)}}$  and the isotypic components  $\mathbb{C}[\mathbf{x}]^{(\ell)}$  of  $\mathbb{C}[\mathbf{x}]$ . One then sees that a set of fundamental equivariants generates  $\mathbb{C}[\mathbf{x}]$  as a  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -module.

**Proposition 2.3** For  $1 \leq \ell \leq l$  and any  $1 \leq k \leq n_\ell$ , the  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -linear maps

$$\begin{aligned} \phi_k: \quad \mathbb{C}[\mathbf{x}]_{\tau^{(\ell)}}^{\mathfrak{G}} &\rightarrow \mathbb{C}[\mathbf{x}]^{(\ell,k)} & \text{and} & \quad \Phi_k: \mathbb{C}[\mathbf{x}]^{(\ell,k)} &\rightarrow \mathbb{C}[\mathbf{x}]_{\tau^{(\ell)}}^{\mathfrak{G}} \\ [q_1, \dots, q_{n_\ell}] &\mapsto q_k & & & q &\mapsto [\pi_{1k}^{(\ell)}(q), \dots, \pi_{n_\ell k}^{(\ell)}(q)] \end{aligned}$$

are well defined and inverse of each other.

PROOF: The fact that  $\phi_k$  and  $\Phi_k$  are  $\mathbb{C}[\mathbf{x}]^{\mathfrak{G}}$ -linear is an easy observation from the definition of the maps  $\pi_{ji}^{(\ell)}$ . We first show that the images of  $\phi_k$  and  $\Phi_k$  are correctly described.

Let  $\mathbf{q} = [q_1, \dots, q_{n_\ell}]$  be an  $\tau^{(\ell)}$ -equivariant. By definition of equivariance  $\mathbf{q} = \rho(g)(\mathbf{q}) \tau^{(\ell)}(g)^{-1}$ . Hence

$$\mathbf{q} = [\rho(g)(q_1), \dots, \rho(g)(q_{n_\ell})] \tau^{(\ell)}(g^{-1}) = \left[ \sum_{j=1}^{n_\ell} \tau_{j1}^{(\ell)}(g^{-1}) \rho(g)(q_j), \dots, \sum_{j=1}^{n_\ell} \tau_{jn_\ell}^{(\ell)}(g^{-1}) \rho(g)(q_j) \right].$$

Summing over  $g \in \mathfrak{G}$  we see

$$|\mathfrak{G}| \mathbf{q} = \left[ \sum_{j=1}^{n_\ell} \sum_{g \in \mathfrak{G}} \tau_{j1}^{(\ell)}(g^{-1}) \rho(g)(q_j), \dots, \sum_{j=1}^{n_\ell} \sum_{g \in \mathfrak{G}} \tau_{jn_\ell}^{(\ell)}(g^{-1}) \rho(g)(q_j) \right] = \frac{|\mathfrak{G}|}{n_\ell} \left[ \sum_{j=1}^{n_\ell} \pi_{1j}^{(\ell)}(q_j), \dots, \sum_{j=1}^{n_\ell} \pi_{n_\ell j}^{(\ell)}(q_j) \right].$$



Since  $\pi_{jk}^{(\ell)} \circ \pi_{kl}^{(\ell)} = \pi_{jl}^{(\ell)}$  we can write

$$q = \frac{1}{n_\ell} \left[ \sum_{j=1}^{n_\ell} \pi_{11}^{(\ell)} \circ \pi_{1j}^{(\ell)}(q_j), \dots, \sum_{j=1}^{n_\ell} \pi_{n_\ell 1}^{(\ell)} \circ \pi_{1j}^{(\ell)}(q_j) \right] = \left[ \pi_{11}^{(\ell)}(\hat{q}), \dots, \pi_{n_\ell 1}^{(\ell)}(\hat{q}) \right], \quad (3)$$

where  $\hat{q} = \frac{1}{n_\ell} \sum_{j=1}^{n_\ell} \pi_{1j}^{(\ell)}(q_j)$ . It follows that  $\phi_k(q) \in \mathbb{C}[x]^{(\ell, k)}$ .

Now let  $q \in \mathbb{C}[x]^{(\ell, k)}$ . By Proposition 2.1.3 we have  $\rho(g) \left( \pi_{jk}^{(\ell)}(q) \right) = \sum_{i=1}^{n_\ell} \tau_{ij}^{(\ell)}(g) \pi_{ik}^{(\ell)}(q)$  so that

$$\begin{aligned} \rho(g) \left( \Phi_k(q) \right) &= \left[ \sum_{i=1}^{n_\ell} \tau_{i1}^{(\ell)}(g) \pi_{ik}^{(\ell)}(q), \dots, \sum_{i=1}^{n_\ell} \tau_{in_\ell}^{(\ell)}(g) \pi_{ik}^{(\ell)}(q) \right] \\ &= \left[ \pi_{1k}^{(\ell)}(q), \dots, \pi_{n_\ell k}^{(\ell)}(q) \right] \tau^{(\ell)}(g) = \Phi_k(q) \tau^{(\ell)}(g). \end{aligned}$$

Hence  $\Phi_k(q) \in \mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ .

To conclude we show that  $\phi_k \circ \Phi_k$  and  $\Phi_k \circ \phi_k$  are the identity maps. Let  $q = [q_1, \dots, q_{n_\ell}] \in \mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ . By Equation (3) there exists  $\hat{q} \in \mathbb{C}[x]$  such that  $q = \left[ \pi_{11}^{(\ell)}(\hat{q}), \dots, \pi_{n_\ell 1}^{(\ell)}(\hat{q}) \right]$ . Hence  $\Phi_k \circ \phi_k(q) = \Phi_k \left( \pi_{k1}^{(\ell)}(\hat{q}) \right) = \left[ \pi_{1k}^{(\ell)} \circ \pi_{k1}^{(\ell)}(\hat{q}), \dots, \pi_{n_\ell k}^{(\ell)} \circ \pi_{k1}^{(\ell)}(\hat{q}) \right] = q$ . If now  $q \in \mathbb{C}[x]^{(\ell, k)}$  then  $\pi_{kk}^{(\ell)}(q) = q$  so that  $\phi_k \circ \Phi_k(q) = q$ .  $\square$

**Corollary 2.4** For  $1 \leq \ell \leq \mathfrak{l}$ , consider a set  $\mathcal{Q} = \{q_1, \dots, q_m\}$  of  $\tau^{(\ell)}$ -equivariants, with  $q_i = [q_{i1}, \dots, q_{in_\ell}]$ . Considering  $\mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ ,  $\mathbb{C}[x]^{(\ell, k)}$ , and  $\mathbb{C}[x]^{(\ell)}$  as  $\mathbb{C}[x]^{\mathfrak{G}}$ -modules, the following statements are equivalent:

1.  $\mathcal{Q}$  is a generating set for  $\mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$
2. for any given  $1 \leq k \leq n_\ell$ ,  $\{q_{ik} \mid 1 \leq i \leq m\}$  is a generating set for  $\mathbb{C}[x]^{(\ell, k)}$
3.  $\{q_{ik} \mid 1 \leq k \leq n_\ell, 1 \leq i \leq m\}$  is a generating set for  $\mathbb{C}[x]^{(\ell)}$ .

PROOF:

1.  $\Rightarrow$  2. Let  $q \in \mathbb{C}[x]^{(\ell, k)}$ . Since  $\Phi_k(q) \in \mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ , there exist  $h_1, \dots, h_m \in \mathbb{C}[x]^{\mathfrak{G}}$  such that  $\Phi_k(q) = \sum_{i=1}^m h_i q_i$ . Applying  $\phi_k$  on both sides we obtain  $q = \sum_{i=1}^m h_i \phi_k(q_i) = \sum h_i q_{ik}$ .

2.  $\Rightarrow$  1. Take  $q \in \mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ . Since  $\phi_k(q) \in \mathbb{C}[x]^{(\ell, k)}$  there are invariant polynomials  $h_1, \dots, h_m$  such that  $\phi_k(q) = \sum_{i=1}^m h_i q_{ik}$ . Applying  $\Phi_k$  on both sides we obtain  $q = \sum_{i=1}^m h_i \Phi_k(q_{ik})$  and therefore  $\Phi_k(q_{1k}) = q_1, \dots, \Phi_k(q_{mk}) = q_m$  form a generating set for  $\mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{G}}$ .

2.  $\Rightarrow$  3. Since  $\mathbb{C}[x]^{(\ell)} = \bigoplus_{j=1}^{n_\ell} \mathbb{C}[x]^{(\ell, j)}$ , and  $\pi_{jk}^{(\ell)} : \mathbb{C}[x]^{(\ell, k)} \rightarrow \mathbb{C}[x]^{(\ell, j)}$  is  $\mathbb{C}[x]^{\mathfrak{G}}$ -linear and bijective, and  $\pi_{jk}^{(\ell)}(q_{ik}) = q_{ij}$ .

3.  $\Rightarrow$  2. Since  $\pi_{kk}^{(\ell)} : \mathbb{C}[x]^{(\ell)} \rightarrow \mathbb{C}[x]^{(\ell, k)}$  is  $\mathbb{C}[x]^{\mathfrak{G}}$ -linear and surjective, and  $\pi_{kk}^{(\ell)}(q_{ij})$  equals to  $q_{ik}$  or 0 according to whether  $j = k$  or not.  $\square$

### 3 Zero dimensional ideals, H-bases and interpolation

This is a section with the premises for our constructions of fundamental invariants and equivariants. It somewhat recapitulates the ingredients and results of [RH21, RH22].

We consider the polynomial ring  $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$  in  $n$  variables and its subspaces  $\mathbb{C}[x]_d$  of homogeneous polynomials of degree  $d$ , and  $\mathbb{C}[x]_{\leq d} = \bigoplus_{e=0}^d \mathbb{C}[x]_e$ .

### 3.1 Orthogonality

$\mathbb{C}[x]$  is naturally endowed with the *apolar product* that can be defined, for  $p, q \in \mathbb{C}[x]$ , as  $\langle p, q \rangle = \bar{p}(\partial)q$ . In the monomial basis, this is spelt out as  $\langle \sum_{\alpha} p_{\alpha} x^{\alpha}, \sum_{\alpha} q_{\alpha} x^{\alpha} \rangle = \sum_{\alpha} \alpha! \bar{p}_{\alpha} q_{\alpha}$ . If  $\mathcal{P} = \{p_1, p_2, \dots\}$  is a basis of  $\mathbb{C}[x]$  we note  $\mathcal{P}^{\dagger}$  the dual basis, that is, the basis  $\{p_1^{\dagger}, p_2^{\dagger}, \dots\}$  such that  $\langle p_i^{\dagger}, p_j \rangle = \delta_{ij}$ . For instance, the dual basis of the monomial basis  $\{x^{\alpha} \mid \alpha \in \mathbb{N}^n\}$  is  $\{\frac{1}{\alpha!} x^{\alpha} \mid \alpha \in \mathbb{N}^n\}$ . The choice of the apolar product as the inner product is crucial to later exploit symmetry.

At many places we shall invoke the orthogonal complement of a subspace  $Q = \langle q_1, \dots, q_r \rangle_{\mathbb{C}}$  in a subspace  $P$  of  $\mathbb{C}[x]$  with basis  $\mathcal{P} = \{p_1, p_2, \dots, p_m\}$ . One way to compute it is to consider the matrix  $A$  whose columns are the coefficients of the  $q_i$  in  $\mathcal{P}$ . A nonzero row vector  $[a_1, \dots, a_m]$  is in the left kernel of  $A$  iff  $\bar{a}_1 p_1^{\dagger} + \dots + \bar{a}_m p_m^{\dagger}$  is in the orthogonal complement of  $Q$  in  $P$ . A basis for the left kernel of  $A$  is for instance obtained through a QR-decomposition [GVL83]. To avoid dealing with both  $\mathcal{P}$  and  $\mathcal{P}^{\dagger}$  one may use an orthonormal polynomial basis.

### 3.2 Zero-dimensional ideals and interpolation.

As expanded on in [CLO05], some properties of the variety of an ideal  $J$  in  $\mathbb{C}[x]$  can be understood from the structure of the quotient algebra  $\mathbb{C}[x]/J$ . For instance, if  $\mathbb{C}[x]/J$  is a finite dimensional  $\mathbb{C}$ -vector space then the variety of  $J$  consists only of points. Such ideals are said to be *zero-dimensional*. When appropriately counted with multiplicities, the number of these points is the dimension of  $\mathbb{C}[x]/J$  as a  $\mathbb{C}$ -vector space. A basis  $\mathcal{Q}$  of a direct complement of  $J$  in  $\mathbb{C}[x]$  can be identified with a basis of  $\mathbb{C}[x]/J$ .

We write  $\mathbb{C}[x]^*$  for the set of linear forms on  $\mathbb{C}[x]$ . If  $r$  is the dimension of  $\mathbb{C}[x]/J$  then  $J$  can be described as the orthogonal of a  $r$ -dimensional  $\mathbb{C}$ -vector subspace  $\Lambda$  of  $\mathbb{C}[x]^*$  of dimension  $r$ :  $J = \Lambda^{\perp} = \cap_{\lambda \in \Lambda} \ker \lambda$ . We shall consider in this article two such descriptions.

First, when  $J$  is radical, its variety consists of  $r$  simple zeros  $\xi_1, \dots, \xi_r \in \mathbb{C}^n$ . Then  $J = \cap_{i=1}^r \ker e_{\xi_i}$  where  $e_{\xi} : \mathbb{C}[x] \rightarrow \mathbb{C}$  is the evaluation at  $\xi \in \mathbb{C}^n$ .

The second scenario is when we have a basis  $\mathcal{Q} = \{q_1, \dots, q_r\}$  of a direct complement of  $J$  in  $\mathbb{C}[x]$ . This allows us to define uniquely  $\lambda_1, \dots, \lambda_r \in \mathbb{C}[x]^*$  by the equation:

$$p \equiv \lambda_1(p)q_1 + \dots + \lambda_r(p)q_r \pmod{J}, \quad p \in \mathbb{C}[x].$$

Then  $J = \Lambda^{\perp}$ , where  $\Lambda = \langle \lambda_1, \dots, \lambda_r \rangle_{\mathbb{C}}$ . This is in particular relevant when we have a Gröbner basis  $\mathcal{B}$  of  $J$ , according to some term order. The terms  $\mathcal{Q} = \{x^{\alpha_1}, \dots, x^{\alpha_r}\}$  that are not multiples of the leading terms of  $\mathcal{B}$  form a basis of  $\mathbb{C}[x]/J$  and the associated forms  $\lambda_1, \dots, \lambda_r$  are computable by Hironaka division [CLO15].

Conversely, if  $\Lambda$  is a  $r$ -dimensional subspace of  $\mathbb{C}[x]^*$  and  $J = \cap_{\lambda \in \Lambda} \ker \lambda$  is an ideal, then, according to [RH22, Section 2],  $\mathcal{Q}$  is a basis of  $\mathbb{C}[x]/J$  if and only if  $\langle \mathcal{Q} \rangle_{\mathbb{C}}$  is an *interpolation space* for  $\Lambda$ , that is: for any linear map  $\phi : \Lambda \rightarrow \mathbb{C}$ , there is a single  $q \in \langle \mathcal{Q} \rangle_{\mathbb{C}}$  such that  $\lambda(p) = \phi(\lambda)$  for all  $\lambda \in \Lambda$ . In these conditions, the *least interpolation space* [DBR92, RH21], is the orthogonal complement of  $J^0$  with respect to the apolar product [RH22, Proposition 2.7].

### 3.3 H-bases

On one hand, *Gröbner bases* are the most established and versatile representations of polynomial ideals [CLO15]. They are defined after a term order is fixed and one then focuses on the leading terms of the polynomials and the initial ideal of  $J$ . The basis of choice for  $\mathbb{C}[x]/J$  then consists of the monomials. On the other hand, monomial bases are incompatible with most symmetries. We hence turn our attention to H-bases, where one focuses on the leading homogeneous forms instead of the leading terms. These were introduced in [Mac16] and revisited in [MS00].

For a nonzero polynomial  $p$  in  $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$  we shall denote  $p^0$  the *leading form* of  $p$ , *i.e.*, the homogeneous polynomial of  $\mathbb{C}[x]$  such that  $\deg(p - p^0) < \deg(p)$ . For a set  $\mathcal{P} \subset \mathbb{C}[x]$ ,  $\mathcal{P}^0$  is the set of the leading forms of the elements in  $\mathcal{P}$ . If  $J$  is an ideal in  $\mathbb{C}[x]$ , then  $J^0$  is a homogeneous ideal in  $\mathbb{C}[x]$ . We have  $\dim \mathbb{C}[x]/J = \dim \mathbb{C}[x]/J^0$  and if  $\mathcal{Q}$  is a linearly independent set of homogeneous polynomials s.t.  $\mathbb{C}[x] = J^0 \oplus \langle \mathcal{Q} \rangle_{\mathbb{C}}$  then  $\mathbb{C}[x] = J \oplus \langle \mathcal{Q} \rangle_{\mathbb{C}}$  [MS00, Theorem 8.6].

**Definition 3.1** A finite set  $\mathcal{H} := \{h_1, \dots, h_m\} \subset \mathbb{C}[x]$  is a H-basis of the ideal  $J := \langle h_1, \dots, h_m \rangle$  if one of the two equivalent conditions holds:

- $J^0 = \langle h_1^0, \dots, h_m^0 \rangle$ ;
- for all  $p \in J$ , there exist  $g_1, \dots, g_m$  such that  $p = \sum_{i=1}^m h_i g_i$  and  $\deg(h_i) + \deg(g_i) \leq \deg(p)$ ,  $i = 1, \dots, m$ .

Any ideal has a finite H-basis since Hilbert basis theorem ensures that  $J^0$  has a finite basis.

We shall now introduce the concepts of minimal, orthogonal and reduced H-basis.

**Definition 3.2** Given a row vector  $h = [h_1, \dots, h_m] \in \mathbb{C}[x]^m$  of homogeneous polynomials and a degree  $d$ , we define the Sylvester map to be the linear map

$$\begin{aligned} \psi_{d,h} : \mathbb{C}[x]_{d-d_1} \times \dots \times \mathbb{C}[x]_{d-d_m} &\rightarrow \mathbb{C}[x]_d \\ (f_1, \dots, f_m) &\rightarrow \sum_{i=1}^m f_i h_i \end{aligned}$$

where  $d_1, \dots, d_m$  are the respective degrees of  $h_1, \dots, h_m$ . If  $\mathcal{H} = \{h_1, \dots, h_m\}$  then  $\Psi_d(\mathcal{H})$  shall denote the image of  $\psi_{d,h}$ , *i.e.*,

$$\Psi_d(\mathcal{H}) = \left\{ \sum_{i=1}^m f_i h_i \mid f_i \in \mathbb{C}[x]_{d-\deg(h_i)} \right\} \subset \mathbb{C}[x]_d.$$

If  $\mathcal{H}$  is a set of polynomials, we shall use the notation  $\mathcal{H}_d^0$  for the set of the degree  $d$  elements of  $\mathcal{H}^0$ . In other words  $\mathcal{H}_d^0 = \mathcal{H}^0 \cap \mathbb{C}[x]_d$ .

**Definition 3.3** We say that a H-basis  $\mathcal{H}$  is minimal if, for any  $d \in \mathbb{N}$ ,  $\mathcal{H}_d^0$  is linearly independent and

$$\Psi_d(J_{d-1}^0) \oplus \langle \mathcal{H}_d^0 \rangle_{\mathbb{C}} = J_d^0. \quad (4)$$

When minimal,  $\mathcal{H}$  is said to be orthogonal if  $\langle \mathcal{H}_d^0 \rangle_{\mathbb{C}}$  is the orthogonal complement of  $\Psi_d(J_{d-1}^0)$  in  $J_d^0$ . Then the reduced H-basis associated to  $\mathcal{H}$  is defined by  $\tilde{\mathcal{H}} = \{h - \tilde{h} \mid h \in \mathcal{H}^0\}$  where, for  $h \in \mathbb{C}[x]$ ,  $\tilde{h}$  is the projection of  $h$  on the orthogonal complement of  $J^0$  parallel to  $J$ .

### 3.4 Symmetry

We consider the representation  $\rho : \mathfrak{G} \rightarrow \text{GL}(\mathbb{C}[x])$  induced by a unitary representation  $\varrho : \mathfrak{G} \rightarrow \mathfrak{U}_n(\mathbb{C})$ , *i.e.*,  $\rho(g)(p) = p \circ \varrho(g^{-1})$ , for  $p \in \mathbb{C}[x]$ ,  $g \in \mathfrak{G}$ . The dual representation is denoted  $\rho^* : \mathfrak{G} \rightarrow \text{GL}(\mathbb{C}[x]^*)$ , with  $\rho^*(g)(\lambda)(p) = \lambda(\rho(g^{-1})(p))$ . It is actually no loss of generality to assume the representation on  $\mathbb{C}^n$  unitary. Indeed, for  $\varrho : \mathfrak{G} \rightarrow \text{GL}_n(\mathbb{C})$ , we can construct an invariant inner product, for instance by averaging over the group the natural inner product. Then, in a basis orthonormal w.r.t. this invariant inner product, the representation becomes unitary.

The apolar product has the property that, for  $a : \mathbb{C}^n \rightarrow \mathbb{C}^n$  a linear map,  $\langle p, q \circ a \rangle = \langle p \circ \bar{a}^t, q \rangle$ . Hence for a unitary representation  $\varrho$ , the apolar product is invariant, *i.e.*,  $\langle \rho(g)(p), \rho(g)(q) \rangle = \langle p, q \rangle$ . Invariance is

furthermore preserved in the constructions we reviewed in this section: the homogeneous subspace  $\mathbb{C}[x]_d$  are invariant, and thus, if an ideal  $J$  is invariant, so are the leading form ideal  $J^0$  together with its orthogonal complement, as well as  $J^\perp = \{\lambda \in \mathbb{C}[x]^* \mid \lambda(p) = 0 \ \forall p \in J\}$ . Conversely if  $\Lambda \subset \mathbb{C}[x]^*$  is invariant, so is  $\Lambda^\perp \subset \mathbb{C}[x]$ . One then sees that if  $J$  is invariant it admits a *symmetry adapted H-basis*, that is, a reduced H-basis that consists of basic equivariants.

A key to exploiting symmetry in algorithms is to exhibit the equivariance of the linear maps involved in the constructions. Their matrices can then be block diagonalized, with identical blocks being repeated according to the dimensions of the irreducible representations of the group. For ideal interpolation [RH22], two maps are relevant: the Vandermonde operator, to obtain a basis of an interpolation space, and the Sylvester map, introduced in Definition 3.2, to obtain the H-basis of the ideal. For the purpose of the algorithm in Section 6, we shall recall the equivariance of this latter.

**Proposition 3.4** [RH22, Proposition 5.2] *Consider  $\Theta : \mathfrak{G} \rightarrow \mathrm{GL}_m(\mathbb{C})$  and  $h = [h_1, \dots, h_m] \in \mathbb{C}[x]_{d_1} \times \dots \times \mathbb{C}[x]_{d_m}$  such that  $h \circ \vartheta(g^{-1}) = h \cdot \Theta(g)$ , for all  $g \in G$ . For any  $d \in \mathbb{N}$ , the map  $\psi_{dh}$  is  $\tau - \rho$  equivariant, i.e.,  $\psi_{dh} \circ \tau(g) = \rho(g) \circ \psi_{dh}$ , for the representation  $\tau$  on  $\mathbb{C}[x]_{d-d_1} \times \dots \times \mathbb{C}[x]_{d-d_m}$  defined by  $\tau(g)(f) = \Theta(g) \cdot f \circ \varrho(g^{-1})$ .*

For  $e \in \mathbb{N}$  and  $1 \leq \ell \leq l$  we introduce the representations

$$\tau_e^{(\ell)} : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{C}[x]_e^{n_e}), \quad \text{given by} \quad \tau_e^{(\ell)}(g)(f) = \mathfrak{r}^{(\ell)}(g) \cdot f \circ \varrho(g^{-1}).$$

If  $\mathcal{H}$  consists of basic equivariants,  $\mathcal{H} = \bigcup_{\ell=1}^l \{h_i^{(\ell)} \mid 1 \leq i \leq m_\ell\}$ , we see that  $\Psi_d(\mathcal{H})$  is the sum of the images of  $\psi_{dh_i^{(\ell)}}$ , which is  $\tau_e^{(\ell)}$ -equivariant, with  $e = d - \deg h_i^{(\ell)}$ . According to [FS92, Theorem 2.5], recalled as [RH22, Proposition 5.4], the matrix of  $\psi_{dh_i^{(\ell)}}$  in  $\tau_e^{(\ell)}$ - and  $\rho_d$ - symmetry adapted bases of  $\mathbb{C}[x]_e^{n_e}$  and  $\mathbb{C}[x]_d$  respectively is

$$\mathrm{diag} \left( \mathbf{I}_{n_k} \otimes A_{\ell,i}^{(k)} \mid k = 1 \dots l \right)$$

where  $A_{\ell,i}^{(k)}$  is a matrix with dimensions the multiplicities of  $\mathfrak{r}^{(k)}$  in  $\mathbb{C}[x]_e^{n_e}$  and  $\mathbb{C}[x]_d$ .

### 3.5 Algorithms

The goal in this article is to identify generating sets of invariants and equivariants. The algorithm for ideal interpolation with symmetry presented in [RH22] is at the heart of our two first constructions, in Section 4 and 5. It also inspired our third algorithm, in Section 6. We recall here the input and output of this algorithm, with the vocabulary that is consistent with the main thread of the present article.

Just as it will be the case of the further three constructions, we shall need symmetry adapted bases up to a certain degree  $d$ . They can be obtained by the projections introduced in Section 2.1. An upper bound on this degree is known at the start of each algorithm, but the termination of the algorithms is based on other criteria that might prove the needed degree to be much smaller. For Algorithm 3.5 the upper bound is  $r = \dim \Lambda$ . Yet it is more likely that we only need to go up to a degree  $d$  such that  $r \leq \dim \mathbb{C}[x]_{\leq d}$ .

**Algorithm 3.5** *Ideal Interpolation with Symmetry* [RH22, Algorithm 3]

**Input:**

- $\mathcal{L} = \bigcup_{\ell=1}^l \mathcal{L}^{(\ell)}$  a symmetry adapted basis for  $\Lambda$ , a  $r$ -dimensional subspace of  $\mathbb{C}[x]^*$ .
- $\mathcal{P} = \bigcup_{\ell=1}^l \bigcup_{d=1}^r \mathcal{P}_d^{(\ell)}$  an orthonormal graded symmetry adapted basis of  $\mathbb{C}[x]_{\leq r}$

- $\mathcal{M}_d^{(\ell)}$  the symmetry adapted basis for the representations  $\tau_d^{(\ell)}$  on  $(\mathbb{C}[x]_d)^{n_\ell}$ ,  $1 \leq \ell \leq l$ ,  $1 \leq d \leq r - 1$

**Output:**

- $\mathcal{H}$  a symmetry adapted  $H$ -basis for  $J := \bigcap_{\lambda \in \Lambda} \ker \lambda$
- $\mathcal{Q} = \bigcup_{\ell=1}^l \mathcal{Q}^{(\ell)}$  a symmetry adapted basis of the orthogonal complement of  $J^0$ .

Though the algorithm calls for the symmetry adapted bases of  $(\mathbb{C}[x]_d)^{n_\ell}$  for the representations  $\tau_d^{(\ell)}$ , in the applications we make of it, in Section 4 and Section 5, the set  $\mathcal{H}$  constructed consists solely of invariants and thus we only need  $\tau_d^{(1)}$ . Hence only  $\mathcal{P}$  is actually needed.

## 4 Invariants & equivariants by interpolation

A reflection group is a subgroup of  $GL_n(\mathbb{C})$  that is generated by matrices that have precisely one eigenvalue different from 1. In this section we show that we can deduce fundamental invariants and equivariants for such group actions from the solution of an ideal interpolation problem as computed in [RH22, Algorithm 3], whose input and output are recalled in Section 3.5.

Reflection groups enjoy extensive properties [BG85, Che55, Kan01]. The most well known is the fact that their rings of invariants  $\mathbb{C}[x]^{\mathfrak{G}}$  are polynomial, *i.e.*, are generated by algebraically independent homogeneous invariants. Concomitantly, if  $N$  is the ideal generated by the invariants of positive degree, that we shall call the Hilbert ideal, the representation induced on the *covariant algebra*  $\mathbb{C}[x]/N$  is equivalent to the regular representation of  $\mathfrak{G}$ . The beautiful properties of reflection groups offer several approaches to compute, first, the invariants whose degrees can be unequivocally read on the Molien series, and then the fundamental equivariants either as  $\mathfrak{G}$ -harmonic polynomials or from the cyclic structure of the covariant algebra. For some classical reflection groups, the fundamental equivariants can be determined through a combinatorial approach [ATY97, Spe35].

In this section we wish to make the rather remarkable observation that the fundamental invariants and equivariants, can be obtained as a direct application of ideal interpolation with symmetry as performed by [RH22, Algorithm 3]. We just need to interpolate along an orbit in general position. This observation was the starting point for the results in this paper. It was later pointed out to us that some points made in [Ber09, Section 8.3] are reminiscent of this observation.

We start with two propositions which are not specific to reflection groups. They lead to the key lemma that is restricted to reflection groups. After another technical lemma we can state the theorem that explains how to apply Algorithm 3.5 to obtain the fundamental invariants and equivariants.

**Proposition 4.1** *For  $\xi \in \mathbb{C}^n$ , consider the evaluation map  $e_\xi : \mathbb{C}[x] \rightarrow \mathbb{C}$  and let  $J = \bigcap_{g \in \mathfrak{G}} \ker e_{g \cdot \xi}$  in  $\mathbb{C}[x]$  be the radical ideal of the orbit of  $\xi$ . Then  $\langle h - e_\xi(h) \mid h \in \mathbb{C}[x]^{\mathfrak{G}} \rangle \subset J$ .*

PROOF:  $e_{g \cdot \xi}(h - e_\xi(h)) = 0$  for all  $h \in \mathbb{C}[x]^{\mathfrak{G}}$  and  $g \in \mathfrak{G}$ .  $\square$

The isotropy subgroup  $\mathfrak{G}_\xi$  of  $\xi \in \mathbb{C}^n$  is the set of elements of  $\mathfrak{G}$  that leave  $\xi$  invariant:  $\mathfrak{G}_\xi = \{g \in \mathfrak{G} \mid g \cdot \xi = \xi\}$

**Proposition 4.2** *If  $J = \bigcap_{g \in \mathfrak{G}} \ker e_{g \cdot \xi}$  then the induced representation on  $\mathbb{C}[x]/J$  is equivalent to the permutation representation of  $\mathfrak{G}$  associated to  $\mathfrak{G}/\mathfrak{G}_\xi$ .*

PROOF: The ideal  $J = \bigcap_{g \in \mathfrak{G}} \ker e_{g \cdot \xi}$  is invariant under the action of  $\mathfrak{G}$ . There is thus an action of  $\mathfrak{G}$  induced on  $\mathbb{C}[x]/J$  that is well defined.

Let  $g_1, \dots, g_r$  be representatives for the classes of  $\mathfrak{G}/\mathfrak{G}_\xi$ . Denoting  $\xi_i = \varrho(g_i)(\xi)$  we have that  $\xi_i \neq \xi_j$  for  $i \neq j$  and  $\{\xi_1, \dots, \xi_r\}$  is the orbit of  $\xi$  and hence the zero set of  $J$ . Therefore for every  $g \in \mathfrak{G}$  there exists a permutation  $\sigma(g)$  of  $\{1, \dots, r\}$  such that  $\varrho(g)(\xi_i) = \xi_{\sigma(g)(i)}$ .

Let  $f_1, \dots, f_r$  be polynomial such  $f_i(\xi_j) = \delta_{ij}$ . Such polynomials can be found via interpolation and we prove next that they form a basis of  $\mathbb{C}[x]/J$ . This latter is of dimension  $r$  hence we only need to prove that they are linearly independent modulo  $J$ . Let  $a_1, \dots, a_r \in \mathbb{C}$  such that  $f = \sum_{i=1}^r a_i f_i \in J$ . Then for any  $1 \leq i \leq r$ ,  $a_i = \mathfrak{e}_{\xi_i}(f) = 0$ .

For any  $1 \leq i, j \leq r$  we have

$$\rho(g)(f_i)(\xi_j) = f_i \circ \varrho(g^{-1})(\xi_j) = f_i(\xi_{\sigma(g^{-1})(j)}) = f_{\sigma(g)(i)}(\xi_j).$$

It follows that  $\rho(g)(f_i) - f_{\sigma(g)(i)}$  vanishes on the orbit of  $\xi$  and thus  $\rho(g)(f_i) - f_{\sigma(g)(i)} \in J$ . Hence, in the basis  $f_1, \dots, f_r$ , the induced matrix representation of  $\mathfrak{G}$  on  $\mathbb{C}[x]/J$  is given by the permutation  $\sigma$ .  $\square$

In particular, if  $\mathfrak{G}_\xi$  is restricted to the identity then the induced representation on  $\mathbb{C}[x]/J$  is equivalent to the regular representation of  $\mathfrak{G}$  and thus has dimension  $|\mathfrak{G}|$ .

**Lemma 4.3** *Assume  $\mathfrak{G}$  is a reflection group and  $\xi \in \mathbb{C}^n$  such that  $\mathfrak{G}_\xi$  is restricted to the identity and define the ideal  $J = \bigcap_{g \in \mathfrak{G}} \ker \mathfrak{e}_{g \cdot \xi} \in \mathbb{C}[x]$ . Then  $J^0$  is the Hilbert ideal, i.e., the ideal generated by all the homogeneous invariants of positive degree.*

PROOF: Let  $N$  be the ideal generated by all the homogeneous invariants of positive degree. From Proposition 4.1 it follows that  $N \subset J^0$ .  $\mathfrak{G}$  being a reflection group, by [Che55] or [Kan01, Theorem 24-1],  $\mathbb{C}[x]/N$  is equivalent to the regular representation and thus of dimension  $|\mathfrak{G}|$ . By Proposition 4.2,  $\mathbb{C}[x]/J$  is also of dimension  $|\mathfrak{G}|$ , and thus so is  $\mathbb{C}[x]/J^0$ . Since  $\mathbb{C}[x]/J^0$  and  $\mathbb{C}[x]/N$  have the same dimension while  $N \subset J^0$ , we can conclude  $J^0 = N$ .  $\square$

**Lemma 4.4** *Let  $J$  be an ideal generated by  $k$  homogeneous invariants of positive degree. Then any homogeneous orthogonal  $H$ -basis of  $J$  consists of invariant polynomials, and of at most  $k$  of them.*

PROOF: We write  $J_d = J \cap \mathbb{C}[x]_d$  and  $J_{\leq d} = J \cap \mathbb{C}[x]_{\leq d}$  and similarly for  $\mathcal{H}$  a set of homogeneous invariants generating  $J$ . Recalling Definition 3.2, we have  $J_d = \Psi_d(\mathcal{H}_{<d}) + \langle \mathcal{H}_d \rangle_{\mathbb{C}}$ .

We show that for any degree  $d$ , any basis for the orthogonal complement  $Q_d$  of  $\Psi_d(J_{<d})$  in  $J_d$  consists of homogeneous invariants in numbers less or equal to the cardinal of  $\mathcal{H}_d$ .

Let  $m$  be the dimension of a complement of  $\Psi_d(\mathcal{H}_{<d})$  in  $J_d$ . Then  $m \leq \text{card}(\mathcal{H}_d)$ . We can select  $h_1, \dots, h_m \in \langle \mathcal{H}_d \rangle \subset \mathbb{C}[x]_d^{\mathfrak{G}}$  s.t.  $J_d = \Psi_d(\mathcal{H}_{<d}) \oplus \langle h_1, \dots, h_m \rangle_{\mathbb{C}}$ .

Consider  $q_1, \dots, q_m$  any basis of  $Q_d$ . There is then a nonsingular matrix  $(a_{ij}) \in \mathbb{C}^{m \times m}$  such that  $q_i = \sum a_{ij} h_j + r_i$  where  $r_i \in \Psi_d(\mathcal{H}_{<d})$ . Since  $\Psi_d(J_{<d})$ ,  $J_d$  and  $Q_d$  are invariant, the polynomial  $q_i - \pi^{(1)}(q_i) = r_i - \pi^{(1)}(r_i)$  belongs to  $Q_d \cap \Psi_d(\mathcal{H}_{<d})$ . It therefore is 0, i.e.,  $q_i = \pi^{(1)}(q_i)$ , that is to say  $q_i \in \mathbb{C}[x]_d^{\mathfrak{G}}$ .  $\square$

**Theorem 4.5** *Consider  $\mathfrak{G} \subset \text{GL}_n(\mathbb{C})$  a reflection group and take  $\xi \in \mathbb{C}^n$  with  $\mathfrak{G}_\xi = \{I_n\}$ . Define*

- $J = \bigcap_{g \in \mathfrak{G}} \ker \mathfrak{e}_{g \cdot \xi}$ , and
- $\mathcal{Q} = \bigcup_{\ell=1}^{\ell} \mathcal{Q}^{(\ell)}$  a symmetry adapted basis of the orthogonal complement of  $J^0$ .

Then :

- Any reduced H-basis of  $J$  is given by a set  $\{h_1 - \mathfrak{e}_\xi(h_1), \dots, h_n - \mathfrak{e}_\xi(h_n)\}$  where  $\{h_1, \dots, h_n\}$  is a set of homogeneous invariants generating  $\mathbb{C}[x]^{\mathfrak{G}}$ ;
- $\mathcal{Q}^{(1)} = \{1\}$  and, for  $2 \leq \ell \leq \mathfrak{l}$ ,  $\mathcal{Q}^{(\ell)}$  consists of precisely  $\mathfrak{n}_\ell$   $\mathfrak{r}^{(\ell)}$ -equivariants  $q_1^{(\ell)}, \dots, q_{\mathfrak{n}_\ell}^{(\ell)}$  freely generating  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$  as a  $\mathbb{C}[x]^{\mathfrak{G}}$ -module. Writing  $q_i^{(\ell)} = [q_{i1}^{(\ell)}, \dots, q_{i\mathfrak{n}_\ell}^{(\ell)}]$ , for  $1 \leq i \leq \mathfrak{n}_\ell$ , we have

$$\mathbb{C}[x] = \bigoplus_{\ell=1}^{\mathfrak{l}} \bigoplus_{i,j=1}^{\mathfrak{n}_\ell} \mathbb{C}[x]^{\mathfrak{G}} q_{ij}^{(\ell)}.$$

PROOF: When  $\mathfrak{G}$  is a reflection group,  $\mathbb{C}[x]^{\mathfrak{G}}$  is generated by  $n$  algebraically independent homogeneous invariants [Che55], [Kan01, Theorem 18-1]. The ideal  $N$  generated by all homogeneous invariants of positive degree is a zero dimensional ideal and thus any of its H-basis cannot have less than  $n$  elements. It thus follows from Lemma 4.4, that a reduced H-basis of  $N$  consists of exactly  $n$  homogeneous invariants.

By Lemma 4.3,  $N = J^0$ . Hence  $\mathcal{H}$  is a H-basis of  $J$  iff  $\mathcal{H}^0$  is a H-basis of  $N$ . It follows from Lemma 4.4 that, for any orthogonal H-basis  $\mathcal{H}$  of  $J$ ,  $\mathcal{H}^0 = \{h_1, \dots, h_n\}$  where  $h_1, \dots, h_n$  form a generating set of homogeneous invariants. With Proposition 4.1 we can conclude that the shape of a reduced H-basis is as stated.

By Lemma 4.3  $J^0 = N$ , and thus  $\mathbb{C}[x] = N \oplus \langle \mathcal{Q} \rangle_{\mathbb{C}}$  since  $\mathcal{Q}$  spans the orthogonal complement of  $J^0$ . Nakayama's lemma for graded algebra [Kan01, Lemma 17-5] thus implies that

$$\mathbb{C}[x] = \sum_{\ell=1}^{\mathfrak{l}} \sum_{k=1}^{\mathfrak{n}_\ell} \left( \sum_i^{\mathfrak{n}_\ell} \mathbb{C}[x]^{\mathfrak{G}} q_{ik}^{(\ell)} \right).$$

By [Kan01, Lemma 18-3] the set  $\{q_{ik}^{(\ell)} \mid 1 \leq \ell \leq \mathfrak{l}, 1 \leq i, k \leq \mathfrak{n}_\ell\}$  is furthermore algebraically independent when  $\mathfrak{G}$  is a reflection group. Moreover  $\mathbb{C}[x]^{(\ell,1)} = \sum_i^{\mathfrak{n}_\ell} \mathbb{C}[x]^{\mathfrak{G}} q_{1i}^{(\ell)}$ , and thus Corollary 2.4 implies that  $\mathcal{Q}^{(\ell)} = \{[q_{i1}^{(\ell)}, \dots, q_{i\mathfrak{n}_\ell}^{(\ell)}] \mid 1 \leq i \leq \mathfrak{l}_\ell\}$  is a freely generating  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$  as a  $\mathbb{C}[x]^{\mathfrak{G}}$ -module.  $\square$

**Example 4.6** The dihedral group  $\mathfrak{D}_m$  is the group of order  $2m$  generated by two elements  $s_1$  and  $s_2$  satisfying the relationships  $s_1^2 = s_2^2 = (s_1 s_2)^m = 1$ . The representation  $\varrho$  of  $\mathfrak{D}_8$  given by

$$\varrho(s_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \varrho(s_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

is a reflection group.  $\mathfrak{D}_8$  has 7 inequivalent irreducible representations, four of dimension 1 and three of dimension 2 [Ser77, Section 5.3].

Applying Algorithm 3.5 to  $\Lambda = \langle \mathfrak{e}_{g\xi} \mid g \in \mathfrak{G} \rangle$  where  $\xi = [a, b]^t$  we obtain the following H-basis  $\mathcal{H}$  of  $J = \Lambda^\perp$  and symmetry adapted basis  $\mathcal{Q} = \bigcup_{\ell=1}^7 \mathcal{Q}^{(\ell)}$  of the orthogonal complement of  $J^0$ .

$$\mathcal{H} = \{x^2 + y^2 - (a^2 + b^2), x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8 - (a^8 - 28a^6b^2 + 70a^4b^4 - 28a^2b^6 + b^8)\}$$

$$\begin{aligned} \mathcal{Q}^{(1)} &= \{1\}, & \mathcal{Q}^{(2)} &= \{xy(x^6 - 7x^4y^2 + 7x^2y^4 - y^6)\}, & \mathcal{Q}^{(3)} &= \{x^4 - 6x^2y^2 + y^4\}, & \mathcal{Q}^{(4)} &= \{xy(x^2 - y^2)\}, \\ \mathcal{Q}^{(5)} &= \{[x, y], [x(x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6), y(-7x^6 + 35x^4y^2 - 21x^2y^4 + y^6)]\}, \\ \mathcal{Q}^{(6)} &= \{[x^2 - y^2, 2xy], [x^6 - 15x^4y^2 + 15x^2y^4 - y^6, 2xy(3x^4 - 10x^2y^2 + 3y^4)]\}, \\ \mathcal{Q}^{(7)} &= \{[x(x^2 - 3y^2), y(3x^2 - y^2)], [x(-x^4 + 10x^2y^2 - 5y^4), y(5x^4 - 10x^2y^2 + y^4)]\}. \end{aligned}$$

As Theorem 4.5 asserts  $\mathcal{H}^0 = \{x^2 + y^2, x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8\}$  is a generating set for  $\mathbb{C}[x]^{\mathfrak{D}_8}$  and, for  $2 \leq \ell \leq 7$ ,  $\mathcal{Q}^{(\ell)}$  is a set of free generators for the  $\mathfrak{r}^{(\ell)}$ -equivariants.

Next example illustrates that Theorem 4.5 fails when not applied to a reflection group. In such cases, fundamental invariants and equivariants can be computed with the algorithm in Section 6.

**Example 4.7** Consider the representation of  $\mathfrak{D}_6$  given by

$$\varrho(s_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \varrho(s_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Contrary to the standard representation of  $\mathfrak{D}_6$  in the plane, this is not a reflection group since  $-1$  is a double eigenvalue of  $\varrho(s_1)$ .

The output of Algorithm 3.5 applied to  $\Lambda = \langle e_{g,\xi} \mid g \in \mathfrak{G} \rangle$  consists of a symmetry adapted  $H$ -basis  $\mathcal{H} = \bigcup_{\ell=1}^6 \mathcal{H}^{(\ell)}$  of  $J$  and a symmetry adapted basis  $\mathcal{Q} = \bigcup_{\ell=1}^6 \mathcal{Q}^{(\ell)}$  of the orthogonal complement of  $J^0$ . For  $\xi = {}^t[a, b, c]$

$$\begin{aligned} \mathcal{Q}^{(1)} &= \{1\}, & \mathcal{Q}^{(2)} &= \{z\}, & \mathcal{Q}^{(3)} &= \{x^3 - 3xy^2\}, & \mathcal{Q}^{(4)} &= \{3x^2y - y^3\}, \\ \mathcal{Q}^{(5)} &= \{[x, y], [yz, -xz]\}, & \mathcal{Q}^{(6)} &= \{[x^2 - y^2, 2xy], [2xyz, y^2z - x^2z]\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}^{(1)} &= \{x^2 + y^2 - (a^2 + b^2), z^2 - c^2\}, \\ \mathcal{H}^{(2)} &= \emptyset, & \mathcal{H}^{(3)} &= \{a(a^2 - b^2)yz(3x^2 - y^2) - bc(3a^2 - b^2)x(x^2 - y^2)\}, \\ \mathcal{H}^{(4)} &= \{b(b^2 - 3a^2)xz(x^2 - 3y^2) - ac(a^2 - 3b^2)y(y^2 - 3x^2)\}, & \mathcal{H}^{(5)} &= \emptyset, \\ \mathcal{H}^{(6)} &= \left\{ \left[ -c(a^2 + b^2)^2(x^4 - 6x^2y^2 + y^4) + 4ab(3a^2 - b^2)(a^2 - 3b^2)xyz + c(a^2 - b^2)(a^4 - 14a^2b^2 + b^4)(x^2 - y^2), \right. \right. \\ & \quad \left. \left. 2c(a^2 + b^2)^2xy(x^2 - y^2) - ab(3a^2 - b^2)(a^2 - 3b^2)(x^2 - y^2)z + c(a^2 - b^2)(a^4 - 14a^2b^2 + b^4)xy \right] \right\} \end{aligned}$$

$\mathcal{Q}$  has 12 elements, as predicted by Proposition 4.2. Yet we shall see in Example 6.5 that a minimal set of fundamental equivariants has 22 elements.

In this case  $\mathcal{H}^0$  has non invariant elements, as for instance the leading form  $yz(3x^2 - y^2)$  of the single element of  $\mathcal{H}^{(3)}$ . One can check that this form does not belong to  $N$ . Hence  $N \not\subseteq J^0$ . Furthermore  $\mathcal{H}^{(1)}$  consists of only 2 elements and thus we can not obtain a fundamental set of invariants directly from  $\mathcal{H}$ .

**Example 4.8** We consider the tetrahedral group  $\mathfrak{T}_h$  defined as the group of symmetry of the tetrahedron whose vertices  $[a, b, c] \in \{-1, 1\}^3$  satisfy  $abc = -1$ . This group is actually a representation of the symmetric group  $\mathfrak{S}_4$ , generated by  $s_1, s_2, s_3$  satisfying  $s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_3)^2 = 1$ . It has order 24 and 5 inequivalent irreducible representations whose dimensions are 1, 1, 2, 3, and 3 [Ser77, Section 5.8].  $\mathfrak{T}_h$  is the reflection group given by the representation:

$$\varrho(s_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varrho(s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varrho(s_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Applying Algorithm 3.5 to  $\Lambda = \langle e_{g,\xi} \mid g \in \mathfrak{T}_h \rangle$  where  $\xi = [a, b, c]^t$  we obtain a  $H$ -basis  $\mathcal{H}$  of the ideal  $J$  of the



orbit of  $\xi$  and a symmetry adapted basis  $\mathcal{Q} = \bigcup_{\ell=1}^5 \mathcal{Q}^{(\ell)}$  of the orthogonal complement of  $J^0$ :

$$\begin{aligned} \mathcal{H} &= \left\{ x^2 + y^2 + z^2 - (a^2 + b^2 + c^2), xyz - abc, x_1^4 + x_2^4 + x_3^4 - \frac{3}{5} (x^2 + y^2 + z^2)^2 - (a^4 + b^4 + c^4) + \frac{3}{5} (a^2 + b^2 + c^2)^2 \right\} \\ \mathcal{Q}^{(1)} &= \{1\}, \\ \mathcal{Q}^{(2)} &= \{(y^2 - z^2)(z^2 - x^2)(x^2 - y^2)\}, \\ \mathcal{Q}^{(3)} &= \left\{ \left[ \sqrt{3}(2x^2 - y^2 - z^2), 3(y^2 - z^2) \right], \left[ \sqrt{3}(2x^4 - y^4 - z^4 - 6x^2(y^2 + z^2) + 12y^2z^2), 3(z^2 - y^2)(6x^2 - y^2 - z^2) \right] \right\} \\ \mathcal{Q}^{(4)} &= \{[x, y, z], [yz, xz, xy], [x(2x^2 - 3y^2 - 3z^2), y(2y^2 - 3z^2 - 3x^2), z(2z^2 - 3x^2 - 3y^2)]\} \\ \mathcal{Q}^{(5)} &= \{[x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)], [yz(y^2 - z^2), xz(x^2 - z^2), xy(x^2 - y^2)], \\ &\quad [x(z^2 - y^2)(z^2 + y^2 - 2x^2), y(x^2 - z^2)(x^2 + z^2 - 2y^2), z(y^2 - x^2)(y^2 + x^2 - 2z^2)]\}. \end{aligned}$$

By Theorem 4.5,  $\mathcal{Q}^{(2)}, \dots, \mathcal{Q}^{(5)}$  are free bases of the basic equivariant modules  $\mathbb{C}[x]_{\tau^{(2)}}^{\mathfrak{S}}$ ,  $\dots$ ,  $\mathbb{C}[x]_{\tau^{(5)}}^{\mathfrak{S}}$  and the leading forms of  $\mathcal{H}$  form a minimal generating set of algebraically independent invariants:

$$\mathcal{H}^0 = \left\{ x^2 + y^2 + z^2, xyz, x_1^4 + x_2^4 + x_3^4 - \frac{3}{5} (x^2 + y^2 + z^2)^2 \right\}$$

## 5 Hironaka decompositions of the equivariant modules

Working with the concepts of *primary and secondary invariants* [Stu93, Section 2.3] allows to have a unique representation of the elements of  $\mathbb{C}[x]^{\mathfrak{S}}$ . Considering additionally a finite number of basic equivariants we obtain unique representations for the elements in the modules  $\mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{S}}$ . Consequently, forming symmetry adapted bases of  $\mathbb{C}[x]_d$ , for any  $d \in \mathbb{N}$ , becomes a task based on comparing degrees. In this section we show how to determine fundamental invariants and equivariants assuming we have sufficiently many invariants already. When applied to a set of *primary invariants*  $h_1, \dots, h_n$  this construction provides a Hironaka decomposition of  $\mathbb{C}[x]^{\mathfrak{S}}$ ,  $\mathbb{C}[x]_{\tau^{(\ell)}}^{\mathfrak{S}}$  and  $\mathbb{C}[x]^{(\ell)}$ , for  $1 \leq \ell \leq \mathfrak{l}$ , as  $\mathbb{C}[h]$ -modules.

**Theorem 5.1** Consider  $\mathcal{H} = \{h_1, \dots, h_k\} \subset \mathbb{C}[x]^{\mathfrak{S}}$  a set of homogeneous invariants of positive degree such that the ideal  $J = \langle h_1, \dots, h_k \rangle$  is a zero dimensional.

Consider  $\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{l}} \mathcal{Q}^{(\ell)}$  a symmetry adapted basis of the orthogonal complement of  $J$  in  $\mathbb{C}[x]$ . Then  $\mathcal{H} \cup \mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}, \dots, \mathcal{Q}^{(\mathfrak{l})}$  form a fundamental set of invariants and equivariants. More precisely, if  $\mathcal{Q}^{(\ell)}$  consists of the  $\tau^{(\ell)}$ -equivariants  $q_i^{(\ell)} = [q_{i1}^{(\ell)}, \dots, q_{in_{\ell}}^{(\ell)}]$ , for  $1 \leq i \leq m_{\ell}$ , then, for  $1 \leq k \leq n_{\ell}$ ,

$$\mathbb{C}[x]^{(\ell, k)} = \sum_{i=1}^{m_{\ell}} \mathbb{C}[h] q_{ik}^{(\ell)}, \quad \mathbb{C}[x]^{(\ell)} = \bigoplus_{k=1}^{n_{\ell}} \left( \sum_{i=1}^{m_{\ell}} \mathbb{C}[h] q_{ik}^{(\ell)} \right) \quad \text{and} \quad \mathbb{C}[x] = \bigoplus_{\ell=1}^{\mathfrak{l}} \bigoplus_{k=1}^{n_{\ell}} \left( \sum_{i=1}^{m_{\ell}} \mathbb{C}[h] q_{ik}^{(\ell)} \right).$$

PROOF: We have

$$\mathbb{C}[x]/J \cong \langle \mathcal{Q} \rangle_{\mathbb{C}} = \bigoplus_{\ell=1}^{\mathfrak{l}} \bigoplus_{k=1}^{n_{\ell}} \bigoplus_i \mathbb{C} q_{ik}^{(\ell)}$$

and thus, by Nakayama's lemma for graded algebra [DK15, Lemma 3.7.1],

$$\mathbb{C}[x] = \sum_{\ell=1}^{\mathfrak{l}} \sum_{k=1}^{n_{\ell}} \left( \sum_i^{m_{\ell}} \mathbb{C}[h] q_{ik}^{(\ell)} \right).$$

For all  $1 \leq \ell \leq \mathfrak{l}$  and  $1 \leq k \leq n_{\ell}$ ,  $q_{ik}^{(\ell)}$  belongs to  $\mathbb{C}[x]^{(\ell, k)} = \pi_{kk}^{(\ell)}(\mathbb{C}[x])$ . These subspaces are in direct sum with each other. Hence two subspaces  $\left( \sum_i \mathbb{C}[h] q_{ik}^{(\ell)} \right)$  for different pairs  $(\ell, k)$  have an intersection restricted to  $\{0\}$ . We can thus replace the two first  $\sum$  by a  $\oplus$  so as to obtain the decompositions stated.  $\square$

When  $\mathcal{H}$  is a generating set of invariants of positive degree,  $\mathcal{Q}^{(1)} = \{1\}$  and the  $\mathfrak{r}^{(\ell)}$ -equivariant  $q_i^{(\ell)}$ ,  $1 \leq i \leq m_\ell$ , generate  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ -module as a  $\mathbb{C}[x]^{\mathfrak{G}}$ -module.

Of particular interest though is the case where  $\mathcal{H} = \{h_1, \dots, h_n\}$  is a set of *primary invariants*, i.e., a set of homogeneous parameters for  $\mathbb{C}[x]^{\mathfrak{G}}$ . Then  $\{h_1, \dots, h_n\}$  are algebraically independent and  $\mathbb{C}[h] = \mathbb{C}[h_1, \dots, h_n]$  is a polynomial algebra.  $\mathbb{C}[x]^{\mathfrak{G}}$  is a Cohen-Macaulay algebra and the *secondary invariants*  $s_1, \dots, s_m$  are defined to be a free basis of  $\mathbb{C}[x]^{\mathfrak{G}}$  viewed as a  $\mathbb{C}[h]$ -module. Any  $p \in \mathbb{C}[x]^{\mathfrak{G}}$  can be written in a unique way as  $p = p_1 s_1 + \dots + p_m s_m$  where  $p_i \in \mathbb{C}[h]$ . This is a so called Hironaka decomposition of  $\mathbb{C}[x]^{\mathfrak{G}}$ . The following results shows how we can actually compute a Hironaka decomposition of  $\mathbb{C}[x]$  as a  $\mathbb{C}[h]$ -module.

**Theorem 5.2** *Consider  $h_1, \dots, h_n$  a set of primary invariants. Let  $m$  be the quotient of  $\prod_{i=1}^n \deg(h_i)$  by  $|\mathfrak{G}|$ . If  $\mathcal{Q} = \bigcup_{\ell=1}^l \mathcal{Q}^{(\ell)}$  is a symmetry adapted basis of the orthogonal complement of the ideal  $\langle h_1, \dots, h_n \rangle$  in  $\mathbb{C}[x]$ , then  $\mathcal{Q}^{(\ell)}$  consists of  $m n_\ell$   $\mathfrak{r}^{(\ell)}$ -equivariants  $q_1, \dots, q_{m n_\ell}$  that form a free basis of  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$  seen as a  $\mathbb{C}[h]$ -module. In particular  $\mathcal{Q}^{(1)}$  consists of a set of secondary invariants for  $h_1, \dots, h_n$ . We have*

$$\mathbb{C}[x]^{(\ell, k)} = \bigoplus_{i=1}^{m_\ell} \mathbb{C}[h] q_{ik}^{(\ell)}, \quad \mathbb{C}[x]^{(\ell)} = \bigoplus_{k=1}^{n_\ell} \left( \bigoplus_{i=1}^{m_\ell} \mathbb{C}[h] q_{ik}^{(\ell)} \right) \quad \text{and} \quad \mathbb{C}[x] = \bigoplus_{\ell=1}^l \bigoplus_{k=1}^{n_\ell} \left( \bigoplus_{i=1}^{m_\ell} \mathbb{C}[h] q_{ik}^{(\ell)} \right).$$

PROOF: By [Stu93, Proposition 2.3.6],  $|\mathfrak{G}|$  divides  $\prod_{i=1}^n \deg(h_i)$  and the number of secondary invariants  $m$  is their quotient. With an argument based on the Molien series of  $\mathbb{C}[x]^{(\ell)}$ , [Sta79, Proposition 4.9] proves that  $\mathbb{C}[x]/\langle h_1, \dots, h_n \rangle$  is equivalent to  $m$  times the regular representation of  $\mathfrak{G}$ . Hence  $m n_\ell^2$  is the dimension of  $\mathbb{C}[x]^{(\ell)}/\langle h_1, \dots, h_n \rangle$  as a  $\mathbb{C}$ -vector space.

From Theorem 5.1 we have

$$\mathbb{C}[x] = \bigoplus_{\ell=1}^l \bigoplus_{k=1}^{n_\ell} \sum_{i=1}^{m n_\ell} \mathbb{C}[h] q_{ik}^{(\ell)}.$$

As  $\mathbb{C}[x]$  is finitely generated over  $\mathbb{C}[x]^{\mathfrak{G}}$ ,  $h_1, \dots, h_n$  also form a homogeneous system of parameters for  $\mathbb{C}[x]$ . Since  $\mathbb{C}[x]$  is Cohen-Macaulay,  $\mathbb{C}[x]$  is a free  $\mathbb{C}[h]$ -module. If  $\eta_1, \dots, \eta_k$  is a free basis, then their images in  $\mathbb{C}[x]/\langle h_1, \dots, h_n \rangle$  span this  $m|\mathfrak{G}|$ -dimensional vector space. Hence the rank of  $\mathbb{C}[x]$  as a  $\mathbb{C}[h]$ -module is at least  $m|\mathfrak{G}|$ . It follows that  $\mathcal{Q}$  is a free basis.  $\square$

With Algorithm 3.5, the above theorems are seen to be straightforwardly constructive: With any term order, compute a Gröbner basis  $\mathcal{B}$  of the homogeneous ideal  $J = \langle h_1, \dots, h_n \rangle$  together with the normal set  $\{x^{\alpha_1}, \dots, x^{\alpha_r}\}$ . This defines the computable linear forms  $\lambda_1, \dots, \lambda_r : \mathbb{C}[x] \rightarrow \mathbb{C}$  such that, for any  $p \in \mathbb{C}[x]$ , the normal form of  $p$  w.r.t.  $\mathcal{B}$  is  $\lambda_1(p)x^{\alpha_1} + \dots + \lambda_r(p)x^{\alpha_r}$ . Applying Algorithm 3.5 to  $\Lambda = \langle \lambda_1, \dots, \lambda_r \rangle_{\mathbb{C}}$ , or alternatively [RH21, Section 5] for the least interpolation space, one computes the desired symmetry adapted basis  $\mathcal{Q}$  of the orthogonal complement of  $J$ .

The advantage of the above approach is that Gröbner bases implementations are widely available and efficient. Yet, as the input invariants  $h_1, \dots, h_k$  are homogeneous, we could alternatively compute a H-basis and its orthogonal complement with linear algebra operations. Such an approach is taken in next section where we actually compute generating invariants on the fly.

There are several algorithms to compute primary invariants. Dade's algorithm [Stu93, Algorithm 2.5.14] constructs primary invariants taking products over  $\mathfrak{G}$ -orbits of linear forms, such that all the degrees of the computed invariants are divisors of the group order. On the other hand [Kem99, Algorithm 2] looks for a set of primary invariants  $\{f_1, \dots, f_n\}$  that minimizes  $\prod_{i=1}^n \deg(f_i)$ , and then  $\sum_{i=1}^n \deg(f_i)$ . The algorithm iterates according to a pre-order over  $\mathbb{N}^n$  and stops when a  $n$ -tuple  $(d_1, \dots, d_n)$  is found such that there exist a set of primary invariants with those degrees. On the other hand [Stu93, Algorithm 2.5.8] and [Gat00, Algorithm 2.1.5] proposed to extract primary invariants from a generating set. In next section we provide a new algorithm to compute generating set of invariants, together with fundamental equivariants.

**Example 5.3** The cyclic group  $\mathfrak{C}_m$  is the group of order  $m$  generated by a single element  $r$ . If we consider its representation in  $\mathbb{C}^n$  given by

$$\varrho(r) = \text{diag} \left( e^{\frac{2i\pi}{m}}, \dots, e^{\frac{2i\pi}{m}} \right)$$

$\mathbb{C}[x]^{\mathfrak{C}_m}$  is generated by all terms of degree  $m$  [Stu93, Proposition 2.1.5] and  $\mathcal{H} = \{x_1^m, \dots, x_n^m\}$  is a set of primary invariants.  $\mathcal{H}$  is also a Gröbner basis with respect to any term order. The normal set consists of the terms  $\mathcal{N} = \{x^i y^j z^k \mid 0 \leq i, j, k \leq m-1\}$ .  $\mathfrak{C}_m$  has  $m$  inequivalent irreducible representations of dimensions 1 [Ser77, Section 5.1]. For  $1 \leq \ell \leq m$ , the fundamental equivariants are given by  $\mathcal{Q}^{(\ell)} = \{x^i y^j z^k \mid i+j+k = m+1-\ell\}$ . Observe  $\mathcal{N} = \bigcup_{\ell=1}^m \mathcal{Q}^{(\ell)}$  and this corroborates Theorem 5.2.

**Example 5.4** The alternating group  $\mathfrak{A}_4$  is the subgroup of  $\mathfrak{S}_4$  of even permutations on 4 elements.  $\mathfrak{A}_4$  has four irreducible representations over  $\mathbb{C}$ , three of dimension 1 and one of dimension 3 [Ser77, Section 5.7]. The pair of non trivial one dimensional representations are conjugate to each other. As mentioned in Section 2.1 we can combine them into one real irreducible representation of dimension 2 in order to keep the computation and the write up over  $\mathbb{R}$ .

Restricting the representation of  $\mathfrak{S}_4$  in Example 4.8 to  $\mathfrak{A}_4$  we obtain the group  $\mathfrak{T}$  of rotational symmetries of the tetrahedron. The fundamental invariants of  $\mathfrak{T}_h$  thus provide a set  $\mathcal{H}$  of primary invariants for this representation:

$$\mathcal{H} = \{x^2 + y^2 + z^2, xyz, x^2y^2 + x^2z^2 + y^2z^2\}.$$

The Gröbner basis and the normal set of the ideal generated by  $\mathcal{H}$  w.r.t a degree reverse lexicographical order are given by

$$\begin{aligned} \mathcal{B} &= \{x^2 + y^2 + z^2, xyz, y^3z + yz^3, y^4 + y^2z^2 + z^4, z^5\} \\ \mathcal{N} &= \{1, z, y, x, z^2, yz, xz, y^2, xy, z^3, yz^2, xz^2, y^2z, y^3, xy^2, z^4, yz^3, xz^3, y^2z^2, xy^3, yz^4, xz^4, y^2z^3, y^2z^4\}. \end{aligned}$$

We define the linear forms  $\lambda_1, \dots, \lambda_{24}$  so that the normal form of  $p \in \mathbb{C}[x]$  w.r.t  $\mathcal{B}$  is  $\lambda_1(p)x^{\alpha_1} + \dots + \lambda_{24}(p)x^{\alpha_{24}}$  where  $x^{\alpha_i} \in \mathcal{N}$ . Then the output of Algorithm 3.5 applied to  $\Lambda = \langle \lambda_i \mid 1 \leq i \leq 24 \rangle$  computes the secondary invariants  $\mathcal{Q}^{(1)}$  and the set of fundamental equivariants  $\mathcal{Q}^{(2)}$  and  $\mathcal{Q}^{(3)}$  that freely generate  $\mathbb{C}[x]_{\mathfrak{r}(2)}^{\mathfrak{C}}$  and  $\mathbb{C}[x]_{\mathfrak{r}(3)}^{\mathfrak{C}}$  as  $\mathbb{C}[h]$ -modules.

$$\begin{aligned} \mathcal{Q}^{(1)} &= \{[1], [(y^2 - z^2)(z^2 - x^2)(x^2 - y^2)]\}, \\ \mathcal{Q}^{(2)} &= \{[2x^2 - y^2 - z^2, \sqrt{3}(y^2 - z^2)], [\sqrt{6}(y^2 - x^2)(6z^2 - x^2 - y^2), (y^2 - z^2)(6x^2 - y^2 + z^2)]\}, \\ \mathcal{Q}^{(3)} &= \{[x, y, z], [yz, xz, xy], [x(x^2 - 3z^2), y(y^2 - 3x^2), z(z^2 - 3y^2)], \\ &\quad [x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)], [yz(y^2 - z^2), zx(z^2 - x^2), xy(x^2 - y^2)], \\ &\quad [2x^3(y^2 - z^2) - x(y^4 - z^4), 2y^3(z^2 - x^2) - y(z^4 - x^4), 2z^3(x^2 - y^2) - z(x^4 - y^4)]\}. \end{aligned}$$

## 6 Simultaneous computation of invariants and equivariants

In this section we present an algorithm that computes, simultaneously, a minimal set of fundamental invariants and equivariants. The construction makes an essential use of the Hilbert ideal and its orthogonal complement in the polynomial ring. The Hilbert ideal is a key concept in several constructions of fundamental invariants, starting with the seminal paper of Hilbert [Hil90] and more recently for reductive groups [Der99] and finite groups [Kin13]. The orthogonal complement is isomorphic to the quotient of the polynomial ring by the Hilbert ideal, which is sometimes called *covariant algebra* [Ber09, Kan01], and plays a significant role in the invariant theory of reflection groups. That it is used for a computational purpose, and more specifically to compute fundamental equivariants of any finite group is new.

Recall that we consider a finite group  $\mathfrak{G}$  with irreducible representations  $\mathfrak{r}^{(1)}, \dots, \mathfrak{r}^{(\iota)}$ , and convention that  $\mathfrak{r}^{(1)}$  is the trivial representation. We assume given a unitary<sup>2</sup> representation  $\varrho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{C})$ , inducing the representation  $\rho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{C}[x])$  on the polynomial ring. The goal of this section is to describe an algorithm to compute minimal sets of generators for, on one hand, the ring of invariants  $\mathbb{C}[x]^{\mathfrak{G}}$  and for, on the other hand, each  $\mathbb{C}[x]^{\mathfrak{G}}$ -module of  $\mathfrak{r}^{(\ell)}$ -equivariants  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ .

We first show how the fundamental equivariants arise from a symmetry adapted basis of the orthogonal complement of the Hilbert ideal. We introduce next the restrictions of the Sylvester map and their properties. They allow to outline an algorithm to compute a minimal generating set of invariants and give our termination criterion. Last we make precise our algorithm to compute simultaneously a minimal generating sets of invariants and equivariants, using the previously presented results.

## 6.1 The Hilbert ideal and its orthogonal complement

The Hilbert ideal  $N$  is the ideal in  $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$  generated by the homogeneous invariants of positive degree. By the original argument of Hilbert on the finite generation of invariant rings, a set of invariants generates  $\mathbb{C}[x]^{\mathfrak{G}}$  if and only if it generates  $N$  [Stu93, Theorem 2.1.3].

The Hilbert ideal  $N$  contains any set of primary invariants, and hence, in the case of a finite group,  $n$  elements that form a regular sequence [Stu93, Section 2.3]. A direct complement of  $N$  in  $\mathbb{C}[x]$  thus has a finite dimension. The orthogonal complement  $Q$  of  $N$ , according to the apolar product on  $\mathbb{C}[x]$ , is an invariant subspace of  $\mathbb{C}[x]$ . It thus makes sense to consider a symmetry adapted basis  $\mathcal{Q} = \bigcup_{\ell=1}^{\iota} \mathcal{Q}^{(\ell)}$  of this space, where  $\mathcal{Q}^{(\ell)}$  is a symmetry adapted basis of the  $\ell$ -th isotypic component of  $Q$ . Recall from Section 2.3 that the elements of  $\mathcal{Q}^{(\ell)}$  can be organized into  $\mathfrak{r}^{(\ell)}$ -equivariants and this is how we choose to present them.

The algorithm to be presented basically exploits the following result.

**Theorem 6.1** *Consider a representation  $\varrho : \mathfrak{G} \rightarrow \mathrm{GL}_n(\mathbb{C})$  of a finite group  $\mathfrak{G}$  and  $N$  the related Hilbert ideal in  $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ . Let  $\mathcal{Q} = \bigcup_{\ell=1}^{\iota} \mathcal{Q}^{(\ell)}$  be a symmetry adapted of an invariant direct complement of  $N$  in  $\mathbb{C}[x]$ . Then*

1. *An orthogonal  $H$ -basis of  $N$  forms a minimal set of generating invariants.*
2.  *$\mathcal{Q}^{(1)} = \{[1]\}$  and, for  $2 \leq \ell \leq \iota$ ,  $\mathcal{Q}^{(\ell)} = \left\{ \left[ q_{i1}^{(\ell)}, \dots, q_{in_{\ell}}^{(\ell)} \right] \mid 1 \leq i \leq m_{\ell} \right\}$  form a minimal generating set for  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ , the  $\mathbb{C}[x]^{\mathfrak{G}}$ -module of  $\mathfrak{r}^{(\ell)}$ -equivariants.*

PROOF: Part 1. follows from Lemma 4.4.

Part 2. If  $\mathcal{Q}$  is a basis of the complement of  $N$   $\mathbb{C}[x]/N \cong \langle \mathcal{Q} \rangle_{\mathbb{C}}$  and then by Nakayama's lemma for graded algebra [DK15, Lemma 3.7.1].

$$\mathbb{C}[x] = \sum_{\ell=1}^{\iota} \sum_{k=1}^{n_{\ell}} \left( \sum_i^{m_{\ell}} \mathbb{C}[x]^{\mathfrak{G}} q_{ik}^{(\ell)} \right).$$

From choosing  $\mathcal{Q}$  as a symmetry adapted basis of an invariant complement of  $N$ , we have that  $q_{ik}^{(\ell)} \in \mathbb{C}[x]^{(\ell,k)}$ . Since  $\mathbb{C}[x] = \bigoplus_{\ell=1}^{\iota} \bigoplus_{k=1}^{n_{\ell}} \mathbb{C}[x]^{(\ell,k)}$  it has to be that  $\mathbb{C}[x]^{(\ell,k)} = \sum_i^{m_{\ell}} \mathbb{C}[x]^{\mathfrak{G}} q_{ik}^{(\ell)}$ . Thus by Corollary 2.4  $\left\{ \left[ q_{i1}^{(\ell)}, \dots, q_{in_{\ell}}^{(\ell)} \right] \mid 1 \leq i \leq m_{\ell} \right\}$  generates  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$  as a  $\mathbb{C}[x]^{\mathfrak{G}}$ -module.

If  $\mathcal{P}^{(\ell)} = \left\{ \left[ p_{i1}^{(\ell)}, \dots, p_{in_{\ell}}^{(\ell)} \right] \mid 2 \leq i \leq \tilde{m}_{\ell} \right\}$ , for  $2 \leq \ell \leq \iota$ , is a minimal generating set for  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ , then, by Corollary 2.4,  $\mathbb{C}[x]^{(\ell)} = \bigoplus_{i=1}^{\tilde{m}_{\ell}} \left( \mathbb{C}[x]^{\mathfrak{G}} p_{i1}^{(\ell)} + \dots + \mathbb{C}[x]^{\mathfrak{G}} p_{in_{\ell}}^{(\ell)} \right)$  and so  $\bigcup_{\ell} \mathcal{P}^{(\ell)}$ , taking  $\mathcal{P}^{(1)} = \{[1]\}$ , spans the quotient  $\mathbb{C}[x]/N$ . It follows that  $m_{\ell} \leq \tilde{m}_{\ell}$ . Hence the module bases  $\mathcal{Q}^{(\ell)}$  are minimal.  $\square$

<sup>2</sup>As argued at the beginning of Section 3.4, this is no loss of generality.

## 6.2 Restrictions of the Sylvester map and a swapping lemma

For a row vector of polynomials  $\mathbf{h} = [h_1 \ \dots \ h_m] \in \mathbb{C}[\mathbf{x}]^m$ , with  $h_i$  of degree  $d_i$ , the Sylvester map to degree  $d$  already introduced in Section 3.3 is

$$\begin{aligned} \psi_{d\mathbf{h}} : \mathbb{C}[\mathbf{x}]_{d-d_1} \times \dots \times \mathbb{C}[\mathbf{x}]_{d-d_m} &\rightarrow \mathbb{C}[\mathbf{x}]_d \\ (g_1, \dots, g_m) &\mapsto \sum_{i=1}^m g_i h_i. \end{aligned}$$

For  $1 \leq \ell \leq \mathfrak{l}$ , and  $1 \leq k \leq n_\ell$  we shall consider the following restriction of this map:

$$\begin{aligned} \psi_{d\mathbf{h}}^{(\ell,k)} : \mathbb{C}[\mathbf{x}]_{d-d_1}^{(\ell,k)} \times \dots \times \mathbb{C}[\mathbf{x}]_{d-d_m}^{(\ell,k)} &\rightarrow \mathbb{C}[\mathbf{x}]_d \\ (g_1, \dots, g_m) &\rightarrow \sum_{i=1}^m g_i h_i. \end{aligned}$$

For the set  $\mathcal{H} = \{h_1, \dots, h_m\}$  we shall denote by  $\Psi_d(\mathcal{H})$  and  $\Psi_d^{(\ell,k)}(\mathcal{H})$  the respective images of  $\psi_{d\mathbf{h}}$  and  $\psi_{d\mathbf{h}}^{(\ell,k)}$ . Note that

$$\Psi_d(\mathcal{H}) = \sum_{\ell=1}^{\mathfrak{l}} \sum_{k=1}^{n_\ell} \Psi_d^{(\ell,k)}(\mathcal{H}). \quad (5)$$

At each step of Algorithm 6.4 we shall deal with  $\mathcal{H} \subset \mathbb{C}[\mathbf{x}]_{\leq d}^{\mathfrak{G}}$  and need to compute a complement of  $\Psi_d(\mathcal{H})$  in  $\mathbb{C}[\mathbf{x}]_d$ . We show that we can use only  $\Psi^{(1)}$  for that purpose, provided we have a symmetry adapted basis of the orthogonal complement. Hence only the bases of  $\mathbb{C}[\mathbf{x}]_{\leq d}^{(1)}$  need to be used to effect the Sylvester maps needed.

**Lemma 6.2** For  $2 \leq \ell \leq \mathfrak{l}$ , consider  $\mathcal{H} \subset \mathbb{C}[\mathbf{x}]_{\leq d}^{(1)} \setminus \mathbb{C}$  and  $\mathcal{Q} \subset \mathbb{C}[\mathbf{x}]_{\leq d-1}^{(\ell,k)}$ . Assume that both

- $\Psi_e^{(1)}(\mathcal{Q}) = \mathbb{C}[\mathbf{x}]_e^{(\ell,k)}$  for all  $1 \leq e \leq d-1$ , and
- $\Psi_e^{(1)}(\mathcal{H}) = \mathbb{C}[\mathbf{x}]_e^{(1)}$  for all  $1 \leq e \leq d$ .

Then  $\Psi_d^{(\ell,k)}(\mathcal{H}) = \Psi_d^{(1)}(\mathcal{Q})$ .

PROOF: Note first that both  $\Psi_d^{(\ell,k)}(\mathcal{H})$  and  $\Psi_d^{(1)}(\mathcal{Q})$  are contained in  $\mathbb{C}[\mathbf{x}]_d^{(\ell,k)}$ .

A polynomial  $q \in \Psi_d^{(\ell,k)}(\mathcal{H})$  can be written  $q = \sum_{h_i \in \mathcal{H}} h_i p_i$  with  $p_i \in \mathbb{C}[\mathbf{x}]_{d-\deg(h_i)}^{(\ell,k)}$ . Since  $\Psi_{d-\deg(h_i)}^{(1)}(\mathcal{Q}) = \mathbb{C}[\mathbf{x}]_{d-\deg(h_i)}^{(\ell,k)}$ , we can write  $p_i = \sum_{q_j \in \mathcal{Q}} r_{ij} q_j$  with  $r_{ij} \in \mathbb{C}[\mathbf{x}]_{d-\deg(h_i q_j)}^{(1)}$ . Hence

$$q = \sum_{h_i \in \mathcal{H}} h_i \sum_{q_j \in \mathcal{Q}} r_{ij} q_j = \sum_{q_j \in \mathcal{Q}} q_j \sum_{h_i \in \mathcal{H}} r_{ij} h_i \in \Psi_d^{(1)}(\mathcal{Q}),$$

so that  $\Psi_d^{(\ell,k)}(\mathcal{H}) \subset \Psi_d^{(1)}(\mathcal{Q})$ . Analogously any  $q \in \Psi_d^{(1)}(\mathcal{Q})$  can be written  $q = \sum_{q_j \in \mathcal{Q}} r_j q_j$  with  $r_i \in \mathbb{C}[\mathbf{x}]_{d-\deg q_j}^{(1)}$ . By hypothesis  $r_i = \sum_{h_i \in \mathcal{H}} p_{ij} h_i$  with  $p_{ij} \in \mathbb{C}[\mathbf{x}]_{d-\deg q_j h_i}^{(1)}$  and thus

$$q = \sum_{q_j \in \mathcal{Q}} \left( \sum_{h_i \in \mathcal{H}} h_i p_{ij} \right) q_j = \sum_{h_i \in \mathcal{H}} h_i \sum_{q_j \in \mathcal{Q}} p_{ij} q_j \in \Psi_d^{(\ell,k)}(\mathcal{H}).$$

Therefore  $\Psi_d^{(\ell,k)}(\mathcal{H}) = \Psi_d^{(1)}(\mathcal{Q})$ .  $\square$

### 6.3 Computing generating sets of invariants.

To compute a minimal generating set  $\mathcal{H}$  of homogeneous invariants one may proceed degree by degree. At degree 0,  $\mathcal{H}_0$  is empty. Assume  $\mathcal{H}_{d-1}$  has been constructed. The space  $\mathbb{C}[x]_d^{\mathfrak{G}}$  can be obtained as the image of  $\mathbb{C}[x]_d$  by the Reynolds operator  $\pi^{(1)}$ . It splits into the space  $\Psi_d^{(1)}(\mathcal{H}_{d-1})$  spanned by the power products of elements of  $\mathcal{H}_{d-1}$  of degree  $d$  and its complement. A basis for this latter is then to be added to  $\mathcal{H}_{d-1}$  to obtain  $\mathcal{H}_d$ .

The ring of invariants  $\mathbb{C}[x]^{\mathfrak{G}}$  is generated by invariants of degree at most  $|\mathfrak{G}|$  [Stu93, Theorem 2.1.4]. Though this bound is not tight in many cases, this provides a first termination criterion. In [Stu93, Algorithm 2.2.5], refined and generalized in [Gat00, Section 2.1], the termination is decided by comparing the Hilbert series of the so far generated ring and the Hilbert series of the ring of invariants determined by Molien's formula. In our algorithm for the simultaneous construction of the fundamental invariants and equivariants we shall use the following termination criterion.

**Theorem 6.3** *Assume  $\mathcal{H}$  is a set of homogeneous invariants of positive degrees such that, for some  $d \in \mathbb{N}$ ,*

1.  $\Psi_c^{(1)}(\mathcal{H}) = \mathbb{C}[x]_c^{(1)}$  for all  $1 \leq c \leq d$
2.  $\Psi_d(\mathcal{H}) = \mathbb{C}[x]_d$

*Then  $\mathcal{H}$  is a generating set for the ring  $\mathbb{C}[x]^{\mathfrak{G}}$ .*

PROOF: Since  $\mathcal{H}$  consists of invariants  $\bigoplus_{c=1}^{d-1} \Psi_c(\mathcal{H})$  is invariant. Let  $Q$  be its orthogonal complements in  $\mathbb{C}[x]_{\leq d-1} = \bigoplus_{c=1}^{d-1} \mathbb{C}[x]_c$ . Due to Hypothesis 1. on  $\mathcal{H}$ ,  $Q \cap \mathbb{C}[x]_{\leq d-1}^{(1)} = \mathbb{C}[x]_0 \cong \mathbb{C}$  and thus  $Q \subset \mathbb{C} \oplus \bigoplus_{\ell=2}^d \mathbb{C}[x]_{\leq d-1}^{(\ell)}$ .

As  $\Psi_d(\mathcal{H}) = \mathbb{C}[x]_d$  implies that  $\Psi_e(\mathcal{H}) = \mathbb{C}[x]_e$  for all  $e \geq d$  we have  $\mathbb{C}[x] = \langle \mathcal{H} \rangle \oplus Q$ . Hence any homogeneous invariant of positive degree must belong to  $\langle \mathcal{H} \rangle$ , meaning that  $\mathcal{H}$  generates the Hilbert ideal. By the argument used in Hilbert's finiteness theorem [Stu93, Theorem 2.1.3],  $\mathcal{H}$  is a generating set for the ring of invariants.  $\square$

### 6.4 Algorithm

The algorithm we outlined for constructing minimal generating sets of invariants is now made precise and enriched with the computation of minimal sets of generating basic equivariants. These latter furthermore serve to check the above termination criterion.

#### Algorithm 6.4 *Fundamental Invariants and Equivariants*

**Input:**  $\mathcal{P} = \bigcup_{\ell=1}^l \bigcup_{d=1}^{|\mathfrak{G}|} \mathcal{P}_d^{(\ell,1)}$  a symmetry adapted basis of  $\mathbb{C}[x]_{|\mathfrak{G}|}$ .

**Output:**  $\mathcal{H} \subset \mathbb{C}[x]^{\mathfrak{G}}$  and  $\mathcal{Q} = \bigcup_{\ell=1}^l \mathcal{Q}^{(\ell)}$  with  $\mathcal{Q}^{(\ell)} \subset \mathbb{C}[x]_{\mathfrak{r}(\ell)}^{\mathfrak{G}}$ .

- $\mathcal{H}$  is a minimal generating set of homogeneous invariants
- $\mathcal{Q}^{(\ell)} = \left\{ \left[ q_{i_1}^{(\ell)}, \dots, q_{i_{n_\ell}}^{(\ell)} \right] \mid 1 \leq i \leq m_\ell \right\}$  is a minimal generating set for  $\mathbb{C}[x]_{\mathfrak{r}(\ell)}^{\mathfrak{G}}$  as a  $\mathbb{C}[x]^{\mathfrak{G}}$ -module

$$\mathcal{Q}_0^{(1,1)} \leftarrow \{[1]\}$$

```

 $\mathcal{H}_0, \mathcal{Q}_0^{(2,1)}, \dots, \mathcal{Q}_0^{(1,1)} \leftarrow \emptyset$ 
 $d \leftarrow 0$ 
 $a_1 \leftarrow 1$       ( $a_d$  shall be the number of degree  $d$  elements in  $\bigcup_{\ell=1}^l \mathcal{Q}_d^{(\ell,1)}$  )
while  $a_d > 0$  and  $d < |\mathfrak{G}|$  do
     $d \leftarrow d + 1$ 
     $\mathcal{K}_d \leftarrow$  a basis of the orthogonal complement of  $\Psi_d^{(1)}(\mathcal{H}_{d-1})$  in  $\mathbb{C}[x]_d^{(1)}$ 
     $\mathcal{H}_d \leftarrow \mathcal{H}_{d-1} \cup \mathcal{K}_d$ 
     $a_d \leftarrow 0$ 
    for  $\ell = 2$  to  $l$  do
         $\mathcal{R}_d^{(\ell,1)} \leftarrow$  a basis of the orthogonal complement of  $\Psi_d^{(1)}(\mathcal{Q}_{d-1}^{(\ell)})$  in  $\mathbb{C}[x]_d^{(\ell,1)}$ 
         $\mathcal{Q}_d^{(\ell,1)} \leftarrow \mathcal{Q}_{d-1}^{(\ell,1)} \cup \mathcal{R}_d^{(\ell,1)}$ 
         $a_d \leftarrow a_d + |\mathcal{R}_d^{(\ell,1)}|$ 
    end-do
end-do
return  $\mathcal{H}_d$  and  $\bigcup_{\ell=1}^l \left\{ \left[ q, \pi_{21}^{(\ell)}(q), \dots, \pi_{n_{\ell 1}}^{(\ell)}(q) \right] \mid q \in \mathcal{Q}_d^{(\ell,1)} \right\}$ 
    
```

PROOF: Thanks to Lemma 6.2, the following three properties are true at the end of each iteration of the *while* loop:

**A**( $d$ ) :  $\Psi_e^{(1)}(\mathcal{H}_d) = \mathbb{C}[x]_e^{\mathfrak{G}}$  for all  $1 \leq e \leq d$ .

**B**( $d$ ) :  $\Psi_e^{(1)}(\mathcal{Q}_d^{(\ell,1)}) = \mathbb{C}[x]_e^{(\ell,1)}$  for all  $1 \leq e \leq d$ .

**C**( $d$ ) :  $\mathcal{Q}_d^{(\ell,1)}$  is a basis of the orthogonal complement of  $\bigoplus_{c=0}^d \Psi_c^{(\ell,1)}(\mathcal{H}_d)$  in  $\mathbb{C}[x]_{\leq d}^{(\ell,1)}$ .

It follows from **C**( $d$ ) and Corollary 2.2 that  $\pi_{k1}^{(\ell)}(\mathcal{Q}_d^{(\ell,1)})$  is a basis of the orthogonal complement of  $\bigoplus_{c=0}^d \Psi_c^{(\ell,k)}(\mathcal{H}_d)$  in  $\mathbb{C}[x]_{\leq d}^{(\ell,k)}$ . Hence

$$\bigcup_{\ell=1}^l \left\{ \left[ q, \pi_{21}^{(\ell)}(q), \dots, \pi_{n_{\ell 1}}^{(\ell)}(q) \right] \mid q \in \mathcal{Q}_d^{(\ell,1)} \right\}$$

is a symmetry adapted basis of the orthogonal complement of  $\Psi_d(\mathcal{H}_d)$  in  $\mathbb{C}[x]_{\leq d}$ .

At the end of an iteration  $a_d = 0$  means that  $\Psi_d^{(\ell,k)}(\mathcal{H}_d) = \mathbb{C}[x]_d^{(\ell,k)}$ , for all  $1 \leq k \leq n_{\ell}$ , and thus  $\Psi_d(\mathcal{H}_d) = \mathbb{C}[x]_d$ . By Theorem 6.3,  $\mathcal{H}_d$  is a generating set of invariants.

By [Stu93, Theorem 2.1.4] (Noether's bound)  $\mathbb{C}[x]_{\leq |\mathfrak{G}|}^{(1)}$  contains a generating set of invariants. Hence, if  $d = |\mathfrak{G}|$ ,  $\mathcal{H}_d$  is a generating set of invariants.

In either cases, from how it is computed,  $\mathcal{H}_d$  is a minimal set of generating invariants: its elements are homogeneous and none can be written as a polynomial in the others. It thus generates the Hilbert ideal and forms a minimal H-basis of it. From the way it is computed, it is furthermore an orthogonal H-basis for the Hilbert ideal. By Theorem 6.1,  $\left\{ \left[ q, \pi_{21}^{(\ell)}(q), \dots, \pi_{n_{\ell 1}}^{(\ell)}(q) \right] \mid q \in \mathcal{Q}_d^{(\ell,1)} \right\}$  is then a minimal generating set of equivariants for  $\mathbb{C}[x]_{\mathfrak{r}^{(\ell)}}^{\mathfrak{G}}$ .  $\square$

**Example 6.5** Consider the representation of  $\mathfrak{D}_6$  given by

$$\varrho(s_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \varrho(s_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This action was examined in [Stu93, Example 2.2.6] and we already considered it in Example 4.7.  $\mathfrak{D}_6$  has six absolutely irreducible representations, four of dimension 1 and two of dimension 2 [Ser77, Section 5.3]. Applying Algorithm 6.4 we obtain the following fundamental invariants

$$\mathcal{H} = \{x^2 + y^2, z^2, x^6 + 15x^2y^2(y^2 - x^2) - y^6, xyz(3x^4 - 10x^2y^2 + 3y^4)\},$$

and equivariants

$$\begin{aligned} \mathcal{Q}^{(2)} &= \{[z], [xy(3x^4 - 10x^2y^2 + 3y^4)]\}, \quad \mathcal{Q}^{(3)} = \{[x(x^2 - 3y^2)], [yz(3x^2 - y^2)]\} \\ \mathcal{Q}^{(4)} &= \{[y(3x^2 - y^2)], [xz(x^2 - 3y^2)]\}, \\ \mathcal{Q}^{(5)} &= \{[x, y], [yz, -xz], [x(x^4 - 10x^2y^2 + 5y^4), y(y^4 - 10y^2x^2 + 5x^4)], \\ &\quad [yz(y^4 - 10x^2y^2 + 5x^4), xz(x^4 - 10y^2x^2 + 5y^4)]\}, \\ \mathcal{Q}^{(6)} &= \{[x^2 - y^2, 2xy], [2xyz, z(y^2 - x^2)], [x^4 - 6x^2y^2 + y^4, xy(x^2 - y^2)], \\ &\quad [xyz(x^2 - y^2), z(x^4 - 6x^2y^2 + y^4)]\}. \end{aligned}$$

**Example 6.6** Consider the three-dimensional representation of the cyclic group  $\mathfrak{C}_4$  of order 4 over  $\mathbb{C}[x, y, z]$  given by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

This action was examined in [Stu93, Example 2.3.7].  $\mathfrak{C}_4$  is an Abelian group with four irreducible representations of dimension 1, two of which can be realized over the reals and the other two are conjugate to each other [Ser77, Section 5.1]. Applying Algorithm 6.4 we get the following generating invariants

$$\mathcal{H} = \{x^2 + y^2, z^2, xyz, z(x^2 - y^2), x^4 + y^4, xy(x^2 - y^2)\}$$

and equivariants

$$\begin{aligned} \mathcal{Q}^{(2)} &= \{z, xy, x^2 - y^2\}, \\ \mathcal{Q}^{(3)} &= \{x - iy, z(x + iy), 3x^2y + y^3 + i(x^3 + 3xy^2)\}, \\ \mathcal{Q}^{(4)} &= \{x + iy, z(x - iy), 3x^2y + y^3 - i(x^3 + 3xy^2)\}. \end{aligned}$$

In the above, the elements of  $\mathcal{Q}^{(3)}$  and  $\mathcal{Q}^{(4)}$  are conjugate of each other. They can be combined as a generating set of equivariants for the underlying real irreducible representation of dimension 2.

**Example 6.7** We consider the 3 dimensional representation of  $\mathfrak{S}_4$ , a group already encountered in Example 4.8, given by

$$\varrho(s_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \varrho(s_2) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \varrho(s_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$



Applying Algorithm 6.4 we obtain the following fundamental invariants

$$\mathcal{H} = \left\{ \begin{array}{l} x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 - \frac{3}{5} (x^2 + y^2 + z^2)^2, \\ x^6 + y^6 + z^6 + \frac{30}{77} (x^2 + y^2 + z^2)^2 - \frac{15}{11} (x^2 + y^2 + z^2)(x^4 + y^4 + z^4), \\ xyz(z^2 - y^2)(x^2 - z^2)(y^2 - x^2) \end{array} \right\},$$

and equivariants

$$\mathcal{Q}^{(2)} = \{[xyz], [(y^2 - z^2)(z^2 - x^2)(x^2 - y^2)]\},$$

$$\mathcal{Q}^{(3)} = \left\{ \begin{array}{l} [2x^2 - y^2 - z^2, \sqrt{3}(y^2 - z^2)], \\ [6x^2(y^2 + z^2) - 12y^2z^2 - 2x^4 + y^4 + z^4, \sqrt{3}(6x^2 - y^2 - z^2)(y^2 - z^2)], \\ [\sqrt{3}xyz(y^2 - z^2), -xyz(2x^2 - y^2 - z^2)], \\ [\sqrt{3}xyz(10x^2 - 3y^2 - 3z^2)(y^2 - z^2), xyz(6x^4 - 3y^4 - 3z^4 - 10x^2(y^2 + z^2) + 20y^2z^2)] \end{array} \right\},$$

$$\mathcal{Q}^{(4)} = \left\{ \begin{array}{l} [yz, xz, xy], \quad [x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)], \\ [yz(y^2 + z^2 - 6x^2), xz(z^2 + x^2 - 6y^2), xy(x^2 + y^2 - 6z^2)], \\ [x(z^2 - y^2)(2x^2 - y^2 - z^2), y(x^2 - z^2)(2y^2 - z^2 - x^2), z(y^2 - z^2)(2z^2 - x^2 - y^2)], \\ [zy(30x^4 - 3y^4 - 3z^4 - 30x^2(y^2 + z^2) + 20y^2z^2), xz(30y^4 - 3z^4 - 3x^4 - 30y^2(z^2 + x^2) + 20x^2z^2), \\ \quad yx(30z^4 - 3x^4 - 3y^4 - 30z^2(x^2 + y^2) + 20x^2y^2)], \\ [x(y^2 - z^2)(3x^4 - 5x^2(z^2 + y^2) + 15z^2y^2), y(z^2 - x^2)(3y^4 - 5y^2(x^2 + z^2) - 15x^2z^2), \\ \quad z(x^2 - y^2)(3z^4 - 5z^2(y^2 + x^2) + 15y^2x^2)], \end{array} \right\},$$

$$\mathcal{Q}^{(5)} = \left\{ \begin{array}{l} [x, y, z], \quad [x(3y^2 + 3z^2 - 2x^2), y(3z^2 + 3x^2 - 2y^2), z(3x^2 + 3y^2 - 2z^2)], \\ [yz(y^2 - z^2), xz(z^2 - x^2), xy(x^2 - y^2)], \\ [x(2x^4 - 5y^4 - 5z^4 - 10x^2(y^2 + z^2) + 60y^2z^2), y(2y^4 - 5x^4 - 5z^4 - 10y^2(x^2 + z^2) + 60x^2z^2), \\ \quad z(2z^4 - 5x^4 - 5y^4 - 10z^2(x^2 + y^2) + 60x^2y^2)], \\ [yz(z^2 - y^2)(10x^2 - y^2 - z^2), xz(x^2 - z^2)(10y^2 - x^2 - z^2), xy(y^2 - z^2)(10z^2 - x^2 - y^2)], \\ [yz(y^2 - z^2)(5x^4 - 3x^2(y^2 + z^2) + y^2z^2), xz(z^2 - x^2)(5y^4 - 3y^2(x^2 + z^2) + x^2z^2), \\ \quad yx(x^2 - y^2)(5z^4 - 3z^2(x^2 + y^2) + x^2y^2)] \end{array} \right\}.$$

We observe that the minimal number of fundamental invariants is 4 and hence we are not dealing here with a reflection group. One can furthermore observe that, except for  $\mathcal{Q}^{(1)} = \{[1]\}$ , each  $\mathcal{Q}^{(\ell)}$  has  $n_\ell$  equivariants.

One might want to compute a Hironaka decomposition of the equivariant modules, and of polynomial ring by direct sum of these. For that one can pick the primary invariants to be the first three fundamental invariants, or  $\mathcal{P} = \{x^2 + y^2 + z^2, x^4 + y^4 + z^4, x^6 + y^6 + z^6\}$ . The product of their degrees is 48, while the group has order 24. Hence a set of secondary invariants has cardinality 2. Applying the strategy of Section 5, one indeed obtains the secondary invariants  $\mathcal{Q}^{(1)} = \{[1], [xyz(z^2 - y^2)(x^2 - z^2)(y^2 - x^2)]\}$  while the rest of the symmetry adapted basis of  $\mathbb{C}[x]/\langle \mathcal{P} \rangle$  is the same as above. A similar result holds for Example 6.5

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## 7 Conclusion and prospects

We presented three constructions for fundamental invariants and equivariants for the representation of a finite group  $\mathfrak{G}$  over  $\mathbb{C}$ . These are the first algorithms to compute fundamental equivariants simultaneously to invariants. They are based on a novel approach, that of using the orthogonal complement to the Hilbert ideal, or other ideals generated by invariants. We believe that this idea can be extended to algebraic groups of positive dimension.

When the representation of our group is over  $\mathbb{R}$ , the output can be recombined to offer the fundamental invariants and equivariants over  $\mathbb{R}$  as well. And similarly for other subfields of  $\mathbb{C}$ , like  $\mathbb{Q}$ .

In the constructions we could have used *invariant direct complements* instead of *orthogonal complements w.r.t. the apolar product*. The algorithms are thus amenable to fields of positive characteristic in the non modular case, when the characteristic of the field does not divide the order of  $\mathfrak{G}$  [Ser77, Chapter 15.5]. This is certainly the case of the independent algorithm in Section 6. The case of the other two constructions is discussed next.

The first construction, that applies to reflection groups, relies entirely on [RH22, Algorithm 3]. This algorithm depends heavily on the apolar product, first off in the definition of the least interpolation space, and the basic linear algebra operation therein is the  $QR$ -decomposition. It would take an effort to remanufacture the algorithm for positive characteristic. On the other hand, in the second construction, that takes as input primary invariants, we can substitute [RH22, Algorithm 3], by [RH22, Section 5, Algorithm 1]. This latter computes a symmetry adapted basis of an appropriate invariant complement, of minimal degree, with LU-factorization. It is thus effective in positive characteristic.

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