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Thin Layer Models for Geophysical Applications

JULIEN DIAZ[1,2], VICTOR PÉRON[2,1]

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Second French-Russian Conference
Novosibirsk, September 25, 2014

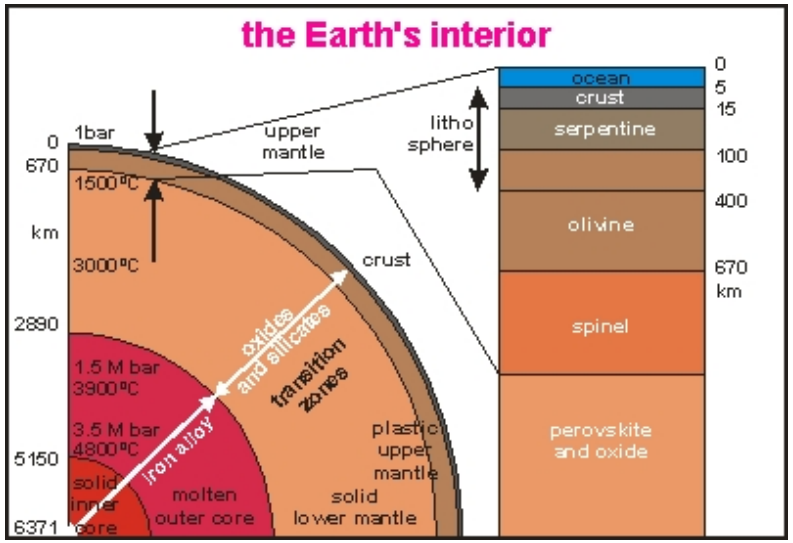
Motivations

HPC-GA Project

- **High Performance Computing for Geophysical Applications**
  FP7 Project : INRIA, BCAM, UFRGS, UNAM
  - Earthquakes
  - Tsunami
- **Numerical Models (WP2)**
  - Asymptotic method : To derive approximate models for the propagation of seismic waves and acoustic waves
  - Elasto-acoustic coupling : To reproduce earthquakes

Motivations

Configurations of interest

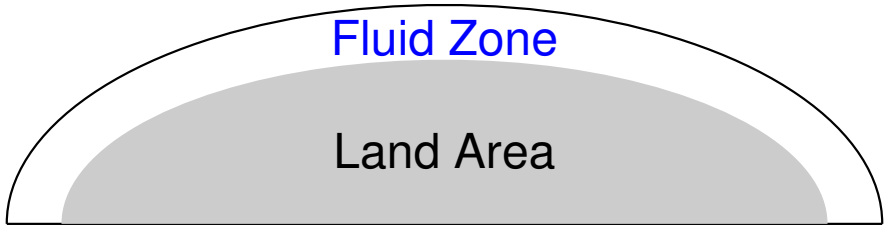


- **Configuration A** : The medium of interest is a Land Area surrounded by a fluid zone
- **Configuration B** : The medium of interest is a Land Area surrounded by a fluid zone and a part of the atmosphere

Motivations

Configuration A

- **Configuration A**:

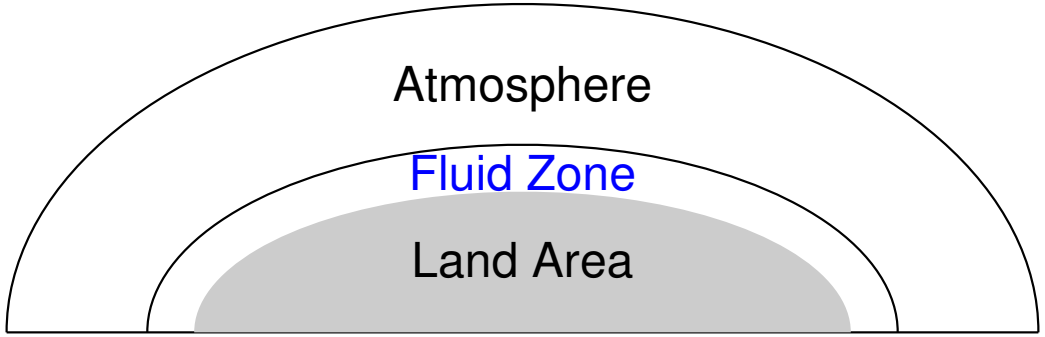


- The thickness of the fluid zone is negligible w.r.t. the wavelength
- **Sub-Configurations**:
  1. **Configuration A1** : A thin layer with a **uniform** thickness
  2. **Configuration A2** : A thin layer with a **variable** thickness

Motivations

Configuration B

- **Configuration B**



- **Difficulty**

  Apply a FEM on a mesh with thin cells in the Fluid Zone and much larger outside

Approach : an Asymptotic Method

1. ”Replace” the thin layer by an Equivalent Boundary Condition
2. Couple this condition with
   - the elastic equation (**Configuration A**)
   - elastic and acoustic equations (**Configuration B**)

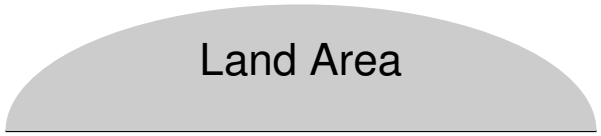


Figure: Configuration A
without the thin layer

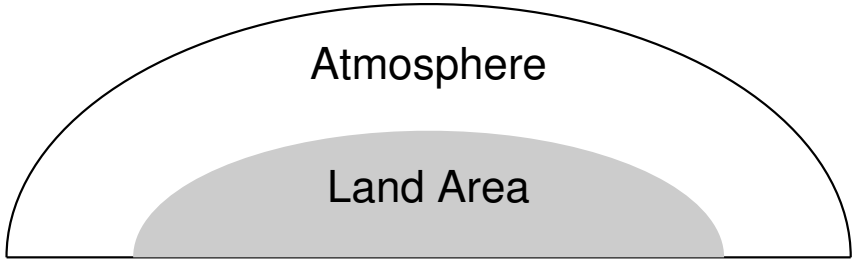


Figure: Configuration B
without the thin layer

3. Apply a (DG) Finite Element Method

Bibliography

- Derivation of Equivalent Conditions for thin layer Problems :

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ABBOUD-AMMARI, 96

BENDALI-LEMRABET, 96

LAFITTE, 99

...

PÉRON, 2014

BUREL, PHD THESIS, 2014

DE CASTRO, DIAZ, PÉRON, 2014

Outline

1. Energy Estimates (Conf. A1)
2. Equivalent Conditions (Conf. A1)
3. Numerical Results (Conf. A1)
4. Equivalent Conditions (Conf. A2 and B)

Outline

1. Energy Estimates (Conf. A1)
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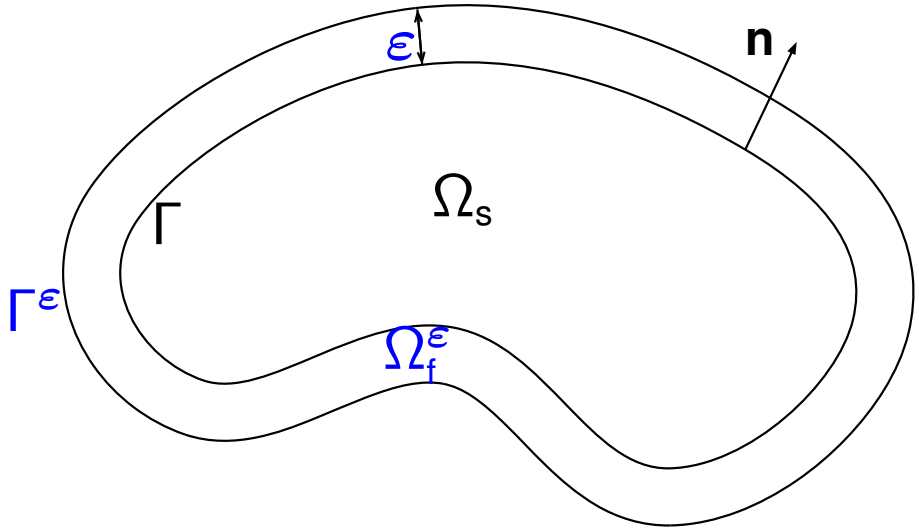
A Model Problem

The Problem $(\mathbf{P}_\varepsilon)$ set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

$$
\begin{array}{lll}
\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in} & \Omega_s \\
\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 & \text{in} & \Omega_f^\varepsilon \\
\partial_{\mathbf{n}} p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} & \text{on} & \Gamma \\
\underline{\underline{\sigma}}(\mathbf{u})\, \mathbf{n} = -p\mathbf{n} & \text{on} & \Gamma \\
p = 0 & \text{on} & \Gamma^\varepsilon
\end{array}
$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of $(\mathbf{P}_\varepsilon)$ as $\varepsilon \to 0$?

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\,\underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

(i) $\underline{\underline{C}}(x) = \big(C_{ijkl}(x)\big)$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$

(ii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ

(iii) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}}\,\underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

(i) $\underline{\underline{C}}(x) = \big(C_{ijkl}(x)\big)$ *is symmetric :* $C_{ijkl} = C_{jikl} = C_{klij}$
(ii) $C_{ijkl}(x)$ *are real valued smooth functions, up to* Γ
(iii) *The tensor* $\underline{\underline{C}}(x)$ *is positive*

Assumption (SA)

The angular frequency ω *is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \textit{in} \quad \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u})\,\mathbf{n} = 0 & \textit{on} \quad \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$*, the problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in \mathrm{H}^1(\Omega_\mathrm{s}) \times \mathrm{H}^1_{0,\Gamma^\varepsilon}(\Omega_\mathrm{f}^\varepsilon)$*, and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathrm{s}} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_\mathrm{f}^\varepsilon} \leqslant C\|\mathbf{f}\|_{0,\Omega_\mathrm{s}}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ as $\varepsilon \to 0$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$, *the problem* $(\mathbf{P}_\varepsilon)$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in \mathrm{H}^1(\Omega_\mathrm{s}) \times \mathrm{H}^1_{0,\Gamma^\varepsilon}(\Omega_\mathrm{f}^\varepsilon)$, *and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathrm{s}} + \|\mathrm{p}_\varepsilon\|_{1,\Omega_\mathrm{f}^\varepsilon} \leqslant C\|\mathbf{f}\|_{0,\Omega_\mathrm{s}}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ as $\varepsilon \to 0$

Key for the proof:

To introduce a "Scaled Problem" by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where} \quad s \in (0, \varepsilon)$$

Outline

1. Energy Estimates (Conf. A1)
2. Equivalent Conditions (Conf. A1)
3. Numerical Results (Conf. A1)
4. Equivalent Conditions (Conf. A2 and B)

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon)$ when $\varepsilon \to 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \cdots$$
$$\mathrm{p}_\varepsilon(x) = \mathfrak{p}_0(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon \mathfrak{p}_1(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon^2 \mathfrak{p}_2(y_\alpha, \frac{s}{\varepsilon}) + \cdots$$

- **Step 2** : Equivalent Conditions of order $k+1 \in \mathbb{N}$.
  Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k \quad \text{up to} \quad \mathcal{O}(\varepsilon^{k+1})$$

- **Step 3** : Prove "Stability" & Convergence results for Equivalent models

Step 1 : Multiscale Expansion - First terms

1. $$\mathfrak{p}_0 = 0$$

2. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on} \quad \Gamma \end{cases}$$

3. $$\mathfrak{p}_1(., S) = (S-1)\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,, \quad S \in (0,1)$$

4. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

5. $$\mathfrak{p}_2(., S) = -(S^2-1)\mathcal{H}\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} + (S-1)\rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

6. $$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} - \mathcal{H}\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} & \text{on} \quad \Gamma \end{cases}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\varepsilon^N := \mathbf{u}_\varepsilon - \sum_{n=0}^{N} \varepsilon^n \mathbf{u}_n \quad \text{in} \quad \Omega_\mathrm{s}\,, \quad \text{and} \quad r_\varepsilon^N := \mathrm{p}_\varepsilon - \sum_{n=0}^{N} \varepsilon^n \mathrm{p}_n \quad \text{in} \quad \Omega_\mathrm{f}^\varepsilon\,.$$

Evaluation of the right hand sides in

$$\begin{cases} \Delta r_\varepsilon^N + \kappa^2 r_\varepsilon^N & = & f_\varepsilon & \text{in} & \Omega_\mathrm{f}^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\varepsilon^N) + \omega^2 \rho \mathbf{r}_\varepsilon^N & = & 0 & \text{in} & \Omega_\mathrm{s} \\ \partial_\mathbf{n} r_\varepsilon^N - \rho_f \omega^2 \mathbf{r}_\varepsilon^N \cdot \mathbf{n} & = & g_\varepsilon & \text{on} & \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^N) + r_\varepsilon^N \mathbf{n} & = & 0 & \text{on} & \Gamma \\ r_\varepsilon^N & = & 0 & \text{on} & \Gamma^\varepsilon\,. \end{cases}$$

The RHS are explicit : $f_\varepsilon = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$ in $\Omega_\mathrm{f}^\varepsilon$, $\quad g_\varepsilon = \mathcal{O}(\varepsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_\mathrm{s}} + \sqrt{\varepsilon}\|r_\varepsilon^N\|_{1,\Omega_\mathrm{f}^\varepsilon} \leqslant C_N \varepsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2, 3\}$, we identify a simpler problem

$$(\mathbf{P}_\varepsilon^k) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k) + \omega^2 \rho\, \mathbf{u}_\varepsilon^k = \mathbf{f} & \text{in} \quad \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathrm{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n})\, \mathbf{n} = 0 & \text{on} \quad \Gamma \end{cases}$$

with $\mathrm{B}_{k,\varepsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

1. Order 1 :

$$\mathbf{T}(\mathbf{u}_0) = 0$$

2. Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon\omega^2\rho_f\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = 0$$

3. Order 3 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon\omega^2\rho_f\left(1 - \varepsilon\mathcal{H}\right)\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = 0$$

4. Order 4 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^3) - \varepsilon\omega^2\rho_f\left(1 - \varepsilon\mathcal{H} + \frac{\varepsilon^2}{3}\left[\Delta_\Gamma + \kappa^2\mathbb{I} + 4\mathcal{H}^2 - \mathcal{K}\right]\right)\mathbf{u}_\varepsilon^3 \cdot \mathbf{n}\,\mathbf{n} = 0$$

Step 3 : Stability and Convergence results

$\mathrm{V}_k = \mathrm{H}^1(\Omega_s)$ when $k = 0, 1, 2$
$\mathrm{V}_k = \{\mathbf{u} \in \mathrm{H}^1(\Omega_s) \mid \quad \mathbf{u} \cdot \mathbf{n}|_\Gamma \in \mathrm{H}^1(\Gamma)\}$ when $k = 3$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in \mathrm{L}^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathrm{V}_k$ and

- *Stability*

$$\|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\|\mathbf{f}\|_{0,\Omega_s}$$

- *Convergence*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1}$$

The case of a Fourier-Robin B.C.

Step 2 : Equivalent Conditions

When considering a Fourier-Robin boundary condition :

$$\partial_{\mathbf{n}} \mathrm{p}_\varepsilon - i\kappa \mathrm{p}_\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon$$

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_\mathrm{f} \left(1 + \varepsilon \left(-2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma\right)\right) (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n})\,\mathbf{n} = 0$$

Outline

1. Energy Estimates (Conf. A1)
2. Equivalent Conditions (Conf. A1)
3. **Numerical Results (Conf. A1)**
4. Equivalent Conditions (Conf. A2 and B)

Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = \mathrm{D}(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($\mathbf{f} = 0$) : $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$-finite elements (Lagrange) in the Library Hou10ni
- Ω_s : Isotropic elastic material with Lamé coefficients :

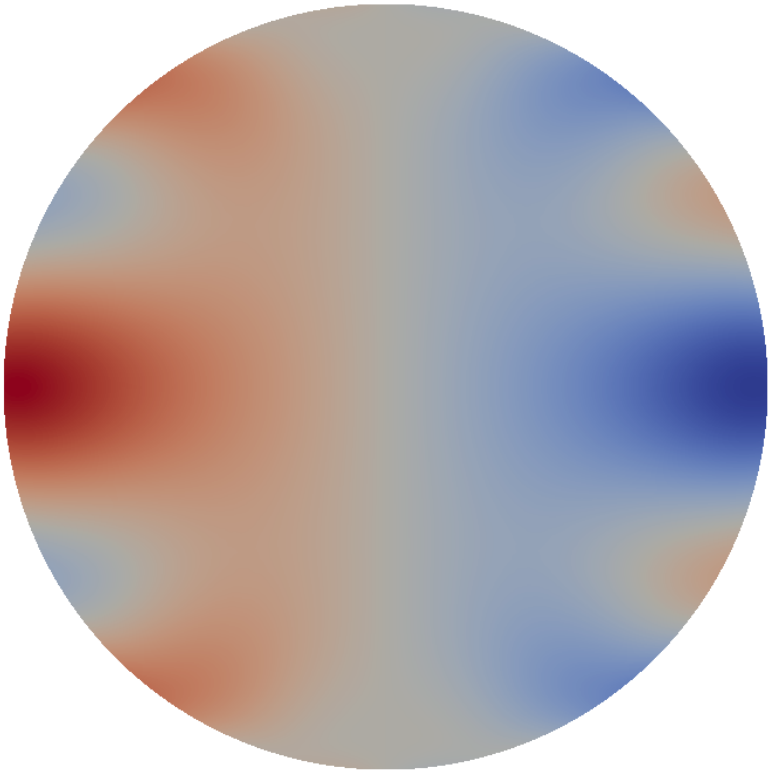
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 \,.$$

- $c = 1500$ m.s^{-1}, $\quad \rho_f = 1000$ kg.m^{-3} , $\quad \rho = 2700$ kg.m^{-3}.

Numerical Simulations

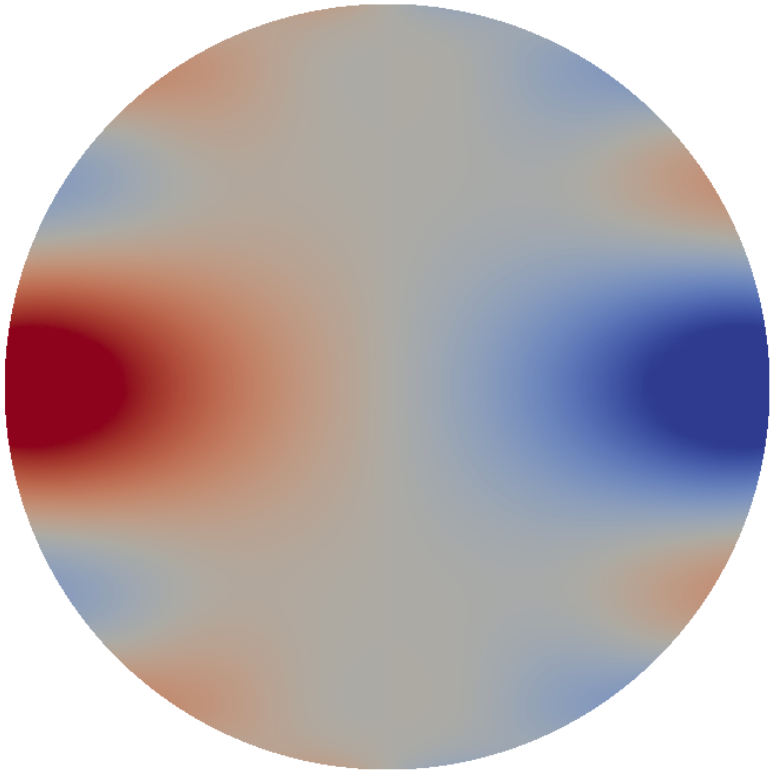
Qualitative Comparisons

$\varepsilon = 0.001;$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$

$\mathbf{u}_\varepsilon$: analytical solution



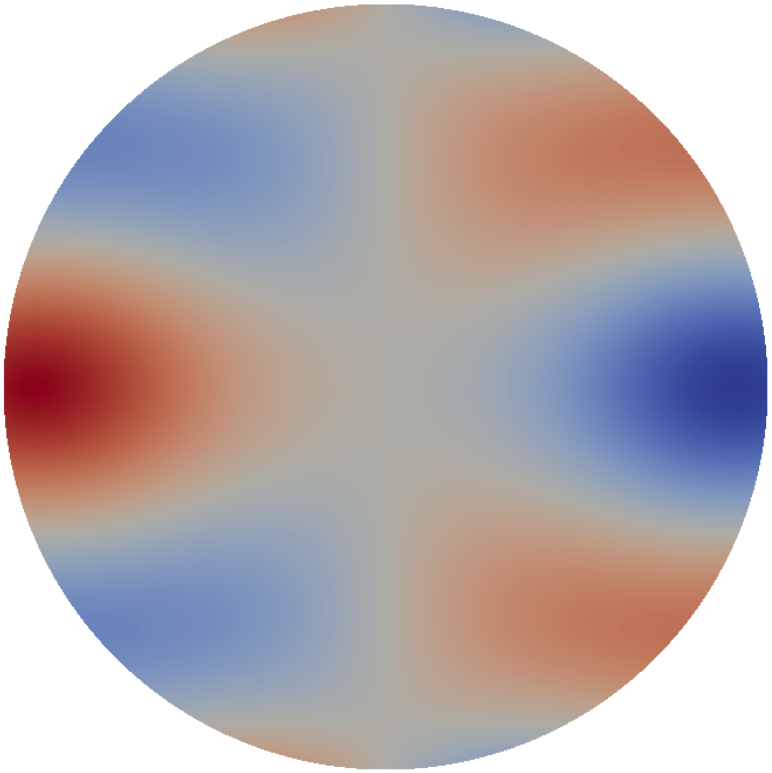
$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

$\mathbf{u}_\varepsilon^2$ (model of order 3) : IPDGM

Numerical Simulations

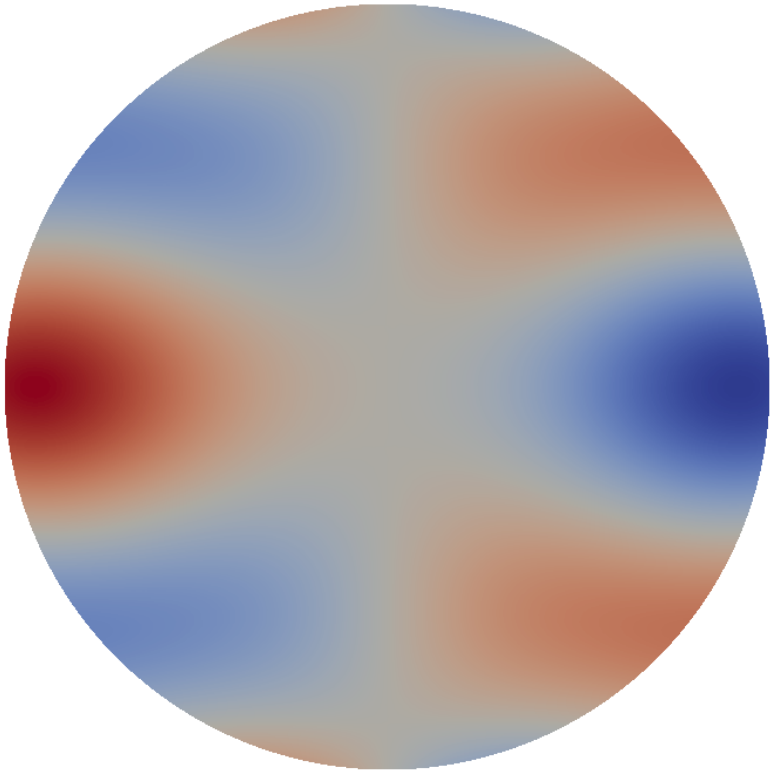
Qualitative Comparisons

$\varepsilon = 0.0001;$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$

$\mathbf{u}_\varepsilon$: analytical solution



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

$\mathbf{u}_\varepsilon^2$ (model of order 3) : IPDGM

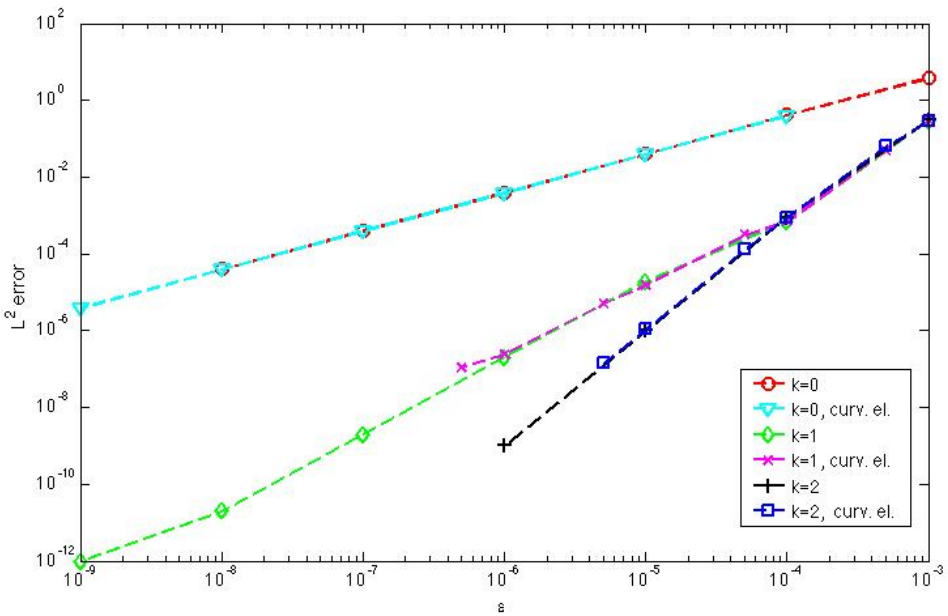
Convergence of the models

Dirichlet case

For $k \in \{0, 1, 2\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution or numerical solution (IPDGM)



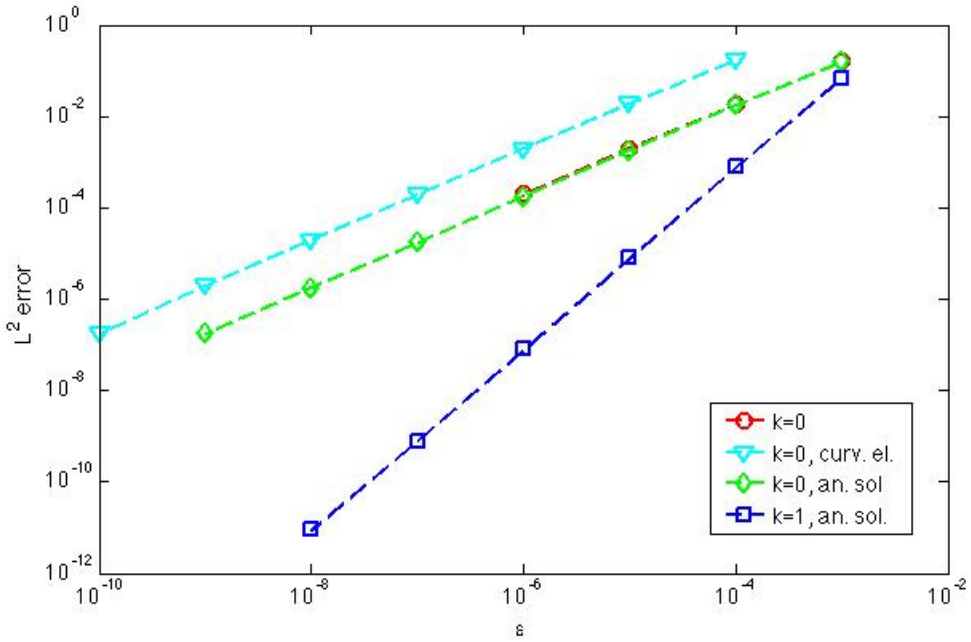
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2-error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

$\mathbf{u}_\varepsilon$: analytical solution

$\mathbf{u}_\varepsilon^k$: analytical solution or numerical solution (IPDGM)



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1. Energy Estimates (Conf. A1)
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Configuration A2

Framework in 2D

- Parameterization of the layer $\Omega_{\mathrm{f}}^{\varepsilon}$

$$\Omega_{\mathrm{f}}^{\varepsilon} = \left\{ \mathbf{x}(t) + s\, f(t)\mathbf{n}(t) \in \mathbb{R}^2 \mid \quad \mathbf{x}(t) \in \Gamma, \quad s \in (0, \varepsilon) \right\}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

Configuration A2

Framework in 2D

- Parameterization of the layer $\Omega_{\mathrm{f}}^{\varepsilon}$

$$\Omega_{\mathrm{f}}^{\varepsilon} = \left\{ \mathbf{x}(t) + s\, f(t)\mathbf{n}(t) \in \mathbb{R}^2 \mid \quad \mathbf{x}(t) \in \Gamma, \quad s \in (0, \varepsilon) \right\}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$: normal vector on Γ

- Euclidean metric of $\Omega_{\mathrm{f}}^{\varepsilon}$ defined in the coordinates (t, s):

$$\begin{pmatrix} \left(1 + s\, f(t)\mathrm{c}(t)\right)^2 + \left(s\, f'(t)\right)^2 & s\, f(t)f'(t) \\ sf(t)f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon f(t)\omega^2 \rho_\mathrm{f} \mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\,\mathbf{n} = 0$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon f(t)\omega^2 \rho_\mathrm{f} \left(1 - \frac{\varepsilon}{2} f(t) \mathrm{c}(t)\right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\,\mathbf{n} = 0$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}^1_\varepsilon) - i\omega c_f \rho_f \left(1 + \varepsilon \left(-f(t)\mathrm{c}(t) + i\kappa^{-1}[g(t)\partial_t + f(t)\partial_t^2]\right)\right) \mathbf{u}^1_\varepsilon \cdot \mathbf{n}\,\mathbf{n} = 0$$

with

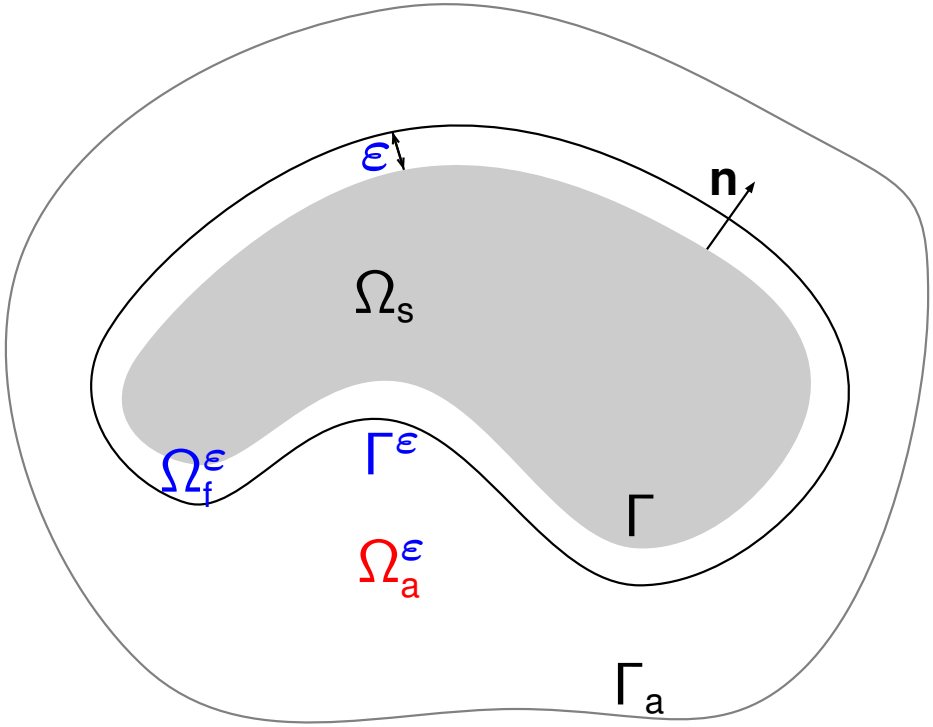
$$g(t) = \left(2 - f^2(t)\right) f'(t)$$

Configuration B

A Model Problem

$(\mathbf{u}, \mathrm{p}, \mathrm{p}^{\mathrm{a}})$ satisfies

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in} \quad \Omega_{\mathrm{s}} \\ \Delta \mathrm{p} + \kappa^2 \mathrm{p} = 0 & \text{in} \quad \Omega_{\mathrm{f}}^{\varepsilon} \\ \Delta \mathrm{p}^{\mathrm{a}} + \kappa_{\mathrm{a}}^2 \mathrm{p}^{\mathrm{a}} = 0 & \text{in} \quad \Omega_{\mathrm{a}}^{\varepsilon} \\ \partial_{\mathbf{n}} \mathrm{p} = \rho_{\mathrm{f}} \omega^2 \mathbf{u} \cdot \mathbf{n} & \text{on} \quad \Gamma \\ \underline{\underline{\sigma}}(\mathbf{u})\, \mathbf{n} = -\mathrm{p}\mathbf{n} & \text{on} \quad \Gamma \\ \mathrm{p}^{\mathrm{a}} = \mathrm{p} & \text{on} \quad \Gamma^{\varepsilon} \\ c_{\mathrm{a}}^2 \partial_{\mathbf{n}} \mathrm{p}^{\mathrm{a}} = c_{\mathrm{f}}^2 \partial_{\mathbf{n}} \mathrm{p} & \text{on} \quad \Gamma^{\varepsilon} \\ \text{ABC} & \text{on} \quad \Gamma_{\mathrm{a}} \end{cases}$$



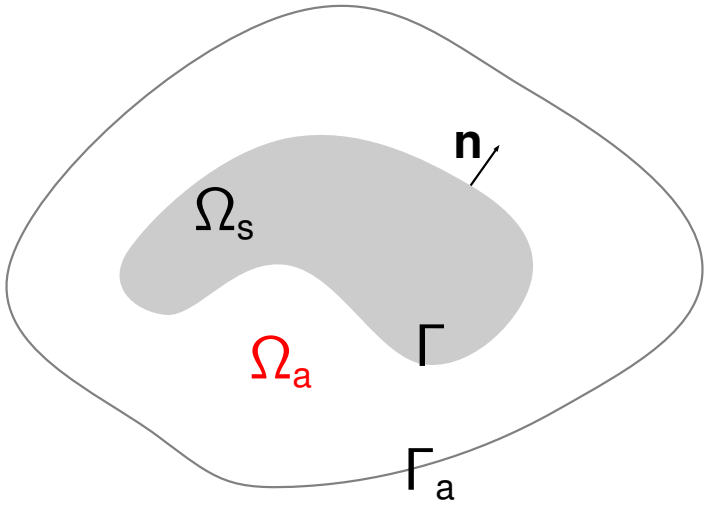
$$\kappa = \frac{\omega}{c_{\mathrm{f}}}, \quad \kappa_{\mathrm{a}} = \frac{\omega}{c_{\mathrm{a}}}$$

Step 2 - Equivalent Conditions

Order 1

$(\mathbf{u}_0, p_0)$ satisfies

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_s \\ \Delta p_0^a + \kappa_a^2 p_0^a = 0 & \text{in} \quad \Omega_a \\ \mathbf{T}(\mathbf{u}_0) = -p_0^a \mathbf{n} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} p_0^a = \left(\frac{c_f}{c_a}\right)^2 \rho_f \, \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_a \end{cases}$$



Step 2 - Equivalent Conditions

Order 2

$(\mathbf{u}_1, \mathrm{p}_1^{\mathrm{a}}) := (\mathbf{u}_1^{\varepsilon}, \mathrm{p}_1^{\varepsilon})$ satisfies

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = \mathbf{f} & \text{in} \quad \Omega_{\mathrm{s}} \\ \Delta \mathrm{p}_1^{\mathrm{a}} + \kappa_{\mathrm{a}}^2 \mathrm{p}_1^{\mathrm{a}} = 0 & \text{in} \quad \Omega_{\mathrm{a}} \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} = -\mathrm{p}_1^{\mathrm{a}}\,\mathbf{n} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} \mathrm{p}_1^{\mathrm{a}} + \varepsilon \left(\left(\frac{c_{\mathrm{f}}}{c_{\mathrm{a}}}\right)^2 - 1 \right) \Delta_{\Gamma} \mathrm{p}_1^{\mathrm{a}} = \left(\frac{c_{\mathrm{f}}}{c_{\mathrm{a}}}\right)^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} & \text{on} \quad \Gamma \\ \text{ABC} & \text{on} \quad \Gamma_{\mathrm{a}} \end{cases}$$

$$\theta_{1,\varepsilon} = \varepsilon \rho_{\mathrm{f}} \omega^2 \left[1 - \left(\frac{c_{\mathrm{f}}}{c_{\mathrm{a}}}\right)^2 \right]$$

$$\theta_{2,\varepsilon} = \rho_{\mathrm{f}} \omega^2 - 2\varepsilon (1 - \rho_{\mathrm{f}} \omega^2) \mathcal{H}$$

Prospect

1. To derive Equivalent Conditions adapted to high order ABCs
2. To implement Equivalent Conditions in configurations A2 and B
3. To implement Equivalent Conditions in 3D and in Time domain
4. To test these asymptotic models with real data

Thank you for your attention

Proof of Uniform Estimates

Variational Problem $(\mathbf{VP}_\varepsilon)$

- $V_\varepsilon = \{(\mathbf{u}, \mathrm{p}) \in \mathrm{H}^1(\Omega_\mathrm{s}) \times \mathrm{H}^1(\Omega_\mathrm{f}^\varepsilon) \,|\, \gamma_0 \mathrm{p} = 0 \quad \text{on } \Gamma^\varepsilon \}$
- For all $\varepsilon > 0$, $(\mathbf{VP}_\varepsilon)$ writes : Find $(\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon) \in V_\varepsilon$ such that

$$\forall (\mathbf{v}, \mathrm{q}) \in V_\varepsilon \,, \qquad a_\varepsilon\left((\mathbf{u}_\varepsilon, \mathrm{p}_\varepsilon), (\mathbf{v}, \mathrm{q})\right) = \langle F, (\mathbf{v}, \mathrm{q})\rangle_{V'_\varepsilon, V_\varepsilon}$$

$$a_\varepsilon\left((\mathbf{u}, \mathrm{p}), (\mathbf{v}, \mathrm{q})\right) := \int_{\Omega_\mathrm{f}^\varepsilon} \left(\nabla \mathrm{p} \cdot \nabla \bar{\mathrm{q}} - \kappa^2 \mathrm{p}\bar{\mathrm{q}}\right) \, \mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_\mathrm{s}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}\right) \mathrm{d}\mathbf{x} + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n}\, \bar{\mathrm{q}} + \mathrm{p}\bar{\mathbf{v}} \cdot \mathbf{n}\right) \, \mathrm{d}s$$

Formulation in a fixed domain

Scaled Problem $(\mathfrak{P}_\varepsilon)$

- $\Omega_f := \Gamma \times (0,1)$
- $V = \{(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(\Omega_f) \,|\, \mathfrak{p}(.\,,1) = 0 \quad \text{on } \Gamma\}$
- $(\mathfrak{P}_\varepsilon)$ writes : Find $(\mathbf{u}, \mathfrak{p}) \in V$ such that $\forall (\mathbf{v}, \mathfrak{q}) \in V$

$$\varepsilon \int_0^1 \int_\Gamma \left[(\mathbb{I} + \varepsilon \mathcal{S}\mathcal{R})^{-2} \nabla_\Gamma \mathfrak{p} \nabla_\Gamma \bar{\mathfrak{q}} + \varepsilon^{-2} \partial_S \mathfrak{p} \partial_S \bar{\mathfrak{q}} - \kappa^2 \mathfrak{p} \bar{\mathfrak{q}} \right] \det(\mathbb{I} + \varepsilon \mathcal{S}\mathcal{R}) \, \mathrm{d}\Omega_f$$
$$+ a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma \left(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \, \bar{\mathfrak{q}} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n} \right) \, \mathrm{d}\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V', V}$$

Scaled Problem $(\mathfrak{P}_\varepsilon)$

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$*, the problem* $(\mathfrak{P}_\varepsilon)$ *with data* $\mathfrak{F}_\varepsilon \in V'$ *has a unique solution* $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ *which satisfies*

$$\sqrt{\varepsilon}\|\nabla_\Gamma \mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \sqrt{\varepsilon}^{-1}\|\partial_s \mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \|\mathfrak{p}_\varepsilon\|_{0,\Omega_f} + \|\mathbf{u}_\varepsilon\|_{1,\Omega_s} \leqslant C\|\mathfrak{F}_\varepsilon\|_{V'} \,.$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$*, any solution* $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ *of problem* $(\mathfrak{P}_\varepsilon)$ *with a data* $\mathfrak{F}_\varepsilon \in V'$ *satisfies*

$$\|(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon)\|_{0,\Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_\varepsilon\|_{0,\Gamma} \leqslant C\|\mathfrak{F}_\varepsilon\|_{V'} \,.$$

Sketch of the proof of the Lemma

Assume : $\exists(\mathbf{u}_m, \mathfrak{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \to 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_{0,\Omega_s\times\Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_m\|_{0,\Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \to 0$$

We prove successively that :

1. $$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_W \leqslant C$$
   with $W = \{(\mathbf{u}, \mathfrak{p}) \in \mathrm{H}^1(\Omega_s) \times \mathrm{H}^1(0,1;\mathrm{L}^2(\Gamma)) \,|\, \mathfrak{p}(.\,,1) = 0 \quad \text{on}\, \Gamma\}$
2. $$\mathbf{u}_m \to \mathbf{u} \quad \text{in} \quad \mathrm{L}^2(\Omega_s) \quad \text{and} \quad \mathfrak{p}_m \to \mathfrak{p} = 0 \quad \text{in} \quad \mathrm{L}^2(\Omega_f)$$
3. $$\|\mathbf{u}\|_{0,\Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1$$
4. $\mathbf{u} = 0$ (using (SA)) : Contradiction.

Step 3 : Proof of Convergence

(i) Derive an expansion of $\mathbf{u}_\varepsilon^k$ and show that :

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k \,,$$
$$\mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k \,.$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on} \quad \Gamma \,,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s} \leqslant C\varepsilon^{k+1} \,.$$