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# ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF PASSIVE LINEAR DESCRIPTOR COMPLEMENTARITY SYSTEMS

Bernard Brogliato\*      Alexandre Rocca†

**Abstract** This article is largely concerned with the time-discretization of descriptor-variable systems coupled with complementarity constraints. They are named **linear descriptor complementarity systems (LDCS)**. More specifically passive **LDCS** with minimal state space representation are studied. The Euler implicit discretization of **LDCS** is analysed: the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-posed under some conditions. Then the convergence of the discretized solutions is studied. Several examples illustrate the applicability and the limitations of the developments.

## 1 INTRODUCTION

The analysis of non-smooth dynamical systems with set-valued right-hand sides satisfying maximal monotone properties has been the object of many contributions, see [26] for a survey. Linear and nonlinear complementarity dynamical systems, various kinds of projected dynamical systems, of differential variational inequalities, differential inclusions with maximal monotone right-hand sides, Moreau's sweeping processes, and some switching dynamics can be recast in this class of set-valued dynamical systems. In this article we focus on singular, or **linear descriptor complementarity systems (LDCS)**. They may be viewed either as an extension of "classical" descriptor-variable systems which are widely used in Automatic Control and Circuit Theory, or of "classical" linear complementarity systems. They can also be seen as differential-algebraic equations (DAEs) coupled with complementarity constraints. Motivations for studying **LDCS** are in circuits with set-valued components [2, Chapters 3 and 5] [32], and in a more abstract framework in state-dependent switching DAEs [71, 70], after a kind of convexification (or filling-in the graph) procedure has been made. They are also closely related to DAEs whose equality constraints are nonsmooth, as they occur in chemistry or thermo-dynamical systems [78, 79, 77].

It has to be noted that the study of **LDCS** has not received a lot of attention yet, and is still at an embryonic stage of analysis. In particular the analysis of the implicit Euler method, which has received a lot of attention in several fields [10, 14, 57, 55, 2, 3, 67, 69], is an open issue for **LDCS**. Our goal is to analyze the well-posedness of the one-step nonsmooth problem (OSNSP) obtained after an implicit (backward) Euler discretization, and the convergence of the discrete-time solutions towards a continuous-time limit which is a solution to the continuous-time system. The second step is useful

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33 not only for proving the existence of solutions to the continuous-time system, but also, perhaps most  
 34 importantly, to show that the proposed time-discretization makes sense for numerical simulations.  
 35 Passivity properties (passive systems, positive real transfer functions, Lur'e equations) are pivotal  
 36 throughout the article. In section 2, a class of passive **LDCS** (which extends descriptor variable systems  
 37 with a positive real transfer function) is introduced. A specific Weierstrass form associated with  
 38 these systems, when the state-space realization is minimal, is used. The well-posedness of the OSNSP  
 39 associated with the implicit Euler discretization, is analysed in section 3. Convergence of the discrete-  
 40 time solutions is studied in sections 4 and 5. Examples that illustrate the theoretical developments, are  
 41 given in section 6. Conclusions are drawn in section 7. The appendix is dedicated to recall various  
 42 mathematical tools.

43 **Notation and definitions:** for any vector  $x \in \mathbb{R}^n$  and any matrix  $M \in \mathbb{R}^{m \times n}$ ,  $\|x\|$  is the Euclidean  
 44 norm and  $\|M\|$  is the Frobenius norm, which are compatible norms [16, Proposition 9.3.5], i.e.,  $\|Mx\| \leq$   
 45  $\|M\| \|x\|$ . Let  $M \in \mathbb{R}^{n \times m}$ , then  $\text{Im}(M)$  is its range,  $\text{Ker}(M)$  is its null space. We use  $\langle x, y \rangle = x^\top y$ , so  
 46  $\langle x, x \rangle = \|x\|^2$ . Positive definite matrix:  $M > 0$  if  $x^\top Mx > 0$  for all  $x \neq 0$ , positive semidefinite matrix:  
 47  $M \geq 0$  if  $x^\top Mx \geq 0$  for all  $x$  (such  $M$  is not necessarily symmetric). The maximum singular value  
 48 is denoted as  $\sigma_{\max}(M)$ , and the minimum and maximum eigenvalues as  $\lambda_{\min}(M)$  and  $\lambda_{\max}(M)$ .  $M_{\bullet i}$   
 49 denotes the  $i$ -th column of  $M$ . More mathematical results and definitions are given in Appendices A, B,  
 50 C, D, E and G.

## 51 2 THE CLASS OF PASSIVE LDCS

52 Let us consider the following singular dynamical system, that we may name a **linear descriptor com-**  
 53 **plementarity system (LDCS):**

$$(2.1) \quad \begin{cases} P\dot{x}(t) = Ax(t) + B\lambda(t) + E(t), & \text{a.e. } t \geq 0 \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + F(t) \geq 0, & \text{for all } t \geq 0 \end{cases}$$

54 with  $x(t) \in \mathbb{R}^n$ ,  $\lambda(t) \in \mathbb{R}^m$ ,  $A, B, C, D$  constant matrices of appropriate dimensions,  $P \in \mathbb{R}^{n \times n}$  has  
 55 rank  $p < n$ . Dissipativity is a fundamental property in Systems and Control [25, 87, 88]. The material  
 56 in this article relies on a result in [28, 32, 51] on a special Weierstrass form for passive descriptor  
 57 variable systems [25, Section 3.1.7] which possess a *minimal state-space realization* [28, Theorem 3.1].  
 58 The passivity of the quintuple  $(P, A, B, C, D)$  means the passivity of the operator  $\lambda \mapsto w$ , see section D.  
 59 Assuming that  $E(t) = 0$  and  $F(t) = 0$ , passivity and minimality, this special form writes as (see [51]  
 60 [32, Equation (16)] [28, Proposition A.3]):

$$(2.2) \quad \begin{cases} (a) \quad \dot{x}_1(t) = A_1x_1(t) + B_1\lambda(t) \\ (b) \quad \dot{x}_2(t) = z(t) + B_2\lambda(t) \\ (c) \quad 0 = x_2(t) + B_3\lambda(t) \\ (d) \quad 0 \leq \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \geq 0, \end{cases}$$

61 where the quadruple  $(A_1, B_1, C_1, \hat{D})$  is itself passive,  $\hat{D} \triangleq D - C_3B_2 - C_2B_3$ ,  $x_1(t) \in \mathbb{R}^{n_1}$ ,  $x_2(t) \in \mathbb{R}^{n_2}$ ,  
 62  $z(t) \in \mathbb{R}^{n_2}$ ,  $n = n_1 + 2n_2$ ,  $\lambda(t) \in \mathbb{R}^m$ , and there exists  $X_3 = X_3^\top \geq 0$  such that  $B_3^\top X_3 = -C_3$ . **The matrix**  
 63  **$X_3$  is unique when the state-space realization is minimal** [28, Theorem 5.4]. The LMI in (D.2) holds.  
 64 Due to the passivity and the complementarity conditions which imply  $\lambda(t)^\top w(t) = 0$  for all times, it  
 65 follows that for all  $t \geq 0$  one obtains  $V(x_1(t), x_2(t)) \triangleq \frac{1}{2}(x_1^\top X_1x_1 + x_2^\top X_3x_2)(t) \leq V(x_1(0), x_2(0))$ , [32]  
 66 and [28, Proof of Theorem 5.1], with  $X_1 > 0$  if the system is strongly SPR and minimal [28, Theorem

67 5.4], see section D. In the latter case  $x_1(\cdot)$  is bounded,  $A_1$  is Hurwitz<sup>1</sup>,  $\hat{D} > 0$  and  $B_3$  is full row rank  
68 ( $\Rightarrow n_2 \leq m$ ) [28, Proposition A.4].

69 **Remark 2.1.** Notice that the **differentiation index (shortly the index in the following)** of the DAE part of  
70 the system (2.2) is equal to 2, since one needs to differentiate the third line twice (assuming  $\lambda(t) = 0$ )  
71 to recover the variable  $\dot{z}(t)$ . It is clear from (2.2) that the **LDCS (2.1)** may be seen as a DAE coupled  
72 with nonsmooth constraints that take the form of complementarity conditions. It is noteworthy that  
73 singular zero order sweeping processes (named in ZOSwP [26]) have been studied [85, 6, 9], but they  
74 form another class of singular systems than (2.1) or (2.2).

75 **Remark 2.2.** The system in (2.1) can be rewritten equivalently as:

$$(2.3) \quad P\dot{x}(t) - Ax(t) - E(t) \in B(\mathcal{N}_{K^*} + D)^{-1}(-Cx(t) - F(t))$$

76 with  $K = K^* = \mathbb{R}_+^m$ , which is, as pointed out in [26, section 3.16], a class of singular systems that has  
77 not yet received much attention. The same applies to (2.2) which is rewritten as:

$$(2.4) \quad \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{z}(t) \end{pmatrix} - \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{pmatrix} \in \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} (\mathcal{N}_{K^*} + D)^{-1}(-w(t)).$$

78 Exogenous terms can also be considered in (2.2) and (2.4), though the analyses in sections 3 and 4  
79 are made without such terms. Also, an alternative way of analysing (2.2) is presented in section 5,  
80 incorporating exogenous signals. An example is treated in section 6.1 where exogenous terms are  
81 taken into account in the special form (2.2), see (6.3) and (6.4). **In the case  $D = 0$ , both (2.4) and (2.3)**  
82 **can be rewritten, under some assumptions, as a singular first-order sweeping process (FOSwP [26]).**  
83 **The optimal control of FOSwP as treated in [42, 41, 36], relies on the analysis of the time-discretization**  
84 **and the convergence of solutions. Such an important issue could be tackled for singular FOSwP, where**  
85 **the results in this article may serve as a starting point.**

### 86 3 THE EULER IMPLICIT SCHEME AND THE OSNSP

87 **In the sequel of the article, we let  $t \in [0, T]$ ,  $0 < T < +\infty$ ,  $h = \frac{T}{N}$ ,  $N \in \mathbb{N} \setminus \{0\}$ ,  $t_k = kh$ ,  $k \in \{0, 1, \dots, N\}$ ,**  
88 **and  $f_k = f(t_k)$ .** The studied implicit Euler time-discretization of (2.2) is as follows:

$$(3.1) \quad \begin{cases} x_{1,k+1} = x_{1,k} + hA_1x_{1,k} + hB_1\lambda_{k+1} \\ x_{2,k+1} = x_{2,k} + hz_{k+1} + hB_2\lambda_{k+1} \\ 0 = x_{2,k+1} + B_3\lambda_{k+1} \\ 0 \leq \lambda_{k+1} \perp w_{k+1} = C_1x_{1,k+1} + C_2x_{2,k+1} + C_3z_{k+1} + D\lambda_{k+1} \geq 0, \end{cases}$$

89 for all  $k \in \{0, 1, \dots, N-1\}$ , where  $A_1x_{1,k}$  could be replaced by  $A_1x_{1,k+1}$  without modifying much the next  
90 developments. Nevertheless we choose to call the method in (3.1) an implicit method despite of the fact  
91 that the linear term is explicitly calculated, because our objective is to focus solely on the set-valued  
92 part of the system (*i.e.*, the complementarity conditions). The set of equations in (3.1) makes a mixed  
93 linear complementarity problem (MLCP). After few calculations from (3.1) the following OSNSP is  
94 obtained (where the variable  $hw_{k+1}$  is used instead of  $w_{k+1}$ ) as an LCP:

$$(3.2) \quad \begin{aligned} 0 \leq \lambda_{k+1} \perp \bar{D}_h\lambda_{k+1} + h\beta_k \geq 0 & \Leftrightarrow \bar{D}_h\lambda_{k+1} + h\beta_k \in -\mathcal{N}_{\mathbb{R}_+^m}(\lambda_{k+1}) \\ & \Leftrightarrow \lambda_{k+1} \in (\bar{D}_h + \mathcal{N}_{\mathbb{R}_+^m})^{-1}(-h\beta_k), \end{aligned}$$

<sup>1</sup>That is, all its eigenvalues have a negative real part.

95 with  $\bar{D}_h \stackrel{\Delta}{=} h\hat{D} + h^2C_1B_1 + B_3^\top X_3B_3$ ,  $\beta_k \stackrel{\Delta}{=} C_1(I_{n_1} + hA_1)x_{1,k} - \frac{1}{h}C_3x_{2,k}$ . It is noteworthy that  $z_{k+1}$  is  
 96 eliminated and does not appear in  $\beta_k$ . The inclusion (3.2) is a VI as (B.2) with  $\varphi(\cdot) = \Psi_{\mathbb{R}_+^m}(\cdot)$ . Obviously  
 97 the well-posedness of the OSNSP depends on the properties of  $\bar{D}_h$ . From the passivity we know that  
 98  $\hat{D} \succcurlyeq 0$  and  $B_3^\top X_3B_3 \succcurlyeq 0$ . Using Theorem A.1 and Corollary A.2 it is possible to characterize the set of  
 99 admissible  $hC_1B_1$  that do not destroy the positive definiteness of  $\bar{D}_h$  in case  $\hat{D} > 0$ .

100 **Remark 3.1.** We do not investigate here whether or not the discrete time system (3.1) is dissipative.  
 101 Such a study is made in [55] for the case of LCS discretized with a  $\theta$ -method.

102 As alluded to in Remark 2.2, the cone  $\mathbb{R}_+^m$  can be replaced in (2.1), (2.2) and (3.1) by any closed non  
 103 empty convex cone  $K \subset \mathbb{R}^m$  and its dual cone  $K^\star$ . Then the complementarity conditions become  
 104  $K^\star \ni \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \in K$ , yielding the inclusion

$$(3.3) \quad \begin{aligned} \bar{D}_h\lambda_{k+1} + h\beta_k &\in -\mathcal{N}_{K^\star}(\lambda_{k+1}) \\ \Leftrightarrow \lambda_{k+1} &\in (\bar{D}_h + \mathcal{N}_{K^\star})^{-1}(-h\beta_k), \end{aligned}$$

105 **which is a linear cone complementarity problem (LCCP)**, instead of (3.2). One can also consider time-  
 106 varying terms in (2.2), in both the dynamics and the complementarity conditions, and adapt  $\beta_k$  accord-  
 107 ingly. Thus in the next proposition a more general complementarity framework is considered where  
 108 the positive orthant is replaced by a closed convex cone.

109 **Proposition 3.2.** *Let  $h > 0$  and  $k \geq 0$  be given. Assume that  $\bar{D}_h \succcurlyeq 0$ , not necessarily symmetric, and let*  
 110  *$K \subset \mathbb{R}^m$  be a non empty closed convex cone (where the complementarity conditions in (2.2) are extended*  
 111 *to  $K^\star \ni \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \in K$ ). Let  $\mathcal{S} = K^\star \cap \text{Ker}(\bar{D}_h + \bar{D}_h^\top) \cap \{w \in$*   
 112  *$\mathbb{R}^m \mid \bar{D}_h w \in K\}$ . Then:*

- 113 1. *If  $\mathcal{S} = \{0\}$ , for each  $h\beta_k$  the LCCP in (3.3) has at least a solution.*
- 114 2. *If  $\mathcal{S} \neq \{0\}$ , and there exists  $\xi_0 \in \mathbb{R}^m$  such that  $\langle h\beta_k - \bar{D}_h^\top \xi_0, v \rangle + \Psi_{K^\star}(v) > 0$  for all  $v \neq 0, v \in \mathcal{S}$ ,*  
 115 *then the LCCP in (3.3) has at least a solution.*
- 116 3. *If  $\lambda_{k+1}^1$  and  $\lambda_{k+1}^2$  are two solutions of the LCCP, then  $\lambda_{k+1}^1 - \lambda_{k+1}^2 \in \text{Ker}(\bar{D}_h + \bar{D}_h^\top)$ .*

117 **Proof:** Let us consider Proposition B.2. Comparing (3.3) and (B.2) one has  $\varphi(\cdot) = \Psi_{K^\star}(\cdot)$ ,  $\mathbf{M} = \bar{D}_h$ ,  
 118  $\mathbf{q} = h\beta_k$ . Using Proposition B.1 c) d), it follows that  $(\text{dom}(\varphi))_\infty = K^\star$ . The second set in  $\mathcal{S}$  is obvious.  
 119 The third set necessitates the calculation of  $(\text{dom}(\varphi_\infty))^\star$ , see (B.3). Again the result of the calculation  
 120 follows from Proposition B.1 c) and d). Then the proof follows from Proposition B.2 a), b) and c).  $\square$

121 Once the existence of  $\lambda_{k+1}$  is established for any  $k \geq 0$ , then the existence of  $x_{1,k+1}$ ,  $x_{2,k+1}$  and  $z_{k+1}$  can  
 122 be inferred from (3.1) for any  $h > 0$ .

123 **Remark 3.3.** If  $K = \mathbb{R}_+^m$  then other results like [43, Theorem 3.8.6] may be used to study the LCP (3.2),  
 124 using for instance the copositivity of  $\bar{D}_h$ .

125 **Let us state a consequence of Proposition 3.2. The definition of a strongly SPR (SSPR) system is recalled**  
 126 **in Appendix D.**

127 **Corollary 3.4.** *Let the quintuple  $(P, A, B, C, D)$  be SSPR. Then, the LCCP in (3.3) has a unique solution for*  
 128 *any  $h\beta_k$ , for  $h > 0$  small enough.*

129 **Proof:** This follows from [28, Theorem 5.4] ( $\Rightarrow \hat{D} > 0$ ), the fact that  $B_3^\top X_3B_3 \succcurlyeq 0$ , Corollary A.2  
 130 ( $\Rightarrow \bar{D}_h > 0$  for  $0 < h < h^* \stackrel{\Delta}{=} \frac{1}{\|C_1B_1\|_2 \|(\frac{\bar{D}_h + \bar{D}_h^\top}{2})^{-1}\|_2} \Rightarrow \mathcal{S} = \{0\}$ ), and items 1 and 3 in Proposition 3.2.  $\square$

131 **It is noteworthy that the SSPRness is only sufficient to guarantee that  $\hat{D} > 0$ , which is itself a sufficient**  
 132 **condition for the existence of solutions to the LCCP (3.3).**

## 133 4 BOUNDEDNESS AND CONVERGENCE ANALYSIS

134 From now on, we assume that the complementarity conditions are given as in (2.2) and (3.1), i.e.,  
 135  $K = K^* = \mathbb{R}_+^m$ , see Remark 4.2. The next result characterizes least-norm solutions of the LCP.

136 **Corollary 4.1.** *Assume that the conditions for existence of a solution in Proposition 3.2 are satisfied. Then*  
 137 *the set of solutions to the LCP in (3.2) is nonempty convex polyhedral and contains a unique least-norm*  
 138 *element  $\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})$  such that  $\|\lambda_{k+1}^{\min}\| \leq \|\lambda_{k+1}\|$  for any other solution  $\lambda_{k+1}$ . Moreover there exists a*  
 139 *constant  $\eta > 0$  depending only on  $\bar{D}_h (= h\hat{D} + h^2C_1B_1 + B_3^\top X_3B_3)$  such that  $\|\lambda_{k+1}^{\min}\| \leq \eta h \|\beta_k\|$ .*

140 **Proof:** Follows from [43, Theorem 3.1.7] and [57, Lemma 1].  $\square$

141 **The constant  $\eta$  generally depends on  $h$  since it depends on  $\bar{D}_h$ , and we may denote it as  $\eta(h)$  (as the**  
 142 **examples in section 6 show, it may also be constant in some cases).** This result certainly is important  
 143 because it will subsequently allow us to get **upper bounds** on the iterates of the implicit scheme (3.1).  
 144 Notice that if  $\bar{D}^h = 0$  then  $\beta_{k,i} > 0 \Rightarrow \lambda_{k+1,i} = 0$ , and  $\beta_{k,i} = 0 \Rightarrow \lambda_{k+1,i} \in \mathbb{R}_+$ , hence  $\lambda_{k+1}^{\min} = 0$  so  
 145 that  $\eta = 0$ . If  $h = 0$  then  $\bar{D}^0 = B_3^\top X_3B_3$ ,  $\beta_0 = -C_3x_{2,k}$ , and  $\eta = \eta_0$  such that  $\|\lambda_{k+1}^{\min}\| \leq \eta_0 \|C_3x_{2,k}\|$  is  
 146 independent of  $h$ . If the LCP( $\bar{D}_h, h\beta_k$ ) enjoys the stability of solutions property [43, Theorem 7.3.12],  
 147 then a small perturbation of  $\bar{D}_h$  implies a small variation of the solution, hence of  $\lambda_{k+1}^{\min}$ , and consequently  
 148 of the constant  $\eta$ . Assume that  $C_3 = 0$  (this is Assumption 2 below). Then solutions of the LCP( $\bar{D}_h, h\beta_k$ )  
 149 are also solutions of LCP( $\hat{D} + hC_1B_1, C_1(I_{n_1} + hA_1)x_{1,k}$ ). If  $\hat{D} + hC_1B_1$  is a P-matrix (which occurs if  $\hat{D}$  is  
 150 a P-matrix and  $h > 0$  is small enough, see Corollary A.2), then it follows from [3, Theorem B.3] [34]  
 151 that the unique solution of LCP( $\hat{D} + hC_1B_1, C_1(I_{n_1} + hA_1)x_{1,k}$ ) (which is also the least-norm solution)  
 152 satisfies  $\|\lambda_{k+1}^{\min}\| \leq \eta \|C_1(I_{n_1} + hA_1)x_{1,k}\|$  for some constant  $\eta > 0$ .

153 **Remark 4.2.** As long as a minimal norm element exists and satisfies the property stated in the corollary,  
 154 one may consider **an LCCP as in (3.3) instead of an LCP (3.2).**

### 155 4.1 CONVERGENCE ANALYSIS

156 From (3.2) and under the conditions of Corollary 4.1, the mapping  $\beta_k \mapsto \lambda_{k+1}(-h\beta_k) = \lambda_{k+1}^{\min}$  with  $\lambda_{k+1}^{\min}$   
 157 the least-norm element of the convex polyhedral set  $(\bar{D}_h + \mathcal{N}_{\mathbb{R}_+^m})^{-1}(-h\beta_k)$ , is a single-valued mapping.  
 158 The implicit scheme (3.1) may be modified as follows (which may be named the “minimal norm” implicit  
 159 scheme):

$$(4.1) \quad \begin{cases} x_{1,k+1} = (I_{n_1} + hA_1)x_{1,k} + hB_1\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ x_{2,k+1} = x_{2,k} + hz_{k+1} + hB_2\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ 0 = x_{2,k+1} + B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ 0 \leq \lambda_{k+1}^{\min} \perp w_{k+1} = C_1x_{1,k+1} + C_2x_{2,k+1} + C_3z_{k+1} + D\lambda_{k+1}^{\min} \geq 0 \\ \|\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})\| \leq \eta \|h\beta_k\|, \quad h\beta_k = hC_1(x_{1,k} + hA_1x_{1,k}) - C_3x_{2,k}, \quad \eta > 0, \end{cases}$$

160 for all  $k \in \{0, \dots, N-1\}$ . The above results explore the well-posedness of the OSNSP, for fixed  $h > 0$ .  
 161 Our goal is now to analyse the boundedness of the iterates on the integration interval  $[0, T]$ ,  $T > 0$ .  
 162 For all  $k \geq 1$  we have  $x_{2,k} = -B_3\lambda_k^{\min}$ , and  $hz_{k+1} = -B_3\lambda_{k+1}^{\min} - x_{2,k} - hB_2\lambda_{k+1}^{\min}$ , thus it is inferred that



163  $x_{2,k+1} = -B_3(\lambda_{k+1}^{\min} + \lambda_k^{\min}) - x_{2,k}$ . Consequently (4.1) is rewritten equivalently as:

$$(4.2) \quad \begin{cases} x_{1,k+1} = (I_{n_1} + hA_1)x_{1,k} + hB_1\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ x_{2,k+1} = -x_{2,k} - B_3(\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) + \lambda_k^{\min}(x_{1,k-1}, x_{2,k-1})) \\ 0 \leq \lambda_{k+1}^{\min} \perp w_{k+1} = C_1(I_{n_1} + hA_1)x_{1,k} - (C_2 + \frac{1}{h}C_3)x_{2,k} + \frac{1}{h}\bar{D}h\lambda_{k+1}^{\min} - C_2B_3\lambda_k^{\min} \geq 0 \\ 0 = x_{2,k+1} + B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}), \\ hz_{k+1} = -B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) - x_{2,k} - hB_2\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ \|\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})\| \leq \eta \|h\beta_k\|, \quad h\beta_k = hC_1(I_{n_1} + hA_1)x_{1,k} - C_3x_{2,k}, \quad \eta > 0, \end{cases}$$

164 for all  $k \in \{1, N-1\}$ , given initial data  $x_{1,0}$  and  $x_{2,0}$ . Starting from (4.2) the next result is obtained:

165 **Lemma 4.3.** Let  $\alpha_{1,k} \stackrel{\Delta}{=} \|x_{1,k}\|$ ,  $\alpha_{2,k} \stackrel{\Delta}{=} \|x_{2,k}\|$ ,  $\alpha_k = (\alpha_{1,k} \quad \alpha_{2,k})^\top$ , then:

$$(4.3) \quad \begin{cases} \alpha_{k+1} \leq (M_i M + M_{i-1} U) \alpha_{k-i} + M_i U \alpha_{k-i-1} \\ M_{i+1} = M_i M + M_{i-1} U, \quad M_0 = I_2, M_{-1} = 0, \quad 0 \leq i \leq k-1, \quad k \geq 1, \end{cases}$$

166 where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $U = \begin{pmatrix} 0 & 0 \\ e & f \end{pmatrix}$ ,  $a = \|I_{n_1} + hA_1\| (1 + h^2 \eta \|B_1\| \|C_1\|)$ ,  $b = h\eta \|B_1\| \|C_3\|$ ,  $c =$   
167  $h\eta \|B_3\| \|C_1\| \|I_{n_1} + hA_1\|$ ,  $d = (1 + \eta \|B_3\| \|C_3\|)$ ,  $e = c$ ,  $f = \eta \|B_3\| \|C_3\|$ . In particular

$$(4.4) \quad \alpha_{k+1} \leq (M_{k-1} M + M_{k-2} U) \alpha_1 + M_{k-1} U \alpha_0$$

168 for all  $k \geq 1$ .

169 **Proof:** The proof starts from (4.2). The following **upper bounds** can be derived:

$$(4.5) \quad \begin{aligned} (a) \quad \|\lambda_{k+1}^{\min}\| &\leq \eta h \|C_1\| \|I_{n_1} + hA_1\| \|x_{1,k}\| + \eta \|C_3\| \|x_{2,k}\| \\ (b) \quad \|x_{1,k+1}\| &\leq \|I_{n_1} + hA_1\| (1 + h^2 \eta \|B_1\| \|C_1\|) \|x_{1,k}\| + h\eta \|B_1\| \|C_3\| \|x_{2,k}\| \\ &= a \|x_{1,k}\| + b \|x_{2,k}\| \\ (c) \quad \|x_{2,k+1}\| &\leq (1 + \eta \|B_3\| \|C_3\|) \|x_{2,k}\| + \eta \|B_3\| \|C_3\| \|x_{2,k-1}\| \\ &\quad + h\eta \|B_3\| \|C_1\| \|I_{n_1} + hA_1\| (\|x_{1,k}\| + \|x_{1,k-1}\|) \\ &= d \|x_{2,k}\| + f \|x_{2,k-1}\| + c (\|x_{1,k}\| + \|x_{1,k-1}\|). \end{aligned}$$

170 One infers that:

$$(4.6) \quad \alpha_{k+1} \leq M \alpha_k + U \alpha_{k-1}$$

171 The **upper bound** in (4.3) therefore holds for  $i = 0$ . Let us prove by induction that it holds for all  
172  $0 \leq i \leq k-1$ . Let it hold for  $i > 0$ , we have  $\alpha_{k-i} \leq M \alpha_{k-i-1} + U \alpha_{k-i-2}$ . Thus  $\alpha_{k+1} \leq (M_i M +$   
173  $M_{i-1} U) M \alpha_{k-i-1} + (M_i M + M_{i-1} U) U \alpha_{k-i-2} + M_i U \alpha_{k-i-1} = (M_{i+1} M + M_i U) \alpha_{k-i-1} + M_{i+1} U \alpha_{k-i-2}$ . The  
174 result is proved.  $\square$

175 Our goal is now to examine (4.4). Let us come back to the matrices  $M$  and  $U$  in Lemma 4.3. It is clear  
176 from (4.4) that the boundedness of the sequence  $\{\alpha_k\}_{k \geq 2}$  depends on the boundedness of  $M_k$ ,  $k \geq 1$ .  
177 Apart from  $d$  and  $f$ , all the other terms in  $M$  and  $U$  are proportional to  $h$ ,  **$\eta h$  or to  $h_1 \stackrel{\Delta}{=} \|I_{n_1} + hA_1\|$** .  
178 Moreover  $d > 1$  whenever  $B_3 \neq 0$  and  $C_3 \neq 0$ . The **term in position (2, 2)** of  $M^n$  is equal to  $d^n + O(h_1 h^2)$ .  
179 **Before stating the next result, let us introduce the following assumptions:**

180 **Assumption 1.** The matrix  $A_1$  is such that  $h_1 \leq 1$  for  $h > 0$  small enough.

181 Let us notice that the induced norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$  [16, Proposition 9.4.9] is compatible with the  
 182 vector norm  $\|\cdot\|_2 = \|\cdot\|$  [16, pp.607-609], and can be used instead of the Frobenius norm to compute  
 183  $h_1$ . The next assumption is crucial in the body of this section.

184 **Assumption 2.** One has  $B_{3,\bullet i} \in \text{Ker}(X_3)$  for all  $1 \leq i \leq m$  ( $\implies C_3 = 0$ ).

185 **Let us remind that  $X_3$  is uniquely defined under minimality and positive-realness [28, Theorem 5.4]. If**  
 186 **Assumption 2 holds then  $d = 1$  and  $b = f = 0$ .** This assumption means that the variable  $z$  does not  
 187 enter the complementarity conditions, see (2.2). Hence in a sense the complementarity part of the  
 188 **LDCS** and the algebraic part are decoupled, **and  $\beta_k$  no longer depends on  $\frac{1}{h}C_3x_{2,k}$ .** **The terms in (4.5)**  
 189 **involve products with  $\eta(h)$ , and it is important to guarantee that they are bounded, when Assumption**  
 190 **2 holds.**

191 **Assumption 3.** **The constant  $\eta = \eta(h)$  is such that the products  $\eta(h)h$  can be made arbitrarily small for**  
 192 **small enough  $h \geq 0$ .**

193 **Notice that this assumption is verified in the examples of section 6, where  $\eta$  is a constant (but these**  
 194 **examples do not verify Assumption 2, however).** The matrices  $M_k$  then involve only products of powers  
 195 of the terms  $a, b, c, d = 1, e$ . These products are bounded for bounded  $k$ , and provided that both  $h$  and  
 196  $h_1$  are small enough so that  $a \leq 1, b \leq 1, c \leq 1, d = 1, e \leq 1$ , they remain bounded as  $k \rightarrow +\infty$ . If the  
 197 inequalities are strict (apart from  $d = 1$ ), the first rows in both  $M_{k-1}M + M_{k-2}U$  and  $M_{k-1}U$  converge  
 198 to zero, while the second rows converge to  $(0, 1)$  (the 1 being the consequence of a term  $d^n$  in matrices  
 199  $M^n$ ). The next proposition is therefore proved:

200 **Proposition 4.4.** **Assume that: a) the conditions of existence of solutions in Proposition 3.2 are verified, b)**  
 201 **Assumptions 1, 2 and 3 hold true, and c) both  $h$  and  $h_1$  are small enough, so that  $a \leq 1, b \leq 1, c \leq 1, d = 1,$   
 202  $e \leq 1$ . **Then the sequences  $\{\alpha_{1,k}\}_k$  and  $\{\alpha_{2,k}\}_k$  are uniformly bounded in  $k$ . If  $A_1 < 0$ , and  $h > 0$  is small**  
 203 **enough such that  $a < 1, c < 1, e < 1$ , then  $\alpha_{1,k} \rightarrow 0$  as  $k \rightarrow +\infty$ .****

204 Recall that  $\alpha_{1,k} = \|x_{1,k}\|$  and  $\alpha_{2,k} = \|x_{2,k}\|$ . The last assertion of the proposition follows from the  
 205 material in Appendix A. Then we have the next corollary.

206 **Corollary 4.5.** **Let the conditions a), b), c) of Proposition 4.4 hold. Then  $\|\lambda_{k+1}^{\min}\|$  is uniformly bounded**  
 207 **in  $k$  and  $h > 0$ , and it converges to zero as  $k \rightarrow +\infty$  if  $\alpha_{1,k} \rightarrow 0$  as  $k \rightarrow +\infty$ . Therefore the sequence**  
 208  **$\left\{\frac{x_{1,k+1} - x_{1,k}}{h}\right\}_k$  is uniformly bounded also in  $k$  and  $h > 0$ .**

209 **Proof:** The boundedness of  $\|\lambda_{k+1}^{\min}\|$  and its convergence follow from (4.5) (a), the boundedness of  
 210  $\|x_{1,k}\|$  and Assumption 2. From the first line in (4.2), one has  $\frac{x_{1,k+1} - x_{1,k}}{h} = A_1x_{1,k} + B_1\lambda_{k+1}^{\min}$ , hence the  
 211 result is proved.  $\square$

212 Let us now study the convergence of the piecewise-linear and step-function approximations defined  
 213 as:

$$(4.7) \quad \begin{cases} x_1^h(t) = x_{1,k+1} + \frac{t_{k+1}-t}{h}(x_{1,k} - x_{1,k+1}) \\ \dot{x}_1^h(t) = \frac{x_{1,k+1} - x_{1,k}}{h} \\ x_{1,h}^*(t) = x_{1,k+1} \end{cases} \quad \text{for all } t \in [t_k, t_{k+1})$$

214 and similarly for the other variables.

215 **Corollary 4.6.** **There exists subsequences denoted as  $\{x_1^h(\cdot)\}_{h \geq 0}$  which converge uniformly to a limit**  
 216  **$x_1(\cdot)$  in  $C^0([0, T]; \mathbb{R}^{n_1})$ , such that almost everywhere one has  $\dot{x}_1(t) = A_1x_1(t) + \lambda_1(t)$ , for some function**  
 217  **$\lambda_1(\cdot)$ . The sequence  $\{x_1^{h,*}(\cdot)\}_{h \geq 0}$  converges to  $x_1(\cdot)$  strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$ . Moreover the sequences**  
 218  **$\{x_{2,h}^*(\cdot)\}_{h \geq 0}$  and  $\{\lambda_h^{\min,*}(\cdot)\}_{h \geq 0}$  converge weakly\* in  $\mathcal{L}_\infty([0, T]; \mathbb{R}^l)$ ,  $l = n_2$  and  $l = m$ , respectively, to**  
 219 **limits  $x_2(\cdot)$  and  $\lambda(\cdot)$  which satisfy the continuous-time constraints.**

220 **Proof:** Both  $\{x_1^h(\cdot)\}_h$  and  $\{\dot{x}_1^h(\cdot)\}_h$  are uniformly bounded on  $[0, T]$ . The sequence of  $C^0([0, T]; \mathbb{R}^{n_1})$   
 221 functions  $\{x_1^h(\cdot)\}_h$  is thus equicontinuous. By the Arzela-Ascoli Theorem this sequence stays in a



222 compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and converges uniformly towards a continuous limit  $x_1(\cdot) : [0, T] \rightarrow$   
 223  $\mathbb{R}^{n_1}$  as  $h \rightarrow 0, h > 0$ . Let us notice that  $x_1(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . Indeed from the above results which  
 224 are stated for  $k \in \{0, \dots, N\}$  and  $h > 0, k \rightarrow +\infty$  only if  $T = Nh \rightarrow +\infty$ . Let us now define the step  
 225 functions  $x_1^{h\star}(t) = x_{1,k+1}$  if  $t \in [t_k, t_{k+1})$ ,  $\lambda_{h\star}^{\min}(t) = \lambda_{k+1}^{\min}$  if  $t \in [t_k, t_{k+1})$ , and

$$(4.8) \quad \dot{x}_1^h(t) = A_1 x_1^{h\star}(t) + B_1 \lambda_{h\star}^{\min}(t), \quad \text{almost everywhere.}$$

226 In addition one has:

$$(4.9) \quad \begin{aligned} \|\dot{x}_1^h(t) - \dot{x}_1^{h\star}(t)\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})}^2 &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 \|\dot{x}_{1,k}\|^2 dt \leq C^2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 dt \\ &\leq C^2 \sum_{k=0}^{N-1} \frac{1}{3} (t_{k+1} - t_k)^3 = \frac{C^2 N h^3}{3} = \frac{C^2 T h^2}{3}, \end{aligned}$$

227 where  $\dot{x}_{1,k} = \frac{x_{1,k+1} - x_{1,k}}{h}$ , and  $\|\dot{x}_{1,k}\| \leq C$  for all  $k \geq 0$  for some  $C > 0$  from Corollary 4.5. It follows that  
 228  $x_1^{h\star}(\cdot) \rightarrow x_1(\cdot)$  strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$ . On the other hand since  $\dot{x}_1^h(\cdot) \in \mathcal{L}_\infty([0, T]; \mathbb{R}^{n_1})$   
 229 it follows from the Banach-Alaoglu-Bourbaki Theorem [18, Theorem III.15] that  $\dot{x}_1^h(\cdot) \rightarrow \dot{x}_{1,\text{lim}}(\cdot)$  in the  
 230 weak $\star$  topology as  $h \rightarrow 0$ . Since the considered domain  $[0, T]$  is bounded, one has  $\mathcal{L}_1([0, T]; \mathbb{R}^{n_1}) \subseteq$   
 231  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$ , thus  $\dot{x}_1^h(\cdot)$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$ . Moreover since  $x_1^h(t) = x_1^h(0) +$   
 232  $\int_0^t \dot{x}_1^h(s) ds$ , one has  $\dot{x}_{1,\text{lim}}(\cdot) = \dot{x}_1(\cdot)$  almost everywhere. Therefore we deduce using (4.8) that  $(B_1 \lambda_{h\star}^{\min})(\cdot)$   
 233 converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$  towards some limit  $\lambda_1(\cdot)$ , and we have  $\dot{x}_1(t) = A_1 x_1(t) +$   
 234  $\lambda_1(t)$  almost everywhere on  $[0, T]$ .

235 Let us consider now the step-functions  $x_{2,h}^\star(\cdot)$  and  $\lambda_h^{\min,\star}(\cdot)$  defined as  $x_{2,h}^\star(t) = x_{2,k+1}$  for all  $t \in$   
 236  $[t_k, t_{k+1})$ , and similarly for the multiplier. It follows from Proposition 4.4 and from Corollary 4.5  
 237 that both sequences  $\{x_{2,h}^\star(\cdot)\}_{h \geq 0}$  and  $\{\lambda_h^{\min,\star}(\cdot)\}_{h \geq 0}$  are uniformly bounded. Hence from the Banach-  
 238 Alaoglu-Bourbaki Theorem, there exist subsequences of both which converge weakly $\star$  to limits  
 239  $x_2 \in \mathcal{L}_\infty([0, T]; \mathbb{R}^{n_2})$  and  $\lambda \in \mathcal{L}_\infty([0, T]; \mathbb{R}^m)$ , respectively. We infer that  $0 = x_{2,h}^\star(t) + B_3 \lambda_h^\star(t) \rightarrow$   
 240  $x_2(t) + B_3 \lambda(t)$  for almost all  $t \geq 0$  in the weak $\star$  sense. Next, using the second line in (4.1) it follows  
 241 that the sequence  $\{hz_k\}_{h \geq 0}$  is also uniformly bounded. Denoting  $\sigma_k = hz_k$ , it follows that the sequence  
 242  $\{\sigma_h^\star\}_{h \geq 0}$  converges weakly $\star$  to a limit denoted as  $\sigma(t) \in \mathcal{L}_\infty([0, T]; \mathbb{R}^{n_2})$ . Moreover one has  $\sigma(t) = 0$   
 243 almost everywhere since  $x_{2,h}^\star(t) - x_{2,h}^\star(t - t_k) - h B_2 \lambda_h^{\min,\star}(t) = \sigma_h^\star(t)$  almost everywhere in  $[0, T]$ .  
 244 Finally, the complementarity conditions in (4.1) are equivalently rewritten as  $0 \leq \lambda_h^\star(t) \perp C_1 x_{1,h}^\star(t) +$   
 245  $C_2 x_{2,h}^\star(t) + D \lambda_h^\star(t) \geq 0$  for all  $t \in [t_k, t_{k+1})$ , since  $C_3 = 0$  by Assumption 2. It is deduced that in the  
 246 limit as  $h \rightarrow 0$  one obtains  $0 \leq \lambda(t) \perp C_1 x_1(t) + C_2 x_2(t) + D \lambda(t) \geq 0$ .  $\square$

## 247 4.2 STABILITY ANALYSIS VIA A LYAPUNOV FUNCTION

248 To complete the foregoing section, it is natural to analyze how the Lyapunov function  $V(x_1, x_2)$  for the  
 249 continuous-time system in (2.2) (see also appendix D), could be used in the discrete-time context for (4.2).  
 250 Indeed implicit Euler methods are known to be able to mimic the continuous-time passivity and stability  
 251 properties, in some cases [55]. It is noteworthy that Assumption 2 implies that  $X_3 x_{2,k} = -X_3 B_3 \lambda_k = 0$   
 252 and  $X_3 x_2(t) = -X_3 B_3 \lambda(t) = 0$  for all  $k$  and  $t$ . Therefore  $V(x_1, x_2) = \frac{1}{2} x_1^\top X_1 x_1$ , so that this Lyapunov  
 253 function is not a positive definite function of the whole state. For the sake of completeness let us  
 254 nevertheless make the calculations in the general setting. We set  $V_k \stackrel{\Delta}{=} \frac{1}{2} x_{1,k}^\top X_1 x_{1,k} + x_{2,k}^\top X_3 x_{2,k}$ . It follows

255 that:

$$\begin{aligned}
V_{k+1} - V_k &= \frac{1}{2}(x_{1,k+1} - x_{1,k})^\top X_1(x_{1,k+1} + x_{1,k}) + (x_{2,k+1} - x_{2,k})^\top X_3(x_{2,k+1} + x_{2,k}) \\
&= \frac{1}{2}(hA_1x_{1,k} + hB_1\lambda_{k+1})^\top X_1(2x_{1,k} + hA_1x_{1,k} + hB_1\lambda_{k+1}) \\
&\quad + (hz_{k+1} + hB_2\lambda_{k+1})^\top X_3(2x_{2,k} + hz_{k+1} + hB_2\lambda_{k+1}) \\
&= h(x_{1,k}^\top A_1^\top X_1 x_{1,k} + \lambda_{k+1}^\top B_1^\top X_1 x_{1,k}) + h^2 \left( \frac{1}{2}x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} + \frac{1}{2}\lambda_{k+1}^\top B_1^\top X_1 B_1 \lambda_{k+1} \right) \\
&\quad + h^2 x_{1,k}^\top A_1^\top X_1 B_1 \lambda_{k+1} + h z_{k+1}^\top C_3 \lambda_{k+1} + h \lambda_{k+1}^\top C_3 B_2 \lambda_{k+1} + h z_{k+1}^\top X_3 x_{2,k} \\
&\quad + h \lambda_{k+1}^\top B_2^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1} - h \lambda_{k+1}^\top w_{k+1} \\
&= \frac{h}{2} x_{1,k}^\top (A_1^\top X_1 + X_1 A_1 + hA_1^\top X_1 A_1) x_{1,k} - h \lambda_{k+1}^\top (\hat{D} + h(C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1)) \lambda_{k+1} \\
&\quad + h x_{1,k}^\top (B_1^\top X_1 - C_1 - hA_1^\top C_1^\top + hA_1^\top X_1 B_1) \lambda_{k+1} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} \\
&\quad + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1}
\end{aligned}
\tag{4.10}$$

$$\begin{aligned}
&\stackrel{\triangleq Q_1}{=} \\
&= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) \begin{pmatrix} A_1^\top X_1 + X_1 A_1 & (B_1^\top X_1 - C_1)^\top \\ B_1^\top X_1 - C_1 & -(\hat{D} + \hat{D}^\top) \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} \\
&\quad + \frac{h^2}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - h^2 \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \\
&\quad + h^2 x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1} \\
&= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) Q_1 \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} + h \lambda_{k+1}^\top w_{k+1} \\
&\quad + h^2 \left\{ \frac{1}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \right. \\
&\quad \left. + x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} \right\} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k}
\end{aligned}$$

256

$$\begin{aligned}
&= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) Q_1 \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} + h \lambda_{k+1}^\top w_{k+1} \\
&\quad + h^2 \left\{ \frac{1}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \right. \\
&\quad \left. + x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} \right\} - \underbrace{\frac{1}{2} x_{2,k}^\top X_3 x_{2,k} + \lambda_{k+1}^\top B_3^\top X_3 B_3 \lambda_{k+1}}_{=0 \text{ if Assumption 2 holds.}}
\end{aligned}
\tag{4.10}$$

257 where the last two terms are obtained using the second equality in (3.1). Remind that  $\lambda_{k+1}^\top w_{k+1} = 0$ .

258 Assume that  $Q_1 < 0$ , then let Assumption 2 hold. Let us rewrite (4.10) as  $V_{k+1} - V_k = \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) [Q_1 +$

259  $hQ_2] \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix}$ , where the indefinite matrix

$$(4.11) \quad Q_2 = \begin{pmatrix} A_1^\top X_1 A_1 & -\frac{1}{2} A_1^\top C_1^\top + \frac{1}{2} A_1^\top X_1 B_1 \\ (-\frac{1}{2} A_1^\top C_1^\top + \frac{1}{2} A_1^\top X_1 B_1)^\top & -C_1 B_1 + B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1 \end{pmatrix},$$

260 and we remind that Assumption 2 implies that  $V_k = \frac{1}{2} x_{1,k}^\top X_1 x_{1,k}$ . **The matrix  $Q_2$  is treated as a**

261 **perturbation to  $Q_1$** . Using Corollary A.2, there exists  $h_{\max} > 0$  such that for all  $h \in [0, h_{\max})$  one

262 has  $Q_3 \stackrel{\triangleq}{=} Q_1 + hQ_2 < 0$ . Using classical bounds on quadratic forms and vector/matrices products

263 ( $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2$  [16, Corollary 9.1.7], if  $Q = Q^\top$  then  $\lambda_{\min}(Q) \|x\|^2 \leq x^\top Q x \leq$

264  $\lambda_{\max}(Q) \|x\|^2$  [16, Corollary 8.4.2],  $\|Qx\| \leq \|Q\| \|x\|$  for compatible norms [16, section 9.3]), and noting

265 that we can replace  $Q_3$  by its negative definite symmetric part in the quadratic form, it follows that:

$$(4.12) \quad \begin{aligned} V_{k+1} &\leq V_k + \frac{h}{2} \lambda_{\max}(Q_3) \|(x_{1,k}, \lambda_{k+1})\|^2 \\ &\leq V_k + \frac{h}{2} \lambda_{\max}(Q_3) \|x_{1,k}\|^2 \\ &\leq V_k + \frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)} V_k = \left(1 + \frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)}\right) V_k, \quad \text{for all } k \geq 0. \end{aligned}$$

266 Since  $\lambda_{\max}(Q_3) < 0$  (which is the reason why the second inequality in (4.12) holds), one has  $1 +$   
 267  $\frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)} < 1$  for all  $h > 0$ . Therefore all  $V_k$  are uniformly bounded for any bounded  $V_0$ , and  $V_k \rightarrow 0$  as  
 268  $k \rightarrow +\infty$  for any  $h \in (0, h_{\max})$ . Let us set  $\lambda_{k+1} = \lambda_{k+1}^{\min}$ . Since the dependence of  $\lambda_{k+1}^{\min}$  in  $x_{2,k}$  through  $\beta_k$   
 269 is  $C_3 x_{2,k}$  (see (4.2)), it follows from Assumptions 2 and 3 that  $\lambda_{k+1}^{\min} = \lambda_{k+1}^{\min}(x_{1,k})$ , and  $\lambda_{k+1}^{\min}(x_{1,k}) \rightarrow 0$  as  
 270  $k \rightarrow +\infty$ . Using the third line in (4.2), we infer that the iterates  $x_{2,k}$  (which are uniformly bounded  
 271 from Proposition 4.4) converge to zero asymptotically also. From the results in section 4.1 and the  
 272 above Lyapunov analysis, we have therefore proved the following:

273 **Lemma 4.7.** *Let Assumptions 2 and 3 hold, and the quintuple  $(P, A, B, C, D)$  be strongly SPR and minimal.*  
 274 *Then there exists  $h_{\max} > 0$  depending on  $A_1, C_1, X_1, X_3, B_1, B_2$ , such that for all  $h \in (0, h_{\max})$ , and for*  
 275 *all bounded initial condition  $x_{1,0}$ , the sequences  $\{x_{1,k}\}_k, \{x_{2,k}\}_k, \{\lambda_k^{\min}\}_k$  are uniformly bounded and*  
 276  *$x_{1,k}, x_{2,k}, \lambda_{k+1}^{\min} \rightarrow 0$  as  $k \rightarrow +\infty$ .*

277 **Remark 4.8.** Concerning Assumption 1, let us examine the constants  $a, b, c, d, f$  in Lemma 4.3, and the  
 278 inequalities in (4.5), as well as the above Lyapunov analysis. The term  $h_1$  is present in both  $a$  and  $c$ .  
 279 The inequality  $a \leq 1$  implies  $h_1 < 1$ , but  $c \leq 1$  does not imply that  $h_1 < 1$ . The constant  $a$  is present in  
 280 (4.5) (b), however the Lyapunov analysis guarantees the boundedness of  $x_{1,k}$  without this assumption  
 281 (the analysis relies on the passivity of  $(A_1, B_1, C_1, \hat{D})$  and  $h \geq 0$  small enough). But, Lemma 4.7 relies on  
 282  $Q_1 < 0$ , which is a strong passivity condition and may be restrictive in applications. There exists some  
 283 cases where SSPRness can be relaxed, as shown next.

284 Let  $(A_1, B_1, C_1, \hat{D})$  be strictly state passive ( $\Rightarrow X_1 A_1 + A_1^T X_1 = -\mu X_1, \mu > 0, X_1 = X_1^T > 0, \mu$  is called the  
 285 dissipation constant) and  $\hat{D} + \hat{D}^T = 0 \Rightarrow X_1 B_1 = C_1^T$ , then  $Q_2$  in (4.11) and  $Q_1$  can be rewritten as:

$$(4.13) \quad Q_2 = \begin{pmatrix} A_1^T X_1 A_1 & 0 \\ 0 & B_2^T X_3 B_2 - \frac{3}{2} B_1^T X_1 B_1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -\mu X_1 & 0 \\ 0 & 0 \end{pmatrix}$$

286 Thus  $Q_1 + hQ_2 < 0$  if and only if  $\mu X_1 - hA_1^T X_1 A_1 > 0$  and  $B_2^T X_3 B_2 - \frac{3}{2} B_1^T X_1 B_1 < 0$ . Let us recall that if  
 287  $(A_1, B_1, C_1, \hat{D})$  is minimal, then strict state passivity and SPRness are equivalent, see Appendix D.

288 **Lemma 4.9.** *Assume that: 1) the quadruple  $(A_1, B_1, C_1, \hat{D})$  is strictly state passive with  $\hat{D} + \hat{D}^T = 0$  and*  
 289 *dissipation constant  $\mu > 0$ , 2)  $B_2^T X_3 B_2 - \frac{3}{2} B_1^T X_1 B_1 < 0$ . Let Assumptions 2 and 3 hold. Then there exists*  
 290  *$h_{\max} > 0$  such that for all  $h \in [0, h_{\max})$ , the sequences  $\{x_{1,k}\}_k, \{x_{2,k}\}_k$  and  $\{\lambda_k^{\min}\}_k$  are uniformly bounded.*  
 291

292 **Proof:** Denote  $\tilde{X}_1 = \mu X_1 - hA_1^T X_1 A_1$ , then  $\tilde{X}_1 > 0$  for  $h \geq 0$  small enough, using Corollary A.2. Using  
 293 2) and Assumption 2, and after few calculations, there exists  $h_{\max} > 0$  such that:

$$(4.14) \quad V_{k+1} \leq \left(1 - \frac{h}{2} \frac{\lambda_{\min}(\tilde{X}_1)}{\lambda_{\max}(X_1)}\right) V_k.$$

294 for all  $h \in [0, h_{\max})$ . Thus  $V_k \rightarrow 0$  as  $k \rightarrow +\infty$ , and  $x_{1,k} \rightarrow 0$  as  $k \rightarrow +\infty$ . It can be inferred that  
 295  $\|x_{1,k}\| \leq \sqrt{\frac{\lambda_{\max}(X_1)}{\lambda_{\min}(X_1)}} \|x_{1,0}\| \alpha^{\frac{k}{2}}$  with  $\alpha < 1$ . From (4.5) (a) and Assumption 3 it follows that  $\|\lambda_{k+1}^{\min}\| \rightarrow 0$  as  
 296  $k \rightarrow +\infty$ , and  $\{\lambda_k^{\min}\}_k$  is uniformly bounded. From (4.1) it is inferred that  $x_{2,k} \rightarrow 0$  as  $k \rightarrow +\infty$  and  
 297  $\{x_{2,k}\}_k$  is uniformly bounded.  $\square$

298 Similar convergence results as in Corollary 4.6 can then be deduced, and are not developed here for  
 299 the sake of brevity. One sees from the above and Corollary A.2 that the degree of strict SPRness

300 (represented by the constant  $\mu$ ) is related to the value of  $h_{\max}$ : the greater  $\mu$ , the greater  $h_{\max}$ . Finally  
 301 it is noteworthy that the strict state passivity in item 1) of lemma 4.9 can again be relaxed, provided  
 302 that the matrix  $A_1^\top X_1 + X_1 A_1 + h A_1^\top X_1 A_1 < 0$ . Consider as an example  $A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $X_1 = I_2$ , then  
 303  $A_1^\top X_1 + X_1 A_1 + h A_1^\top X_1 A_1 = \begin{pmatrix} -2+h & 0 \\ 0 & 0 \end{pmatrix} < 0$  for all  $0 \leq h < 2$ .

## 304 5 ANOTHER APPROACH FOR PASSIVE SYSTEMS

305 The LDCS in (2.2) are suitably transformed in [32] to enable their analysis. The transformation may be  
 306 seen as a kind of relative degree reduction, involving the multiplier derivative. As will become clear  
 307 later, this new dynamics allows us to somewhat relax the stringent Assumption 2 (see Assumption 4 c)  
 308 below). More accurately, a close relationship is shown in [32, Lemma 3, Theorem 3] between (2.2) and  
 309 the following system:

$$(5.1) \quad \begin{cases} (a) \begin{pmatrix} I_{n_1} & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} w(t) \\ (b) 0 \leq \lambda(t) \perp w(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 z(t) + D \lambda(t) \geq 0 \\ (c) x_2(t) = -B_3 \lambda(t), z(t) = -B_2 \lambda(t) - B_3 \dot{\lambda}(t). \end{cases}$$

310 It is noteworthy that this dynamics is obtained from the original one by treating the multiplier  $\lambda$  as a  
 311 state variable. The "equivalence" between both systems is to be understood first as follows [32, Lemma  
 312 3]:

313  $\{(x_1, \lambda, w) \mid (x_1, x_2, \lambda, w) \text{ is a solution of (2.2) (a, b, c)}\} = \{(x_1, \lambda, w) \mid (x_1, \lambda, w) \text{ is a solution of (5.1) (a)}$   
 314  $\text{and } B_3 \lambda \text{ is absolutely continuous}\}.$

315 By solution of (2.2) it is meant that  $(x_1, x_2)$  is absolutely continuous and the dynamics holds Lebesgue  
 316 almost everywhere [32, Definition 3]. The initial conditions are chosen compatible with the constraints.  
 317 It is noteworthy that the variables  $z$  and  $\lambda$  are not shown to be absolutely continuous, and examples in  
 318 sections 6.1 and 6.3 will illustrate this fact. One has  $w(t) \in -\mathcal{N}_{\mathbb{R}_+^m}(\lambda(t))$ , therefore one obtains the  
 319 differential inclusion (DI):

$$(5.2) \quad \begin{aligned} \begin{pmatrix} I_{n_1} & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix} - \begin{pmatrix} A_1 & B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} &\in - \begin{pmatrix} 0 \\ I_m \end{pmatrix} \mathcal{N}_{\mathbb{R}_+^m}(\lambda(t)) \\ &= - \begin{pmatrix} 0 \\ I_m \end{pmatrix} \mathcal{N}_{\mathbb{R}_+^m} \left( \begin{pmatrix} 0 & I_m \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right) \\ &= -\mathcal{N}_\Phi \left( \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right) \end{aligned}$$

320 with  $\Phi \triangleq \{(x_1, \lambda) \in \mathbb{R}^{n_1} \times \mathbb{R}^m \mid \lambda \geq 0\}$ . If this DI has a solution  $(x_1(\cdot), \lambda(\cdot))$ , then  $x_2(\cdot)$ ,  $z(\cdot)$ , and  $w(\cdot)$   
 321 can be obtained (provided indeed that  $B_3 \lambda(\cdot)$  is absolutely continuous). Multiplying the first line in  
 322 (5.2) by  $X_1$  and adding an exogenous excitation  $G(t)$  to the left-hand side, one obtains

$$(5.3) \quad \underbrace{\begin{pmatrix} X_1 & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix}}_{\triangleq \mathcal{P}} - \underbrace{\begin{pmatrix} X_1 A_1 & X_1 B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix}}_{\triangleq \mathcal{A}} + G(t) \in -\mathcal{N}_\Phi \left( \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right)$$

323 which is similar to the form obtained in [32, Theorem 3]. Let us assume that an exogenous input  
324  $V(t) = (v_{(a)}(t), v_{(b)}(t), v_{(c)}(t), v_{(d)}(t))^T$  is added to (2.2), then:

$$(5.4) \quad G(t) = \begin{pmatrix} v_{(a)}(t) \\ C_3 (v_{(b)}(t) + \dot{v}_{(c)}(t)) + C_2 v_{(c)}(t) - v_{(d)}(t) \end{pmatrix}.$$

325 Notice that  $B_3^T X_3 B_3 \geq 0$ , therefore  $\mathcal{P} = \mathcal{P}^T \geq 0$ . However  $\mathcal{P}$  has a specific block-diagonal structure.  
326 Under strong SPRness and minimality,  $\mathcal{A} < 0$ , since  $\mathcal{A} + \mathcal{A}^T < 0$  from the KYP Lemma [25] (then  $A_1$   
327 is a Hurwitz matrix,  $X_1 = X_1^T > 0$  and  $\hat{D} > 0$ ). The exogeneous term  $G(t)$  stems from the exogeneous  
328 terms  $E(t)$  and  $F(t)$  in (2.1) and their time-derivatives (see (5.4) and (6.4) for an example).

329 In a second step the "equivalence" between both systems is to be understood as follows [32, Theorem  
330 3]:

331  $\{(x_1, \lambda) \mid (x_1, x_2, \lambda, w) \text{ is a solution of (2.2) } (a, b, c, d)\} = \{(x_1, \lambda) \mid (x_1, \lambda) \text{ is a solution of (5.3) and}$   
332  $B_3 \lambda (= -x_2) \text{ is absolutely continuous}\}.$

333 As will become clear later, the results that are obtained using this approach, and the approach in the  
334 foregoing section, are not the same (especially concerning the convergence of the piecewise-linear  
335 discrete solutions).

### 336 5.1 THE TIME-DISCRETIZATION AND THE OSNSP

337 As alluded to in (5.3), we can enlarge the analysis by considering time-varying terms, so as to embed  
338 (5.3) into the DI:  $\mathcal{P} \dot{\xi}(t) - \mathcal{A} \xi(t) + G(t) \in -\mathcal{N}_\Phi(\xi(t))$ . The Euler implicit scheme associated with (5.3) is:

$$(5.5) \quad \mathcal{P} \xi_{k+1} - \underbrace{(\mathcal{P} + h\mathcal{A})}_{\triangleq \mathcal{P}_h} \xi_k + hG_k \in -\mathcal{N}_\Phi(\xi_{k+1}),$$

339 It is noteworthy that this scheme is not equal to the implicit scheme in (3.1), as can be shown after few  
340 manipulations, see Appendix H. Though the difference between both approaches is not analysed further  
341 in this paper, it will be tackled numerically through an example in section 6.3. In addition, a slightly  
342 more general convex set than in (5.5) is considered, for the sake of generality. It corresponds to changing  
343 the complementarity conditions in (5.1) (b) to cone complementarity conditions  $K^* \ni \lambda(t) \perp w(t) \in K$ ,  
344  $K \subseteq \mathbb{R}^m$  a nonempty closed convex cone, similarly to what is done in Proposition 3.2. Thus  $\Phi = \mathbb{R}^{n_1} \times K^*$ .  
345

346 **Proposition 5.1.** *Let  $h > 0$  and  $k \geq 0$  be given. Let us consider the DI in (5.3) and its discretization in*  
347 *(5.5), with  $\Phi = \mathbb{R}^{n_1} \times K^*$ , with  $K$  a nonempty closed convex cone. Let us denote  $q_k \triangleq -\mathcal{P}_h \xi_k + hG_k$ . Let us*  
348 *consider the set:*

$$(5.6) \quad \mathcal{S} \triangleq \{(x_1, \lambda) \in \mathbb{R}^{n_1} \times \mathbb{R}^m \mid x_1 \in \text{Ker}(X_1), \lambda \in \text{Ker}(B_3^T X_3 B_3) \cap (B_3^T X_3 B_3)^{-1}(K) \cap K^*\}.$$

349 *Then:*

- 350 1. *If  $\mathcal{S} = \{0\}$ , for each  $q_k \in \mathbb{R}^{n_1 \times m}$  the problem  $VI(\mathcal{P}, q_k, \Psi_\Phi)$  has at least one solution.*
- 351 2. *Assume that  $\mathcal{S} \neq \{0\}$ . If there exists  $\xi^0 \in \text{dom}(\Phi)$  such that  $\langle q_k - \mathcal{P}^T \xi^0, v \rangle > 0$  for all  $v \in \mathcal{S}, v \neq 0$ ,*  
352 *then the problem  $VI(\mathcal{P}, q_k, \Psi_\Phi)$  has at least one solution. In case  $X_1 = X_1^T$ , one can take  $\xi_0 = 0$ .*
- 353 3. *If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of  $VI(\mathcal{P}, q_k, \psi_\Phi)$ , then  $\xi_{k+1}^1 - \xi_{k+1}^2 \in \text{Ker}(\mathcal{P})$ , hence  $x_{1,k+1}$  is unique.*  
354 *Moreover  $\langle q_k, \begin{pmatrix} 0 \\ \lambda_{k+1}^1 - \lambda_{k+1}^2 \end{pmatrix} \rangle = 0$ .*

- 355 4. If  $X_1 > 0$  and  $B_3^\top X_3 B_3 > 0$ , then the problem  $VI(\mathcal{P}, q_k, \Psi_\Phi)$  always has a unique solution.
- 356 5. Assume that the quintuple  $(P, A, B, C, D)$  is minimal and strongly SPR, so that the quadruple  
 357  $(A_1, B_1, C_1, \hat{D})$  is minimal and strongly SPR. If  $B_3^\top X_3 B_3 > 0$ , then the problem  $VI(\mathcal{P}, q_k, \Psi_\Phi)$  always  
 358 has a unique solution.
- 359 6. Let  $K = K^\star = \mathbb{R}_+^m$ . Then  $\mathcal{S} = \{(x_1, \lambda) \mid x_1 \in \text{Ker}(X_1), \lambda \in \mathbb{R}_+^m \cap \text{Ker}(B_3^\top X_3 B_3)\}$ .

360 **Proof:** From Proposition B.1 d) one has  $(K^\star)_\infty = K^\star$  and  $\Phi_\infty = \Phi$ . Also  $\varphi = \Psi_\Phi$  hence from  
 361 Proposition B.1 c) and d),  $\varphi_\infty = \Psi_{\mathbb{R}^{n_1} \times K^\star}$  and  $(\text{dom}(\varphi_\infty))^\star = \Phi^\star = \mathbb{R}^{n_1} \times K$ . Then one has  $\mathcal{S} \stackrel{\Delta}{=} \Phi \cap$   
 362  $\text{Ker}(\mathcal{P}) \cap \{\xi \in \mathbb{R}^{n_1+m} \mid \mathcal{P} \begin{pmatrix} x_1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \alpha \in K\} = \Phi \cap \text{Ker}(\mathcal{P}) \cap \{\xi \in \mathbb{R}^{n_1+m} \mid x_1 \in \text{Ker}(X_1), B_3^\top X_3 B_3 \lambda \in$   
 363  $K\}$ , where we used the block diagonal structure of  $\mathcal{P}$ . Therefore  $\mathcal{S} = \{(x_1, \lambda) \in \mathbb{R}^{n_1} \times \mathbb{R}^m \mid x_1 \in$   
 364  $\text{Ker}(X_1), B_3^\top X_3 B_3 \lambda \in K, \lambda \in \text{Ker}(B_3^\top X_3 B_3), \lambda \in K^\star\}$ . In item 2) one uses that  $\Psi_\Phi(v) = 0$  when  $v \in \mathcal{S}$ ,  
 365 and item b') in Proposition B.2. Let us now deal with item 4. In this case  $\mathcal{P} = \mathcal{P}^\top > 0$  and the result  
 366 follows from items 1 and 3. Let us define  $\mathcal{R} = \mathcal{R}^\top > 0$ ,  $\mathcal{R}^2 = \mathcal{P}$ , and  $\zeta_k = \mathcal{R}\xi_k$ . Then the VI in (5.5) is  
 367 equivalently rewritten as:

$$(5.7) \quad \zeta_{k+1} - \mathcal{R}^{-1} \mathcal{P}_h \mathcal{R}^{-1} \zeta_k + \mathcal{R}^{-1} G_k \in -\mathcal{R}^{-1} \mathcal{N}_\Phi(\mathcal{R}^{-1} \zeta_{k+1}) = -\mathcal{N}_{\tilde{\Phi}}(\zeta_{k+1}),$$

368 where the last equality stems from the chain rule, with  $\tilde{\Phi} = \{z \in \mathbb{R}^{n_1+m} \mid \mathcal{R}^{-1} z \in \Phi\}$ . One infers  
 369 from (5.7) that  $\zeta_{k+1} = \text{proj}[\tilde{\Phi}; \mathcal{R}^{-1} \mathcal{P}_h \mathcal{R}^{-1} \zeta_k - \mathcal{R}^{-1} G_k] \Rightarrow \xi_{k+1} = \mathcal{R}^{-1} \text{proj}[\tilde{\Phi}; \mathcal{R}^{-1} \mathcal{P}_h \xi_k - \mathcal{R}^{-1} G_k]$ . The first  
 370 assertion in item 5 about strong SPRness follows from [28, Theorem 5.4 (A')], which guarantees that  
 371  $X_1 > 0$  (see Appendix D). The last assertion of item 5 is a consequence of item 4. Consider finally item  
 372 6. One has  $(B_3^\top X_3 B_3)^{-1}(K) = \{\lambda \in \mathbb{R}^m \mid B_3^\top X_3 B_3 \lambda = \alpha, \alpha_i \geq 0, 1 \leq i \leq m\}$ , see e.g., [53, Example 15].  
 373 Hence  $\text{Ker}(B_3^\top X_3 B_3) \cap (B_3^\top X_3 B_3)^{-1}(K) = \text{Ker}(B_3^\top X_3 B_3)$ .  $\square$

374 In item 2, the sufficient condition  $\langle q_k, v \rangle > 0$  for all  $v \in \mathcal{S}$  is equivalent to  $q_k \in \mathcal{S}^\star \setminus \mathcal{S}^\perp$ . One notices  
 375 that under the conditions of item 4 in Proposition 5.1, the DI in (5.3) is equivalent to a DI with maximal  
 376 monotone set-valued term:

$$(5.8) \quad \dot{\zeta}(t) - \mathcal{R}^{-1} \mathcal{A} \mathcal{R}^{-1} \zeta(t) + \mathcal{R}^{-1} G(t) \in -\mathcal{N}_{\tilde{\Phi}}(\zeta(t))$$

377 with  $\zeta = \mathcal{R}\xi$ . This fits within the class of DIs analysed for instance in [14]. Under basic assumption on  
 378 the regularity of  $G(\cdot)$ , (5.8) has Lipschitz continuous solutions with uniqueness.

379 **Proposition 5.2.** Assume that  $X_1 > 0$ , and  $G(\cdot)$  is bounded. The problem  $VI(\mathcal{P}, q_k, \Psi_\Phi)$  always has at least  
 380 one solution for bounded  $\xi_k$ .

381 **Proof:** Consider item 2 in Proposition 5.1, with  $\xi^0 = \begin{pmatrix} x^0 \\ \lambda^0 \end{pmatrix}$ ,  $v = \begin{pmatrix} v_x \\ v_\lambda \end{pmatrix}$ . Then:

$$(5.9) \quad \begin{aligned} \langle q_k - \mathcal{P}^\top \xi^0, v \rangle &= \langle -\mathcal{P}_h \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} + hG_k, v \rangle - \langle B_3^\top X_3 B_3 \lambda^0, v_\lambda \rangle \\ &\quad - \langle X_1 x^0, v_x \rangle \\ &= \langle - \begin{pmatrix} X_1 + hX_1 A_1 & hX_1 B_1 \\ -hC_1 & B_3^\top X_3 B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} + hG_k, v \rangle \\ &\quad - \langle B_3^\top X_3 B_3 \lambda^0, v_\lambda \rangle - \langle X_1 x^0, v_x \rangle. \end{aligned}$$

382 Since  $\xi^0 \in \Phi$  it follows that  $x^0 \in \mathbb{R}^{n_1}$  while  $\lambda^0 \in \mathbb{R}_+^m$ , and similarly  $v_x \in \mathbb{R}^{n_1}$ ,  $v_\lambda \in \mathbb{R}_+^m$ . Since  $v \neq 0$  let



383 us take  $x^0 = -\alpha X_1^{-1} \frac{v_x}{\|v_x\|} - \beta X_1^{-1} \frac{v_x}{\|v_x\|^2} \|v_\lambda\|$  for some  $\alpha \geq, \beta \geq 0$ , and  $\lambda^0 = 0$ . It follows that:

$$\begin{aligned}
 \langle q_k - \mathcal{P}^\top \xi^0, v \rangle &= -\left\langle \begin{pmatrix} X_1 + hX_1A_1 & hX_1B_1 \\ -hC_1 & B_3^\top X_3B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix}, v \right\rangle + \langle hG_k, v \rangle \\
 &\geq -\left\| \begin{pmatrix} X_1 + hX_1A_1 & hX_1B_1 \\ -hC_1 & B_3^\top X_3B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} \right\| \|v\| - \|hG_k\| \|v\| \\
 &\quad + \alpha \|v_x\| + \beta \|v_\lambda\| \\
 &\geq -\left\| \begin{pmatrix} X_1 + hX_1A_1 & hX_1B_1 \\ -hC_1 & B_3^\top X_3B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} \right\| (\|v_x\| + \|v_\lambda\|) \\
 &\quad - \|hG_k\| (\|v_x\| + \|v_\lambda\|) + \alpha \|v_x\| + \beta \|v_\lambda\|.
 \end{aligned}
 \tag{5.10}$$

384 **Since  $G(\cdot)$  is bounded**, for large enough  $\alpha$  and  $\beta$  it follows that  $\langle q_k - \mathcal{P}^\top \xi^0, v \rangle > 0$  and this holds for  
 385 any  $v \neq 0$  inside  $\mathcal{S}$ .  $\square$

## 386 5.2 BOUNDEDNESS AND CONVERGENCE ANALYSIS

387 In the next analysis we assume that  $B_3^\top X_3B_3$  is singular, which is the most general case (see the example  
 388 in section 6.2, see also section 5.3 for the invertible case), **and that  $K = \mathbb{R}_+^m$  and that  $X_1 > 0$** . From the  
 389 structure of  $\mathcal{P}$  and of the set-valued right-hand side, it follows from (5.5) that  $x_{1,k+1}$  is calculable from  
 390  $x_{1,k}$  and  $\lambda_k$ :

$$\begin{aligned}
 X_1 x_{1,k+1} &= (X_1 + hX_1A_1)x_{1,k} + hX_1B_1\lambda_k - hG_k^x \\
 \Leftrightarrow x_{1,k+1} &= (I_{n_1} + hA_1)x_{1,k} + hB_1\lambda_k - hX_1^{-1}G_k^x, \quad \text{for all } k \geq 0.
 \end{aligned}
 \tag{5.11}$$

391 where  $G_k = \begin{pmatrix} G_k^x \\ G_k^\lambda \end{pmatrix}$ .

392 **Lemma 5.3.** *From (5.11), for all  $k \geq 0$  one has:*

$$(5.12) \quad x_{1,k+1} = (I_{n_1} + hA_1)^{k+1}x_{1,0} + h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} B_1 \lambda_i - h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} X_1^{-1} G_i^x.$$

393

394 **Proof:** The relation is true for  $k = 0$ . Let it be true for  $k \geq 1$ . Then using (5.11) it follows that:

$$\begin{aligned}
 x_{1,k+2} &= (I_{n_1} + hA_1)x_{1,k+1} + hB_1\lambda_{k+1} - hX_1^{-1}G_{k+1}^x \\
 &= (I_{n_1} + hA_1)\left\{ (I_{n_1} + hA_1)^{k+1}x_{1,0} + h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} B_1 \lambda_i \right. \\
 &\quad \left. - h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} X_1^{-1} G_i^x \right\} + hB_1\lambda_{k+1} - hX_1^{-1}G_{k+1}^x \\
 &= (I_{n_1} + hA_1)^{k+2}x_{1,0} + h \sum_{i=0}^{k+1} (I_{n_1} + hA_1)^{k+1-i} B_1 \lambda_i \\
 &\quad - h \sum_{i=0}^{k+1} (I_{n_1} + hA_1)^{k+1-i} X_1^{-1} G_i^x.
 \end{aligned}
 \tag{5.13}$$

395 By induction the relation in (5.12) is true for all  $k \geq 0$ .  $\square$

396 Using (5.5) we can also express  $\lambda_{k+1}$  as the solution of a generalised equation (GE):

$$\begin{aligned}
 B_3^\top X_3B_3 \lambda_{k+1} + \underbrace{(h\hat{D} - B_3^\top X_3B_3)}_{\triangleq q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)} \lambda_k + hC_1x_{1,k} + hG_k^\lambda &\in -\mathcal{N}_{\mathbb{R}_+^m}(\lambda_{k+1}) \\
 \Downarrow \\
 \lambda_{k+1} &\in (B_3^\top X_3B_3 + \mathcal{N}_{\mathbb{R}_+^m})^{-1} \left( (B_3^\top X_3B_3 - h\hat{D})\lambda_k - hC_1x_{1,k} - hG_k^\lambda \right)
 \end{aligned}
 \tag{5.14}$$

397 for all  $k \geq 0$ . The second formulation shows the iterative feature of this GE for  $\lambda_k$ , which makes it  
 398 quite different from the LCCP (3.3). It is equivalent to the LCP( $B_3^\top X_3 B_3, q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)$ ). Under the  
 399 conditions stated in Proposition 5.1 and Corollary 4.1, the set of solutions to this LCP is non empty, and it  
 400 contains a least-norm element  $\lambda_{k+1}^{\min}, k \geq 0$ , satisfying  $\|\lambda_{k+1}^{\min}(\lambda_k, x_{1,k}, G_k^\lambda)\| \leq \eta \|q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)\|, k \geq 0$ ,  
 401 where  $\eta > 0$  depends only on  $B_3^\top X_3 B_3$ . Therefore  $\|\lambda_{k+1}^{\min}(\lambda_k, x_{1,k}, G_k^\lambda)\| \leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \|\lambda_k^{\min}\| +$   
 402  $h\eta \|C_1 x_{1,k}\| + h\eta \|G_k^\lambda\|$ . Let us denote  $h_3 \stackrel{\Delta}{=} \|h\hat{D} - B_3^\top X_3 B_3\|$ .

403 **Lemma 5.4.** *One has for all  $k \geq 0$ :*

$$(5.15) \quad \begin{aligned} \|\lambda_{k+2}^{\min}(\lambda_{k+1}, x_{1,k+1}, G_{k+1}^\lambda)\| &\leq (\eta h_3)^{k+1} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\ &+ h\eta \sum_{i=0}^k (\eta h_3)^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|) \end{aligned}$$

404

405 **Proof:** The inequality is satisfied for  $k = 0$ . Assume that it is true for some  $k \geq 1$ . Then

$$(5.16) \quad \begin{aligned} \|\lambda_{k+3}^{\min}(\lambda_{k+2}, x_{1,k+2}, G_{k+2}^\lambda)\| &\leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \|\lambda_{k+2}^{\min}\| + h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\ &\leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \{\eta^{k+1} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\ &+ h\eta \sum_{i=0}^k \eta^{k-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|)\} \\ &+ h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\ &= \eta^{k+2} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+2} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\ &+ h\eta \sum_{i=0}^k \eta^{k+1-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|) \\ &+ h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\ &= \eta^{k+2} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+2} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\ &+ h\eta \sum_{i=0}^{k+1} \eta^{k+1-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|). \end{aligned}$$

406 The result follows by induction.  $\square$

407 The next step is to use Lemma 5.4 and Lemma 5.3 to get a new **upper bound**. Let us denote  $h_1 \stackrel{\Delta}{=} \|I_{n_1} + hA_1\|$ .

408 Then using (5.12) and (5.15) one has:

$$(5.17) \quad \begin{cases} \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h \sum_{i=0}^k h_1^{k-i} \|B_1\| \|\lambda_{i+1}^{\min}\| + h \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| \\ \|\lambda_{k+2}^{\min}\| \leq \eta^{k+1} h_3^{k+1} \|\lambda_1^{\min}\| + h\eta \sum_{i=0}^k \eta^{k-i} h_3^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|) \end{cases}$$

409 Therefore we obtain:

$$(5.18) \quad \begin{aligned} \|x_{1,k+1}\| &\leq h_1^{k+1} \|x_{1,0}\| + h \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| \\ &+ h \sum_{i=0}^k h_1^{k-i} \|B_1\| \eta^i h_3^i \|\lambda_1^{\min}\| + h h_1^k \|B_1\| \|\lambda_1^{\min}\| \\ &+ h \sum_{i=1}^k h_1^{k-i} \|B_1\| h\eta \sum_{j=0}^{i-1} \eta^{i-1-j} h_3^{i-1-j} \|C_1\| \|G_{j+1}^\lambda\| \\ &+ h \sum_{i=1}^k h_1^{k-i} \|B_1\| h\eta \sum_{j=0}^{i-1} \eta^{i-1-j} h_3^{i-1-j} \|C_1\| \|x_{1,j+1}\|, \end{aligned}$$

410 which we rewrite as:

$$(5.19) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$$

411 with

$$(5.20) \quad \begin{aligned} f_k &= \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| + \|B_1\| \|\lambda_1^{\min}\| \sum_{i=0}^k h_1^{k-i} (\eta h_3)^i + h_1^k \|B_1\| \|\lambda_1^{\min}\| \\ &+ h \eta \|B_1\| \|C_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|G_{j+1}^\lambda\|. \end{aligned}$$

412 The third term in the right-hand side of (5.19) involves terms from  $\|x_{1,1}\|$  to  $\|x_{1,k}\|$ . The inequality (5.19)  
413 can be rewritten as:

$$(5.21) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \sum_{i=1}^k \beta_i \|x_{1,i}\|,$$

414 for some positive sequence of real numbers  $\{\beta_i\}$ , each  $\beta_i$  being calculated from the expression in  
415 (5.19)  $\eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$ , and thus being made of powers of  $h_1$  and of  $h_3 \eta$ .  
416 Calculations show that (see Appendix F)

$$(5.22) \quad \beta_i = \eta \|C_1\| \|B_1\| \sum_{j=i}^k h_1^{k-j} (\eta h_3)^{j-i}, \quad i \geq 1.$$

417 Let us now apply a version of the discrete-time Gronwall inequality, due to Bachpatte [13] [47, Lemma  
418 100], see section E. It is deduced that for all  $k \geq 0$ :

$$(5.23) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \sum_{i=0}^k (h_1^{i+1} \beta_i \|x_{1,0}\| + h \beta_i f_i) \prod_{j=i+1}^{k+1} (1 + h^2 \beta_j).$$

419 Let us state the next assumption (recall that  $h_3 = \|h\hat{D} - B_3^\top X_3 B_3\|$  and  $h_1 = \|I_{n_1} + hA_1\|$ ).

420 **Assumption 4.** One has a)  $h_1 \leq 1$  for  $h > 0$  small enough, b)  $\|G(t)\|$  is uniformly bounded by  $G_{\max} < +\infty$   
421 on  $[0, T]$  for any  $T > 0$ , and  $G(\cdot)$  is Lipschitz continuous with Lipschitz constant  $k_G \geq 0$ , c)  $\eta h_3 \leq 1$   
422 for  $h > 0$  small enough.

423 Inequality a) is similar to Assumption 1. As illustrated through examples in section 6, it **may hold** in  
424 practice for reasonable values of the **time step**. See also the developments in Appendix A for conditions  
425 on its validity, **and Remark 5.7 for its relaxation**. Assumption 4 b) is a standard property of exogenous  
426 time-dependent terms. The inequality c) is the central assumption of Assumption 4, which quantifies  
427 the couplings between the DAE part and the complementarity part of the **LDCS**. Some examples of  
428 circuits in section 6 will prove that it can be satisfied in practice. Notice from (5.14) that if  $B_3^\top X_3 B_3 = 0$   
429 (Assumption 2), then the minimal-norm element of the set of solutions to the LCP is always  $\lambda_k^{\min} = 0$ ,  
430 hence  $\eta = 0$  in that case. Using (5.20) the next **upper bound** can be found:

$$(5.24) \quad f_k \leq (k+1)(G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|) + \|B_1\| \|\lambda_1^{\min}\| + h \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j,$$

431 Since  $h = \frac{T}{N}$  and  $k \in \{0, N-1\}$ :

$$(5.25) \quad h f_k \leq T \frac{(k+1)}{N} (G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|) + \frac{T}{N} \|B_1\| \|\lambda_1^{\min}\| + \frac{T^2}{n^2} \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j.$$

432 Now we have that  $\sum_{i=1}^k \sum_{j=0}^{i-1} j = (k-1) + 2(k-1) + 3(k-3) + \dots + (k-3)3 + (k-2)2 + (k-1)$ , where  
 433 the sum consists of  $k-1$  terms. Hence  $\sum_{i=1}^k \sum_{j=0}^{i-1} j = \mathcal{O}(k^2)$ , and  $\frac{T^2}{N^2} \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j =$   
 434  $T^2 \eta \|B_1\| \|C_1\| G_{\max} \frac{\mathcal{O}(k^2)}{N^2}$ . Since  $k \leq N$  we infer that this term is uniformly bounded in  $k$ , for each  
 435 fixed bounded  $T$ . The same holds for  $T \frac{(k+1)}{N} (G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|)$ . Thus  $hf_k \leq \alpha_1 \max(T, T^2)$   
 436 for some  $\alpha_1$  which depends on  $\eta, \|B_1\|, \|C_1\|, G_{\max}, \|\lambda_1^{\min}\|$ .

437 The next step consists of computing an **upper bound** from (5.23) as:

$$(5.26) \quad \|x_{1,k+1}\| \leq \left( \|x_{1,0}\| + \alpha_1 \max(T, T^2) \right) \left( 1 + \frac{T^2}{N^2} \sum_{i=0}^k \beta_i \Pi_{j=i+1}^{k+1} \left( 1 + \frac{T^2}{N^2} \beta_j \right) \right).$$

438 In view of Assumption 4 and (5.22), one has  $\beta_i \leq \eta \|C_1\| \|B_1\| \sum_{j=i}^k 1 = \eta \|C_1\| \|B_1\| (k-i) \leq$   
 439  $\eta \|C_1\| \|B_1\| N, i \geq 1, k \leq N$ .

440 Consequently:

$$(5.27) \quad \begin{aligned} \|x_{1,k+1}\| &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \frac{T^2}{N} \sum_{i=0}^k \eta \|C_1\| \|B_1\| \Pi_{j=i+1}^{k+1} \left( 1 + \frac{T^2}{N} \eta \|C_1\| \|B_1\| \right) \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} \sum_{i=0}^k \Pi_{j=i+1}^{k+1} \left( 1 + \frac{T^2}{N} \eta \|C_1\| \|B_1\| \right) \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} \sum_{i=0}^k \left( 1 + \frac{T^2}{N} \eta \|C_1\| \|B_1\| \right)^{k-i} \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} \sum_{i=0}^k \left( 1 + \frac{\alpha_2}{N} \right)^{k-i} \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} \sum_{i=0}^N \left( 1 + \frac{\alpha_2}{N} \right)^N \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} (N+1) \left( 1 + \frac{\alpha_2}{N} \right)^N \right) \\ &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left( 1 + T^2 \eta \|C_1\| \|B_1\| \frac{N+1}{N} \left( 1 + \frac{\alpha_2}{N} \right)^N \right), \end{aligned}$$

441 where  $\alpha_2 = T^2 \eta \|C_1\| \|B_1\|$ . One has  $(1 + \frac{\alpha_2}{N})^N \rightarrow e^{\alpha_2}$  as  $N \rightarrow +\infty$  [73, p.196]. Therefore the sequence  
 442  $\{x_{1,k}\}_k$  is uniformly bounded over  $k \in \{0, N-1\}$ . It directly follows from Lemma 5.4 that the sequence  
 443  $\{\lambda_k^{\min}\}_k$  is uniformly bounded as well. It follows from (4.1) that the sequence  $\{\frac{x_{1,k+1} - x_{1,k}}{h}\}_k$  is uniformly  
 444 bounded also. Therefore we are ready to prove the following result. The piece-wise linear approxima-  
 445 tions are defined in (4.7), and the step approximations  $x_1^{h*}(t) = x_{1,k+1}$  if  $t \in [t_k, t_{k+1})$ ,  $\lambda_{h*}^{\min}(t) = \lambda_k^{\min}$  if  
 446  $t \in [t_k, t_{k+1})$ .

447 **Proposition 5.5.** *Let Assumption 4 hold true, as well as the conditions for the well-posedness of the*  
 448 *OSNSP. Then the sequence  $\{x_1^h(\cdot)\}_{h \geq 0}$  converges uniformly towards a continuous limit  $x_1(\cdot)$ . Moreover*  
 449  *$x_1^{h*}(\cdot)$  converges strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  towards  $x_1(\cdot)$ , and  $\dot{x}_1^h(\cdot)$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$*   
 450 *towards  $\dot{x}_1(\cdot)$ . Therefore  $\{(B_1 \lambda_{h*}^{\min})(\cdot)\}_{h \geq 0}$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  towards a limit  $\lambda_1(\cdot)$ , and*  
 451  *$\dot{x}_1(t) = A_1 x_1(t) + G^x(t) + \lambda_1(t)$  almost everywhere on  $[0, T]$ .*

452 **Proof:** Both sequences  $\{x_1^h(\cdot)\}$  and  $\{\dot{x}_1^h(\cdot)\}$  are uniformly bounded on  $[0, T]$ , the sequence  $\{x_1^h(\cdot)\}$   
 453 of  $C^0([0, T]; \mathbb{R}^{n_1})$  functions is thus equicontinuous. From the Ascoli-Arzelà Theorem,  $\{x_1^h(\cdot)\}$  stays  
 454 in a compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and indeed it converges uniformly towards a continuous limit  
 455  $x_1(\cdot)$ . Let us now examine the sequences of piecewise-linear functions  $\{G^h(\cdot)\}_{h \geq 0}$  and  $\{\dot{G}^h(\cdot)\}_{h \geq 0}$ ,  
 456 designed similarly as (4.7). From Assumption 4 b) one infers that  $G^h(t)$  is uniformly bounded on  
 457  $[0, T]$ ,  $\frac{\|G_{k+1} - G_k\|}{h} \leq k_E$  implies that  $\dot{G}^h(t)$  is also uniformly bounded in  $h$  on  $[0, T]$ , thus  $\{G^h(\cdot)\}_{h \geq 0}$  is  
 458 a sequence of Lipschitz continuous, uniformly bounded, equicontinuous functions. From the Ascoli-  
 459 Arzelà Theorem,  $\{G^h(\cdot)\}$  stays in a compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and it converges uniformly

460 towards the continuous limit  $G(\cdot)$  as  $h \rightarrow 0$ . The remaining part of the proof is similar to the proof of  
 461 Corollary 4.6 (adding  $G^{x,h}(\cdot)$  with the required properties) and it is omitted.  $\square$

462 **Remark 5.6.** Assumption 4 c) is clearly less restrictive than Assumption 2. It still imposes some constraint  
 463 on the matrix  $C_3 B_3$ . Examples are treated next which show that it is satisfied in some practical cases.

464 Let us now prove the uniform boundedness of the sequences  $\{\lambda_k^{\min}\}_k$  and  $\{x_{2,k}\}_k$ . To this aim we shall  
 465 use first the second line in (5.17) and (5.27). Let us denote the right-hand side of (5.27) as  $x_{1,\max}(T, \eta, x_{1,0})$ .  
 466 Then from (5.17) and Assumption 4 we obtain:

$$(5.28) \quad \begin{aligned} \|\lambda_{k+1}^{\min}\| &\leq \|\lambda_0^{\min}\| + h\eta \sum_{i=0}^k (\|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \|G_{i+1}^\lambda\|) \\ &\leq \|\lambda_0^{\min}\| + h\eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0})(k+1) + h\eta \sum_{i=0}^k \|G_{i+1}^\lambda\| \\ &\leq \|\lambda_0^{\min}\| + \frac{T(k+1)}{N} \eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \eta \frac{T(k+1)}{N} G_{\max} \\ &\leq \|\lambda_0^{\min}\| + \frac{T(N+1)}{N} \eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \eta \frac{T(N+1)}{N} G_{\max}. \end{aligned}$$

467 Therefore  $\|\lambda_{k+1}^{\min}\|$  is uniformly bounded for any bounded  $T$  and initial data. The boundedness of  $x_{2,k}$   
 468 follows from the equality constraint in (4.1). A result similar to Corollary 4.6 holds, but this time  
 469 verifying the complementarity conditions is less easy due to the presence of  $z$  in the complementarity  
 470 variable  $w$ .

471 **Remark 5.7.** [Relaxation of assumptions] Assumption 4 a) is convenient to simplify the upper bounds  
 472 calculations, however it is somewhat restrictive. One path to relax it, is to study the terms  $h_1^k$  another way,  
 473 since for any  $a_1 > 0$  one has  $(1 + \frac{T}{N} a_1)^N \rightarrow e^{a_1 T}$  as  $N \rightarrow +\infty$ , while this sequence is increasing. Consider  
 474 first (5.20). Taking into account Assumption 4 b) and c), one sees that the first two terms in  $h f_k$  are upper  
 475 bounded by constants times  $h \sum_{i=0}^k h_1^{k-i} = h \sum_{i=0}^k h_1^i \leq h k (1 + h a_1)^k$ , where  $h_1 \leq \|I_m\| + h \|A_1\| = 1 + h a_1$   
 476 for some  $a_1 \geq 0$ . We have  $h k (1 + h a_1)^k \leq \frac{T}{N} k (1 + \frac{T a_1}{N})^N \rightarrow T e^{a_1 T}$  as  $N \rightarrow +\infty$ . The third term of  $h f_k$   
 477 is upper bounded by a constant times  $h h_1^k \leq \frac{T}{N} (1 + h a_1)^N \rightarrow 0$  as  $N \rightarrow +\infty$ . The last term of  $h f_k$   
 478 is upper bounded by a constant times  $h^2 \sum_{i=1}^k h_1^{k-i} k \leq \frac{T^2}{N^2} k^2 (1 + h a_1)^k \leq \frac{T^2}{N^2} N^2 (1 + h a_1)^N \rightarrow T^2 e^{T a_1}$   
 479 as  $N \rightarrow +\infty$ . These first calculations allow us to upper bound the term  $h f_k$  in (5.23). The first term  
 480  $h_1^{k+1} \|x_{1,0}\| = (1 + h a_1)^{k+1} \|x_{1,0}\| \leq (1 + h a_1)^N \|x_{1,0}\| \rightarrow \|x_{1,0}\| e^{a_1 T}$  as  $N \rightarrow +\infty$ .

481 The third term in (5.23) is more complex:  $h^2 \sum_{i=0}^k (h_1^{i+1} \beta_i \|x_{1,0}\| + h \beta_i f_i) \prod_{j=i+1}^{k+1} (1 + h^2 \beta_j)$ . Since from As-  
 482 sumption 4 c)  $\eta h_3 \leq 1$ , it follows that  $\beta_i \leq \eta \|C_1\| \|B_1\| \sum_{j=1}^k h_1^{k-j} \leq \eta \|C_1\| \|B_1\| \sum_{j=1}^k (1 + h a_1)^{k-j}$   
 483 for some  $a_1 > 0$ . Thus  $h^2 \beta_j \leq \eta \|C_1\| \|B_1\| h^2 k (1 + h a_1)^{k-1} \leq \eta \|C_1\| \|B_1\| \frac{T^2}{N^2} N (1 + \frac{T}{N} a_1)^N =$   
 484  $\eta \|C_1\| \|B_1\| \frac{T^2}{N} (1 + \frac{T}{N} a_1)^N$ . Hence  $\prod_{j=i+1}^{k+1} (1 + h^2 \beta_j) \leq (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{N} (1 + \frac{T}{N} a_1)^N)^N \rightarrow e^{\eta \|C_1\| \|B_1\| T^2 e^{a_1 T}}$ .  
 485 The term  $h^2 \sum_{i=0}^k \beta_i h f_i = \sum_{i=0}^k (h^2 \beta_i) (h f_i) \leq N \eta \|C_1\| \|B_1\| \frac{T^2}{N} (\alpha_1 T + \alpha_2 \frac{T}{N} + \alpha_3 T^2) (1 + \frac{T}{N} a_1)^{2N} \rightarrow$   
 486  $\eta \|C_1\| \|B_1\| T^2 (\alpha_1 T + \alpha_3 T^2) e^{2 a_1 T}$ , hence is bounded for any bounded  $T$ . The term  $h^2 \sum_{i=0}^k h_1^{i+1} \beta_i \|x_{1,0}\| =$   
 487  $\|x_{1,0}\| \sum_{i=0}^k h_1^{i+1} h^2 \beta_i \leq \eta \|C_1\| \|B_1\| \|x_{1,0}\| \sum_{i=0}^k (1 + h a_1)^k h^2 k (1 + h a_1)^{k-1} \leq \eta \|C_1\| \|B_1\| \|x_{1,0}\| \sum_{i=0}^k (1 +$   
 488  $h a_1)^{2N} \frac{T^2}{N^2} N \leq \eta \|C_1\| \|B_1\| \|x_{1,0}\| T^2 e^{2 a_1 T}$ . Consequently from (5.23) it is inferred that if Assumption  
 489 4 b) and c) holds, then  $\|x_{1,k+1}\|$  is bounded by a term that depends only on  $T$ , initial conditions and  
 490 constants.

491 Let us examine now Assumption 4 c) and the terms which involve  $h_3^k$ . We have  $h_3 = \|h \hat{D} -$   
 492  $B_3^T X_3 B_3\|$ . Let us assume that  $\hat{D}$  is invertible. Then  $\|h \hat{D} - B_3^T X_3 B_3\| = \|\hat{D}(h - \hat{D}^{-1} B_3^T X_3 B_3)\| \leq$   
 493  $\|\hat{D}\| \|(h I_m - \hat{D}^{-1} B_3^T X_3 B_3)\| \leq \|\hat{D}\| (h + \|\hat{D}^{-1} B_3^T X_3 B_3\|) \leq \|\hat{D}\| (h + \|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|) =$   
 494  $\|\hat{D}\| \|\hat{D}^{-1}\| \|B_3^T X_3 B_3\| \left(1 + \frac{1}{\|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|} h\right)$ .

495 Therefore it follows that  $h_3^k \leq (\|\hat{D}\| \|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|)^k \left(1 + \frac{T}{\|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|} \frac{1}{N}\right)^k$ . Consequently  
 496  $(\eta h_3)^k \leq (\eta \|\hat{D}\| \|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|)^k \left(1 + \frac{T}{\|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|} \frac{1}{N}\right)^k$  which converges as  $N \rightarrow +\infty$  to  
 497  $(\eta \|\hat{D}\| \|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|)^N e^{\frac{T}{\|\hat{D}^{-1}\| \|B_3^T X_3 B_3\|}}$ . A necessary condition for the limit to be bounded is

498  $\eta \|\hat{D}\| \|\hat{D}^{-1}\| \|B_3^\top X_3 B_3\| \leq 1$ : it is unclear whether or not this inequality is less conservative than the  
 499 one in Assumption 4 c).

500 We conclude that while the relaxation of Assumption 4 a) is reasonable, the relaxation of Assumption  
 501 4 c) is not obvious.

### 502 5.3 CASE $X_1 > 0$ AND $B_3^\top X_3 B_3 > 0$

503 This is the case studied in item 4 of Proposition 5.1 and may be verified in practice, see the example in  
 504 section 6.1. Then the DI in (5.3) is equivalently rewritten as in (5.8). Consequently one can apply the  
 505 implicit Euler scheme and all the existence, convergence and order results presented in [14] (see that  
 506 article for details on the discretization method and theoretical results), to (5.8). It is inferred that in  
 507 this case (5.8) possesses unique solutions in  $C^0([0, T]; \mathbb{R}^{n_1+m})$  with essentially bounded derivatives [14,  
 508 Proposition 2.6]. It is deduced that  $x_2(\cdot)$  is also unique in  $C^0([0, T]; \mathbb{R}^{n_1+m})$  with essentially bounded  
 509 derivative, while the algebraic variable  $z(\cdot)$  (which depends on  $\lambda(\cdot)$ , see (5.1)) is essentially bounded  
 510 (hence *a priori* it may be discontinuous).

## 511 6 EXAMPLES AND NUMERICAL SIMULATIONS

512 Two examples of circuits with ideal diodes, and one academic example, are developed in this section.  
 513 Let us comment that the analysis of circuits applies to hydraulics, due to the strong analogy between  
 514 hydraulic and electrical circuits [68, 56] [52, Chapter 1], where check valves play the role of diodes.

### 515 6.1 FIRST EXAMPLE

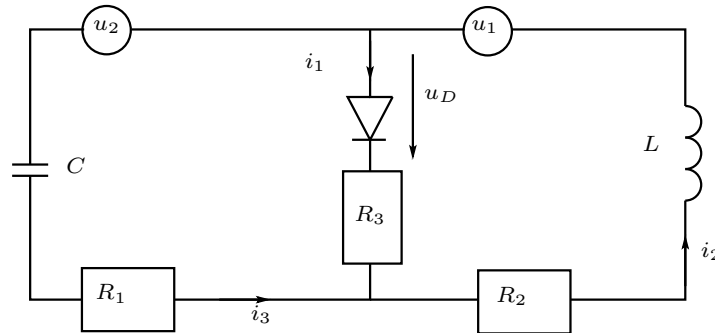


Figure 1: An RLCD passive circuit.

516 Let us consider the circuit in Figure 1. Let us assume first that  $R_1 = R_3 = 0 \Omega$ . Let  $\xi_1 = i_2$ ,  $\xi_2 =$   
 517  $\int_0^t i_3(t) dt$ ,  $\xi_3 = i_1$ . The dynamics is given by:

$$(6.1) \quad \begin{cases} \dot{\xi}_1(t) = -\frac{R_2}{L} \xi_1(t) + \frac{1}{L} u_D + \frac{1}{L} u_1(t) \\ \dot{\xi}_2(t) = \xi_1(t) - \xi_3(t) \\ 0 = \frac{1}{C} \dot{\xi}_2(t) + u_D(t) + u_2(t) \\ 0 \leq u_D \perp \xi_3(t) \geq 0. \end{cases}$$

518 It is noteworthy that this circuit dynamics agrees with [2, Theorem 3.6] [48] about the index (indeed it  
 519 cannot be of index 1 due to the loop with a capacitive element). In our notations we have  $x = (\xi_1, \xi_2)^\top$ ,  
 520  $z = \xi_3$ ,  $C = (0 \ 0 \ 1)$ ,  $B = (\frac{1}{L} \ 0 \ 1)^\top$ . The transfer function  $H(s) = \frac{z(s)}{u_D(s)} = \frac{Cs^2 + \frac{CR}{L}s + \frac{1}{L}}{s + \frac{R}{L}} = Cs + \frac{1}{Ls + R}$ . It has a



521 relative degree  $r = -1$ . It can be checked that  $H(s)$  is positive real (PR) as long as  $R_2 > 0$  [25, Theorem  
 522 2.45], but it is not strictly positive real (SPR) (the first condition in item 3.b in [25, Theorem 2.61] does  
 523 not hold, despite of the fact that the second condition in the same item holds), and hence it is not  
 524 strongly SPR. Let us verify the minimality of (6.1):

$$525 \quad 1. \text{rank}((A \ B) - sP \ B) = \text{rank} \begin{pmatrix} \frac{-R}{L} - s & 0 & 0 & \frac{1}{L} \\ 1 & -s & -1 & 0 \\ 0 & \frac{1}{C} & 0 & 1 \end{pmatrix} = 3 \text{ for all } s \in \mathbb{C}.$$

$$526 \quad 2. \text{rank}(P \ B) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{L} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3.$$

$$527 \quad 3. \text{rank}(A^\top - sP^\top \ C^\top) = \text{rank} \begin{pmatrix} \frac{-R}{L} - s & 1 & 0 & 0 \\ 0 & -s & \frac{1}{C} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 3 \text{ for all } s \in \mathbb{C}.$$

$$528 \quad 4. \text{rank}(P^\top \ C^\top) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3.$$

$$529 \quad 5. (A \ B) \text{Ker}(P) = \begin{pmatrix} \frac{-R}{L} & 0 & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{C} & 0 \end{pmatrix} \text{Ker}(P) = \{\xi \in \mathbb{R}^3 \mid \xi_1 = \xi_3 = 0, \xi_2 \in \mathbb{R}\} \subset \text{Im}(P) = \{\xi \in \\ 530 \quad \mathbb{R}^3 \mid \xi_3 = 0, \xi_2 \in \mathbb{R}, \xi_1 \in \mathbb{R}\},$$

531 Therefore we infer that (6.1) admits a special Weiertrass form as (2.2). Let us choose  $L = 4.7 \cdot 10^{-4}$

$$532 \quad H, R = 1\Omega, C = 10^{-1} \text{ F. Let us define } S = \begin{pmatrix} -1.414 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-4} \end{pmatrix}, T = \begin{pmatrix} -0.707 & 0 & 0 \\ 0 & 0 & 1 \\ -0.707 & -1 & 0 \end{pmatrix}. \text{ Then } SAT =$$

$$533 \quad \begin{pmatrix} -2.12 \cdot 10^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, SB = \begin{pmatrix} -3 \cdot 10^{-3} \\ 0 \\ 10^{-4} \end{pmatrix}, CT = (-0.707 \ -1 \ 0), S \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.301 \cdot 10^4 & 0 \\ 0 & 0 \\ 0 & 10^{-4} \end{pmatrix}. \text{ Let us make}$$

534 the state variable change  $\zeta = T^{-1}\xi$ , then we obtain the transformed dynamics:

$$(6.2) \quad \begin{cases} \dot{\zeta}_1(t) = -2.12 \cdot 10^3 \zeta_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ \dot{\zeta}_3(t) = \zeta_2(t) \\ 0 = \zeta_3(t) + 10^{-4} \lambda(t) + 10^{-4} u_2(t) \\ 0 \leq \lambda(t) \perp w(t) = -0.707 \zeta_1(t) - \zeta_2(t) \geq 0. \end{cases}$$

535 Let us now define  $x_1 = \zeta_1, x_2 = \zeta_3, x_3 = \zeta_2$ , then we obtain

$$(6.3) \quad \begin{cases} \dot{x}_1(t) = -2.12 \cdot 10^3 x_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ \dot{x}_2(t) = x_3(t) \\ 0 = x_2(t) + 10^{-4} \lambda(t) + 10^{-4} u_2(t) \\ 0 \leq \lambda(t) \perp w(t) = -0.707 x_1(t) - x_3(t) \geq 0, \end{cases}$$

536 which is the canonical form in (2.2) with  $z = x_3, B_1 = -3 \cdot 10^{-3}, B_2 = 0, B_3 = 10^{-4}, C_1 = -0.707,$   
 537  $C_2 = 0, C_3 = -1, D = 0$ . We have  $X_3 = 10^4, A_1 = -2.12 \cdot 10^3, X_1 = \frac{C_1}{B_1} > 0, \hat{D} = D - C_3 B_2 - C_2 B_3 = 0,$   
 538  $\bar{D}_h = h^2 C_1 B_1 + B_3^\top X_3 B_3 = 2.121 \cdot 10^{-3} h^2 + 10^{-4}, h_3 = B_3^\top X_3 B_3 = 10^{-4}, h_1 = |1 + h A_1| = |1 - 2.12 \cdot 10^3 h|$   
 539 which verifies Assumption 4 for  $h > 0$  small enough. Notice that the set  $\mathcal{S}$  in Propositions 3.2 and 5.1

540 verifies  $\mathcal{S} = \{0\}$ , hence the OSNSP is well-posed with unique solution. Provided that Assumption 4 is  
 541 satisfied, the material in section 5 applies. In a similar way to (5.1), the system (6.3) is transformed into:

$$(6.4) \quad \begin{cases} \dot{x}_1(t) = -2.12 \cdot 10^3 x_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ 10^{-4} \dot{\lambda}(t) = 0.707 x_1(t) - 10^{-4} \dot{u}_2(t) + w(t) \\ x_2(t) = -10^{-4} \lambda(t) - 10^{-4} u_2(t) \\ z(t) = -10^{-4} \dot{\lambda}(t) - 10^{-4} \dot{u}_2(t). \end{cases}$$

542 From (5.14), the following LCP is obtained:

$$(6.5) \quad \begin{cases} 0 \leq 10^{-4} \lambda_{k+1} + q_k \perp \lambda_{k+1} \geq 0 \\ q_k = -10^{-4} \lambda_k - 0.707 h x_{1,k} - 10^{-4} h u_{2,k}, \end{cases}$$

543 and  $x_{1,k+1}, x_{2,k+1}, z_{k+1}$  as in (H.1). Let us check Assumptions 1 and 4 c). Since  $\hat{D} = 0$  it is easy to show that  
 544  $\eta = 10^4$  while  $h_3 = 10^{-4}$ , using the LCP in (5.14) that can be solved explicitly. Thus  $\eta h_3 \leq 1$  for all  $h \geq 0$ .  
 545 However,  $h_1 = \|1 - 2.12 \cdot 10^3 h\| \leq 1$  if and only if  $h < 4.7 \cdot 10^{-4}$ . It is also noteworthy that this system  
 546 verifies the conditions of section 5.3, and item 4 of Proposition 5.1, since  $X_1 > 0$  and  $B_3^\top X_3 B_3 = 10^{-4} > 0$ .  
 547 Hence the state space form (5.8), that is equivalent to (5.3) since  $\mathcal{P} > 0$ , can be calculated. Also the  
 548 generalized equation (5.14) always has a unique solution  $\lambda_{k+1}$ . As we shall see through numerical  
 549 simulations, the variable  $z(\cdot) = x_3(\cdot)$  may be discontinuous. Actually this is in agreement with the  
 550 theoretical conclusions from section 5.3, which do not state that this variable is continuous. It also  
 551 shows that a complete convergence analysis, including  $z$ , should rely on other tools than those used in  
 552 this paper.

553 In Figure 2, the results of the numerical simulation of (6.3) with an implicit Euler scheme are depicted<sup>2</sup>.  
 554 Numerical simulations are performed with the INRIA software package SICONOS [1]. At each time step,  
 555 the Lemke solver [65, 64] is applied to solve the LCP (3.1) resulting from the discretization scheme. The  
 556 initial conditions are set to  $\mathbf{x}(0) = (0.01, -10^{-4}, -7.07 \cdot 10^{-3})^\top$ , and the time step is  $h = 5 \cdot 10^{-6}$ s. The  
 557 input functions  $u_1(t)$  and  $u_2(t)$  are taken as sinusoidal functions:  $u_1(t) = \sin(t)$  and  $u_2(t) = \cos(10t)$ .  
 558 Let us observe that the variable  $z(t) = x_3(t)$  appears to be discontinuous at  $t \approx 5.9$ s. The multiplier  
 559  $\lambda(t)$  remains continuous, see Figure 3. Finally, an order of convergence equal to one is deduced from  
 560 the data in Figure 4. The error formula used is:

$$(6.6) \quad \text{error} = \|\mathbf{x}(t) - \mathbf{x}^h(t)\|_\infty.$$

561

562 **Remark 6.1 (Modelling).** One can obtain a relative degree one LCS by eliminating the variable  $u_D(t)$   
 563 in (6.1). However, for the sake of generality, and since in practice equality constraints are not always  
 564 eliminated, we consider here a poorly formulated version of the circuit's dynamics, where the variables  
 565 are badly matched to the constraints. Such bad matching could be avoided using structural analysis.

## 566 6.2 SECOND EXAMPLE

567 This circuit example depicted in Figure 5 is inspired from [32]. The so-called modified nodal analysis is  
 568 used to derive the dynamics [2, chapter 3]. It is assumed that the voltage source delivers a constant volt-  
 569 age  $u(t) = u_c$ , while the other voltage source  $u_1(t)$  is time-varying Lipschitz continuous and bounded.  
 570 Notice that the current/voltage characteristic of this device can be written as  $i_5(t) \in -\partial\psi_{\{u_c\}}(u(t))$ , or

<sup>2</sup>Source code of the examples can be found at <https://github.com/siconos/siconos-tutorials/tree/master/sandbox/mlcspaper>

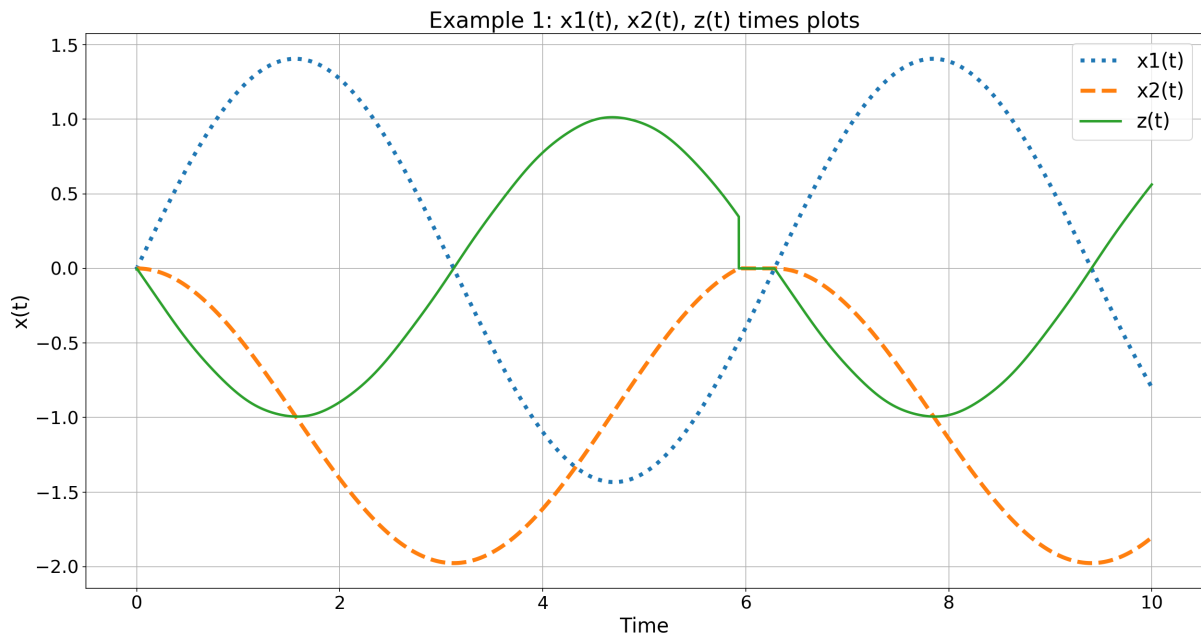


Figure 2: Implicit Euler numerical simulation of (6.3) with  $h = 5 \cdot 10^{-6}$ s. The numerical simulations of the variables  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t) = z(t)$  are depicted.

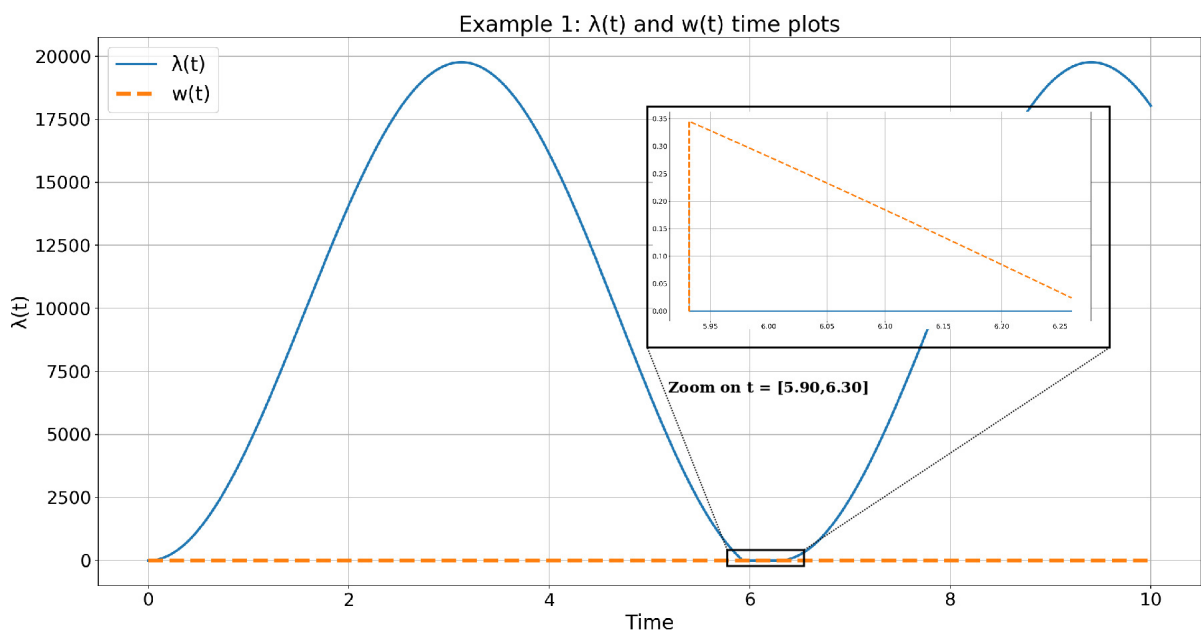


Figure 3: Implicit Euler numerical simulation of (6.3) with  $h = 5 \cdot 10^{-6}$ s. The numerical simulations of the variables  $\lambda(t)$  and  $w(t)$  are depicted.

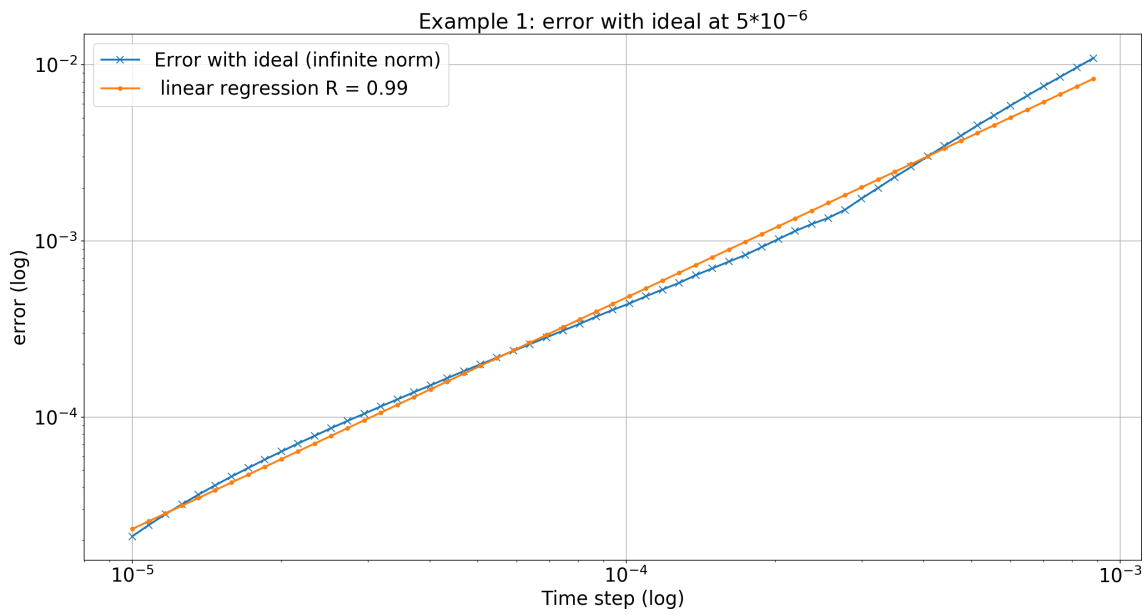


Figure 4: Numerical convergence of the implicit Euler scheme applied to (6.3). The reference curve is taken for  $h = 5 \cdot 10^{-6}$  s. The error formula used is given in (6.6).

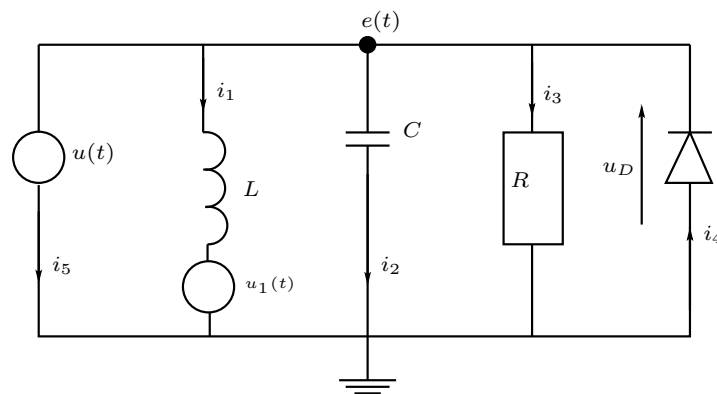


Figure 5: An RLCD passive circuit.

571 equivalently as  $\mathbb{R} \ni i_5(t) \perp u(t) - u_c \in \{0\}$ . The dynamics is given by:

$$(6.7) \quad \begin{cases} \mathbf{C}\dot{e}(t) = -R^{-1}e(t) + i_4(t) - i_1(t) - i_5(t) \\ L\frac{di_1}{dt}(t) = e(t) + u_1(t) \\ 0 = e(t) + u(t) \\ 0 \leq u_D(t) \perp i_4(t) \geq 0 \\ \mathbb{R} \ni i_5(t) \perp u(t) - u_c \in \{0\}. \end{cases}$$

572 The state vector is  $\xi = (\xi_1, \xi_2, \xi_3)^\top = (e, i_1, i_5)^\top$ , and  $u_D(t) = e(t)$ . The complementarity conditions can  
573 be written as  $K = \mathbb{R}^+ \times \mathbb{R} \ni (u_D(t), i_5(t)) \perp (i_4(t), u(t) - u_c) \in \mathbb{R}^+ \times \{0\} = K^*$ . Setting  $\lambda_1 = i_4$  and

574  $\lambda_2 = u(t) - u_c$ , we obtain  $P = \begin{pmatrix} \mathbf{C} & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $A = \begin{pmatrix} -R^{-1} & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $D = 0$ ,

575 and the **LDCS** form in (2.1) is equal to:

$$(6.8) \quad \begin{cases} \dot{\xi}_1(t) = -\frac{1}{RC}\xi_1(t) - \frac{1}{C}\xi_2(t) - \frac{1}{C}\xi_3(t) + \frac{1}{C}\lambda_1(t) \\ \dot{\xi}_2(t) = \frac{1}{L}\xi_1(t) + \frac{u_1(t)}{L} \\ 0 = \xi_1(t) + \lambda_2(t) + u_c \\ 0 \leq \xi_1(t) \perp \lambda_1(t) \geq 0 \\ \mathbb{R} \ni \xi_3(t) \perp \lambda_2(t) \in \{0\}. \end{cases}$$

576 Let  $S = \begin{pmatrix} 0 & \frac{1}{L} & -\frac{1}{L} \\ -1 & 0 & -\frac{1}{R} \\ 0 & 0 & -\mathbf{C} \end{pmatrix}$ ,  $T = \begin{pmatrix} 0 & 0 & -\frac{1}{C} \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ , then  $SAT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $SPT = \begin{pmatrix} \frac{1}{L} & 0 & 0 \\ 0 & 0 & \frac{1}{C} \\ 0 & 0 & 0 \end{pmatrix}$ ,  $CT = \begin{pmatrix} 0 & 0 & -\frac{1}{C} \\ -1 & 1 & 0 \end{pmatrix}$ ,

577  $SB = \begin{pmatrix} 0 & -\frac{1}{L} \\ 1 & -\frac{1}{R} \\ 0 & -\mathbf{C} \end{pmatrix}$ . Using the state variable change  $\zeta = T^{-1}\xi$  with  $T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -\mathbf{C} & 0 & 0 \end{pmatrix}$ , one obtains:

$$(6.9) \quad \begin{cases} \dot{\zeta}_1(t) = -(\lambda_2(t) + u_c(t)) + \frac{u_1(t)}{L} \\ \frac{1}{C}\dot{\zeta}_3(t) = \zeta_2(t) + \lambda_1(t) - \frac{1}{R}(\lambda_2(t) + u_c(t)) \\ 0 = -\mathbf{C}(\lambda_2(t) + u_c(t)) + \zeta_3(t) \\ K^* \ni \lambda(t) \perp w(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \zeta_1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta_2(t) + \begin{pmatrix} -\frac{1}{C} \\ 0 \end{pmatrix} \zeta_3(t) \in K. \end{cases}$$

578 Let  $x_1 = \zeta_1$ ,  $x_2 = \frac{1}{C}\zeta_3$ ,  $z = \zeta_2$ ,  $B_1 = (0 \ -1)$ ,  $B_2 = (1 \ -\frac{1}{R})$ ,  $B_3 = (0 \ -1)$ ,  $C_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ,  $C_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $C_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ,

579  $D = 0$ . This dynamics is rewritten in the canonical form as:

$$(6.10) \quad \begin{cases} \dot{x}_1(t) = -\lambda_2(t) - u_c(t) + \frac{u_1(t)}{L} \\ \dot{x}_2(t) = z(t) + \lambda_1(t) - \frac{1}{R}(\lambda_2(t) + u_c(t)) \\ 0 = -\lambda_2(t) - u_c(t) + x_2(t) \\ K^* \ni \lambda(t) \perp w(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_2(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z(t) \in K. \end{cases}$$

580 In the simulations depicted in Figures 6a and 6b, it has been chosen  $u_c(t) = \sin(10t) - 1 \leq 0$ , which  
581 complies with the system's intrinsic constraints. Then (when  $u_1(t) = 0$ ) one recovers the standard  
582 form in (2.2), with a different class of sets  $K$ , however. One has  $A_1 = 0$ ,  $B_3^\top X_3 = -C_3$  with  $X_3 = 1$ , and

583  $X_1 B_1 = C_1^\top$  (this follows necessarily from (D.2) and  $Q_1 \leq 0$  and using [25, Lemma A.69]), with  $X_1 = 1$   
 584 (hence the quadruple  $(A_1, B_1, C_1, \hat{D})$  is passive but it is not strictly passive and its associated transfer  
 585 function is not strongly SPR). Also  $\hat{D} = -C_3 B_2 - C_2 B_3 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{R} \end{pmatrix} \geq 0$ . Also  $B_3^\top X_3 B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \geq 0$ .

586 It is computed that  $\bar{D}_h = \begin{pmatrix} 0 & h \\ -h & \frac{h}{R} + h^2 + 1 \end{pmatrix}$ . The various sets for the set  $\mathcal{S}$  in Proposition 3.2 are  
 587 calculated as:  $K^* = \mathbb{R}^+ \times \{0\}$ ,  $\text{Ker}(\bar{D}_h + \bar{D}_h^\top) = \begin{pmatrix} 0 & 0 \\ 0 & 2(\frac{h}{R} + h^2 + 1) \end{pmatrix} = \mathbb{R} \times \{0\}$ ,  $\{w \mid \bar{D}_h w \in K =$   
 588  $\mathbb{R}^+ \times \mathbb{R}\} = \mathbb{R} \times \mathbb{R}^+$ . So  $S = \mathbb{R}^+ \times \{0\} \neq 0$  and item 2 of Proposition 3.2 has to be applied. It is calculated  
 589 that  $\langle h\beta_k - \bar{D}_h \xi_0, v \rangle = \langle \begin{pmatrix} -h\xi_0^1 \\ -hx_{1,k} - x_{2,k} + h\xi_0^1 - (\frac{h}{2} + h^2 + 1)\xi_0^2 \end{pmatrix}, v \rangle$ . Also  $v \in K^*$  implies that  $v_1 \geq 0$  and  
 590  $v_2 = 0$ , so it follows that  $\langle h\beta_k - \bar{D}_h \xi_0, v \rangle + \psi_{K^*}(v) > 0$  is guaranteed, if  $v_1 > 0$ , by taking  $\xi_0^1 < 0$ . It is  
 591 inferred that item 2 in Proposition 3.2 is verified and the OSNSP has at least one solution.

592 Any two solutions  $\lambda_{k+1}^1$  and  $\lambda_{k+1}^2$  satisfy  $\lambda_{2,k+1}^1 = \lambda_{2,k+1}^2$  for all  $h \geq 0$ , in view of the form of  $\bar{D}_h$ . Then  
 593 both  $x_{1,k+1}$  and  $x_{2,k+1}$  are unique due to the form of  $B_1$  and  $B_3$ . For the convergence analysis, let us note  
 594 that this system does not satisfy Assumption 2, therefore let us use the material in section 5. Using the

595 Frobenius norm one finds that  $h_3 = \sqrt{2h^2 + \left(\frac{h}{R} - 1\right)^2}$ , using the Hölder induced norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$

596 one finds a smaller  $h_3 = \sqrt{h^2 + \left(\frac{h}{R} - 1\right)^2}$ , and  $0 \leq h_3 \leq 1$  for all  $h \in [0, \frac{2R}{1+R^2}]$ . The material in section 5 is  
 597 based on the LCP in (5.14), for which the existence of a minimal norm solution satisfying the inequalities  
 598 stated below (5.14) is guaranteed. However in this example  $K = \mathbb{R}_+ \times \mathbb{R} \neq \mathbb{R}_+^2$ . The following problem  
 599 has to be studied instead (it is noteworthy that the rest of the analysis remains unchanged as long as  
 600 the minimal norm solution can be characterised):

$$(6.11) \quad K \ni B_3^\top X_3 B_3 \lambda_{k+1} + q_k^\lambda \perp \lambda_{k+1} \in K^*,$$

601 (where  $q_k^\lambda$  is defined in (5.14)), i.e.,

$$(6.12) \quad \mathbb{R}_+ \times \{0\} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} 0 \\ \lambda_{2,k+1} \end{pmatrix} + q_k^\lambda \in \mathbb{R}_+ \times \mathbb{R}.$$

602 Therefore  $\lambda_{2,k+1} = 0$ , and  $\lambda_{1,k+1} = 0$  is always a solution (notice that the external excitation is constrained  
 603 to be non negative to comply with the system's constraints). This result agrees with the numerical  
 604 simulations in Figure 6b.

### 605 6.3 THIRD EXAMPLE

606 Let us consider now an academic toy-example directly in the form of (2.2), whose dynamics are given  
 607 as:

$$(6.13) \quad \begin{cases} \dot{x}_1(t) = -2x_1(t) + 2\lambda_1(t) - \lambda_2(t) \\ \dot{x}_2(t) = z(t) \\ 0 = x_2(t) + \lambda_1(t) - 2\lambda_2(t) \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} 2 \\ -1 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 3 \end{pmatrix} x_2(t) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} z(t) + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \geq 0 \end{cases}$$

608 It can be verified that the transfer matrix  $H(s) = D + C(sP - A)^{-1}B$  is positive real [25, Definition  
 609 2.34], and the state-space realization is minimal (see section G). In fact  $H(s)$  is even strongly (or  
 610 extended) positive real [25, Definition 2.78]. Thus [28, Theorem 5.4 (A') (B')] applies. It is noteworthy



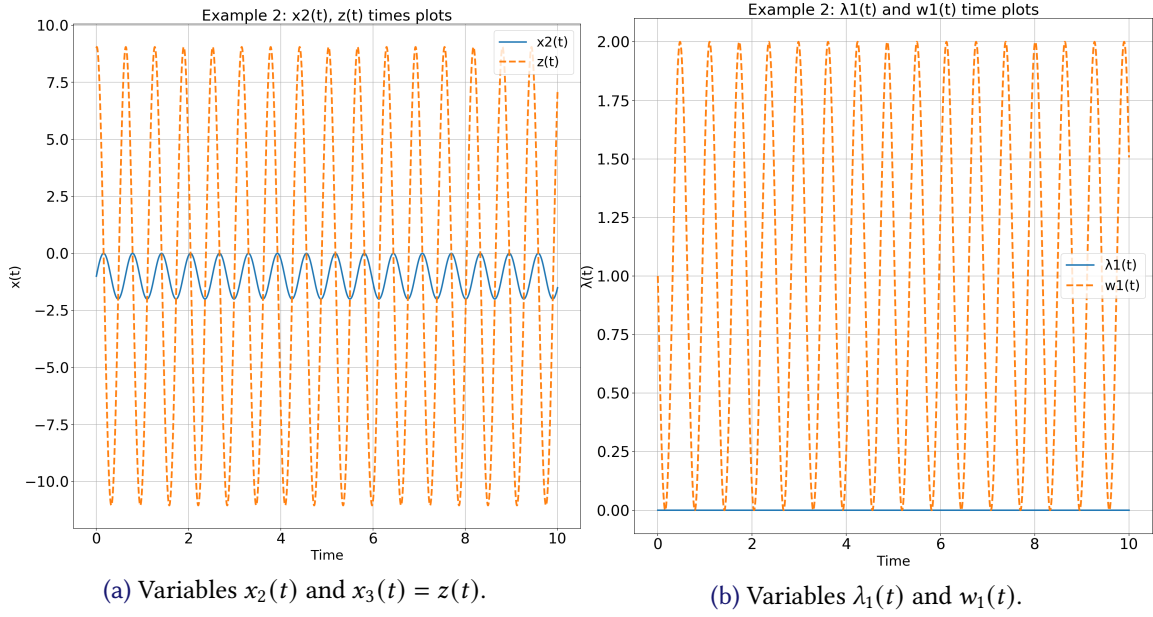


Figure 6: Implicit Euler numerical simulation of (6.9) with  $h = 5 \cdot 10^{-6}$ s. The solutions  $x_1(t)$  are not displayed as they are independent of  $z(t)$ ,  $\lambda_1(t)$ . The variable  $\lambda_2(t)$  trivially equals 0.

611 that since  $D = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} > 0$ , it follows that the multipliers  $\lambda$  are Lipschitz continuous functions of  
 612 the state  $(x_1, x_2, z)$ . Thus the LDCS is a DAE with Lipschitz continuous nonlinear and nonsmooth  
 613 right-hand side. From (6.13) it follows that:  $A_1 = -2 < 0$ ,  $C_1 = X_1 B_1^T = (2, -1)^T$  with  $X_1 = 1 > 0$ ,  $X_3 = I_2$ ,  
 614  $B_3 = (1 \quad -2) = -C_3^T$ , and  $\hat{D} = \begin{pmatrix} 1 & 1 \\ -3 & 7 \end{pmatrix} > 0$  with  $(\hat{D} + \hat{D}^T) > 0$ . It follows from [28, Theorem 5.4  
 615 (A'), Theorem 4.6] that the quadruple  $(A_1, B_1, C_1, \hat{D})$  is strictly passive. It can be checked that the set  
 616  $\mathcal{S} = \{0\}$  in Proposition 3.2 for any  $h > 0$ , thus there exists a unique solution  $\lambda_{k+1}$  to the implicit Euler  
 617 discretization of (6.13). However  $C_3 \neq 0$  and Assumption 2 does not hold, preventing the convergence  
 618 analysis in section 4 and the stability analysis in section 4.1 to apply. Consequently, the "equivalent"  
 619 system (5.1) from Section 5 is analysed:

$$(6.14) \quad \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \end{pmatrix} = \begin{pmatrix} -2 & 2 & -1 \\ -2 & -1 & -1 \\ 1 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ w_1(t) \\ w_2(t) \end{pmatrix} \\ w(t) \in -\mathcal{N}_{\mathbb{R}_+^2}(\lambda(t)) \end{cases}$$

620 Let us check if Assumption 4 (c) is verified. Using the Frobenius norm, one can obtain that  $0 \leq h_3 =$   
 621  $\sqrt{60h^2 + 25 - 66h} \leq 1$  if  $0 < h \leq 0.7$ . In the context of (6.14), the LCP (5.14) becomes:

$$(6.15) \quad \begin{aligned} 0 \leq \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} + \begin{pmatrix} (h-1) & (h+3) \\ (2-3h) & (7h-4) \end{pmatrix} \begin{pmatrix} \lambda_{1,k} \\ \lambda_{2,k} \end{pmatrix} + \begin{pmatrix} 2hx_{1,k} \\ -hx_{1,k} \end{pmatrix} \geq 0 \\ \iff \\ 0 \leq \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \lambda_{1,k+1} - 2\lambda_{2,k+1} \\ 4\lambda_{2,k+1} - 2\lambda_{1,k+1} \end{pmatrix} + \begin{pmatrix} q_{1,k} \\ q_{2,k} \end{pmatrix} \geq 0. \end{aligned}$$

622 From Corollary 4.1, there exists a solution to (6.15). We can compute  $\lambda_{k+1}$  in each mode of the LCP  
 623 where  $\lambda_{k+1} \neq 0$ :

- 624 1. If  $\lambda_{1,k+1} = 0$  and  $\lambda_{2,k+1} \geq 0$  then  $\|\lambda_{2,k+1}\| \leq \frac{1}{2}\|q_{1,k}\|$ .
- 625 2. If  $\lambda_{2,k+1} = 0$  and  $\lambda_{1,k+1} \geq 0$  then  $\|\lambda_{1,k+1}\| \leq \frac{1}{2}\|q_{2,k}\|$ .
- 626 3. If  $\lambda_{1,k+1} > 0$  and  $\lambda_{2,k+1} > 0$  there is infinitely many solutions. Indeed, we obtain :

$$(6.16) \quad \begin{cases} \lambda_{1,k+1} = 2\lambda_{2,k+1} - q_{1,k} \\ 0 = 2q_{1,k} + q_{2,k} \end{cases}$$

627 From [16, Proposition 6.1.7], there exists a minimal solution  $\lambda_{\min,k+1}$ . Then, (6.16)'s particular solution  
 628  $\lambda_{2,k+1} = -\lambda_{1,k+1}$  leads to  $\|\lambda_{\min,k+1}\| \leq (1/3)\|q_{k+1}\|$ . It follows that  $\eta = \frac{1}{2} < 1$  for all  $h > 0$ , and  $\eta h_3 \leq 1$   
 629 for all  $0 < h \leq 1.4$ , and finally, the results from Section 5.2 hold if small enough time steps are chosen.  
 630 Problem (6.14) is simulated using the discretization (6.15). The LCP (6.15) is solved using the numerical  
 631 solver Lemke provided in the SICONOS Toolbox<sup>3</sup> [1]. In Figures 7a and 7b the simulated variables  $(x_1, x_2, z)$   
 632 and  $(\lambda_1, \lambda_2)$  are displayed for time step  $h = 5 \cdot 10^{-6}$ s. The initial conditions are  $(x_{1,0} = 5, \lambda_{1,0} = 5, \lambda_{2,0} = 0)$   
 633 and the discrete variables  $x_{2,k}$  and  $z_k$  are given by the discretization of (5.1) (c):

$$(6.17) \quad \begin{cases} x_{2,k} = -B_3 \lambda_k \\ z_k = -B_3 \frac{\lambda_{k+1} - \lambda_k}{h}, \end{cases}$$

634 with  $h$  the time step. Let us notice in Figure 7a that the variable  $z$  possesses a discontinuity. In Figure  
 635 8a we plot the error provided in (6.6), with respect to a reference solution taken for  $h = 5 \cdot 10^{-6}$ s. We  
 636 confirm the order one convergence rate for  $h > 0$  sufficiently small. Finally, let us also notice that  
 637 the solutions of the schemes in (3.1) and in (5.5), are not exactly the same (recall that the continuous  
 638 systems (2.2) and (5.1) are "equivalent" in the sense explained after (5.3), see also section H). This can  
 639 be observed in Figure 8b where the differences between the two schemes are shown for  $(x_1, x_2)$ .

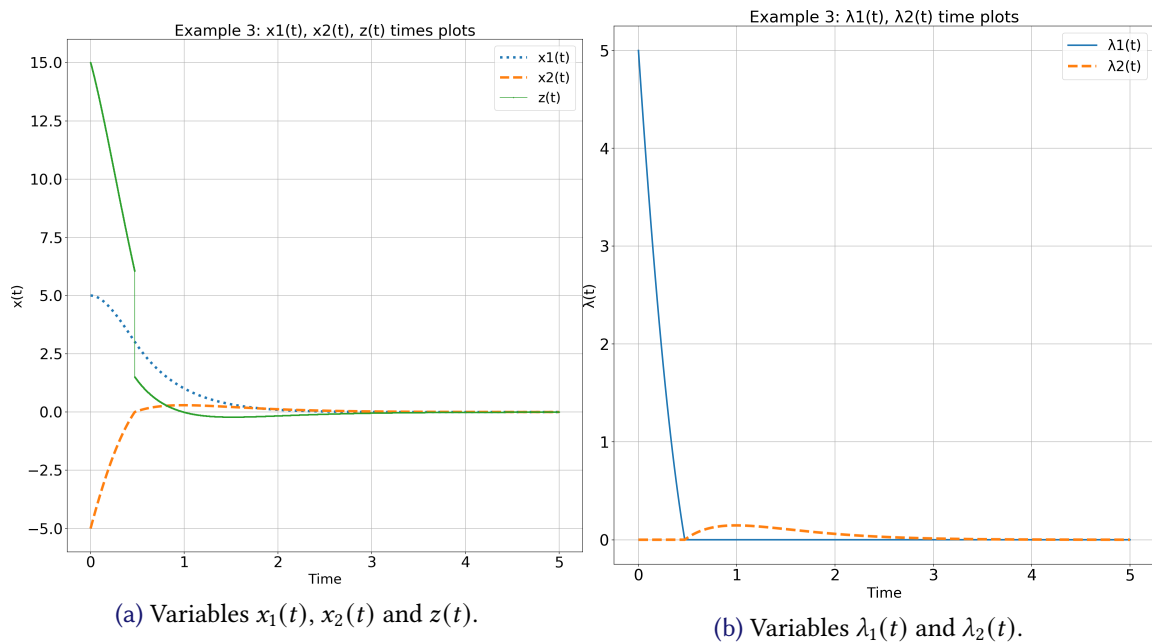
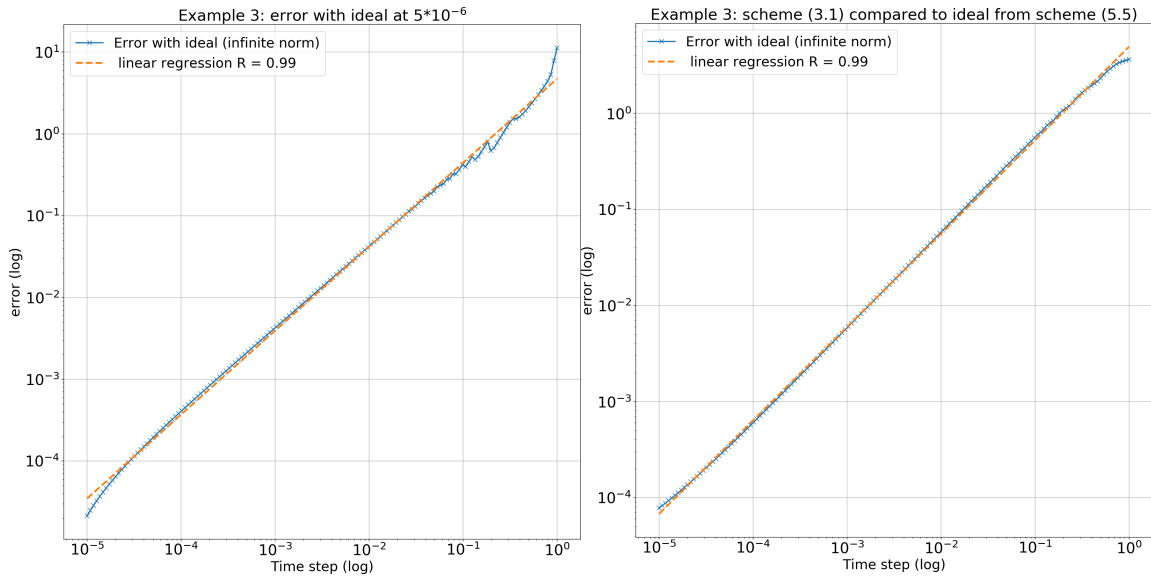


Figure 7: Implicit Euler numerical simulation of (6.13) with  $h = 5 \cdot 10^{-6}$ s

<sup>3</sup><https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>



(a) Convergence of scheme (5.5) applied to (6.14). (b) Convergence of scheme (3.1) applied to (6.13).

Figure 8: Numerical convergence studies, with error formulas (6.6), of schemes (3.1) and (5.5). The reference is computed with the scheme (5.5) and  $h = 5 \cdot 10^{-6}$ s.

## 640 7 CONCLUSION

641 This article deals with the time-discretization of a class of singular linear complementarity systems  
 642 (descriptor variable linear complementarity systems) which satisfy a passivity constraint. The implicit  
 643 Euler method is analysed. First the one-step nonsmooth problem is studied, and conditions which  
 644 guarantee its well-posedness are given. Then the approximate solutions convergence is analysed, and  
 645 it is shown that under some reasonable assumptions, convergence holds. Our results are however  
 646 partial in the sense that we are not able to prove the convergence of all the variables. One academic  
 647 example and two examples of electrical circuits with ideal diodes (equivalently hydraulic circuits with  
 648 check valves) illustrate the developments, with numerical simulations.

## 649 APPENDIX A PRESERVATION OF POSITIVE DEFINITENESS WITH PERTURBATION

650 We give here an excerpt of [37, Theorem 2.11], and a corollary of it. Let us recall that for a given  
 651  $M \in \mathbb{R}^{n \times n}$ ,  $\|M\|_{2,2}$  is the induced matricial norm such that  $\|M\|_{2,2} = \sigma_{\max}(M)$  (the largest singular  
 652 value).

**Theorem A.1.** [37] *Let  $M \in \mathbb{R}^{n \times n}$  be a positive definite matrix. Then every matrix*

$$A \in \left\{ A \in \mathbb{R}^{n \times n} \mid \left\| \left( \frac{M+M^T}{2} \right)^{-1} \right\|_{2,2} \|M - A\|_{2,2} < 1 \right\}$$

653 *is positive definite.*

654 **Corollary A.2.** [21] *Let  $D = P + N$ , where  $D, P$  and  $N$  are  $n \times n$  real matrices, and  $P > 0$ , not necessarily  
 655 symmetric. If*

$$(A.1) \quad \|N\|_{2,2} < \frac{1}{\left\| \left( \frac{P+P^T}{2} \right)^{-1} \right\|_{2,2}}$$

656 *then  $D > 0$ .*

657 Consider the term  $h_1$  in Assumptions 1 and 4. If  $h < \frac{1}{\sigma_{\max}(A_1)}$  then  $I_{n_1} + hA_1 > 0$  from Corollary A.2.  
 658 If in addition  $A_1 \preceq (<) 0$  then  $0 < I_{n_1} + hA_1 \preceq (<) I_{n_1}$ . Consider now  $\bar{A}_1 = (I_{n_1} + hA_1)^\top (I_{n_1} + hA_1) =$   
 659  $I_{n_1} + h(A_1 + A_1^\top) + h^2 A_1^\top A_1 = I_{n_1} + h\tilde{A}_1$ , with  $\tilde{A}_1 = A_1 + A_1^\top + hA_1^\top A_1 = \tilde{A}_1^\top$ . If  $A_1 < 0$  ( $\Rightarrow A_1 + A_1^\top < 0$ ), then  
 660  $\tilde{A}_1 < 0$  for  $h > 0$  small enough, still using Corollary A.2. Thus  $\bar{A}_1 < I_{n_1}$  for  $h > 0$  small enough. Moreover  
 661 for  $h > 0$  small enough it follows that  $\bar{A}_1 > 0$  by Corollary A.2. It is deduced that there exists  $h_{\max} > 0$   
 662 such that  $0 < \bar{A}_1 < I_{n_1}$  for all  $h \in (0, h_{\max})$ . From [16, Lemma 8.4.1],  $0 < \lambda_{\min}(\bar{A}_1)$  and  $\lambda_{\max}(\bar{A}_1) < 1$ . It  
 663 is inferred that  $\sigma_{\max}(I_{n_1} + hA_1) = \|I_{n_1} + hA_1\|_{2,2} = h_1 < 1$ .

## 664 APPENDIX B WELL-POSEDNESS OF VARIATIONAL INEQUALITIES

665 The next results use the notions of recession functions and cones, which we briefly introduce now (see  
 666 [24, 53] for illustrating examples), [72, 53, 89]. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex and lower  
 667 semi-continuous function, we denote by  $\text{dom}(f) \stackrel{\Delta}{=} \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$  the domain of the function  
 668  $f(\cdot)$ . The Fenchel transform  $f^*(\cdot)$  of  $f(\cdot)$  is the proper, convex and lower semicontinuous function  
 669 defined by

$$(B.1) \quad (\text{for all } z \in \mathbb{R}^n) \mid f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}.$$

The subdifferential  $\partial f(x)$  of  $f(\cdot)$  at  $x \in \mathbb{R}^n$  is defined by

$$\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq \langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}.$$

670 We denote by  $\text{Dom}(\partial f) \stackrel{\Delta}{=} \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$  the domain of the subdifferential operator  $\partial f : \mathbb{R}^n \rightarrow$   
 671  $\mathbb{R}^n$ . Recall that (see, e.g., Theorem 2, Chapter 10, Section 3 in [11]):  $\text{Dom}(\partial f) \subset \text{dom}(f)$ .

Let  $x_0$  be any element in the domain  $\text{dom}(f)$  of  $f(\cdot)$ , the recession function  $f_\infty(\cdot)$  of  $f(\cdot)$  is defined by

$$(\text{for all } x \in \mathbb{R}^n) : f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x).$$

672 The function  $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper convex and lower semicontinuous function which  
 673 describes the asymptotic behavior of  $f(\cdot)$ .

Let  $K \subset \mathbb{R}^n$  be a nonempty closed convex set. Let  $x_0$  be any element in  $K$ . The recession cone of  $K$  is defined by [72] [89, Definition 1.11]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

674 The set  $K_\infty$  is a nonempty closed convex cone that is described in terms of the directions which  
 675 recede from  $K$ . The indicator function of a set  $K \subseteq \mathbb{R}^n$  is  $\Psi_K(x) = 0$  if  $x \in K$ ,  $\Psi_K(x) = +\infty$  if  $x \notin K$ .  
 676 If  $K$  is closed non empty convex, we have  $\partial \Psi_K(x) = \mathcal{N}_K(x)$ , the so-called normal cone to  $K$  at  $x$ ,  
 677 defined as  $\mathcal{N}_K(x) = \{v \in \mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$ . When  $K$  is finitely represented, i.e.,  
 678  $K = \{x \in \mathbb{R}^n \mid k_i(x) \geq 0, 1 \leq i \leq m\}$ , and if the functions  $k_i(\cdot)$  satisfy some constraint qualification  
 679 (like, independency, or extensions like the MFCQ), then  $\mathcal{N}_K(x)$  is generated by the outwards normals  
 680 at the active constraints  $k_i(x) = 0$ , i.e.,  $\mathcal{N}_K(x) = \{v \in \mathbb{R}^n \mid v = -\lambda_i \nabla k_i(x), k_i(x) = 0, \lambda_i \geq 0\}$ .

681 Let us here recall some important properties of the recession function and recession cone (see e.g., [17,  
 682 Proposition 1.4.8]):

683 **Proposition B.1.** *The following statements hold:*

684 *a) Let  $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be two proper, convex and lower semicontinuous*  
 685 *functions. Suppose that  $f_1 + f_2$  is proper. Then for all  $x \in \mathbb{R}^n$ :  $(f_1 + f_2)_\infty(x) = (f_1)_\infty(x) + (f_2)_\infty(x)$ .*

686 *b) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function and let  $K$  be a nonempty  
687 closed convex set, such that  $f + \Psi_K$  is proper (equivalently  $\text{dom}(f) \cap K$  is non empty). Then for all  $x \in \mathbb{R}^n$ :  
688  $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$ .*

689 *c) Let  $K \subset \mathbb{R}^n$  be a nonempty, closed and convex set. Then for all  $x \in \mathbb{R}^n$ :  $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$ . Moreover  
690 for all  $x \in K$  and  $e \in K_\infty$ :  $x + e \in K$ .*

691 *d) If  $K \subseteq \mathbb{R}^n$  is a nonempty closed and convex cone, then  $K_\infty = K$ .*

692 *e) Let  $K = P(A, b) \triangleq \{x \in \mathbb{R}^n | Ax \geq b\}$  for  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ . If  $K \neq \emptyset$  then  $K_\infty = P(A, 0) = \{x \in$   
693  $\mathbb{R}^n | Ax \geq 0\}$ .*

694 *f)  $K \subset \mathbb{R}^n$  is a non-empty closed convex bounded set if and only if  $K_\infty = \{0_n\}$ .*

695 *g) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function. Then  $\text{epi}(f_\infty) =$   
696  $(\text{epi}(f))_\infty$ .*

697 Sets as in item e) are called  $\mathcal{H}$ -polyhedra, and there is an equivalence between sets  $P(A, 0)$  and finitely  
698 generated convex cones [89, Theorem 1.3]. Let us now concatenate [5, Theorem 3, Corollaries 3 and 4].  
699 They concern variational inequalities (VIs) of the form: Find  $u \in \mathbb{R}^n$  such that

$$(B.2) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in \mathbb{R}^n$$

700 where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is a real matrix,  $\mathbf{q} \in \mathbb{R}^n$  a vector and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  a proper convex and  
701 lower semicontinuous function. The VI in (B.2) is equivalent to the inclusion  $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u) \Leftrightarrow u \in$   
702  $(\mathbf{M} + \partial\varphi)^{-1}(-\mathbf{q})$ .

703 The problem in (B.2) is denoted as  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  in the next proposition. We also set:

$$(B.3) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in \mathbb{R}^n | \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

704 Note that  $(\text{dom}(\varphi_\infty))^*$  is the dual cone of the domain of the recession function  $\varphi_\infty$  while  $(\text{dom}(\varphi))_\infty$   
705 (that we may denote also as  $\text{dom}(\varphi)_\infty$ ) is the recession cone of  $\text{dom}(\varphi)$ .

706 **Proposition B.2.** [5] *Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function with  
707 closed domain,  $\mathbf{M} \in \mathbb{R}^{n \times n}$ , and suppose that  $\mathbf{M} \succcurlyeq 0$  (not necessarily symmetric).*

708 *a) If  $(\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$  then for each  $\mathbf{q} \in \mathbb{R}^n$ , problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at  
709 least one solution.*

710 *b) Suppose that  $(\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$ . If there exists  $x_0 \in \text{dom}(\varphi)$  such that*

$$(B.4) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \forall v \in (\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0,$$

711 *then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.*

712 *b') If  $\mathbf{M} = \mathbf{M}^\top$  then one can take  $x_0 = 0$  in b).*

713 *c) If  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  then  $u_1 - u_2 \in \ker\{\mathbf{M} + \mathbf{M}^\top\}$ .*

714 *d) If  $\mathbf{M} = \mathbf{M}^\top$  and  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , then  $\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$ .  
715*

716 *e) If  $\mathbf{M} = \mathbf{M}^\top$  and  $\varphi(x + z) = \varphi(x)$  for all  $x \in \text{dom}(\varphi)$  and  $z \in \ker\{\mathbf{M}\}$  and  $\langle \mathbf{q}, e \rangle \neq 0$  for all  
717  $e \in \ker\{\mathbf{M}\}, e \neq 0$ , then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at most one solution.*

718 *f) If  $\mathbf{M} = \mathbf{M}^\top$ , then  $u$  is a solution of  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  if and only if it is a solution of the optimization problem  
719  $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$ .*

720 Notice that the function  $\varphi(\cdot)$  will never be strictly convex in our case (it is an indicator function) so  
 721 that the strict convexity argument of [5, Theorem 5] which applies when  $\mathbf{M}$  is a  $P_0$ -matrix never holds.  
 722 The study of VIs as in (B.2) can be traced back to [76].

### 723 APPENDIX C SOME CONVEX ANALYSIS AND COMPLEMENTARITY THEORY TOOLS

724 If  $K \subset \mathbb{R}^n$  is a set, then  $K^\star = \{z \in \mathbb{R}^n \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$  is its dual cone. Its closure is denoted  
 725  $\bar{K}$ . Let  $K$  be a **nonempty** closed convex cone, then:

$$(c.1) \quad K^\star \ni x \perp y \in K \iff x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^\star}(x).$$

726 Let  $M = M^\top > 0$ ,  $x$  and  $y$  two vectors, then

$$(c.2) \quad M(x - y) \in -\mathcal{N}_K(x) \iff x = \text{proj}_M[K; y] \iff x = \min_{z \in K} \frac{1}{2}(z - y)^\top M(z - y).$$

727 We note that this is a particular case of (B.2), so that Proposition B.2 can be considered as the charac-  
 728 terization of a generalized projection operator  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ . The first equivalence is [15, Proposition  
 729 6.46].

### 730 APPENDIX D PASSIVE DESCRIPTOR VARIABLE SYSTEMS

731 **Dissipative systems have been formalized rigorously by J.C. Willems in his two seminal articles [87, 88].**  
 732 **Passive systems are a particular case of dissipative systems.** A quadruple  $(A, B, C, D)$  is said passive if  
 733 the next LMI is satisfied:

$$(D.1) \quad \begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \geq 0, \quad X = X^\top \geq 0.$$

734 It is said *strictly* passive if the inequalities are satisfied with  $> 0$ . This is not to be confused with the  
 735 strict *state* passivity [25, Definition 4.54], since strict passivity implies strict state passivity, but not the  
 736 inverse in general. In fact strict passivity is directly related to strongly SPR transfer matrices [25, p.62  
 737 and section 3.12.2], while SPR transfer matrices **with minimal realisations** are strictly state passive [25,  
 738 Theorem 4.73] [66], i.e.,  $-XA - A^\top X + \mu X \geq 0$  for some  $\mu > 0$ , and one may have  $D + D^\top \geq 0$  but not  
 739  $> 0$ .

740 Positive real transfer matrices [25, Definition 2.34] with a state space realization represented by  
 741 quintuples  $(P, A, B, C, D)$  which are minimal (see Appendix G for conditions of minimality of a state-  
 742 space realization for descriptor systems) can be rewritten as in (2.2) [28, 51], and they have the positive  
 743 semidefinite storage function  $V(x_1, x_2) = \frac{1}{2}(x_1^\top X_1 x_1 + x_2^\top X_3 x_2)$ , with supply rate  $\lambda^\top w$ . Indeed, by  
 744 differentiating  $V(x_1(t), x_2(t))$  along the trajectories of (2.2), adding  $\pm \lambda^\top w$  and completing the squares,  
 745 one can compute that:

$$(D.2) \quad \dot{V}(t) = \frac{1}{2}(x_1^\top, \lambda^\top) \underbrace{\begin{pmatrix} X_1 A_1 + A_1^\top X_1 & X_1 B_1 - C_1^\top \\ B_1^\top X_1 - C_1 & -(\hat{D} + \hat{D}^\top) \end{pmatrix}}_{\triangleq Q_1} \begin{pmatrix} x_1 \\ \lambda \end{pmatrix} + \lambda^\top(t) w(t),$$

746 where  $X_1 = X_1^\top \geq 0$ ,  $Q_1 \leq 0$ , and if the system is strongly SPR then  $X_1 > 0$ ,  $Q_1 < 0$  [28, Theorem  
 747 5.4]. The equality in (D.2) is an infinitesimal dissipation equality, showing that the system is passive  
 748 [25, Definition 2.1, Theorem 4.53]. Such systems possess particular structural properties [30, 33] [25,

749 section 3.8]. In the SSPR case it follows from (D.2) and the complementarity conditions that  $x_1(\cdot)$  and  
 750  $x_2^\top X_3 x_2(\cdot)$  are bounded for any bounded initial condition.

751 Let us recall some definitions, where  $A^*$  denotes the conjugate transpose of  $A$  (see [25, Chapter 2] for  
 752 details):

753 **Definition D.1.** A transfer matrix  $H(s) \in \mathbb{C}^{m \times m}$ ,  $s \in \mathbb{C}$ , is positive real (PR) if: 1)  $H(s)$  has no poles in  
 754  $\text{Re}(H(s)) > 0$ , 2)  $H(s)$  is real for positive real  $s$ , 3)  $H(s) + H^*(s) \succeq 0$  for all  $\text{Re}(s) > 0$ . Let  $H(s)$  be  
 755 regular, then it is strictly positive real (SPR) if  $H(s - \epsilon)$  is PR for some  $\epsilon > 0$ . It is strongly SPR (SSPR)  
 756 if 1)  $H(s)$  is analytic in  $\text{Re}(s) \geq 0$ , 2)  $\text{Re}(H(j\omega)) \geq \delta > 0$  for all  $\omega \in [-\infty, +\infty]$  and some real  $\delta$ .

## 757 APPENDIX E GRONWALL-BACHPATTE INEQUALITY [13, 47]

Let us consider a positive real numbers sequence  $\{x_k\}$  such that  $x_k \leq \alpha_k + \sum_{i=0}^{k-1} \beta_i x_i$ , where  $\{\alpha_k\}$  and  
 $\{\beta_k\}$  are two real numbers sequences with  $\{\beta_k\}$  positive. Then one has

$$x_k \leq \alpha_k + \sum_{i=0}^{k-1} \beta_i \alpha_i \prod_{j=i+1}^k (1 + \beta_j).$$

## 758 APPENDIX F CALCULATION OF $\beta_i$

759 The starting point is the term  $\eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$  in (5.19). Let us consider  
 760 the factors of the terms  $\|x_{1,j+1}\|$  separately:

761 • Factor of  $\|x_{1,1}\|$  (all terms with  $j = 0$ ):  $h_1^{k-1} + h_1^{k-2}(\eta h_3) + h_1^{k-3}(\eta h_3)^2 + \dots + h_1(\eta h_3)^{k-2} + (\eta h_3)^{k-1} =$   
 762  $\sum_{i=1}^k h_1^{k-i} (\eta h_3)^{i-1}$ .

763 • Factor of  $\|x_{1,2}\|$  (all terms with  $j = 1$ ):  $h_1^{k-2} + h_1^{k-3}(\eta h_3) + h_1^{k-4}(\eta h_3)^2 + \dots + h_1(\eta h_3)^{k-3} + (\eta h_3)^{k-2} =$   
 764  $\sum_{i=2}^k h_1^{k-i} (\eta h_3)^{i-2}$ .

765 • ...

766 • Factor of  $\|x_{1,k-1}\|$  (all terms with  $j = k - 2$ ):  $h_1 + (\eta h_3) = \sum_{i=k-1}^k h_1^{k-i} (\eta h_3)^{i-k+1}$ .

767 • Factor of  $\|x_{1,k}\|$  (all terms with  $j = k$ ):  $1 = \sum_{i=k}^k h_1^{k-i} (\eta h_3)^{i-k}$ .

768 Therefore one infers (5.22).

## 769 APPENDIX G MINIMALITY OF STATE SPACE REALIZATIONS

770 Let us report the results stated in [28, Theorem 3.1] [44, Theorems 2-2.1, 2-3.1, 2-6.3].

771 **Theorem G.1.** Let  $H(s) = C(sP - A)^{-1}B + D$ ,  $s \in \mathbb{C}$ , be a rational transfer function where  $P$  and  $A$  are  
 772  $n \times n$  matrices. Then  $P\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t) + Du(t)$  is a minimal realization of  $H(s)$  if,  
 773 and only if, the following conditions are satisfied:

774 •  $\text{rank}(A - sP \ B) = n$  for all  $s \in \mathbb{C}$ ,

775 •  $\text{rank}(P \ B) = n$ ,

776 •  $\text{rank}(A^\top - sP^\top \ C^\top) = n$ ,

777 •  $\text{rank}(P^\top \ C^\top) = n$ ,

778 •  $A \text{Ker}(P) \subseteq \text{Im}(P)$ .



## 779 APPENDIX H DIFFERENCES BETWEEN (3.1) AND IN (5.5)

780 Let us consider the discrete scheme (5.5) with  $E(t) = 0$  and  $X_1$  full rank, this is rewritten as:

$$(H.1) \quad \begin{cases} (a) & x_{1,k+1} = x_{1,k} + hA_1x_{1,k} + hB_1\lambda_k \\ (b) & B_3^T X_3 B_3 \lambda_{k+1} = B_3^T X_3 B_3 \lambda_k - hC_1x_{1,k} - h\hat{D}\lambda_k + w_{k+1} \\ (c) & 0 \leq \lambda_{k+1} \perp w_{k+1} = C_1x_{1,k+1} + C_2x_{2,k+1} + C_3z_{k+1} + D\lambda_{k+1} \geq 0 \\ (d) & x_{2,k} = -B_3\lambda_k \\ (e) & z_k = -B_2\lambda_k - B_3 \frac{\lambda_{k+1} - \lambda_k}{h}. \end{cases}$$

781 This is the numerical scheme with which the examples have been simulated. Combining (H.1) (d) and  
782 (e), gives:

$$(H.2) \quad x_{2,k+1} = x_{2,k} + hz_k + hB_2\lambda_k.$$

783 Notice that this is the choice made in (6.17). The equations in (H.1) (a) and (H.2) are not equal to their  
784 counterparts in (3.1), because  $\lambda_k$  and  $z_k$  are used in (H.1) (a) and (H.2) instead of  $\lambda_{k+1}$  and  $z_{k+1}$  in (3.1).  
785 This explains that the numerical data obtained from both schemes, differ. This also explains why the  
786 convergence results obtained for both schemes, are not the same neither.

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