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# Effective Controllability Test for Fast Oscillating Control Systems. Application to Solar Sailing

Alesia Herasimenka,<sup>a</sup> Jean-Baptiste Caillau,<sup>a</sup> Lamberto Dell’Elce,<sup>b</sup> and Jean-Baptiste Pomet<sup>b</sup>

**Abstract**—Geometric tools for the assessment of local controllability often require that the control set has the origin within its interior. This study gets rid of this assumption, and investigates the controllability of fast-oscillating dynamical systems subject to positivity constraints on the control variable, *i.e.*, the convex exterior approximation of the control set is a cone with vertex at the origin. A constructive methodology is offered to determine whether the averaged state of the system with controls in the exterior convex cone can be locally moved to an arbitrary direction of the tangent manifold. The controllability of a solar sail in orbit about a planet is analysed to illustrate the contribution. It is shown that, given an initial orbit, a minimum cone angle exists which allows the sail to move slow orbital elements to any arbitrary direction.

## I. INTRODUCTION

A classical approach to study controllability of a control system is to evaluate the rank of its Lie algebra. The so called Lie algebra rank condition (LARC) requires that this rank be equal to the dimension of the state space. It is always necessary, at least in the real analytic case, but sufficiency requires additional conditions. For a general control affine system of the form  $\dot{x} = X^0(x) + u_1 X^1(x) + \dots + u_m X^m(x)$ , a well known condition (classical, stated *e.g.* as [1, theorem 5, chap. 4]) requires that the drift  $X^0$  be recurrent, a property that is true if all solutions of  $\dot{x} = X^0(x)$  are periodic, but is more general. The LARC plus this recurrence property imply controllability if there are no constraints on the control in  $\mathbb{R}^m$ ; when the control  $u = (u_1, \dots, u_m)$  is constrained to a subset  $U$  of  $\mathbb{R}^m$ , the origin has to be in  $U$  for the condition to be relevant, but [1, theorem 5, chap. 4] asserts controllability under the condition that  $U$  not only contains the origin but is a neighborhood of the origin. Here, we are interested in systems where the origin is on the boundary of  $U$ ; this is motivated by solar sail control, see Fig. 1 where  $U$  is the set in blue.

The scope of the study is limited to so called fast-oscillating dynamical systems, of the form<sup>1</sup>

$$\begin{cases} \frac{dI}{dt} = \varepsilon \sum_{i=1}^m u_i F_i(I, \varphi) \\ \frac{d\varphi}{dt} = \omega(I) \end{cases} \quad (1)$$

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<sup>1</sup>It would be natural to also have a “small” term as a perturbation on the dynamics of the fast variable. In order to facilitate the notation, we do not consider this possibility here.

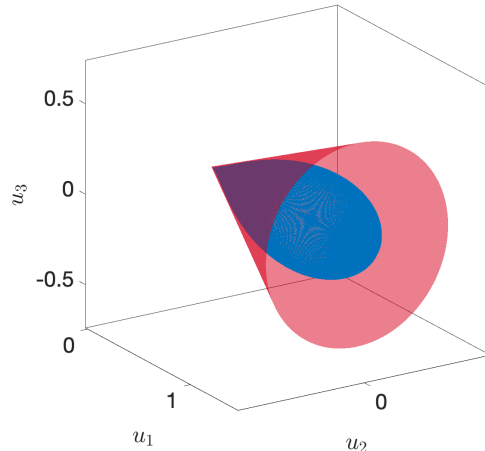


Fig. 1. Example of orbital control with solar sails. Equations are of the form of System (1), the control  $u = (u_1, u_2, u_3)$  is homogeneous to a force, and the solar sail only allows forces contained in the set  $U$  figured in blue in the picture (for some characteristics of the sail). The minimal convex cone containing the control set  $U$  is depicted in red. Neither  $U$  nor this cone are neighborhoods of the origin.

where  $\varepsilon > 0$  is a small parameter,  $I \in M$  denotes the slow component of the state and the angle  $\varphi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$  the fast component.  $M$  is a real analytic manifold of dimension  $n$  and the state space is naturally  $M \times \mathbb{S}^1$ . Each  $F_i$ ,  $1 \leq i \leq m$ , can be considered either as a smooth map  $M \times \mathbb{S}^1 \rightarrow TM$  or as a vector field on  $M \times \mathbb{S}^1$  whose projection on the second factor of the product is zero, there will be no ambiguity. From the smooth map  $\omega : M \rightarrow \mathbb{R}$ , one defines the drift  $F_0 = \omega \partial/\partial\varphi$ , it is a vector field on  $M \times \mathbb{S}^1$  whose projection on the first factor of the product is zero. The control  $u = (u_1, \dots, u_m)$  is constrained to belong to some fixed bounded subset  $U$  of  $\mathbb{R}^m$ . When the control is a function of time, it has to have values in  $U$  for all time. We assume  $U$  bounded to ensure that the variable  $I$  is slow. The solutions of the differential equation associated to  $F_0$  are obviously all periodic (the one starting from  $(I_0, \varphi_0)$  has period  $2\pi/\omega(I_0)$ ), so [1, theorem 5, chap. 4] yields controllability of System (1) if the LARC holds and  $U$  is a neighborhood of 0. As stated above, we are interested in cases where the LARC does hold but 0 is on the boundary of  $U$ .

The target here is to apply this methodology to orbital control of a spacecraft in orbit around a planet using solar sails. The possibility of using solar radiation pressure (SRP) as an inexhaustible source of propulsion intrigued researchers

since decades and triggered research in that direction. The potential exploitation of these devices in space missions of various nature, *e.g.*, interplanetary transfers, planet escapes, de-orbiting was proposed, see for instance [2]. Most available contributions offer numerical solutions to optimal transfers and locally-optimal feedback strategies, but a thorough analysis of the controllability of solar sails is not available, yet. Orbital control with SRP is challenging because the sail cannot generate a force with positive component toward the direction of the Sun (just like one cannot sail exactly against the wind in marine sailing). When incoming SRP is only partially reflected (which is systematically true in real-life missions), the possible directions of the control force are contained in a convex cone with revolution symmetry with respect to an axis through the origin, see Fig. 1, and its angle approaches zero as the portion of reflected SRP is decreased. This is a typical case of systems of the type (1), where the LARC is satisfied, recurrence of the drift is met from the structure of the system, as noticed above, but the control set  $U$ , or the convex cone it generates, is *not* a neighborhood of the zero control. In [3], we have outlined non-controllability of these sails for some reflectivity coefficients. Here, it is shown that, for a given orbit, a minimum cone angle exists which makes the system controllable. In turn, this result may serve as a mission design requirement.

In Section II, we give a sufficient condition for controllability, that is proved elsewhere [4], and introduce an optimisation problem whose solution is equivalent to checking whether that condition is satisfied. Numerical solution of the problem is tackled in Section III. Finally, the application of the methodology to solar sails orbital control is discussed in Section IV.

## II. CONTROLLABILITY OF FAST-OSCILLATING SYSTEMS

### A. A condition for controllability

Consider System (1) and the associated vector fields  $F_0, \dots, F_m$  on  $M \times \mathbb{S}^1$  defined right after Eq. (1). Let  $\varepsilon$  be a small positive parameter, the drift vector field is  $F_0$  and the control vector fields are  $\varepsilon F_1, \dots, \varepsilon F_m$ . We consider the following conditions:

- (i) the LARC holds everywhere, i.e.  $\{F_0, F_1, \dots, F_m\}$  is bracket generating,
- (ii) the control set  $U$  contains the origin, and
- (iii) for all  $I \in M$

$$\text{cone} \left\{ \sum_{i=1}^m u_i F_i(I, \varphi), u \in U, \varphi \in \mathbb{S}^1 \right\} = T_I M.$$

The following holds [4]:

*Proposition 1:* Under assumptions (i) to (iii), System (1) is controllable in the following sense: for any  $(I_0, \varphi_0)$ , there is an open and dense subset  $\mathcal{A} \subset M \times \mathbb{S}^1$ , with the property that for each  $(I_f, \varphi_f)$  in  $\mathcal{A}$ , there is a time  $T$  and an integrable control  $u(\cdot) : [0, T] \rightarrow U$  that drives initial condition  $(I_0, \varphi_0)$  at time 0 to  $(I_f, \varphi_f)$  at time  $T$ .

We refer to the preprint [4] for a proof. This is essentially a consequence of [1, theorem 9, chapter 3] using the fact that

periodicity makes the drift a “normalizer” and [1, theorems 2 and 3, chapter 3] (sometimes called Krener’s theorem) on the topology of the accessible set;  $\mathcal{A}$  is the topological interior of the accessible set from  $(I_0, \varphi_0)$  in arbitrary time.

*Remark 1:* In the real analytic case, (iii) implies (i).

*Remark 2:* Another point of view would be to introduce the averaged system as defined in [5]. It has state  $I$ , its control at each time is an integrable function  $u : \mathbb{S}^1 \rightarrow U$  (since  $U$  is bounded,  $u$  is then in any  $L^p$ ,  $1 \leq p \leq \infty$ ; we write  $u \in L^2(\mathbb{S}^1, U)$ ), and its dynamics read

$$\frac{dI}{dt} = \mathcal{F}(I) u(\cdot) \quad (2)$$

where the equation is to be understood as

$$\frac{dI(t)}{dt} = \mathcal{F}(I(t)) u(t)(\cdot)$$

and where  $\mathcal{F}(I)$  is the linear map  $L^2(\mathbb{S}^1, U) \rightarrow T_I M$

$$\mathcal{F}(I) u(\cdot) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \sum_{i=1}^m u_i(\varphi) F_i(I, \varphi) d\varphi \quad (3)$$

(read  $\int_{\mathbb{S}^1}$  as  $\int_0^{2\pi}$ ). Conditions (i) and (iii) then imply controllability of System (2) in the sense that, for each  $I_0$  in  $M$ , there is an open dense subset  $\mathcal{A}$  of  $M$  such that, for all  $I_f \in \mathcal{A}$ , there is a time  $T$  and a control  $u : [0, T] \rightarrow L^2(\mathbb{S}^1, U)$  such that the solution of Eq. (2) starting at  $I_0$  at time 0 arrives at  $I_f$  at time  $T$ . Here, the results from [1] are not needed,  $T$  can be taken small as  $I_f$  gets close to  $I_0$ , and  $\mathcal{A} = M$  if  $U$  is convex. The relation with controllability of System (1) in time  $\approx T/\varepsilon$  occurs for small  $\varepsilon$  only and under some conditions, according to the convergence result in [5] (established there only if  $U$  is an Euclidean ball centered at the origin, but extendable *mutatis mutandis*).

*Remark 3 (Localisation):* Assume that (i) holds everywhere (it is the case of our satellite application). If condition (iii) is only known to hold at one point  $I$  in  $M$ , it remains true in a neighbourhood  $O$  of  $I$ . Although the results from [1] that we invoked cannot be localized in general, this very special configuration allows one to have a local result on  $O \times \mathbb{S}^1$  in that case because the augmented family of vector fields in the sense of [1, chapter 3, section 2] allows to go in all directions and is augmented by transporting only along solutions of the drift, that does *not* move the slow variable  $I$ . See again [4] for details.

Condition (i) can be checked via a finite number of differentiations, and (ii) by inspection. One goal of this paper is to give a verifiable check, relying on convex optimisation, of the property (iii) at a point  $I$ , and to give consequences on solar sailing.

### B. Formulation as an optimisation problem

Assume condition (iii) does not hold at some point  $I$  in  $M$ . Then the set generating the convex cone in (iii) is contained in some half-space, and there exists a nonzero  $p_I$  in  $T_I^* M$

such that, for all  $u$  in  $U$  and all  $\varphi$  in  $\mathbb{S}^1$ ,

$$\left\langle p_I, \sum_{i=1}^m u_i F_i(I, \varphi) \right\rangle \leq 0.$$

Actually this property even holds true for all  $u$  in

$$K := \text{cone}(U). \quad (4)$$

This implies the weaker property that, for any square integrable control function  $u : \mathbb{S}^1 \rightarrow K$ ,

$$\langle p_I, \mathcal{F}(I) u \rangle \leq 0 \quad (5)$$

with  $\mathcal{F}$  defined in Eq. (3). Let  $e_0, \dots, e_n$  in  $T_I M$  be the vertices of an  $n$ -simplex containing 0 in its interior. The negation of Eq. (5) is that, for all  $k = 0, \dots, n$ , there exists a control  $u$  in  $L^2(\mathbb{S}^1, \mathbb{R}^m)$  valued in  $K$  such that

$$2\pi \mathcal{F}(I) u = e_k.$$

This condition together with the cone constraint of Eq. (4) define feasibility conditions for the optimal control problem over  $T_I M$  (remember that  $I$  is fixed) with quadratic cost

$$\int_{\mathbb{S}^1} |u(\varphi)|^2 d\varphi \rightarrow \min. \quad (6)$$

This is indeed a control problem associated with the following dynamics with state  $\delta I \in T_I M$  and time  $\varphi$ , where  $I$  appears as a fixed parameter:

$$\frac{d\delta I}{d\varphi} = \sum_{i=1}^m u_i F_i(I, \varphi),$$

with control constraint  $u \in K$  and boundary conditions  $\delta I(0) = 0$  and  $\delta I(2\pi) = e_k$ . Note that we have chosen a quadratic cost in order to preserve the convexity of the problem.

*Remark 4:* Given  $I_k$  such that there is some curve in  $M$  connecting  $I_0$  to  $I_k$  with tangent vector  $e_k$  at  $I_0$ , Problem (6) can serve as a proxy to find an admissible curve for the averaged dynamics of System (2); this proxy being all the more accurate as  $I_k$  is close to  $I_0$  in the manifold of slow variables.

We show in the next section that this problem is accurately approximated by a convex optimisation after approximating  $K$  by a polyhedral cone and truncating the Fourier series of the control. Eventually, by proving feasibility of these control problems for every vertex  $e_k$ , one has an effective check of local controllability around  $I$  of the original problem.

### III. DISCRETISATION OF THE SEMI-INFINITE OPTIMISATION PROBLEM

The discretisation of Problem (6) is achieved in two steps. First,  $K$  is approximated by the polyhedral cone  $K_g \subset K$  generated by  $G_1, \dots, G_g$  chosen in  $\partial K$ : admissible controls are given by a conical combination of the form

$$u(\varphi) = \sum_{j=1}^g \gamma_j(\varphi) G_j, \quad \gamma_j(\varphi) \geq 0, \quad \varphi \in \mathbb{S}^1, \quad j = 1, \dots, g.$$

Second, an  $N$ -dimensional basis of trigonometric polynomials,  $\Phi(\varphi) = (1, e^{i\varphi}, e^{2i\varphi}, \dots, e^{(N-1)i\varphi})$ , is used to model functions  $\gamma_j$  as

$$\gamma_j(\varphi) = (\Phi(\varphi)|c_j)_H$$

where  $c_j \in \mathbb{C}^N$  are complex-valued vectors (serving as design variables of the finite-dimensional problem), and where  $(\cdot|\cdot)_H$  is the Hermitian product on  $\mathbb{C}^N$ . Positivity constraints on the functions  $\gamma_j$  define a semi-infinite optimisation problem; these constraints are enforced by leveraging on the formalism of squared functional systems outlined in [6] which allows to recast continuous positivity constraints into linear matrix inequalities (LMI). Specifically, given a trigonometric polynomial  $p(\varphi) = (\Phi(\varphi)|c)_H$  and the linear operator  $\Lambda^* : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^N$  associated to  $\Phi(\varphi)$  (more details are provided in Appendix A), it holds

$$(\forall \varphi \in \mathbb{S}^1) : p(\varphi) \geq 0 \iff (\exists Y \geq 0) : \Lambda^*(Y) = c.$$

For an admissible control  $u$  valued in  $K_g$ , one has

$$\int_{\mathbb{S}^1} \sum_{i=1}^m u_i(\varphi) F_i(\varphi) d\varphi = \sum_{j=1}^g (L_j c_j + \bar{L}_j \bar{c}_j)$$

with  $L_j(I)$  in  $\mathbb{C}^{n \times N}$  defined by

$$L_j(I) = \frac{1}{2} \sum_{i=1}^m \int_{\mathbb{S}^1} G_{ij} F_i(I, \varphi) \Phi^H(\varphi) d\varphi,$$

where  $\Phi^H(\varphi)$  denotes the Hermitian transpose and where  $G_j = (G_{ij})_{i=1, \dots, m}$ . We note that the components of  $L_j(I)$  are Fourier coefficients of the function  $\sum_{i=1}^m G_{ij} F_i(I, \varphi)$ . The discrete Fourier transform (DFT) can be used to approximate  $L_j(I)$ . Since vector fields  $F_i$  are smooth, truncation of the series is justified by the fast decrease of the coefficients. Finally, for a control  $u$  valued in  $K_g$  with coefficients  $\gamma_j$  that are truncated Fourier series of order  $N-1$ , the  $L^2$  norm over  $\mathbb{S}^1$  is easily expressed in terms the coefficients  $c_j$  using orthogonality of the family of complex exponentials:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{S}^1} |u(\varphi)|^2 d\varphi &= \frac{1}{2} \sum_{j,l=1}^g \sum_{k=0}^{N-1} G_l^T G_j (\bar{c}_{lk} c_{jk} + c_{lk} \bar{c}_{jk}) \\ &= \sum_{l,j=1}^g G_l^T G_j (c_j | c_l)_H. \end{aligned}$$

As a result, for every vertex  $e_k$ , the finite-dimensional convex programming approximation of Problem (6) is

$$\begin{aligned} \min_{c_j \in \mathbb{C}^N, Y_j \in \mathbb{C}^{N \times N}} \sum_{j,l=1}^g G_l^T G_j (c_j | c_l)_H \quad \text{subject to} \\ \sum_{j=1}^g (L_j c_j + \bar{L}_j \bar{c}_j) = e_k \\ Y_j \geq 0, \quad \Lambda^*(Y_j) = c_j, \quad j = 1, \dots, g. \end{aligned} \quad (7)$$

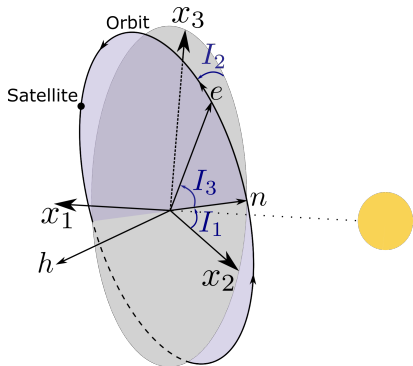


Fig. 2. Orbital orientation using Euler angles  $I_1, I_2, I_3$ . Here,  $h$  and  $e$  denote the angular momentum and eccentricity vectors of the orbit.

#### IV. CONTROLLABILITY OF A NON-IDEAL SOLAR SAIL

##### A. Orbital dynamics

The equations of motion of a solar sail in orbit about a planet are now introduced. Consider a reference frame with origin at the center of the planet,  $x_1$  toward the Sun-planet direction,  $x_2$  toward an arbitrary direction orthogonal to  $x_1$ , and  $x_3$  completes the right-hand frame. Slow variables consist of Euler angles denoted  $I_1, I_2, I_3$  orienting the orbital plane and perigee via a  $1-2-1$  rotation as shown in Fig. 2. Then,  $I_4$  and  $I_5$  are semi-major axis and eccentricity of the orbit, respectively. These coordinates define on an open set of  $\mathbb{R}^5$  a standard local chart of the five-dimensional configuration manifold  $M$  [7]. The fast variable,  $\varphi \in \mathbb{S}^1$ , is the mean anomaly of the satellite. The motion of the sail is governed by Eq. (1). Vector fields  $F_i(I, \varphi)$  are detailed in Appendix B, and  $m = 3$ . These fields are deduced by assuming that:

- (i) Solar eclipses are neglected
- (ii) SRP is the only perturbation
- (iii) Orbit semi-major axis,  $I_4$ , is much smaller than the Sun-planet distance (so that radiation pressure has reasonably constant magnitude)
- (iv) The period of the heliocentric orbit of the planet is much larger than the orbital period of the sail (so that motion of the reference frame is neglected).

We note that removing the first assumption may be problematic, since eclipses would introduce discontinuities (or very sharp variations) in the vector fields, which could jeopardise the convergence of DFT coefficients. Other assumptions are only introduced to facilitate the presentation of the results and are not critical for the methodology.

##### B. Solar sail models

Solar sails are satellites that leverage on SRP to modify their orbit. Interaction between photons and sail's surface results in a thrust applied to the satellites. Its magnitude and direction depend on several variables, namely distance from Sun, orientation of the sail, cross-sectional area, optical properties (reflectivity and absorptivity coefficients of the surface) [8]. A realistic sail model combines both absorptive and reflective forces. Here, a simplified model is used by assuming that the sail is flat with surface  $A$ , and that only a

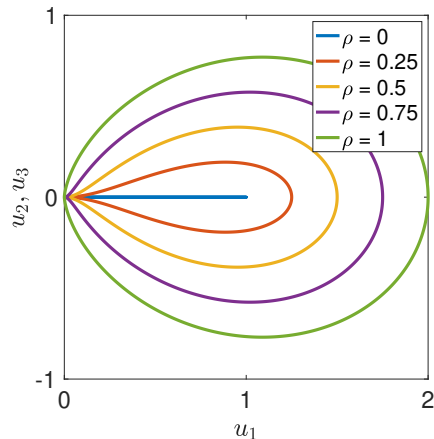


Fig. 3. Control sets for different reflectivity coefficients  $\rho$

portion  $\rho$  of the incoming radiation is reflected in a specular way ( $\rho \in [0, 1]$  is referred to as reflectivity coefficient in the remainder). Hence, denoting  $n$  the unit vector orthogonal to the sail,  $\delta$  the angle between  $n$  and  $x_1$  (recall that  $x_1$  is the direction of the Sun),  $t = \sin^{-1} \delta x_1 \times (n \times x_1)$  a unit vector orthogonal to  $x_1$  in the plane generated by  $n$  and  $x_1$ , the force per mass unit,  $m$ , of the sail is given by

$$F(n) = \frac{AP}{m} \cos \delta [(1 + \rho \cos 2\delta) x_1 + \rho \sin 2\delta t]$$

where  $P$  is the SRP magnitude and is a function of the Sun-sail distance. By virtue of assumption (iii),  $P$  is assumed to be constant, and the small parameter  $\varepsilon$  is set to  $\varepsilon = AP/m$ . Control set is thus given by

$$U = \left\{ \frac{F(n)}{\varepsilon}, \forall n \in \mathbb{R}^3, |n| = 1, (n|x_1) \geq 0 \right\}$$

Fig. 3 shows control sets for different reflectivity coefficients of the sail. When  $\rho = 0$  the sail is perfectly absorptive. In this particular case, Lie algebra of the system is not full rank. Conversely,  $\rho = 1$  represents a perfectly-reflective sail, which is the ideal case. This set is symmetric with respect to  $x_1$ , and  $K = \text{cone}(U)$  is a circular cone with angle obtained by solving

$$\tan \alpha = \min_{\delta \in [0, \pi/2]} \frac{(F(n) | t)}{(F(n) | x_1)} = \min_{\delta \in [0, \pi/2]} \frac{\rho \sin 2\delta}{(1 + \rho \cos 2\delta)}$$

which yields

$$\alpha = \tan^{-1} \left( \frac{\rho}{\sqrt{1 - \rho^2}} \right) \quad (8)$$

Fig. 1 and 4 show the minimal convex cone of angle  $\alpha$  including the control set.

##### C. Simulation and results

Problem (7) is solved by means of CVX, a package for specifying and solving convex programs [9], [10]. Table I lists initial conditions and parameters used for the simulations. Because of the symmetries of the problem, the results do not depend on  $I_1$  (first Euler angle) or  $I_4$  (semi-major axis), so

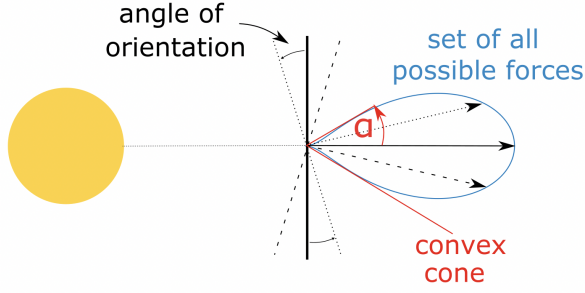


Fig. 4. Convexification of the control set

TABLE I  
SIMULATION PARAMETERS

Initial conditions	
$I_2$	20 deg
$I_3$	30 deg
$I_5$	0.5
Constants for Figs. 7 and 8	
DFT order, $N$	10
Number of generators, $g$	10
Direction of displacement	$e_5$
Cone angle, $\alpha$	80 deg

we have not included them in the table. Figure 5 shows the magnitude of Fourier coefficients of vector fields. Polynomials are truncated at order 10. At this order, the magnitude of coefficients is reduced of a factor  $10^3$  with respect to zeroth-order terms. The possibility to truncate polynomials at low-order is convenient when multiple instances of Problem (7) need to be solved.

A major takeoff of the proposed methodology is the assessment of a minimum cone angle required to have local controllability of the system. To this purpose, Problem (7) is solved for various  $\alpha$  between 0 and 90 deg, and for all  $e_{\pm k} = \pm \partial / \partial I_k$ ,  $k = 1, \dots, n$  (these vertices do not define a simplex, but the resulting computation is obviously sufficient to test local controllability). The minimum cone angle necessary for local controllability is the smallest angle such that Problem (7) is feasible for all vertices. When it is the case, we define  $\zeta^*(e_{\pm k})$  to be the inverse of the value function,

$$\zeta^*(e_{\pm k}) = \frac{2}{\|u\|_2^2},$$

and set  $\zeta^*(e_{\pm k}) = 0$  when the problem is not feasible. For the orbit at hand, feasibility occurs for  $\alpha = 19.4$  deg as depicted in Fig. 6. This angle may serve as a minimal requirement for the design of the sail. Specifically, the reflectivity coefficient associated to this cone angle can be evaluated by inverting Eq. (8), namely

$$\rho = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}.$$

In the example at hand,  $\rho \simeq 0.3$  is the minimum reflectivity that satisfies the controllability criterion (indeed, the precise

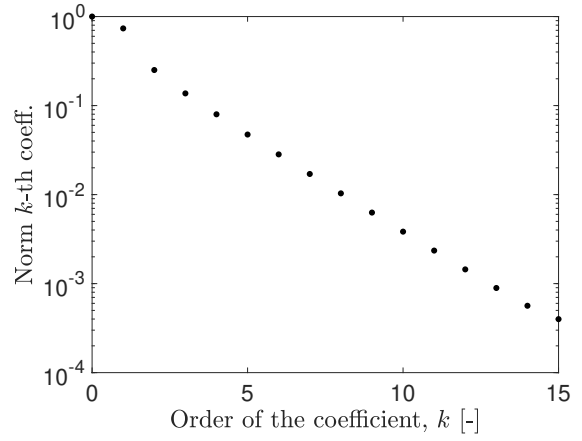


Fig. 5. Convergence of coefficients of the DFT. The norm is evaluated as  $\sqrt{\sum_i |\int_{\mathbb{S}^1} F_i e^{ik\varphi} d\varphi|^2} / \sqrt{\sum_i |\int_{\mathbb{S}^1} F_i d\varphi|^2}$ .

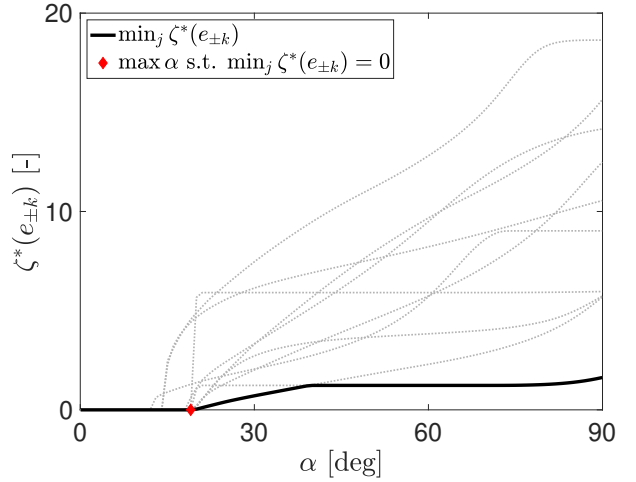
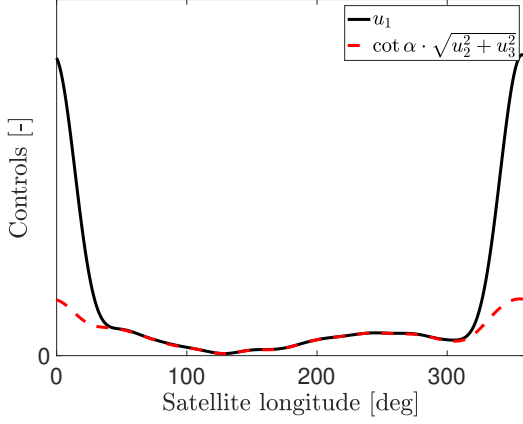


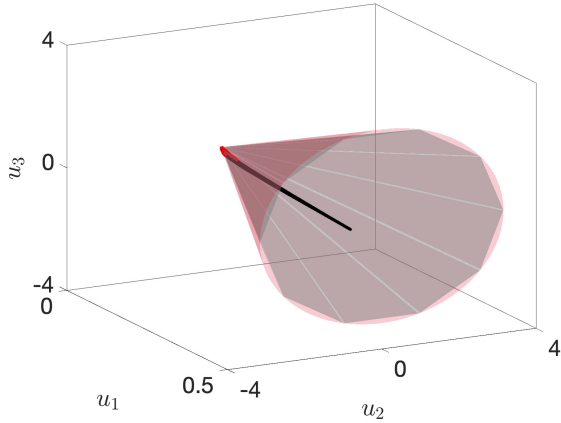
Fig. 6. Grey lines show the resulting displacement  $\zeta^*$  toward all positive and negative base vectors, i.e.,  $e_{\pm k} = \pm \partial / \partial I_k$ ,  $k = 1, \dots, n$ . Black line shows the minimum of these curves. The minimal angle  $\alpha$  ensuring local controllability is highlighted in red. One can notice that some curves do not strictly increase, but are constant instead. It means that the control is inside the cone, and increasing  $\alpha$  does not change the result.

value of the minimum  $\rho$  depends on orbital conditions). In addition, optical properties degrade in time [11], so that this result may be also used to investigate degradation of the controllability of a sail during its lifetime.

Figures 7 and 8 show controls and trajectory for  $e_5 = \partial / \partial I_5$  (i.e., increase of orbital eccentricity) with  $\alpha = 80$  deg. Periodic control obtained as solution of Problem (7) is applied for several orbits. The displacement of the averaged state is clearly toward the desired direction, namely all slow variables but  $I_5$  exhibit periodic variations, while  $I_5$  has a positive secular drift. The structure of the control arcs is such that control is on the surface of the cone in the middle of the maneuver whereas it is at the interior at the beginning and end. We note that no initial guess is required to solve Problem (7). As such, *a priori* knowledge of this structure is not necessary.



(a) Black line shows control in Sun direction, the red one combines the two other components. When they coincide, the control is on the cone's boundary.



(b) The polyhedral approximation  $K_g$  of  $K$ .

Fig. 7. Control force solution of Problem (7).

## V. CONCLUSIONS

A methodology to verify local controllability of a system with conical constraints on the control set was proposed. A convex optimisation problem needs to be solved to this purpose. Controllability of solar sails is investigated as case study, and it is shown that a minimum cone angle  $\alpha$  exists that satisfies the proposed criterion. This angle yields a minimum requirement for the surface reflectivity of the sail.

## VI. ACKNOWLEDGMENTS

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## APPENDIX

### A. POSITIVE POLYNOMIALS

Consider the basis of trigonometric polynomials  $\Phi = (1, e^{i\varphi}, e^{2i\varphi}, \dots, e^{(N-1)i\varphi})$ . Its corresponding squared functional system is  $S^2(\varphi) = \Phi(\varphi)\Phi^H(\varphi)$  where  $S^H(\varphi)$  denotes conjugate transpose of  $S(\varphi)$ . Let  $\Lambda_H : \mathbb{C}^N \rightarrow \mathbb{C}^{N \times N}$  be a

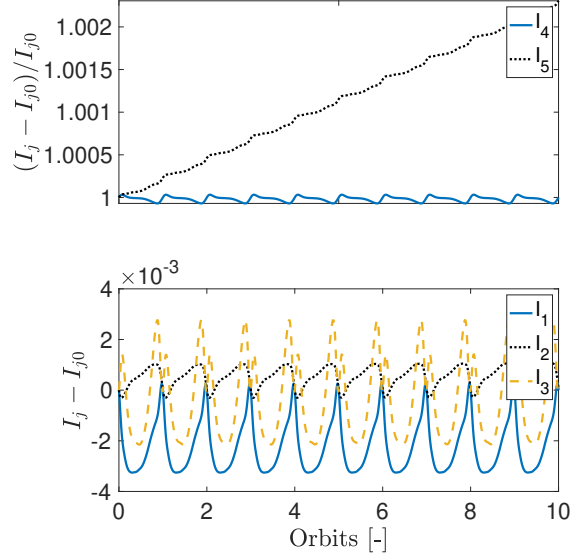


Fig. 8. For verification, controls resulting from the optimisation problem are injected into real dynamical equations. Plots of trajectories of slow variables correspond to the desired movement (increase of eccentricity,  $I_5$ ). Moreover, this trajectory is stable over multiple orbits.

linear operator mapping coefficients of polynomials in  $\Phi(\varphi)$  to the squared base, so that application of  $\Lambda_H$  on  $\Phi(\varphi)$  yields

$$\Lambda_H(\Phi(\varphi)) = \Phi(\varphi)\Phi^H(\varphi)$$

and define its adjoint operator  $\Lambda_H^* : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^N$  as

$$(Y|\Lambda_H(c))_H \equiv (\Lambda_H^*(Y)|c)_H, \quad Y \in \mathbb{C}^{N \times N}, \quad c \in \mathbb{C}^N.$$

Theory of squared functional system postulated by Nesterov [6] proves that polynomial  $(\Phi(\varphi)|c)_H$  is non-negative for all  $\varphi \in \mathbb{S}^1$  if and only if there is a Hermitian positive semidefinite matrix  $Y$  such that  $c = \Lambda_H^*(Y)$ , namely

$$(\forall \varphi \in \mathbb{S}^1) : (\Phi(\varphi)|c)_H \geq 0 \iff (\exists Y \geq 0) : c = \Lambda_H^*(Y).$$

In fact in this case it holds

$$\begin{aligned} (\Phi(\varphi)|c)_H &= (\Phi(\varphi)|\Lambda_H^*(Y))_H = (\Lambda_H(\Phi(\varphi))|Y)_H, \\ &= (\Phi(\varphi)\Phi^H(\varphi)|Y)_H = \Phi^H(\varphi)Y\Phi(\varphi) \geq 0. \end{aligned}$$

For trigonometric polynomials  $\Lambda^*$  is given by

$$\Lambda^*(Y) = \begin{bmatrix} (Y|T_0) \\ \vdots \\ (Y|T_{N-1}) \end{bmatrix}$$

where  $T_j$   $j = 0, \dots, N-1$  are Toeplitz matrices such that

$$T_0 = I, \quad T_j^{(k,l)} = \begin{cases} 2 & \text{if } k-l=j \\ 0 & \text{otherwise} \end{cases} \quad j = 1, \dots, N-1$$

### B. EQUATIONS OF MOTION OF SOLAR SAILS

Slow component of the state vector consists of three Euler angles, which position the orbital plane and perigee in space,  $I_1$ ,  $I_2$ ,  $I_3$ , and of the semi-major axis and eccentricity,  $I_4$  and  $I_5$ , respectively. Mean anomaly is the fast variable,  $\varphi$ .



Kepler's equation is used to relate  $\varphi$  to the eccentric anomaly,  $\psi$ , and then to the true anomaly,  $\theta$ , as

$$\varphi = \psi - I_5 \sin \psi, \quad \tan \frac{\theta}{2} = \sqrt{\frac{1+I_5}{1-I_5}} \tan \frac{\psi}{2}$$

Vector fields of the equations of motion are given by

$$F_i = \sum_{j=1}^3 R_{ij} F_j^{(LV LH)}$$

where,  $R_{ij}$  are components of the rotation matrix from the reference to the local-vertical local-horizontal frames,

$$R = R_1(I_3 + \theta)R_2(I_2)R_1(I_1)$$

( $R_i(x)$  denoting a rotation of angle  $x$  about the  $i$ -th axis), and vector fields  $F_j^{(LV LH)}$  can be deduced from Gauss variational equations (GVE) expressed with classical orbital element [12] by replacing the right ascension of the ascending node, inclination, and argument of perigee with  $I_1$ ,  $I_2$ , and  $I_3$ , respectively. Rescaling time such that the planetary constant equals 1, these fields are

$$F_1^{(LV LH)} = \sqrt{I_4(1-I_5^2)} \begin{bmatrix} 0 \\ 0 \\ \frac{\cos \theta}{I_5} \\ 2 \frac{I_4 I_5}{1-I_5^2} \sin \theta \\ \sin \theta \end{bmatrix}$$

$$F_2^{(LV LH)} = \sqrt{I_4(1-I_5^2)} \begin{bmatrix} 0 \\ 0 \\ \frac{2+I_5 \cos \theta \sin \theta}{1+I_5 \cos \theta} \frac{I_5}{I_5} \\ 2 \frac{I_4 I_5}{1-I_5^2} (1+I_5 \cos \theta) \\ \frac{I_5 \cos^2 \theta + 2 \cos \theta + I_5}{1+I_5 \cos \theta} \end{bmatrix}$$

$$F_3^{(LV LH)} = \frac{\sqrt{I_4(1-I_5^2)}}{1+I_5 \cos \theta} \begin{bmatrix} \frac{\sin(I_3 + \theta)}{\sin I_2} \\ \frac{\cos(I_3 + \theta)}{\sin(I_3 + \theta)} \frac{\cos I_2}{\sin I_2} \\ 0 \\ 0 \end{bmatrix}$$

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