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# ANALYSIS OF THE IMPLICIT EULER TIME-DISCRETIZATION OF PASSIVE DESCRIPTOR-VARIABLE LINEAR COMPLEMENTARITY SYSTEMS

Bernard Brogliato\*      Alexandre Rocca†

**Abstract** This article is largely concerned with the time-discretization of descriptor-variable systems coupled to with complementarity constraints. They are named descriptor-variable linear complementarity systems (DVLCS). More specifically passive DVLCS with minimal state space representation are studied. The Euler implicit discretization of DVLCS is analysed: the one-step non-smooth problem (OSNSP), that is a generalized equation, is shown to be well-posed under some conditions. Then the convergence of the discretized solutions is studied. Several examples illustrate the applicability and the limitations of the developments.

## 1 INTRODUCTION

The analysis of non-smooth dynamical systems with set-valued right-hand sides satisfying maximal monotone properties has been the object of many contributions, see [26] for a survey. Linear and nonlinear complementarity dynamical systems, various kinds of projected dynamical systems, of differential variational inequalities, differential inclusions with maximal monotone right-hand sides, Moreau's sweeping processes, and some switching dynamics can be recast in this class of set-valued dynamical systems. In this article we focus on singular, or descriptor-variable linear complementarity systems (DVLCS). They may be viewed either as an extension of "classical" descriptor-variable systems which are widely used in Automatic Control and Circuit Theory, or of "classical" linear complementarity systems. They can also be seen as differential-algebraic equations (DAEs) coupled with complementarity constraints. Motivations for studying DVLCS are in circuits with set-valued components [2, Chapters 3 and 5] [32], and in a more abstract framework in state-dependent switching DAEs [68, 67], after a kind of convexification (or filling-in the graph) procedure has been made. They are also closely related to DAEs whose equality constraints are nonsmooth, as they occur in chemistry or thermo-dynamical systems [75, 76, 74].

It has to be noted that the study of DVLCS has not received a lot of attention yet, and is still at an embryonic stage of analysis. In particular the analysis of the implicit Euler method, which has received a lot of attention in several fields [10, 14, 54, 52, 2, 3, 64, 66], is an open issue for DVLCS. Our goal is to analyze the well-posedness of the one-step nonsmooth problem (OSNSP) obtained after an implicit (backward) Euler discretization, and the convergence of the discrete-time solutions towards a continuous-time limit which is a solution to the continuous-time system. The second step is useful

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33 not only for proving the existence of solutions to the continuous-time system, but also, perhaps most  
 34 importantly, to show that the proposed time-discretization makes sense for numerical simulations.  
 35 Passivity properties (passive systems, positive real transfer functions, Lur'e equations) are pivotal  
 36 throughout the article. In section 2, a class of passive DVLCS (which extends descriptor variable  
 37 systems with a positive real transfer function) is introduced. A specific Weierstrass form associated  
 38 with these systems, when the state-space realization is minimal, is used. The well-posedness of the  
 39 one-step non-smooth problem (OSNSP) associated with the implicit Euler discretization, is analysed  
 40 in section 3. Convergence of the discrete-time solutions is studied in sections 4 and 5. Examples that  
 41 illustrate the theoretical developments, are given in section 6. Conclusions are drawn in section 7. The  
 42 appendix is dedicated to recall various mathematical tools.

43 **Notation and definitions:** for any vector  $x \in R^n$  and any matrix  $M \in R^{m \times n}$ ,  $\|x\|$  is the Euclidean  
 44 norm and  $\|M\|$  is the Frobenius norm, which are compatible norms [16, Proposition 9.3.5], i.e.,  $\|Mx\| \leq$   
 45  $\|M\| \|x\|$ . Let  $M \in R^{n \times m}$ , then  $\text{Im}(M)$  is its range,  $\text{Ker}(M)$  is its null space. We use  $\langle x, y \rangle = x^\top y$ , so  
 46  $\langle x, x \rangle = \|x\|^2$ . Positive definite matrix:  $M > 0$  if  $x^\top Mx > 0$  for all  $x \neq 0$ , positive semidefinite matrix:  
 47  $M \geq 0$  if  $x^\top Mx \geq 0$  for all  $x$  (such  $M$  is not necessarily symmetric). The maximum singular value  
 48 is denoted as  $\sigma_{\max}(M)$ , and the minimum and maximum eigenvalues as  $\lambda_{\min}(M)$  and  $\lambda_{\max}(M)$ . More  
 49 mathematical results and definitions are given in Appendices A, B, C, D, E and G.

## 50 2 THE CLASS OF PASSIVE DVLCS

51 Let us consider the following singular dynamical system, that we may name a descriptor-variable  
 52 linear complementarity system (DVLCS):

$$(2.1) \quad \begin{cases} P\dot{x}(t) = Ax(t) + B\lambda(t) + E(t), & \text{a.e. } t \geq 0 \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + F(t) \geq 0, & \text{for all } t \geq 0 \end{cases}$$

53 with  $x(t) \in R^n$ ,  $\lambda(t) \in R^m$ ,  $A, B, C, D$  constant matrices of appropriate dimensions,  $P \in R^{n \times n}$  has rank  
 54  $p < n$ . Dissipativity is a fundamental property in Systems and Control [25]. The material in this article  
 55 relies on a result in [28, 32, 48] on a special Weierstrass form for passive descriptor variable systems  
 56 [25, Section 3.1.7] which possess a *minimal state-space realization* [28, Theorem 3.1]. The passivity of  
 57 the quintuple  $(P, A, B, C, D)$  means the passivity of the operator  $\lambda \mapsto w$ , see section D. Assuming that  
 58  $E(t) = 0$  and  $F(t) = 0$ , passivity and minimality, this special form writes as (see [48] [32, Equation (16)]  
 59 [28, Proposition A.3]):

$$(2.2) \quad \begin{cases} (a) \quad \dot{x}_1(t) = A_1x_1(t) + B_1\lambda(t) \\ (b) \quad \dot{x}_2(t) = z(t) + B_2\lambda(t) \\ (c) \quad 0 = x_2(t) + B_3\lambda(t) \\ (d) \quad 0 \leq \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \geq 0, \end{cases}$$

60 where the quadruple  $(A_1, B_1, C_1, \hat{D})$  is itself passive,  $\hat{D} \triangleq D - C_3B_2 - C_2B_3$ ,  $x_1(t) \in R^{n_1}$ ,  $x_2(t) \in R^{n_2}$ ,  
 61  $z(t) \in R^{n_2}$ ,  $n = n_1 + 2n_2$ ,  $\lambda(t) \in R^m$ , and there exists  $X_3 = X_3^\top \geq 0$  such that  $B_3^\top X_3 = -C_3$ . The LMI in (D.2)  
 62 holds. Due to the passivity and the complementarity conditions which imply  $\lambda(t)^\top w(t) = 0$  for all times,  
 63 it follows that for all  $t \geq 0$  one obtains  $V(x_1(t), x_2(t)) \triangleq \frac{1}{2}(x_1^\top X_1 x_1 + x_2^\top X_3 x_2)(t) \leq V(x_1(0), x_2(0))$ ,  
 64 [32] and [28, Proof of Theorem 5.1], with  $X_1 > 0$  if the system is strongly SPR and minimal [28, Theorem  
 65 5.4], see section D. In the latter case  $x_1(\cdot)$  is bounded,  $A_1$  is Hurwitz<sup>1</sup>,  $\hat{D} > 0$  and  $B_3$  is full row rank  
 66 ( $\Rightarrow n_2 \leq m$ ) [28, Proposition A.4].

<sup>1</sup>That is, all its eigenvalues have a negative real part.

67 **Remark 2.1.** Notice that the index of the DAE part of the system (2.2) is equal to 2, since one needs to  
 68 differentiate the third line twice (assuming  $\lambda(t) = 0$ ) to recover the variable  $\dot{z}(t)$ . It is clear from (2.2)  
 69 that the DVLCS (2.1) may be seen as a DAE coupled with nonsmooth constraints that take the form of  
 70 complementarity conditions. It is noteworthy that singular zero order sweeping processes (named in  
 71 ZOSwP [26]) have been studied [82, 6, 9], but they form another class of singular systems than (2.1) or  
 72 (2.2).

73 **Remark 2.2.** The system in (2.1) can be rewritten equivalently as:

$$(2.3) \quad P\dot{x}(t) - Ax(t) - E(t) \in B(\mathcal{N}_{K^*} + D)^{-1}(-Cx(t) - F(t))$$

74 with  $K = K^* = \mathbf{R}_+^m$ , which is, as pointed out in [26, section 3.16], a class of singular systems that has  
 75 not yet received much attention. The same applies to (2.2) which is rewritten as:

$$(2.4) \quad \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{z}(t) \end{pmatrix} - \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{pmatrix} \in \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} (\mathcal{N}_{K^*} + D)^{-1}(-w(t)).$$

76 Exogenous terms can also be considered in (2.2) and (2.4), though the analyses in sections 3 and 4  
 77 are made without such terms. Also, an alternative way of analysing (2.2) is presented in section 5,  
 78 incorporating exogenous signals. An example is treated in section 6.1 where exogenous terms are  
 79 taken into account in the special form (2.2), see (6.3) and (6.4).

### 80 3 THE EULER IMPLICIT SCHEME AND THE OSNSP

81 The studied implicit Euler time-discretization of (2.2) is as follows:

$$(3.1) \quad \begin{cases} x_{1,k+1} = x_{1,k} + hA_1x_{1,k} + hB_1\lambda_{k+1} \\ x_{2,k+1} = x_{2,k} + hz_{k+1} + hB_2\lambda_{k+1} \\ 0 = x_{2,k+1} + B_3\lambda_{k+1} \\ 0 \leq \lambda_{k+1} \perp w_{k+1} = C_1x_{1,k+1} + C_2x_{2,k+1} + C_3z_{k+1} + D\lambda_{k+1} \geq 0, \end{cases}$$

82 where  $A_1x_{1,k}$  could be replaced by  $A_1x_{1,k+1}$  without modifying much the next developments. Nevertheless  
 83 we choose to call the method in (3.1) an implicit method despite of the fact that the linear term is  
 84 explicitly calculated, because our objective is to focus solely on the set-valued part of the system (*i.e.*,  
 85 the complementarity conditions). The set of equations in (3.1) makes a mixed linear complementarity  
 86 problem (MLCP). After few calculations from (3.1) the following OSNSP is obtained (where the variable  
 87  $hw_{k+1}$  is used instead of  $w_{k+1}$ ) as an LCP:

$$(3.2) \quad \begin{aligned} 0 \leq \lambda_{k+1} \perp \bar{D}_h\lambda_{k+1} + h\beta_k \geq 0 &\Leftrightarrow \bar{D}_h\lambda_{k+1} + h\beta_k \in -\mathcal{N}_{\mathbf{R}_+^m}(\lambda_{k+1}) \\ &\Leftrightarrow \lambda_{k+1} \in (\bar{D}_h + \mathcal{N}_{\mathbf{R}_+^m})^{-1}(-h\beta_k), \end{aligned}$$

88 with  $\bar{D}_h \triangleq h\hat{D} + h^2C_1B_1 + B_3^\top X_3B_3$ ,  $\beta_k \triangleq C_1(I_{n_1} + hA_1)x_{1,k} - \frac{1}{h}C_3x_{2,k}$ . It is noteworthy that  $z_{k+1}$  is  
 89 eliminated and does not appear in  $\beta_k$ . The inclusion (3.2) is a VI as (B.2) with  $\varphi(\cdot) = \psi_{\mathbf{R}_+^m}(\cdot)$ . Obviously  
 90 the well-posedness of the OSNSP depends on the properties of  $\bar{D}_h$ . From the passivity we know that  
 91  $\hat{D} \succcurlyeq 0$  and  $B_3^\top X_3B_3 \succcurlyeq 0$ . Using Theorem A.1 and Corollary A.2 it is possible to characterize the set of  
 92 admissible  $hC_1B_1$  that do not destroy the positive definiteness in case  $\hat{D} > 0$ .

93 **Remark 3.1.** We do not investigate here whether or not the discrete time system (3.1) is dissipative.  
 94 Such a study is made in [52] for the case of LCS discretized with a  $\theta$ -method.

95 It follows from [28, Theorem 5.4] that if the quintuple  $(P, A, B, C, D)$  is strongly positive real (which is  
 96 a strong form of strict passivity), then  $\hat{D} > 0$ . As alluded to in Remark 2.2, the cone  $R_+^m$  can be replaced  
 97 in (2.1), (2.2) and (3.1) by any closed non empty convex cone  $K \subset R^m$  and its dual cone  $K^\star$ . Then the  
 98 complementarity conditions become  $K^\star \ni \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \in K$ ,  
 99 yielding the inclusion

$$(3.3) \quad \begin{aligned} \bar{D}_h \lambda_{k+1} + h\beta_k &\in -\mathcal{N}_{K^\star}(\lambda_{k+1}) \\ &\Leftrightarrow \lambda_{k+1} \in (\bar{D}_h + \mathcal{N}_{K^\star})^{-1}(-h\beta_k), \end{aligned}$$

100 instead of (3.2). One can also consider time-varying terms in (2.2), in both the dynamics and the  
 101 complementarity conditions, and adapt  $\beta_k$  accordingly. Thus in the next proposition a more general  
 102 complementarity framework is considered where the positive orthant is replaced by a closed convex  
 103 cone.

104 **Proposition 3.2.** *Let  $h > 0$  and  $k \geq 0$  be given. Assume that  $\bar{D}_h \geq 0$ , not necessarily symmetric, and let  
 105  $K \subset R^m$  be a non empty closed convex cone (where the complementarity conditions in (2.2) are extended  
 106 to  $K^\star \ni \lambda(t) \perp w(t) = C_1x_1(t) + C_2x_2(t) + C_3z(t) + D\lambda(t) \in K$ ). Let  $\mathcal{S} = K^\star \cap \text{Ker}(\bar{D}_h + \bar{D}_h^\top) \cap \{w \in$   
 107  $R^m \mid \bar{D}_h w \in K\}$ . Then:*

- 108 1. *If  $\mathcal{S} = \{0\}$  then for each  $h\beta_k$  the LCP in (3.2) has at least a solution.*
- 109 2. *If  $\mathcal{S} \neq \{0\}$ , and there exists  $\xi_0 \in R^m$  such that  $\langle h\beta_k - \bar{D}_h^\top \xi_0, v \rangle + \psi_{K^\star}(v) > 0$  for all  $v \neq 0, v \in \mathcal{S}$ ,  
 110 then the LCP in (3.2) has at least a solution.*
- 111 3. *If  $\lambda_{k+1}^1$  and  $\lambda_{k+1}^2$  are two solutions of the LCP, then  $\lambda_{k+1}^1 - \lambda_{k+1}^2 \in \text{Ker}(\bar{D}_h + \bar{D}_h^\top)$ .*

112 **Proof:** Let us consider Proposition B.2. Comparing (3.3) and (B.2) one has  $\varphi(\cdot) = \psi_{K^\star}(\cdot)$ ,  $\mathbf{M} = \bar{D}_h$ ,  
 113  $\mathbf{q} = h\beta_k$ . Using Proposition B.1 c) d), it follows that  $(\text{dom}(\varphi))_\infty = K^\star$ . The second set in  $\mathcal{S}$  is obvious.  
 114 The third set necessitates the calculation of  $(\text{dom}(\varphi_\infty))^\star$ , see (B.3). Again the result of the calculation  
 115 follows from Proposition B.1 c) and d). Then the proof follows from Proposition B.2 a), b) and c).  $\square$

116 Once the existence of  $\lambda_{k+1}$  is established for any  $k \geq 0$ , then the existence of  $x_{1,k+1}$ ,  $x_{2,k+1}$  and  $z_{k+1}$  can  
 117 be inferred from (3.1) for any  $h > 0$ .

118 **Remark 3.3.** If  $K = R_+^m$  then other results like [40, Theorem 3.8.6] may be used to study the LCP (3.2),  
 119 using for instance the copositivity of  $\bar{D}_h$ .

## 120 4 BOUNDEDNESS AND CONVERGENCE ANALYSIS

121 From now on, we assume that the complementarity conditions are given as in (2.2) and (3.1), i.e.,  
 122  $K = K^\star = R_+^m$ . Let us state a consequence of Proposition 3.2.

123 **Corollary 4.1.** *Let the quintuple  $(P, A, B, C, D)$  be strongly positive real. Then the LCP in (3.2) has a unique  
 124 solution for any  $h\beta_k$ , for  $h > 0$  small enough.*

125 **Proof:** This follows from [28, Theorem 5.4] ( $\Rightarrow \hat{D} > 0$ ), the fact that  $B_3^\top X_3 B_3 \geq 0$ , Corollary A.2  
 126 ( $\Rightarrow \bar{D}_h > 0$  for  $0 < h < h^* \triangleq \frac{1}{\|C_1 B_1\|_2 \|(\frac{\bar{D}_h + \bar{D}_h^\top}{2})^{-1}\|_2}$ ), and items 1 and 3 in Proposition 3.2.  $\square$

127 The next result characterizes least-norm solutions of the LCP.

128 **Corollary 4.2.** *Let the conditions of Proposition 3.2 hold true, and assume that the conditions of item 1 or of  
 129 item 2 for existence of a solution are satisfied. Then the set of solutions to the LCP in (3.2) is convex and  
 130 contains a unique least-norm element  $\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})$  such that  $\|\lambda_{k+1}^{\min}\| \leq \|\lambda_{k+1}\|$  for any other solution  
 131  $\lambda_{k+1}$ . Moreover there exists a constant  $\eta > 0$  depending only on  $\bar{D}^h$  such that  $\|\lambda_{k+1}^{\min}\| \leq \eta h \|\beta_k\|$ .*

132 **Proof:** Follows from [40, Theorem 3.1.7] and [54, Lemma 1].  $\square$

133 This result certainly is important because it will subsequently allow us to get upperbounds on the iterates  
 134 of the implicit scheme (3.1). Notice that if  $\bar{D}^h = 0$  then  $\beta_{k,i} > 0 \Rightarrow \lambda_{k+1,i} = 0$ , and  $\beta_{k,i} = 0 \Rightarrow \lambda_{k+1,i} \in \mathbb{R}_+$ ,  
 135 hence  $\lambda_{k+1}^{\min} = 0$  so that  $\eta = 0$ . If  $h = 0$  then  $\bar{D}^0 = B_3^\top X_3 B_3$ ,  $\beta_0 = -C_3 x_{2,k}$ , and  $\eta = \eta_0$  such that  
 136  $\|\lambda_{k+1}^{\min}\| \leq \eta_0 \|C_3 x_{2,k}\|$  is independent of  $h$ . If the LCP( $\bar{D}_h, h\beta_k$ ) enjoys the stability of solutions property  
 137 [40, Theorem 7.3.12], then a small perturbation of  $\bar{D}_h$  implies a small variation of the solution, hence  
 138 of  $\lambda_{k+1}^{\min}$ , and consequently of the constant  $\eta$ . Assume that  $C_3 = 0$  (this is Assumption 2 below). Then  
 139 solutions of the LCP( $\bar{D}_h, h\beta_k$ ) are also solutions of LCP( $\hat{D} + hC_1 B_1, C_1(I_{n_1} + hA_1)x_{1,k}$ ). If  $\hat{D} + hC_1 B_1$  is a  
 140 P-matrix (which occurs if  $\hat{D}$  is a P-matrix and  $h > 0$  is small enough, see Corollary A.2), then it follows  
 141 from [3, Theorem B.3] [34] that the unique solution of LCP( $\hat{D} + hC_1 B_1, C_1(I_{n_1} + hA_1)x_{1,k}$ ) (which is also  
 142 the least-norm solution) satisfies  $\|\lambda_{k+1}^{\min}\| \leq \eta \|C_1(I_{n_1} + hA_1)x_{1,k}\|$  for some constant  $\eta > 0$ .

143 **Remark 4.3.** As long as a minimal norm element exists and satisfies the property stated in the corollary,  
 144 one may consider a cone LCP as in (3.3) instead of (3.2).

#### 145 4.1 CONVERGENCE ANALYSIS

146 From (3.2) and under the conditions of Corollary 4.2, the mapping  $\beta_k \mapsto \lambda_{k+1}(-h\beta_k) = \lambda_{k+1}^{\min}$  with  $\lambda_{k+1}^{\min}$   
 147 the least-norm element of the convex polyhedral set  $(\bar{D}_h + \mathcal{N}_{\mathbb{R}_+^m})^{-1}(-h\beta_k)$ , is a single-valued mapping.  
 148 The implicit scheme (3.1) may be modified as follows (which may be named the ‘‘minimal norm’’ implicit  
 149 scheme):

$$(4.1) \quad \begin{cases} x_{1,k+1} = (I_{n_1} + hA_1)x_{1,k} + hB_1\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ x_{2,k+1} = x_{2,k} + hz_{k+1} + hB_2\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ 0 = x_{2,k+1} + B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ 0 \leq \lambda_{k+1}^{\min} \perp w_{k+1} = C_1 x_{1,k+1} + C_2 x_{2,k+1} + C_3 z_{k+1} + D\lambda_{k+1}^{\min} \geq 0 \\ \|\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})\| \leq \eta \|h\beta_k\|, \quad h\beta_k = hC_1(x_{1,k} + hA_1 x_{1,k}) - C_3 x_{2,k}, \quad \eta > 0, \end{cases}$$

150 for all  $k \in \{0, \dots, n-1\}$ . The above results explore the well-posedness of the OSNSP, for fixed  $h > 0$ .  
 151 Our goal is now to analyse the boundedness of the iterates on the integration interval  $[0, T]$ ,  $T > 0$ .  
 152 For all  $k \geq 1$  we have  $x_{2,k} = -B_3\lambda_k^{\min}$ , and  $hz_{k+1} = -B_3\lambda_{k+1}^{\min} - x_{2,k} - hB_2\lambda_{k+1}^{\min}$ , thus it is inferred that  
 153  $x_{2,k+1} = -B_3(\lambda_{k+1}^{\min} + \lambda_k^{\min}) - x_{2,k}$ . Consequently (4.1) is rewritten equivalently as:

$$(4.2) \quad \begin{cases} x_{1,k+1} = (I_{n_1} + hA_1)x_{1,k} + hB_1\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ x_{2,k+1} = -x_{2,k} - B_3(\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) + \lambda_k^{\min}(x_{1,k-1}, x_{2,k-1})) \\ 0 \leq \lambda_{k+1}^{\min} \perp w_{k+1} = C_1(I_{n_1} + hA_1)x_{1,k} - (C_2 + \frac{1}{h}C_3)x_{2,k} + \frac{1}{h}\bar{D}_h\lambda_{k+1}^{\min} - C_2B_3\lambda_k^{\min} \geq 0 \\ 0 = x_{2,k+1} + B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}), \\ hz_{k+1} = -B_3\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) - x_{2,k} - hB_2\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k}) \\ \|\lambda_{k+1}^{\min}(x_{1,k}, x_{2,k})\| \leq \eta \|h\beta_k\|, \quad h\beta_k = hC_1(I_{n_1} + hA_1)x_{1,k} - C_3x_{2,k}, \quad \eta > 0, \end{cases}$$

154 for all  $k \in \{1, n-1\}$ , given initial data  $x_{1,0}$  and  $x_{2,0}$ . Starting from (4.2) the next result is obtained:

155 **Lemma 4.4.** Let  $\alpha_{1,k} \triangleq \|x_{1,k}\|$ ,  $\alpha_{2,k} \triangleq \|x_{2,k}\|$ ,  $\alpha_k = (\alpha_{1,k} \quad \alpha_{2,k})^\top$ , then:

$$(4.3) \quad \begin{cases} \alpha_{k+1} \leq (M_i M + M_{i-1} N)\alpha_{k-i} + M_i N \alpha_{k-i-1} \\ M_{i+1} = M_i M + M_{i-1} N, \quad M_0 = I_2, M_{-1} = 0, \quad 0 \leq i \leq k-1, \quad k \geq 1, \end{cases}$$



156 where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $N = \begin{pmatrix} 0 & 0 \\ e & f \end{pmatrix}$ ,  $a = \|I_{n_1} + hA_1\| (1 + h^2\eta \|B_1\| \|C_1\|)$ ,  $b = h\eta \|B_1\| \|C_3\|$ ,  $c =$   
 157  $h\eta \|B_3\| \|C_1\| \|I_{n_1} + hA_1\|$ ,  $d = (1 + \eta \|B_3\| \|C_3\|)$ ,  $e = c$ ,  $f = \eta \|B_3\| \|C_3\|$ . In particular

$$(4.4) \quad \alpha_{k+1} \leq (M_{k-1}M + M_{k-2}N)\alpha_1 + M_{k-1}N\alpha_0$$

158 for all  $k \geq 1$ .

159 **Proof:** The proof starts from (4.2). The following upperbounds can be derived:

$$(4.5) \quad \begin{aligned} \|\xi_k^{\min}\| &\leq \eta h \|C_1\| \|I_{n_1} + hA_1\| \|x_{1,k}\| + \eta \|C_3\| \|x_{2,k}\| \\ \|x_{1,k+1}\| &\leq \|I_{n_1} + hA_1\| (1 + h^2\eta \|B_1\| \|C_1\|) \|x_{1,k}\| + h\eta \|B_1\| \|C_3\| \|x_{2,k}\| \\ \|x_{2,k+1}\| &\leq (1 + \eta \|B_3\| \|C_3\|) \|x_{2,k}\| + \eta \|B_3\| \|C_3\| \|x_{2,k-1}\| \\ &\quad + h\eta \|B_3\| \|C_1\| \|I_{n_1} + hA_1\| (\|x_{1,k}\| + \|x_{1,k-1}\|). \end{aligned}$$

160 One infers that:

$$(4.6) \quad \alpha_{k+1} \leq M\alpha_k + N\alpha_{k-1}$$

161 The upperbound in (4.3) therefore holds for  $i = 0$ . Let us prove by induction that it holds for all  
 162  $0 \leq i \leq k - 1$ . Let it hold for  $i > 0$ , we have  $\alpha_{k-i} \leq M\alpha_{k-i-1} + N\alpha_{k-i-2}$ . Thus  $\alpha_{k+1} \leq (M_iM +$   
 163  $M_{i-1}N)M\alpha_{k-i-1} + (M_iM + M_{i-1}N)N\alpha_{k-i-2} + M_iN\alpha_{k-i-1} = (M_{i+1}M + M_iN)\alpha_{k-i-1} + M_{i+1}N\alpha_{k-i-2}$ . The  
 164 result is proved.  $\square$

165 Our goal is now to examine (4.4). Let us come back to the matrices  $M$  and  $N$  in Lemma 4.4. It is  
 166 clear from (4.4) that the boundedness of the sequence  $\{\alpha_k\}_{k \geq 2}$  depends on the boundedness of  $M_k$ ,  
 167  $k \geq 1$ . Apart from  $d$  and  $f$ , all the other terms in  $M$  and  $N$  are proportional to  $h$  or to  $h_1 \triangleq \|I_{n_1} + hA_1\|$ .  
 168 Moreover  $d > 1$  whenever  $B_3 \neq 0$  and  $C_3 \neq 0$ . The (2, 2) term of  $M^n$  is equal to  $d^n + O(h_1h^2)$ . We are  
 169 led to make the following assumptions:

170 **Assumption 1.** The matrix  $A_1$  is such that  $h_1 \leq 1$  for  $h > 0$  small enough.

171 Let us notice that the induced norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$  [16, Proposition 9.4.9] is compatible with the  
 172 vector norm  $\|\cdot\|_2 = \|\cdot\|$  [16, pp.607-609], and can be used instead of the Frobenius norm to compute  $h_1$ .  
 173 The fact that small enough timestep is needed stems from the explicit discretization of the term  $A_1x_1$  in  
 174 (2.2) and is not essential in our developments since, as alluded to above, it is the implicit discretization  
 175 of the set-valued terms which is our focus. The next assumption is crucial in the body of this section.

176 **Assumption 2.** One has  $B_{3,\bullet i} \in \text{Ker}(X_3)$  for all  $1 \leq i \leq m$  ( $\implies C_3 = 0$ ).

177 Then  $d = 1$  and  $f = 0$ . This assumption means that the variable  $z$  does not enter the complementarity  
 178 conditions, see (2.2). Hence in a sense the complementarity part of the DVLCS and the algebraic part,  
 179 are decoupled. The matrices  $M_k$  then involve only products of powers of the terms  $a, b, c, d = 1, e$ .  
 180 These products are bounded for bounded  $k$ , and provided that both  $h$  and  $h_1$  are small enough so that  
 181  $a \leq 1, b \leq 1, c \leq 1, d = 1, e \leq 1$ , they remain bounded as  $k \rightarrow +\infty$ . If the inequalities are strict (apart  
 182 from  $d = 1$ ), the first rows in both  $M_{k-1}M + M_{k-2}N$  and  $M_{k-1}N$  converge to zero, while the second rows  
 183 converge to  $(0, 1)$  (the 1 being the consequence of a term  $d^n$  in matrices  $M^n$ ). The next proposition is  
 184 therefore proved:

185 **Proposition 4.5.** Assume that the conditions of existence in Proposition 3.2 are verified. Let Assumptions 1  
 186 and 2 hold true, and let both  $h$  and  $h_1$  be small enough so that  $a \leq 1, b \leq 1, c \leq 1, d = 1, e \leq 1$ . Then the  
 187 sequences  $\{\alpha_{1,k}\}_k$  and  $\{\alpha_{2,k}\}_k$  are uniformly bounded in  $k$ . If  $A_1 < 0$ , and  $h > 0$  is small enough such that  
 188  $a < 1, b < 1, c < 1, e < 1$ , then  $\alpha_{1,k} \rightarrow 0$  as  $k \rightarrow +\infty$ .

189 Recall that  $\alpha_{1,k} = \|x_{1,k}\|$  and  $\alpha_{2,k} = \|x_{2,k}\|$ . The last assertion of the proposition follows from the  
 190 material in Appendix A. It is important to notice that all the terms depending on  $\eta$  that remain in (4.5),  
 191 are of the form  $h\eta$  or  $h^2\eta$ , hence they remain bounded even in case  $\eta$  depends on  $h^{-1}$ . Then we have  
 192 the next corollary.

193 **Corollary 4.6.** *Let the conditions of Proposition 4.5 hold. Then  $\|\lambda_{k+1}^{\min}\|$  is uniformly bounded in  $k$  and  
 194  $h > 0$ , and it converges to 0 as  $k \rightarrow +\infty$  if  $\alpha_{1,k} \rightarrow 0$  as  $k \rightarrow +\infty$ . Therefore the sequence  $\{\frac{x_{1,k+1}-x_{1,k}}{h}\}_k$  is  
 195 uniformly bounded also in  $k$  and  $h > 0$ .*

196 **Proof:** The boundedness of  $\|\lambda_{k+1}^{\min}\|$  and its convergence follow from its upperbound, see the last  
 197 line in (4.2). From the first line in (4.2), one has  $\frac{x_{1,k+1}-x_{1,k}}{h} = A_1x_{1,k} + B_1\lambda_{k+1}^{\min}$ , hence the result is proved.  $\square$

198 Let us now study the convergence of the piecewise-linear and step-function approximations defined  
 199 as:

$$(4.7) \quad \begin{cases} x_1^h(t) = x_{1,k+1} + \frac{t_{k+1}-t}{h}(x_{1,k} - x_{1,k+1}) \\ \dot{x}_1^h(t) = \frac{x_{1,k+1}-x_{1,k}}{h} \\ x_{1,h}^*(t) = x_{1,k+1} \end{cases} \quad \text{for all } t \in [t_k, t_{k+1})$$

200 and similarly for the other variables.

201 **Corollary 4.7.** *There exists subsequences denoted as  $\{x_1^h(\cdot)\}_{h \geq 0}$  which converge uniformly to a limit  $x_1(\cdot)$   
 202 in  $C^0([0, T]; \mathbb{R}^{n_1})$ , such that almost everywhere one has  $\dot{x}_1(t) = A_1x_1(t) + \lambda_1(t)$ , for some function  $\lambda_1(\cdot)$ .  
 203 Moreover the sequences  $\{x_{2,h}^*(\cdot)\}_{h \geq 0}$  and  $\{\lambda_h^{\min,*}(\cdot)\}_{h \geq 0}$  converge weakly $\star$  in  $\mathcal{L}_\infty([0, T]; \mathbb{R}^l)$ ,  $l = n_2$  and  
 204  $l = m$ , respectively, to limits  $x_2(\cdot)$  and  $\lambda(\cdot)$  which satisfy the continuous-time constraints.*

205 **Proof:** Both  $\{x_1^h(\cdot)\}_h$  and  $\{\dot{x}_1^h(\cdot)\}_h$  are uniformly bounded on  $[0, T]$ . The sequence of  $C^0([0, T]; \mathbb{R}^{n_1})$   
 206 functions  $\{x_1^h(\cdot)\}_h$  is thus equicontinuous. By the Arzela-Ascoli Theorem this sequence stays in a  
 207 compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and converges uniformly towards a continuous limit  $x_1(\cdot) : [0, T] \rightarrow$   
 208  $\mathbb{R}^{n_1}$  as  $h \rightarrow 0$ ,  $h > 0$ . Let us notice that  $x_1(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . Indeed from the above results which  
 209 are stated for  $k \in \{0, \dots, n-1\}$  and  $h > 0$ ,  $k \rightarrow +\infty$  only if  $T = nh \rightarrow +\infty$ . Let us now define the step  
 210 functions  $x_1^{h,*}(t) = x_{1,k+1}$  if  $t \in [t_k, t_{k+1})$ ,  $\lambda_{h,*}^{\min}(t) = \lambda_{k+1}^{\min}$  if  $t \in [t_k, t_{k+1})$ , and

$$(4.8) \quad \dot{x}_1^h(t) = A_1x_1^{h,*}(t) + B_1\lambda_{h,*}^{\min}(t), \quad \text{almost everywhere.}$$

211 In addition one has:

$$(4.9) \quad \begin{aligned} \|x_1^h(t) - x_1^{h,*}(t)\|_{\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})}^2 &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 \|\dot{x}_{1,k}\|^2 dt \leq C^2 \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 dt \\ &\leq C^2 \sum_{k=0}^{n-1} \frac{1}{3} (t_{k+1} - t_k)^3 = \frac{C^2 nh^3}{3} = \frac{C^2 Th^2}{3}, \end{aligned}$$

212 where  $\dot{x}_{1,k} = \frac{x_{1,k+1}-x_{1,k}}{h}$ , and  $\|\dot{x}_{1,k}\| \leq C$  for all  $k \geq 0$  for some  $C > 0$  from Corollary 4.6. It fol-  
 213 lows that  $x_1^{h,*}(\cdot) \rightarrow x_1(\cdot)$  strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$ . On the other hand since  $\dot{x}_1^h(\cdot) \in$   
 214  $\mathcal{L}_\infty([0, T]; \mathbb{R}^{n_1})$  it follows from the Banach-Alaoglu-Bourbaki Theorem [18, Theorem III.15] that  $\dot{x}_1^h(\cdot) \rightarrow$   
 215  $\dot{x}_{1,\lim}(\cdot)$  in the weak $\star$  topology as  $h \rightarrow 0$ . Since the considered domain  $[0, T]$  is bounded, one has  
 216  $\mathcal{L}_1([0, T]; \mathbb{R}^{n_1}) \subseteq \mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$ , thus  $\dot{x}_1^h(\cdot)$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$ . Moreover since  
 217  $x_1^h(t) = x_1^h(0) + \int_0^t \dot{x}_1^h(t) dt$ , one has  $\dot{x}_{1,\lim}(\cdot) = \dot{x}_1(\cdot)$  almost everywhere. Therefore we deduce using  
 218 (4.8) that  $(B_1\lambda_{h,*}^{\min})(\cdot)$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  as  $h \rightarrow 0$  towards some limit  $\lambda_1(\cdot)$ , and we  
 219 have  $\dot{x}_1(t) = A_1x_1(t) + \lambda_1(t)$  almost everywhere on  $[0, T]$ .

220 Let us consider now the step-functions  $x_{2,h}^*(\cdot)$  and  $\lambda_h^{\min,*}(\cdot)$  defined as  $x_{2,h}^*(t) = x_{2,k+1}$  for all  $t \in$   
 221  $[t_k, t_{k+1})$ , and similarly for the multiplier. It follows from Proposition 4.5 and from Corollary 4.6



222 that both sequences  $\{x_{2,h}^\star(\cdot)\}_{h \geq 0}$  and  $\{\lambda_h^{\min,\star}(\cdot)\}_{h \geq 0}$  are uniformly bounded. Hence from the Banach-  
 223 Alaoglu-Bourbaki Theorem, there exist subsequences of both which converge weakly $\star$  to limits  
 224  $x_2 \in \mathcal{L}_\infty([0, T]; \mathbb{R}^{n_2})$  and  $\lambda \in \mathcal{L}_\infty([0, T]; \mathbb{R}^m)$ , respectively. We infer that  $0 = x_{2,h}^\star(t) + B_3 \lambda_h^\star(t) \rightarrow$   
 225  $x_2(t) + B_3 \lambda(t)$  for almost all  $t \geq 0$  in the weak $\star$  sense. Next, using the second line in (4.1) it follows  
 226 that the sequence  $\{hz_k\}_{h \geq 0}$  is also uniformly bounded. Denoting  $\sigma_k = hz_k$ , it follows that the sequence  
 227  $\{\sigma_h^\star\}_{h \geq 0}$  converges weakly $\star$  to a limit denoted as  $\sigma(t) \in \mathcal{L}_\infty([0, T]; \mathbb{R}^{n_2})$ . Moreover one has  $\sigma(t) = 0$   
 228 almost everywhere since  $x_{2,h}^\star(t) - x_{2,h}^\star(t - t_k) - hB_2 \lambda_h^{\min,\star}(t) = \sigma_h^\star(t)$  almost everywhere in  $[0, T]$ .  
 229 Finally, the complementarity conditions in (4.1) are equivalently rewritten as  $0 \leq \lambda_h^\star(t) \perp C_1 x_{1,h}^\star(t) +$   
 230  $C_2 x_{2,h}^\star(t) + D \lambda_h^\star(t) \geq 0$  for all  $t \in [t_k, t_{k+1})$ , since  $C_3 = 0$  by Assumption 2. It is deduced that in the  
 231 limit as  $h \rightarrow 0$  one obtains  $0 \leq \lambda(t) \perp C_1 x_1(t) + C_2 x_2(t) + D \lambda(t) \geq 0$ .  $\square$

## 232 4.2 STABILITY ANALYSIS VIA A LYAPUNOV FUNCTION

233 To complete the foregoing section, it is natural to analyze how the Lyapunov function  $V(x_1, x_2)$  for the  
 234 continuous-time system in (2.2) (see also appendix D), could be used in the discrete-time context for (4.2).  
 235 Indeed implicit Euler methods are known to be able to mimic the continuous-time passivity and stability  
 236 properties, in some cases [52]. It is noteworthy that Assumption 2 implies that  $X_3 x_{2,k} = -X_3 B_3 \lambda_k = 0$   
 237 and  $X_3 x_2(t) = -X_3 B_3 \lambda(t) = 0$  for all  $k$  and  $t$ . Therefore  $V(x_1, x_2) = \frac{1}{2} x_1^\top X_1 x_1$ , so that this Lyapunov  
 238 function is not a positive definite function of the whole state. For the sake of completeness let us  
 239 nevertheless make the calculations in the general setting. We set  $V_k \triangleq \frac{1}{2} x_{1,k}^\top X_1 x_{1,k} + x_{2,k}^\top X_3 x_{2,k}$ . It follows  
 240 that:

$$\begin{aligned}
 V_{k+1} - V_k &= \frac{1}{2} (x_{1,k+1} - x_{1,k})^\top X_1 (x_{1,k+1} + x_{1,k}) + (x_{2,k+1} - x_{2,k})^\top X_3 (x_{2,k+1} + x_{2,k}) \\
 &= \frac{1}{2} (hA_1 x_{1,k} + hB_1 \lambda_{k+1})^\top X_1 (2x_{1,k} + hA_1 x_{1,k} + hB_1 \lambda_{k+1}) \\
 &\quad + (hz_{k+1} + hB_2 \lambda_{k+1})^\top X_3 (2x_{2,k} + hz_{k+1} + hB_2 \lambda_{k+1}) \\
 &= h(x_{1,k}^\top A_1^\top X_1 x_{1,k} + \lambda_{k+1}^\top B_1^\top X_1 x_{1,k}) + h^2 (\frac{1}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} + \frac{1}{2} \lambda_{k+1}^\top B_1^\top X_1 B_1 \lambda_{k+1}) \\
 &\quad + h^2 x_{1,k}^\top A_1^\top X_1 B_1 \lambda_{k+1} + hz_{k+1}^\top C_3 \lambda_{k+1} + h \lambda_{k+1}^\top C_3 B_2 \lambda_{k+1} + hz_{k+1}^\top X_3 x_{2,k} \\
 &\quad + h \lambda_{k+1}^\top B_2^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1} - h \lambda_{k+1}^\top w_{k+1} \\
 &= \frac{h}{2} x_{1,k}^\top (A_1^\top X_1 + X_1 A_1 + hA_1^\top X_1 A_1) x_{1,k} - h \lambda_{k+1}^\top (\hat{D} + h(C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1)) \lambda_{k+1} \\
 &\quad + h x_{1,k}^\top (B_1^\top X_1 - C_1 - hA_1^\top C_1^\top + hA_1^\top X_1 B_1) \lambda_{k+1} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} \\
 &\quad + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1} \\
 (4.10) \quad &= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) \begin{pmatrix} A_1^\top X_1 + X_1 A_1 & (B_1^\top X_1 - C_1)^\top \\ B_1^\top X_1 - C_1 & -(\hat{D} + \hat{D}^\top) \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} \\
 &\quad + \frac{h^2}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - h^2 \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \\
 &\quad + h^2 x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k} + h \lambda_{k+1}^\top w_{k+1} \\
 &= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) Q_1 \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} + h \lambda_{k+1}^\top w_{k+1} \\
 &\quad + h^2 \left\{ \frac{1}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \right. \\
 &\quad \left. + x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} \right\} + \frac{h}{2} x_{2,k}^\top X_3 B_2 \lambda_{k+1} + \frac{h}{2} z_{k+1}^\top X_3 x_{2,k}
 \end{aligned}$$

241

$$\begin{aligned}
(4.10) \quad &= \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) Q_1 \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix} + h \lambda_{k+1}^\top w_{k+1} \\
&+ h^2 \left\{ \frac{1}{2} x_{1,k}^\top A_1^\top X_1 A_1 x_{1,k} - \lambda_{k+1}^\top (C_1 B_1 - B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1) \lambda_{k+1} \right. \\
&\left. + x_{1,k}^\top (-A_1^\top C_1^\top + A_1^\top X_1 B_1) \lambda_{k+1} \right\} - \underbrace{\frac{1}{2} x_{2,k}^\top X_3 x_{2,k} + \lambda_{k+1}^\top B_3^\top X_3 B_3 \lambda_k}_{=0 \text{ if Assumption 2 holds.}}
\end{aligned}$$

242 where the last two terms are obtained using the second equality in (3.1). Remind that  $\lambda_{k+1}^\top w_{k+1} = 0$ .  
243 Assume that  $Q_1 < 0$ , then let Assumption 2 hold. Let us rewrite (4.10) as  $V_{k+1} - V_k = \frac{h}{2} (x_{1,k}^\top, \lambda_{k+1}^\top) [Q_1 +$   
244  $hQ_2] \begin{pmatrix} x_{1,k} \\ \lambda_{k+1} \end{pmatrix}$ , where the indefinite matrix

$$(4.11) \quad Q_2 = \begin{pmatrix} A_1^\top X_1 A_1 & -\frac{1}{2} A_1^\top C_1^\top + \frac{1}{2} A_1^\top X_1 B_1 \\ (-\frac{1}{2} A_1^\top C_1^\top + \frac{1}{2} A_1^\top X_1 B_1)^\top & -C_1 B_1 + B_2^\top X_3 B_2 - \frac{1}{2} B_1^\top X_1 B_1 \end{pmatrix},$$

245 and we remind that Assumption 2 implies that  $V_k = \frac{1}{2} x_{1,k}^\top X_1 x_{1,k}$ . Using Corollary A.2, there exists  
246  $h_{\max} > 0$  such that for all  $h \in [0, h_{\max})$  one has  $Q_3 \triangleq Q_1 + hQ_2 < 0$ . Using classical bounds on quadratic  
247 forms and vector/matrices products ( $x^\top y \leq \|x\| \|y\| \leq \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2$  [16, Corollary 9.1.7], if  $Q = Q^\top$   
248 then  $\lambda_{\min}(Q) \|x\|^2 \leq x^\top Q x \leq \lambda_{\max}(Q) \|x\|^2$  [16, Corollary 8.4.2],  $\|Qx\| \leq \|Q\| \|x\|$  for compatible  
249 norms [16, section 9.3]), and noting that we can replace  $Q_3$  by its negative definite symmetric part in  
250 the quadratic form, it follows that:

$$\begin{aligned}
(4.12) \quad V_{k+1} &\leq V_k + \frac{h}{2} \lambda_{\max}(Q_3) \|(x_{1,k}, \lambda_{k+1})\|^2 \\
&\leq V_k + \frac{h}{2} \lambda_{\max}(Q_3) \|x_{1,k}\|^2 \\
&\leq V_k + \frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)} V_k = \left(1 + \frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)}\right) V_k, \quad \text{for all } k \geq 0.
\end{aligned}$$

251 Since  $\lambda_{\max}(Q_3) < 0$  (which is the reason why the second inequality in (4.12) holds), one has  $1 +$   
252  $\frac{h}{2} \frac{\lambda_{\max}(Q_3)}{\lambda_{\max}(X_1)} < 1$  for all  $h > 0$ . Therefore all  $V_k$  are uniformly bounded for any bounded  $V_0$ , and  $V_k \rightarrow 0$  as  
253  $k \rightarrow +\infty$  for any  $h \in (0, h_{\max})$ . Let us set  $\lambda_{k+1} = \lambda_{k+1}^{\min}$ . Since the dependence of  $\lambda_{k+1}^{\min}$  in  $x_{2,k}$  through  $\beta_k$  is  
254  $C_3 x_{2,k}$  (see (4.2)), it follows from Assumption 2 that  $\lambda_{k+1}^{\min} = \lambda_{k+1}^{\min}(x_{1,k})$ , and  $\lambda_{k+1}^{\min}(x_{1,k}) \rightarrow 0$  as  $k \rightarrow +\infty$ .  
255 Using the third line in (4.2), we infer that the iterates  $x_{2,k}$  (which are uniformly bounded from  
256 Proposition 4.5) converge to zero asymptotically also. From the results in section 4.1 and the above  
257 Lyapunov analysis, we have therefore proved the following:

258 **Lemma 4.8.** *Let Assumptions 1 and 2 hold, and the quintuple  $(P, A, B, C, D)$  be strongly SPR and minimal.*  
259 *Then there exists  $h_{\max} > 0$  depending on  $A_1, C_1, X_1, X_3, B_1, B_2$ , such that for all  $h \in (0, h_{\max})$ , and for*  
260 *all bounded initial condition  $x_{1,0}$ , the sequences  $\{x_{1,k}\}_k, \{x_{2,k}\}_k, \{\lambda_k^{\min}\}_k$  are uniformly bounded and*  
261  *$x_{1,k}, x_{2,k}, \lambda_{k+1}^{\min} \rightarrow 0$  as  $k \rightarrow +\infty$ .*

## 262 5 ANOTHER APPROACH FOR PASSIVE SYSTEMS

263 The DVLCs in (2.2) are suitably transformed in [32] to enable their analysis. The transformation may  
264 be seen as a kind of relative degree reduction, involving the multiplier derivative. As will become clear  
265 later, this new dynamics allows us to somewhat relax the stringent Assumption 2 (see Assumption 3 c)  
266 below). More accurately, a close relationship is shown in [32, Lemma 3, Theorem 3] between (2.2) and

267 the following system:

$$(5.1) \quad \begin{cases} (a) \quad \begin{pmatrix} I_{n_1} & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} w(t) \\ (b) \quad 0 \leq \lambda(t) \perp w(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 z(t) + D \lambda(t) \geq 0 \\ (c) \quad x_2(t) = -B_3 \lambda(t), \quad z(t) = -B_2 \lambda(t) - B_3 \dot{\lambda}(t). \end{cases}$$

268 It is noteworthy that this dynamics is obtained from the original one by treating the multiplier  $\lambda$  as a  
269 state variable. The "equivalence" between both systems is to be understood first as follows [32, Lemma  
270 3]:

271  $\{(x_1, \lambda, w) \mid (x_1, x_2, \lambda, w) \text{ is a solution of (2.2) (a, b, c)}\} = \{(x_1, \lambda, w) \mid (x_1, \lambda, w) \text{ is a solution of (5.1) (a)}$   
272  $\text{and } B_3 \lambda \text{ is absolutely continuous}\}.$

273 By solution of (2.2) it is meant that  $(x_1, x_2)$  is absolutely continuous and the dynamics holds Lebesgue  
274 almost everywhere [32, Definition 3]. The initial conditions are chosen compatible with the constraints.  
275 It is noteworthy that the variables  $z$  and  $\lambda$  are not shown to be absolutely continuous, and examples  
276 in sections 6.1 and 6.3 will illustrate this fact. One has  $w(t) \in -\mathcal{N}_{R_+^m}(\lambda(t))$ , therefore one obtains the  
277 differential inclusion (DI):

$$(5.2) \quad \begin{aligned} \begin{pmatrix} I_{n_1} & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix} - \begin{pmatrix} A_1 & B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} &\in - \begin{pmatrix} 0 \\ I_m \end{pmatrix} \mathcal{N}_{R_+^m}(\lambda(t)) \\ &= - \begin{pmatrix} 0 \\ I_m \end{pmatrix} \mathcal{N}_{R_+^m} \left( \begin{pmatrix} 0 & I_m \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right) \\ &= -\mathcal{N}_\Phi \left( \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right) \end{aligned}$$

278 with  $\Phi \stackrel{\Delta}{=} \{(x_1, \lambda) \in R^{n_1} \times R^m \mid \lambda \geq 0\}$ . If this DI has a solution  $(x_1(\cdot), \lambda(\cdot))$ , then  $x_2(\cdot)$ ,  $z(\cdot)$ , and  $w(\cdot)$   
279 can be obtained (provided indeed that  $B_3 \lambda(\cdot)$  is absolutely continuous). Multiplying the first line in  
280 (5.2) by  $X_1$  and adding an exogenous excitation  $G(t)$  to the left-hand side, one obtains

$$(5.3) \quad \underbrace{\begin{pmatrix} X_1 & 0 \\ 0 & B_3^\top X_3 B_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}(t) \end{pmatrix}}_{\triangleq \mathcal{P}} - \underbrace{\begin{pmatrix} X_1 A_1 & X_1 B_1 \\ -C_1 & -\hat{D} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix}}_{\triangleq \mathcal{A}} + G(t) \in -\mathcal{N}_\Phi \left( \begin{pmatrix} x_1(t) \\ \lambda(t) \end{pmatrix} \right)$$

281 which is similar to the form obtained in [32, Theorem 3]. Let us assume we add an exogenous input  
282  $V(t) = (v_{(a)}(t), v_{(b)}(t), v_{(c)}(t), v_{(d)}(t))^\top$  to (2.2), then:

$$(5.4) \quad G(t) = \begin{pmatrix} v_{(a)}(t) \\ C_3 (v_{(b)}(t) + \dot{v}_{(c)}(t)) + C_2 v_{(c)}(t) - v_{(d)}(t) \end{pmatrix}.$$

283 Notice that  $B_3^\top X_3 B_3 \geq 0$ , therefore  $\mathcal{P} = \mathcal{P}^\top \geq 0$ . However  $\mathcal{P}$  has a specific block-diagonal structure.  
284 Under strong SPRness and minimality,  $\mathcal{A} < 0$ , since  $\mathcal{A} + \mathcal{A}^\top < 0$  from the KYP Lemma [25] (then  $A_1$   
285 is a Hurwitz matrix,  $X_1 = X_1^\top > 0$  and  $\hat{D} > 0$ ). The exogeneous term  $G(t)$  stems from the exogeneous  
286 terms  $E(t)$  and  $F(t)$  in (2.1) and their time-derivatives (see (6.4) for an example).

287 In a second step the "equivalence" between both systems is to be understood first as follows [32,  
288 Theorem 3]:

289  $\{(x_1, \lambda) \mid (x_1, x_2, \lambda, w) \text{ is a solution of (2.2) (a, b, c, d)}\} = \{(x_1, \lambda) \mid (x_1, \lambda) \text{ is a solution of (5.3) and } B_3 \lambda \text{ is}$   
290  $\text{absolutely continuous}\}.$

291 As will become clear later, the results that are obtained using this approach, and the approach in the  
 292 foregoing section, are not the same (especially concerning the convergence of the piecewise-linear  
 293 discrete solutions). The differences between both numerical schemes are illustrated through a numerical  
 294 example in section 6.3, see also Appendix H.

### 295 5.1 THE TIME-DISCRETIZATION AND THE OSNSP

296 As alluded to in (5.3), we can enlarge the analysis by considering time-varying terms, so as to embed  
 297 (5.3) into the DI:  $\mathcal{P}\dot{\xi}(t) - \mathcal{A}\xi(t) + G(t) \in -\mathcal{N}_{\Phi}(\xi(t))$ . The Euler implicit scheme associated with (5.3) is:

$$(5.5) \quad \mathcal{P}\xi_{k+1} - \underbrace{(\mathcal{P} + h\mathcal{A})}_{\triangleq \mathcal{P}_h} \xi_k + hG_k \in -\mathcal{N}_{\Phi}(\xi_{k+1}),$$

298 It is noteworthy that this scheme is not equal to the implicit scheme in (3.1), as can be shown after  
 299 few manipulations, see Appendix H. Though the difference between both approaches is not analysed  
 300 further in this paper, it will be tackled numerically through an example in section 6.3. The following  
 301 holds.

302 **Proposition 5.1.** *Let  $h > 0$  and  $k \geq 0$  be given. Assume that the quintuple  $(P, A, B, C, D)$  is minimal and  
 303 strongly SPR, so that the quadruple  $(A_1, B_1, C_1, \hat{D})$  is minimal and strongly SPR. Let us consider the DI in  
 304 (5.3) and its discretization in (5.5), with  $q_k \triangleq -\mathcal{P}_h \xi_k + hG_k$ . Let us consider the set:*

$$(5.6) \quad \mathcal{S} \triangleq \{(x_1, \lambda) \mid x_1 = 0, \lambda \in \mathbb{R}_+^m \cap \text{Ker}(B_3^\top X_3 B_3)\}.$$

305 *Then:*

- 306 1. *If  $\mathcal{S} = \{0\}$ , then for each  $q_k \in \mathbb{R}^{n_1 \times m}$  the problem  $VI(\mathcal{P}, q_k, \psi_{\Phi})$  has at least one solution.*
- 307 2. *Assume that  $\mathcal{S} \neq \{0\}$ . If there exists  $\xi^0 \in \text{dom}(\Phi)$  such that  $\langle q_k - \mathcal{P}^\top \xi^0, v \rangle > 0$  for all  $v \in \mathcal{S}, v \neq 0$ ,  
 308 then the problem  $VI(\mathcal{P}, q_k, \psi_{\Phi})$  has at least one solution.*
- 309 3. *If  $\xi_{k+1}^1$  and  $\xi_{k+1}^2$  are two solutions of  $VI(\mathcal{P}, q_k, \psi_{\Phi})$ , then  $\xi_{k+1}^1 - \xi_{k+1}^2 \in \text{Ker}(\mathcal{P})$ , hence  $x_{1,k+1}$  is unique.  
 310 Moreover  $\langle q_k, \begin{pmatrix} 0 \\ \lambda_{k+1}^1 - \lambda_{k+1}^2 \end{pmatrix} \rangle = 0$ .*
- 311 4. *If in addition  $B_3^\top X_3 B_3 > 0$ , then the VI in (5.5) always has a unique solution.*

312 **Proof:** The first assertion about strong SPRness follows from [28, Theorem 5.4 (A')], and it guarantees  
 313 that  $X_1 > 0$  (see Appendix D). Let us pass to the analysis of (5.6). From Proposition B.1 e) one has  
 314  $\Phi_\infty = \Phi$ . Then one has  $\mathcal{S} \triangleq \Phi \cap \text{Ker}(\mathcal{P}) \cap \{\xi \in \mathbb{R}^{n_1+m} \mid \mathcal{P} \begin{pmatrix} x_1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R}^m, \alpha_i \geq 0, 1 \leq i \leq m\} =$   
 315  $\Phi \cap \text{Ker}(\mathcal{P}) \cap \{\xi \in \mathbb{R}^{n_1+m} \mid x_1 = 0, B_3^\top X_3 B_3 \lambda = \alpha, \alpha \in \mathbb{R}^m, \alpha_i \geq 0, 1 \leq i \leq m\}$ . The third subset  
 316 in  $\mathcal{S}$  follows from [50, Example 15], and the structure of  $\mathcal{P}$  with  $X_1 > 0$ . The result follows noting  
 317 that  $\text{Ker}(B_3^\top X_3 B_3) \cap \{\lambda \in \mathbb{R}_+^m \mid B_3^\top X_3 B_3 \lambda = \alpha, \alpha \in \mathbb{R}^m, \alpha_i \geq 0, 1 \leq i \leq m\} = \text{Ker}(B_3^\top X_3 B_3)$ . Then  
 318 the remaining part follows from Proposition B.2, where the symmetry of  $\mathcal{P}$  is used (see item b') in  
 319 Proposition B.2). In item 2) one uses that  $\psi_{\Phi}(v) = 0$  when  $v \in \mathcal{S}$ . Let us now deal with item 4. In this  
 320 case  $\mathcal{P} = \mathcal{P}^\top > 0$ . Let us define  $\mathcal{R} = \mathcal{R}^\top > 0, \mathcal{R}^2 = \mathcal{P}$ , and  $\zeta_k = \mathcal{R}\xi_k$ . Then the VI in (5.5) is equivalently  
 321 rewritten as:

$$(5.7) \quad \zeta_{k+1} - \mathcal{R}^{-1} \mathcal{P}_h \mathcal{R}^{-1} \zeta_k + \mathcal{R}^{-1} G_k \in -\mathcal{R}^{-1} \mathcal{N}_{\Phi}(\mathcal{R}^{-1} \zeta_{k+1}) = -\mathcal{N}_{\tilde{\Phi}}(\zeta_{k+1}),$$

322 where the last equality stems from the chain rule, with  $\tilde{\Phi} = \{z \in \mathbb{R}^{n_1+m} \mid \mathcal{R}^{-1} z \in \Phi\}$ . One infers from  
 323 (5.7) that  $\zeta_{k+1} = \text{proj}[\tilde{\Phi}; \mathcal{R}^{-1} \mathcal{P}_h \mathcal{R}^{-1} \zeta_k - \mathcal{R}^{-1} G_k] \Rightarrow \xi_{k+1} = \mathcal{R}^{-1} \text{proj}[\tilde{\Phi}; \mathcal{R}^{-1} \mathcal{P}_h \xi_k - \mathcal{R}^{-1} G_k]$ .  $\square$

324

325 **Corollary 5.2.** Assume that  $X_1 > 0$ , and  $G(\cdot)$  is bounded. The problem  $VI(\mathcal{P}, q_k, \psi_\Phi)$  always has at least  
 326 one solution for bounded  $\xi_k$ .

327 **Proof:** Consider item 2 in Proposition 5.1, with  $\xi^0 = \begin{pmatrix} x^0 \\ \lambda^0 \end{pmatrix}$ ,  $v = \begin{pmatrix} v_x \\ v_\lambda \end{pmatrix}$ . Then:

$$(5.8) \quad \begin{aligned} \langle q_k - \mathcal{P}^\top \xi^0, v \rangle &= \langle -\mathcal{P}_h \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} + hG_k, v \rangle - \langle B_3^\top X_3 B_3 \lambda^0, v_\lambda \rangle \\ &\quad - \langle X_1 x^0, v_x \rangle \\ &= \langle - \begin{pmatrix} X_1 + hX_1 A_1 & hX_1 B_1 \\ -hC_1 & B_3^\top X_3 B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} + hG_k, v \rangle \\ &\quad - \langle B_3^\top X_3 B_3 \lambda^0, v_\lambda \rangle - \langle X_1 x^0, v_x \rangle. \end{aligned}$$

328 Since  $\xi^0 \in \Phi$  it follows that  $x^0 \in \mathbb{R}^{n_1}$  while  $\lambda^0 \in \mathbb{R}_+^m$ , and similarly  $v_x \in \mathbb{R}^{n_1}$ ,  $v_\lambda \in \mathbb{R}_+^m$ . Since  $v \neq 0$  let us  
 329 take  $x^0 = -\alpha X_1^{-1} \frac{v_x}{\|v_x\|} - \beta X_1^{-1} \frac{v_\lambda}{\|v_\lambda\|^2} \|v_\lambda\|$  for some  $\alpha \geq 0$ ,  $\beta \geq 0$ , and  $\lambda^0 = 0$ . It follows that:

$$(5.9) \quad \begin{aligned} \langle q_k - \mathcal{P}^\top \xi^0, v \rangle &= - \left\langle \begin{pmatrix} X_1 + hX_1 A_1 & hX_1 B_1 \\ -hC_1 & B_3^\top X_3 B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix}, v \right\rangle + \langle hG_k, v \rangle \\ &\geq - \left\| \begin{pmatrix} X_1 + hX_1 A_1 & hX_1 B_1 \\ -hC_1 & B_3^\top X_3 B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} \right\| \|v\| - \|hG_k\| \|v\| \\ &\quad + \alpha \|v_x\| + \beta \|v_\lambda\| \\ &\geq - \left\| \begin{pmatrix} X_1 + hX_1 A_1 & hX_1 B_1 \\ -hC_1 & B_3^\top X_3 B_3 - h\hat{D} \end{pmatrix} \begin{pmatrix} x_{1,k} \\ \lambda_k \end{pmatrix} \right\| (\|v_x\| + \|v_\lambda\|) \\ &\quad - \|hG_k\| (\|v_x\| + \|v_\lambda\|) + \alpha \|v_x\| + \beta \|v_\lambda\|. \end{aligned}$$

330 For large enough  $\alpha$  and  $\beta$  it follows that  $\langle q_k - \mathcal{P}^\top \xi^0, v \rangle > 0$  and this holds for any  $v \neq 0$  inside  $\mathcal{S}$ .  $\square$

331 In item 2, the condition  $\langle q_k, v \rangle > 0$  for all  $v \in \mathcal{S}$  is equivalent to  $q_k \in \mathcal{S}^\star \setminus \mathcal{S}^\perp$ . Obviously instead of  
 332 assuming the strong SPRness we could have just assumed that  $X_1 > 0$ . Most importantly one notices  
 333 that under the conditions of item 4, the DI in (5.3) is equivalent to a DI with maximal monotone  
 334 set-valued term:

$$(5.10) \quad \dot{\zeta}(t) - \mathcal{R}^{-1} \mathcal{A} \mathcal{R}^{-1} \zeta(t) + \mathcal{R}^{-1} G(t) \in -\mathcal{N}_\Phi(\zeta(t))$$

335 with  $\zeta = \mathcal{R}\xi$ . This fits within the class of DIs analysed for instance in [14]. Under basic assumption on  
 336 the regularity of  $G(\cdot)$ , (5.10) has Lipschitz continuous solutions with uniqueness.

## 337 5.2 BOUNDEDNESS AND CONVERGENCE ANALYSIS

338 In the next analysis we assume that  $B_3^\top X_3 B_3$  is singular, which is the most general case (see the example  
 339 in section 6.2, see also section 5.3 for the invertible case). However we still assume that  $X_1 > 0$ . From  
 340 the structure of  $\mathcal{P}$  and of the set-valued right-hand side, it follows from (5.5) that  $x_{1,k+1}$  is calculable  
 341 from  $x_{1,k}$  and  $\lambda_k$ :

$$(5.11) \quad \begin{aligned} X_1 x_{1,k+1} &= (X_1 + hX_1 A_1) x_{1,k} + hX_1 B_1 \lambda_k - hG_k^x \\ \Leftrightarrow x_{1,k+1} &= (I_{n_1} + hA_1) x_{1,k} + hB_1 \lambda_k - hX_1^{-1} G_k^x, \quad \text{for all } k \geq 0. \end{aligned}$$

342 where  $G_k = \begin{pmatrix} G_k^x \\ G_k^\lambda \end{pmatrix}$ .

343 **Lemma 5.3.** From (5.11), for all  $k \geq 0$  one has:

$$(5.12) \quad x_{1,k+1} = (I_{n_1} + hA_1)^{k+1} x_{1,0} + h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} B_1 \lambda_i - h \sum_{i=0}^k (I_{n_1} + hA_1)^{k-i} X_1^{-1} G_i^x.$$

344

345 **Proof:** The relation is true for  $k = 0$ . Let it be true for  $k \geq 1$ . Then using (5.11) it follows that:

$$\begin{aligned}
 x_{1,k+2} &= (I_{n_1} + hA_1)x_{1,k+1} + hB_1\lambda_{k+1} - hX_1^{-1}G_{k+1}^x \\
 &= (I_{n_1} + hA_1)\{(I_{n_1} + hA_1)^{k+1}x_{1,0} + h\sum_{i=0}^k (I_{n_1} + hA_1)^{k-i}B_1\lambda_i \\
 (5.13) \quad &\quad - h\sum_{i=0}^k (I_{n_1} + hA_1)^{k-i}X_1^{-1}G_i^x\} + hB_1\lambda_{k+1} - hX_1^{-1}G_{k+1}^x \\
 &= (I_{n_1} + hA_1)^{k+2}x_{1,0} + h\sum_{i=0}^{k+1} (I_{n_1} + hA_1)^{k+1-i}B_1\lambda_i \\
 &\quad - h\sum_{i=0}^{k+1} (I_{n_1} + hA_1)^{k+1-i}X_1^{-1}G_i^x.
 \end{aligned}$$

346 By induction the relation in (5.12) is true for all  $k \geq 0$ .  $\square$

347 Using (5.5) we can also express  $\lambda_{k+1}$  as the solution of a generalised equation (GE):

$$\begin{aligned}
 B_3^\top X_3 B_3 \lambda_{k+1} + \underbrace{(h\hat{D} - B_3^\top X_3 B_3)}_{\triangleq q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)} \lambda_k + hC_1 x_{1,k} + hG_k^\lambda &\in -\mathcal{N}_{\mathbb{R}_+^m}(\lambda_{k+1}) \\
 (5.14) \quad &\quad \updownarrow \\
 \lambda_{k+1} \in (B_3^\top X_3 B_3 + \mathcal{N}_{\mathbb{R}_+^m})^{-1} \left( (B_3^\top X_3 B_3 - h\hat{D})\lambda_k - hC_1 x_{1,k} - hG_k^\lambda \right)
 \end{aligned}$$

348 for all  $k \geq 0$ . The second formulation shows the iterative feature of this GE for  $\lambda_k$ , which makes it  
 349 quite different from the LCCP (3.3). It is equivalent to the LCP( $B_3^\top X_3 B_3, q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)$ ). Under the  
 350 conditions stated in Proposition 5.1 and Corollary 4.2, the set of solutions to this LCP is non empty, and it  
 351 contains a least-norm element  $\lambda_{k+1}^{\min}, k \geq 0$ , satisfying  $\|\lambda_{k+1}^{\min}(\lambda_k, x_{1,k}, G_k^\lambda)\| \leq \eta \|q_k^\lambda(\lambda_k, x_{1,k}, G_k^\lambda)\|, k \geq 0$ ,  
 352 where  $\eta > 0$  depends only on  $B_3^\top X_3 B_3$ . Therefore  $\|\lambda_{k+1}^{\min}(\lambda_k, x_{1,k}, G_k^\lambda)\| \leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \|\lambda_k^{\min}\| +$   
 353  $h\eta \|C_1 x_{1,k}\| + h\eta \|G_k^\lambda\|$ . Let us denote  $h_3 \triangleq \|h\hat{D} - B_3^\top X_3 B_3\|$ .

354 **Lemma 5.4.** One has for all  $k \geq 0$ :

$$\begin{aligned}
 (5.15) \quad \|\lambda_{k+2}^{\min}(\lambda_{k+1}, x_{1,k+1}, G_{k+1}^\lambda)\| &\leq (\eta h_3)^{k+1} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\
 &\quad + h\eta \sum_{i=0}^k (\eta h_3)^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|)
 \end{aligned}$$

355



356 **Proof:** The inequality is satisfied for  $k = 0$ . Assume that it is true for some  $k \geq 1$ . Then

$$\begin{aligned}
(5.16) \quad & \|\lambda_{k+3}^{\min}(\lambda_{k+2}, x_{1,k+2}, G_{k+2}^\lambda)\| \leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \|\lambda_{k+2}^{\min}\| + h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\
& \leq \eta \|h\hat{D} - B_3^\top X_3 B_3\| \{\eta^{k+1} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\
& \quad + h\eta \sum_{i=0}^k \eta^{k-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|)\} \\
& \quad + h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\
& = \eta^{k+2} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+2} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\
& \quad + h\eta \sum_{i=0}^k \eta^{k+1-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|) \\
& \quad + h\eta \|C_1 x_{1,k+2}\| + h\eta \|G_{k+2}^\lambda\| \\
& = \eta^{k+2} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+2} \|\lambda_1^{\min}(\lambda_0, x_{1,0}, G_0^\lambda)\| \\
& \quad + h\eta \sum_{i=0}^{k+1} \eta^{k+1-i} \|h\hat{D} - B_3^\top X_3 B_3\|^{k+1-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|).
\end{aligned}$$

357 The result follows by induction.  $\square$

358 The next step is to use Lemma 5.4 and Lemma 5.3 to get a new upperbound. Let us denote  $h_1 \triangleq \|I_{n_1} + hA_1\|$ .  
359 Then using (5.12) and (5.15) one has:

$$(5.17) \quad \begin{cases} \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h \sum_{i=0}^k h_1^{k-i} \|B_1\| \|\lambda_{i+1}^{\min}\| + h \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| \\ \|\lambda_{k+2}^{\min}\| \leq \eta^{k+1} h_3^{k+1} \|\lambda_1^{\min}\| + h\eta \sum_{i=0}^k \eta^{k-i} h_3^{k-i} (\|C_1 x_{1,i+1}\| + \|G_{i+1}^\lambda\|) \end{cases}$$

360 Therefore we obtain:

$$\begin{aligned}
(5.18) \quad & \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| \\
& \quad + h \sum_{i=0}^k h_1^{k-i} \|B_1\| \|\eta^i h_3^i \|\lambda_1^{\min}\| + h h_1^k \|B_1\| \|\lambda_1^{\min}\| \\
& \quad + h \sum_{i=1}^k h_1^{k-i} \|B_1\| h\eta \sum_{j=0}^{i-1} \eta^{i-1-j} h_3^{i-1-j} \|C_1\| \|G_{j+1}^\lambda\| \\
& \quad + h \sum_{i=1}^k h_1^{k-i} \|B_1\| h\eta \sum_{j=0}^{i-1} \eta^{i-1-j} h_3^{i-1-j} \|C_1\| \|x_{1,j+1}\|,
\end{aligned}$$

361 which we rewrite as:

$$(5.19) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$$

362 with

$$\begin{aligned}
(5.20) \quad & f_k = \sum_{i=0}^k h_1^{k-i} \|X_1^{-1}\| \|G_i^x\| + \|B_1\| \|\lambda_1^{\min}\| \sum_{i=0}^k h_1^{k-i} (\eta h_3)^i + h_1^k \|B_1\| \|\lambda_1^{\min}\| \\
& \quad + h\eta \|B_1\| \|C_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|G_{j+1}^\lambda\|.
\end{aligned}$$

363 The third term in the right-hand side of (5.19) involves terms from  $\|x_{1,1}\|$  to  $\|x_{1,k}\|$ . The inequality (5.19)  
364 can be rewritten as:

$$(5.21) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \sum_{i=1}^k \beta_i \|x_{1,i}\|,$$

365 for some positive sequence of real numbers  $\{\beta_i\}$ , each  $\beta_i$  being calculated from the expression in  
 366 (5.19)  $\eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$ , and thus being made of powers of  $h_1$  and of  $h_3 \eta$ .  
 367 Calculations show that (see Appendix F)

$$(5.22) \quad \beta_i = \eta \|C_1\| \|B_1\| \sum_{j=i}^k h_1^{k-j} (\eta h_3)^{j-i}, \quad i \geq 1.$$

368 Let us now apply a version of the discrete-time Gronwall inequality, due to Bachpatte [13] [44, Lemma  
 369 100], see section E. It is deduced that for all  $k \geq 0$ :

$$(5.23) \quad \|x_{1,k+1}\| \leq h_1^{k+1} \|x_{1,0}\| + h f_k + h^2 \sum_{i=0}^k (h_1^{i+1} \beta_i \|x_{1,0}\| + h \beta_i f_i) \prod_{j=i+1}^{k+1} (1 + h^2 \beta_j).$$

370 Let us state the next assumption (recall that  $h_3 = \|h\hat{D} - B_3^\top X_3 B_3\|$  and  $h_1 = \|I_{n_1} + hA_1\|$ ).

371 **Assumption 3.** One has a)  $h_1 \leq 1$  for  $h > 0$  small enough, b)  $\|G(t)\|$  is uniformly bounded by  $G_{\max} < +\infty$   
 372 on  $[0, T]$  for any  $T > 0$ , and  $G(\cdot)$  is Lipschitz continuous with Lipschitz constant  $k_G \geq 0$ , c)  $\eta h_3 \leq 1$   
 373 for  $h \geq 0$  small enough.

374 Inequality a) is similar to Assumption 1. As alluded to above, this inequality stems from the explicit  
 375 discretisation of the term  $A_1 x_1$  in (2.2) (a). As illustrated through examples in section 6, it holds in  
 376 practice for reasonable values of the timestep. See also the developments in Appendix A. Assumption 3  
 377 b) is a standard property of exogenous time-dependent terms. The inequality c) is the central assumption  
 378 of Assumption 3, which quantifies the couplings between the DAE part and the complementarity part  
 379 of the DVLCS. Some examples of circuits in section 6 will prove that it can be satisfied in practice.  
 380 Notice from (5.14) that if  $B_3^\top X_3 B_3 = 0$  (Assumption 2), then the minimal-norm element of the set of  
 381 solutions to the LCP is always  $\xi_k^{\min} = 0$ , hence  $\eta = 0$  in that case. Assumption 3 implies using (5.22)  
 382 that  $\beta_i \leq \eta \|C_1\| \|B_1\| \sum_{j=i}^k (\eta h_3)^{j-i} \leq \eta \|C_1\| \|B_1\| (k - i + 1)$ . Consequently using (5.20) the next  
 383 upperbound can be found:

$$(5.24) \quad f_k \leq (k+1)(G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|) + \|B_1\| \|\lambda_1^{\min}\| + h \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j,$$

384 Since  $h = \frac{T}{n}$  and  $k \in \{0, n\}$ :

$$(5.25) \quad h f_k \leq T \frac{(k+1)}{n} (G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|) + \frac{T}{n} \|B_1\| \|\lambda_1^{\min}\| + \frac{T^2}{n^2} \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j.$$

385 Now we have that  $\sum_{i=1}^k \sum_{j=0}^{i-1} j = (k-1) + 2(k-1) + 3(k-2) + \dots + (k-3)3 + (k-2)2 + (k-1)$ , where  
 386 the sum consists of  $k-1$  terms. Hence  $\sum_{i=1}^k \sum_{j=0}^{i-1} j = \mathcal{O}(k^2)$ , and  $\frac{T^2}{n^2} \eta \|B_1\| \|C_1\| G_{\max} \sum_{i=1}^k \sum_{j=0}^{i-1} j =$   
 387  $T^2 \eta \|B_1\| \|C_1\| G_{\max} \frac{\mathcal{O}(k^2)}{n^2}$ . Since  $k \leq n$  we infer that this term is uniformly bounded in  $k$ , for each  
 388 fixed bounded  $T$ . The same holds for  $T \frac{(k+1)}{n} (G_{\max} \|X_1^{-1}\| + \|B_1\| \|\lambda_1^{\min}\|)$ . Thus  $h f_k \leq \alpha_1 \max(T, T^2)$   
 389 for some  $\alpha_1$  which depends on  $\eta, \|B_1\|, \|C_1\|, G_{\max}, \|\lambda_1^{\min}\|$ .

390 The next step consists of computing an upperbound from (5.23) as:

$$(5.26) \quad \|x_{1,k+1}\| \leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) \left(1 + \frac{T^2}{n^2} \sum_{i=0}^k \beta_i \prod_{j=i+1}^{k+1} (1 + \frac{T^2}{n^2} \beta_j)\right).$$

391 In view of Assumption 3 and (5.22), one has  $\beta_i \leq \eta \|C_1\| \|B_1\| \sum_{j=i}^k 1 = \eta \|C_1\| \|B_1\| (k - i) \leq$   
 392  $\eta \|C_1\| \|B_1\| n, i \geq 1, k \leq n$ . Consequently:

$$\begin{aligned}
 \|x_{1,k+1}\| &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \frac{T^2}{n} \sum_{i=0}^k \eta \|C_1\| \|B_1\| \Pi_{j=i+1}^{k+1} (1 + \frac{T^2}{n} \eta \|C_1\| \|B_1\|)) \\
 &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{n} \sum_{i=0}^k \Pi_{j=i+1}^{k+1} (1 + \frac{T^2}{n} \eta \|C_1\| \|B_1\|)) \\
 &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{n} \sum_{i=0}^k (1 + \frac{T^2}{n} \eta \|C_1\| \|B_1\|)^{k-i}) \\
 (5.27) \quad &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{n} \sum_{i=0}^k (1 + \frac{\alpha_2}{n})^{k-i}) \\
 &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{n} \sum_{i=0}^n (1 + \frac{\alpha_2}{n})^n) \\
 &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + \eta \|C_1\| \|B_1\| \frac{T^2}{n} (n+1) (1 + \frac{\alpha_2}{n})^n) \\
 &\leq (\|x_{1,0}\| + \alpha_1 \max(T, T^2)) (1 + T^2 \eta \|C_1\| \|B_1\| \frac{n+1}{n} (1 + \frac{\alpha_2}{n})^n),
 \end{aligned}$$

393 where  $\alpha_2 = T^2 \eta \|C_1\| \|B_1\|$ . One has  $(1 + \frac{\alpha_2}{n})^n \rightarrow e^{\alpha_2}$  as  $n \rightarrow +\infty$  [70, p.196]. Therefore the sequence  
 394  $\{x_{1,k}\}$  is uniformly bounded over  $k \in \{0, n\}$ . It directly follows from Lemma 5.4 that the sequence  $\{\lambda_k^{\min}\}$   
 395 is uniformly bounded as well. It follows from (4.1) that the sequence  $\{\frac{x_{1,k+1} - x_{1,k}}{h}\}$  is uniformly bounded  
 396 also. Therefore we are ready to prove the following result. The piece-wise linear approximations  
 397 are defined in (4.7), and the step approximations  $x_1^{h*}(t) = x_{1,k+1}$  if  $t \in [t_k, t_{k+1})$ ,  $\lambda_{h*}^{\min}(t) = \lambda_k^{\min}$  if  
 398  $t \in [t_k, t_{k+1})$ .

399 **Proposition 5.5.** *Let Assumption 3 hold true, as well as the conditions for the well-posedness of the OSNSP in*  
 400 *Proposition 3.2. Then the sequence  $\{x_1^h(\cdot)\}$  converges uniformly towards a continuous limit  $x_1(\cdot)$ . Moreover*  
 401  *$x_1^{h*}(\cdot)$  converges strongly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  towards  $x_1(\cdot)$ , and  $\dot{x}_1^h(\cdot)$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$*   
 402 *towards  $\dot{x}_1(\cdot)$ . Therefore  $\{(B_1 \lambda_{h*}^{\min})(\cdot)\}$  converges weakly in  $\mathcal{L}_2([0, T]; \mathbb{R}^{n_1})$  towards a limit  $\lambda_1(\cdot)$ , and*  
 403  *$\dot{x}_1(t) = A_1 x_1(t) + G^x(t) + \lambda_1(t)$  almost everywhere on  $[0, T]$ .*

404 **Proof:** Both sequences  $\{x_1^h(\cdot)\}$  and  $\{\dot{x}_1^h(\cdot)\}$  are uniformly bounded on  $[0, T]$ , the sequence  $\{x_1^h(\cdot)\}$   
 405 of  $C^0([0, T]; \mathbb{R}^{n_1})$  functions is thus equicontinuous. From the Ascoli-Arzelà Theorem,  $\{x_1^h(\cdot)\}$  stays  
 406 in a compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and indeed it converges uniformly towards a continuous limit  
 407  $x_1(\cdot)$ . Let us now examine the sequences of piecewise-linear functions  $\{G^h(\cdot)\}_{h \geq 0}$  and  $\{\dot{G}^h(\cdot)\}_{h \geq 0}$ ,  
 408 designed similarly as (4.7). From Assumption 3 b) one infers that  $G^h(t)$  is uniformly bounded on  
 409  $[0, T]$ ,  $\frac{\|G_{k+1} - G_k\|}{h} \leq k_E$  implies that  $\dot{G}^h(t)$  is also uniformly bounded in  $h$  on  $[0, T]$ , thus  $\{G^h(\cdot)\}_{h \geq 0}$  is  
 410 a sequence of Lipschitz continuous, uniformly bounded, equicontinuous functions. From the Ascoli-  
 411 Arzelà Theorem,  $\{G^h(\cdot)\}$  stays in a compact subset of  $C^0([0, T]; \mathbb{R}^{n_1})$  and it converges uniformly  
 412 towards the continuous limit  $G(\cdot)$  as  $h \rightarrow 0$ . The remaining part of the proof is similar to the proof of  
 413 Corollary 4.7 (adding  $G^{x,h}(\cdot)$  with the required properties) and it is omitted.  $\square$

414 **Remark 5.6.** Assumption 3 c) is clearly less restrictive than Assumption 2. It still imposes some constraint  
 415 on the matrix  $C_3 B_3$ . Examples are treated next which show that it is satisfied in some practical cases.

416 Let us now prove the uniform boundedness of the sequences  $\{\lambda_k^{\min}\}$  and  $\{x_{2,k}\}$ . To this aim we shall  
 417 use first the second line in (5.17) and (5.27). Let us denote the right-hand side of (5.27) as  $x_{1,\max}(T, \eta, x_{1,0})$ .  
 418 Then from (5.17) and Assumption 3 we obtain:

$$\begin{aligned}
 \|\lambda_{k+1}^{\min}\| &\leq \|\lambda_0^{\min}\| + h\eta \sum_{i=0}^k (\|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \|G_{i+1}^\lambda\|) \\
 (5.28) \quad &\leq \|\lambda_0^{\min}\| + h\eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0}) (k+1) + h\eta \sum_{i=0}^k \|G_{i+1}^\lambda\| \\
 &\leq \|\lambda_0^{\min}\| + \frac{T(k+1)}{n} \eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \eta \frac{T(k+1)}{n} G_{\max} \\
 &\leq \|\lambda_0^{\min}\| + \frac{T(n+1)}{n} \eta \|C_1\| x_{1,\max}(T, \eta, x_{1,0}) + \eta \frac{T(n+1)}{n} G_{\max}.
 \end{aligned}$$

419 Therefore  $\|\lambda_{k+1}^{\min}\|$  is uniformly bounded for any bounded  $T$  and initial data. The boundedness of  $x_{2,k}$   
 420 follows from the equality constraint in (4.1). A result similar to Corollary 4.7 holds, but this time  
 421 verifying the complementarity conditions is less easy due to the presence of  $z$  in the complementarity  
 422 variable  $w$ .

### 423 5.3 CASE $X_1 > 0$ AND $B_3^\top X_3 B_3 > 0$

424 This is the case studied in item 4 of Proposition 5.1 and may be verified in practice, see the example in  
 425 section 6.1. Then the DI in (5.3) is equivalently rewritten as in (5.10). Consequently one can apply the  
 426 implicit Euler scheme and all the existence, convergence and order results presented in [14] (see that  
 427 article for details on the discretization method and theoretical results), to (5.10). It is inferred that in this  
 428 case (5.10) possesses unique solutions in  $C^0([0, T]; \mathbb{R}^{n_1+m})$  with essentially bounded derivatives [14,  
 429 Proposition 2.6]. It is deduced that  $x_2(\cdot)$  is also unique in  $C^0([0, T]; \mathbb{R}^{n_1+m})$  with essentially bounded  
 430 derivative, while the algebraic variable  $z(\cdot)$  (which depends on  $\dot{\lambda}(\cdot)$ , see (5.1)) is essentially bounded  
 431 (hence *a priori* it may be discontinuous).

## 432 6 EXAMPLES AND NUMERICAL SIMULATIONS

433 Two examples of circuits with ideal diodes, and one academic example, are developed in this section.  
 434 Let us comment that the analysis of circuits applies to hydraulics, due to the strong analogy between  
 435 hydraulic and electrical circuits [65, 53] [49, Chapter 1], where check valves play the role of diodes.

### 436 6.1 FIRST EXAMPLE

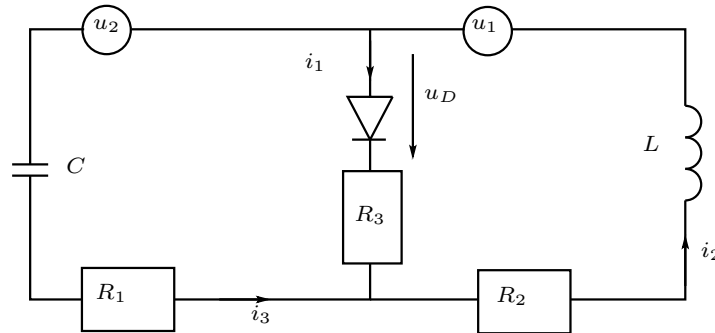


Figure 1: An RLCD passive circuit.

437 Let us consider the circuit in Figure 1. Let us assume first that  $R_1 = R_3 = 0 \Omega$ . Let  $\xi_1 = i_2$ ,  $\xi_2 =$   
 438  $\int_0^t i_3(t)dt$ ,  $\xi_3 = i_1$ . The dynamics is given by:

$$(6.1) \quad \begin{cases} \dot{\xi}_1(t) = -\frac{R_2}{L} \xi_1(t) + \frac{1}{L} u_D + \frac{1}{L} u_1(t) \\ \dot{\xi}_2(t) = \xi_1(t) - \xi_3(t) \\ 0 = \frac{1}{C} \xi_2(t) + u_D(t) + u_2(t) \\ 0 \leq u_D \perp \xi_3(t) \geq 0. \end{cases}$$

439 It is noteworthy that this circuit dynamics agrees with [2, Theorem 3.6] [45] about the index (indeed it  
 440 cannot be of index 1 due to loop with a capacitive element). In our notations we have  $x = (\xi_1, \xi_2)^\top$ ,  
 441  $z = \xi_3$ ,  $C = (0 \ 0 \ 1)$ ,  $B = (\frac{1}{L} \ 0 \ 1)^\top$ . The transfer function  $H(s) = \frac{z(s)}{u_D(s)} = \frac{Cs^2 + \frac{CR}{L}s + \frac{1}{L}}{s + \frac{R}{L}} = Cs + \frac{1}{Ls + R}$ . It has a

442 relative degree  $r = -1$ . It can be checked that  $H(s)$  is positive real (PR) as long as  $R_2 > 0$  [25, Theorem  
 443 2.45], but it is not strictly positive real (SPR) (the first condition in item 3.b in [25, Theorem 2.61] does  
 444 not hold, despite of the fact that the second condition in the same item holds), and hence it is not  
 445 strongly SPR. Let us verify the minimality of (6.1):

$$446 \quad 1. \text{rank}((A \ B) - sP \ B) = \text{rank} \begin{pmatrix} \frac{-R}{L} - s & 0 & 0 & \frac{1}{L} \\ 1 & -s & -1 & 0 \\ 0 & \frac{1}{C} & 0 & 1 \end{pmatrix} = 3 \text{ for all } s \in \mathbb{C}.$$

$$447 \quad 2. \text{rank}(P \ B) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{L} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3.$$

$$448 \quad 3. \text{rank}(A^\top - sP^\top \ C^\top) = \text{rank} \begin{pmatrix} \frac{-R}{L} - s & 1 & 0 & 0 \\ 0 & -s & \frac{1}{C} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 3 \text{ for all } s \in \mathbb{C}.$$

$$449 \quad 4. \text{rank}(P^\top \ C^\top) = \text{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3.$$

$$450 \quad 5. (A \ B) \text{Ker}(P) = \begin{pmatrix} \frac{-R}{L} & 0 & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{C} & 0 \end{pmatrix} \text{Ker}(P) = \{\xi \in \mathbb{R}^3 \mid \xi_1 = \xi_3 = 0, \xi_2 \in \mathbb{R}\} \subset \text{Im}(P) = \{\xi \in$$

$$451 \quad \mathbb{R}^3 \mid \xi_3 = 0, \xi_2 \in \mathbb{R}, \xi_1 \in \mathbb{R}\},$$

452 Therefore we infer that (6.1) admits a special Weiertrass form as (2.2). Let us choose  $L = 4.7 \cdot 10^{-4}$

$$453 \quad H, R = 1\Omega, C = 10^{-1} \text{ F. Let us define } S = \begin{pmatrix} -1.414 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-4} \end{pmatrix}, T = \begin{pmatrix} -0.707 & 0 & 0 \\ 0 & 0 & 1 \\ -0.707 & -1 & 0 \end{pmatrix}. \text{ Then } SAT =$$

$$454 \quad \begin{pmatrix} -2.12 \cdot 10^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, SB = \begin{pmatrix} -3 \cdot 10^{-3} \\ 0 \\ 10^{-4} \end{pmatrix}, CT = (-0.707 \ -1 \ 0), S \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.301 \cdot 10^4 & 0 \\ 0 & 0 \\ 0 & 10^{-4} \end{pmatrix}. \text{ Let us make}$$

455 the state variable change  $\zeta = T^{-1}\xi$ , then we obtain the transformed dynamics:

$$(6.2) \quad \begin{cases} \dot{\zeta}_1(t) = -2.12 \cdot 10^3 \zeta_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ \dot{\zeta}_3(t) = \zeta_2(t) \\ 0 = \zeta_3(t) + 10^{-4} \lambda(t) + 10^{-4} u_2(t) \\ 0 \leq \lambda(t) \perp w(t) = -0.707 \zeta_1(t) - \zeta_2(t) \geq 0. \end{cases}$$

456 Let us now define  $x_1 = \zeta_1, x_2 = \zeta_3, x_3 = \zeta_2$ , then we obtain

$$(6.3) \quad \begin{cases} \dot{x}_1(t) = -2.12 \cdot 10^3 x_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ \dot{x}_2(t) = x_3(t) \\ 0 = x_2(t) + 10^{-4} \lambda(t) + 10^{-4} u_2(t) \\ 0 \leq \lambda(t) \perp w(t) = -0.707 x_1(t) - x_3(t) \geq 0, \end{cases}$$

457 which is the canonical form in (2.2) with  $z = x_3, B_1 = -3 \cdot 10^{-3}, B_2 = 0, B_3 = 10^{-4}, C_1 = -0.707,$   
 458  $C_2 = 0, C_3 = -1, D = 0$ . We have  $X_3 = 10^4, A_1 = -2.12 \cdot 10^3, X_1 = \frac{C_1}{B_1} > 0, \hat{D} = D - C_3 B_2 - C_2 B_3 = 0,$   
 459  $\bar{D}_h = h^2 C_1 B_1 + B_3^\top X_3 B_3 = 2.121 \cdot 10^{-3} h^2 + 10^{-4}, h_3 = B_3^\top X_3 B_3 = 10^{-4}, h_1 = |1 + h A_1| = |1 - 2.12 \cdot 10^3 h|$   
 460 which verifies Assumption 3 for  $h > 0$  small enough. Notice that the set  $\mathcal{S}$  in Propositions 3.2 and 5.1

461 verifies  $\mathcal{S} = \{0\}$ , hence the OSNSP is well-posed with unique solution. Provided that Assumption 3 is  
 462 satisfied, the material in section 5 applies. In a similar way to (5.1), the system (6.3) is transformed into:

$$(6.4) \quad \begin{cases} \dot{x}_1(t) = -2.12 \cdot 10^3 x_1(t) - 3 \cdot 10^{-3} \lambda(t) + 0.301 \cdot 10^4 u_1(t) \\ 10^{-4} \dot{\lambda}(t) = 0.707 x_1(t) - 10^{-4} \dot{u}_2(t) + w(t) \\ x_2(t) = -10^{-4} \lambda(t) - 10^{-4} u_2(t) \\ z(t) = -10^{-4} \dot{\lambda}(t) - 10^{-4} \dot{u}_2(t). \end{cases}$$

463 From (5.14), the following LCP is obtained:

$$(6.5) \quad \begin{cases} 0 \leq 10^{-4} \lambda_{k+1} + q_k \perp \lambda_{k+1} \geq 0 \\ q_k = -10^{-4} \lambda_k - 0.707 h x_{1,k} - 10^{-4} h u_{2,k}, \end{cases}$$

464 and  $x_{1,k+1}, x_{2,k+1}, z_{k+1}$  as in (H.1). Let us check Assumption 3 c). Since  $\hat{D} = 0$  it is easy to show that  
 465  $\eta = 10^4$  while  $h_3 = 10^{-4}$ , using the LCP in (5.14) that can be solved explicitly. Thus  $\eta h_3 \leq 1$  for all  $h \geq 0$ .  
 466 However,  $h_1 = \|1 - 2.12 \cdot 10^3 h\| \leq 1$  if and only if  $h < 4.7 \cdot 10^{-4}$ . It is also noteworthy that this system  
 467 verifies the conditions of section 5.3, and item 4 of Proposition 5.1, since  $X_1 > 0$  and  $B_3^\top X_3 B_3 = 10^{-4} > 0$ .  
 468 Hence the state space form (5.10), that is equivalent to (5.3) since  $\mathcal{P} > 0$ , can be calculated. Also the  
 469 generalized equation (5.14) always has a unique solution  $\lambda_{k+1}$ . As we shall see through numerical  
 470 simulations, the variable  $z(\cdot) = x_3(\cdot)$  may be discontinuous. Actually this is in agreement with the  
 471 theoretical conclusions from section 5.3, which do not state that this variable is continuous. It also  
 472 shows that a complete convergence analysis, including  $z$ , should rely on other tools than those used in  
 473 this paper.

474 In Figure 2, the results of the numerical simulation of (6.3) with an implicit Euler scheme are depicted<sup>2</sup>.  
 475 Numerical simulations are performed with the software SICONOS [1]. At each time step, the Lemke  
 476 solver [62, 61] is applied to solve the LCP (3.1) resulting from the discretization scheme. The initial  
 477 conditions are set to  $\mathbf{x}(0) = (0.01, -10^{-4}, -7.07 \cdot 10^{-3})^\top$ , and the time step is  $h = 5 \cdot 10^{-6}$ s. The input  
 478 functions  $u_1(t)$  and  $u_2(t)$  are taken as sinusoidal functions:  $u_1(t) = \sin(t)$  and  $u_2(t) = \cos(10t)$ . Let  
 479 us observe that the variable  $z(t) = x_3(t)$  appears to be discontinuous at  $t \approx 5.9$ s. The multiplier  $\lambda(t)$   
 480 remains continuous, see Figure 3. Finally, an order of convergence equal to one is deduced from the  
 481 data in Figure 4. The error formula used is:

$$(6.6) \quad \text{error} = \|\mathbf{x}(t) - \mathbf{x}^h(t)\|_\infty.$$

482

## 483 6.2 SECOND EXAMPLE

484 This circuit example depicted in Figure 5 is inspired from [32]. The so-called modified nodal analysis is  
 485 used to derive the dynamics [2, chapter 3]. It is assumed that the voltage source delivers a constant volt-  
 486 age  $u(t) = u_c$ , while the other voltage source  $u_1(t)$  is time-varying Lipschitz continuous and bounded.  
 487 Notice that the current/voltage characteristic of this device can be written as  $i_5(t) \in -\partial\psi_{\{u_c\}}(u(t))$ , or  
 488 equivalently as  $R \ni i_5(t) \perp u(t) - u_c \in \{0\}$ . The dynamics is given by:

$$(6.7) \quad \begin{cases} C \dot{e}(t) = -R^{-1} e(t) + i_4(t) - i_1(t) - i_5(t) \\ L \frac{di_1}{dt}(t) = e(t) + u_1(t) \\ 0 = e(t) + u(t) \\ 0 \leq u_D(t) \perp i_4(t) \geq 0 \\ R \ni i_5(t) \perp u(t) - u_c \in \{0\}. \end{cases}$$

<sup>2</sup>Source code of the examples can be found at <https://github.com/siconos/siconos-tutorials/tree/master/sandbox/mlcspaper>



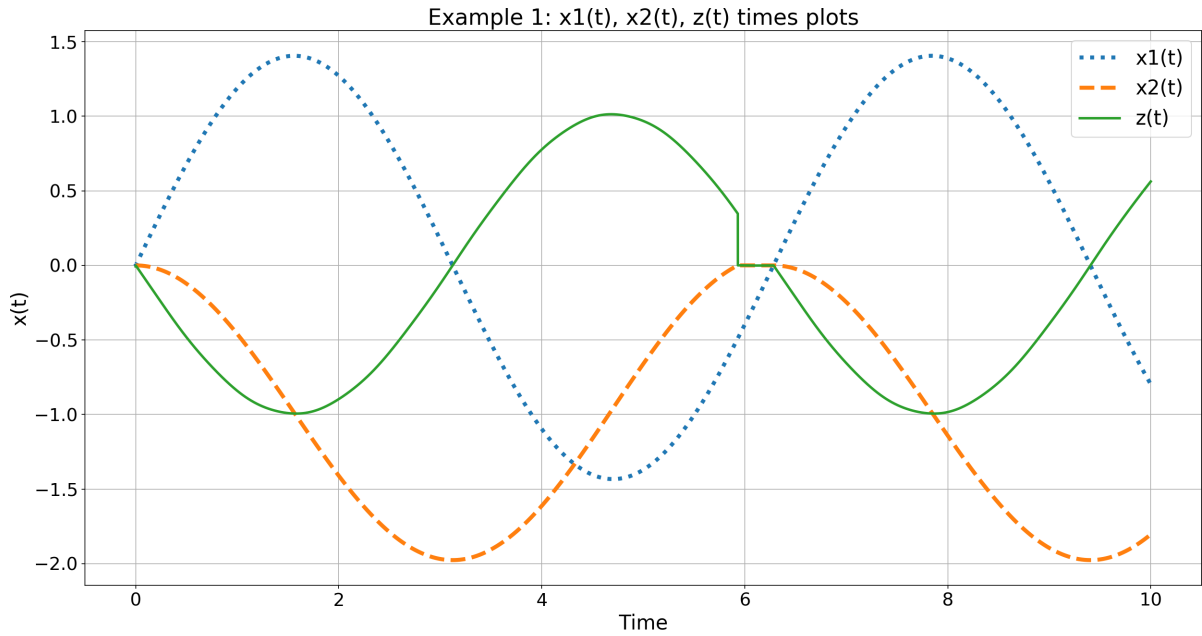


Figure 2: Implicit Euler numerical simulation of (6.3) with  $h = 5 \cdot 10^{-6}$ s. The numerical simulations of the variables  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t) = z(t)$  are depicted.

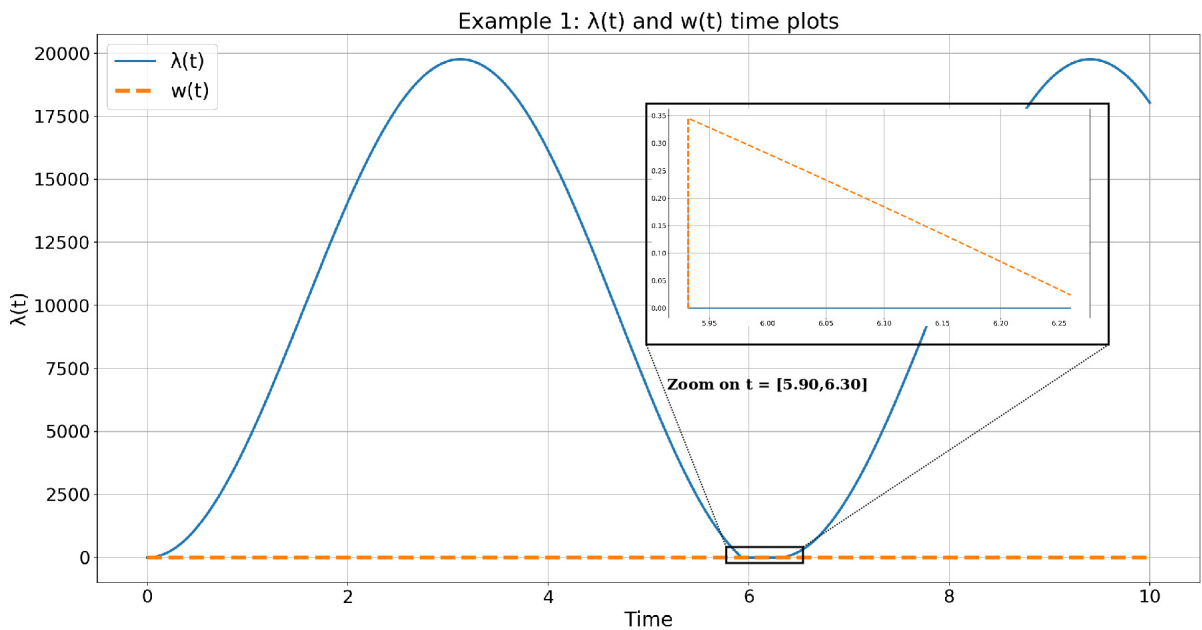
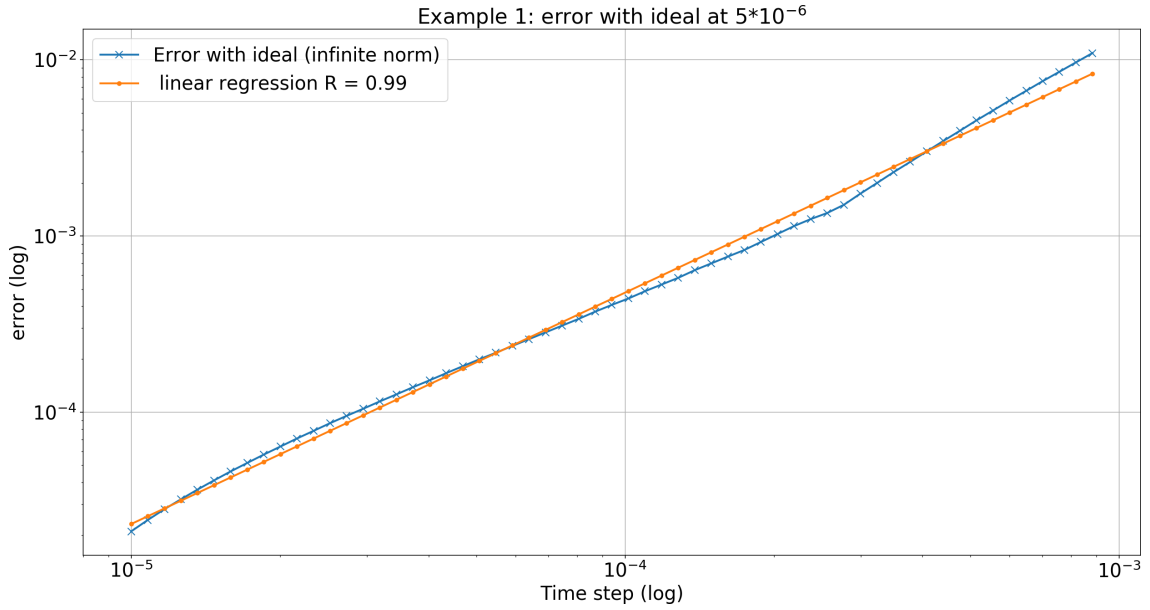


Figure 3: Implicit Euler numerical simulation of (6.3) with  $h = 5 \cdot 10^{-6}$ s. The numerical simulations of the variables  $\lambda(t)$  and  $w(t)$  are depicted.



**Figure 4:** Numerical convergence of the implicit Euler scheme applied to (6.3). The reference curve is taken for  $h = 5 \cdot 10^{-6}$ s. The error formula used is given in (6.6).

489 The state vector is  $\xi = (\xi_1, \xi_2, \xi_3)^\top = (e, i_1, i_5)^\top$ , and  $u_D(t) = e(t)$ . The complementarity conditions can  
 490 be written as  $K = R^+ \times R \ni (u_D(t), i_5(t)) \perp (i_4(t), u(t) - u_c) \in R^+ \times \{0\} = K^*$ . Setting  $\lambda_1 = i_4$  and  
 491  $\lambda_2 = u(t) - u_c$ , we obtain  $P = \begin{pmatrix} C & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $A = \begin{pmatrix} -R^{-1} & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $D = 0$ ,  
 492 and the DVLCs form in (2.1) is equal to:

$$(6.8) \quad \begin{cases} \dot{\xi}_1(t) = -\frac{1}{RC}\xi_1(t) - \frac{1}{C}\xi_2(t) - \frac{1}{C}\xi_3(t) + \frac{1}{C}\lambda_1(t) \\ \dot{\xi}_2(t) = \frac{1}{L}\xi_1(t) + \frac{u_1(t)}{L} \\ 0 = \xi_1(t) + \lambda_2(t) + u_c \\ 0 \leq \xi_1(t) \perp \lambda_1(t) \geq 0 \\ R \ni \xi_3(t) \perp \lambda_2(t) \in \{0\}. \end{cases}$$

493 Let  $S = \begin{pmatrix} 0 & \frac{1}{L} & -\frac{1}{L} \\ -1 & 0 & -\frac{1}{R} \\ 0 & 0 & -C \end{pmatrix}$ ,  $T = \begin{pmatrix} 0 & 0 & -\frac{1}{C} \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ , then  $SAT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $SPT = \begin{pmatrix} \frac{1}{L} & 0 & 0 \\ 0 & 0 & \frac{1}{C} \\ 0 & 0 & 0 \end{pmatrix}$ ,  $CT = \begin{pmatrix} 0 & 0 & -\frac{1}{C} \\ -1 & 1 & 0 \end{pmatrix}$ ,  
 494  $SB = \begin{pmatrix} 0 & -\frac{1}{L} \\ 1 & -\frac{1}{R} \\ 0 & -C \end{pmatrix}$ . Using the state variable change  $\zeta = T^{-1}\xi$  with  $T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -C & 0 & 0 \end{pmatrix}$ , one obtains:

$$(6.9) \quad \begin{cases} \dot{\zeta}_1(t) = -(\lambda_2(t) + u_c(t)) + \frac{u_1(t)}{L} \\ \frac{1}{C}\dot{\zeta}_3(t) = \zeta_2(t) + \lambda_1(t) - \frac{1}{R}(\lambda_2(t) + u_c(t)) \\ 0 = -C(\lambda_2(t) + u_c(t)) + \zeta_3(t) \\ K^* \ni \lambda(t) \perp w(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \zeta_1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \zeta_2(t) + \begin{pmatrix} -\frac{1}{C} \\ 0 \end{pmatrix} \zeta_3(t) \in K. \end{cases}$$

495 Let  $x_1 = \zeta_1$ ,  $x_2 = \frac{1}{C}\zeta_3$ ,  $z = \zeta_2$ ,  $B_1 = (0 \ -1)$ ,  $B_2 = (1 \ -\frac{1}{R})$ ,  $B_3 = (0 \ -1)$ ,  $C_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ,  $C_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $C_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ,  
 496  $D = 0$ . This dynamics is rewritten in the canonical form as:

$$(6.10) \quad \begin{cases} \dot{x}_1(t) = -\lambda_2(t) - u_c(t) + \frac{u_1(t)}{L} \\ \dot{x}_2(t) = z(t) + \lambda_1(t) - \frac{1}{R}(\lambda_2(t) + u_c(t)) \\ 0 = -\lambda_2(t) - u_c(t) + x_2(t) \\ K^* \ni \lambda(t) \perp w(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} x_1(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_2(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z(t) \in K. \end{cases}$$

497 In the simulations in Figures 6a and 6b it has been chosen  $u_c(t) = \sin(10t) - 1 \leq 0$ , which complies  
 498 with the system's intrinsic constraints. Then (when  $u_1(t) = 0$ ) one recovers the standard form in (2.2),  
 499 with a different class of sets  $K$ , however. One has  $A_1 = 0$ ,  $B_3^T X_3 = -C_3$  with  $X_3 = 1$ , and  $X_1 B_1 = C_1^T$   
 500 (this follows necessarily from (D.2) and  $Q_1 \leq 0$  and using [25, Lemma A.69]), with  $X_1 = 1$  (hence the  
 501 quadruple  $(A_1, B_1, C_1, \hat{D})$  is passive but it is not strictly passive and its associated transfer function is  
 502 not strongly SPR). Also  $\hat{D} = -C_3 B_2 - C_2 B_3 = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{R} \end{pmatrix} \succcurlyeq 0$ . Also  $B_3^T X_3 B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \succcurlyeq 0$ .

503 It is computed that  $\bar{D}_h = \begin{pmatrix} 0 & h \\ -h & \frac{h}{R} + h^2 + 1 \end{pmatrix}$ . The various sets for the set  $\mathcal{S}$  in Proposition 3.2 are

504 calculated as:  $K^* = \mathbb{R}^+ \times \{0\}$ ,  $\text{Ker}(\bar{D}_h + \bar{D}_h^T) = \begin{pmatrix} 0 & 0 \\ 0 & 2(\frac{h}{R} + h^2 + 1) \end{pmatrix} = \mathbb{R} \times \{0\}$ ,  $\{w \mid \bar{D}_h w \in K =$   
 505  $\mathbb{R}^+ \times \mathbb{R}\} = \mathbb{R} \times \mathbb{R}^+$ . So  $S = \mathbb{R}^+ \times \{0\} \neq 0$  and item 2 of Proposition 3.2 has to be applied. It is calculated

506 that  $\langle h\beta_k - \bar{D}_h \xi_0, v \rangle = \langle \begin{pmatrix} -h\xi_0^1 \\ -hx_{1,k} - x_{2,k} + h\xi_0^1 - (\frac{h}{2} + h^2 + 1)\xi_0^2 \end{pmatrix}, v \rangle$ . Also  $v \in K^*$  implies that  $v_1 \geq 0$  and  
 507  $v_2 = 0$ , so it follows that  $\langle h\beta_k - \bar{D}_h \xi_0, v \rangle + \psi_{K^*}(v) > 0$  is guaranteed, if  $v_1 > 0$ , by taking  $\xi_0^1 < 0$ . It is  
 508 inferred that item 2 in Proposition 3.2 is verified and the OSNSP has at least one solution.

509 Any two solutions  $\lambda_{k+1}^1$  and  $\lambda_{k+1}^2$  satisfy  $\lambda_{2,k+1}^1 = \lambda_{2,k+1}^2$  for all  $h \geq 0$ , in view of the form of  $\bar{D}_h$ . Then  
 510 both  $x_{1,k+1}$  and  $x_{2,k+1}$  are unique due to the form of  $B_1$  and  $B_3$ . For the convergence analysis, let us note  
 511 that this system does not satisfy Assumption 2, therefore let us use the material in section 5. Using the

512 Frobenius norm one finds that  $h_3 = \sqrt{2h^2 + \left(\frac{h}{R} - 1\right)^2}$ , using the Hölder induced norm  $\|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$

513 one finds a smaller  $h_3 = \sqrt{h^2 + \left(\frac{h}{R} - 1\right)^2}$ , and  $0 \leq h_3 \leq 1$  for all  $h \in [0, \frac{2R}{1+R^2}]$ . The material in section

514 5 is based on the LCP in (5.14), for which the existence of a minimal norm solution satisfying the  
 515 inequalities stated below (5.14) is guaranteed. However in this example  $K = \mathbb{R}_+ \times \mathbb{R} \neq \mathbb{R}_+^2$ . The following  
 516 problem has to be studied instead (it is noteworthy that the rest of the analysis remains unchanged as  
 517 long as the minimal norm solution can be characterised):

$$(6.11) \quad K \ni B_3^T X_3 B_3 \lambda_{k+1} + q_k^\lambda \perp \lambda_{k+1} \in K^*,$$

518 *i.e.*,

$$(6.12) \quad \mathbb{R}_+ \times \{0\} \ni \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} 0 \\ \lambda_{2,k+1} \end{pmatrix} + q_k^\lambda \in \mathbb{R}_+ \times \mathbb{R}.$$

519 Therefore  $\lambda_{2,k+1} = 0$ , and the value of  $\lambda_{1,k+1} = 0$  is always a solution (notice that the external excitation  
 520 is constrained to be non negative to comply with the system's constraints). This result agrees with the  
 521 numerical simulations in Figure 6b.

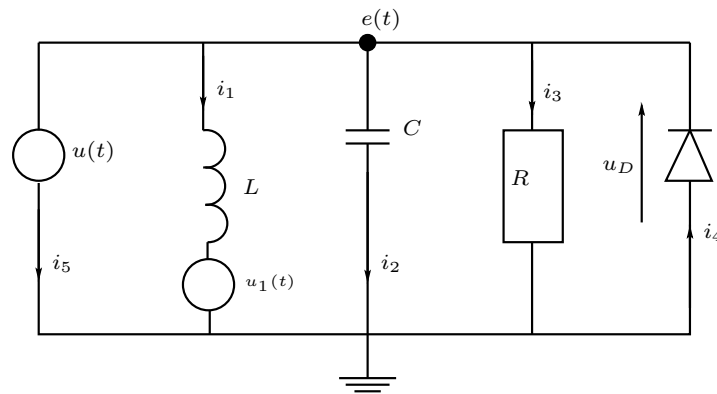
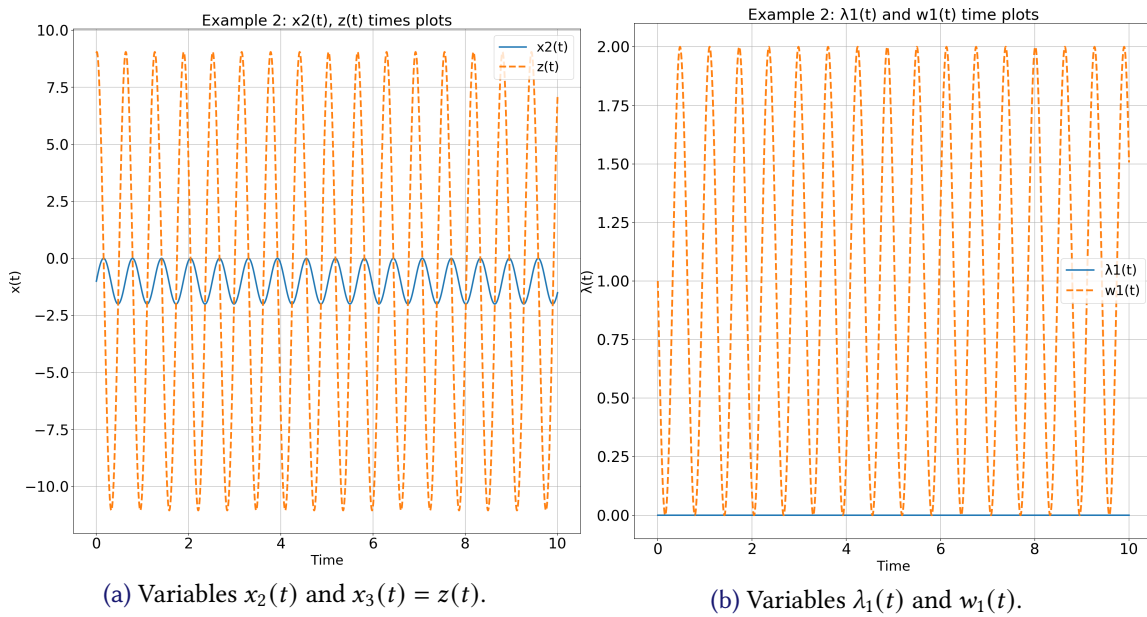


Figure 5: An *RLCD* passive circuit.



(a) Variables  $x_2(t)$  and  $x_3(t) = z(t)$ .

(b) Variables  $\lambda_1(t)$  and  $w_1(t)$ .

Figure 6: Implicit Euler numerical simulation of (6.9) with  $h = 5 \cdot 10^{-6}$ s. The solutions of  $x_1(t)$  are not displayed as they are independent of  $z(t), \lambda_1(t)$ . The variable  $\lambda_2(t)$  trivially equals 0, and is not computed.

## 522 6.3 THIRD EXAMPLE

523 Let us consider now an academic toy-example directly in the form of (2.2), whose dynamics are given  
524 as:

$$(6.13) \quad \begin{cases} \dot{x}_1(t) = -2x_1(t) + 2\lambda_1(t) - \lambda_2(t) \\ \dot{x}_2(t) = z(t) \\ 0 = x_2(t) + \lambda_1(t) - 2\lambda_2(t) \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} 2 \\ -1 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 3 \end{pmatrix} x_2(t) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} z(t) + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \geq 0 \end{cases}$$

525 It can be verified that the transfer matrix  $H(s) = D + C(sP - A)^{-1}B$  is positive real [25, Definition  
526 2.34], and the state space realization is minimal (see section G). In fact  $H(s)$  is even strongly (or  
527 extended) positive real [25, Definition 2.78]. Thus [28, Theorem 5.4 (A') (B')] applies. It is noteworthy  
528 that since  $D = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} > 0$ , it follows that the multipliers  $\lambda$  are Lipschitz continuous functions of  
529 the state  $(x_1, x_2, z)$ . Thus the DVLCS is a DAE with Lipschitz continuous nonlinear and nonsmooth  
530 right-hand side. From (6.13) it follows that:  $A_1 = -2 < 0$ ,  $C_1 = X_1 B_1^T = (2, -1)^T$  with  $X_1 = 1 > 0$ ,  $X_3 = I_2$ ,  
531  $B_3 = (1 \ -2) = -C_3^T$ , and  $\hat{D} = \begin{pmatrix} 1 & 1 \\ -3 & 7 \end{pmatrix} > 0$  with  $(\hat{D} + \hat{D}^T) > 0$ . It follows from [28, Theorem 5.4  
532 (A'), Theorem 4.6] that the quadruple  $(A_1, B_1, C_1, \hat{D})$  is strictly passive. It can be checked that the set  
533  $\mathcal{S} = \{0\}$  in Proposition 3.2 for any  $h > 0$ , thus there exists a unique solution  $\lambda_{k+1}$  to the implicit Euler  
534 discretization of (6.13). However  $C_3 \neq 0$  and Assumption 2 does not hold, preventing the convergence  
535 analysis in section 4 and the stability analysis in section 4.1 to apply. Consequently, the "equivalent"  
536 system (5.1) from Section 5 is analysed:

$$(6.14) \quad \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \end{pmatrix} = \begin{pmatrix} -2 & 2 & -1 \\ -2 & -1 & -1 \\ 1 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ w_1(t) \\ w_2(t) \end{pmatrix} \\ w(t) \in -\mathcal{N}_{\mathbb{R}_+^2}(\lambda(t)) \end{cases}$$

537 Let us check if Assumption 3 (c) is verified. Using the Frobenius norm, one can obtain that  $0 \leq h_3 =$   
538  $\sqrt{60h^2 + 25 - 66h} \leq 1$  if  $0 < h \leq 0.7$ . In the context of (6.14), the LCP (5.14) becomes:

$$(6.15) \quad \begin{aligned} 0 \leq \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} + \begin{pmatrix} (h-1) & (h+3) \\ (2-3h) & (7h-4) \end{pmatrix} \begin{pmatrix} \lambda_{1,k} \\ \lambda_{2,k} \end{pmatrix} + \begin{pmatrix} 2hx_{1,k} \\ -hx_{1,k} \end{pmatrix} \geq 0 \\ \iff \\ 0 \leq \begin{pmatrix} \lambda_{1,k+1} \\ \lambda_{2,k+1} \end{pmatrix} \perp \begin{pmatrix} \lambda_{1,k+1} - 2\lambda_{2,k+1} \\ 4\lambda_{2,k+1} - 2\lambda_{1,k+1} \end{pmatrix} + \begin{pmatrix} q_{1,k} \\ q_{2,k} \end{pmatrix} \geq 0. \end{aligned}$$

539 From Corollary 4.2, there exists a solution to (6.15). We can compute  $\lambda_{k+1}$  in each mode of the LCP  
540 where  $\lambda_{k+1} \neq 0$ :

- 541 1. If  $\lambda_{1,k+1} = 0$  and  $\lambda_{2,k+1} \geq 0$  then  $\|\lambda_{2,k+1}\| \leq \frac{1}{2}\|q_{1,k}\|$ .
- 542 2. If  $\lambda_{2,k+1} = 0$  and  $\lambda_{1,k+1} \geq 0$  then  $\|\lambda_{1,k+1}\| \leq \frac{1}{2}\|q_{2,k}\|$ .
- 543 3. If  $\lambda_{1,k+1} > 0$  and  $\lambda_{2,k+1} > 0$  there is infinitely many solutions. Indeed, we obtain:

$$(6.16) \quad \begin{cases} \lambda_{1,k+1} = 2\lambda_{2,k+1} - q_{1,k} \\ 0 = 2q_{1,k} + q_{2,k} \end{cases}$$

544 From [16, Proposition 6.1.7], there exists a minimal solution  $\lambda_{\min,k+1}$ . Then, (6.16)'s particular solution  
 545  $\lambda_{2,k+1} = -\lambda_{1,k+1}$  leads to  $\|\lambda_{\min,k+1}\| \leq (1/3)\|q_{k+1}\|$ . It follows that  $\eta = \frac{1}{2} < 1$  for all  $h > 0$ , and  $\eta h_3 \leq 1$   
 546 for all  $0 < h \leq 1.4$ , and finally, the results from Section 5.2 hold if small enough time steps are chosen.  
 547 Problem (6.14) is simulated using the discretization (6.15). The LCP (6.15) is solved using the numerical  
 548 solver Lemke provided in the SICONOS Toolbox<sup>3</sup> [1]. In Figures 7a and 7b the simulated variables  $(x_1, x_2, z)$   
 549 and  $(\lambda_1, \lambda_2)$  are displayed for time step  $h = 5 \cdot 10^{-6}$ s. The initial conditions are  $(x_{1,0} = 5, \lambda_{1,0} = 5, \lambda_{2,0} = 0)$   
 550 and the discrete variables  $x_{2,k}$  and  $z_k$  are given by the discretization of (5.1) (c):

$$(6.17) \quad \begin{cases} x_{2,k} = -B_3 \lambda_k \\ z_k = -B_3 \frac{\lambda_{k+1} - \lambda_k}{h}, \end{cases}$$

551 with  $h$  the time step. Let us notice in Figure 7a that the variable  $z$  possesses a discontinuity. In Figure  
 552 8a we plot the error provided in (6.6), with respect to a reference solution taken for  $h = 5 \cdot 10^{-6}$ s. We  
 553 confirm the order one convergence rate for  $h > 0$  sufficiently small. Finally, let us also notice that  
 554 the solutions of the schemes in (3.1) and in (5.5), are not exactly the same (recall that the continuous  
 555 systems (2.2) and (5.1) are "equivalent" in the sense explained after (5.3), see also section H). This can  
 556 be observed in Figure 8b where the differences between the two schemes are shown for  $(x_1, x_2)$ .

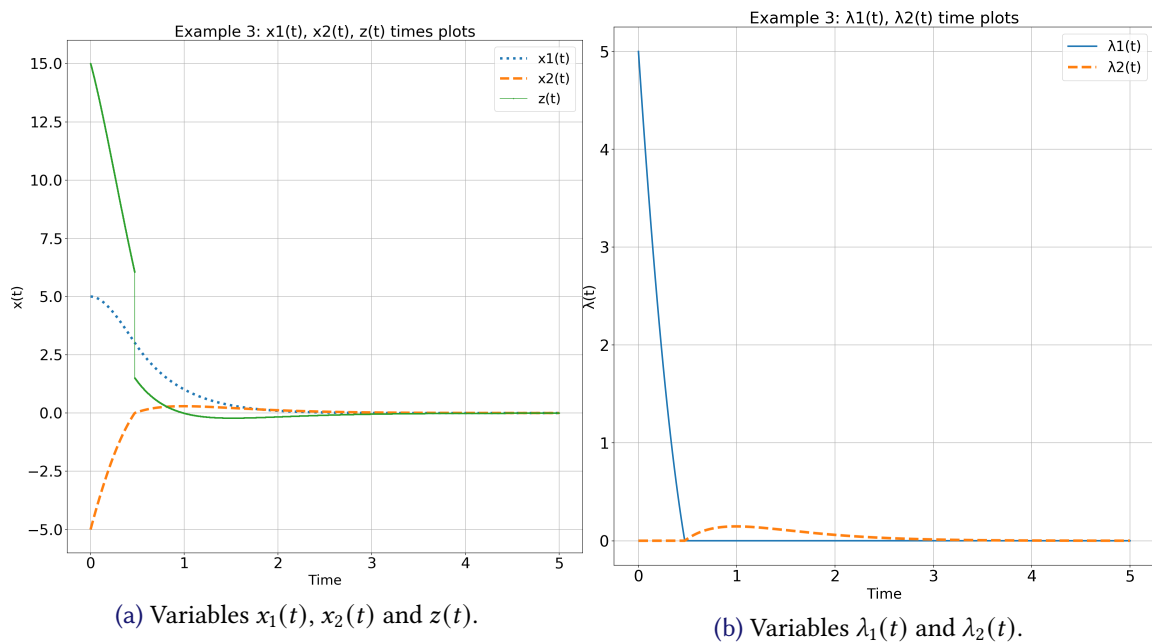


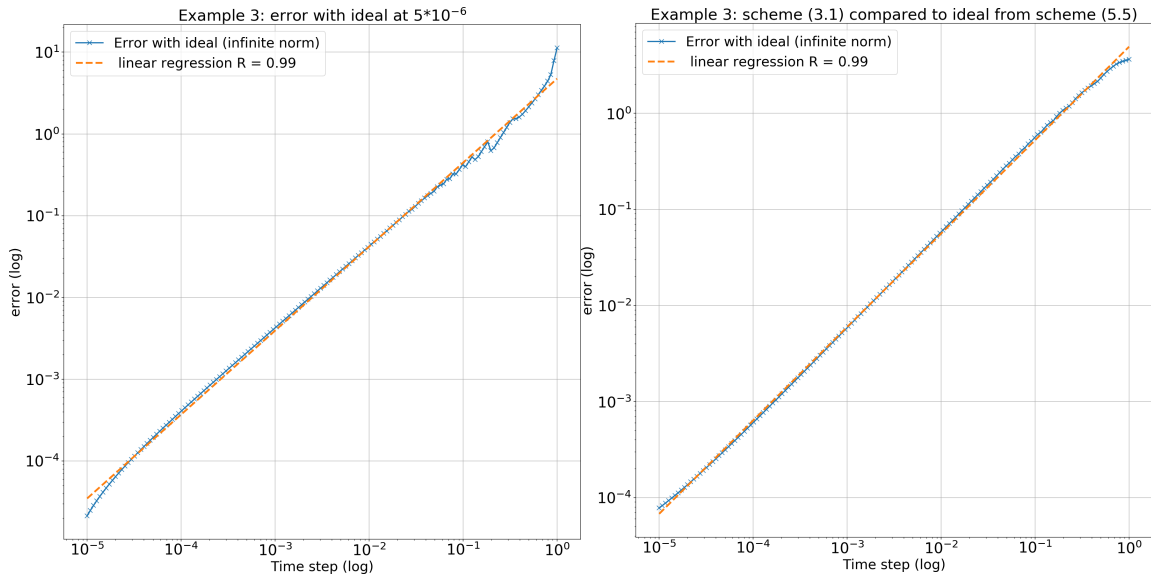
Figure 7: Implicit Euler numerical simulation of (6.13) with  $h = 5 \cdot 10^{-6}$ s

## 557 7 CONCLUSION

558 This article deals with the time-discretization of a class of singular linear complementarity systems  
 559 (descriptor variable linear complementarity systems) which satisfy a passivity constraint. The implicit  
 560 Euler method is analysed. First the one-step nonsmooth problem is studied, and conditions which  
 561 guarantee its well-posedness are given. Then the approximate solutions convergence is analysed, and  
 562 it is shown that under some reasonable assumptions, convergence holds. Our results are however

<sup>3</sup><https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>





(a) Convergence of scheme (5.5) applied to (6.14). (b) Convergence of scheme (3.1) applied to (6.13).

**Figure 8:** Numerical convergence studies, with error formulas (6.6), of schemes (3.1) and (5.5). The reference is computed with the scheme (5.5) and  $h = 5 \cdot 10^{-6}$  s.

563 partial in the sense that we are not able to prove the convergence of all the variables. One academic  
 564 example and two examples of electrical circuits with ideal diodes (equivalently hydraulic circuits with  
 565 check valves) illustrate the developments, with numerical simulations.

## 566 APPENDIX A PRESERVATION OF POSITIVE DEFINITENESS WITH PERTURBATION

567 We give here an excerpt of [36, Theorem 2.11], and a corollary of it. Let us recall that for a given  
 568  $M \in \mathbf{R}^{n \times n}$ ,  $\|M\|_{2,2}$  is the induced matricial norm such that  $\|M\|_{2,2} = \sigma_{\max}(M)$  (the largest singular  
 569 value).

**Theorem A.1.** [36] *Let  $M \in \mathbf{R}^{n \times n}$  be a positive definite matrix. Then every matrix*

$$A \in \{A \in \mathbf{R}^{n \times n} \mid \left\| \left( \frac{M+M^T}{2} \right)^{-1} \right\|_{2,2} \|M - A\|_{2,2} < 1\}$$

570 *is positive definite.*

571 **Corollary A.2.** [21] *Let  $D = P + N$ , where  $D$ ,  $P$  and  $N$  are  $n \times n$  real matrices, and  $P > 0$ , not necessarily  
 572 symmetric. If*

$$(A.1) \quad \|N\|_2 < \frac{1}{\left\| \left( \frac{P+P^T}{2} \right)^{-1} \right\|_{2,2}}$$

573 *then  $D > 0$ .*

574 Consider the term  $h_1$  in Assumptions 1 and 3. If  $h < \frac{1}{\sigma_{\max}(A_1)}$  then  $I_{n_1} + hA_1 > 0$  from Corollary A.2. If  
 575 in addition  $A_1 \preccurlyeq (<) 0$  then  $0 < I_{n_1} + hA_1 \preccurlyeq (<) I_{n_1}$ . Consider now  $\bar{A}_1 = (I_{n_1} + hA_1)^T (I_{n_1} + hA_1) =$   
 576  $I_{n_1} + h(A_1 + A_1^T) + h^2 A_1^T A_1 = I_{n_1} + h\bar{A}_1$ , with  $\bar{A}_1 = A_1 + A_1^T + hA_1^T A_1 = \bar{A}_1^T$ . If  $A_1 < 0$  ( $\Rightarrow A_1 + A_1^T < 0$ ), then  
 577  $\bar{A}_1 < 0$  for  $h > 0$  small enough, still using Corollary A.2. Thus  $\bar{A}_1 < I_{n_1}$  for  $h > 0$  small enough. Moreover  
 578 for  $h > 0$  small enough it follows that  $\bar{A}_1 > 0$  by Corollary A.2. It is deduced that there exists  $h_{\max} > 0$   
 579 such that  $0 < \bar{A}_1 < I_{n_1}$  for all  $h \in (0, h_{\max})$ . From [16, Lemma 8.4.1],  $0 < \lambda_{\min}(\bar{A}_1)$  and  $\lambda_{\max}(\bar{A}_1) < 1$ . It  
 580 is inferred that  $\sigma_{\max}(I_{n_1} + hA_1) = \|I_{n_1} + hA_1\|_{2,2} = h_1 < 1$ .

581 **APPENDIX B WELL-POSEDNESS OF VARIATIONAL INEQUALITIES**

582 The next results use the notions of recession functions and cones, which we briefly introduce now (see  
 583 [24, 50] for illustrating examples), [69, 50, 84]. Let  $f : \mathbb{R}^n \rightarrow R \cup \{+\infty\}$  be a proper convex and lower  
 584 semi-continuous function, we denote by  $\text{dom}(f) \triangleq \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$  the domain of the function  
 585  $f(\cdot)$ . The Fenchel transform  $f^*(\cdot)$  of  $f(\cdot)$  is the proper, convex and lower semicontinuous function  
 586 defined by

$$(B.1) \quad (\text{for all } z \in \mathbb{R}^n) \mid f^*(z) = \sup_{x \in \text{dom}(f)} \{\langle x, z \rangle - f(x)\}.$$

The subdifferential  $\partial f(x)$  of  $f(\cdot)$  at  $x \in \mathbb{R}^n$  is defined by

$$\partial f(x) = \{\omega \in \mathbb{R}^n \mid f(v) - f(x) \geq \langle \omega, v - x \rangle, \forall v \in \mathbb{R}^n\}.$$

587 We denote by  $\text{Dom}(\partial f) \triangleq \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$  the domain of the subdifferential operator  $\partial f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .  
 588 Recall that (see e.g. Theorem 2, Chapter 10, Section 3 in [11]):  $\text{Dom}(\partial f) \subset \text{dom}(f)$ .

Let  $x_0$  be any element in the domain  $\text{dom}(f)$  of  $f(\cdot)$ , the recession function  $f_\infty(\cdot)$  of  $f(\cdot)$  is defined by

$$(\text{for all } x \in \mathbb{R}^n) : f_\infty(x) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f(x_0 + \lambda x).$$

589 The function  $f_\infty : \mathbb{R}^n \rightarrow R \cup \{+\infty\}$  is a proper convex and lower semi-continuous function which  
 590 describes the asymptotic behavior of  $f(\cdot)$ .

Let  $K \subset \mathbb{R}^n$  be a nonempty closed convex set. Let  $x_0$  be any element in  $K$ . The recession cone of  $K$  is defined by [69] [84, Definition 1.11]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda} (K - x_0) = \{u \in \mathbb{R}^n \mid x + \lambda u \in K, \forall \lambda \geq 0, \forall x \in K\}.$$

591 The set  $K_\infty$  is a nonempty closed convex cone that is described in terms of the directions which  
 592 recede from  $K$ . The indicator function of a set  $K \subseteq \mathbb{R}^n$  is  $\psi_K(x) = 0$  if  $x \in K$ ,  $\psi_K(x) = +\infty$  if  $x \notin K$ .  
 593 If  $K$  is closed non empty convex, we have  $\partial \psi_K(x) = \mathcal{N}_K(x)$ , the so-called normal cone to  $K$  at  $x$ ,  
 594 defined as  $\mathcal{N}_K(x) = \{v \in \mathbb{R}^n \mid v^\top (s - x) \leq 0 \text{ for all } s \in K\}$ . When  $K$  is finitely represented, i.e.,  
 595  $K = \{x \in \mathbb{R}^n \mid k_i(x) \geq 0, 1 \leq i \leq m\}$ , and if the functions  $k_i(\cdot)$  satisfy some constraint qualification  
 596 (like, independency, or extensions like the MFCQ), then  $\mathcal{N}_K(x)$  is generated by the outwards normals  
 597 at the active constraints  $k_i(x) = 0$ , i.e.,  $\mathcal{N}_K(x) = \{v \in \mathbb{R}^n \mid v = -\lambda_i \nabla k_i(x), k_i(x) = 0, \lambda_i \geq 0\}$ .

598 Let us here recall some important properties of the recession function and recession cone (see e.g., [17,  
 599 Proposition 1.4.8]):

600 **Proposition B.1.** *The following statements hold:*

601 a) Let  $f_1 : \mathbb{R}^n \rightarrow R \cup \{+\infty\}$  and  $f_2 : \mathbb{R}^n \rightarrow R \cup \{+\infty\}$  be two proper, convex and lower semi-continuous  
 602 functions. Suppose that  $f_1 + f_2$  is proper. Then for all  $x \in \mathbb{R}^n$ :  $(f_1 + f_2)_\infty(x) = (f_1)_\infty(x) + (f_2)_\infty(x)$ .

603 b) Let  $f : \mathbb{R}^n \rightarrow R \cup \{+\infty\}$  be a proper, convex and lower semi-continuous function and let  $K$  be a  
 604 nonempty closed convex set, such that  $f + \Psi_K$  is proper (equivalently  $\text{dom}(f) \cap K$  is non empty). Then for  
 605 all  $x \in \mathbb{R}^n$ :  $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$ .

606 c) Let  $K \subset \mathbb{R}^n$  be a nonempty, closed and convex set. Then for all  $x \in \mathbb{R}^n$ :  $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$ . Moreover  
 607 for all  $x \in K$  and  $e \in K_\infty$ :  $x + e \in K$ .

608 d) If  $K \subseteq \mathbb{R}^n$  is a nonempty closed and convex cone, then  $K_\infty = K$ .

609 e) Let  $K = P(A, b) \stackrel{\Delta}{=} \{x \in R^n | Ax \geq b\}$  for  $A \in R^{m \times n}$  and  $b \in R^m$ . If  $K \neq \emptyset$  then  $K_\infty = P(A, 0) = \{x \in$   
 610  $R^n | Ax \geq 0\}$ .

611 f)  $K \subset R^n$  is a non-empty closed convex bounded set if and only if  $K_\infty = \{0_n\}$ .

612 g) Let  $f : R^n \rightarrow R \cup \{+\infty\}$  be a proper, convex and lower semi-continuous function. Then  $\text{epi}(f_\infty) =$   
 613  $(\text{epi}(f))_\infty$ .

614 Sets as in item e) are called  $\mathcal{H}$ -polyhedra, and there is an equivalence between sets  $P(A, 0)$  and  
 615 finitely generated convex cones [84, Theorem 1.3]. Let us now concatenate [5, Theorem 3, Corollaries 3  
 616 and 4]. They concern variational inequalities (VIs) of the form: Find  $u \in R^n$  such that

$$(B.2) \quad \langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in R^n$$

617 where  $\mathbf{M} \in R^{n \times n}$  is a real matrix,  $\mathbf{q} \in R^n$  a vector and  $\varphi : R^n \rightarrow R \cup \{+\infty\}$  a proper convex and  
 618 lower semicontinuous function. The VI in (B.2) is equivalent to the inclusion  $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u) \Leftrightarrow u \in$   
 619  $(\mathbf{M} + \partial\varphi)^{-1}(-\mathbf{q})$ .

620 The problem in (B.2) is denoted as  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  in the next proposition. We also set:

$$(B.3) \quad \mathcal{K}(\mathbf{M}, \varphi) = \{x \in R^n | \mathbf{M}x \in (\text{dom}(\varphi_\infty))^*\}.$$

621 Note that  $(\text{dom}(\varphi_\infty))^*$  is the dual cone of the domain of the recession function  $\varphi_\infty$  while  $(\text{dom}(\varphi))_\infty$   
 622 (that we may denote also as  $\text{dom}(\varphi)_\infty$ ) is the recession cone of  $\text{dom}(\varphi)$ .

623 **Proposition B.2.** [5] Let  $\varphi : R^n \rightarrow R \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function with  
 624 closed domain,  $\mathbf{M} \in R^{n \times n}$ , and suppose that  $\mathbf{M} \succcurlyeq 0$  (not necessarily symmetric).

625 a) If  $(\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$  then for each  $\mathbf{q} \in R^n$ , problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at  
 626 least one solution.

627 b) Suppose that  $(\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$ . If there exists  $x_0 \in \text{dom}(\varphi)$  such that

$$(B.4) \quad \langle \mathbf{q} - \mathbf{M}^\top x_0, v \rangle + \varphi_\infty(v) > 0, \forall v \in (\text{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi), v \neq 0,$$

628 then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at least one solution.

629 b') If  $\mathbf{M} = \mathbf{M}^\top$  then one can take  $x_0 = 0$  in b).

630 c) If  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  then  $u_1 - u_2 \in \ker\{\mathbf{M} + \mathbf{M}^\top\}$ .

631 d) If  $\mathbf{M} = \mathbf{M}^\top$  and  $u_1$  and  $u_2$  denote two solutions of problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$ , then  $\langle \mathbf{q}, u_1 - u_2 \rangle = \varphi(u_2) - \varphi(u_1)$ .

632

633 e) If  $\mathbf{M} = \mathbf{M}^\top$  and  $\varphi(x + z) = \varphi(x)$  for all  $x \in \text{dom}(\varphi)$  and  $z \in \ker\{\mathbf{M}\}$  and  $\langle \mathbf{q}, e \rangle \neq 0$  for all  
 634  $e \in \ker\{\mathbf{M}\}, e \neq 0$ , then problem  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  has at most one solution.

635 f) If  $\mathbf{M} = \mathbf{M}^\top$ , then  $u$  is a solution of  $VI(\mathbf{M}, \mathbf{q}, \varphi)$  if and only if it is a solution of the optimization problem  
 636  $\min_{x \in R^n} \frac{1}{2} x^\top \mathbf{M}x + \langle \mathbf{q}, x \rangle + \varphi(x)$ .

637 Notice that the function  $\varphi(\cdot)$  will never be strictly convex in our case (it is an indicator function) so  
 638 that the strict convexity argument of [5, Theorem 5] which applies when  $\mathbf{M}$  is a  $P_0$ -matrix never holds.  
 639 The study of VIs as in (B.2) can be traced back to [73].

## 640 APPENDIX C SOME CONVEX ANALYSIS AND COMPLEMENTARITY THEORY TOOLS

641 If  $K \subset R^n$  is a set, then  $K^\star = \{z \in R^n | \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$  is its dual cone. Its closure is denoted  $\bar{K}$ .  
 642 Let  $K$  be a closed convex cone, then:

$$(C.1) \quad K^\star \ni x \perp y \in K \Leftrightarrow x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^\star}(x).$$

643 Let  $M = M^\top > 0$ ,  $x$  and  $y$  two vectors, then

$$(C.2) \quad M(x - y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \text{proj}_M[K; y] \Leftrightarrow x = \min_{z \in K} \frac{1}{2} (z - y)^\top M (z - y).$$

644 We note that this is a particular case of (B.2), so that Proposition B.2 can be considered as the charac-  
645 terization of a generalized projection operator  $VI(M, q, \varphi)$ . The first equivalence is [15, Proposition  
646 6.46].

## 647 APPENDIX D PASSIVE DESCRIPTOR VARIABLE SYSTEMS

648 In this paper we call a quadruple  $(A, B, C, D)$  passive if the next LMI is satisfied:

$$(D.1) \quad \begin{pmatrix} -XA - A^\top X & -XB + C^\top \\ -B^\top X + C & D + D^\top \end{pmatrix} \geq 0, \quad X = X^\top > 0.$$

649 It is said *strictly* passive if the first matrix inequality is satisfied with  $> 0$ . This is not to be confused  
650 with the strict *state* passivity [25, Definition 4.54], since strict passivity implies strict state passivity, but  
651 not the inverse in general. In fact strict passivity is directly related to strongly SPR transfer matrices  
652 [25, p.62 and section 3.12.2], while SPR transfer matrices are strictly state passive [25, Theorem 4.73]  
653 [63].

654 Positive real transfer matrices [25, Definition 2.34] with a state space realization represented by  
655 quintuples  $(P, A, B, C, D)$  which are minimal (see Appendix G for conditions of minimality of a state-  
656 space realization for descriptor systems) can be rewritten as in (2.2) [28, 48], and they have the positive  
657 semidefinite storage function  $V(x_1, x_2) = \frac{1}{2}(x_1^\top X_1 x_1 + x_2^\top X_3 x_2)$ , with supply rate  $\lambda^\top w$ . Indeed, by  
658 differentiating  $V(x_1(t), x_2(t))$  along the trajectories of (2.2), adding  $\pm \lambda^\top w$  and completing the squares,  
659 one can compute that:

$$(D.2) \quad \dot{V}(t) = \frac{1}{2} (x_1^\top, \lambda^\top) \underbrace{\begin{pmatrix} X_1 A_1 + A_1^\top X_1 & X_1 B_1 - C_1^\top \\ B_1^\top X_1 - C_1 & -(\hat{D} + \hat{D}^\top) \end{pmatrix}}_{\triangleq Q_1} \begin{pmatrix} x_1 \\ \lambda \end{pmatrix} + \lambda^\top(t) w(t),$$

660 where  $X_1 = X_1^\top \geq 0$ ,  $Q_1 \leq 0$ , and if the system is strongly SPR then  $X_1 > 0$ ,  $Q_1 < 0$  [28, Theorem  
661 5.4]. The equality in (D.2) is an infinitesimal dissipation equality, showing that the system is passive  
662 [25, Definition 2.1, Theorem 4.53]. Such systems possess particular structural properties [30, 33] [25,  
663 section 3.8]. In the SSPR case it follows from (D.2) and the complementarity conditions that  $x_1(\cdot)$  and  
664  $x_2^\top X_3 x_2(\cdot)$  are bounded for any bounded initial condition.

## 665 APPENDIX E GRONWALL-BACHPATTE INEQUALITY [13, 44]

Let us consider a positive real numbers sequence  $\{x_k\}$  such that  $x_k \leq \alpha_k + \sum_{i=0}^{k-1} \beta_i x_i$ , where  $\{\alpha_k\}$  and  
 $\{\beta_k\}$  are two real numbers sequences with  $\{\beta_k\}$  positive. Then one has

$$x_k \leq \alpha_k + \sum_{i=0}^{k-1} \beta_i \alpha_i \prod_{j=i+1}^k (1 + \beta_j).$$

## 666 APPENDIX F CALCULATION OF $\beta_i$

667 The starting point is the term  $\eta \|C_1\| \|B_1\| \sum_{i=1}^k h_1^{k-i} \sum_{j=0}^{i-1} (\eta h_3)^{i-1-j} \|x_{1,j+1}\|$  in (5.19). Let us consider  
668 the factors of the terms  $\|x_{1,j+1}\|$  separately:

- 669 • Factor of  $\|x_{1,1}\|$  (all terms with  $j = 0$ ):  $h_1^{k-1} + h_1^{k-2}(\eta h_3) + h_1^{k-3}(\eta h_3)^2 + \dots + h_1(\eta h_3)^{k-2} + (\eta h_3)^{k-1} =$   
670  $\sum_{i=1}^k h_1^{k-i}(\eta h_3)^{i-1}$ .
- 671 • Factor of  $\|x_{1,2}\|$  (all terms with  $j = 1$ ):  $h_1^{k-2} + h_1^{k-3}(\eta h_3) + h_1^{k-4}(\eta h_3)^2 + \dots + h_1(\eta h_3)^{k-3} + (\eta h_3)^{k-2} =$   
672  $\sum_{i=2}^k h_1^{k-i}(\eta h_3)^{i-2}$ .
- 673 • ...
- 674 • Factor of  $\|x_{1,k-1}\|$  (all terms with  $j = k - 2$ ):  $h_1 + (\eta h_3) = \sum_{i=k-1}^k h_1^{k-i}(\eta h_3)^{i-k+1}$ .
- 675 • Factor of  $\|x_{1,k}\|$  (all terms with  $j = k$ ):  $1 = \sum_{i=k}^k h_1^{k-i}(\eta h_3)^{i-k}$ .
- 676 Therefore one infers (5.22).

## 677 APPENDIX G MINIMALITY OF STATE SPACE REALIZATIONS

678 Let us report the results stated in [28, Theorem 3.1] [41, Theorems 2-2.1, 2-3.1, 2-6.3].

679 **Theorem G.1.** *Let  $H(s) = C(sP - A)^{-1}B + D$ ,  $s \in \mathbb{C}$ , be a rational transfer function where  $P$  and  $A$  are*  
680  *$n \times n$  matrices. Then  $P\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t) + Du(t)$  is a minimal realization of  $H(s)$  if,*  
681 *and only if, the following conditions are satisfied:*

- 682 •  $\text{rank}(A - sP \ B) = n$  for all  $s \in \mathbb{C}$ ,
- 683 •  $\text{rank}(P \ B) = n$ ,
- 684 •  $\text{rank}(A^\top - sP^\top \ C^\top) = n$ ,
- 685 •  $\text{rank}(P^\top \ C^\top) = n$ ,
- 686 •  $A \text{Ker}(P) \subseteq \text{Im}(P)$ .

## 687 APPENDIX H DIFFERENCES BETWEEN (3.1) AND IN (5.5)

688 Let us consider the discrete scheme (5.5) with  $E(t) = 0$  and  $X_1$  full rank, this is rewritten as:

$$(H.1) \quad \begin{cases} (a) \ x_{1,k+1} = x_{1,k} + hA_1x_{1,k} + hB_1\lambda_k \\ (b) \ B_3^\top X_3 B_3 \lambda_{k+1} = B_3^\top X_3 B_3 \lambda_k - hC_1x_{1,k} - h\hat{D}\lambda_k + w_{k+1} \\ (c) \ 0 \leq \lambda_{k+1} \perp w_{k+1} = C_1x_{1,k+1} + C_2x_{2,k+1} + C_3z_{k+1} + D\lambda_{k+1} \geq 0 \\ (d) \ x_{2,k} = -B_3\lambda_k \\ (e) \ z_k = -B_2\lambda_k - B_3 \frac{\lambda_{k+1} - \lambda_k}{h}. \end{cases}$$

689 This is the numerical scheme with which the examples have been simulated. Combining (H.1) (d) and  
690 (e), gives:

$$(H.2) \quad x_{2,k+1} = x_{2,k} + hz_k + hB_2\lambda_k.$$

691 Notice that this is the choice made in (6.17). The equations in (H.1) (a) and (H.2) are not equal to their  
692 counterparts in (3.1), because  $\lambda_k$  and  $z_k$  are used in (H.1) (a) and (H.2) instead of  $\lambda_{k+1}$  and  $z_{k+1}$  in (3.1).  
693 This explains that the numerical data obtained from both schemes, differ. This also explains why the  
694 convergence results obtained for both schemes, are not the same neither.

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