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# Implicit Implementation of Nonsmooth Controllers to Nonsmooth Actuators

Ryo Kikuuwe<sup>\*†</sup>, Yuki Yamamoto<sup>\*</sup>, and Bernard Brogliato<sup>‡</sup>

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## Abstract

This paper presents an approach to implement a sliding-mode position controller to a plant equipped with a nonsmooth actuator. The actuator is modeled as a set-valued function from the control input and the velocity to the actuator force, which is motivated by quasistatic characteristics of hydraulic actuators shown in a previous study. The implementation of the sliding-mode controller is performed with the implicit discretization of the nominal plant model and the controller, which copes with the difficulties caused by set-valuedness, such as numerical chattering. Stability analyses both in the continuous-time and discrete-time domains are presented. Simulation results illustrate the theoretical findings.

*Keywords:* hydraulic actuators, sliding-mode control, set-valuedness, differential inclusions, implicit discretization

## 1 Introduction

Recently Kikuuwe et al. [1] presented a quasistatic modeling approach for hydraulic actuators used in commercial excavators. The actuator model presented in [1] is described as a nonsmooth function from the velocity to the force that depends on the valve openings. The actuator force is set-valued at the zero velocity because the closed valves prevent the oil flow as well as the motion of the piston, and the actuator force is subject to velocity-dependent limits because of the relief valves and the pressure drop at the control valves. This quasistatic actuator model has a form that is quite unfamiliar to the control community, and thus poses a new, interesting class of control problems. Such problems would be worth tackling not only for hydraulic actuators but also for yet-unknown future actuators.

This paper deals with the position control problem of a second-order plant driven by a nonsmooth actuator of a class that includes Kikuuwe et al.'s [1] quasistatic model of hydraulic actuators. The whole control system discussed in this paper is illustrated in Fig. 1. The actuator is assumed to be set-valued and subject to velocity-dependent

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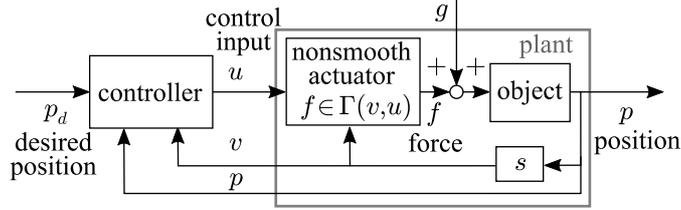


Figure 1: Position control problem discussed in this paper.

saturation. We consider implementing a sliding-mode controller for the convenience of dealing with the saturation and also of designing the convergent behavior to the target position. Practically speaking, this problem setting is important for semi-automatic or remote-controlled hydraulic excavators that may receive target position commands set far from the current position. It cannot be easily handled by many of the existing techniques, which are based on simple proportional-integral (PI) control [2–6] or linearized actuator models [7, 8] neglecting the actuator saturation brought about by the relief valves. This paper intends to prepare theoretical foundations for this problem without restricting its scope to hydraulic systems.

In the use of actuators of particular characteristics, one of the common approaches is to use the inverse models of the actuator models, which map the control input to the actuator force. This approach has been employed mainly for actuators with hysteresis [9–11]. Previous studies have investigated applications to piezoactuators [9], magnetostrictive actuators [11], shape memory alloys [11], reluctance motors [12], and electro-hydraulic systems [13]. Other types of nonlinearity, such as dead-zone and backlash [10] and creep [9], have also been considered. The application of such an approach to our problem is not straightforward because of the set-valuedness of the actuator, which makes the whole plant dynamics governed by a differential inclusion. Moreover, using sliding-mode control injects another set-valuedness into the closed-loop system, posing additional difficulty.

The mathematical treatment of nonsmooth dynamical systems involving the set-valuedness has been investigated by some researchers. For simulation purposes, implicit discretization has been known to be useful for a long time [14–16]. With the implicit discretization, the set-valuedness, such as Coulomb friction, rigid-body contact, and an ideal sliding-mode controller, is enclosed within an *algebraic loop*, which is an algebraic constraint between the input and the output and can be seen as a feedback loop without latency. At every timestep, the algebraic constraints are solved either analytically or numerically. This scheme is extended into an implementation scheme of nonsmooth controllers, of which the outputs are set-valued [17–22]. In the implicit implementation scheme, an algebraic loop is formed between a nominal plant model and the nonsmooth controller, and the solution of the algebraic constraint is used as the control input at every sampling step. In other words, this implementation scheme employs a one-step state predictor based on the nominal plant model.

This paper presents an approach to implement nonsmooth controllers to plants equipped with intrinsically nonsmooth actuators. The actuator force to be generated is determined by a one-step predictor based on the nominal model of the plant. The determined force is converted into the control input through the inverse map of the qua-

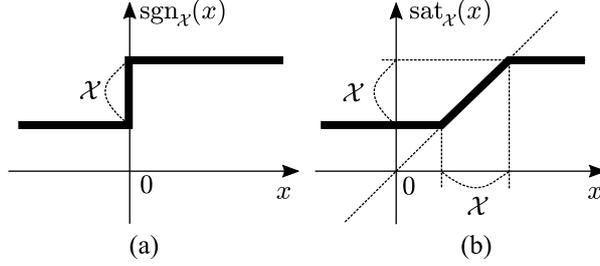


Figure 2: Graphs of  $\text{sgn}_{\mathcal{X}}(x)$  and  $\text{sat}_{\mathcal{X}}(x)$ .

sistatic actuator model using the predicted velocity obtained by the nominal plant model. Some theoretical analysis and illustrative simulation results are presented.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries. Section 3 shows problem settings including the details of the nonsmooth plant model. Section 4 proposes a simple sliding-mode controller and its discrete-time implementation to the nonsmooth plant. Section 5 shows some illustrative simulation results. Section 6 provides concluding remarks.

## 2 Mathematical Preliminaries

In this paper,  $\mathbb{R}$  denotes the set of all real numbers,  $\mathbb{R}_+$  denotes the set of all non-negative real numbers,  $\mathbb{N}$  denotes the set of all non-negative integers, and  $\mathcal{B}$  denotes the unit closed ball in  $\mathbb{R}$ , i.e.,  $\mathcal{B} \triangleq [-1, 1] \subset \mathbb{R}$ . For a set  $\mathcal{Z}$ ,  $\text{conv}\mathcal{Z}$  stands for the convex hull of  $\mathcal{Z}$ . It should be recalled that a closed and bounded interval in  $\mathbb{R}$  means a compact convex subset of  $\mathbb{R}$ .

This paper uses the following functions:

$$\text{sgn}_{\mathcal{X}}(x) \triangleq \begin{cases} \min \mathcal{X} & \text{if } x < 0 \\ \mathcal{X} & \text{if } x = 0 \\ \max \mathcal{X} & \text{if } x > 0 \end{cases} \quad (1)$$

$$\text{sat}_{\mathcal{X}}(x) \triangleq \begin{cases} \min \mathcal{X} & \text{if } x < \min \mathcal{X} \\ x & \text{if } x \in \mathcal{X} \\ \max \mathcal{X} & \text{if } x > \max \mathcal{X} \end{cases} \quad (2)$$

where  $\mathcal{X}$  is a closed and bounded interval in  $\mathbb{R}$ . These functions are illustrated in Fig. 2. For brevity, we write

$$\text{sgn}(x) \triangleq \text{sgn}_{\mathcal{B}}(x), \quad \text{sat}(x) \triangleq \text{sat}_{\mathcal{B}}(x). \quad (3)$$

The functions have the following properties:

**Lemma 1.** *Let  $\mathcal{X}$  be a closed and bounded interval in  $\mathbb{R}$  and  $x, y \in \mathbb{R}$  be real numbers. Then, the following statement holds true:*

$$y \in \text{sgn}_{\mathcal{X}}(x - y) \iff y = \text{sat}_{\mathcal{X}}(x). \quad (4)$$

**Proof.** Let  $[A, B] \triangleq \mathcal{X}$ . Then,  $y \in \text{sgn}_{\mathcal{X}}(x - y) \iff (y = A \wedge x - y < 0) \vee (y \in \mathcal{X} \wedge x - y = 0) \vee (y = B \wedge x - y > 0) \iff (y = A \wedge x < A) \vee (y = x \wedge x \in \mathcal{X}) \vee (y = B \wedge x > B) \iff y = \text{sat}_{\mathcal{X}}(x)$ .  $\square$

**Lemma 2.** Let  $\mathcal{X}$  be a closed and bounded interval in  $\mathbb{R}$  and let  $a > 0$  and  $b, x \in \mathbb{R}$  be real numbers. Then, the following statement holds true:

$$x = \text{sat}_{\mathcal{X}}(b - ax) \iff x = \text{sat}_{\mathcal{X}}(b/(a + 1)). \quad (5)$$

**Proof.** Let  $[A, B] = \mathcal{X}$ . Then,  $x = \text{sat}_{\mathcal{X}}(b - ax) \iff (b - ax < A \wedge x = A) \vee (x = b - ax \wedge x \in \mathcal{X}) \vee (b - ax > B \wedge x = B) \iff (x = A \wedge b/(a + 1) < A) \vee (x = b/(1 + a) \wedge b/(a + 1) \in \mathcal{X}) \vee (x = B \wedge b/(a + 1) < B) \iff x = \text{sat}_{\mathcal{X}}(b/(a + 1))$ .  $\square$

It should be noted that these proof can be greatly simplified by using the tools of convex analysis, such as normal cones. For example, Lemma 1 can be simply proven as  $y \in \text{sgn}_{\mathcal{X}}(x - y) \iff x - y \in N_{\mathcal{X}}(y) \iff y = \text{proj}(\mathcal{X}; x) = \text{sat}_{\mathcal{X}}(x)$ . (See Appendix B of [23] for definitions and see Proposition 6.47 of [24] for a detailed proof.)

This paper uses the inequality signs ( $<$ ,  $>$ ,  $\leq$  and  $\geq$ ) with the subscript  $\forall$  to write inequalities involving sets, which should be read as follows:

$$x <_{\forall} \mathcal{Y} \iff (x < y, \forall y \in \mathcal{Y}) \quad (6)$$

$$\mathcal{X} \forall < y \iff (x < y, \forall x \in \mathcal{X}) \quad (7)$$

$$\mathcal{X} \forall <_{\forall} \mathcal{Y} \iff (x < y, \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}) \quad (8)$$

where  $\mathcal{X}$  and  $\mathcal{Y}$  are sets of real numbers.

A set-valued function  $\Phi$  is said to be monotone if  $(\Phi(x_1) - \Phi(x_2))(x_1 - x_2) \forall \geq 0$  for all  $x_1$  and  $x_2$ . It is said to be strictly monotone if  $(\Phi(x_1) - \Phi(x_2))(x_1 - x_2) \forall > 0$  for all  $x_1$  and  $x_2$  satisfying  $x_1 \neq x_2$ . A set-valued function  $\Phi$  maps a set  $\mathcal{Z}$  in the following manner:

$$\Phi(\mathcal{Z}) = \bigcup_{x \in \mathcal{Z}} \Phi(x). \quad (9)$$

As a consequence of (9), expressions involving nested set-valued functions (i.e., compositions of set-valued functions) should be read as follows:

$$\Phi_1(\Phi_2(x)) = \bigcup_{y \in \Phi_2(x)} \Phi_1(y), \quad (10)$$

as is the case also in [22].

## 3 Nonsmooth Actuator

### 3.1 Actuator and Plant

Now, let  $\mathcal{F}$  be a closed and bounded interval in  $\mathbb{R}$  including zero in its interior, i.e.,  $\mathcal{F} = [\min \mathcal{F}, \max \mathcal{F}]$  and  $\min \mathcal{F} < 0 < \max \mathcal{F}$ . The core of the problem discussed in this paper is a class of set-valued functions  $\Gamma : \mathbb{R} \times \mathcal{B} \rightrightarrows \mathcal{F}$  that possess the following properties:

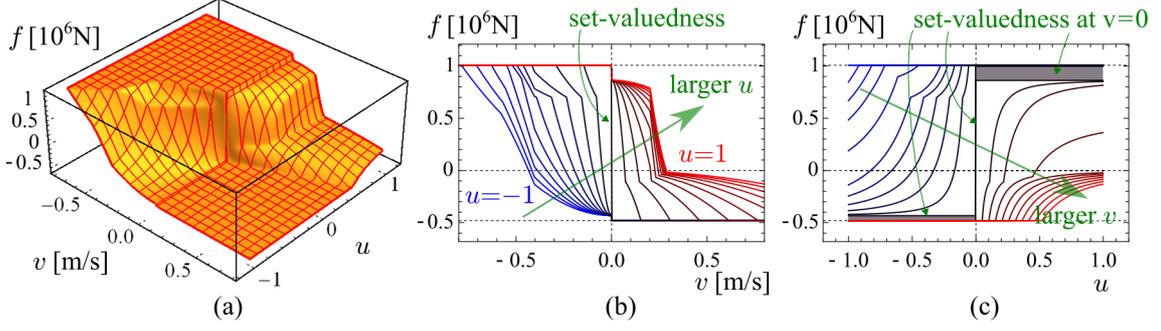


Figure 3: Graph of  $\Gamma(v, u)$  presented in [1]. It is set-valued at  $v = 0$ .

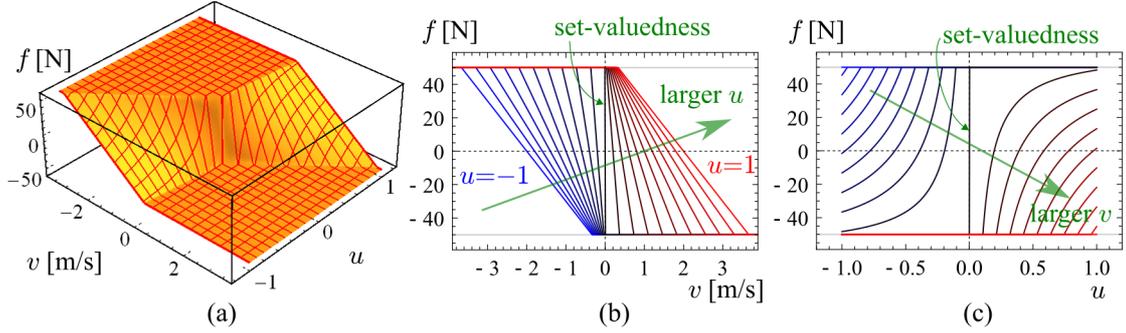


Figure 4: Graph of  $\Gamma_{\text{ex}}(v, u)$  defined in (11) with  $\{F, F_m, B\} = \{50 \text{ N}, 60 \text{ N}, 30 \text{ Ns/m}\}$ . It is set-valued at  $\{v, u\} = \{0, 0\}$ .

- P1** For all  $v \in \mathbb{R}$  and  $u \in \mathcal{B}$ ,  $\Gamma(v, u)$  is a closed and bounded interval in  $\mathbb{R}$ .
- P2** For all  $v \in \mathbb{R}$ ,  $\Gamma(v, \mathcal{B}) = \text{conv}(\Gamma(v, -1) \cup \Gamma(v, 1)) \subseteq \mathcal{F}$ , which is a closed and bounded interval in  $\mathbb{R}$ .
- P3**  $\Gamma$  is upper semicontinuous [25, p.32] in the set-valued sense.
- P4** For all  $u \in \mathcal{B}$ ,  $\Gamma(-v, u)$  is monotone with respect to  $v$ .
- P5** For all  $v \in \mathbb{R}$ , both  $\max \Gamma(v, u)$  and  $\min \Gamma(v, u)$  are monotone functions of  $u$ . (Note that  $\Gamma(v, u) = \max \Gamma(v, u) = \min \Gamma(v, u)$  if  $\Gamma(v, u)$  is a singleton.)
- P6**  $\Gamma(v, 0) = \text{sgn}_{\mathcal{F}}(-v)$  for all  $v \in \mathbb{R}$ .
- P7**  $\Gamma(0, u) \searrow < 0$  for all  $u \in [-1, 0)$  and  $\Gamma(0, u) \nearrow > 0$  for all  $u \in (0, 1]$ .

From the properties P1 to P7, the following properties are deduced:

**PD1**  $\Gamma(0, 0) = \mathcal{F}$ . ( $\because$  P6)

**PD2**  $\Gamma(v, u) = \max \mathcal{F}$  if  $v < 0$  and  $u > 0$ , i.e., in the second quadrant of the  $v$ - $u$  plane, and  $\Gamma(v, u) = \min \mathcal{F}$  if  $v > 0$  and  $u < 0$ , i.e., in the fourth quadrant of the  $v$ - $u$  plane. ( $\because$  P4, P5, P6, and P7)

**PD3** The set  $\Gamma(v, \mathcal{B})$  coincides with  $\mathcal{F}$  when  $v = 0$  ( $\because$  P6), and becomes smaller as  $|v|$  increases ( $\because$  P4).

This paper uses this class of functions  $\Gamma$  as a quasistatic model of an actuator that produces a force satisfying  $f \in \Gamma(v, u)$  at the velocity  $v \in \mathbb{R}$  according to the dimensionless control input  $u \in \mathcal{B}$ . It may or may not be set-valued when  $\{v, u\} \neq \{0, 0\}$ . The actuator force  $f$  belongs to the velocity-dependent set  $\Gamma(v, \mathcal{B})$ . It can be increased by increasing the control input  $u$  (due to P5) and by decreasing the velocity  $v$  (due to P4), in such a way as if the actuator includes an internal nonlinear viscous resistance. When the control input  $u$  is zero, the actuator acts exactly like the Coulomb friction (due to P6), of which the force always opposes the velocity.

One example of  $\Gamma$  is illustrated in Fig. 3, which has been presented in [1]. Another one, which is much simpler, can be given as follows:

$$\Gamma_{\text{ex}}(v, u) \triangleq \begin{cases} \text{sat}_{F\mathcal{B}}(F_m \text{sgn}(u) - Bv/|u|) & \text{if } |u| \in (0, 1] \\ \text{sgn}_{F\mathcal{B}}(-v) & \text{if } u = 0 \end{cases} \quad (11)$$

where  $F_m \geq F > 0$  and  $B > 0$ . This function  $\Gamma_{\text{ex}}$  is illustrated in Fig. 4. In Fig. 3(a) and Fig. 4(a), for example, the property P6 is visible as the vertical line at  $\{v, u\} = \{0, 0\}$ , and the property PD2 is visible as the highest and lowest plateaus.

This paper discusses the position control problem of the plant with the following dynamics:

$$M\dot{v} \stackrel{\text{a.e.}}{=} f + g, \quad \dot{p} = v \quad (12a)$$

$$f \stackrel{\text{a.e.}}{\in} \Gamma(v, u). \quad (12b)$$

Here, ‘‘a.e.’’ stands for ‘‘almost everywhere in time.’’ The plant (12) in combination with a position controller is illustrated in Fig. 1. Here, we refer to (12a) as the controlled object, which consists of a mass  $M > 0$  with position  $p \in \mathbb{R}$  and velocity  $v \in \mathbb{R}$ . The mass is subjected to the external force  $g \in \mathbb{R}$  and the actuator force  $f \in \mathbb{R}$ . We assume that  $g$  is bounded and is a piecewise-continuous function of time  $t$ . The actuator is modeled as  $\Gamma$  in (12b) and it applies the force  $f$  to the plant, depending on the plant velocity  $v$  and the control input  $u$ . The control input  $u$  is provided by the controller to be combined with this plant, as illustrated in Fig. 1. Because of the properties P1 and P3, the differential inclusion (12) always has an absolutely continuous solution  $v$  if  $u \in \mathcal{B}$  is a measurable function of time  $t$ . (See, e.g., [25, Theorem 4.7], the three conditions therein being satisfied.)

One important feature of the plant (12) is that, when the external force  $g$  is zero,  $\dot{v} = 0$  can be established at the velocity  $v_{ss}$  satisfying  $0 \in \Gamma(v_{ss}, u)$ , which means that the steady-state velocity  $v_{ss}$  can be manipulated through the control input  $u$ . In this sense, the actuator (12b) can be said to be a force-saturated, weakly-velocity-commanded system. In the special case where  $\Gamma = \Gamma_{\text{ex}}$  in (11), we have  $v_{ss} = F_m u / B$ .

**Remark 1.** *The dependence of  $\Gamma(v, u)$  on the velocity  $v$  is crucial in our problem setting. For example, if  $\Gamma$  is defined as  $\Gamma(v, u) = F \text{sgn}(u)$  with  $F > 0$  and  $u \equiv 0$ , the differential inclusion (12) reduces to  $M\dot{v} \in [-F, F]$ , which does not have a unique solution obviously. According to Theorem 1 in [26], with  $\Gamma(v, u)$  possessing the property P4, the differential inclusion (12) has unique solutions from almost all initial values of  $v \in \mathbb{R}$ .*

### 3.2 Inverse Model of Actuator

When an actuator is used in a control system, a controller should usually be constructed so that it determines the actuator force to be generated. In order to use such a controller with an actuator of the form of (12b), one needs to convert the desired actuator force  $\hat{f}$  to a control input  $u$  sent to the actuator using an inverse model of the actuator. For this purpose, we define a set-valued function  $\Theta : \mathbb{R} \times \mathcal{F} \rightrightarrows \mathcal{B}$  that satisfies the following:

$$u \in \Theta(v, \hat{f}) \iff \hat{f} \in \Gamma(v, u). \quad (13)$$

It means that  $\Theta$  is the inverse function of  $\Gamma$  with respect to its second argument. Note that  $\Theta$  is set-valued because of the property PD2 of  $\Gamma$ , implying that multiple values of  $u$  correspond to a given pair  $\{v, \hat{f}\}$ , especially if either  $\hat{f} = \max \mathcal{F}$  or  $\hat{f} = \min \mathcal{F}$ . It is worth noting that the highest and lowest plateaus in Fig. 3(a) and Fig. 4(a) correspond to the set-valuedness of  $\Theta(v, \hat{f})$ . With the case of  $\Gamma$  illustrated in Fig. 3,  $\Theta(v, \hat{f})$  can be set-valued also when  $v = 0$  as can be visible as the gray rectangle areas in Fig. 3(c). In addition,  $\Theta$  is not a total function, i.e., its domain is not the whole  $\mathbb{R} \times \mathbb{R}$ , because  $\Theta(v, \hat{f}) = \emptyset$  if  $\hat{f} \notin \Gamma(v, \mathcal{B}) \subseteq \mathcal{F}$ .

The fact that  $\Theta$  is set-valued and non-total causes inconvenience in the use of  $\Theta$  in the controller. To avoid the inconvenience, it is better to prepare a total single-valued function  $\Theta_s : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{B}$  that satisfies the following:

$$\Theta_s(v, \hat{f}) \in \Theta\left(v, \text{sat}_{\Gamma(v, \mathcal{B})}(\hat{f})\right), \quad \forall \hat{f} \in \mathbb{R}, \quad \forall v \in \mathbb{R}, \quad (14)$$

which means that, if  $\hat{f} \in \Gamma(v, \mathcal{B})$ ,  $\Theta_s(v, \hat{f})$  is a single-valued selection of  $\Theta(v, \hat{f})$ , and otherwise,  $f = \Gamma(v, \Theta_s(v, \hat{f}))$  is the projection of  $\hat{f}$  onto  $\Gamma(v, \mathcal{B})$  due to the property P1. In the special case where  $\Gamma = \Gamma_{\text{ex}}$  in (11), one example of  $\Theta_s$  can be given as follows:

$$\Theta_{s, \text{ex}}(v, \hat{f}) \triangleq \begin{cases} \text{sat}\left(Bv/(F_m - \text{sgn}(v)\hat{f})\right) & \text{if } v \neq 0 \wedge F_m - \text{sgn}(v)\hat{f} > 0 \\ \text{sgn}(v) & \text{if } v \neq 0 \wedge F_m - \text{sgn}(v)\hat{f} \leq 0 \\ 0 & \text{if } v = 0. \end{cases} \quad (15)$$

This expression can be derived by carefully examining (11) under different conditions, e.g., the sign of  $u$  and whether  $|\hat{f}| < F$  or  $|\hat{f}| \geq F$ .

### 3.3 Implementation of a Controller

Here, one should note that the equivalence in (13) is somewhat misleading. In order to exert a non-zero force  $\hat{f}$  at the velocity  $v = 0$ , (13) allows for the control input  $u = 0$  because  $0 \in \Theta(0, \hat{f})$  for all  $\hat{f} \in \mathcal{F}$ , but the control input  $u = 0$  at  $v = 0$  obviously does not drive the actuator.

The source of the problem is that, for some velocities  $v$ , the controller input  $u$  allowed by  $\Theta(v, \hat{f})$  may not actually result in  $f = \hat{f}$ , e.g., when  $v = 0$  and  $\min \mathcal{F} < \hat{f} < \max \mathcal{F}$ . One alternative is to attempt to realize  $f = \hat{f}$  at a small time  $T$  later. That is, with a desired force  $\hat{f}$ , one can consider determining  $u$  so that

$$\hat{f} \in \Gamma(v + T(\hat{f} + g)/M, u), \quad (16)$$

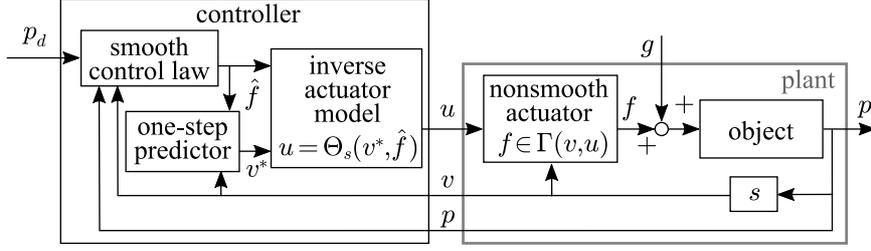


Figure 5: Implementation of a smooth controller to a nonsmooth actuator.

or equivalently,

$$u \in \Theta(v + T(\hat{f} + g)/M, \hat{f}). \quad (17)$$

Here,  $v + T(\hat{f} + g)/M$  can be viewed as a predicted velocity based on the nominal model (12a) of the plant. One may also use a single-valued function  $\Theta_s$  satisfying (14) to uniquely determine  $u$  as follows:

$$u = \Theta_s(v + T(\hat{f} + g)/M, \hat{f}). \quad (18)$$

It should be noted that, if  $T$  is viewed as the sampling interval of the discrete-time device to which the controller should be implemented, (18) can be seen as a solution of the implicit (backward) Euler discretization of the plant model (12), which is

$$M(v_{k+1}^* - v_k)/T = \hat{f}_k + g_k, \quad \hat{f}_k \in \Gamma(v_{k+1}^*, u_k). \quad (19)$$

This expression is derived by substituting  $\dot{v} \approx (v_{k+1}^* - v_k)/T$  to (12). Here,  $k \in \mathbb{N}$  denotes the discrete-time index,  $v_k$  is the current measured velocity, and  $v_{k+1}^*$  is the velocity predicted by the plant model (12) or the velocity intended to be realized in the next timestep. We are assuming that the plant parameter  $M$  and the external force  $g$  are accurately known. The algebraic constraints in (19) can be solved as follows:

$$v_{k+1}^* := v_k + T(\hat{f}_k + g_k)/M \quad (20a)$$

$$u_k := \Theta_s(v_{k+1}^*, \hat{f}_k). \quad (20b)$$

which can be implemented as an algorithm to convert the desired actuator force  $\hat{f}_k$  to the control input  $u_k$  to the controller. To be more accurate, the control input  $u$  is to be kept constant at the value  $u_k$  for the time period  $t \in [kT, (k+1)T)$ . Fig. 5 illustrates this implementation scheme. It can be seen as a variant of the implicit implementation scheme [17–22], in which the implicit discretizations of the nominal plant model and a controller are algebraically combined. It should be emphasized that the original scheme [17–22] is to deal with a nonsmooth controller, while the scheme in Fig. 5 is to deal with a plant including a nonsmooth actuator.

One justification of using (16) and (17), with the predicted velocity  $v + T(\hat{f} + g)/M$  instead of the current velocity  $v$ , is summarized as follows.

**Theorem 1.** *Assume that  $v$  is an absolutely continuous function of time that satisfies (12). Let  $\mathcal{T}$  be a sufficiently long time period starting from the time  $t = t_0$ . At the time*

$t = t_0$ , let  $v = v_0$  and  $g = g_0$ . Let us also assume that  $g \equiv g_0$  for the period  $\mathcal{T}$ . Let  $T > 0$  and  $v^* = v_0 + T(\hat{f} + g_0)/M$ . Then, setting  $u \in \Theta(v^*, \hat{f})$  for the period  $\mathcal{T}$  results in  $v$  changing monotonically from  $v = v_0$  toward  $v = v^*$  until  $v = v^*$  is achieved.

A proof of Theorem 1 is given in Appendix A. It should be noted that this result also justifies the use of a non-unique selection function  $\Theta_s$  in (18) and (20b) because this result does not depend on a particular choice of the value of  $u$  within the set  $\Theta(v^*, \hat{f})$ .

## 4 Implementation of Nonsmooth Controller

### 4.1 Controller in the Continuous-Time Domain

One important feature of the actuator (12b) is that the actuator force  $f$  is subject to a velocity-dependent bound  $\Gamma(v, \mathcal{B})$ . sliding-mode control is a convenient choice for the use with bounded actuator force because it has modes in which the actuator force is at its upper or lower limits. Another beneficial aspect of sliding-mode controllers is that they allow for the design of the convergent behavior toward a desired state, through the design of the so-called sliding surface. This article considers the following controller, which is intended to be a sliding-mode controller for the plant (12):

$$\hat{f} \stackrel{\text{a.e.}}{\in} \Gamma(v, \text{sgn}(p_d - p - Hv)). \quad (21)$$

Note that  $\Gamma(v, \pm 1)$  may be set-valued as can be seen in Fig. 3. In the use of this controller, the control input  $u$  to the actuator should be determined so that the actuator generates the force  $\hat{f}$  satisfying (21).

Now let us consider the closed-loop system composed of (12) and (21) in the ideal situation where the desired force  $\hat{f}$  is realized by the actuator, i.e.,  $\hat{f} \equiv f$ . With  $\sigma \triangleq p + Hv - p_d$ , it can be written as follows:

$$\begin{bmatrix} \dot{\sigma} \\ \dot{v} \end{bmatrix} \stackrel{\text{a.e.}}{=} \begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} H/M \\ 1/M \end{bmatrix} (f + g) \quad (22a)$$

$$f \stackrel{\text{a.e.}}{\in} \Gamma(v, \text{sgn}(-\sigma)) \quad (22b)$$

$$u \stackrel{\text{a.e.}}{\in} \Theta(v, f) \quad (22c)$$

where the nested set-valued functions should be read as in (10). The expression (22) means that, if  $u$  is chosen so that (22c) is satisfied almost everywhere in time, the closed-loop system described by (22a) and (22b) is realized. Because the set  $\Gamma(v, \text{sgn}(-\sigma)) \subseteq \mathcal{F}$  is compact and convex for all  $[\sigma, v]^T \in \mathbb{R}^2$  (due to the properties P1 and P2), the differential inclusion composed of (22a) and (22b) always has a solution with respect to  $[\sigma, v]^T$ . (See, e.g., [25, Theorem 4.7] and the three conditions therein.) It should be noted that (22c) is algebraically equivalent to  $f \stackrel{\text{a.e.}}{\in} \Gamma(v, u)$ , but (22b) and (22c) do not necessarily mean that  $u = \text{sgn}(-\sigma)$  or  $u \in \text{sgn}(-\sigma)$  because different  $u$  values may provide the same  $f$  values in (22c).

Properties of the closed-loop system composed of (22a) and (22b) are summarized as follows:

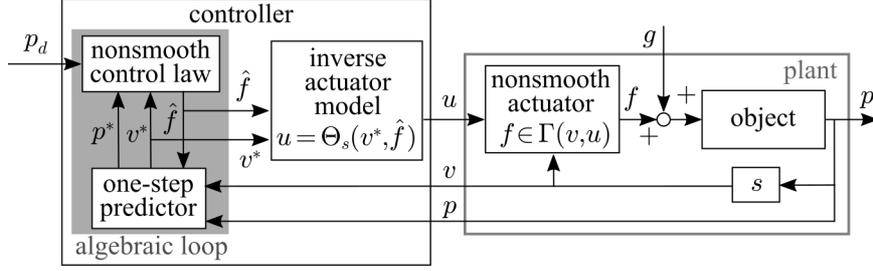


Figure 6: Proposed scheme: implicit implementation of a nonsmooth controller to a nonsmooth actuator.

**Theorem 2.** Consider the system composed of (22a) and (22b) and assume that  $g$  is appropriately bounded, as indicated in the proof, for all  $t \in \mathbb{R}_+$ . Then, a subset of the subspace  $\mathcal{S} \triangleq \{[\sigma, v]^T \mid \sigma = 0\}$  is finite-time stable and positively invariant, and the origin  $[\sigma, v]^T = 0$  is asymptotically stable. In addition, the origin is globally asymptotically stable if  $g \equiv 0$ .

A proof is presented in Appendix B. This result means that the controller (21) can be seen as a sliding-mode controller with the switching surface  $\mathcal{S}$  that makes the position  $p$  converge to the desired position  $p_d$ .

## 4.2 Proposed Implementation Scheme

The difficulty in the implementation of the controller (21) in discrete time is that both the actuator (12b) and the controller (21) are nonsmooth. The predictor-based implementation scheme detailed in Section 3.3, and in Fig. 5, is not suited for this purpose because the nonsmooth controller (21) does not uniquely designate the actuator force  $\hat{f}$  when  $\sigma = 0$ . To deal with the difficulty, we consider using a predicted value of  $\sigma$ , instead of the current value of  $\sigma$ , as the input to the controller (21). It can be realized in the same way as the original implicit implementation scheme [17–22], which is based on the implicit discretization of the controller and the nominal plant model.

With the implicit Euler discretization, an approximation of the system composed of the plant (12) and the controller (21) under the ideal assumption of  $f = \hat{f}$  can be written as follows:

$$M(v_{k+1}^* - v_k)/T = \hat{f}_k + g_k \quad (23a)$$

$$p_{k+1}^* = p_k + T v_{k+1}^* \quad (23b)$$

$$\hat{f}_k \in \Gamma(v_{k+1}^*, \text{sgn}(p_{d,k} - p_{k+1}^* - H v_{k+1}^*)) \quad (23c)$$

$$u_k \in \Theta(v_{k+1}^*, \hat{f}_k). \quad (23d)$$

Eliminating  $v_{k+1}^*$  and  $p_{k+1}^*$  from (23c) and (23d) by using (23a) and (23b) yields the following:

$$\hat{f}_k \in \Gamma(v_{f,k} + \eta \hat{f}_k, \text{sgn}(v_{s,k} - v_{f,k} - \eta \hat{f}_k)) \quad (24a)$$

$$u_k \in \Theta(v_{f,k} + \eta \hat{f}_k, \hat{f}_k) \quad (24b)$$

where

$$v_{f,k} \triangleq v_k + \eta g_k \quad (24c)$$

$$v_{s,k} \triangleq (p_{d,k} - p_k)/(H + T) \quad (24d)$$

$$\eta \triangleq T/M. \quad (24e)$$

Possible interpretations for the intermediate variables  $v_{f,k}$  and  $v_{s,k}$  are that  $v_{f,k}$  is the velocity that would be achieved if the actuator force  $\hat{f}_k$  is zero, and that  $v_{s,k}$  is the velocity that would be achieved in the ideal sliding motion if the actuator force was unbounded.

The expression (24) shows that, once  $\hat{f}_k$  is obtained as the solution of (24a), one can obtain  $u_k$  from (24b). In order to provide a solution of (24a), we define a single-valued function  $\Gamma_\eta : \mathbb{R} \times \mathcal{B} \rightarrow \mathcal{F}$  that satisfies the following relation:

$$f = \Gamma_\eta(v, u) \iff f \in \Gamma(v + \eta f, u). \quad (25)$$

The uniqueness of the single-valued function  $\Gamma_\eta$  satisfying (25) is given as follows:

**Theorem 3.** *The function  $\Gamma_\eta$  defined by (25) is a total, single-valued function of  $v$  and  $u$ .*

**Proof.** For any  $v \in \mathbb{R}$  and  $u \in \mathcal{B}$ , there exists a unique  $v_1 \in \mathbb{R}$  that satisfies  $0 \in \Gamma(v_1, u) - (v_1 - v)/\eta$  because the right-hand side is unbounded and strictly monotone with respect to  $-v_1$  due to the property P4. With such  $v_1$ , one can obtain  $f = (v_1 - v)/\eta$ , which satisfies  $f \in \Gamma(v + \eta f, u)$ .  $\square$

For example, the function  $\Gamma_\eta$  for the function  $\Gamma_{\text{ex}}$  defined in (11) can be obtained as follows:

$$\Gamma_{\eta, \text{ex}}(v, u) \triangleq \text{sat}_{FB} \left( \frac{F_m u - Bv}{\eta B + |u|} \right). \quad (26)$$

This function can be derived from (11) by using Lemmas 1 and 2. A closed-form expression of the function  $\Lambda$  that satisfies  $v = \Lambda(\beta, f, u) \iff \beta v + f \in \Gamma(v, u)$  has been presented in [1] for the function  $\Gamma$  in Fig. 3. It can be easily converted into  $\Gamma_\eta$  through  $\Gamma_\eta(v, u) = (\Lambda(1/\eta, -v/\eta, u) - v)/\eta$  or  $\Lambda(\beta, f, u) = (\Gamma_{1/\beta}(-f/\beta, u) - f)/\beta$ .

With the function  $\Gamma_\eta$  defined by (25), one can rewrite (24a) as follows:

$$\hat{f}_k = \text{sat}_{\Gamma_\eta(v_{f,k}, \mathcal{B})} ((v_{s,k} - v_{f,k})/\eta) \quad (27)$$

because we have the following relations:

$$\begin{aligned} & f \in \Gamma(v_b + \eta f, \text{sgn}(v_a - \eta f)) \\ \iff & (f \in \Gamma(v_b + \eta f, \mathcal{B}) \wedge \eta f = v_a) \\ & \vee (f \in \Gamma(v_b + \eta f, 1) \wedge \eta f < v_a) \\ & \vee (f \in \Gamma(v_b + \eta f, -1) \wedge \eta f > v_a) \\ \iff & (f \in \Gamma_\eta(v_b, \mathcal{B}) \wedge f = v_a/\eta) \\ & \vee (f = \Gamma_\eta(v_b, 1) \wedge f < v_a/\eta) \\ & \vee (f = \Gamma_\eta(v_b, -1) \wedge f > v_a/\eta) \\ \iff & f = \text{sat}_{\Gamma_\eta(v_b, \mathcal{B})} (v_a/\eta) \end{aligned} \quad (28)$$

for all  $v_a$ ,  $v_b$  and  $f$ . Here, the fact that

$$\begin{aligned} f \in \Gamma(v + \eta f, \mathcal{B}) &\iff \exists u \in \mathcal{B} \text{ s.t. } f \in \Gamma(v + \eta f, u) \\ &\iff \exists u \in \mathcal{B} \text{ s.t. } f = \Gamma_\eta(v, u) \\ &\iff f \in \Gamma_\eta(v, \mathcal{B}) \end{aligned} \quad (29)$$

is used. By using (24) and (27), one can obtain an algorithm to calculate  $u_k$  satisfying (23) with given inputs  $\{v_k, p_k, g_k\}$  as follows:

$$v_{f,k} := v_k + \eta g_k \quad (30a)$$

$$v_{s,k} := (p_{d,k} - p_k)/(H + T) \quad (30b)$$

$$\hat{f}_k := \text{sat}_{\Gamma_\eta(v_{f,k}, \mathcal{B})}((v_{s,k} - v_{f,k})/\eta) \quad (30c)$$

$$u_k := \Theta_s(v_{f,k} + \eta \hat{f}_k, \hat{f}_k). \quad (30d)$$

The obtained  $u_k$  is to be used as the control input  $u$  to the actuator for the period  $t \in [kT, (k+1)T)$ .

One illustration of the proposed controller (30) is presented in Fig. 6. This algorithm also uses a one-step predictor to obtain  $v_{k+1}^*$  and  $p_{k+1}^*$ . The predicted velocity  $v_{k+1}^*$  appears as  $v_{f,k} + \eta \hat{f}_k$  in (30d) although the predicted position  $p_{k+1}^*$  does not explicitly appear in the algorithm. The ‘algebraic loop’ is formed between the nominal model (i.e., one-step predictor) (12a) and the sliding-mode controller (24a), as has been the case with the original implicit implementation scheme [17, 18]. The analytical solution of the algebraic loop appears in (30c). After the algebraic constraint is solved, the predicted velocity  $v_{k+1}^*$  is used as the input to the function  $\Theta_s$  in (30d) to obtain the control input  $u_k$ , as is in the case discussed in Section 3.3.

### 4.3 Errors in Nominal Model

It should be noted that the proposed scheme heavily depends on the nominal model of the plant including the external force. This section analyzes the effect of modeling errors in the discrete-time controller (30) applied to the plant (12).

First, one can see that the real plant (12), from which  $p$  is eliminated by using  $\sigma \triangleq p + Hv - p_d$ , can be approximated by the following discrete-time form:

$$\begin{bmatrix} \sigma_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_k \\ v_k \end{bmatrix} + \begin{bmatrix} \eta H_t \\ \eta \end{bmatrix} (f_k + g_k) + T^2 \begin{bmatrix} \varepsilon_{\sigma, k+1} \\ \varepsilon_{v, k+1} \end{bmatrix} \quad (31a)$$

$$f_k \in \text{conv}(\Gamma(v_k, u_k) \cup \Gamma(v_{k+1}, u_k)) \quad (31b)$$

where  $\eta \triangleq T/M$  and  $H_t \triangleq H + T$ . In the last term of the right-hand side of (31a),  $T^2 \varepsilon_{\sigma, k+1}$  and  $T^2 \varepsilon_{v, k+1}$  are errors induced by the discretization, which are scaled by  $T^2$  considering the fact that the errors of the Euler discretization are generally of the order of  $\mathcal{O}(T^2)$ . Because the forces  $f$  and  $g$  may vary within a timestep,  $f_k$  and  $g_k$  are set as average values within the timestep  $k$ . The errors caused by this approximation are also included in  $\varepsilon_{\sigma, k+1}$  and  $\varepsilon_{v, k+1}$ . The actuator force  $f_k$  at the timestep  $k$  cannot be determined strictly but one can assume that it resides within the range indicated in (31b). The control input  $u_k$  is assumed to be kept constant within every timestep.

Next, by strictly distinguishing the nominal model from the real plant, one can rewrite (23), which is an algebraically equivalent form of the discrete-time controller (30), in the following form:

$$\begin{bmatrix} \sigma_{k+1}^* \\ v_{k+1}^* \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_k \\ v_k \end{bmatrix} + \begin{bmatrix} \hat{\eta}H_t \\ \hat{\eta} \end{bmatrix} (f_k + g_k) \quad (32a)$$

$$\hat{f}_k \in \hat{\Gamma}(v_{k+1}^*, \text{sgn}(-\sigma_{k+1}^*)) \quad (32b)$$

$$\hat{f}_k \in \hat{\Gamma}(v_{k+1}^*, u_k) \quad (32c)$$

where  $\hat{\eta} \triangleq T/\hat{M}$ . Here, the symbols with hats are values and functions of the nominal plant model, which may differ from those in the real plant (31). It involves the set-valuedness in (32b) and (32c), but they always have unique and closed-form solutions as follows:

$$\hat{f}_k = \text{sat}_{\hat{\Gamma}_{\hat{\eta}}(\hat{v}_{f,k}, \mathcal{B})} (-\hat{\sigma}_{f,k}/(\hat{\eta}H_t)) \quad (33a)$$

$$u_k = \hat{\Theta}(\hat{v}_{f,k} + \hat{\eta}\hat{f}_k, \hat{f}_k) \quad (33b)$$

where  $\hat{v}_{f,k} \triangleq v_k + \hat{\eta}\hat{g}_k$  and  $\hat{\sigma}_{f,k} \triangleq \sigma_k + Tv_k + \hat{\eta}H_t\hat{g}_k$ .

One can see that (32), which is an implicit form the controller, involves the predicted values  $\sigma_{k+1}^*$  and  $v_{k+1}^*$ , and they may differ from the true values  $\sigma_{k+1}$  and  $v_{k+1}$  of the plant (31). The error between the predicted and the real states can be written as follows:

$$\begin{aligned} \begin{bmatrix} \tilde{\sigma}_{k+1} \\ \tilde{v}_{k+1} \end{bmatrix} &\triangleq \begin{bmatrix} \sigma_{k+1}^* \\ v_{k+1}^* \end{bmatrix} - \begin{bmatrix} \sigma_{k+1} \\ v_{k+1} \end{bmatrix} \\ &= T \begin{bmatrix} H_t \\ 1 \end{bmatrix} \tilde{a}_k - T^2 \begin{bmatrix} \varepsilon_{\sigma,k+1} \\ \varepsilon_{v,k+1} \end{bmatrix} \end{aligned} \quad (34a)$$

where

$$\tilde{a}_k \triangleq (\hat{f}_k + \hat{g}_k)/\hat{M} - (f_k + g_k)/M. \quad (34b)$$

By using this, one can write the closed-loop system in the following discrete-time form:

$$\begin{bmatrix} \sigma_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_k \\ v_k \end{bmatrix} + \begin{bmatrix} \hat{\eta}H_t \\ \hat{\eta} \end{bmatrix} (f_k + g_k) + \begin{bmatrix} \tilde{\sigma}_{k+1} \\ \tilde{v}_{k+1} \end{bmatrix} \quad (35a)$$

$$\hat{f}_k \in \hat{\Gamma}(v_{k+1} - \tilde{v}_{k+1}, \text{sgn}(-\sigma_{k+1} + \tilde{\sigma}_{k+1})). \quad (35b)$$

The discrete-time system (35) have the following properties:

**Theorem 4.** *Consider the system (35) and assume that  $\hat{g}_k$ ,  $\tilde{a}_k$ ,  $\varepsilon_{\sigma,k}$ , and  $\varepsilon_{v,k}$  are appropriately bounded, as indicated in the proof, for all  $k \in \mathbb{N}$ . Then, a neighborhood of a subset of the subspace  $\mathcal{S} \triangleq \{[\sigma, v]^T \mid \sigma = 0\}$  is finite-time stable and positively invariant. Moreover, a neighborhood of the origin of the system (35) is asymptotically stable.*

Appendix C presents a proof of this theorem, of which the structure is very similar to that of the proof of Theorem 2 in Appendix B. Theorem 4 means that a subset of the subspace  $\mathcal{S}$  approximately acts as a sliding surface, and that the state approximately converges to the origin. The theorem implies that the proposed discrete-time algorithm (30) implemented to the plant (12) well approximates the continuous-time closed-loop system (22) as long as the prediction errors  $\tilde{\sigma}_{k+1}$  and  $\tilde{v}_{k+1}$  are small enough. One can also see that, from (34), the errors are of the order of the sampling interval  $T$ .

## 5 Numerical Examples

Some simulation results to illustrate the proposed scheme are presented. We consider the plant (12) where a mass  $M = 1$  kg is driven by an actuator modeled by  $\Gamma_{\text{ex}}$  defined by (11). The external force  $g$  is given as a function  $g(t)$  of time. The plant dynamics (12) is realized in the simulator with the backward Euler discretization, which provides the following closed-form algorithm:

$$f_i := \Gamma_{h/M}(v_i + hg(ih)/M, u_i) \quad (36a)$$

$$v_{i+1} := v_i + h(f_i + g(ih))/M \quad (36b)$$

$$p_{i+1} := p_i + hv_{i+1}. \quad (36c)$$

Here, the subscript  $i \in \mathbb{N}$  stands for the discrete-time index in the simulator, and the timestep size is set as  $h = 0.0001$  s. The values of the actuator's parameters are set as  $F = 50$  N,  $F_m = 60$  N, and  $B = 30$  Ns/m, which are the same as those in Fig. 4.

The controller is constructed as in (30) with  $\Gamma_\eta$  and  $\Theta_s$  being replaced by  $\Gamma_{\eta,\text{ex}}$  in (26) and  $\Theta_{s,\text{ex}}$  in (15), respectively. The controller's sampling interval is set as  $T = 0.01$  s, the time constant for the sliding surface is set as  $H = 1$  s, and the desired position is fixed as  $p_d \equiv 1$  m. Here, notice that the sampling interval  $T$  of the controller is set much larger than the timestep size  $h$  of the plant simulation. It is for approximating the discrete-time controller applied to the continuous-time plant. The functions  $\Gamma_{\eta,\text{ex}}$  and  $\Theta_{s,\text{ex}}$  depend on the nominal values of the plant parameters  $\{M, F, F_m, B\}$  and the disturbance  $g(t)$ . Their values used in the controller are hereafter denoted as  $\{\hat{M}, \hat{F}, \hat{F}_m, \hat{B}, \hat{g}(t)\}$ , respectively. Some different parameter settings for the controller are used in the simulations.

Explicit implementation of the controller (21) (i.e., the implementation of (21) with the inverse function  $\Theta$ ) is not attempted because it has long been known that it causes chattering (see, e.g., [17, 18, 21]). It is also because there is no unique way to resolve the set-valuedness in (21) and because  $\Theta_s$  does not drive the actuator from the zero velocity as pointed out in Section 3.3.

Fig. 7 shows simulation results of ideal situations where the external force is set as  $g(t) \equiv 0$  and the plant parameters are assumed to be exactly known (i.e.,  $\hat{M} = M$ ,  $\hat{F} = F$ ,  $\hat{F}_m = F_m$ ,  $\hat{B} = B$ , and  $\hat{g}(t) \equiv g(t) \equiv 0$ ). The results with various initial states are shown. It can be seen that, from all initial states, the state  $(\sigma, v)$  reaches the set  $\mathcal{S}$  (defined in Theorem 2) in finite time and reaches the origin asymptotically, illustrating Theorem 2 and Theorem 4. More precisely, the subset satisfying  $|v| \leq v_s \triangleq F_m H / (BH - M) = 2.11$  m/s of  $\mathcal{S}$  is reached, being consistent with the content in the proof of Theorem 2. (A simple calculus shows that the maximum  $v_s$  satisfying (49) with  $R = 0$  and  $\Gamma_{\text{ex}}$  defined in (26) is  $v_s \triangleq F_m H / (BH - M)$  when  $BH > M$ .)

In simulations reported hereafter, the external force  $g$  is given as

$$g(t) = g_0(t) + \begin{cases} -80 & \text{if } t \in [2 \text{ s}, 2.2 \text{ s}] \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

where

$$g_0(t) \triangleq -20 \sin(4\pi t), \quad (38)$$

which is the sum of a sinusoidal force and an impulse-like force. Fig. 8 shows some results. The thick gray curves represent an almost ideal case, where the external force is

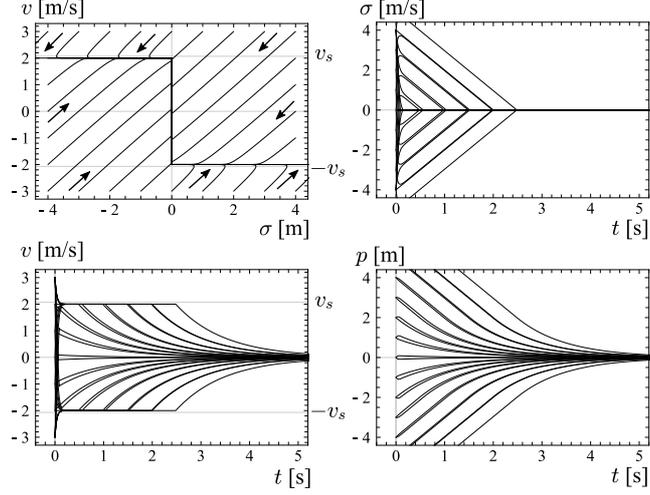


Figure 7: Simulation results of the ideal case ( $\hat{M} = M$ ,  $\hat{F} = F$ ,  $\hat{F}_m = F_m$ ,  $\hat{B} = B$ , and  $\hat{g}(t) \equiv g(t) \equiv 0$ ) with various initial states. The value  $v_s$  is defined as  $v_s \triangleq F_m H / (BH - M) = 2.11$  m/s, which coincides with  $v_s$  appearing in Appendix B.

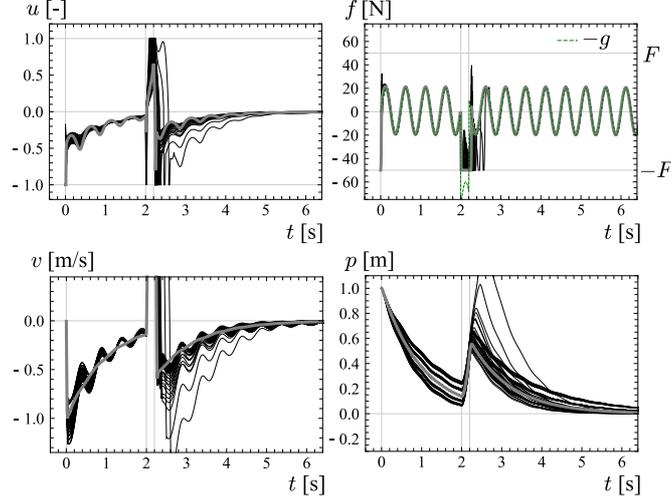


Figure 8: Simulation results with various modeling errors, in which the values of  $\{\hat{M}, \hat{F}, \hat{F}_m, \hat{B}\}$  are varied between  $-23\%$  to  $+30\%$  of their true values  $\{M, F, F_m, B\}$  of the plant. The thick gray curves represent an almost ideal case with no modeling errors and  $\hat{g}(t) \equiv g_0(t)$ .

known except the impulse-like component (i.e.,  $\hat{g}(t) \equiv g_0(t)$ ) and the plant parameters are exactly known (i.e.,  $\hat{M} = M$ ,  $\hat{F} = F$ ,  $\hat{F}_m = F_m$ , and  $\hat{B} = B$ ). The black curves represent the results of controllers whose parameter values are between  $-23\%$  to  $+30\%$  of the true plant parameter values. It is shown that, in all cases, the position  $p$  moves toward the target position  $p_d = 1.0$  m and it actually converges after the large disturbance in  $t \in [2 \text{ s}, 2.2 \text{ s}]$ . The sinusoidal oscillations of the actuator force  $f$  indicate that the force  $f$  reacts to the sinusoidal disturbance  $g$  even if  $\hat{g}$  is not set to be consistent with  $g$ . It can be seen that, however, the velocity  $v$  is more influenced by the external force  $g$  when  $\hat{g} \neq g_0$  (the black curves) than when  $\hat{g} \equiv g_0$  (the gray thick curve), although the position  $p$  eventually converges to the desired position  $p_d$ . These results exhibit a certain level of

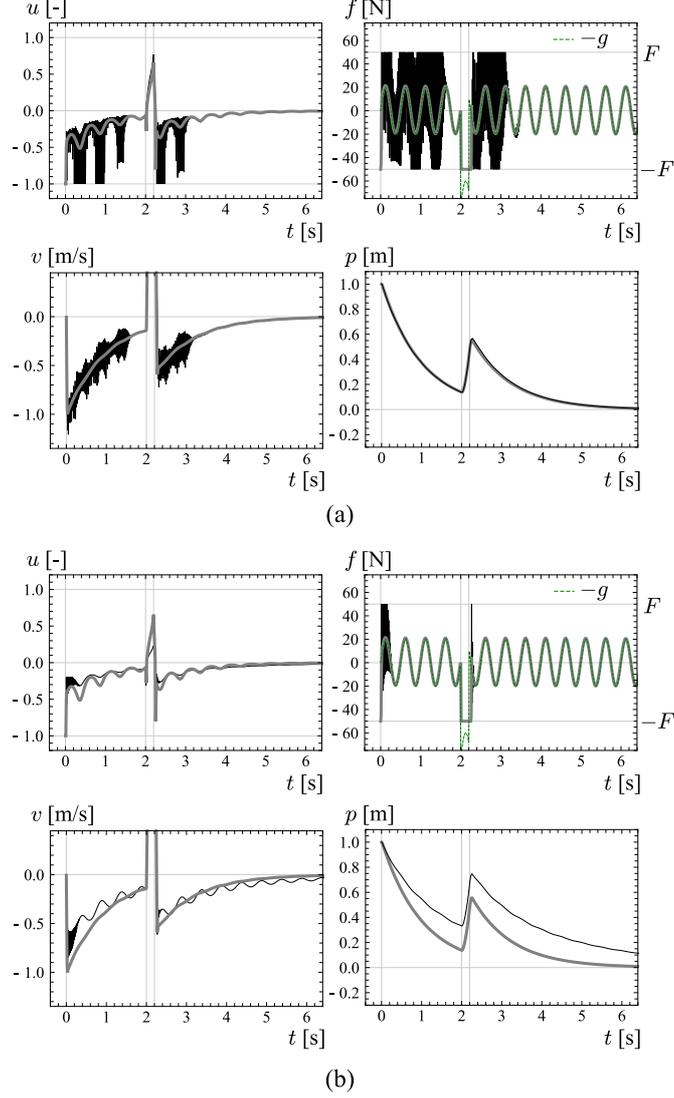


Figure 9: Simulation results with chattering caused by (a) an excessively large  $\hat{M}$  and (b) an excessively small  $\hat{B}$ . The thick gray curves represent an almost ideal case with no modeling errors and  $\hat{g}(t) \equiv g_0(t)$ .

robustness of the controller against the disturbance and modeling errors.

Fig. 9 shows results with much larger  $\hat{M}$  ( $= 3.0M$ ) and much smaller  $\hat{B}$  ( $= 0.33B$ ). It is shown that the control input  $u$  and the actuator force  $f$  exhibit chattering although the position  $p$  converges to  $p_d$ . The parameter  $B$  is an actuator parameter that can be understood as the slope of the velocity-force curve. These results suggest that the inertia of the plant should not be overestimated and that the slope of the force-velocity curve of the actuator should not be underestimated.

In order to emulate the spool dynamics of the control valves of hydraulic actuators (see, e.g., [27]), we consider the case where the control input  $u$  generated by the controller is lagged by a low-pass filtering effect of the plant. Specifically, we consider the plant (12)

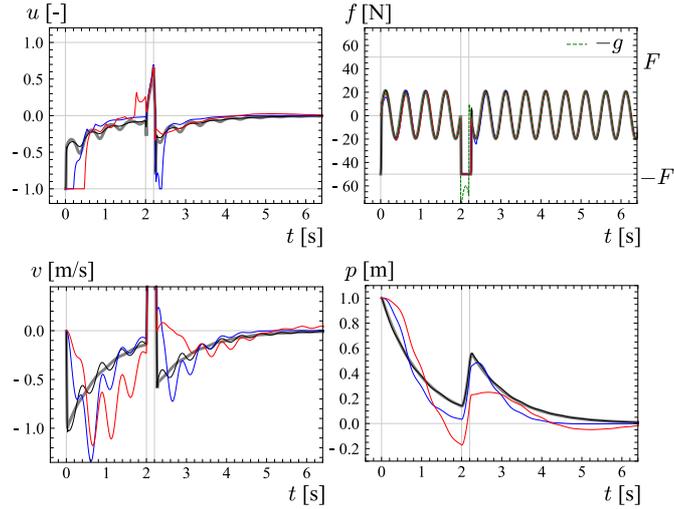


Figure 10: Simulation results with low-pass filtered inputs:  $\tau = 0.4$  s (red),  $\tau = 0.2$  s (blue), and  $\tau = 0$  (black). The thick gray curves represent an almost ideal case with no modeling errors and  $\hat{g}(t) \equiv g_0(t)$ .

with  $u$  being replaced by  $u_f$  that is determined as follows:

$$u_f = \mathcal{L}^{-1}[\mathcal{L}[u]/(\tau s + 1)^2] \quad (39)$$

where  $u$  is the input from the controller,  $\mathcal{L}$  stands for the Laplace transform, and  $\tau$  is a positive constant representing the time constant of the filtering effect. Simulation results with different values of  $\tau$  are shown in Fig. 10. It is shown that, although the lag in the plant affects the behavior, the position  $p$  still converges to the desired position  $p_d$ . These results suggest that the proposed controller may be applicable to hydraulic actuators with spool dynamics, while some extensions to take the lag into account would improve the performance.

## 6 Conclusions

This article has presented an implementation scheme of a class of nonsmooth position controllers (i.e., sliding-mode controllers) to a class of nonsmooth plants, being motivated by a recently-proposed quasistatic model of hydraulic actuators. The implementation is performed with the implicit discretization of the nominal plant model and the nonsmooth controller. Some stability proofs have been provided both in the continuous- and discrete-time domains. Some illustrative simulation results have also been presented.

Future improvements to the proposed controller would include the robustification of the controller against unmodeled actuator dynamics or the incorporation of the actuator dynamics model into the controller.

## A Proof of Theorem 1

As a preparation for the proof of Theorem 1, we can use the following lemma:

**Lemma 3.** Let  $\Gamma : \mathbb{R} \times \mathcal{B} \rightrightarrows \mathbb{R}$  be a set-valued map possessing the properties P1 to P7. Then, the following statement holds true:

$$(v_1 - v_2) (\Gamma(v_2, \Theta_s(v_1, f)) - f) \underset{\forall \geq 0}{\geq} 0 \quad \forall f \in \Gamma(v_1, \mathcal{B}), \quad \forall v_1, v_2 \in \mathbb{R}. \quad (40)$$

**Proof of Lemma 3.** The property P4 of  $\Gamma$  implies that

$$(v_1 - v_2)(\Gamma(v_2, u) - f) \underset{\forall \geq 0}{\geq} 0 \quad (41)$$

is satisfied for all  $\{f, u, v_1, v_2\} \in \mathcal{X}_0$  where

$$\mathcal{X}_0 \triangleq \{\{f, u, v_1, v_2\} \mid f \in \Gamma(v_1, u), u \in \mathcal{B}, v_1, v_2 \in \mathbb{R}\}. \quad (42)$$

Because, using (13),  $u = \Theta_s(v_1, f) \in \mathcal{B}$  implies  $f \in \Gamma(v_1, u) \subseteq \Gamma(v_1, \mathcal{B})$ , we have

$$\mathcal{X}_1 \triangleq \{\{f, u, v_1, v_2\} \mid u = \Theta_s(v_1, f), f \in \Gamma(v_1, \mathcal{B}), v_1, v_2 \in \mathbb{R}\} \subseteq \mathcal{X}_0. \quad (43)$$

Therefore, (41) is satisfied for all  $\{f, u, v_1, v_2\} \in \mathcal{X}_1$ , and it implies (40).  $\square$

Based on this lemma, we can prove Theorem 1.

**Proof of Theorem 1.** For the time period  $\mathcal{T}$ , the velocity  $v$  satisfies the following:

$$M\dot{v} \stackrel{\text{a.e.}}{\in} \Gamma(v, \Theta_s(v^*, \hat{f})) + g_0, \quad v(t_0) = v_0. \quad (44)$$

Let us define  $W(v - v^*) \triangleq M(v - v^*)^2/2$ . Then, considering Lemma 3 and the fact that  $\hat{f} + g_0 = M(v^* - v_0)/T$  by definition, we have the following:

$$\begin{aligned} \dot{W} &\stackrel{\text{a.e.}}{\in} (v - v^*) \left( \Gamma(v, \Theta(v^*, \hat{f})) + g_0 \right) \\ &= -(v^* - v) \left( \Gamma(v, \Theta(v^*, \hat{f})) - \hat{f} \right) + (\hat{f} + g_0)(v - v^*) \\ &\underset{\forall \leq}{\leq} -M(v_0 - v^*)(v - v^*)/T. \end{aligned} \quad (45)$$

Therefore,  $\dot{W} \stackrel{\text{a.e.}}{\leq} -|v_0 - v^*|\sqrt{2MW}/T$  is satisfied as long as  $t \in \mathcal{T}$  and  $v \in \text{conv}\{v_0, v^*\}$ . Considering that  $W$  is an absolutely continuous function of time and that  $v = v_0$  at the time instant  $t = t_0$ , one can see that  $W$  monotonically decreases and thus  $v$  monotonically approaches  $v^*$  until it reaches  $v^*$  in finite time.  $\square$

## B Stability Proofs: Continuous Time

**Proof of Theorem 2.** Let us assume that there exist  $R > 0$  and  $\delta > 0$  that satisfy  $|g| < R - \delta$  for all  $t$  and

$$R < \min\{f \mid f \in (-\Gamma(0, -1)) \wedge f \in \Gamma(0, 1)\}, \quad (46)$$

of which the right-hand side is positive because of the property P7. (See Fig. 11(a).) The proof proceeds in three steps: (a), (b) and (c).

(a) *Finite-time stability of a subset of  $\mathcal{S}$ .*

From (22), if  $\sigma \neq 0$ , we have the following:

$$\frac{d}{dt}|\sigma| \stackrel{\text{a.e.}}{\in} \begin{cases} \gamma_+(v, g) & \text{if } \sigma > 0 \\ \gamma_-(v, g) & \text{if } \sigma < 0 \end{cases} \quad (47)$$

where

$$\gamma_+(v, g) \triangleq (H/M)(Mv/H + \Gamma(v, -1) + g) \quad (48a)$$

$$\gamma_-(v, g) \triangleq (H/M)(-Mv/H - \Gamma(v, 1) - g). \quad (48b)$$

Here, note that  $\gamma_{\pm}(v, g)$  may be set-valued because  $\Gamma(v, \pm 1)$  may be set-valued as illustrated in Fig. 3.

Let us choose  $v_s > 0$  so that

$$\forall v \in v_s \mathcal{B}, \quad -Mv/H + R\mathcal{B} \subset \Gamma(v, \mathcal{B}), \quad (49)$$

or equivalently,

$$\begin{aligned} \forall v \in v_s \mathcal{B}, \quad & Mv/H + R + \Gamma(v, -1) \preceq 0 \\ & \wedge -Mv/H + R - \Gamma(v, 1) \preceq 0, \end{aligned} \quad (50)$$

is satisfied. Here, recall that  $\Gamma(v, \mathcal{B}) = \text{conv}(\Gamma(v, -1) \cup \Gamma(v, 1))$ . The existence of  $v_s$  satisfying (49) is implied by the condition (46) (See Fig. 11(a)). With such a choice of  $v_s$ , we can see that  $\gamma_{\pm}(v, g) \preceq -H\delta/M$  for all  $v \in v_s \mathcal{B}$  because of (48) and (50). Therefore,  $d|\sigma|/dt \stackrel{\text{a.e.}}{<} -H\delta/M$  is satisfied when  $[\sigma, v]^T \in \mathcal{V} \setminus \mathcal{S}$  where

$$\mathcal{V} \triangleq \{[\sigma, v]^T \mid v \in v_s \mathcal{B}\}. \quad (51)$$

This implies that the subset  $\mathcal{S} \cap \mathcal{V}$  is finite-time stable and the sliding mode at  $\sigma = 0$  is established in this subset (See Fig. 11(b)).

Assume that  $[\sigma, v]^T \in \mathcal{S} \cap \mathcal{V}$ ,  $\sigma = 0$  and

$$-Mv/H - g \in \Gamma(v, \mathcal{B}) \quad (52)$$

are satisfied. In this situation, (22) implies that  $f = -Mv/H - g$ , which results in  $\dot{\sigma} = 0$  and  $\dot{v} = -v/H$ . Therefore, when  $[\sigma, v]^T \in \mathcal{S} \cap \mathcal{V}$ ,  $\sigma$  remains zero and  $|v|$  monotonically decreases, and because  $0 \in \mathcal{S} \cap \mathcal{V}$ ,  $[\sigma, v]^T$  never deviates from  $\mathcal{S} \cap \mathcal{V}$ . Therefore,  $\mathcal{S} \cap \mathcal{V}$  is positively invariant.

(b) *Asymptotic stability of the origin.*

Let us define a Lyapunov function candidate as follows:

$$V(\sigma, v) = \kappa|\sigma| + Hv^2/2 \quad (53)$$

where  $\kappa$  is a positive constant. When  $\sigma = 0$  and (52) are satisfied, the system (22) is in the sliding mode, i.e.,  $\dot{\sigma} = 0$ , and (22) results in  $\dot{v} = -v/H$  and  $\dot{V} = -v^2$ . On the other hand, when  $\sigma = 0$  is satisfied but (52) is not satisfied,  $\dot{\sigma} = 0$  cannot be true and  $\dot{V}$  may not exist. Such a case may happen only at the time instants at which the state  $[\sigma, v]^T$  instantaneously penetrates the surface  $\sigma = 0$ . Therefore, one can say that  $\dot{V} = -v^2$  is satisfied for almost all  $t$  if  $\sigma = 0$ .

Including the case of  $\sigma \neq 0$ , one can see that (22) implies the following:

$$\dot{V} \stackrel{\text{a.e.}}{\in} -v^2 + \begin{cases} \gamma_+(v, g)(\kappa + v) & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma = 0 \\ \gamma_-(v, g)(\kappa - v) & \text{if } \sigma < 0. \end{cases} \quad (54)$$

Recall that  $\gamma_{\pm}(v, g) \vee < 0$  if  $|v| \leq v_s$ . Therefore, by setting  $\kappa > v_s$ , one can see that  $[\sigma, v]^T \in \mathcal{V}$  is a sufficient condition for  $\dot{V} \stackrel{\text{a.e.}}{\leq} -v^2$ . When  $v = 0 \wedge \sigma \neq 0$ ,  $\dot{v} = 0$  cannot happen because of the definition (22) of the system and the condition (46). Thus, invoking LaSalle's invariance principle, one can see that the origin  $(\sigma, v) = (0, 0)$  is asymptotically stable if (46) is satisfied.

Because  $V(\sigma, v)$  monotonically decreases in  $\mathcal{V}$ , a subset of the region of attraction of the origin can be given as follows:

$$\mathcal{A} \triangleq \{[\sigma, v]^T \mid V(\sigma, v) \leq H v_s^2 / 2\}. \quad (55)$$

Moreover, because  $|\sigma|$  monotonically decreases in  $\mathcal{A}$  and it satisfies  $(\mathcal{S} \cap \mathcal{V}) \subset \mathcal{A} \subset \mathcal{V}$  (see Fig. 11(b)),  $\mathcal{A}$  is also a subset of the region of attraction of the set  $\mathcal{S} \cap \mathcal{V}$ . From an initial state  $[\sigma(0), v(0)]^T \in \mathcal{A}$ , the maximum reaching time to  $\mathcal{S} \cap \mathcal{V}$  is  $M|\sigma(0)|/(H\delta)$  because  $d|\sigma|/dt < -H\delta/M$ .

(c) *Global asymptotic stability in the case of  $g \equiv 0$ .*

Let us set a Lyapunov function candidate as follows:

$$V_g(\sigma, v) = \max(-\kappa_n \sigma, \kappa_p \sigma) + H v^2 / 2 \quad (56)$$

where  $\kappa_n$  and  $\kappa_p$  are positive constants. When  $g \equiv 0$ , its time derivative can be written as follows:

$$\dot{V}_g \stackrel{\text{a.e.}}{\in} \begin{cases} \gamma_{g+}(v) - \kappa_n^2 & \text{if } \sigma > 0 \\ -v^2 & \text{if } \sigma = 0 \wedge -Mv/H \in \Gamma(v, \mathcal{B}) \\ \gamma_{g-}(v) - \kappa_p^2 & \text{if } \sigma < 0 \end{cases} \quad (57)$$

where

$$\gamma_{g+}(v) \triangleq (\kappa_n + v)(\kappa_n + H\Gamma(v, -1)/M) \quad (58)$$

$$\gamma_{g-}(v) \triangleq (\kappa_p - v)(\kappa_p - H\Gamma(v, 1)/M). \quad (59)$$

The functions  $\gamma_{g\pm}(v)$  can be made negative everywhere by setting  $\kappa_n$  and  $\kappa_p$  so that the following is satisfied:

$$\Gamma(-\kappa_n, -1) \ni -M\kappa_n/H, \quad \Gamma(\kappa_p, 1) \ni M\kappa_p/H. \quad (60)$$

The properties P3 and P4 of  $\Gamma$  imply that such  $\kappa_n$  and  $\kappa_p$  always exist and satisfy  $\Gamma(0, -1) \vee \leq -M\kappa_n/H < 0 < M\kappa_p/H \leq \vee \Gamma(0, 1)$ . Therefore, with the choice of  $\kappa_n$  and  $\kappa_p$  satisfying (60), (57) implies that  $\dot{V}_g \stackrel{\text{a.e.}}{<} 0$  is satisfied except at the origin. Therefore, one can conclude that the origin  $[\sigma, v]^T = 0$  is globally asymptotically stable if  $g \equiv 0$ .  $\square$

**Remark 2.** *The set  $\mathcal{A}$  is a conservative estimate of the region of attraction. It does not include initial states from which the state converges to the origin after penetrating the subspace  $\mathcal{S}$  several times. A more accurate estimate of the region of attraction may be obtained by using a more complicated Lyapunov function based on the solution trajectory under the most destabilizing external force as in [28].*

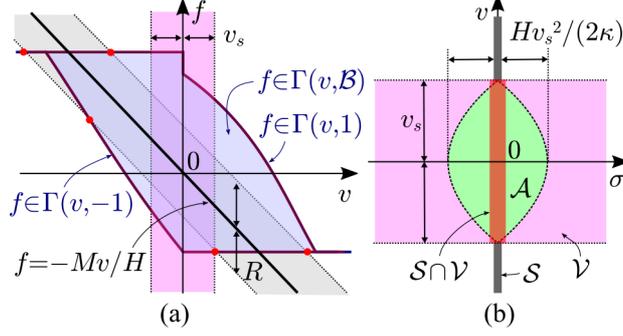


Figure 11: Illustrations for the proof of Theorem 2. (a)  $\Gamma$  and  $v_s$ . (b) Subsets of the state space.

## C Stability Proofs: Discrete Time

**Proof of Theorem 4.** Let us assume that  $\hat{g}_k$ ,  $\tilde{a}_k$ ,  $\varepsilon_{\sigma,k}$ , and  $\varepsilon_{v,k}$  are bounded as  $|\hat{g}_k| < \hat{R} - \delta$ ,  $|\tilde{a}_k| < U_a$ ,  $|\varepsilon_{\sigma,k}| < U_\sigma$ , and  $|\varepsilon_{v,k}| < U_v$ , respectively, for all  $k \geq 0$ , where  $\hat{R}$ ,  $\delta$ ,  $U_a$ ,  $U_\sigma$ , and  $U_v$  are positive constants. Let  $w_{\sigma,k} \triangleq (H+T)\tilde{a}_{k-1} + T\varepsilon_{\sigma,k}$  and  $w_{v,k} \triangleq \tilde{a}_{k-1} + \varepsilon_{\sigma,k} - H\varepsilon_{v,k}$ . They are bounded as  $|w_{\sigma,k}| < W_\sigma$  and  $|w_{v,k}| < W_v$  where  $W_\sigma \triangleq (H+T)U_a + TU_\sigma$  and  $W_v \triangleq U_a + U_\sigma + HU_v$ . Let us assume the following:

$$\hat{R} + \hat{M}W_\sigma/H < \min((-\hat{\Gamma}(0, -1)) \cup \hat{\Gamma}(0, 1)). \quad (61)$$

Let us define  $x \triangleq [\sigma, v]^T$ , which is the real state vector, and  $x^* \triangleq [\sigma^*, v^*]^T$ , which is the predicted state vector. In the following, we consider two sequences:  $\{x_k\}_{k \in \mathbb{N}}$  and  $\{x_k^*\}_{k \in \mathbb{N}}$ . The proof proceeds in two steps: (a) and (b).

(a) *Finite-time stability of a subset of  $\mathcal{S}$ .*

From (32) and (34), one can see that the predicted states  $\{x_k^*\}_{k \in \mathbb{N}}$  are recursively obtained as

$$x_{k+1}^* = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k^* + \begin{bmatrix} \hat{\eta}H_t \\ \hat{\eta} \end{bmatrix} (\hat{f}_k + \hat{g}_k) + \begin{bmatrix} \tilde{\sigma}_k + T\tilde{v}_k \\ \tilde{v}_k \end{bmatrix} \quad (62a)$$

$$\hat{f}_k \in \hat{\Gamma}(v_{k+1}^*, \text{sgn}(-\sigma_{k+1}^*)) \quad (62b)$$

and that the real states  $\{x_k\}_{k \in \mathbb{N}}$  are in neighborhoods of the predicted states as can be written as  $x_k \in x_k^* + T\mathcal{E}$  where

$$\mathcal{E} \triangleq \left\{ \begin{bmatrix} H_t\tilde{a} - T\varepsilon_\sigma \\ \tilde{a} - T\varepsilon_v \end{bmatrix} \mid |\tilde{a}| < U_a, |\varepsilon_\sigma| < U_\sigma, |\varepsilon_v| < U_v \right\}. \quad (63)$$

Therefore, the remainder of the proof focuses on the properties of the sequence  $\{x_k^*\}_{k \in \mathbb{N}}$  of the predicted states.

Equations (62) and (34) yield the following:

$$\frac{|\sigma_{k+1}^*| - |\sigma_k^*|}{T} \leq \vee \begin{cases} \gamma_+(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) & \text{if } \sigma_{k+1}^* > 0 \\ -|\sigma_k^*|/T & \text{if } \sigma_{k+1}^* = 0 \\ \gamma_-(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) & \text{if } \sigma_{k+1}^* < 0 \end{cases} \quad (64)$$

where

$$\hat{\gamma}_+(v, g, w_\sigma) \triangleq \frac{H}{\hat{M}} \left( \frac{\hat{M}v}{H} + \hat{\Gamma}(v, -1) + g + \frac{\hat{M}w_\sigma}{H} \right) \quad (65)$$

$$\hat{\gamma}_-(v, g, w_\sigma) \triangleq \frac{H}{\hat{M}} \left( -\frac{\hat{M}v}{H} - \hat{\Gamma}(v, 1) - g - \frac{\hat{M}w_\sigma}{H} \right). \quad (66)$$

Let us choose  $\hat{v}_s$  so that

$$\forall v \in \hat{v}_s \mathcal{B}, -\frac{\hat{M}v}{H} + \left( \hat{R} + \frac{\hat{M}W_\sigma}{H} \right) \mathcal{B} \subset \hat{\Gamma}(v, \mathcal{B}). \quad (67)$$

The existence of  $\hat{v}_s$  satisfying (67) is implied by the condition (61). Then, these functions satisfy  $\hat{\gamma}_+(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) \leq -H\delta/\hat{M}$  and  $\hat{\gamma}_-(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) \leq -H\delta/\hat{M}$  if  $x_{k+1}^* \in \hat{\mathcal{V}}$  where

$$\hat{\mathcal{V}} \triangleq \{[\sigma, v]^T \mid v \in \hat{v}_s \mathcal{B}\}. \quad (68)$$

This means that  $|\sigma_{k+1}^*| < |\sigma_k^*| - H\delta/\hat{M}$  is satisfied for all  $\sigma_{k+1}^* \neq 0$ . Therefore, the subset  $\mathcal{S} \cap \hat{\mathcal{V}}$  can be reached from a neighborhood of it by the sequence  $\{x_k^*\}_{k \in \mathbb{N}}$  in a finite number of timesteps. It is also the case with the subset  $(\mathcal{S} \cap \hat{\mathcal{V}}) + T\mathcal{E}$  and the sequence  $\{x_k\}_{k \in \mathbb{N}}$ .

A careful observation on (62) and (34) shows that, if  $x_{k+1}^* \in \mathcal{S} \cap \hat{\mathcal{V}}$ , we have  $\sigma_{k+1}^* = 0$  and

$$\frac{v_{k+1}^{*2} - v_k^{*2}}{2T} < -\frac{v_{k+1}^*}{H}(v_{k+1}^* + Tw_{v,k} + \sigma_k^*/T). \quad (69)$$

This implies that, if  $x_{k+1}^* \in \mathcal{S} \cap \hat{\mathcal{V}}$  and  $x_k^* \in \mathcal{S}$  (i.e.,  $\sigma_k^* = 0$ ),  $|v_{k+1}^*| \leq |v_k^*|$  if  $|v_{k+1}^*| \geq TW_v$ . Let us define

$$\mathcal{T} \triangleq \{[\sigma, v]^T \mid |v| < TW_v\} \quad (70)$$

and assume that  $W_v$  is small enough to satisfy  $\mathcal{T} \subset \hat{\mathcal{V}}$ . Then, one can see that, once  $x^* \in \mathcal{S} \cap \hat{\mathcal{V}}$  is achieved,  $\{x_k^*\}_{k \in \mathbb{N}}$  does not deviate from the subset  $\mathcal{S} \cap \hat{\mathcal{V}}$  (which thus can be said to be positively invariant) and eventually reaches the terminal attractor  $\mathcal{S} \cap \mathcal{T}$ . It means that the real state sequence  $\{x_k\}_{k \in \mathbb{N}}$  reaches  $(\mathcal{S} \cap \hat{\mathcal{V}}) + T\mathcal{E}$  in finite time, stays there after that, and eventually converges to the terminal attractor  $(\mathcal{S} \cap \mathcal{T}) + T\mathcal{E}$ .

(b) *Asymptotic stability of the origin.*

Let us define

$$\hat{V}([\sigma, v]^T) = \kappa|\sigma| + Hv^2/2 \quad (71)$$

where  $\kappa$  is a positive constant. Careful derivation shows that

$$\begin{aligned} \frac{\hat{V}(x_{k+1}^*) - \hat{V}(x_k^*)}{T} &\leq -v_{k+1}^*(v_{k+1}^* + TW_v) \\ &+ \begin{cases} \hat{\gamma}_+(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) (\kappa + v_{k+1}^*) & \text{if } \sigma_{k+1}^* > 0 \\ 0 & \text{if } \sigma_{k+1}^* = 0 \\ \hat{\gamma}_-(v_{k+1}^*, \hat{g}_k, w_{\sigma,k}) (\kappa - v_{k+1}^*) & \text{if } \sigma_{k+1}^* < 0. \end{cases} \end{aligned} \quad (72)$$

Let us choose  $\kappa > \hat{v}_s$ . Then, one can see that  $\hat{V}(x_{k+1}^*) < \hat{V}(x_k^*)$  if  $\sigma_{k+1}^* \neq 0$  and  $x_{k+1}^* \in \hat{\mathcal{V}}$  and if  $\sigma_{k+1}^* = 0$  and  $x_{k+1}^* \notin \mathcal{T}$ . Therefore,  $V(x_k^*)$  monotonically decreases in a neighborhood of  $\mathcal{S} \cap \mathcal{T}$  and thus a level set including  $\mathcal{S} \cap \mathcal{T}$  is asymptotically stable in terms of the predicted state  $x^*$ . It also implies that the real state arrives in a neighborhood of a level set including  $(\mathcal{S} \cap \mathcal{T}) + T\mathcal{E}$ , which thus can be said to be asymptotically stable.  $\square$

**Remark 3.** *The approach of the above proof is similar to that adopted in Sections V.B and V.C of [20] in that they involve the predicted states, the real states, and the error between them, and that the sliding surface is achieved by the predicted states. One feature of the above proof is that a relatively complicated Lyapunov function constructed for the continuous-time system is reused in a discretized form.*

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