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# Thin Layer Models for Geophysical Applications

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## Thin Layer Models for Geophysical Applications

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Second French-Russian Conference

Novosibirsk, September 25, 2014

# Motivations

## HPC-GA Project

- **High Performance Computing for Geophysical Applications**

FP7 Project : INRIA, BCAM, UFRGS, UNAM

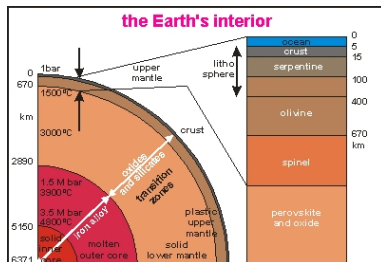
- Earthquakes
- Tsunami

- **Numerical Models (WP2)**

- Asymptotic method : To derive approximate models for the propagation of seismic waves and acoustic waves
- Elasto-acoustic coupling : To reproduce earthquakes

# Motivations

## Configurations of interest



- **Configuration A** : The medium of interest is a Land Area surrounded by a fluid zone
- **Configuration B** : The medium of interest is a Land Area surrounded by a fluid zone and a part of the atmosphere

# Motivations

## Configuration A

- **Configuration A:**

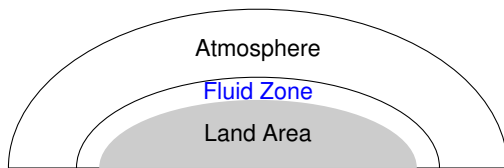


- The thickness of the **fluid zone** is negligible w.r.t. the wavelength
- **Sub-Configurations:**
  - 1 **Configuration A1** : A **thin layer** with a **uniform** thickness
  - 2 **Configuration A2** : A **thin layer** with a **variable** thickness

# Motivations

## Configuration B

- **Configuration B**



- **Difficulty**

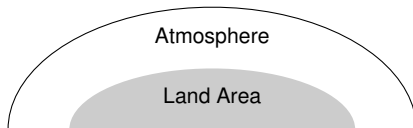
Apply a FEM on a mesh with **thin cells** in the **Fluid Zone** and much larger outside

## Approach : an Asymptotic Method

- 1 "Replace" the **thin layer** by an Equivalent Boundary Condition
- 2 Couple this condition with
  - the elastic equation (**Configuration A**)
  - elastic and acoustic equations (**Configuration B**)



**Figure: Configuration A**  
without the **thin layer**



**Figure: Configuration B**  
without the **thin layer**

- 3 Apply a (DG) Finite Element Method

# Bibliography

- Derivation of Equivalent Conditions for **thin layer** Problems :



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ABBOUD-AMMARI, 96



BENDALI-LEMRAËT, 96



LAFITTE, 99

...



PÉRON, 2014



BUREL, PHD THESIS, 2014



DE CASTRO, DIAZ, PÉRON, 2014



# Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)

# Outline

- 1 Energy Estimates (Conf. A1)**
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## A Model Problem

The Problem  $(\mathbf{P}_\varepsilon)$  set in a smooth domain  $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$ :

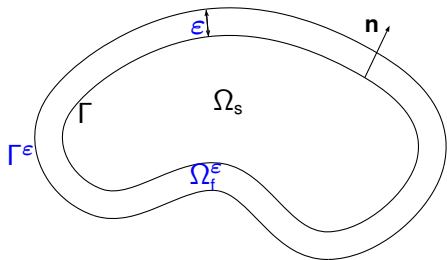
$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in } \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} \quad \text{on } \Gamma$$

$$p = 0 \quad \text{on } \Gamma^\varepsilon$$



**Issue** : Uniform Estimates for solutions  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  of  $(\mathbf{P}_\varepsilon)$  as  $\varepsilon \rightarrow 0$ ?

# Framework

$$\text{Hooke's law : } \underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$$

## Assumption

- (i)  $\underline{\underline{C}}(x) = (C_{ijkl}(x))$  is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii)  $C_{ijkl}(x)$  are real valued smooth functions, up to  $\Gamma$
- (iii) The tensor  $\underline{\underline{C}}(x)$  is positive

# Framework

Hooke's law :  $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

## Assumption

- (i)  $\underline{\underline{C}}(x) = (C_{ijkl}(x))$  is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii)  $C_{ijkl}(x)$  are real valued smooth functions, up to  $\Gamma$
- (iii) The tensor  $\underline{\underline{C}}(x)$  is positive

## Assumption (SA)

The angular frequency  $\omega$  is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

# Uniform Estimates

## Theorem

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathbf{P}_\varepsilon)$  has a unique solution  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H_{0,\Gamma_\varepsilon}^1(\Omega_f^\varepsilon)$ , and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

**Application** : Convergence of an asymptotic expansion for  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  as  $\varepsilon \rightarrow 0$

# Uniform Estimates

## Theorem

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathbf{P}_\varepsilon)$  has a unique solution  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H_{0,\Gamma_\varepsilon}^1(\Omega_f^\varepsilon)$ , and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C \|\mathbf{f}\|_{0,\Omega_s}$$

**Application** : Convergence of an asymptotic expansion for  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  as  $\varepsilon \rightarrow 0$   
**Key for the proof:**

To introduce a "Scaled Problem" by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where } s \in (0, \varepsilon)$$

# Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)**
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)



# Methodology

- **Step 1** : Derive an Asymptotic Expansion for  $(\mathbf{u}_\epsilon, p_\epsilon)$  when  $\epsilon \rightarrow 0$

$$\begin{aligned}\mathbf{u}_\epsilon(x) &= \mathbf{u}_0(x) + \epsilon \mathbf{u}_1(x) + \epsilon^2 \mathbf{u}_2(x) + \dots \\ p_\epsilon(x) &= p_0(y_\alpha, \frac{s}{\epsilon}) + \epsilon p_1(y_\alpha, \frac{s}{\epsilon}) + \epsilon^2 p_2(y_\alpha, \frac{s}{\epsilon}) + \dots\end{aligned}$$

- **Step 2** : Equivalent Conditions of order  $k + 1 \in \mathbb{N}$ .  
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\epsilon} := \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \dots + \epsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\epsilon^{k+1})$$

- **Step 3** : Prove "Stability" & Convergence results for Equivalent models

## Step 1 : Multiscale Expansion - First terms

1

$$p_0 = 0$$

2

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on } \Gamma \end{cases}$$

3

$$p_1(\cdot, S) = (S - 1) \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}, \quad S \in (0, 1)$$

4

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma \end{cases}$$

5

$$p_2(\cdot, S) = -(S^2 - 1) \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} + (S - 1) \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

6

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} - \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma \end{cases}$$

## Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^N := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^N := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon.$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} \Delta r_\epsilon^N + \kappa^2 r_\epsilon^N & = f_\epsilon \quad \text{in } \Omega_f^\epsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\epsilon^N) + \omega^2 \rho \mathbf{r}_\epsilon^N & = 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} r_\epsilon^N - \rho_f \omega^2 \mathbf{r}_\epsilon^N \cdot \mathbf{n} & = g_\epsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\epsilon^N) + r_\epsilon^N \mathbf{n} & = 0 \quad \text{on } \Gamma \\ r_\epsilon^N & = 0 \quad \text{on } \Gamma^\epsilon. \end{array} \right.$$

The RHS are explicit :  $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$  in  $\Omega_f^\epsilon$ ,  $g_\epsilon = \mathcal{O}(\epsilon^N)$  on  $\Gamma$ .

**Optimal Estimates :**  $\|\mathbf{r}_\epsilon^N\|_{1,\Omega_s} + \sqrt{\epsilon} \|r_\epsilon^N\|_{1,\Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

## Step 2 : Equivalent Conditions on $\Gamma$

For all  $k \in \{0, 1, 2, 3\}$ , we identify a simpler problem

$$(\mathbf{P}_{\epsilon}^k) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\epsilon}^k) + \omega^2 \rho \mathbf{u}_{\epsilon}^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_{\epsilon}^k) + \mathbf{B}_{k,\epsilon}(\mathbf{u}_{\epsilon}^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

with  $\mathbf{B}_{k,\epsilon}$  a surfacic differential operator.

## Step 2 : Equivalent Conditions

1 Order 1 :

$$\mathbf{T}(\mathbf{u}_0) = 0$$

2 Order 2 :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - \epsilon \omega^2 \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

3 Order 3 :

$$\mathbf{T}(\mathbf{u}_\epsilon^2) - \epsilon \omega^2 \rho_f (1 - \epsilon \mathcal{H}) \mathbf{u}_\epsilon^2 \cdot \mathbf{n} \mathbf{n} = 0$$

4 Order 4 :

$$\mathbf{T}(\mathbf{u}_\epsilon^3) - \epsilon \omega^2 \rho_f \left( 1 - \epsilon \mathcal{H} + \frac{\epsilon^2}{3} [\Delta_\Gamma + \kappa^2 \mathbb{I} + 4\mathcal{H}^2 - \mathcal{K}] \right) \mathbf{u}_\epsilon^3 \cdot \mathbf{n} \mathbf{n} = 0$$

## Step 3 : Stability and Convergence results

$$V_k = H^1(\Omega_s) \text{ when } k = 0, 1, 2$$

$$V_k = \{\mathbf{u} \in H^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_{\Gamma} \in H^1(\Gamma)\} \text{ when } k = 3$$

### Proposition

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathbf{P}_\varepsilon^k)$  with data  $\mathbf{f} \in L^2(\Omega_s)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in V_k$  and

- Stability

$$\|\mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C \|\mathbf{f}\|_{0, \Omega_s}$$

- Convergence

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C\varepsilon^{k+1}$$

# The case of a Fourier-Robin B.C.

## Step 2 : Equivalent Conditions

When considering a Fourier-Robin boundary condition :

$$\partial_{\mathbf{n}} p_{\varepsilon} - i\kappa p_{\varepsilon} = 0 \quad \text{on } \Gamma^{\varepsilon}$$

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^1) - i\omega c_f \rho_f \left(1 + \varepsilon \left(-2\mathcal{H} + i\kappa^{-1} \Delta_{\Gamma}\right)\right) (\mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n}) \mathbf{n} = 0$$

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## Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain :  $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on  $\Gamma$  ( $\mathbf{f} = 0$ ) :  $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$  with  $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$ -finite elements (Lagrange) in the Library [Hou10ni](#)
- $\Omega_s$  : Isotropic elastic material with Lamé coefficients :

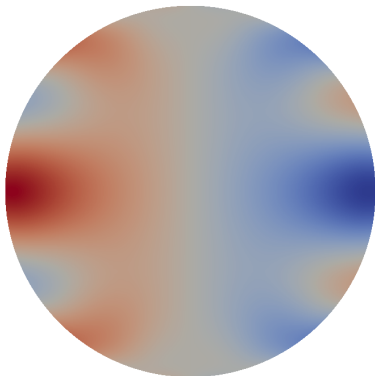
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 .$$

- $c = 1500 \text{ m.s}^{-1}$ ,  $\rho_f = 1000 \text{ kg.m}^{-3}$ ,  $\rho = 2700 \text{ kg.m}^{-3}$ .

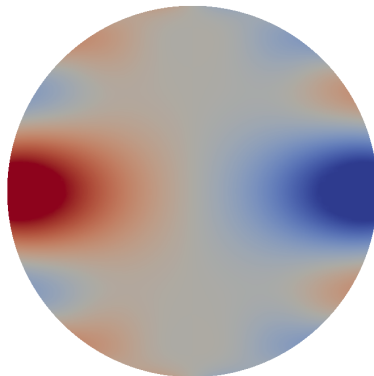
# Numerical Simulations

## Qualitative Comparisons

$$\varepsilon = 0.001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$   
 $\mathbf{u}_\varepsilon$ : analytical solution

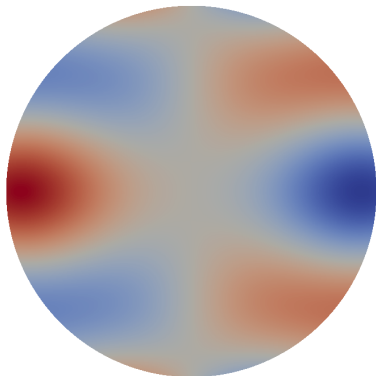


$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$   
 $\mathbf{u}_\varepsilon^2$  (model of order 3) : IPDGM

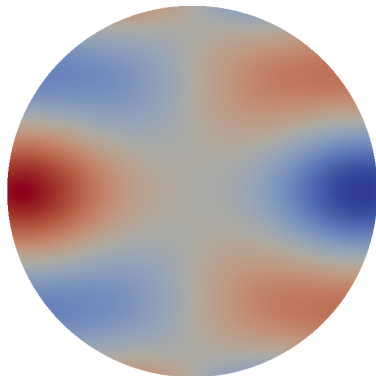
# Numerical Simulations

## Qualitative Comparisons

$$\varepsilon = 0.0001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$   
 $\mathbf{u}_\varepsilon$ : analytical solution



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$   
 $\mathbf{u}_\varepsilon^2$  (model of order 3) : IPDGM

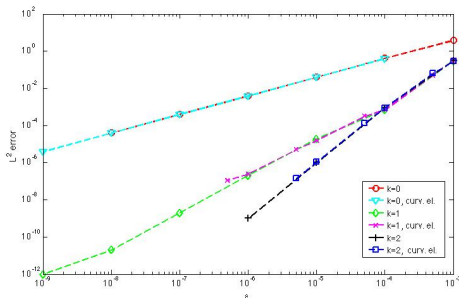
# Convergence of the models

## Dirichlet case

For  $k \in \{0, 1, 2\}$ , we plot the  $L^2$ -error  $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$  w.r.t.  $\epsilon$

$\mathbf{u}_\epsilon$ : analytical solution

$\mathbf{u}_\epsilon^k$ : analytical solution or numerical solution (IPDGM)



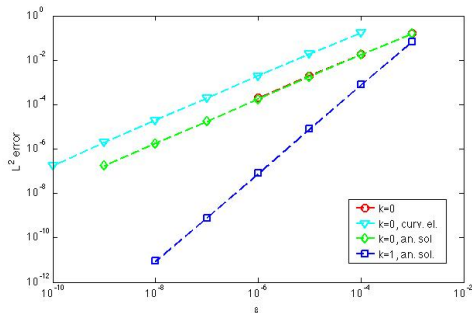
# Convergence of the models

## Fourier-Robin b.c.

For  $k \in \{0, 1\}$ , we plot the  $L^2$ -error  $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0,\Omega_s}$  w.r.t.  $\varepsilon$

$\mathbf{u}_\varepsilon$ : analytical solution

$\mathbf{u}_\varepsilon^k$ : analytical solution or numerical solution (IPDGM)



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- 1 Energy Estimates (Conf. A1)
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## Configuration A2

### Framework in 2D

- Parameterization of the layer  $\Omega_\Gamma^\epsilon$

$$\Omega_\Gamma^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

$t$  : arc-length on  $\Gamma$

$f$  : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$  : normal vector on  $\Gamma$

## Configuration A2

### Framework in 2D

- Parameterization of the layer  $\Omega_f^\epsilon$

$$\Omega_f^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

$t$  : arc-length on  $\Gamma$

$f$  : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$  : normal vector on  $\Gamma$

- Euclidean metric of  $\Omega_f^\epsilon$  defined in the coordinates  $(t, s)$ :

$$\begin{pmatrix} (1 + s f(t) c(t))^2 + (s f'(t))^2 & s f(t) f'(t) \\ s f(t) f'(t) & f(t)^2 \end{pmatrix}$$



## Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$ :

$$\mathbf{T}(\mathbf{u}_0) = 0$$

- $k = 1$ :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon f(t) \omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

- $k = 2$ :

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon f(t) \omega^2 \rho_f \left( 1 - \frac{\varepsilon}{2} f(t) c(t) \right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n} \mathbf{n} = 0$$

## Step 2 : Equivalent Conditions

### Fourier-Robin b.c.

- $k = 0$  :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

- $k = 1$  :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c_f \rho_f \left( 1 + \epsilon \left( -f(t)c(t) + i\kappa^{-1} [g(t)\partial_t + f(t)\partial_t^2] \right) \right) \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

with

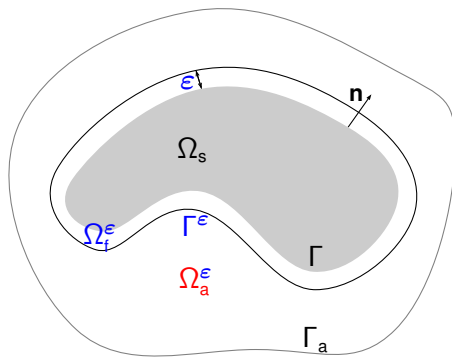
$$g(t) = (2 - f^2(t)) f'(t)$$

# Configuration B

## A Model Problem

$(\mathbf{u}, p, p^a)$  satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in } \Omega_s \\ \Delta p + \kappa^2 p = 0 & \text{in } \Omega_f^\epsilon \\ \Delta p^a + \kappa_a^2 p^a = 0 & \text{in } \Omega_a^\epsilon \\ \partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} & \text{on } \Gamma \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} & \text{on } \Gamma \\ p^a = p & \text{on } \Gamma^\epsilon \\ c_a^2 \partial_n p^a = c_f^2 \partial_n p & \text{on } \Gamma^\epsilon \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$



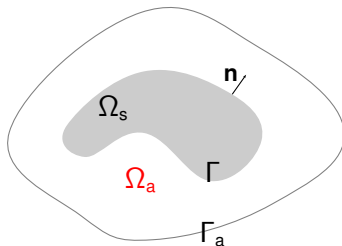
$$\kappa = \frac{\omega}{c_f}, \quad \kappa_a = \frac{\omega}{c_a}$$

## Step 2 - Equivalent Conditions

Order 1

$(\mathbf{u}_0, p_0)$  satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \Delta p_0^a + \kappa_a^2 p_0^a = 0 & \text{in } \Omega_a \\ \mathbf{T}(\mathbf{u}_0) = -p_0^a \mathbf{n} & \text{on } \Gamma \\ \partial_n p_0^a = \left(\frac{\alpha}{c_a}\right)^2 \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on } \Gamma \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$



## Step 2 - Equivalent Conditions

Order 2

$(\mathbf{u}_1, \mathbf{p}_1^a) := (\mathbf{u}_1^\varepsilon, \mathbf{p}_1^\varepsilon)$  satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = \mathbf{f} & \text{in } \Omega_s \\ \Delta \mathbf{p}_1^a + \kappa_a^2 \mathbf{p}_1^a = 0 & \text{in } \Omega_a \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = -\mathbf{p}_1^a \mathbf{n} & \text{on } \Gamma \\ \partial_n \mathbf{p}_1^a + \varepsilon \left( \left( \frac{c_f}{c_a} \right)^2 - 1 \right) \Delta_\Gamma \mathbf{p}_1^a = \left( \frac{c_f}{c_a} \right)^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} & \text{on } \Gamma \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$

$$\theta_{1,\varepsilon} = \varepsilon \rho_f \omega^2 \left[ 1 - \left( \frac{c_f}{c_a} \right)^2 \right]$$

$$\theta_{2,\varepsilon} = \rho_f \omega^2 - 2\varepsilon (1 - \rho_f \omega^2) \mathcal{H}$$

# Prospect

- 1 To derive Equivalent Conditions adapted to high order ABCs
- 2 To implement Equivalent Conditions in configurations A2 and B
- 3 To implement Equivalent Conditions in 3D and in Time domain
- 4 To test these asymptotic models with real data

Thank you for your attention

# Proof of Uniform Estimates

## Variational Problem ( $\mathbf{VP}_\varepsilon$ )

- $V_\varepsilon = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\varepsilon\}$
- For all  $\varepsilon > 0$ , ( $\mathbf{VP}_\varepsilon$ ) writes : Find  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$  such that

$$\forall (\mathbf{v}, q) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, p_\varepsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V'_\varepsilon, V_\varepsilon}$$

$$\begin{aligned}
 a_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) &:= \int_{\Omega_f^\varepsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, dx \\
 &+ \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}) \, dx + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, ds
 \end{aligned}$$



## Formulation in a fixed domain

### Scaled Problem ( $\mathfrak{P}_\varepsilon$ )

- $\Omega_f := \Gamma \times (0, 1)$
- $V = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f) \mid p(\cdot, 1) = 0 \text{ on } \Gamma\}$
- ( $\mathfrak{P}_\varepsilon$ ) writes : Find  $(\mathbf{u}, p) \in V$  such that  $\forall (\mathbf{v}, q) \in V$

$$\varepsilon \int_0^1 \int_\Gamma [(\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_\Gamma p \nabla_\Gamma \bar{q} + \varepsilon^{-2} \partial_S p \partial_S \bar{q} - \kappa^2 p \bar{q}] \det(\mathbb{I} + \varepsilon S\mathcal{R}) d\Omega_f$$

$$+ a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, q) \rangle_{V', V}$$

# Scaled Problem ( $\mathfrak{P}_\varepsilon$ )

## Uniform estimates

### Theorem

Under Assumption (SA),  $\exists \varepsilon_0 > 0$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , the problem ( $\mathfrak{P}_\varepsilon$ ) with data  $\mathfrak{F}_\varepsilon \in V'$  has a unique solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  which satisfies

$$\sqrt{\varepsilon} \|\nabla_{\Gamma} \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

### Lemma

Under Assumption (SA),  $\exists \varepsilon_0 > 0$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , any solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  of problem ( $\mathfrak{P}_\varepsilon$ ) with a data  $\mathfrak{F}_\varepsilon \in V'$  satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

## Sketch of the proof of the Lemma

Assume :  $\exists (\mathbf{u}_m, \mathbf{p}_m) \in V^{\mathbb{N}}$  satisfying  $(\mathfrak{P}_{\varepsilon_m})$  with  $\varepsilon_m \rightarrow 0$  and  $\mathfrak{F}_m \in V'$  s.t.

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

We prove successively that :

1

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C$$

with  $W = \{(\mathbf{u}, \mathbf{p}) \in H^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma)) \mid \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}$

2

$$\mathbf{u}_m \rightarrow \mathbf{u} \text{ in } L^2(\Omega_s) \quad \text{and} \quad \mathbf{p}_m \rightarrow \mathbf{p} = 0 \text{ in } L^2(\Omega_f)$$

3

$$\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} = 1$$

4  $\mathbf{u} = 0$  (using (SA)) : Contradiction.

## Step 3 : Proof of Convergence

(i) Derive an expansion of  $\mathbf{u}_\varepsilon^k$  and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k, \varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s} \leq C \varepsilon^{k+1}.$$