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# ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF PASSIVE DESCRIPTOR-VARIABLE LINEAR COMPLEMENTARITY SYSTEMS 

Bernard Brogliato* Alexandre Rocca


#### Abstract

This article is largely concerned with the time-discretization of descriptor-variable systems coupled to with complementarity constraints. The are named descriptor-variable linear complementarity systems (DVLCS). More specifically passive DVLCS with minimal state space representation are studied. The Euler implicit discretization of DVLCS is analysed: the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-posed under some conditions. Then the convergence of the discretized solutions is studied. Several examples illustrate the applicability and the limitations of the developments.


## 1 INTRODUCTION

The analysis of non-smooth dynamical systems with set-valued right-hand sides satisfying maximal monotone properties has been the object of many contributions, see [26] for a survey. Linear and nonlinear complementarity dynamical systems, various kinds of projected dynamical systems, of differential variational inequalities, differential inclusions with maximal monotone right-hand sides, Moreau's sweeping processes, and some switching dynamics can be recast in this class of set-valued dynamical systems. In this article we focus on singular, or descriptor-variable linear complementarity systems (DVLCS). They may be viewed either as an extension of "classical" descriptor-variable systems which are widely used in Automatic Control and Circuit Theory, or of "classical" linear complementarity systems. They can also be seen as differential-algebraic equations (DAEs) coupled with complementarity constraints. Motivations for studying DVLCS are in circuits with set-valued components [2, Chapters 3 and 5] [32], and in a more abstract framework in state-dependent switching DAEs [68, 67], after a kind of convexification (or filling-in the graph) procedure has been made. They are also closely related to DAEs whose equality constraints are nonsmooth, as they occur in chemistry or thermo-dynamical systems [75, 76, 74].
It has to be noted that the study of DVLCS has not received a lot of attention yet, and is still at an embryonic stage of analysis. In particular the analysis of the implicit Euler method, which has received a lot of attention in several fields [ $10,14,54,52,2,3,64,66$ ], is an open issue for DVLCS. Our goal is to analyze the well-posedness of the one-step nonsmooth problem (OSNSP) obtained after an implicit (backward) Euler discretization, and the convergence of the discrete-time solutions towards a continuous-time limit which is a solution to the continuous-time system. The second step is useful

[^0]\[

\left\{$$
\begin{array}{l}
P \dot{x}(t)=A x(t)+B \lambda(t)+E(t), \text { a.e. } t \geq 0  \tag{2.1}\\
0 \leq \lambda(t) \perp w(t)=C x(t)+D \lambda(t)+F(t) \geq 0, \text { for all } t \geq 0
\end{array}
$$\right.
\]

with $x(t) \in R^{n}, \lambda(t) \in R^{m}, A, B, C, D$ constant matrices of appropriate dimensions, $P \in R^{n \times n}$ has rank $p<n$. Dissipativity is a fundamental property in Systems and Control [25]. The material in this article relies on a result in $[28,32,48]$ on a special Weierstrass form for passive descriptor variable systems [25, Section 3.1.7] which possess a minimal state-space realization [28, Theorem 3.1]. The passivity of the quintuple $(P, A, B, C, D)$ means the passivity of the operator $\lambda \mapsto w$, see section D . Assuming that $E(t)=0$ and $F(t)=0$, passivity and minimality, this special form writes as (see [48] [32, Equation (16)] [28, Proposition A.3]):

$$
\begin{align*}
& \text { (a) } \dot{x}_{1}(t)=A_{1} x_{1}(t)+B_{1} \lambda(t) \\
& \text { (b) } \dot{x}_{2}(t)=z(t)+B_{2} \lambda(t)  \tag{2.2}\\
& \text { (c) } 0=x_{2}(t)+B_{3} \lambda(t) \\
& \text { (d) } 0 \leq \lambda(t) \perp w(t)=C_{1} x_{1}(t)+C_{2} x_{2}(t)+C_{3} z(t)+D \lambda(t) \geq 0,
\end{align*}
$$

where the quadruple $\left(A_{1}, B_{1}, C_{1}, \hat{D}\right)$ is itself passive, $\hat{D} \triangleq D-C_{3} B_{2}-C_{2} B_{3}, x_{1}(t) \in R^{n_{1}}, x_{2}(t) \in R^{n_{2}}$, $z(t) \in R^{n_{2}}, n=n_{1}+2 n_{2}, \lambda(t) \in R^{m}$, and there exists $X_{3}=X_{3}^{\top} \geqslant 0$ such that $B_{3}^{\top} X_{3}=-C_{3}$. The LMI in (D.2) holds. Due to the passivity and the complementarity conditions which imply $\lambda(t)^{\top} w(t)=0$ for all times, it follows that for all $t \geq 0$ one obtains $V\left(x_{1}(t), x_{2}(t)\right) \triangleq \frac{1}{2}\left(x_{1}^{\top} X_{1} x_{1}+x_{2}^{\top} X_{3} x_{2}\right)(t) \leq V\left(x_{1}(0), x_{2}(0)\right)$, [32] and [28, Proof of Theorem 5.1], with $X_{1}>0$ if the system is strongly SPR and minimal [28, Theorem 5.4], see section D. In the latter case $x_{1}(\cdot)$ is bounded, $A_{1}$ is Hurwitz ${ }^{1}, \hat{D}>0$ and $B_{3}$ is full row rank $\left(\Rightarrow n_{2} \leq m\right)$ [28, Proposition A.4].

[^1]\[

$$
\begin{equation*}
P \dot{x}(t)-A x(t)-E(t) \in B\left(\mathcal{N}_{K^{\star}}+D\right)^{-1}(-C x(t)-F(t)) \tag{2.3}
\end{equation*}
$$

\]

with $K=K^{\star}=R_{+}^{m}$, which is, as pointed out in [26, section 3.16], a class of singular systems that has not yet received much attention. The same applies to (2.2) which is rewritten as:

$$
\left(\begin{array}{ccc}
I_{n_{1}} & 0 & 0  \tag{2.4}\\
0 & I_{n_{2}} & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t) \\
\dot{z}(t)
\end{array}\right)-\left(\begin{array}{ccc}
A_{1} & 0 & 0 \\
0 & 0 & I_{n_{2}} \\
0 & I_{n_{2}} & 0
\end{array}\right)\left(\begin{array}{c}
x_{1}(t) \\
x_{2}(t) \\
z(t)
\end{array}\right) \in\left(\begin{array}{l}
B_{1} \\
B_{2} \\
B_{3}
\end{array}\right)\left(\mathcal{N}_{K^{\star}}+D\right)^{-1}(-w(t)) .
$$

Exogenous terms can also be considered in (2.2) and (2.4), though the analyses in sections 3 and 4 are made without such terms. Also, an alternative way of analysing (2.2) is presented in section 5, incorporating exogenous signals. An example is treated in section 6.1 where exogenous terms are taken into account in the special form (2.2), see (6.3) and (6.4).

## 3 THE EULER IMPLICIT SCHEME AND THE OSNSP

The studied implicit Euler time-discretization of (2.2) is as follows:

$$
\left\{\begin{array}{l}
x_{1, k+1}=x_{1, k}+h A_{1} x_{1, k}+h B_{1} \lambda_{k+1}  \tag{3.1}\\
x_{2, k+1}=x_{2, k}+h z_{k+1}+h B_{2} \lambda_{k+1} \\
0=x_{2, k+1}+B_{3} \lambda_{k+1} \\
0 \leq \lambda_{k+1} \perp w_{k+1}=C_{1} x_{1, k+1}+C_{2} x_{2, k+1}+C_{3} z_{k+1}+D \lambda_{k+1} \geq 0
\end{array}\right.
$$

where $A_{1} x_{1, k}$ could be replaced by $A_{1} x_{1, k+1}$ without modifying much the next developments. Nevertheless we choose to call the method in (3.1) an implicit method despite of the fact that the linear term is explicitly calculated, because our objective is to focus solely on the set-valued part of the system (i.e., the complementarity conditions). The set of equations in (3.1) makes a mixed linear complementarity problem (MLCP). After few calculations from (3.1) the following OSNSP is obtained (where the variable $h w_{k+1}$ is used instead of $\left.w_{k+1}\right)$ as an LCP:

$$
0 \leq \lambda_{k+1} \perp \bar{D}_{h} \lambda_{k+1}+h \beta_{k} \geq 0 \quad \Leftrightarrow \quad \bar{D}_{h} \lambda_{k+1}+h \beta_{k} \in-\mathcal{N}_{R_{+}^{m}}\left(\lambda_{k+1}\right)
$$

$$
\begin{equation*}
\Leftrightarrow \quad \lambda_{k+1} \in\left(\bar{D}_{h}+\mathcal{N}_{R_{+}^{m}}\right)^{-1}\left(-h \beta_{k}\right), \tag{3.2}
\end{equation*}
$$

with $\bar{D}_{h} \triangleq h \hat{D}+h^{2} C_{1} B_{1}+B_{3}^{\top} X_{3} B_{3}, \beta_{k} \triangleq C_{1}\left(I_{n_{1}}+h A_{1}\right) x_{1, k}-\frac{1}{h} C_{3} x_{2, k}$. It is noteworthy that $z_{k+1}$ is eliminated and does not appear in $\beta_{k}$. The inclusion (3.2) is a VI as (в.2) with $\varphi(\cdot)=\psi_{R_{+}^{m}}(\cdot)$. Obviously the well-posedness of the OSNSP depends on the properties of $\bar{D}_{h}$. From the passivity we know that $\hat{D} \geqslant 0$ and $B_{3}^{\top} X_{3} B_{3} \geqslant 0$. Using Theorem A. 1 and Corollary A. 2 it is possible to characterize the set of admissible $h C_{1} B_{1}$ that do not destroy the positive definiteness in case $\hat{D}>0$.
Remark 3.1. We do not investigate here whether or not the discrete time system (3.1) is dissipative. Such a study is made in [52] for the case of LCS discretized with a $\theta$-method.

It follows from [28, Theorem 5.4] that if the quintuple ( $P, A, B, C, D$ ) is strongly positive real (which is a strong form of strict passivity), then $\hat{D}>0$. As alluded to in Remark 2.2, the cone $R_{+}^{m}$ can be replaced in (2.1), (2.2) and (3.1) by any closed non empty convex cone $K \subset R^{m}$ and its dual cone $K^{\star}$. Then the complementarity conditions become $K^{\star} \ni \lambda(t) \perp w(t)=C_{1} x_{1}(t)+C_{2} x_{2}(t)+C_{3} z(t)+D \lambda(t) \in K$, yielding the inclusion
(3.3)

$$
\begin{aligned}
& \bar{D}_{h} \lambda_{k+1}+h \beta_{k} \in-\mathcal{N}_{K^{\star}}\left(\lambda_{k+1}\right) \\
& \Leftrightarrow \lambda_{k+1} \in\left(\bar{D}_{h}+\mathcal{N}_{K^{\star}}\right)^{-1}\left(-h \beta_{k}\right)
\end{aligned}
$$

instead of (3.2). One can also consider time-varying terms in (2.2), in both the dynamics and the complementarity conditions, and adapt $\beta_{k}$ accordingly. Thus in the next proposition a more general complementarity framework is considered where the positive orthant is replaced by a closed convex cone.

Proposition 3.2. Let $h>0$ and $k \geq 0$ be given. Assume that $\bar{D}_{h} \geqslant 0$, not necessarily symmetric, and let $K \subset R^{m}$ be a non empty closed convex cone (where the complementarity conditions in (2.2) are extended to $\left.K^{\star} \ni \lambda(t) \perp w(t)=C_{1} x_{1}(t)+C_{2} x_{2}(t)+C_{3} z(t)+D \lambda(t) \in K\right)$. Let $\mathcal{S}=K^{\star} \cap \operatorname{Ker}\left(\bar{D}_{h}+\bar{D}_{h}^{\top}\right) \cap\{w \in$ $\left.R^{m} \mid \bar{D}_{h} w \in K\right\}$. Then:

1. If $\mathcal{S}=\{0\}$ then for each $h \beta_{k}$ the LCP in (3.2) has at least a solution.
2. If $\mathcal{S} \neq\{0\}$, and there exists $\xi_{0} \in R^{m}$ such that $\left\langle h \beta_{k}-\bar{D}_{h}^{\top} \xi_{0}, v\right\rangle+\psi_{K^{\star}}(v)>0$ for all $v \neq 0, v \in \mathcal{S}$, then the LCP in (3.2) has at least a solution.
3. If $\lambda_{k+1}^{1}$ and $\lambda_{k+1}^{2}$ are two solutions of the LCP, then $\lambda_{k+1}^{1}-\lambda_{k+1}^{2} \in \operatorname{Ker}\left(\bar{D}_{h}+\bar{D}_{h}^{\top}\right)$.

Proof: Let us consider Proposition в.2. Comparing (3.3) and (в.2) one has $\varphi(\cdot)=\psi_{K^{\star}}(\cdot), \mathbf{M}=\bar{D}_{h}$, $\mathrm{q}=h \beta_{k}$. Using Proposition B.1 c) d), it follows that $(\operatorname{dom}(\varphi))_{\infty}=K^{\star}$. The second set in $\mathcal{S}$ is obvious. The third set necessitates the calculation of $\left(\operatorname{dom}\left(\varphi_{\infty}\right)\right)^{\star}$, see (в.3). Again the result of the calculation follows from Proposition B. 1 c) and d). Then the proof follows from Proposition B. 2 a), b) and c). $\boxtimes$

Once the existence of $\lambda_{k+1}$ is established for any $k \geq 0$, then the existence of $x_{1, k+1}, x_{2, k+1}$ and $z_{k+1}$ can be inferred from (3.1) for any $h>0$.
Remark 3.3. If $K=R_{+}^{m}$ then other results like [40, Theorem 3.8.6] may be used to study the LCP (3.2), using for instance the copositivity of $\bar{D}_{h}$.

## 4 BOUNDEDNESS AND CONVERGENCE ANALYSIS

From now on, we assume that the complementarity conditions are given as in (2.2) and (3.1), i.e., $K=K^{\star}=R_{+}^{m}$. Let us state a consequence of Proposition 3.2.
Corollary 4.1. Let the quintuple ( $P, A, B, C, D$ ) be strongly positive real. Then the LCP in (3.2) has a unique solution for any $h \beta_{k}$, for $h>0$ small enough.

Proof: This follows from [28, Theorem 5.4] ( $\Rightarrow \hat{D}>0$ ), the fact that $B_{3}^{\top} X_{3} B_{3} \geqslant 0$, Corollary A. 2 $\left(\Rightarrow \bar{D}_{h}>0\right.$ for $\left.0<h<h^{*} \triangleq \frac{1}{\left\|C_{1} B_{1}\right\|_{2}\left\|\left(\frac{\dot{D}+\hat{D}^{\top}}{2}\right)^{-1}\right\|_{2}}\right)$, and items 1 and 3 in Proposition 3.2. $\boxtimes$ The next result characterizes least-norm solutions of the LCP.
Corollary 4.2. Let the conditions of Proposition 3.2 hold true, and assume that the conditions of item 1 or of item 2 for existence of a solution are satisfied. Then the set of solutions to the LCP in (3.2) is convex and contains a unique least-norm element $\lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right)$ such that $\left\|\lambda_{k+1}^{\min }\right\| \leq\left\|\lambda_{k+1}\right\|$ for any other solution $\lambda_{k+1}$. Moreover there exists a constant $\eta>0$ depending only on $\bar{D}^{h}$ such that $\left\|\lambda_{k+1}^{\min }\right\| \leq \eta h\left\|\beta_{k}\right\|$.
for all $k \in\{0, \ldots, n-1\}$. The above results explore the well-posedness of the OSNSP, for fixed $h>0$. Our goal is now to analyse the boundedness of the iterates on the integration interval $[0, T], T>0$. For all $k \geq 1$ we have $x_{2, k}=-B_{3} \lambda_{k}^{\min }$, and $h z_{k+1}=-B_{3} \lambda_{k+1}^{\min }-x_{2, k}-h B_{2} \lambda_{k+1}^{\min }$, thus it is inferred that $x_{2, k+1}=-B_{3}\left(\lambda_{k+1}^{\min }+\lambda_{k}^{\min }\right)-x_{2, k}$. Consequently (4.1) is rewritten equivalently as:

$$
\left\{\begin{array}{l}
x_{1, k+1}=\left(I_{n_{1}}+h A_{1}\right) x_{1, k}+h B_{1} \lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right)  \tag{4.2}\\
x_{2, k+1}=-x_{2, k}-B_{3}\left(\lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right)+\lambda_{k}^{\min }\left(x_{1, k-1}, x_{2, k-1}\right)\right) \\
0 \leq \lambda_{k+1}^{\min } \perp w_{k+1}=C_{1}\left(I_{n_{1}}+h A_{1}\right) x_{1, k}-\left(C_{2}+\frac{1}{h} C_{3}\right) x_{2, k}+\frac{1}{h} \bar{D}_{h} \lambda_{k+1}^{\min }-C_{2} B_{3} \lambda_{k}^{\min } \geq 0 \\
0=x_{2, k+1}+B_{3} \lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right), \\
h z_{k+1}=-B_{3} \lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right)-x_{2, k}-h B_{2} \lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right) \\
\left\|\lambda_{k+1}^{\min }\left(x_{1, k}, x_{2, k}\right)\right\| \leq \eta\left\|h \beta_{k}\right\|, \quad h \beta_{k}=h C_{1}\left(I_{n_{1}}+h A_{1}\right) x_{1, k}-C_{3} x_{2, k}, \quad \eta>0
\end{array}\right.
$$

for all $k \in\{1, n-1\}$, given initial data $x_{1,0}$ and $x_{2,0}$. Starting from (4.2) the next result is obtained: Lemma 4.4. Let $\alpha_{1, k} \triangleq\left\|x_{1, k}\right\|, \alpha_{2, k} \triangleq\left\|x_{2, k}\right\|, \alpha_{k}=\left(\begin{array}{ll}\alpha_{1, k} & \alpha_{2, k}\end{array}\right)^{\top}$, then:

$$
\left\{\begin{array}{l}
\alpha_{k+1} \leq\left(M_{i} M+M_{i-1} N\right) \alpha_{k-i}+M_{i} N \alpha_{k-i-1} \\
M_{i+1}=M_{i} M+M_{i-1} N, M_{0}=I_{2}, M_{-1}=0,0 \leq i \leq k-1, k \geq 1
\end{array}\right.
$$

$$
\alpha_{k+1} \leq\left(M_{k-1} M+M_{k-2} N\right) \alpha_{1}+M_{k-1} N \alpha_{0}
$$

for all $k \geq 1$.
Proof: The proof starts from (4.2). The following upperbounds can be derived:

$$
\begin{align*}
\left\|\xi_{k}^{\min }\right\| \leq & \eta h\left\|C_{1}\right\|\left\|I_{n_{1}}+h A_{1}\right\|\left\|x_{1, k}\right\|+\eta\left\|C_{3}\right\|\left\|x_{2, k}\right\| \\
\left\|x_{1, k+1}\right\| \leq & \left\|I_{n_{1}}+h A_{1}\right\|\left(1+h^{2} \eta\left\|B_{1}\right\|\left\|C_{1}\right\|\right)\left\|x_{1, k}\right\|+h \eta\left\|B_{1}\right\|\left\|C_{3}\right\|\left\|x_{2, k}\right\|  \tag{4.5}\\
\left\|x_{2, k+1}\right\| \leq & \left(1+\eta\left\|B_{3}\right\|\left\|C_{3}\right\|\right)\left\|x_{2, k}\right\|+\eta\left\|B_{3}\right\|\left\|C_{3}\right\|\left\|x_{2, k-1}\right\| \\
& +h \eta\left\|B_{3}\right\|\left\|C_{1}\right\|\left\|I_{n_{1}}+h A_{1}\right\|\left(\left\|x_{1, k}\right\|+\left\|x_{1, k-1}\right\|\right) .
\end{align*}
$$

One infers that:

$$
\begin{equation*}
\alpha_{k+1} \leq M \alpha_{k}+N \alpha_{k-1} \tag{4.6}
\end{equation*}
$$

The upperbound in (4.3) therefore holds for $i=0$. Let us prove by induction that it holds for all $0 \leq i \leq k-1$. Let it hold for $i>0$, we have $\alpha_{k-i} \leq M \alpha_{k-i-1}+N \alpha_{k-i-2}$. Thus $\alpha_{k+1} \leq\left(M_{i} M+\right.$ $\left.M_{i-1} N\right) M \alpha_{k-i-1}+\left(M_{i} M+M_{i-1} N\right) N \alpha_{k-i-2}+M_{i} N \alpha_{k-i-1}=\left(M_{i+1} M+M_{i} N\right) \alpha_{k-i-1}+M_{i+1} N \alpha_{k-i-2}$. The result is proved. $\boxtimes$
Our goal is now to examine (4.4). Let us come back to the matrices $M$ and $N$ in Lemma 4.4. It is clear from (4.4) that the boundedness of the sequence $\left\{\alpha_{k}\right\}_{k \geq 2}$ depends on the boundedness of $M_{k}$, $k \geq 1$. Apart from $d$ and $f$, all the other terms in $M$ and $N$ are proportional to $h$ or to $h_{1} \triangleq\left\|I_{n_{1}}+h A_{1}\right\|$. Moreover $d>1$ whenever $B_{3} \neq 0$ and $C_{3} \neq 0$. The (2,2) term of $M^{n}$ is equal to $d^{n}+O\left(h_{1} h^{2}\right)$. We are led to make the following assumptions:
Assumption 1. The matrix $A_{1}$ is such that $h_{1} \leq 1$ for $h>0$ small enough.
Let us notice that the induced norm $\|\cdot\|_{2,2}=\sigma_{\max }(\cdot)$ [16, Proposition 9.4.9] is compatible with the vector norm $\|\cdot\|_{2}=\|\cdot\|\left[16\right.$, pp.6o7-6o9] , and can be used instead of the Frobenius norm to compute $h_{1}$. The fact that small enough timestep is needed stems from the explicit discretization of the term $A_{1} x_{1}$ in (2.2) and is not essential in our developments since, as alluded to above, it is the implicit discretization of the set-valued terms which is our focus. The next assumption is crucial in the body of this section. Assumption 2. One has $B_{3, \bullet} \in \operatorname{Ker}\left(X_{3}\right)$ for all $1 \leq i \leq m\left(\Longrightarrow C_{3}=0\right)$.

Then $d=1$ and $f=0$. This assumption means that the variable $z$ does not enter the complementarity conditions, see (2.2). Hence in a sense the complementarity part of the DVLCS and the algebraic part, are decoupled. The matrices $M_{k}$ then involve only products of powers of the terms $a, b, c, d=1, e$. These products are bounded for bounded $k$, and provided that both $h$ and $h_{1}$ are small enough so that $a \leq 1, b \leq 1, c \leq 1, d=1, e \leq 1$, they remain bounded as $k \rightarrow+\infty$. If the inequalities are strict (apart form $d=1$ ), the first rows in both $M_{k-1} M+M_{k-2} N$ and $M_{k-1} N$ converge to zero, while the second rows converge to $(0,1)$ (the 1 being the consequence of a term $d^{n}$ in matrices $M^{n}$ ). The next proposition is therefore proved:
Proposition 4.5. Assume that the conditions of existence in Proposition 3.2 are verified. Let Assumptions 1 and 2 hold true, and let both $h$ and $h_{1}$ be small enough so that $a \leq 1, b \leq 1, c \leq 1, d=1, e \leq 1$. Then the sequences $\left\{\alpha_{1, k}\right\}_{k}$ and $\left\{\alpha_{2, k}\right\}_{k}$ are uniformly bounded in $k$. If $A_{1}<0$, and $h>0$ is small enough such that $a<1, b<1, c<1, e<1$, then $\alpha_{1, k} \rightarrow 0$ as $k \rightarrow+\infty$.

Recall that $\alpha_{1, k}=\left\|x_{1, k}\right\|$ and $\alpha_{2, k}=\left\|x_{2, k}\right\|$. The last assertion of the proposition follows from the material in Appendix A. It is important to notice that all the terms depending on $\eta$ that remain in (4.5), are of the form $h \eta$ or $h^{2} \eta$, hence they remain bounded even in case $\eta$ depends on $h^{-1}$. Then we have the next corollary.
Corollary 4.6. Let the conditions of Proposition 4.5 hold. Then $\left\|\lambda_{k+1}^{\min }\right\|$ is uniformly bounded in $k$ and $h>0$, and it converges to o as $k \rightarrow+\infty$ if $\alpha_{1, k} \rightarrow 0$ as $k \rightarrow+\infty$. Therefore the sequence $\left\{\frac{x_{1, k+1}-x_{1, k}}{h}\right\}_{k}$ is uniformly bounded also in $k$ and $h>0$.

Proof: The boundedness of $\left\|\lambda_{k+1}^{\min }\right\|$ and its convergence follow from its upperbound, see the last line in (4.2). From the first line in (4.2), one has $\frac{x_{1, k+1}-x_{1, k}}{h}=A_{1} x_{1, k}+B_{1} \lambda_{k+1}^{\min }$, hence the result is proved. $\boxtimes$ Let us now study the convergence of the piecewise-linear and step-function approximations defined as:
(4.7)

$$
\begin{cases}x_{1}^{h}(t)=x_{1, k+1}+\frac{t_{k+1}-t}{h}\left(x_{1, k}-x_{1, k+1}\right) \\ \dot{x}_{1}^{h}(t)=\frac{x_{1, k+1}-x_{1, k}}{h} \\ x_{1, h}^{\star}(t)=x_{1, k+1} & \text { for all } t \in\left[t_{k}, t_{k+1}\right)\end{cases}
$$

and similarly for the other variables.
Corollary 4.7. There exists subsequences denoted as $\left\{x_{1}^{h}(\cdot)\right\}_{h \geq 0}$ which converge uniformly to a limit $x_{1}(\cdot)$ in $C^{0}\left([0, T] ; R^{n_{1}}\right)$, such that almost everywhere one has $\dot{x}_{1}(t)=A_{1} x_{1}(t)+\lambda_{1}(t)$, for some function $\lambda_{1}(\cdot)$. Moreover the sequences $\left\{x_{2, h}^{\star}(\cdot)\right\}_{h \geq 0}$ and $\left\{\lambda_{h}^{\min , \star}(\cdot)\right\}_{h \geq 0}$ converge weaklyぇ in $\mathcal{L}_{\infty}\left([0, T] ; R^{l}\right), l=n_{2}$ and $l=m$, respectively, to limits $x_{2}(\cdot)$ and $\lambda(\cdot)$ which satisfy the continuous-time constraints.

Proof: Both $\left\{x_{1}^{h}(\cdot)\right\}_{h}$ and $\left\{\dot{x}_{1}^{h}(\cdot)\right\}_{h}$ are uniformly bounded on $[0, T]$. The sequence of $C^{0}\left([0, T] ; R^{n_{1}}\right)$ functions $\left\{x_{1}^{h}(\cdot)\right\}_{h}$ is thus equicontinuous. By the Arzela-Ascoli Theorem this sequence stays in a compact subset of $C^{0}\left([0, T] ; R^{n_{1}}\right)$ and converges uniformly towards a continuous limit $x_{1}(\cdot):[0, T] \rightarrow$ $R^{n_{1}}$ as $h \rightarrow 0, h>0$. Let us notice that $x_{1}(t) \rightarrow 0$ as $t \rightarrow+\infty$. Indeed from the above results which are stated for $k \in\{0, \ldots, n-1\}$ and $h>0, k \rightarrow+\infty$ only if $T=n h \rightarrow+\infty$. Let us now define the step functions $x_{1}^{h \star}(t)=x_{1, k+1}$ if $t \in\left[t_{k}, t_{k+1}\right), \lambda_{h \star}^{\min }(t)=\lambda_{k+1}^{\min }$ if $t \in\left[t_{k}, t_{k+1}\right)$, and

$$
\begin{equation*}
\dot{x}_{1}^{h}(t)=A_{1} x_{1}^{h \star}(t)+B_{1} \lambda_{h \star}^{\min }(t), \quad \text { almost everywhere } . \tag{4.8}
\end{equation*}
$$

In addition one has:
(4.9)

$$
\begin{aligned}
\left\|x_{1}^{h}(t)-x_{1}^{h \star}(t)\right\|_{\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)}^{2} & =\sum_{k=0}^{n-1} \int_{t_{k}}^{t_{k+1}}\left(t_{k+1}-t\right)^{2}\left\|\dot{x}_{1, k}\right\|^{2} d t \leq C^{2} \sum_{k=0}^{n-1} \int_{t_{k}}^{t_{k+1}}\left(t_{k+1}-t\right)^{2} d t \\
& \leq C^{2} \sum_{k=0}^{n-1} \frac{1}{3}\left(t_{k+1}-t_{k}\right)^{3}=\frac{C^{2} n h^{3}}{3}=\frac{C^{2} T h^{2}}{3}
\end{aligned}
$$

where $\dot{x}_{1, k}=\frac{x_{1, k+1}-x_{1, k}}{h}$, and $\left\|\dot{x}_{1, k}\right\| \leq C$ for all $k \geq 0$ for some $C>0$ from Corollary 4.6. It follows that $x_{1}^{h \star}(\cdot) \rightarrow x_{1}(\cdot)$ strongly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$ as $h \rightarrow 0$. On the other hand since $\dot{x}_{1}^{h}(\cdot) \in$ $\mathcal{L}_{\infty}\left([0, T] ; R^{n_{1}}\right)$ it follows from the Banach-Alaoglu-Bourbaki Theorem [18, Theorem III.15] that $\dot{x}_{1}^{h}(\cdot) \rightarrow$ $\dot{x}_{1, \lim }(\cdot)$ in the weak topology as $h \rightarrow 0$. Since the considered domain $[0, T]$ is bounded, one has $\mathcal{L}_{1}\left([0, T] ; R^{n_{1}}\right) \subseteq \mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$, thus $\dot{x}_{1}^{h}(\cdot)$ converges weakly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$. Moreover since $x_{1}^{h}(t)=x_{1}^{h}(0)+\int_{0}^{t} \dot{x}_{1}^{h}(t) d t$, one has $\dot{x}_{1, \lim }(\cdot)=\dot{x}_{1}(\cdot)$ almost everywhere. Therefore we deduce using (4.8) that $\left(B_{1} \lambda_{h \star}^{\min }\right)(\cdot)$ converges weakly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$ as $h \rightarrow 0$ towards some limit $\lambda_{1}(\cdot)$, and we have $\dot{x}_{1}(t)=A_{1} x_{1}(t)+\lambda_{1}(t)$ almost everywhere on $[0, T]$.
Let us consider now the step-functions $x_{2, h}^{\star}(\cdot)$ and $\lambda_{h}^{\min , \star}(\cdot)$ defined as $x_{2, h}^{\star}(t)=x_{2, k+1}$ for all $t \in$ [ $t_{k}, t_{k+1}$ ), and similarly for the multiplier. It follows from Proposition 4.5 and from Corollary 4.6
that both sequences $\left\{x_{2, h}^{\star}(\cdot)\right\}_{h \geq 0}$ and $\left\{\lambda_{h}^{\min , \star}(\cdot)\right\}_{h \geq 0}$ are uniformly bounded. Hence from the Banach-Alaoglu-Bourbaki Theorem, there exist subsequences of both which converge weakly $\star$ to limits $x_{2} \in \mathcal{L}_{\infty}\left([0, T] ; R^{n_{2}}\right)$ and $\lambda \in \mathcal{L}_{\infty}\left([0, T] ; R^{m}\right.$, respectively. We infer that $0=x_{2, h}^{\star}(t)+B_{3} \lambda_{h}^{\star}(t) \rightarrow$ $x_{2}(t)+B_{3} \lambda(t)$ for almost all $t \geq 0$ in the weak $\star$ sense. Next, using the second line in (4.1) it follows that the sequence $\left\{h z_{k}\right\}_{h \geq 0}$ is also uniformly bounded. Denoting $\sigma_{k}=h z_{k}$, it follows that the sequence $\left\{\sigma_{h}^{\star}\right\}_{h \geq 0}$ converges weakly $\star$ to a limit denoted as $\sigma(t) \in \mathcal{L}_{\infty}\left([0, T] ; R^{n_{2}}\right)$. Moreover one has $\sigma(t)=0$ almost everywhere since $x_{2, h}^{\star}(t)-x_{2, h}^{\star}\left(t-t_{k}\right)-h B_{2} \lambda_{h}^{\min , \star}(t)=\sigma_{h}^{\star}(t)$ almost everywhere in $[0, T]$. Finally, the complementarity conditions in (4.1) are equivalently rewritten as $0 \leq \lambda_{h}^{\star}(t) \perp C_{1} x_{1, h}^{\star}(t)+$ $C_{2} x_{2, h}^{\star}(t)+D \lambda_{h}^{\star}(t) \geq 0$ for all $t \in\left[t_{k}, t_{k+1}\right)$, since $C_{3}=0$ by Assumption 2. It is deduced that in the limit as $h \rightarrow 0$ one obtains $0 \leq \lambda(t) \perp C_{1} x_{1}(t)+C_{2} x_{2}(t)+D \lambda(t) \geq 0$. $\boxtimes$

### 4.2 STABILITY ANALYSIS VIA A LYAPUNOV FUNCTION

To complete the foregoing section, it is natural to analyze how the Lyapunov function $V\left(x_{1}, x_{2}\right)$ for the continuous-time system in (2.2) (see also appendix D), could be used in the discrete-time context for (4.2). Indeed implicit Euler methods are known to be able to mimic the continuous-time passivity and stability properties, in some cases [52]. It is noteworthy that Assumption 2 implies that $X_{3} x_{2, k}=-X_{3} B_{3} \lambda_{k}=0$ and $X_{3} x_{2}(t)=-X_{3} B_{3} \lambda(t)=0$ for all $k$ and $t$. Therefore $V\left(x_{1}, x_{2}\right)=\frac{1}{2} x_{1}^{\top} X_{1} x_{1}$, so that this Lyapunov function is not a positive definite function of the whole state. For the sake of completeness let us nevertheless make the calculations in the general setting. We set $V_{k} \triangleq \frac{1}{2} x_{1, k}^{\top} X_{1} x_{1, k}+x_{2, k}^{\top} X_{3} x_{2, k}$. It follows that:

$$
\begin{aligned}
V_{k+1}-V_{k}= & \frac{1}{2}\left(x_{1, k+1}-x_{1, k}\right)^{\top} X_{1}\left(x_{1, k+1}+x_{1, k}\right)+\left(x_{2, k+1}-x_{2, k}\right)^{\top} X_{3}\left(x_{2, k+1}+x_{2, k}\right) \\
= & \frac{1}{2}\left(h A_{1} x_{1, k}+h B_{1} \lambda_{k+1}\right)^{\top} X_{1}\left(2 x_{1, k}+h A_{1} x_{1, k}+h B_{1} \lambda_{k+1}\right) \\
& +\left(h z_{k+1}+h B_{2} \lambda_{k+1}\right)^{\top} X_{3}\left(2 x_{2, k}+h z_{k+1}+h B_{2} \lambda_{k+1}\right) \\
= & h\left(x_{1, k}^{\top} A_{1}^{\top} X_{1} x_{1, k}+\lambda_{k+1}^{\top} B_{1}^{\top} X_{1} x_{1, k}\right)+h^{2}\left(\frac{1}{2} x_{1, k}^{\top} A_{1}^{\top} X_{1} A_{1} x_{1, k}+\frac{1}{2} \lambda_{k+1}^{\top} B_{1}^{\top} X_{1} B_{1} \lambda_{k+1}\right) \\
& +h^{2} x_{1, k}^{\top} A_{1}^{\top} X_{1} B_{1} \lambda_{k+1}+h z_{k+1}^{\top} C_{3} \lambda_{k+1}+h \lambda_{k+1}^{\top} C_{3} B_{2} \lambda_{k+1}+h z_{k+1}^{\top} X_{3} x_{2, k} \\
& +h \lambda_{k+1}^{\top} B_{2}^{\top} X_{3} x_{2, k}+h \lambda_{k+1}^{\top} w_{k+1}-h \lambda_{k+1}^{\top} w_{k+1} \\
= & \frac{h}{2} x_{1, k}^{\top}\left(A_{1}^{\top} X_{1}+X_{1} A_{1}+h A_{1}^{\top} X_{1} A_{1}\right) x_{1, k}-h \lambda_{k+1}^{\top}\left(\hat{D}+h\left(C_{1} B_{1}-B_{2}^{\top} X_{3} B_{2}-\frac{1}{2} B_{1}^{\top} X_{1} B_{1}\right)\right) \lambda_{k+1} \\
& +h x_{1, k}^{\top}\left(B_{1}^{\top} X_{1}-C_{1}-h A_{1}^{\top} C_{1}^{\top}+h A_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}+\frac{h}{2} x_{2, k}^{\top} X_{3} B_{2} \lambda_{k+1} \\
& +\frac{h}{2} z_{k+1}^{\top} X_{3} x_{2, k}+h \lambda_{k+1}^{\top} w_{k+1}
\end{aligned}
$$

(4.10)

$$
=\frac{h}{2}\left(x_{1, k}^{\top}, \lambda_{k+1}^{\top}\right)\left(\begin{array}{cc}
A_{1}^{\top} X_{1}+X_{1} A_{1} & \left(B_{1}^{\top} X_{1}-C_{1}\right)^{\top} \\
B_{1}^{\top} X_{1}-C_{1} & -\left(\hat{D}+\hat{D}^{\top}\right)
\end{array}\right)\binom{x_{1, k}}{\lambda_{k+1}}
$$

$$
+\frac{h^{2}}{2} x_{1, k}^{\top} A_{1}^{\top} X_{1} A_{1} x_{1, k}-h^{2} \lambda_{k+1}^{\top}\left(C_{1} B_{1}-B_{2}^{\top} X_{3} B_{2}-\frac{1}{2} B_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}
$$

$$
+h^{2} x_{1, k}^{\top}\left(-A_{1}^{\top} C_{1}^{\top}+A_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}+\frac{h}{2} x_{2, k}^{\top} X_{3} B_{2} \lambda_{k+1}+\frac{h}{2} z_{k+1}^{\top} X_{3} x_{2, k}+h \lambda_{k+1}^{\top} w_{k+1}
$$

$$
=\frac{h}{2}\left(x_{1, k}^{\top}, \lambda_{k+1}^{\top}\right) Q_{1}\binom{x_{1, k}}{\lambda_{k+1}}+h \lambda_{k+1}^{\top} w_{k+1}
$$

$$
+h^{2}\left\{\frac{1}{2} x_{1, k}^{\top} A_{1}^{\top} X_{1} A_{1} x_{1, k}-\lambda_{k+1}^{\top}\left(C_{1} B_{1}-B_{2}^{\top} X_{3} B_{2}-\frac{1}{2} B_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}\right.
$$

$$
\left.+x_{1, k}^{\top}\left(-A_{1}^{\top} C_{1}^{\top}+A_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}\right\}+\frac{h}{2} x_{2, k}^{\top} X_{3} B_{2} \lambda_{k+1}+\frac{h}{2} z_{k+1}^{\top} X_{3} x_{2, k}
$$

(4.10)

$$
\begin{aligned}
= & \frac{h}{2}\left(x_{1, k}^{\top}, \lambda_{k+1}^{\top}\right) Q_{1}\binom{x_{1, k}}{\lambda_{k+1}}+h \lambda_{k+1}^{\top} w_{k+1} \\
& +h^{2}\left\{\frac{1}{2} x_{1, k}^{\top} A_{1}^{\top} X_{1} A_{1} x_{1, k}-\lambda_{k+1}^{\top}\left(C_{1} B_{1}-B_{2}^{\top} X_{3} B_{2}-\frac{1}{2} B_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}\right. \\
& \left.+x_{1, k}^{\top}\left(-A_{1}^{\top} C_{1}^{\top}+A_{1}^{\top} X_{1} B_{1}\right) \lambda_{k+1}\right\}-\underbrace{\frac{1}{2} x_{2, k}^{\top} X_{3} x_{2, k}+\lambda_{k+1}^{\top} B_{3}^{\top} X_{3} B_{3} \lambda_{k}}_{=0 \text { if Assumption 2 holds. }}
\end{aligned}
$$

where the last two terms are obtained using the second equality in (3.1). Remind that $\lambda_{k+1}^{\top} w_{k+1}=0$. Assume that $Q_{1}<0$, then let Assumption 2 hold. Let us rewrite (4.10) as $V_{k+1}-V_{k}=\frac{h}{2}\left(x_{1, k}^{\top}, \lambda_{k+1}^{\top}\right)\left[Q_{1}+\right.$ $\left.h Q_{2}\right]\binom{x_{1, k}}{\lambda_{k+1}}$, where the indefinite matrix
(4.11)

$$
Q_{2}=\left(\begin{array}{cc}
A_{1}^{\top} X_{1} A_{1} & -\frac{1}{2} A_{1}^{\top} C_{1}^{\top}+\frac{1}{2} A_{1}^{\top} X_{1} B_{1} \\
\left(-\frac{1}{2} A_{1}^{\top} C_{1}^{\top}+\frac{1}{2} A_{1}^{\top} X_{1} B_{1}\right)^{\top} & -C_{1} B_{1}+B_{2}^{\top} X_{3} B_{2}-\frac{1}{2} B_{1}^{\top} X_{1} B_{1}
\end{array}\right)
$$

and we remind that Assumption 2 implies that $V_{k}=\frac{1}{2} x_{1, k}^{\top} X_{1} x_{1, k}$. Using Corollary A.2, there exists $h_{\max }>0$ such that for all $h \in\left[0, h_{\max }\right)$ one has $Q_{3} \triangleq Q_{1}+h Q_{2}<0$. Using classical bounds on quadratic forms and vector/matrices products $\left(x^{\top} y \leq\|x\|\|y\| \leq \frac{1}{2}\|x\|^{2}+\frac{1}{2}\|y\|^{2}\right.$ [16, Corollary 9.1.7], if $Q=Q^{\top}$ then $\lambda_{\min }(Q)\|x\|^{2} \leq x^{\top} Q x \leq \lambda_{\max }(Q)\|x\|^{2}[16$, Corollary 8.4.2], $\|Q x\| \leq\|Q\|\|x\|$ for compatible norms [16, section 9.3]), and noting that we can replace $Q_{3}$ by its negative definite symmetric part in the quadratic form, it follows that:

$$
\begin{align*}
V_{k+1} & \leq V_{k}+\frac{h}{2} \lambda_{\max }\left(Q_{3}\right)\left\|\left(x_{1, k}, \lambda_{k+1}\right)\right\|^{2} \\
& \leq V_{k}+\frac{h}{2} \lambda_{\max }\left(Q_{3}\right)\left\|x_{1, k}\right\|^{2}  \tag{4.12}\\
& \leq V_{k}+\frac{h}{2} \frac{\lambda_{\max }\left(Q_{3}\right)}{\lambda_{\max }\left(X_{1}\right)} V_{k}=\left(1+\frac{h}{2} \frac{\lambda_{\max }\left(Q_{3}\right)}{\lambda_{\max }\left(X_{1}\right)}\right) V_{k}, \text { for all } k \geq 0
\end{align*}
$$

Since $\lambda_{\max }\left(Q_{3}\right)<0$ (which is the reason why the second inequality in (4.12) holds), one has $1+$ $\frac{h}{2} \frac{\lambda_{\max }\left(Q_{3}\right)}{\lambda_{\max }\left(X_{1}\right)}<1$ for all $h>0$. Therefore all $V_{k}$ are uniformly bounded for any bounded $V_{0}$, and $V_{k} \rightarrow 0$ as $k \rightarrow+\infty$ for any $h \in\left(0, h_{\max }\right)$. Let us set $\lambda_{k+1}=\lambda_{k+1}^{\min }$. Since the dependence of $\lambda_{k+1}^{\min }$ in $x_{2, k}$ through $\beta_{k}$ is $C_{3} x_{2, k}$ (see (4.2)), it follows from Assumption 2 that $\lambda_{k+1}^{\min }=\lambda_{k+1}^{\min }\left(x_{1, k}\right)$, and $\lambda_{k+1}^{\min }\left(x_{1, k}\right) \rightarrow 0$ as $k \rightarrow+\infty$. Using the third line in (4.2), we infer that the iterates $x_{2, k}$ (which are are uniformly bounded from Proposition 4.5 ) converge to zero asymptotically also. From the results in section 4.1 and the above Lyapunov analysis, we have therefore proved the following:
Lemma 4.8. Let Assumptions 1 and 2 hold, and the quintuple ( $P, A, B, C, D$ ) be strongly SPR and minimal. Then there exists $h_{\max }>0$ depending on $A_{1}, C_{1}, X_{1}, X_{3}, B_{1}, B_{2}$, such that for all $h \in\left(0, h_{\max }\right)$, and for all bounded initial condition $x_{1,0}$, the sequences $\left\{x_{1, k}\right\}_{k},\left\{x_{2, k}\right\}_{k},\left\{\lambda_{k}^{\min }\right\}_{k}$ are uniformly bounded and $x_{1, k}, x_{2, k}, \lambda_{k+1}^{\min } \rightarrow 0$ as $k \rightarrow+\infty$.

## 5 ANOTHER APPROACH FOR PASSIVE SYSTEMS

The DVLCS in (2.2) are suitably transformed in [32] to enable their analysis. The transformation may be seen as a kind of relative degree reduction, involving the multiplier derivative. As will become clear later, this new dynamics allows us to somewhat relax the stringent Assumption 2 (see Assumption 3 c) below). More accurately, a close relationship is shown in [32, Lemma 3, Theorem 3] between (2.2) and
the following system:
(a) $\left(\begin{array}{cc}I_{n_{1}} & 0 \\ 0 & B_{3}^{\top} X_{3} B_{3}\end{array}\right)\binom{\dot{x}_{1}(t)}{\dot{\lambda}(t)}=\left(\begin{array}{cc}A_{1} & B_{1} \\ -C_{1} & -\hat{D}\end{array}\right)\binom{x_{1}(t)}{\lambda(t)}+\binom{0}{I_{m}} w(t)$
(b) $0 \leq \lambda(t) \perp w(t)=C_{1} x_{1}(t)+C_{2} x_{2}(t)+C_{3} z(t)+D \lambda(t) \geq 0$
(c) $x_{2}(t)=-B_{3} \lambda(t), z(t)=-B_{2} \lambda(t)-B_{3} \dot{\lambda}(t)$.

It is noteworthy that this dynamics is obtained from the original one by treating the multiplier $\lambda$ as a state variable. The "equivalence" between both systems is to be understood first as follows [32, Lemma 3]:
$\left\{\left(x_{1}, \lambda, w\right) \mid\left(x_{1}, x_{2}, \lambda, w\right)\right.$ is a solution of $\left.(2.2)(a, b, c)\right\}=\left\{\left(x_{1}, \lambda, w\right) \mid\left(x_{1}, \lambda, w\right)\right.$ is a solution of (5.1) (a) and $B_{3} \lambda$ is absolutely continuous $\}$.

By solution of (2.2) it is meant that ( $x_{1}, x_{2}$ ) is absolutely continuous and the dynamics holds Lebesgue almost everywhere [32, Definition 3]. The initial conditions are chosen compatible with the constraints. It is noteworthy that the variables $z$ and $\lambda$ are not shown to be absolutely continuous, and examples in sections 6.1 and 6.3 will illustrate this fact. One has $w(t) \in-\mathcal{N}_{R_{+}^{m}}(\lambda(t))$, therefore one obtains the differential inclusion (DI):

$$
\left(\begin{array}{cc}
I_{n_{1}} & 0 \\
0 & B_{3}^{\top} X_{3} B_{3}
\end{array}\right)\binom{\dot{x}_{1}(t)}{\dot{\lambda}(t)}-\left(\begin{array}{cc}
A_{1} & B_{1} \\
-C_{1} & -\hat{D}
\end{array}\right)\binom{x_{1}(t)}{\lambda(t)} \quad \in-\binom{0}{I_{m}} \mathcal{N}_{R_{+}^{m}}(\lambda(t))
$$

(5.2)

$$
\begin{aligned}
& =-\binom{0}{I_{m}} \mathcal{N}_{\boldsymbol{R}_{+}^{m}}\left(\left(\begin{array}{ll}
0 & I_{m}
\end{array}\right)\binom{x_{1}(t)}{\lambda(t)}\right) \\
& =-\mathcal{N}_{\Phi}\left(\binom{x_{1}(t)}{\lambda(t)}\right)
\end{aligned}
$$

with $\Phi \triangleq\left\{\left(x_{1}, \lambda\right) \in R^{n_{1}} \times R^{m} \mid \lambda \geq 0\right\}$. If this DI has a solution $\left(x_{1}(\cdot), \lambda(\cdot)\right)$, then $x_{2}(\cdot), z(\cdot)$, and $w(\cdot)$ can be obtained (provided indeed that $B_{3} \lambda(\cdot)$ is absolutely continuous). Multiplying the first line in (5.2) by $X_{1}$ and adding an exogenous excitation $G(t)$ to the left-hand side, one obtains

$$
\underbrace{\left(\begin{array}{cc}
X_{1} & 0 \\
0 & B_{3}^{\top} X_{3} B_{3}
\end{array}\right)}_{\triangleq \mathcal{P}}\binom{\dot{x}_{1}(t)}{\dot{\lambda}(t)}-\underbrace{\left(\begin{array}{cc}
X_{1} A_{1} & X_{1} B_{1} \\
-C_{1} & -\hat{D}
\end{array}\right)}_{\triangleq \mathcal{A}}\binom{x_{1}(t)}{\lambda(t)}+G(t) \in-\mathcal{N}_{\Phi}\left(\binom{x_{1}(t)}{\lambda(t)}\right)
$$

1 which is similar to the form obtained in [32, Theorem 3]. Let us assume we add an exogenous input $V(t)=\left(v_{(a)}(t), v_{(b)}(t), v_{(c)}(t), v_{(d)}(t)\right)^{\mathrm{T}}$ to (2.2), then:

$$
G(t)=\binom{v_{(a)}(t)}{C_{3}\left(v_{(b)}(t)+\dot{v}_{(c)}(t)\right)+C_{2} v_{(c)}(t)-v_{(d)}(t)}
$$

Notice that $B_{3}^{\top} X_{3} B_{3} \geqslant 0$, therefore $\mathcal{P}=\mathcal{P}^{\top} \geqslant 0$. However $\mathcal{P}$ has a specific block-diagonal structure. Under strong SPRness and minimality, $\mathcal{A}<0$, since $\mathcal{A}+\mathcal{A}^{\top}<0$ from the KYP Lemma [25] (then $A_{1}$ is a Hurwitz matrix, $X_{1}=X_{1}^{\top}>0$ and $\left.\hat{D}>0\right)$. The exogeneous term $G(t)$ stems from the exogeneous terms $E(t)$ and $F(t)$ in (2.1) and their time-derivatives (see (6.4) for an example).
In a second step the "equivalence" between both systems is to be understood first as follows [32, Theorem 3]:
$\left\{\left(x_{1}, \lambda\right) \mid\left(x_{1}, x_{2}, \lambda, w\right)\right.$ is a solution of $\left.(2.2)(a, b, c, d)\right\}=\left\{\left(x_{1}, \lambda\right) \mid\left(x_{1}, \lambda\right)\right.$ is a solution of $(5 \cdot 3)$ and $B_{3} \lambda$ is absolutely continuous $\}$.

$$
\begin{equation*}
\mathcal{S} \triangleq\left\{\left(x_{1}, \lambda\right) \mid x_{1}=0, \lambda \in R_{+}^{m} \cap \operatorname{Ker}\left(B_{3}^{\top} X_{3} B_{3}\right)\right\} . \tag{5.6}
\end{equation*}
$$

Then:

1. If $\mathcal{S}=\{0\}$, then for each $q_{k} \in R^{n_{1} \times m}$ the $\operatorname{problem} \operatorname{VI}\left(\mathcal{P}, q_{k}, \psi_{\Phi}\right)$ has at least one solution.
2. Assume that $\mathcal{S} \neq\{0\}$. If there exists $\xi^{0} \in \operatorname{dom}(\Phi)$ such that $\left\langle q_{k}-\mathcal{P}^{\top} \xi^{0}, v\right\rangle>0$ for all $v \in \mathcal{S}, v \neq 0$, then the problem $V I\left(\mathcal{P}, q_{k}, \psi_{\Phi}\right)$ has at least one solution.
3. If $\xi_{k+1}^{1}$ and $\xi_{k+1}^{2}$ are two solutions of $\operatorname{VI}\left(\mathcal{P}, q_{k}, \psi_{\Phi}\right)$, then $\xi_{k+1}^{1}-\xi_{k+1}^{2} \in \operatorname{Ker}(\mathcal{P})$, hence $x_{1, k+1}$ is unique. Moreover $\left\langle q_{k},\binom{0}{\lambda_{k+1}^{1}-\lambda_{k+1}^{2}}\right\rangle=0$.
4. If in addition $B_{3}^{\top} X_{3} B_{3}>0$, then the VI in (5.5) always has a unique solution.

Proof: The first assertion about strong SPRness follows from [28, Theorem $5 \cdot 4$ (A')], and it guarantees that $X_{1}>0$ (see Appendix d). Let us pass to the analysis of (5.6). From Proposition B. 1 e) one has $\Phi_{\infty}=\Phi$. Then one has $\mathcal{S} \triangleq \Phi \cap \operatorname{Ker}(\mathcal{P}) \cap\left\{\xi \in R^{n_{1}+m} \left\lvert\, \mathcal{P}\binom{x_{1}}{\lambda}=\binom{0}{\alpha}\right., \alpha \in R^{m}, \alpha_{i} \geq 0,1 \leq i \leq m\right\}=$ $\Phi \cap \operatorname{Ker}(\mathcal{P}) \cap\left\{\xi \in R^{n_{1}+m} \mid x_{1}=0, B_{3}^{\top} X_{3} B_{3} \lambda=\alpha, \alpha \in R^{m}, \alpha_{i} \geq 0,1 \leq i \leq m\right\}$. The third subset in $\mathcal{S}$ follows from [50, Example 15], and the structure of $\mathcal{P}$ with $X_{1}>0$. The result follows noting that $\operatorname{Ker}\left(B_{3}^{\top} X_{3} B_{3}\right) \cap\left\{\lambda \in R_{+}^{m} \mid B_{3}^{\top} X_{3} B_{3} \lambda=\alpha, \alpha \in R^{m}, \alpha_{i} \geq 0,1 \leq i \leq m\right\}=\operatorname{Ker}\left(B_{3}^{\top} X_{3} B_{3}\right)$. Then the remaining part follows from Proposition b.2, where the symmetry of $\mathcal{P}$ is used (see item b') in Proposition b.2). In item 2) one uses that $\psi_{\Phi}(v)=0$ when $v \in \mathcal{S}$. Let us now deal with item 4 . In this case $\mathcal{P}=\mathcal{P}^{\top}>0$. Let us define $\mathcal{R}=\mathcal{R}^{\top}>0, \mathcal{R}^{2}=\mathcal{P}$, and $\zeta_{k}=\mathcal{R} \xi_{k}$. Then the VI in (5.5) is equivalently rewritten as:

$$
\begin{equation*}
\zeta_{k+1}-\mathcal{R}^{-1} \mathcal{P}_{h} \mathcal{R}^{-1} \zeta_{k}+\mathcal{R}^{-1} G_{k} \in-\mathcal{R}^{-1} \mathcal{N}_{\Phi}\left(\mathcal{R}^{-1} \zeta_{k+1}\right)=-\mathcal{N}_{\tilde{\Phi}}\left(\zeta_{k+1}\right) \tag{5.7}
\end{equation*}
$$

where the last equality stems from the chain rule, with $\tilde{\Phi}=\left\{z \in R^{n_{1}+m} \mid \mathcal{R}^{-1} z \in \Phi\right\}$. One infers from (5.7) that $\zeta_{k+1}=\operatorname{proj}\left[\tilde{\Phi} ; \mathcal{R}^{-1} \mathcal{P}_{h} \mathcal{R}^{-1} \zeta_{k}-\mathcal{R}^{-1} G_{k}\right] \Rightarrow \xi_{k+1}=\mathcal{R}^{-1} \operatorname{proj}\left[\tilde{\Phi} ; \mathcal{R}^{-1} \mathcal{P}_{h} \xi_{k}-\mathcal{R}^{-1} G_{k}\right] . \boxtimes$

Corollary 5.2. Assume that $X_{1}>0$, and $G(\cdot)$ is bounded. The problem $\operatorname{VI}\left(\mathcal{P}, q_{k}, \psi_{\Phi}\right)$ always has at least one solution for bounded $\xi_{k}$.

Proof: Consider item 2 in Proposition $5 \cdot 1$, with $\xi^{0}=\binom{x^{0}}{\lambda^{0}}, v=\binom{v_{x}}{v_{\lambda}}$. Then:

$$
\begin{align*}
\left\langle q_{k}-\mathcal{P}^{\top} \xi^{0}, v\right\rangle= & \left\langle-\mathcal{P}_{h}\binom{x_{1, k}}{\lambda_{k}}+h G_{k}, v\right\rangle-\left\langle B_{3}^{\top} X_{3} B_{3} \lambda^{0}, v_{\lambda}\right\rangle \\
& -\left\langle X_{1} x^{0}, v_{x}\right\rangle \\
= & \left\langle-\left(\begin{array}{cc}
X_{1}+h X_{1} A_{1} & h X_{1} B_{1} \\
-h C_{1} & B_{3}^{\top} X_{3} B_{3}-h \hat{D}
\end{array}\right)\binom{x_{1, k}}{\lambda_{k}}+h G_{k}, v\right\rangle  \tag{5.8}\\
& -\left\langle B_{3}^{\top} X_{3} B_{3} \lambda^{0}, v_{\lambda}\right\rangle-\left\langle X_{1} x^{0}, v_{x}\right\rangle .
\end{align*}
$$

Since $\xi^{0} \in \Phi$ it follows that $x^{0} \in R^{n_{1}}$ while $\lambda^{0} \in R_{+}^{m}$, and similarly $v_{x} \in R^{n_{1}}, v_{\lambda} \in R_{+}^{m}$. Since $v \neq 0$ let us take $x^{0}=-\alpha X_{1}^{-1} \frac{v_{x}}{\left\|v_{x}\right\|}-\beta X_{1}^{-1} \frac{v_{x}}{\left\|v_{x}\right\|^{2}}\left\|v_{\lambda}\right\|$ for some $\alpha \geq, \beta \geq 0$, and $\lambda^{0}=0$. It follows that:

$$
\begin{align*}
\left\langle q_{k}-\mathcal{P}^{\top} \xi^{0}, v\right\rangle & =-\left\langle\left(\begin{array}{cc}
X_{1}+h X_{1} A_{1} & h X_{1} B_{1} \\
-h C_{1} & B_{3}^{\top} X_{3} B_{3}-h \hat{D}
\end{array}\right)\binom{x_{1, k}}{\lambda_{k}}, v\right\rangle+\left\langle h G_{k}, v\right\rangle \\
& \geq-\left\|\left(\begin{array}{cc}
X_{1}+h X_{1} A_{1} & h X_{1} B_{1} \\
-h C_{1} & B_{3}^{\top} X_{3} B_{3}-h \hat{D}
\end{array}\right)\binom{x_{1, k}}{\lambda_{k}}\right\|\|v\|-\left\|h G_{k}\right\|\|v\|  \tag{5.9}\\
& +\alpha\left\|v_{x}\right\|+\beta\left\|v_{\lambda}\right\| \\
& \geq-\left\|\left(\begin{array}{cc}
X_{1}+h X_{1} A_{1} & h X_{1} B_{1} \\
-h C_{1} & B_{3}^{\top} X_{3} B_{3}-h \hat{D}
\end{array}\right)\binom{x_{1, k}}{\lambda_{k}}\right\|\left(\left\|v_{x}\right\|+\left\|v_{\lambda}\right\|\right) \\
& -\left\|h G_{k}\right\|\left(\left\|v_{x}\right\|+\left\|v_{\lambda}\right\|\right)+\alpha\left\|v_{x}\right\|+\beta\left\|v_{\lambda}\right\| .
\end{align*}
$$

For large enough $\alpha$ and $\beta$ it follows that $\left\langle q_{k}-\mathcal{P}^{\top} \xi^{0}, v\right\rangle>0$ and this holds for any $v \neq 0$ inside $\mathcal{S}$. $\boxtimes$
In item 2, the condition $\left\langle q_{k}, v\right\rangle>0$ for all $v \in \mathcal{S}$ is equivalent to $q_{k} \in \mathcal{S}^{\star} \backslash \mathcal{S}^{\perp}$. Obviously instead of assuming the strong SPRness we could have just assumed that $X_{1}>0$. Most importantly one notices that under the conditions of item 4 , the DI in (5.3) is equivalent to a DI with maximal monotone set-valued term:

$$
\begin{equation*}
\dot{\zeta}(t)-\mathcal{R}^{-1} \mathcal{A} \mathcal{R}^{-1} \zeta(t)+\mathcal{R}^{-1} G(t) \in-\mathcal{N}_{\tilde{\Phi}}(\zeta(t)) \tag{5.10}
\end{equation*}
$$

with $\zeta=\mathcal{R} \xi$. This fits within the class of DIs analysed for instance in [14]. Under basic assumption on the regularity of $G(\cdot)$, (5.10) has Lipschitz continuous solutions with uniqueness.

### 5.2 BOUNDEDNESS AND CONVERGENCE ANALYSIS

In the next analysis we assume that $B_{3}^{\top} X_{3} B_{3}$ is singular, which is the most general case (see the example in section 6.2, see also section 5.3 for the invertible case). However we still assume that $X_{1}>0$. From the structure of $\mathcal{P}$ and of the set-valued right-hand side, it follows from (5.5) that $x_{1, k+1}$ is calculable from $x_{1, k}$ and $\lambda_{k}$ :

$$
X_{1} x_{1, k+1}=\left(X_{1}+h X_{1} A_{1}\right) x_{1, k}+h X_{1} B_{1} \lambda_{k}-h G_{k}^{x}
$$

$$
\begin{equation*}
\Leftrightarrow \quad x_{1, k+1} \quad=\left(I_{n_{1}}+h A_{1}\right) x_{1, k}+h B_{1} \lambda_{k}-h X_{1}^{-1} G_{k}^{x}, \text { for all } k \geq 0 \tag{5.11}
\end{equation*}
$$

where $G_{k}=\binom{G_{k}^{x}}{G_{k}^{\lambda}}$.
Lemma 5.3. From (5.11), for all $k \geq 0$ one has:

$$
\begin{equation*}
x_{1, k+1}=\left(I_{n_{1}}+h A_{1}\right)^{k+1} x_{1,0}+h \sum_{i=0}^{k}\left(I_{n_{1}}+h A_{1}\right)^{k-i} B_{1} \lambda_{i}-h \sum_{i=0}^{k}\left(I_{n_{1}}+h A_{1}\right)^{k-i} X_{1}^{-1} G_{i}^{x} \tag{5.12}
\end{equation*}
$$

(5.13)

$$
\begin{aligned}
x_{1, k+2} & =\left(I_{n_{1}}+h A_{1}\right) x_{1, k+1}+h B_{1} \lambda_{k+1}-h X_{1}^{-1} G_{k+1}^{x} \\
& =\left(I_{n_{1}}+h A_{1}\right)\left\{\left(I_{n_{1}}+h A_{1}\right)^{k+1} x_{1,0}+h \sum_{i=0}^{k}\left(I_{n_{1}}+h A_{1}\right)^{k-i} B_{1} \lambda_{i}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.-h \sum_{i=0}^{k}\left(I_{n_{1}}+h A_{1}\right)^{k-i} X_{1}^{-1} G_{i}^{x}\right\}+h B_{1} \lambda_{k+1}-h X_{1}^{-1} G_{k+1}^{x} \\
= & \left(I_{n_{1}}+h A_{1}\right)^{k+2} x_{1,0}+h \sum_{i=0}^{k+1}\left(I_{n_{1}}+h A_{1}\right)^{k+1-i} B_{1} \lambda_{i} \\
& -h \sum_{i=0}^{k+1}\left(I_{n_{1}}+h A_{1}\right)^{k+1-i} X_{1}^{-1} G_{i}^{x} .
\end{aligned}
$$

Lemma 5.4. One has for all $k \geq 0$ :

$$
\begin{equation*}
\left\|\lambda_{k+2}^{\min }\left(\lambda_{k+1}, x_{1, k+1}, G_{k+1}^{\lambda}\right)\right\| \leq\left(\eta h_{3}\right)^{k+1}\left\|\lambda_{1}^{\min }\left(\lambda_{0}, x_{1,0}, G_{0}^{\lambda}\right)\right\| \tag{5.15}
\end{equation*}
$$

for all $k \geq 0$. The second formulation shows the iterative feature of this GE for $\lambda_{k}$, which makes it quite different from the $\operatorname{LCCP}$ (3.3). It is equivalent to the $\operatorname{LCP}\left(B_{3}^{\top} X_{3} B_{3}, q_{k}^{\lambda}\left(\lambda_{k}, x_{1, k}, G_{k}^{\lambda}\right)\right)$. Under the conditions stated in Proposition 5.1 and Corollary 4.2, the set of solutions to this LCP is non empty, and it contains a least-norm element $\lambda_{k+1}^{\min }, k \geq 0$, satisfiying $\left\|\lambda_{k+1}^{\min }\left(\lambda_{k}, x_{1, k}, G_{k}^{\lambda}\right)\right\| \leq \eta\left\|q_{k}^{\lambda}\left(\lambda_{k}, x_{1, k}, G_{k}^{\lambda}\right)\right\|, k \geq 0$, where $\eta>0$ depends only on $B_{3}^{\top} X_{3} B_{3}$. Therefore $\left\|\lambda_{k+1}^{\min }\left(\lambda_{k}, x_{1, k}, G_{k}^{\lambda}\right)\right\| \leq \eta\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|\left\|\lambda_{k}^{\min }\right\|+$ $h \eta\left\|C_{1} x_{1, k}\right\|+h \eta\left\|G_{k}^{\lambda}\right\|$. Let us denote $h_{3} \triangleq\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|$.

$$
+h \eta \sum_{i=0}^{k}\left(\eta h_{3}\right)^{k-i}\left(\left\|C_{1} x_{1, i+1}\right\|+\left\|G_{i+1}^{\lambda}\right\|\right)
$$

By induction the relation in (5.12) is true for all $k \geq 0$. $\boxtimes$

Using (5.5) we can also express $\lambda_{k+1}$ as the solution of a generalised equation (GE):
(5.14)

$$
\begin{gathered}
B_{3}^{\top} X_{3} B_{3} \lambda_{k+1}+\underbrace{\left(h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right) \lambda_{k}+h C_{1} x_{1, k}+h G_{k}^{\lambda}}_{\triangleq q_{k}^{\lambda}\left(\lambda_{k}, x_{1, k}, G_{k}^{\lambda}\right)} \in-\mathcal{N}_{R_{+}^{m}}\left(\lambda_{k+1}\right) \\
\lambda_{k+1} \in\left(B_{3}^{\top} X_{3} B_{3}+\mathcal{N}_{R_{+}^{m}}\right)^{-1}\left(\left(B_{3}^{\top} X_{3} B_{3}-h \hat{D}\right) \lambda_{k}-h C_{1} x_{1 k}-h G_{k}^{\lambda}\right)
\end{gathered}
$$ (5.16)

$$
\begin{aligned}
\left\|\lambda_{k+3}^{\min }\left(\lambda_{k+2}, x_{1, k+2}, G_{k+2}^{\lambda}\right)\right\| & \leq \eta\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|\left\|\lambda_{k+2}^{\min }\right\|+h \eta\left\|C_{1} x_{1, k+2}\right\|+h \eta\left\|G_{k+2}^{\lambda}\right\| \\
& \leq \eta\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|\left\{\eta^{k+1}\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|^{k+1}\left\|\lambda_{1}^{\min }\left(\lambda_{0}, x_{1,0}, G_{0}^{\lambda}\right)\right\|\right. \\
& \left.+h \eta \sum_{i=0}^{k} \eta^{k-i}\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|^{k-i}\left(\left\|C_{1} x_{1, i+1}\right\|+\left\|G_{i+1}^{\lambda}\right\|\right)\right\} \\
& +h \eta\left\|C_{1} x_{1, k+2}\right\|+h \eta\left\|G_{k+2}^{\lambda}\right\| \\
& =\eta^{k+2}\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|^{k+2}\left\|\lambda_{1}^{\min }\left(\lambda_{0}, x_{1,0}, G_{0}^{\lambda}\right)\right\| \\
& +h \eta \sum_{i=0}^{k} \eta^{k+1-i}\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\| \|^{k+1-i}\left(\left\|C_{1} x_{1, i+1}\right\|+\left\|G_{i+1}^{\lambda}\right\|\right) \\
& +h \eta\left\|C_{1} x_{1, k+2}\right\|+h \eta\left\|G_{k+2}^{\lambda}\right\| \\
& =\eta^{k+2}\left\|h \hat{D}-{ }_{3}^{\top} X_{3} B_{3}\right\|^{k+2}\left\|\lambda_{1}^{\min }\left(\lambda_{0}, x_{1,0}, G_{0}^{\lambda}\right)\right\| \\
& +h \eta \sum_{i=0}^{k+1} \eta^{k+1-i}\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|^{k+1-i}\left(\left\|C_{1} x_{1, i+1}\right\|+\left\|G_{i+1}^{\lambda}\right\|\right) .
\end{aligned}
$$

$$
\begin{equation*}
\left\|x_{1, k+1}\right\| \leq h_{1}^{k+1}\left\|x_{1,0}\right\|+h f_{k}+h^{2} \eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{i=1}^{k} h_{1}^{k-i} \sum_{j=0}^{i-1}\left(\eta h_{3}\right)^{i-1-j}\left\|x_{1, j+1}\right\| \tag{5.19}
\end{equation*}
$$

with
(5.20)

$$
\begin{aligned}
f_{k} & =\sum_{i=0}^{k} h_{1}^{k-i}\left\|X_{1}^{-1}\right\|\left\|G_{i}^{x}\right\|+\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\| \sum_{i=0}^{k} h_{1}^{k-i}\left(\eta h_{3}\right)^{i}+h_{1}^{k}\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\| \\
& +h \eta\left\|B_{1}\right\|\left\|C_{1}\right\| \sum_{i=1}^{k} h_{1}^{k-i} \sum_{j=0}^{i-1}\left(\eta h_{3}\right)^{i-1-j}\left\|G_{j+1}^{\lambda}\right\|
\end{aligned}
$$

The third term in the right-hand side of (5.19) involves terms from $\left\|x_{1,1}\right\|$ to $\left\|x_{1, k}\right\|$. The inequality (5.19) can be rewritten as:

$$
\begin{equation*}
\left\|x_{1, k+1}\right\| \leq h_{1}^{k+1}\left\|x_{1,0}\right\|+h f_{k}+h^{2} \sum_{i=1}^{k} \beta_{i}\left\|x_{1, i}\right\| \tag{5.21}
\end{equation*}
$$

for some positive sequence of real numbers $\left\{\beta_{i}\right\}$, each $\beta_{i}$ being calculated from the expression in (5.19) $\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{i=1}^{k} h_{1}^{k-i} \sum_{j=0}^{i-1}\left(\eta h_{3}\right)^{i-1-j}\left\|x_{1, j+1}\right\|$, and thus being made of powers of $h_{1}$ and of $h_{3} \eta$. Calculations show that (see Appendix F )

$$
\begin{equation*}
\beta_{i}=\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{j=i}^{k} h_{1}^{k-j}\left(\eta h_{3}\right)^{j-i}, \quad i \geq 1 . \tag{5.22}
\end{equation*}
$$

Let us now apply a version of the discrete-time Gronwall inequality, due to Bachpatte [13] [44, Lemma 100], see section E . It is deduced that for all $k \geq 0$ :

$$
\begin{equation*}
\left\|x_{1, k+1}\right\| \leq h_{1}^{k+1}\left\|x_{1,0}\right\|+h f_{k}+h^{2} \sum_{i=0}^{k}\left(h_{1}^{i+1} \beta_{i}\left\|x_{1,0}\right\|+h \beta_{i} f_{i}\right) \Pi_{j=i+1}^{k+1}\left(1+h^{2} \beta_{j}\right) \tag{5.23}
\end{equation*}
$$

Let us state the next assumption (recall that $h_{3}=\left\|h \hat{D}-B_{3}^{\top} X_{3} B_{3}\right\|$ and $\left.h_{1}=\left\|I_{n_{1}}+h A_{1}\right\|\right)$.
Assumption 3 . One has a) $h_{1} \leq 1$ for $h>0$ small enough, b) $\|G(t)\|$ is uniformly bounded by $G_{\max }<+\infty$ on $[0, T]$ for any $T>0$, and $G(\cdot)$ is Lipschitz continuous with Lipschitz constant $k_{G} \geq 0$, c) $\eta h_{3} \leq 1$ for $h \geq 0$ small enough.

Inequality a) is similar to Assumption 1. As alluded to above, this inequality stems from the explicit discretisation of the term $A_{1} x_{1}$ in (2.2) (a). As illustrated through examples in section 6, it holds in practice for reasonable values of the timestep. See also the developments in Appendix A. Assumption 3 b) is a standard property of exogenous time-dependent terms. The inequality c) is the central assumption of Assumption 3, which quantifies the couplings between the DAE part and the complementarity part of the DVLCS. Some examples of circuits in section 6 will prove that it can be satisfied in practice. Notice from (5.14) that if $B_{3}^{\top} X_{3} B_{3}=0$ (Assumption 2), then the minimal-norm element of the set of solutions to the LCP is always $\xi_{k}^{\min }=0$, hence $\eta=0$ in that case. Assumption 3 implies using (5.22) that $\beta_{i} \leq \eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{j=i}^{k}\left(\eta h_{3}\right)^{j-i} \leq \eta\left\|C_{1}\right\|\left\|B_{1}\right\|(k-i+1)$. Consequently using (5.20) the next upperbound can be found:

$$
\begin{equation*}
f_{k} \leq(k+1)\left(G_{\max }\left\|X_{1}^{-1}\right\|+\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\|\right)+\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\|+h \eta\left\|B_{1}\right\|\left\|C_{1}\right\| G_{\max } \sum_{i=1}^{k} \sum_{j=0}^{i-1} j \tag{5.24}
\end{equation*}
$$

Since $h=\frac{T}{n}$ and $k \in\{0, n\}$ :
(5.25) $h f_{k} \leq T \frac{(k+1)}{n}\left(G_{\max }\left\|X_{1}^{-1}\right\|+\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\|\right)+\frac{T}{n}\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\|+\frac{T^{2}}{n^{2}} \eta\left\|B_{1}\right\|\left\|C_{1}\right\| G_{\max } \sum_{i=1}^{k} \sum_{j=0}^{i-1} j$.

Now we have that $\sum_{i=1}^{k} \sum_{j=0}^{i-1} j=(k-1)+2(k-1)+3(k-3)+\ldots+(k-3) 3+(k-2) 2+(k-1)$, where the sum consists of $k-1$ terms. Hence $\sum_{i=1}^{k} \sum_{j=0}^{i-1} j=O\left(k^{2}\right)$, and $\frac{T^{2}}{n^{2}} \eta\left\|B_{1}\right\|\left\|C_{1}\right\| G_{\max } \sum_{i=1}^{k} \sum_{j=0}^{i-1} j=$ $T^{2} \eta\left\|B_{1}\right\|\left\|C_{1}\right\| G_{\max } \frac{O\left(k^{2}\right)}{n^{2}}$. Since $k \leq n$ we infer that this term is uniformly bounded in $k$, for each fixed bounded $T$. The same holds for $T \frac{(k+1)}{n}\left(G_{\max }\left\|X_{1}^{-1}\right\|+\left\|B_{1}\right\|\left\|\lambda_{1}^{\min }\right\|\right)$. Thus $h f_{k} \leq \alpha_{1} \max \left(T, T^{2}\right)$ for some $\alpha_{1}$ which depends on $\eta,\left\|B_{1}\right\|,\left\|C_{1}\right\|, G_{\max },\left\|\lambda_{1}^{\min }\right\|$.
The next step consists of computing an upperbound from (5.23) as:

$$
\begin{equation*}
\left\|x_{1, k+1}\right\| \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\frac{T^{2}}{n^{2}} \sum_{i=0}^{k} \beta_{i} \Pi_{j=i+1}^{k+1}\left(1+\frac{T^{2}}{n^{2}} \beta_{j}\right)\right) \tag{5.26}
\end{equation*}
$$

In view of Assumption 3 and (5.22), one has $\beta_{i} \leq \eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{j=i}^{k} 1=\eta\left\|C_{1}\right\|\left\|B_{1}\right\|(k-i) \leq$ $\eta\left\|C_{1}\right\|\left\|B_{1}\right\| n, i \geq 1, k \leq n$. Consequently:

$$
\begin{aligned}
\left\|x_{1, k+1}\right\| & \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\frac{T^{2}}{n} \sum_{i=0}^{k} \eta\left\|C_{1}\right\|\left\|B_{1}\right\| \Pi_{j=i+1}^{k+1}\left(1+\frac{T^{2}}{n} \eta\left\|C_{1}\right\|\left\|B_{1}\right\|\right)\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{T^{2}}{n} \sum_{i=0}^{k} \Pi_{j=i+1}^{k+1}\left(1+\frac{T^{2}}{n} \eta\left\|C_{1}\right\|\left\|B_{1}\right\|\right)\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{T^{2}}{n} \sum_{i=0}^{k}\left(1+\frac{T^{2}}{n} \eta\left\|C_{1}\right\|\left\|B_{1}\right\|\right)^{k-i}\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{T^{2}}{n} \sum_{i=0}^{k}\left(1+\frac{\alpha_{2}}{n}\right)^{k-i}\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{T^{2}}{n} \sum_{i=0}^{n}\left(1+\frac{\alpha_{2}}{n}\right)^{n}\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{T^{2}}{n}(n+1)\left(1+\frac{\alpha_{2}}{n}\right)^{n}\right) \\
& \leq\left(\left\|x_{1,0}\right\|+\alpha_{1} \max \left(T, T^{2}\right)\right)\left(1+T^{2} \eta\left\|C_{1}\right\|\left\|B_{1}\right\| \frac{n+1}{n}\left(1+\frac{\alpha_{2}}{n}\right)^{n}\right),
\end{aligned}
$$

where $\alpha_{2}=T^{2} \eta\left\|C_{1}\right\|\left\|B_{1}\right\|$. One has $\left(1+\frac{\alpha_{2}}{n}\right)^{n} \rightarrow e^{\alpha_{2}}$ as $n \rightarrow+\infty$ [70, p.196]. Therefore the sequence $\left\{x_{1, k}\right\}$ is uniformly bounded over $k \in\{0, n\}$. It directly follows from Lemma 5.4 that the sequence $\left\{\lambda_{k}^{\min }\right\}$ is uniformly bounded as well. It follows from (4.1) that the sequence $\left\{\frac{x_{1, k+1}-x_{1, k}}{h}\right\}$ is uniformly bounded also. Therefore we are ready to prove the following result. The piece-wise linear approximations are defined in (4.7), and the step approximations $x_{1}^{h \star}(t)=x_{1, k+1}$ if $t \in\left[t_{k}, t_{k+1}\right), \lambda_{h \star}^{\min }(t)=\lambda_{k}^{\min }$ if $t \in\left[t_{k}, t_{k+1}\right)$.
Proposition 5.5. Let Assumption 3 hold true, as well as the conditions for the well-posedness of the OSNSP in Proposition 3.2. Then the sequence $\left\{x_{1}^{h}(\cdot)\right\}$ converges uniformly towards a continuous limit $x_{1}(\cdot)$. Moreover $x_{1}^{h \star}(\cdot)$ converges strongly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$ towards $x_{1}(\cdot)$, and $\dot{x}_{1}^{h}(\cdot)$ converges weakly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$ towards $\dot{x}_{1}(\cdot)$. Therefore $\left\{\left(B_{1} \lambda_{h \star}^{\min }\right)(\cdot)\right\}$ converges weakly in $\mathcal{L}_{2}\left([0, T] ; R^{n_{1}}\right)$ towards a limit $\lambda_{1}(\cdot)$, and $\dot{x}_{1}(t)=A_{1} x_{1}(t)+G^{x}(t)+\lambda_{1}(t)$ almost everywhere on $[0, T]$.

Proof: Both sequences $\left\{x_{1}^{h}(\cdot)\right\}$ and $\left\{\dot{x}_{1}^{h}(\cdot)\right\}$ are uniformly bounded on $[0, T]$, the sequence $\left\{x_{1}^{h}(\cdot)\right\}$ of $C^{0}\left([0, T] ; R^{n_{1}}\right)$ functions is thus equicontinuous. From the Ascoli-Arzela Theorem, $\left\{x_{1}^{h}(\cdot)\right\}$ stays in a compact subset of $C^{0}\left([0, T] ; R^{n_{1}}\right)$ and indeed it converges uniformly towards a continuous limit $x_{1}(\cdot)$. Let us now examine the sequences of piecewise-linear functions $\left\{G^{h}(\cdot)\right\}_{h \geq 0}$ and $\left\{\dot{G}^{h}(\cdot)\right\}_{h \geq 0}$, designed similarly as (4.7). From Assumption 3 b ) one infers that $G^{h}(t)$ is uniformly bounded on $[0, T], \frac{\left\|G_{k+1}-G_{k}\right\|}{h} \leq k_{E}$ implies that $\dot{G}^{h}(t)$ is also uniformly bounded in $h$ on $[0, T]$, thus $\left\{G^{h}(\cdot)\right\}_{h \geq 0}$ is a sequence of Lipschitz continuous, uniformly bounded, equicontinuous functions. From the AscoliArzela Theorem, $\left\{G^{h}(\cdot)\right\}$ stays in a compact subset of $C^{0}\left([0, T] ; R^{n_{1}}\right)$ and it converges uniformly towards the continuous limit $G(\cdot)$ as $h \rightarrow 0$. The remaining part of the proof is similar to the proof of Corollary $4 \cdot 7$ (adding $G^{x, h}(\cdot)$ with the required properties) and it is omitted. $\boxtimes$
Remark 5.6. Assumption 3 c) is clearly less restrictive than Assumption 2. It still imposes some constraint on the matrix $C_{3} B_{3}$. Examples are treated next which show that it is satisfied in some practical cases.

Let us now prove the uniform boundedness of the sequences $\left\{\lambda_{k}^{\min }\right\}$ and $\left\{x_{2, k}\right\}$. To this aim we shall use first the second line in (5.17) and (5.27). Let us denote the right-hand side of (5.27) as $x_{1, \max }\left(T, \eta, x_{1,0}\right)$. Then from (5.17) and Assumption 3 we obtain:

$$
\begin{align*}
\left\|\lambda_{k+1}^{\min }\right\| & \leq\left\|\lambda_{0}^{\min }\right\|+h \eta \sum_{i=0}^{k}\left(\left\|C_{1}\right\| x_{1, \max }\left(T, \eta, x_{1,0}\right)+\left\|G_{i+1}^{\lambda}\right\|\right) \\
& \leq\left\|\lambda_{0}^{\min }\right\|+h \eta\left\|C_{1}\right\| x_{1, \max }\left(T, \eta, x_{1,0}\right)(k+1)+h \eta \sum_{i=0}^{k}\left\|G_{i+1}^{\lambda}\right\| \\
& \leq\left\|\lambda_{0}^{\min }\right\|+\frac{T(k+1)}{n} \eta\left\|C_{1}\right\| x_{1, \max }\left(T, \eta, x_{1,0}\right)+\eta \frac{T(k+1)}{n} G_{\max }  \tag{5.28}\\
& \leq\left\|\lambda_{0}^{\min }\right\|+\frac{T(n+1)}{n} \eta\left\|C_{1}\right\| x_{1, \max }\left(T, \eta, x_{1,0}\right)+\eta \frac{T(n+1)}{n} G_{\max } .
\end{align*}
$$

Therefore $\left\|\lambda_{k+1}^{\min }\right\|$ is uniformly bounded for any bounded $T$ and initial data. The boundedness of $x_{2, k}$ follows from the equality constraint in (4.1). A result similar to Corollary 4.7 holds, but this time verifying the complementarity conditions is less easy due to the presence of $z$ in the complementarity variable $w$.

### 5.3 CASE $X_{1}>0$ and $B_{3}^{\top} X_{3} B_{3}>0$

This is the case studied in item 4 of Proposition 5.1 and may be verified in practice, see the example in section 6.1. Then the DI in (5.3) is equivalently rewritten as in (5.10). Consequently one can apply the implicit Euler scheme and all the existence, convergence and order results presented in [14] (see that article for details on the discretization method and theoretical results), to (5.10). It is inferred that in this case (5.10) possesses unique solutions in $C^{0}\left([0, T] ; R^{n_{1}+m}\right)$ with essentially bounded derivatives [14, Proposition 2.6]. It is deduced that $x_{2}(\cdot)$ is also unique in $C^{0}\left([0, T] ; R^{n_{1}+m}\right)$ with essentially bounded derivative, while the algebraic variable $z(\cdot)$ (which depends on $\dot{\lambda}(\cdot)$, see (5.1)) is essentially bounded (hence a priori it may be discontinuous).

## 6 EXAMPLES AND NUMERICAL SIMULATIONS

Two examples of circuits with ideal diodes, and one academic example, are developed in this section. Let us comment that the analysis of circuits applies to hydraulics, due to the strong analogy between hydraulic and electrical circuits [65,53] [49, Chapter 1], where check valves play the role of diodes.

### 6.1 FIRST EXAMPLE



Figure 1: An RLCD passive circuit.

Let us consider the circuit in Figure 1. Let us assume first that $R_{1}=R_{3}=0 \Omega$. Let $\xi_{1}=i_{2}, \xi_{2}=$ $\int_{0}^{t} i_{3}(t) d t, \xi_{3}=i_{1}$. The dynamics is given by:

$$
\left\{\begin{array}{l}
\dot{\xi}_{1}(t)=-\frac{R_{2}}{L} \xi_{1}(t)+\frac{1}{L} u_{D}+\frac{1}{L} u_{1}(t)  \tag{6.1}\\
\dot{\xi}_{2}(t)=\xi_{1}(t)-\xi_{3}(t) \\
0=\frac{1}{C} \xi_{2}(t)+u_{D}(t)+u_{2}(t) \\
0 \leq u_{D} \perp \xi_{3}(t) \geq 0
\end{array}\right.
$$

It is noteworthy that this circuit dynamics agrees with [2, Theorem 3.6] [45] about the index (indeed it cannot be of index 1 due to loop with a capacitive element). In our notations we have $x=\left(\xi_{1}, \xi_{2}\right)^{\top}$, $z=\xi_{3}, C=\left(\begin{array}{lll}0 & 0 & 1\end{array}\right), B=\left(\begin{array}{lll}\frac{1}{L} & 0 & 1\end{array}\right)^{\top}$. The transfer function $H(s)=\frac{z(s)}{u_{D}(s)}=\frac{\mathrm{C} s^{2}+\frac{\mathrm{CR}}{L} s+\frac{1}{L}}{s+\frac{R}{L}}=\mathrm{C} s+\frac{1}{L s+R}$. It has a
$\mathrm{H}, R=1 \Omega, C=10^{-1} \mathrm{~F}$. Let us define $S=\left(\begin{array}{ccc}-1.414 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-4}\end{array}\right), T=\left(\begin{array}{ccc}-0.707 & 0 & 0 \\ 0 & 0 & 1 \\ -0.707 & -1 & 0\end{array}\right)$. Then $S A T=$ $\left(\begin{array}{ccc}-2.1210^{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right), S B=\left(\begin{array}{c}-310^{-3} \\ 0 \\ 10^{-4}\end{array}\right), C T=\left(\begin{array}{lll}-0.707 & -1 & 0\end{array}\right), S\left(\begin{array}{cc}\frac{1}{L} & 0 \\ 0 & 0 \\ 0 & 1\end{array}\right)=\left(\begin{array}{cc}0.30110^{4} & 0 \\ 0 & 0 \\ 0 & 10^{-4}\end{array}\right)$. Let us make the state variable change $\zeta=T^{-1} \xi$, then we obtain the transformed dynamics:

$$
\left\{\begin{array}{l}
\dot{\zeta}_{1}(t)=-2.1210^{3} \zeta_{1}(t)-310^{-3} \lambda(t)+0.30110^{4} u_{1}(t)  \tag{6.2}\\
\dot{\zeta}_{3}(t)=\zeta_{2}(t) \\
0=\zeta_{3}(t)+10^{-4} \lambda(t)+10^{-4} u_{2}(t) \\
0 \leq \lambda(t) \perp w(t)=-0.707 \zeta_{1}(t)-\zeta_{2}(t) \geq 0 .
\end{array}\right.
$$

Let us now define $x_{1}=\zeta_{1}, x_{2}=\zeta_{3}, x_{3}=\zeta_{2}$, then we obtain

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=-2.1210^{3} x_{1}(t)-310^{-3} \lambda(t)+0.30110^{4} u_{1}(t)  \tag{6.3}\\
\dot{x}_{2}(t)=x_{3}(t) \\
0=x_{2}(t)+10^{-4} \lambda(t)+10^{-4} u_{2}(t) \\
0 \leq \lambda(t) \perp w(t)=-0.707 x_{1}(t)-x_{3}(t) \geq 0,
\end{array}\right.
$$

which is the canonical form in (2.2) with $z=x_{3}, B_{1}=-3.10^{-3}, B_{2}=0, B_{3}=10^{-4}, C_{1}=-0.707$, $C_{2}=0, C_{3}=-1, D=0$. We have $X_{3}=10^{4}, A_{1}=-2.1210^{3}, X_{1}=\frac{C_{1}}{B_{1}}>0, \hat{D}=D-C_{3} B_{2}-C_{2} B_{3}=0$, $\bar{D}_{h}=h^{2} C_{1} B_{1}+B_{3}^{\top} X_{3} B_{3}=2.12110^{-3} h^{2}+10^{-4}, h_{3}=B_{3}^{\top} X_{3} B_{3}=10^{-4}, h_{1}=\left|1+h A_{1}\right|=\left|1-2.1210^{3} h\right|$ which verifies Assumption 3 for $h>0$ small enough. Notice that the set $\mathcal{S}$ in Propositions 3.2 and 5.1
verifies $\mathcal{S}=\{0\}$, hence the OSNSP is well-posed with unique solution. Provided that Assumption 3 is satisfied, the material in section 5 applies. In a similar way to (5.1), the system (6.3) is transformed into:

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=-2.1210^{3} x_{1}(t)-310^{-3} \lambda(t)+0.30110^{4} u_{1}(t)  \tag{6.4}\\
10^{-4} \dot{\lambda}(t)=0.707 x_{1}(t)-10^{-4} \dot{u}_{2}(t)+w(t) \\
x_{2}(t)=-10^{-4} \lambda(t)-10^{-4} u_{2}(t) \\
z(t)=-10^{-4} \dot{\lambda}(t)-10^{-4} \dot{u}_{2}(t) .
\end{array}\right.
$$

From (5.14), the following LCP is obtained:

$$
\left\{\begin{array}{l}
0 \leq 10^{-4} \lambda_{k+1}+q_{k} \perp \lambda_{k+1} \geq 0  \tag{6.5}\\
\left.q_{k}=-10^{-4} \lambda_{k}-0.707 h x_{1, k}-10^{-4} h u_{2, k}\right),
\end{array}\right.
$$

and $x_{1, k+1}, x_{2, k+1}, z_{k+1}$ as in (H.1). Let us check Assumption 3 c). Since $\hat{D}=0$ it is easy to show that $\eta=10^{4}$ while $h_{3}=10^{-4}$, using the LCP in (5.14) that can be solved explicitly. Thus $\eta h_{3} \leq 1$ for all $h \geq 0$. However, $h_{1}=\left\|1-2.1210^{3} h\right\| \leq 1$ if and only if $h<4.710^{-4}$. It is also noteworthy that this system verifies the conditions of section $5 \cdot 3$, and item 4 of Proposition 5.1 , since $X_{1}>0$ and $B_{3}^{\top} X_{3} B_{3}=10^{-4}>0$. Hence the state space form (5.10), that is equivalent to (5.3) since $\mathcal{P}>0$, can be calculated. Also the generalized equation (5.14) always has a unique solution $\lambda_{k+1}$. As we shall see through numerical simulations, the variable $z(\cdot)=x_{3}(\cdot)$ may be discontinuous. Actually this is in agreement with the theoretical conclusions from section $5 \cdot 3$, which do not state that this variable is continuous. It also shows that a complete convergence analysis, including $z$, should rely on other tools than those used in this paper.
In Figure 2, the results of the numerical simulation of (6.3) with an implicit Euler scheme are depicted ${ }^{2}$. Numerical simulations are performed with the software Siconos [1]. At each time step, the Lemke solver [62, 61] is applied to solve the LCP (3.1) resulting from the discretization scheme. The initial conditions are set to $\mathbf{x}(0)=\left(0.01,-10^{-4},-7.07 \cdot 10^{-3}\right)^{\mathrm{T}}$, and the time step is $h=5 \cdot 10^{-6} \mathrm{~s}$. The input functions $u_{1}(t)$ and $u_{2}(t)$ are taken as sinusoidal functions: $u_{1}(t)=\sin (t)$ and $u_{2}(t)=\cos (10 t)$. Let us observe that the variable $z(t)=x_{3}(t)$ appears to be discontinuous at $t \approx 5.9 \mathrm{~s}$. The multiplier $\lambda(t)$ remains continuous, see Figure 3. Finally, an order of convergence equal to one is deduced from the data in Figure 4. The error formula used is:

$$
\begin{equation*}
\text { error }=\left\|\mathbf{x}(t)-\mathbf{x}^{h}(t)\right\|_{\infty} \tag{6.6}
\end{equation*}
$$

### 6.2 SECOND EXAMPLE

This circuit example depicted in Figure 5 is inspired from [32]. The so-called modified nodal analysis is used to derive the dynamics [2, chapter 3]. It is assumed that the voltage source delivers a constant voltage $u(t)=u_{c}$, while the other voltage source $u_{1}(t)$ is time-varying Lipschitz continuous and bounded. Notice that the current/voltage characteristic of this device can be written as $i_{5}(t) \in-\partial \psi_{\left\{u_{c}\right\}}(u(t))$, or equivalently as $R \ni i_{5}(t) \perp u(t)-u_{c} \in\{0\}$. The dynamics is given by:

$$
\left\{\begin{array}{l}
\mathrm{C} \dot{e}(t)=-R^{-1} e(t)+i_{4}(t)-i_{1}(t)-i_{5}(t)  \tag{6.7}\\
L \frac{d i_{1}}{d t}(t)=e(t)+u_{1}(t) \\
0=e(t)+u(t) \\
0 \leq u_{D}(t) \perp i_{4}(t) \geq 0 \\
R \ni i_{5}(t) \perp u(t)-u_{c} \in\{0\}
\end{array}\right.
$$

[^2]

Figure 2: Implicit Euler numerical simulation of (6.3) with $h=5 \cdot 10^{-6} \mathrm{~s}$. The numerical simulations of the variables $x_{1}(t), x_{2}(t)$ and $x_{3}(t)=z(t)$ are depicted.

Example $1: \lambda(t)$ and $w(t)$ time plots


Figure 3: Implicit Euler numerical simulation of (6.3) with $h=5 \cdot 10^{-6} \mathrm{~s}$. The numerical simulations of the variables $\lambda(t)$ and $w(t)$ are depicted.


Figure 4: Numerical convergence of the implicit Euler scheme applied to (6.3). The reference curve is taken for $h=5.10^{-6} \mathrm{~s}$. The error formula used is given in (6.6).

The state vector is $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right)^{\top}=\left(e, i_{1}, i_{5}\right)^{\top}$, and $u_{D}(t)=e(t)$. The complementarity conditions can be written as $K=R^{+} \times R \ni\left(u_{D}(t), i_{5}(t)\right) \perp\left(i_{4}(t), u(t)-u_{c}\right) \in R^{+} \times\{0\}=K^{\star}$. Setting $\lambda_{1}=i_{4}$ and $\lambda_{2}=u(t)-u_{c}$, we obtain $P=\left(\begin{array}{ccc}C & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0\end{array}\right), A=\left(\begin{array}{ccc}-R^{-1} & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0\end{array}\right), B=\left(\begin{array}{ll}1 & 0 \\ 0 & 0 \\ 0 & 1\end{array}\right), C=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right), D=0$, and the DVLCS form in (2.1) is equal to:

$$
\left\{\begin{array}{l}
\dot{\xi}_{1}(t)=-\frac{1}{R \mathrm{C}} \xi_{1}(t)-\frac{1}{\mathrm{C}} \xi_{2}(t)-\frac{1}{\mathrm{C}} \xi_{3}(t)+\frac{1}{\mathrm{C}} \lambda_{1}(t)  \tag{6.8}\\
\dot{\xi}_{2}(t)=\frac{1}{L} \xi_{1}(t)+\frac{u_{1}(t)}{L} \\
0=\xi_{1}(t)+\lambda_{2}(t)+u_{c} \\
0 \leq \xi_{1}(t) \perp \lambda_{1}(t) \geq 0 \\
R \ni \xi_{3}(t) \perp \lambda_{2}(t) \in\{0\} .
\end{array}\right.
$$

Let $S=\left(\begin{array}{ccc}0 & \frac{1}{L} & -\frac{1}{L} \\ -1 & 0 & -\frac{1}{R} \\ 0 & 0 & -\mathrm{C}\end{array}\right), T=\left(\begin{array}{ccc}0 & 0 & -\frac{1}{\mathrm{C}} \\ 1 & 0 & 0 \\ -1 & 1 & 0\end{array}\right)$, then $S A T=\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right), S P T=\left(\begin{array}{ccc}\frac{1}{L} & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{C}} \\ 0 & 0 & 0\end{array}\right), C T=\left(\begin{array}{ccc}0 & 0 & -\frac{1}{\mathrm{C}} \\ -1 & 1 & 0\end{array}\right)$, $S B=\left(\begin{array}{ll}0 & -\frac{1}{L} \\ 1 & -\frac{1}{R} \\ 0 & -\mathbf{C}\end{array}\right)$. Using the state variable change $\zeta=T^{-1} \xi$ with $T^{-1}=\left(\begin{array}{ccc}0 & 1 & 0 \\ 0 & 1 & 1 \\ -\mathbf{C} & 0 & 0\end{array}\right)$, one obtains:
(6.9)

$$
\left\{\begin{aligned}
\dot{\zeta}_{1}(t) & =-\left(\lambda_{2}(t)+u_{c}(t)\right)+\frac{u_{1}(t)}{L} \\
\frac{1}{\mathrm{C}} \dot{\zeta}_{3}(t) & =\zeta_{2}(t)+\lambda_{1}(t)-\frac{1}{R}\left(\lambda_{2}(t)+u_{c}(t)\right) \\
0 & =-\mathrm{C}\left(\lambda_{2}(t)+u_{c}(t)\right)+\zeta_{3}(t) \\
K^{\star} \ni \lambda(t) & \perp w(t)=\binom{0}{-1} \zeta_{1}(t)+\binom{0}{1} \zeta_{2}(t)+\binom{-\frac{1}{\mathrm{C}}}{0} \zeta_{3}(t) \in K .
\end{aligned}\right.
$$

Let $x_{1}=\zeta_{1}, x_{2}=\frac{1}{\mathrm{C}} \zeta_{3}, z=\zeta_{2}, B_{1}=\left(\begin{array}{ll}0 & -1\end{array}\right), B_{2}=\left(\begin{array}{ll}1 & -\frac{1}{R}\end{array}\right), B_{3}=\left(\begin{array}{ll}0 & -1\end{array}\right), C_{1}=\binom{0}{-1}, C_{3}=\binom{0}{1}, C_{2}=\binom{-1}{0}$, $D=0$. This dynamics is rewritten in the canonical form as:

$$
\left\{\begin{aligned}
\dot{x}_{1}(t) & =-\lambda_{2}(t)-u_{c}(t)+\frac{u_{1}(t)}{L} \\
\dot{x}_{2}(t) & =z(t)+\lambda_{1}(t)-\frac{1}{R}\left(\lambda_{2}(t)+u_{c}(t)\right) \\
0 & =-\lambda_{2}(t)-u_{c}(t)+x_{2}(t) \\
K^{\star} \ni \lambda(t) & \perp w(t)=\binom{0}{-1} x_{1}(t)+\binom{-1}{0} x_{2}(t)+\binom{0}{1} z(t) \in K .
\end{aligned}\right.
$$

In the simulations in Figures 6a and 6b it has been chosen $u_{c}(t)=\sin (10 t)-1 \leq 0$, which complies with the system's intrinsic constraints. Then (when $u_{1}(t)=0$ ) one recovers the standard form in (2.2), with a different class of sets $K$, however. One has $A_{1}=0, B_{3}^{\top} X_{3}=-C_{3}$ with $X_{3}=1$, and $X_{1} B_{1}=C_{1}^{\top}$ (this follows necessarily from (D.2) and $Q_{1} \leqslant 0$ and using [25, Lemma A.69]), with $X_{1}=1$ (hence the quadruple $\left(A_{1}, B_{1}, C_{1}, \hat{D}\right)$ is passive but it is not strictly passive and its associated transfer function is not strongly SPR). Also $\hat{D}=-C_{3} B_{2}-C_{2} B_{3}=\left(\begin{array}{cc}0 & 1 \\ -1 & \frac{1}{R}\end{array}\right) \geqslant 0$. Also $B_{3}^{\top} X_{3} B_{3}=\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right) \geqslant 0$.
It is computed that $\bar{D}_{h}=\left(\begin{array}{cc}0 & h \\ -h & \frac{h}{R}+h^{2}+1\end{array}\right)$. The various sets for the set $\mathcal{S}$ in Proposition 3.2 are calculated as: $K^{*}=\mathbb{R}^{+} \times\{0\}, \operatorname{Ker}\left(\bar{D}_{h}+\bar{D}_{h}^{\mathrm{T}}\right)=\left(\begin{array}{cc}0 & 0 \\ 0 & 2\left(\frac{h}{R}+h^{2}+1\right)\end{array}\right)=\mathbb{R} \times\{0\},\left\{w \mid \bar{D}_{h} w \in K=\right.$ $\left.\mathbb{R}^{+} \times \mathbb{R}\right\}=\mathbb{R} \times \mathbb{R}^{+}$. So $S=\mathbb{R}^{+} \times\{0\} \neq 0$ and item 2 of Proposition 3.2 has to be applied. It is calculated that $\left\langle h \beta_{k}-\bar{D}_{h} \xi_{0}, v\right\rangle=\left\langle\binom{-h \xi_{0}^{1}}{-h x_{1, k}-x_{2, k}+h \xi_{0}^{1}-\left(\frac{h}{2}+h^{2}+1\right) \xi_{0}^{2}}\right.$. Also $v \in K^{\star}$ implies that $v_{1} \geq 0$ and $v_{2}=0$, so it follows that $\left\langle h \beta_{k}-\bar{D}_{h} \xi_{0}, v\right\rangle+\psi_{K^{\star}}(v)>0$ is guaranteed, if $v_{1}>0$, by taking $\xi_{0}^{1}<0$. It is inferred that item 2 in Proposition 3.2 is verified and the OSNSP has at least one solution.
Any two solutions $\lambda_{k+1}^{1}$ and $\lambda_{k+1}^{2}$ satisfy $\lambda_{2, k+1}^{1}=\lambda_{2, k+1}^{2}$ for all $h \geq 0$, in view of the form of $\bar{D}_{h}$. Then both $x_{1, k+1}$ and $x_{2, k+1}$ are unique due to the form of $B_{1}$ and $B_{3}$. For the convergence analysis, let us note that this system does not satisfy Assumption 2, therefore let us use the material in section 5 . Using the Frobenius norm one finds that $h_{3}=\sqrt{2 h^{2}+\left(\frac{h}{R}-1\right)^{2}}$, using the Hölder induced norm $\|\cdot\|_{2,2}=\sigma_{\max }(\cdot)$ one finds a smaller $h_{3}=\sqrt{h^{2}+\left(\frac{h}{R}-1\right)^{2}}$, and $0 \leq h_{3} \leq 1$ for all $h \in\left[0, \frac{2 R}{1+R^{2}}\right]$. The material in section 5 is based on the LCP in (5.14), for which the existence of a minimal norm solution satisfying the inequalities stated below (5.14) is guaranteed. However in this example $K=R_{+} \times R \neq R_{+}^{2}$. The following problem has to be studied instead (it is noteworthy that the rest of the analysis remains unchanged as long as the minimal norm solution can be characterised):

$$
\begin{equation*}
K \ni B_{3}^{\top} X_{3} B_{3} \lambda_{k+1}+q_{k}^{\lambda} \perp \lambda_{k+1} \in K^{\star} \tag{6.11}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
R_{+} \times\{0\} \ni\binom{\lambda_{1, k+1}}{\lambda_{2, k+1}} \perp\binom{0}{\lambda_{2, k+1}}+q_{k}^{\lambda} \in R_{+} \times R \tag{6.12}
\end{equation*}
$$

Therefore $\lambda_{2, k+1}=0$, and the value of $\lambda_{1, k+1}=0$ is always a solution (notice that the external excitation is constrained to be non negative to comply with the system's constraints). This result agrees with the numerical simulations in Figure 6b.


Figure 5: An RLCD passive circuit.


Figure 6: Implicit Euler numerical simulation of (6.9) with $h=5 \cdot 10^{-6} \mathrm{~s}$. The solutions of $x_{1}(t)$ are not displayed as they are independent of $z(t), \lambda_{1}(t)$. The variable $\lambda_{2}(t)$ trivially equals 0 , and is not computed. as:

### 6.3 THIRD EXAMPLE

Let us consider now an academic toy-example directly in the form of (2.2), whose dynamics are given

$$
\left\{\begin{aligned}
\dot{x}_{1}(t) & =-2 x_{1}(t)+2 \lambda_{1}(t)-\lambda_{2}(t) \\
\dot{x}_{2}(t) & =z(t) \\
0 & =x_{2}(t)+\lambda_{1}(t)-2 \lambda_{2}(t) \\
0 \leq\binom{\lambda_{1}(t)}{\lambda_{2}(t)} & \perp\binom{2}{-1} x_{1}(t)+\binom{1}{3} x_{2}(t)+\binom{-1}{2} z(t)+\left(\begin{array}{cc}
2 & -1 \\
0 & 1
\end{array}\right)\binom{\lambda_{1}(t)}{\lambda_{2}(t)} \geq 0
\end{aligned}\right.
$$

It can be verified that the transfer matrix $H(s)=D+C(s P-A)^{-1} B$ is positive real [25, Definition 2.34], and the state space realization is minimal (see section G). In fact $H(s)$ is even strongly (or extended) positive real [25, Definition 2.78]. Thus [28, Theorem 5.4 (A') ( $\left.\mathrm{B}^{\prime}\right)$ ] applies. It is noteworthy that since $D=\left(\begin{array}{cc}2 & -1 \\ 0 & 1\end{array}\right)>0$, it follows that the multipliers $\lambda$ are Lipschitz continuous functions of the state $\left(x_{1}, x_{2}, z\right)$. Thus the DVLCS is a DAE with Lipschitz continuous nonlinear and nonsmooth right-hand side. From (6.13) it follows that: $A_{1}=-2<0, C_{1}=X_{1} B_{1}^{\mathrm{T}}=(2,-1)^{\mathrm{T}}$ with $X_{1}=1>0, X_{3}=I_{2}$, $B_{3}=\left(\begin{array}{ll}1 & -2\end{array}\right)=-C_{3}^{\top}$, and $\hat{D}=\left(\begin{array}{cc}1 & 1 \\ -3 & 7\end{array}\right)>0$ with $\left(\hat{D}+\hat{D}^{\mathrm{T}}\right)>0$. It follows from [28, Theorem $5 \cdot 4$ (A'), Theorem 4.6] that the quadruple $\left(A_{1}, B_{1}, C_{1}, \hat{D}\right)$ is strictly passive. It can be checked that the set $\mathcal{S}=\{0\}$ in Proposition 3.2 for any $h>0$, thus there exists a unique solution $\lambda_{k+1}$ to the implicit Euler discretization of (6.13). However $C_{3} \neq 0$ and Assumption 2 does not hold, preventing the convergence analysis in section 4 and the stability analysis in section 4.1 to apply. Consequently, the "equivalent" system (5.1) from Section 5 is analysed:

$$
\left\{\begin{align*}
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & -2 \\
0 & -2 & 4
\end{array}\right)\left(\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{\lambda}_{1}(t) \\
\dot{\lambda}_{2}(t)
\end{array}\right) & =\left(\begin{array}{ccc}
-2 & 2 & -1 \\
-2 & -1 & -1 \\
1 & 3 & -7
\end{array}\right)\left(\begin{array}{c}
x_{1}(t) \\
\lambda_{1}(t) \\
\lambda_{2}(t)
\end{array}\right)+\left(\begin{array}{c}
0 \\
w_{1}(t) \\
w_{2}(t)
\end{array}\right)  \tag{6.14}\\
w(t) & \in-\mathcal{N}_{\mathbb{R}_{+}^{2}}(\lambda(t))
\end{align*}\right.
$$

Let us check if Assumption 3 (c) is verified. Using the Frobenius norm, one can obtain that $0 \leq h_{3}=$ $\sqrt{60 h^{2}+25-66 h} \leq 1$ if $0<h \leq 0.7$. In the context of (6.14), the LCP (5.14) becomes:

$$
0 \leq\binom{\lambda_{1, k+1}}{\lambda_{2, k+1}} \perp\left(\begin{array}{cc}
1 & -2 \\
-2 & 4
\end{array}\right)\binom{\lambda_{1, k+1}}{\lambda_{2, k+1}}+\left(\begin{array}{cc}
(h-1) & (h+3) \\
(2-3 h) & (7 h-4)
\end{array}\right)\binom{\lambda_{1, k}}{\lambda_{2, k}}+\binom{2 h x_{1, k}}{-h x_{1, k}} \geq 0
$$

$$
\begin{equation*}
\Longleftrightarrow \tag{6.15}
\end{equation*}
$$

$$
0 \leq\binom{\lambda_{1, k+1}}{\lambda_{2, k+1}} \perp\binom{\lambda_{1, k+1}-2 \lambda_{2, k+1}}{4 \lambda_{2, k+1}-2 \lambda_{1, k+1}}+\binom{q_{1, k}}{q_{2, k}} \geq 0
$$

From Corollary 4.2, there exists a solution to (6.15). We can compute $\lambda_{k+1}$ in each mode of the LCP where $\lambda_{k+1} \neq 0$ :

1. If $\lambda_{1, k+1}=0$ and $\lambda_{2, k+1} \geq 0$ then $\left\|\lambda_{2, k+1}\right\| \leq \frac{1}{2}\left\|q_{1, k}\right\|$.
2. If $\lambda_{2, k+1}=0$ and $\lambda_{1, k+1} \geq 0$ then $\left\|\lambda_{1, k+1}\right\| \leq \frac{1}{2}\left\|q_{2, k}\right\|$.
3. If $\lambda_{1, k+1}>0$ and $\lambda_{2, k+1}>0$ there is infinitely many solutions. Indeed, we obtain :

$$
\left\{\begin{align*}
\lambda_{1, k+1} & =2 \lambda_{2, k+1}-q_{1, k}  \tag{6.16}\\
0 & =2 q_{1, k}+q_{2, k}
\end{align*}\right.
$$

From [16, Proposition 6.1.7], there exists a minimal solution $\lambda_{\min , k+1}$. Then, (6.16)'s particular solution $\lambda_{2, k+1}=-\lambda_{1, k+1}$ leads to $\left\|\lambda_{\text {min, } k+1}\right\| \leq(1 / 3)\left\|q_{k+1}\right\|$. It follows that $\eta=\frac{1}{2}<1$ for all $h>0$, and $\eta h_{3} \leq 1$ for all $0<h \leq 1.4$, and finally, the results from Section 5.2 hold if small enough time steps are chosen. Problem (6.14) is simulated using the discretization (6.15). The LCP (6.15) is solved using the numerical solver Lemke provided in the Siconos Toolbox ${ }^{3}$ [1]. In Figures 7 a and 7 b the simulated variables $\left(x_{1}, x_{2}, z\right)$ and $\left(\lambda_{1}, \lambda_{2}\right)$ are displayed for time step $h=5 \cdot 10^{-6} \mathrm{~s}$. The initial conditions are ( $x_{1,0}=5, \lambda_{1,0}=5, \lambda_{2,0}=0$ ) and the discrete variables $x_{2, k}$ and $z_{k}$ are given by the discretization of (5.1) (c):

$$
\left\{\begin{align*}
x_{2, k} & =-B_{3} \lambda_{k}  \tag{6.17}\\
z_{k} & =-B_{3} \frac{\lambda_{k+1}-\lambda_{k}}{h}
\end{align*}\right.
$$

with $h$ the time step. Let us notice in Figure 7a that the variable $z$ possesses a discontinuity. In Figure 8 a we plot the error provided in (6.6), with respect to a reference solution taken for $h=5.10^{-6}$ s. We confirm the order one convergence rate for $h>0$ sufficiently small. Finally, let us also notice that the solutions of the schemes in (3.1) and in (5.5), are not exactly the same (recall that the continuous systems (2.2) and (5.1) are "equivalent" in the sense explained after (5.3), see also section H ). This can be observed in Figure 8b where the differences between the two schemes are shown for ( $x_{1}, x_{2}$ ).


Figure 7: Implicit Euler numerical simulation of (6.13) with $h=5 \cdot 10^{-6} \mathrm{~S}$

## 7 CONCLUSION

This article deals with the time-discretization of a class of singular linear complementarity systems (descriptor variable linear complementarity systems) which satisfy a passivity constraint. The implicit Euler method is analysed. First the one-step nonsmooth problem is studied, and conditions which guarantee its well-posedness are given. Then the approximate solutions convergence is analysed, and it is shown that under some reasonable assumptions, convergence holds. Our results are however

[^3]

Figure 8: Numerical convergence studies, with error formulas (6.6), of schemes (3.1) and (5.5). The reference is computed with the scheme (5.5) and $h=5 \cdot 10^{-6} \mathrm{~s}$.
partial in the sense that we are not able to prove the convergence of all the variables. One academic example and two examples of electrical circuits with ideal diodes (equivalently hydraulic circuits with check valves) illustrate the developments, with numerical simulations.

## APPENDIX A PRESERVATION OF POSITIVE DEFINITENESS WITH PERTURBATION

We give here an excerpt of [36, Theorem 2.11], and a corollary of it. Let us recall that for a given $M \in R^{n \times n},\|M\|_{2,2}$ is the induced matricial norm such that $\|M\|_{2,2}=\sigma_{\max }(M)$ (the largest singular value).
Theorem A.1. [36] Let $M \in R^{n \times n}$ be a positive definite matrix. Then every matrix

$$
A \in\left\{A \in R^{n \times n} \left\lvert\,\left\|\left(\frac{M+M^{\top}}{2}\right)^{-1}\right\|_{2,2}\|M-A\|_{2,2}<1\right.\right\}
$$

is positive definite.
Corollary A.2. [21] Let $D=P+N$, where $D, P$ and $N$ are $n \times n$ real matrices, and $P>0$, not necessarily symmetric. If
(A.1)

$$
\|N\|_{2}<\frac{1}{\left\|\left(\frac{P+P^{T}}{2}\right)^{-1}\right\|_{2,2}}
$$

then $D>0$.
Consider the term $h_{1}$ in Assumptions 1 and 3. If $h<\frac{1}{\sigma_{\max }\left(A_{1}\right)}$ then $I_{n_{1}}+h A_{1}>0$ from Corollary A.2. If in addition $A_{1} \leqslant(<) 0$ then $0<I_{n_{1}}+h A_{1} \leqslant(<) I_{n_{1}}$. Consider now $\bar{A}_{1}=\left(I_{n_{1}}+h A_{1}\right)^{\top}\left(I_{n_{1}}+h A_{1}\right)=$ $I_{n_{1}}+h\left(A_{1}+A_{1}^{\top}\right)+h^{2} A_{1}^{\top} A_{1}=I_{n_{1}}+h \tilde{A}_{1}$, with $\tilde{A}_{1}=A_{1}+A_{1}^{\top}+h A_{1}^{\top} A_{1}=\tilde{A}_{1}^{\top}$. If $A_{1}<0\left(\Rightarrow A_{1}+A_{1}^{\top}<0\right)$, then $\tilde{A}_{1}<0$ for $h>0$ small enough, still using Corollary A.2. Thus $\bar{A}_{1}<I_{n_{1}}$ for $h>0$ small enough. Morover for $h>0$ small enough it follows that $\bar{A}_{1}>0$ by Corollary A.2. It is deduced that there exists $h_{\max }>0$ such that $0<\bar{A}_{1}<I_{n_{1}}$ for all $h \in\left(0, h_{\max }\right)$. From [16, Lemma 8.4.1], $0<\lambda_{\min }\left(\bar{A}_{1}\right)$ and $\lambda_{\max }\left(\bar{A}_{1}\right)<1$. It is inferred that $\sigma_{\max }\left(I_{n_{1}}+h A_{1}\right)=\left\|I_{n_{1}}+h A_{1}\right\|_{2,2}=h_{1}<1$.

## APPENDIX B WELL-POSEDNESS OF VARIATIONAL INEQUALITIES

The next results use the notions of recession functions and cones, which we briefly introduce now (see [24,50] for illustrating examples), [69,50, 84]. Let $f: R^{n} \rightarrow R \cup\{+\infty\}$ be a proper convex and lower semi-continuous function, we denote by $\operatorname{dom}(f) \stackrel{\Delta}{=}\left\{x \in R^{n} \mid f(x)<+\infty\right\}$ the domain of the function $f(\cdot)$. The Fenchel transform $f^{\star}(\cdot)$ of $f(\cdot)$ is the proper, convex and lower semicontinuous function defined by

$$
\begin{equation*}
\text { (for all } \left.z \in R^{n}\right) \mid f^{\star}(z)=\sup _{x \in \operatorname{dom}(f)}\{\langle x, z\rangle-f(x)\} \tag{B.1}
\end{equation*}
$$

The subdifferential $\partial f(x)$ of $f(\cdot)$ at $x \in R^{n}$ is defined by

$$
\partial f(x)=\left\{\omega \in R^{n} \mid f(v)-f(x) \geq\langle\omega, v-x\rangle, \forall v \in R^{n}\right\} .
$$

We denote by $\operatorname{Dom}(\partial f) \stackrel{\Delta}{=}\left\{x \in R^{n} \mid \partial f(x) \neq \emptyset\right\}$ the domain of the subdifferential operator $\partial f: R^{n} \rightarrow R^{n}$. Recall that (see e.g. Theorem 2, Chapter 10, Section 3 in $[11]): \operatorname{Dom}(\partial f) \subset \operatorname{dom}(f)$.

Let $x_{0}$ be any element in the domain $\operatorname{dom}(f)$ of $f(\cdot)$, the recession function $f_{\infty}(\cdot)$ of $f(\cdot)$ is defined by

$$
\text { (for all } \left.x \in R^{n}\right): f_{\infty}(x)=\lim _{\lambda \rightarrow+\infty} \frac{1}{\lambda} f\left(x_{0}+\lambda x\right)
$$

The function $f_{\infty}: R^{n} \rightarrow R \cup\{+\infty\}$ is a proper convex and lower semi-continuous function which describes the asymptotic behavior of $f(\cdot)$.

Let $K \subset R^{n}$ be a nonempty closed convex set. Let $x_{0}$ be any element in $K$. The recession cone of $K$ is defined by [69] [84, Definition 1.11]:

$$
K_{\infty}=\bigcap_{\lambda>0} \frac{1}{\lambda}\left(K-x_{0}\right)=\left\{u \in R^{n} \mid x+\lambda u \in K, \forall \lambda \geq 0, \forall x \in K\right\} .
$$

The set $K_{\infty}$ is a nonempty closed convex cone that is described in terms of the directions which recede from $K$. The indicator function of a set $K \subseteq R^{n}$ is $\psi_{K}(x)=0$ if $x \in K, \psi_{K}(x)=+\infty$ if $x \notin K$. If $K$ is closed non empty convex, we have $\partial \psi_{K}(x)=\mathcal{N}_{K}(x)$, the so-called normal cone to $K$ at $x$, defined as $\mathcal{N}_{K}(x)=\left\{v \in R^{n} \mid v^{\top}(s-x) \leq 0\right.$ for all $\left.s \in K\right\}$. When $K$ is finitely represented, i.e., $K=\left\{x \in R^{n} \mid k_{i}(x) \geq 0,1 \leq i \leq m\right\}$, and if the functions $k_{i}(\cdot)$ satisfy some constraint qualification (like, independency, or extensions like the MFCQ), then $\mathcal{N}_{K}(x)$ is generated by the outwards normals at the active constraints $k_{i}(x)=0$, i.e., $\mathcal{N}_{K}(x)=\left\{v \in R^{n} \mid v=-\lambda_{i} \nabla k_{i}(x), k_{i}(x)=0, \lambda_{i} \geq 0\right\}$.
Let us here recall some important properties of the recession function and recession cone (see e.g., [17, Proposition 1.4.8]):
Proposition B.1. The following statements hold:
a) Let $f_{1}: R^{n} \rightarrow R \cup\{+\infty\}$ and $f_{2}: R^{n} \rightarrow R \cup\{+\infty\}$ be two proper, convex and lower semi-continuous functions. Suppose that $f_{1}+f_{2}$ is proper. Then for all $x \in R^{n}:\left(f_{1}+f_{2}\right)_{\infty}(x)=\left(f_{1}\right)_{\infty}(x)+\left(f_{2}\right)_{\infty}(x)$.
b) Let $f: R^{n} \rightarrow R \cup\{+\infty\}$ be a proper, convex and lower semi-continuous function and let $K$ be $a$ nonempty closed convex set, such that $f+\Psi_{K}$ is proper (equivalently $\operatorname{dom}(f) \cap K$ is non empty). Then for all $x \in R^{n}:\left(f+\Psi_{K}\right)_{\infty}(x)=f_{\infty}(x)+\left(\Psi_{K}\right)_{\infty}(x)$.
c) Let $K \subset R^{n}$ be a nonempty, closed and convex set. Then for all $x \in R^{n}:\left(\Psi_{K}\right)_{\infty}(x)=\Psi_{K_{\infty}}(x)$. Moreover for all $x \in K$ and $e \in K_{\infty}: x+e \in K$.
d) If $K \subseteq R^{n}$ is a nonempty closed and convex cone, then $K_{\infty}=K$.
e) Let $K=P(A, b) \triangleq\left\{x \in R^{n} \mid A x \geq b\right\}$ for $A \in R^{m \times n}$ and $b \in R^{m}$. If $K \neq \emptyset$ then $K_{\infty}=P(A, 0)=\{x \in$ $\left.R^{n} \mid A x \geq 0\right\}$.
f) $K \subset R^{n}$ is a non-empty closed convex bounded set if and only if $K_{\infty}=\left\{0_{n}\right\}$.
g) Let $f: R^{n} \rightarrow R \cup\{+\infty\}$ be a proper, convex and lower semi-continuous function. Then epi $\left(f_{\infty}\right)=$ $(\text { epi }(f))_{\infty}$.

Sets as in item e) are called $\mathcal{H}$-polyhedra, and there is an equivalence between sets $P(A, 0)$ and finitely generated convex cones [84, Theorem 1.3]. Let us now concatenate [5, Theorem 3, Corollaries 3 and 4]. They concern variational inequalities (VIs) of the form: Find $u \in R^{n}$ such that
$\langle\mathbf{M} u+\mathbf{q}, v-u\rangle+\varphi(v)-\varphi(u) \geq 0$, for all $v \in R^{n}$
where $\mathrm{M} \in R^{n \times n}$ is a real matrix, $\mathrm{q} \in R^{n}$ a vector and $\varphi: R^{n} \rightarrow R \cup\{+\infty\}$ a proper convex and lower semicontinuous function. The VI in (в.2) is equivalent to the inclusion $\mathbf{M} u+\mathbf{q} \in-\partial \varphi(u) \Leftrightarrow u \in$ $(\mathbf{M}+\partial \varphi)^{-1}(-\mathbf{q})$.

The problem in (в.2) is denoted as $V I(\mathbf{M}, \mathbf{q}, \varphi)$ in the next proposition. We also set:

$$
\begin{equation*}
\mathcal{K}(\mathbf{M}, \varphi)=\left\{x \in R^{n} \mid \mathbf{M} x \in\left(\operatorname{dom}\left(\varphi_{\infty}\right)\right)^{\star}\right\} . \tag{в.3}
\end{equation*}
$$

Note that $\left(\operatorname{dom}\left(\varphi_{\infty}\right)\right)^{\star}$ is the dual cone of the domain of the recession function $\varphi_{\infty}$ while $(\operatorname{dom}(\varphi))_{\infty}$ (that we may denote also as $\left.\operatorname{dom}(\varphi)_{\infty}\right)$ is the recession cone of $\operatorname{dom}(\varphi)$.

Proposition B.2. [5] Let $\varphi: R^{n} \rightarrow R \cup\{+\infty\}$ be a proper, convex and lower semicontinuous function with closed domain, $\mathrm{M} \in R^{n \times n}$, and suppose that $\mathbf{M} \geqslant 0$ (not necessarily symmetric).
a) If $(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi)=\{0\}$ then for each $\mathbf{q} \in R^{n}$, problem $V I(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.
b) Suppose that $(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq\{0\}$. If there exists $x_{0} \in \operatorname{dom}(\varphi)$ such that
(в.4) $\quad\left\langle\mathbf{q}-\mathbf{M}^{\top} x_{0}, v\right\rangle+\varphi_{\infty}(v)>0, \forall v \in(\operatorname{dom}(\varphi))_{\infty} \cap \operatorname{ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0$,
then problem $V I(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.
$b^{\prime}$ ) If $\mathrm{M}=\mathrm{M}^{\top}$ then one can take $x_{0}=0$ in $b$ ).
c) If $u_{1}$ and $u_{2}$ denote two solutions of problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ then $u_{1}-u_{2} \in \operatorname{ker}\left\{\mathbf{M}+\mathbf{M}^{\top}\right\}$.
d) If $\mathbf{M}=\mathbf{M}^{\top}$ and $u_{1}$ and $u_{2}$ denote two solutions of problem $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$, then $\left\langle\mathbf{q}, u_{1}-u_{2}\right\rangle=\varphi\left(u_{2}\right)-\varphi\left(u_{1}\right)$.
e) If $\mathbf{M}=\mathbf{M}^{\top}$ and $\varphi(x+z)=\varphi(x)$ for all $x \in \operatorname{dom}(\varphi)$ and $z \in \operatorname{ker}\{\mathbf{M}\}$ and $\langle\mathbf{q}, e\rangle \neq 0$ for all $e \in \operatorname{ker}\{\mathbf{M}\}, e \neq 0$, then problem $V I(\mathbf{M}, \mathbf{q}, \varphi)$ has at most one solution.
f) If $\mathbf{M}=\mathbf{M}^{\top}$, then $u$ is a solution of $\operatorname{VI}(\mathbf{M}, \mathbf{q}, \varphi)$ if and only if it is a solution of the optimization problem $\min _{x \in \boldsymbol{R}^{n}} \frac{1}{2} x^{\top} \mathbf{M} x+\langle\mathbf{q}, x\rangle+\varphi(x)$.

Notice that the function $\varphi(\cdot)$ will never be strictly convex in our case (it is an indicator function) so that the strict convexity argument of [5, Theorem 5] which applies when $\mathbf{M}$ is a $P_{0}$-matrix never holds. The study of VIs as in (в.2) can be traced back to [73].

## APPENDIX C SOME CONVEX ANALYSIS AND COMPLEMENTARITY THEORY TOOLS

If $K \subset R^{n}$ is a set, then $K^{\star}=\left\{z \in R^{n} \mid\langle z, x\rangle \geq 0\right.$ for all $\left.x \in K\right\}$ is its dual cone. Its closure is denoted $\bar{K}$. Let $K$ be a closed convex cone, then:

$$
\begin{equation*}
K^{\star} \ni x \perp y \in K \Leftrightarrow x \in-\mathcal{N}_{K}(y) \Longleftrightarrow y \in-\mathcal{N}_{K^{\star}}(x) . \tag{c.1}
\end{equation*}
$$

(c.2)

$$
M(x-y) \in-\mathcal{N}_{K}(x) \Leftrightarrow x=\operatorname{proj}_{M}[K ; y] \Leftrightarrow x=\min _{z \in K} \frac{1}{2}(z-y)^{\top} M(z-y)
$$

We note that this is a particular case of (в.2), so that Proposition B. 2 can be considered as the characterization of a generalized projection operator $V I(\mathbf{M}, \mathbf{q}, \varphi)$. The first equivalence is [15, Proposition 6.46].

## APPENDIX D PASSIVE DESCRIPTOR VARIABLE SYSTEMS

In this paper we call a quadruple $(A, B, C, D)$ passive if the next LMI is satisfied:
(D.1)

$$
\left(\begin{array}{cc}
-X A-A^{\top} X & -X B+C^{\top} \\
-B^{\top} X+C & D+D^{\top}
\end{array}\right) \geqslant 0, \quad X=X^{\top}>0
$$

It is said strictly passive if the first matrix inequality is satisfied with $>0$. This is not to be confused with the strict state passivity [25, Definition 4.54 ], since strict passivity implies strict state passivity, but not the inverse in general. In fact strict passivity is directy related to strongly SPR transfer matricess [25, p. 62 and section 3.12.2], while SPR transfer matrices are strictly state passive [25, Theorem 4.73] [63].
Positive real transfer matrices [25, Definition 2.34] with a state space realization represented by quintuples ( $P, A, B, C, D$ ) which are minimal (see Appendix G for conditions of minimality of a statespace realization for descriptor systems) can be rewritten as in $(2.2)$ [28, 48], and they have the positive semidefinite storage function $V\left(x_{1}, x_{2}\right)=\frac{1}{2}\left(x_{1}^{\top} X_{1} x_{1}+x_{2}^{\top} X_{3} x_{2}\right)$, with supply rate $\lambda^{\top} w$. Indeed, by differentiating $V\left(x_{1}(t), x_{2}(t)\right)$ along the trajectories of (2.2), adding $\pm \lambda^{\top} w$ and completing the squares, one can compute that:
(D.2)

$$
\dot{V}(t)=\frac{1}{2}\left(x_{1}^{\top}, \lambda^{\top}\right) \underbrace{\left(\begin{array}{cc}
X_{1} A_{1}+A_{1}^{\top} X_{1} & X_{1} B_{1}-C_{1}^{\top} \\
B_{1}^{\top} X_{1}-C_{1} & -\left(\hat{D}+\hat{D}^{\top}\right)
\end{array}\right)}_{\triangleq Q_{1}}\binom{x_{1}}{\lambda}+\lambda^{\top}(t) w(t)
$$

where $X_{1}=X_{1}^{\top} \geqslant 0, Q_{1} \leqslant 0$, and if the system is strongly SPR then $X_{1}>0, Q_{1} \prec 0$ [28, Theorem 5.4]. The equality in (D.2) is an infinitesimal dissipation equality, showing that the system is passive [25, Definition 2.1, Theorem 4.53]. Such systems possess particular structural properties [30, 33] [25, section 3.8]. In the SSPR case it follows from (D.2) and the complementarity conditions that $x_{1}(\cdot)$ and $x_{2}^{\top} X_{3} x_{2}(\cdot)$ are bounded for any bounded initial condition.

## APPENDIX E GRONWALL-BACHPATTE INEQUALITY [13, 44]

Let us consider a positive real numbers sequence $\left\{x_{k}\right\}$ such that $x_{k} \leq \alpha_{k}+\sum_{i=0}^{k-1} \beta_{i} x_{i}$, where $\left\{\alpha_{k}\right\}$ and $\left\{\beta_{k}\right\}$ are two real numbers sequences with $\left\{\beta_{k}\right\}$ positive. Then one has

$$
x_{k} \leq \alpha_{k}+\sum_{i=0}^{k-1} \beta_{i} \alpha_{i} \Pi_{j=i+1}^{k}\left(1+\beta_{i}\right)
$$

## APPENDIX F CALCULATION OF $\beta_{i}$

The starting point is the term $\eta\left\|C_{1}\right\|\left\|B_{1}\right\| \sum_{i=1}^{k} h_{1}^{k-i} \sum_{j=0}^{i-1}\left(\eta h_{3}\right)^{i-1-j}\left\|x_{1, j+1}\right\|$ in (5.19). Let us consider the factors of the terms $\left\|x_{1, j+1}\right\|$ separately:

- Factor of $\left\|x_{1,1}\right\|$ (all terms with $\left.j=0\right): h_{1}^{k-1}+h_{1}^{k-2}\left(\eta h_{3}\right)+h_{1}^{k-3}\left(\eta h_{3}\right)^{2}+\ldots+h_{1}\left(\eta h_{3}\right)^{k-2}+\left(\eta h_{3}\right)^{k-1}=$ $\sum_{i=1}^{k} h_{1}^{k-i}\left(\eta h_{3}\right)^{i-1}$
- Factor of $\left\|x_{1,2}\right\|($ all terms with $j=1): h_{1}^{k-2}+h_{1}^{k-3}\left(\eta h_{3}\right)+h_{1}^{k-4}\left(\eta h_{3}\right)^{2}+\ldots+h_{1}\left(\eta h_{3}\right)^{k-3}+\left(\eta h_{3}\right)^{k-2}=$ $\sum_{i=2}^{k} h_{1}^{k-i}\left(\eta h_{3}\right)^{i-2}$.
-...
- Factor of $\left\|x_{1, k-1}\right\|($ all terms with $j=k-2): h_{1}+\left(\eta h_{3}\right)=\sum_{i=k-1}^{k} h_{1}^{k-i}\left(\eta h_{3}\right)^{i-k+1}$.
- Factor of $\left\|x_{1, k}\right\|($ all terms with $j=k): 1=\sum_{i=k}^{k} h_{1}^{k-i}\left(\eta h_{3}\right)^{i-k}$.

Therefore one infers (5.22).

## APPENDIX G MINIMALITY OF STATE SPACE REALIZATIONS

Let us report the results stated in [28, Theorem 3.1] [41, Theorems 2-2.1, 2-3.1, 2-6.3].
Theorem G.1. Let $H(s)=C(s P-A)^{-1} B+D, s \in \mathbb{C}$, be a rational transfer function where $P$ and $A$ are $n \times n$ matrices. Then $P \dot{x}(t)=A x(t)+B u(t), y(t)=C x(t)+D u(t)$ is a minimal realization of $H(s)$ if, and only if, the following conditions are satisfied:

- $\operatorname{rank}(A-s P B)=n$ for all $s \in \mathbb{C}$,
- $\operatorname{rank}\left(\begin{array}{ll}P & B\end{array}\right)=n$,
- $\operatorname{rank}\left(A^{\top}-s P^{\top} C^{\top}\right)=n$,
- $\operatorname{rank}\left(P^{\top} C^{\top}\right)=n$,
- $A \operatorname{Ker}(P) \subseteq \operatorname{Im}(P)$.


## APPENDIX H DIFFERENCES BETWEEN (3.1) AND IN (5.5)

Let us consider the discrete scheme (5.5) with $E(t)=0$ and $X_{1}$ full rank, this is rewritten as:
(H.1)

$$
\begin{aligned}
& \text { (a) } x_{1, k+1}=x_{1, k}+h A_{1} x_{1, k}+h B_{1} \lambda_{k} \\
& \text { (b) } B_{3}^{\top} X_{3} B_{3} \lambda_{k+1}=B_{3}^{\top} X_{3} B_{3} \lambda_{k}-h C_{1} x_{1, k}-h \hat{D} \lambda_{k}+w_{k+1} \\
& \text { (c) } 0 \leq \lambda_{k+1} \perp w_{k+1}=C_{1} x_{1, k+1}+C_{2} x_{2, k+1}+C_{3} z_{k+1}+D \lambda_{k+1} \geq 0 \\
& \text { (d) } x_{2, k}=-B_{3} \lambda_{k} \\
& \text { (e) } z_{k}=-B_{2} \lambda_{k}-B_{3} \frac{\lambda_{k+1}-\lambda_{k}}{h} \text {. }
\end{aligned}
$$

This is the numerical scheme with which the examples have been simulated. Combining (н.1) (d) and (e), gives:

$$
\begin{equation*}
x_{2, k+1}=x_{2, k}+h z_{k}+h B_{2} \lambda_{k} . \tag{H.2}
\end{equation*}
$$

Notice that this is the choice made in (6.17). The equations in (н.1) (a) and (н.2) are not equal to their counterparts in (3.1), because $\lambda_{k}$ and $z_{k}$ are used in (н.1) (a) and (н.2) instead of $\lambda_{k+1}$ and $z_{k+1}$ in (3.1). This explains that the numerical data obtained from both schemes, differ. This also explains why the convergence results obtained for both schemes, are not the same neither.

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[^1]:    ${ }^{1}$ That is, all its eigenvalues have a negative real part.

[^2]:    ${ }^{2}$ Source code of the examples can be found at https://github.com/siconos/siconos-tutorials/tree/master/.sandbox/mlcspaper

[^3]:    ${ }^{3}$ https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html

