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Abstract

We study an opinion formation game between a Designer and an Adversary. While the Designer creates the network, both these players can influence network nodes (agents) initially, with ties being broken in favor of the Designer. Final opinions of agents are a convex combination of own opinions and the average network peer opinion. The optimal influence strategy shows threshold effects with non-empty equilibrium networks having star type architectures. By contrast, when the tie-breaking rule favors the Adversary, non-empty equilibrium networks are regular networks. The effect of random interactions between network nodes altering the network is also studied.

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1 Introduction

Social networks have become key drivers of information transmission and opinion formation. Frequently wars on the political, economic and business front are now fought on social networks. Such competition could be about decisions to adopt a particular technology, eating a certain type of food or choosing a political position. Clearly, this creates strong incentives to design (and modify) social networks and influence agents that belong to them. In this paper, we examine one aspect of this theme: What are the network architectures that a planner/designer should build and which agents should she influence in order to ensure that a certain proportion of agents adopt her preferred action given that there is a strategic adversary who wishes to promote a different opinion?

Given that it is costly to create networks and/or influence individuals, the insights from this research can be used to modify existing networks. In contrast to the typical diffusion or opinion formation models, it can be used to understand how opinions form when there is competition for such opinions in the network. Our models provides insights about how the costs of different strategies of the designer and the adversary can affect equilibrium behavior. For instance, we find that if the designer is more persuasive than the adversary, then the equilibrium network architectures are quite different from those that arise when this condition is reversed. Competing influences or opinions may also exist in the creation of teams or organizational structures that may have the possibility of adversarial outcomes.

In our model of competition among influencers, the Designer (De) makes two decisions. First, unlike the typical model of opinion formation where the network is exogenously given, the Designer sets up the network by forming links between the n agents. Second, she chooses a set of agents in the network that she wants to influence. She faces an Adversary (Ad) who just makes one decision : which set of agents to influence in the network. There are three possibilities concerning the opinion of agents: no opinion – denoted by $\emptyset$, or opinion 0, or opinion 1.[1] In the benchmark model, when an agent is influenced by both De and Ad, as a tie-breaking rule we assume that the Designer's influence prevails. The goal of De is to get the agents to vote 1, while the Adversary's (Ad) goal is to prevent the success of De. Thus, De and Ad play a zero sum type game, where (i) it is costly for De to form links and to influence agents, and (ii) it is costly for Ad to influence agents. However, our analysis differs from the typical Designer-Adversary games by incorporating a role for the social network. Specifically, the final opinion of each agent i is a convex combination of two ingredients: (i) his initial opinion that depends on the way De and Ad have influenced him, and (ii) the opinion of his neighbors in the network formed by De.

[1] We may interpret the vote of each agent as the action he chooses, for example, the adoption of a new technology, or a new product or the endorsement of a political idea while $\emptyset$ denotes the notion that the agent does not chose either of the options.

The convex combination allows us to investigate the varying role of peers in opinion formation. Observe, also that both the Designer and the Adversary may be able to use these peers to their advantage.

Our analysis begins with a benchmark model where (i) every agent, who is influenced simultaneously by De and Ad, follows the opinion of De, i.e. opinion 1, and (ii) in order to win, De must have all agents vote 1. Our main results concern the Subgame Perfect Equilibrium of this sequential move game. In particular, assuming Ad plays a best response we provide the optimal strategy of De. We show that in any optimal strategy, De must influence a minimal number of agents after forming the desired network. Moreover, only three types of architectures may occur in equilibrium: the empty network – network without links, the star network – all agents influenced by De are linked with the centre also influenced by De, or core-periphery network – each peripheral agent is involved in a single link with an agent influenced by De belonging to the core.[2] Note that each agent in the core must satisfy a condition regarding its number of neighbors in the core and the periphery. Next, we relax the assumption that all network relationships between agents exist because of De. More precisely, we allow for every unlinked pair of agents to be linked with some probability. Here we provide two types of results. First, we provide bounds on the probability under which results obtained in the benchmark model continue to hold. Second, we discuss an example with four firms and show that for intermediate probability values, the results of the benchmark model do not hold.

In the benchmark model we make two assumptions that are relaxed in the extensions. In the first extension of the model, we assume that De only wants half the population of agents to vote 1. We establish that the results are partially unchanged when this assumption is introduced. Indeed, some equilibrium architectures obtained in the benchmark model are robust to the threshold that De requires to win the game: the partial star and the empty network. However, the number of agents that De has to influence in a SPNE is modified when she does not have to obtain that all agents vote 1. Second, in the benchmark model, we give an advantage to De since an agent who is influenced both by De and Ad follows the opinion of De. By contrast in the second extension, the tie-breaking rule favors Ad and an agent influenced by both De and Ad has the opinion 0. Another way to think of this rule is to say that De has lower technology or ability to influence opinions. We show that it is necessary for the cost incurred by Ad for influencing agents to be sufficiently high to allow De to build a non-empty network and to influence agents. Second, we characterize the optimal strategies of De which depend on her cost to influence agents and form links, and the optimal strategies of Ad. There are two possibilities for the best response of Ad: (i) either his best response consists in influencing only the neighbors of the agent he wishes to

[2]Core-periphery networks are close to star networks if we identify the core with a node. In this case, the resulting network is similar to a star network where the core is the centre. For this reason, core-periphery networks can be seen as a type of star network.

persuade, or (ii) in influencing the neighbors of the agent whose vote he wishes to get as well as the agent himself. Hence, De has two types of strategies in equilibrium. Either De constructs a regular network: she provides the same kind of neighborhood to all agents and she influences all agents, or a network in which she does not influence certain agents, and agents she does not influence have types of neighborhoods different from the types of neighborhoods of the agents she influences. This result is in sharp contrast to earlier results in Designer-Adversary type models where the Designer always forms a star type network.

The rest of the paper is organized as follows. In Section 2, we present the related literature. In Section 3, we introduce the model setup. In Section 4, we provide intuition to understand the role of network formation, i.e. we provide results for situations where the network is exogenously given. In Section 5, we establish results for the benchmark model where the Designer is a better influencer than the Adversary. In Section 6, we extend the model in two directions. First, we consider situations where De only needs to influence a majority of the population to vote 1. Second, we study situations where the tie-breaking rule favors the Adversary. In Section 8, we show that our results form the upper bound on costs incurred by De when agents can interact repeatedly and change their opinions.

2 Related literature

Our paper relates to several different aspects of the networks literature.

Exogenous threats. Our paper is closely related to models where social networks face an exogenous threat. There are two types of models in this literature. One strand has centralized network protection carried out by a Designer as in our model, while in the other stand of the literature protection is decentralized and is carried out by the agents comprising the network. In the first type of models Dziubiński and Goyal (2013, 2017) study the optimal design and defense of networks assuming an intelligent attacker or Adversary as in our model. The Designer forms links between the n agents, and must protect them to ensure their survival. The Designer's objective is to maximize the size of connected components.
Goyal and Vigier (2014) extend the work of Dziubiński and Goyal by allowing the attacks (or threats) to spread like a contagion. Bravard, Charroin and Touati (2016) extend Dziubiński and Goyal (2013) where the Adersary targets links and the Designer has the possibility to protect them. Hoyer and De Jaegher (2016) consider a framework where the Designer has to shape the network by forming enough links to retain connectivity under attacks. In this framework, certain parts of the network are always vulnerable and cannot be protected by the Designer. They study the optimal networks under the possibility of link or node removal at different cost ranges. Note that unlike these models in our model, network peers can affect opinions and the Designer's ob-

jective is not network connectivity but (directly or indirectly) influencing at least a majority of the agents.

In the second type of models Cabrales, Gottardi and Vega-Redondo (2017) and Baccara and Bar-Isaac (2008) study the propagation of attacks in networks respectively in financial firms where financial risk can spread between connected firms and in criminal networks where connectivity increases vulnerability because of external threats. So individidual nodes make connectivity related decisions. In Acemoglu, Malekian and Ozdaglar (2016) agents are connected but in a random network. Agents have to invest in protection to be immune which depends on their links and the probability of being infected in the random network. In Haller and Hoyer (2019) group members individually sponsor costly links and form an information network. An Adversary aims to disrupt the information flow within the network by deleting some of the links. The authors study how the group responds to such common enemy. In our paper, we do not focus on the strategic aspect of the choice of agents. Unlike these papers we have a two player game between the Designer and Adversary who care about influencing agents and not maintaining connectivity. As already mentioned, agents can affect the opinions of their neighbors to varying degrees in our model.

Interaction on Exogenous Networks. Our paper is also related to models where the interaction structure defined by the *given network* affects the behavior of other agents. There are two main types of interaction models in the literature: discrete and linear models. In the discrete models actions are typically binary. They describe the choice of a location or the adoption of a new technology (Schelling, 1969, Frankel, Morris and Pauzner, 2003[3]). As in our paper, in these models there is a threshold effect: starting with the default action, an agent adopts the alternative action when the number (or proportion) of his neighbors adopting exceeds a specified threshold. The planner's objective is to maximize the number (or the expected) of people adopting an action or a new technology.

In the linear type of models, agents interact in a strategic game and have quadratic payoffs. In their seminal paper, Ballester et al. (2006) adress the question of the "best" player, i.e. *the key player*, to remove for minimizing the total output of agents. This makes her the most influencial agent in the network. Since then, there has been a significant literature using this framework. For instance, Demange (2017) analyzes the optimal targeting strategies of a planner who aims to increase the aggregate action of a population. The agents interact through a social network and react to their exposure to neighbors' actions given that there is complementarity between actions of agents. In Zhou and Chen (2015) the model incorporates payoff externalities and strategic complementarity where players belong to two different groups. The authors analyze a two-stage game in which players in the leader group make contributions before the follower group. Zhou and Chen provide an index to identify the key leader. They show that this player can differ

[3]Watts and Dodds (2007) provides an application where there exist opinion leaders.

from the key player in the simultaneous-move game. Joshi et al. (2019) examine the interactions across two networks under strategic complementarities when the interaction structure in one network is fixed. Unlike our paper, in this literature the influence game is not modeled as a zero-sum game between a Designer and an Adversary. This feature allows us to focus on the properties that networks should have to maximize a player's opportunities to influence agents. Moreover, in our model, the Designer has the freedom to design the network.

Non-strategic influence Models. Finally, our paper is related to models of influence, diffusion and social learning where typically the goal is to study the spread of influence among a set of non-strategic agents. In Golub and Jackson (2010), and Grabisch et al. (2018), for instance, each agent adopts an opinion (between 0 and 1) by computing a weighted average of the opinions of the agents with whom he is linked in a given social network. Players have one of two competing opinions (0 and 1). Each player chooses an agent and exerts an influence on him. The authors are interested in the convergence of opinions among agents. Jackson and Yariv (2007) analyze games on social networks where agents select one of two actions. Agents' payoffs from each of the two actions depend on how many neighbors she has, the distribution of actions among her neighbors, and a cost for each of the actions. They analyze the diffusion of behavior when in each period agents choose a myopic best response.

3 Model Setup

Let $[\![a,b]\!] = \{\ell \in \mathbb{N}, a \leq \ell \leq b\}$. Moreover, $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the largest integer less than x and the smallest integer greater than x. Further, for every set X, $\sharp X$ is its cardinality.

Agents and Interaction. A finite set of agents or nodes, $\mathcal{N} = [\![1,n]\!]$, $n \geq 4$, interact according to an undirected network g. An undirected network g is a pair $(\mathcal{N}, E(g))$, where $E(g) \subset \mathcal{N} \times \mathcal{N}$ is the set of links. A link between two agents i and j is interpreted as the existence of a relationship between these agents. With slight abuse of notation, we denote by ij the link between agents i and j in g, i.e. $ij \in E(g)$. Let $\mathbb{A}(g)$ be an $n \times n$ adjacency matrix. We have for every $(i,j) \in \mathcal{N}^2$, $\mathbb{A}_{i,j}(g) \in [\![0,1]\!]$, where $\mathbb{A}_{i,j}(g) = 1$ iff $ij \in E(g)$.[4] Let $\mathcal{N}_i(g) = \{j \in \mathcal{N} : ij \in E(g)\}$ be the set of neighbors of agent $i \in \mathcal{N}$. We denote by $G[\mathcal{N}]$ the set of all networks that have $\mathcal{N}$ as the set of agents. A path between agents $i = i_0$ and $j = i_m$ in g is a sequence $i, i_1, \ldots, i_{m-1}, j$ where $i_k i_{k+1} \in E(g)$, with $k \in [\![0, m-1]\!]$. A network g is connected if there exists a path between $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$ for every pair (i,j). A subnetwork $g[\mathcal{N}'] = (\mathcal{N}', E(g[\mathcal{N}']))$ of network g is a network, such that $\mathcal{N}' \subseteq \mathcal{N}$ and for $i, j \in \mathcal{N}'$, we have $ij \in E(g[\mathcal{N}'])$ if and only if $ij \in E(g)$. We say that i is an isolated agent in g when $\sharp\mathcal{N}_i(g) = 0$.

[4]Obviously, $\mathbb{A}(g)$ is symmetric.

Strategies of the Primary Influencers/Players. We assume that there are two *Primary influencers/players*: the *Designer*, De/she, and the *Adversary*, Ad/he. In this sequential move game De moves first and Ad moves second. First, De forms the network, i.e. sets up the links between the agents. Given that our objective is to study how a designer and adversary can drive opinion formation in a network, as a starting point we assume that agents have no opinions before they are influenced by either one or both primary influencers. Second, each primary influencer may influence each agent $i \in \mathcal{N}$ into forming their initial opinion. For simplicity, there are only three possible opinions (or actions) for every agent $i \in \mathcal{N}$: $\emptyset$, 0, or 1. De's influence on agent $i \in \mathcal{N}$ takes the value 1 while Ad's influence takes the value 0. The set of agents influenced by De is denoted by $\mathcal{I}_{De} \subseteq \mathcal{N}$. Similarly, the set of agents influenced by Ad is denoted by $\mathcal{I}_{Ad} \subseteq \mathcal{N}$. Formally, a strategy for De, s^{De}, is a mapping that assigns to $\mathcal{N}$ a pair $(g[s^{De}], \mathcal{I}_{De}[s^{De}]) \in G[\mathcal{N}] \times 2^{\mathcal{N}}$. When there is no ambiguity, we write $(g, \mathcal{I}_{De})$ instead of $(g[s^{De}], \mathcal{I}_{De}[s^{De}])$. In other words, De has to choose both the architecture of the network but also the location of agents in $\mathcal{I}_{De}$ on the network

Similarly, a strategy for Ad is a mapping, s^{Ad} that assigns to each pair $(g, \mathcal{I}_{De})$ a set of agents $\mathcal{I}_{Ad} \subseteq \mathcal{N}$. For every pair of strategies (s^{De}, s^{Ad}) we define a triple $(g[s^{De}], \mathcal{I}_{De}[s^{De}], \mathcal{I}_{Ad}[s^{Ad}])$. Again, when there is no ambiguity, we write $(g, \mathcal{I}_{De}, \mathcal{I}_{Ad})$ instead of $(g[s^{De}], \mathcal{I}_{De}[s^{De}], \mathcal{I}_{Ad}[s^{Ad}])$.[5]

Initial Opinion of Agents. Initially, each agent $i \in \mathcal{N}$ has no opinion, $\emptyset$. When an agent is influenced neither by De, nor by Ad, his initial opinion continues to be $\emptyset$. When De (resp. Ad) is the only one who influences agent i, then the initial opinion of i is 1 (resp. 0). When agent i is influenced by both primary influencers, his initial opinion depends on who has the greater ability to influence. Formally, the initial opinion of each agent i, θ_i^{In}, is given by Ψ that assigns to each $(g, \mathcal{I}_{De}, \mathcal{I}_{Ad})$ the pair $(g, \boldsymbol{\theta}^{\text{In}})$, such that:

$$\theta_i^{\text{In}} = \begin{cases} 1 & \text{if } i \in \mathcal{I}_{De} \setminus \mathcal{I}_{Ad} \\ 0 & \text{if } i \in \mathcal{I}_{Ad} \setminus \mathcal{I}_{De} \\ \emptyset & \text{if } i \notin \mathcal{I}_{De} \cup \mathcal{I}_{Ad} \end{cases}$$

If agent i is influenced both by De and Ad (i.e. $i \in \mathcal{I}_{De} \cup \mathcal{I}_{Ad}$), then there are three possibilities. First, both De and Ad cancel out each other influence. This is akin to analyzing the model where these players belong to the set of uninfluenced agents and therefore is ignored here. Second, De has a greater ability to influence than Ad (for instance because of better technology) or Ad has a greater ability to influence. These two possibilities are explored in our paper.

Benchmark Model: When both De and Ad influence an agent, De is the winner, i.e. $\theta_i^{\text{In}} = 1$. We denote the corresponding mapping by Ψ_1.

[5]Since Ad best responds against the strategy chosen by De, his strategy can be interpreted as the worst possibility that De would face. Hence, De can be seen as an infinitely risk-averse player.

Extension: Ad is a better influencer than De. When both De and Ad influence an agent, Ad is the winner, i.e. $\theta_i^{\text{In}} = 0$. We denote the corresponding mapping by Ψ_2.

For brevity, we use Ψ when the subscript precision is not necessary. The n-uple $\boldsymbol{\theta}^{\text{In}} = (\theta_1^{\text{In}}, \ldots, \theta_n^{\text{In}})$ provides the initial opinion of every agent $i \in \mathcal{N}$.

Final Opinion of Agents. Agents form their final opinion by taking into account their own initial opinion and the weighted average of their neighbors' opinion. This captures the fact that an agent's peers also influence his opinions. We have three possible cases. (a) None of agent i's neighbors have been influenced, then agent i's initial opinion is her final opinion. (b) Next, some of agent i's neighbors have been influenced by De or Ad. Then there are two possibilities. Either agent i has not been influenced and his final opinion is determined by the average opinion of his neighbors, or agent i has been influenced and his final opinion of agent i is determined both by his initial opinion and the average opinion of his neighbors. Formally, let $\mathcal{N}(k, \boldsymbol{\theta}^{\text{In}}) = \{j \in \mathcal{N} : \theta_j^{\text{In}} = k, k \in \{\emptyset, 0, 1\}\}$ be the set of agents with initial opinion k. The set of neighbors of agent i with $k \in \{\emptyset, 0, 1\}$ as initial opinion is denoted by $\mathcal{N}_i^k(g) = \{j \in \mathcal{N}_i(g) \cap \mathcal{N}(k, \boldsymbol{\theta}^{\text{In}})\}$. Moreover, when $\mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) \neq \emptyset$, let

$$\bar{\Theta}_i = \frac{1}{\sharp\mathcal{N}_i^0(g) + \sharp\mathcal{N}_i^1(g)} \sum_{j \in \mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g)} \theta_j^{\text{In}}$$

be the initial average opinion of i's neighbors. In $\bar{\Theta}_i$, we assume that i does not take into account his neighbors j that do not have an initial opinion, i.e those for whom $\theta_j^{\text{In}} = \emptyset$.

We now deal with θ_i^{Fi} the final opinion of agent $i \in \mathcal{N}$. If $\mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) = \emptyset$, then $\theta_i^{\text{Fi}} = \theta_i^{\text{In}}$. Otherwise, when $i \notin \mathcal{N}(\emptyset, \theta^{\text{In}})$, we have:

$$\theta_i^{\text{Fi}} = \begin{cases} 1 & \text{if } (1-\alpha)\theta_i^{\text{In}} + \alpha\overline{\Theta}_i \geq \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\alpha \in (\frac{1}{2}, 1]$. When $i \in \mathcal{N}(\emptyset, \theta^{\text{In}})$, we have

$$\theta_i^{\text{Fi}} = \begin{cases} 1 & \text{if } \overline{\Theta}_i \geq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In other words, the final opinion of agent i depends on both his own opinion and the opinion of his peers who are captured in the model by his network peers. Parameter $\alpha \in (\frac{1}{2}, 1]$ determines the weight placed on peer opinion and own opinion; it measures the degree of peer effect.[6] More precisely, when the initial opinion of agent i and the average initial opinion of his peers is not $\emptyset$, then θ_i^{Fi} is a convex combination of these opinions. Moreover i only takes into account initial opinions that are not equal to $\emptyset$; when all initial opinions are equal to $\emptyset$ the final opinion of agent

[6]Note that if $\alpha < \frac{1}{2}$, then there is no possibility for an agent i to modify its initial opinion. Hence we do not consider this case.

i is $\emptyset$. Note that when $\alpha = 1$, agent i does not take into account his initial opinion to form his final opinion. The n-uple $\boldsymbol{\theta}^{\text{Fi}} = (\theta_1^{\text{Fi}}, \ldots, \theta_n^{\text{Fi}})$ provides the final opinion of every agent $i \in \mathcal{N}$. Finally, we define function $\Phi : (g, \boldsymbol{\theta}^{\text{In}}) \mapsto \Phi(g, \boldsymbol{\theta}^{\text{In}}) = (g, \boldsymbol{\theta}^{\text{Fi}})$ which maps every $(g, \boldsymbol{\theta}^{\text{In}})$ to $(g, \boldsymbol{\theta}^{\text{Fi}})$ according to the previous rules.

Opinion Network and Vote of Agents. An opinion network is a pair $(g, \boldsymbol{\theta}^{\text{Fi}}) = [\Phi \circ \Psi](g, \mathcal{I}_{De}, \mathcal{I}_{Ad})$, where $\Phi \circ \Psi$ refers to the composition of Φ and Ψ. We assume that after all possible influences are taken into account, each agent truthfully "votes" for an outcome. Since our focus is on the game between the Designer and the Adversary, the role of the agents is kept to a minimum. More precisely, each agent i abstains ($\emptyset$) or votes, 0 or 1 according to his opinion θ_i^{Fi}. We denote by $\mathcal{N}(k, \boldsymbol{\theta}^{\text{Fi}}) = \{j \in \mathcal{N} : \theta_j^{\text{Fi}} = k\}$ for $k \in \{\emptyset, 0, 1\}$. The vote cast by agent i can be interpreted as his realized/chosen action.

Cost functions. Forming links and influencing agents are costly actions. More precisely, De's cost function depends on the number of links she forms and the number of agents she influences. We have

$$C(\sharp E(g), \sharp \mathcal{I}_{De}), \tag{3}$$

where $C(\cdot, \cdot)$ is strictly increasing and convex in each of its argument.
Given that the overall cost function is convex, for simplicity, we sometimes assume that it is linear in its two components.

$$C(\sharp E(g), \sharp \mathcal{I}_{De}) = \sharp E(g) c_L + \sharp \mathcal{I}_{De} c_{De}, \tag{4}$$

where $c_L > 0$ is the unit cost of forming each link, and $c_{De} > 0$ is the cost of influence that De incurs for each agent she influences.
Moreover, we let $c_{Ad} > 0$ be the cost incurred by Ad for each agent he influences. In other words, the cost function of Ad is linear.

Payoff of Primary Influencers. The benefits of players only depend on opinion of each agent while the costs incurred by each player $k \in \{De, Ad\}$ only depend on the strategy of k. In the benchmark model, we consider that De wins only if every agent votes 1.[7] The payoff of De, for choosing s^{De} when the Ad responds with $s^{Ad}(s^{De}(\mathcal{N}))$ is $U^{De}(s^{De}, s^{Ad}(s^{De}(\mathcal{N}))) = u^{De}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi])$, where

$$u^{De}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi]) = \begin{cases} 1 - C(\sharp E(g), \sharp \mathcal{I}_{De}) & \text{if } \mathcal{N}(1, \boldsymbol{\theta}^{\text{Fi}}) = \mathcal{N}, \\ \\ -C(\sharp E(g), \sharp \mathcal{I}_{De}) & \text{otherwise.} \end{cases} \tag{5}$$

[7] In Section 6, we relax this assumption.

We assume that the maximal cost incurred by De is lower than 1, i.e. $C(\frac{n(n-1)}{2}, n) < 1$.[8] Similarly, the payoff of Ad when he responds with s^{Ad} to $s^{De}(\mathcal{N})$ is given by $U^{Ad}(s^{De}, s^{Ad}(s^{De}(\mathcal{N})))$ $= u^{Ad}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi])$ with

$$u^{Ad}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi]) = \begin{cases} 1 - c_{Ad}\sharp\mathcal{I}_{Ad} & \text{if } \mathcal{N}(1, \boldsymbol{\theta}^{\text{Fi}}) \neq \mathcal{N}, \\ \\ -c_{Ad}\sharp\mathcal{I}_{Ad} & \text{otherwise.} \end{cases} \tag{6}$$

Here, we assume that the cost function is linear. The results are not qualitatively changed if we assume that the cost of influencing agents is strictly increasing.

Let $k_{Ad} = \lfloor 1/c_{Ad} \rfloor$. Clearly, k_{Ad} is the maximal number of agents that player Ad has an incentive to influence in our benchmark model. We assume that $k_{Ad} > 1$.[9] The payoff of Ad is positive for any number of influenced agents in $[\![0, k_{Ad}]\!]$ when at least one agent votes 0.

Structure of the Game. We now provide the timing of the game for the sake of clarity:

Stage 1. De chooses her strategy, that is

1. De builds the network, i.e. she forms links between the agents, and
2. De influences a set of agents.

Stage 2. Ad influences a set of agents, including possibly the same agents influenced by De.

Stage 3. Agents form their opinion and payoffs are provided to players:

1. each agent influences and is in turn influenced by her neighborhood using the rule Φ,
2. every agent i votes for $\emptyset$, 0, or 1 according to θ_i^{Fi},
3. payoffs are determined for players De and Ad.

Subgame Perfect Equilibrium (SPNE). An SPNE is a pair $(s_\star^{De}, s_\star^{Ad}(s_\star^{De}(\mathcal{N})))$ that prescribes the following strategic choices. In Stage 2, given network g, Ad plays a best response $s_\star^{Ad}(s^{De}(\mathcal{N}))$ to $s^{De}(\mathcal{N})$:

$$s_\star^{Ad}(s^{De}(\mathcal{N})) \in \arg\max_{y \subseteq \mathcal{N}} \{u^{Ad}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, y; \Psi, \Phi])\}.$$

Given the relative influence of each influencer and the peer influence (driven respectively by Ψ and Φ) De obtains $u^{De}(g, \boldsymbol{\theta}^{\text{Fi}}[s^{De}, s_\star^{Ad}; \Psi, \Phi])$ when she chooses s^{De}. Let $S(\mathcal{N})$ be the set of all mappings between $\mathcal{N}$ and pairs $(g, \mathcal{I}_{De})$. Then, in Stage 1, De plays $s_\star^{De}$ such that

$$s_\star^{De} \in \arg\max_{x \in S(\mathcal{N})} \{u^{De}(g, \boldsymbol{\theta}^{\text{Fi}}[x, s_\star^{Ad}(x); \Psi, \Phi])\}.$$

Specific Networks. The *empty network*, g^e, is a network where all agents are isolated. A *star* is a network where there is a central agent, denoted by i_c, who has formed links with all other agents

[8]Consequently, for the linear case, we have $\frac{n(n-1)}{2}c_L + \sharp\mathcal{N}c_{De} < 1$.

[9]Since $k_{Ad} > 1$, Ad always has an incentive to influence at least one agent to vote 0.

and there are no links between i and j when $i, j \in \mathcal{N} \setminus \{i_c\}$. A *partial-star* g is a network where $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$ and subnetwork $g[\mathcal{N}']$ is a star and subnetwork $g[\mathcal{N}'']$ is an empty network. For $q \leq 1$,[10] g is a $(q, \mathcal{X})$-*core-periphery* network, denoted by $(q, \mathcal{X})$-*cp*, when it satisfies the three following properties:

(P1) $\mathcal{N}$ is partitioned into two subsets, $\mathcal{X}$ called the core and $\mathcal{Y}$ called the periphery;

(P2) for every $i \in \mathcal{X}$,

$$\sum_{j \in \mathcal{X}} \mathbb{A}_{i,j}(g) \geq q \sum_{j \in \mathcal{Y}} \mathbb{A}_{i,j}(g);$$

(P3) for every $i \in \mathcal{Y}$, $\sum_{j \in \mathcal{X}} \mathbb{A}_{i,j}(g) = 1$, and $\sum_{j \in \mathcal{Y}} \mathbb{A}_{i,j}(g) = 0$.

A network g is a $(q, \mathcal{X})$-*minimal-cp*, denoted by $(q, \mathcal{X})$-mcp, if it is a $(q, \mathcal{X})$-*cp* network, and satisfies the following additional property:

(P4) there is no $(q, \mathcal{X})$-cp network g' such that $\sum_{(i,j) \in \mathcal{N}^2} \mathbb{A}_{i,j}(g') < \sum_{(i,j) \in \mathcal{N}^2} \mathbb{A}_{i,j}(g)$, i.e. g' contains contains at least as many links as g.

The previous definition does not imply anything about the existence of $(q, \mathcal{X})$-mcp networks. In Appendix A.1, we define a class of networks that are $(q, \mathcal{X})$-mcp and provide a constructive algorithm that ensures their existence. We provide an example of a (1, $[\![1, 4]\!]$)-mcp network in Figure 1 where agents in $[\![1, 4]\!]$, colored blue, belong to $\mathcal{X}$.

A $(q, p, \mathcal{X})$-*partial-star*, denoted by $(q, p, \mathcal{X})$-ps, is a network g where

- $g[\mathcal{X}]$ is a partial star where i_c is the centre of the star,
- p agents, with ℓ a typical one of them, are peripheral belong to $\mathcal{X}$, and satisfy $q \sum_{j \in \mathcal{N} \setminus \mathcal{X}} \mathbb{A}_{\ell,j}(g) \leq 1$,
- for i_c, we have $q \sum_{j \in \mathcal{N} \setminus \mathcal{X}} \mathbb{A}_{i_c,j}(g) \leq p$,
- $\sharp \mathcal{X} - p - 1$ agents are isolated,
- for every $i \in \mathcal{N} \setminus \mathcal{X}$, $\sum_{j \in \mathcal{X}} \mathbb{A}_{i,j}(g) = 1$, and $\sum_{j \in \mathcal{N} \setminus \mathcal{X}} \mathbb{A}_{i,j}(g) = 0$.

Network g^2 in Figure 2 is a $(1, 5, [\![1, 8]\!])$-ps network where agents colored blue belong to $\mathcal{X} = [\![1, 8]\!]$; note that $p = \sharp [\![2, 6]\!] = 5$.

Specific Strategies. In the game both De and Ad choose their decision to influence based on the network g. Here we introduce some notation to help distinguish between different strategies that play an important role in our analysis. In the influenced empty network strategy, infl-$\emptyset$, De forms the empty network and influences all agents. We denote this strategy by $s_{\emptyset}^{\text{De}}$. In a $(q, \mathcal{X})$-*influenced-mcp network strategy*, $(q, \mathcal{X})$-*imcp*, De forms a $(q, \mathcal{X})$-*mcp* network with $\mathcal{X} = \mathcal{I}_{De}$. In a $(q, p, \mathcal{X})$-*influenced-ps strategy*, $(q, p, \mathcal{X})$-*ips*, De forms a $(q, p, \mathcal{X})$-*ps* where $\mathcal{X} = \mathcal{I}_{De}$. We denote by $s_{\emptyset}^{Ad}$ the strategy of player Ad where he influences no agent, that is $\mathcal{I}_{Ad} = \emptyset$.

[10]We will see below that since $\alpha \leq 1$, $(q, \mathcal{X})$-cp networks have to be defined only for $q \leq 1$.

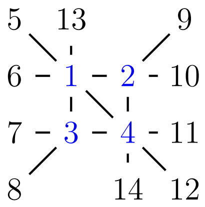

Figure 1: Network g^1

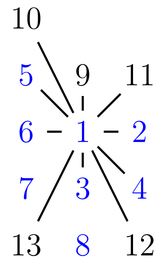

Figure 2: Network g^2

4 First Intuition: Fixed Social Networks

In this section we provide some examples to illustrate the importance of network architecture in determining the infuence strategies chosen by players De and Ad. We examine the strategies of players De and Ad when both of them take the network g as given to help us understand the importance of studying network formation in such games. Consider the following cost function:

$$C(\sharp\mathcal{I}_{De}) = \sharp\mathcal{I}_{De}c_{De},$$

with $1-nc_{De} > 0$. This cost function only takes into account the number of agents De influences. Moreover, for expository purposes in these examples we consider that $\alpha = 1$ in Equation (1). The rest of the model setup remains unchanged.

Example 1 *(A Star Network)* Let g be a star network, with i_c the centre of this star.

Benchmark model. In equilibrium, De has to influence i_c, otherwise all neighbors of i_c will vote 0 when Ad is the only player who influences i_c. Obviously, De must influence i_c to ensure that every agent in $\mathcal{N} \setminus \{i_c\}$ votes 1. Moreover, De has to ensure that the number of neighbors of i_c she influences is no lower than the number of neighbors of i_c that Ad may influence. Otherwise i_c votes 0. In other words, it is necessary that $\sharp\mathcal{N}_{i_c}(g) \cap \mathcal{I}_{De} \geq \min\{k_{Ad}, \sharp(\mathcal{N}_{i_c}(g) \setminus \mathcal{I}_{De})\}$. Note that when this inequality holds Ad has no incentive to influence anyone since it is costly, and due to the strategy used by De he cannot persuade any agent to vote 0. It follows that in an SPNE, De chooses $\sharp\mathcal{I}_{De} = 1 + \min\{k_{Ad}, \sharp(\mathcal{N}_{i_c}(g) \setminus \mathcal{I}_{De})\}$ and $i_c \in \mathcal{I}_{De}$, and Ad chooses to influence no agents.

When Ad is a better influencer than De. Then there does not exist a situation under which De can ensure that all agents vote 1. Indeed, Ad always has the option to influence i_c. In such a case, all neighbors of i_c vote 0. It follows that if we allow for network formation, in an SPNE, De would form the empty network and influence no agents, and Ad influences no agents.

Example 2 *(The Complete Network)* In the complete network g, $\mathbb{A}_{i,j}(g) = 1$ for every $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$.

Benchmark model. We claim that De has to influence $\nu = 1 + \min\{\lfloor \frac{n}{2} \rfloor, k_{Ad}\}$ agents. To show this, let μ be the number of agents that De influences. Consider agent $i \notin \mathcal{I}_{De}$. Since g

is complete, i has $n-1$ neighbors. In equilibrium a strategy of De must lead to $\overline{\Theta}_i \geq 1/2$. But $\overline{\Theta}_i = \dfrac{\mu}{\mu + \min(n-1-\mu, k_{Ad})} = \dfrac{\mu}{\min(n-1; \mu + k_{Ad})}$. Thus $\overline{\Theta}_i \geq 1/2 \Leftrightarrow 2\mu \geq \min(n-1; \mu + k_{Ad}) \Leftrightarrow \left\{\mu \geq \dfrac{n-1}{2} \text{ or } \mu \geq k_{Ad}\right\} \Leftrightarrow \mu \geq \min\left(\left\lfloor \dfrac{n}{2} \right\rfloor, k_{Ad}\right)$.

Now, consider agent j such that $j \in \mathcal{I}_{De}$.

Then $\overline{\Theta}_j = \dfrac{\mu - 1}{\mu - 1 + \min(n-\mu, k_{Ad})} = \dfrac{\mu - 1}{\min(n, \mu + k_{Ad}) - 1}$. Thus $\overline{\Theta}_j \geq 1/2 \Leftrightarrow 2\mu - 2 \geq \min(n, \mu + k_{Ad}) - 1 \Leftrightarrow \left\{\mu \geq \dfrac{n+1}{2} \text{ or } \mu \geq 1 + k_{Ad}\right\} \Leftrightarrow \mu \geq 1 + \min\left(\left\lfloor \dfrac{n}{2} \right\rfloor; k_{Ad}\right)$. To sum up in an SPNE, De influences $1 + \min\left(\left\lfloor \frac{n}{2} \right\rfloor; k_{Ad}\right)$ and Ad influences no agents.

When Ad is a better influencer than De. Clearly, if the maximum number of agents that Ad may influence, k_{Ad}, is sufficiently high, $k_{Ad} \geq \left\lceil \frac{n}{2} \right\rceil$, then De cannot obtain a positive payoff when she influences agents. In that case, in an SPNE, De and Ad do not influence any agents.

When $k_{Ad} < \left\lceil \frac{n}{2} \right\rceil$, De has to influence $2k_{Ad}+1$ agents since in that case every agent i has at least k_{Ad} neighbors in $\mathcal{I}_{De}$ who are not influenced by Ad and at most k_{Ad} neighbors who are influenced by Ad. Consequently, in an SPNE, De influences $2k_{Ad}+1$ and Ad influences no agents.

Example 3 *(The Empty Network)* In the empty network, there is no link, so the initial influences determines the vote of agents.

Benchmark model. Clearly, in an SPNE, De has to influence all agents and Ad does not influence any agent.

When Ad is a better influencer than De. De cannot obtain a positive payoff, since at least one agent votes 0. Basically, Ad always has a strategy where he can get at least one agent to vote 0 since $k_{Ad} \geq 1$. Consequently, in an SPNE, De influences no agents and Ad influences no agents.

The key insights from the fixed network problem emerge from looking at the two extreme cases of the empty and the complete network. It is easy to see that when De can create the network, some strategies will never be a part of the SPNE. For instance, if the cost of forming links is low relative to the cost of influencing agents, De will not build the empty network and influence all agents. Similarly, the complete network seems very costly since it requires both a large number of costly links and a large number of agents to influence. More precisely, even if the cost of forming links is zero, De will not form the complete network since she has to influence a high number of agents. We now investigate what are the strategies that De and Ad should choose when De has the option to form the network.

5 The Benchmark Model: Advantage Designer

In this section, we assume that De has greater influence than Ad. Recall that in our model if both De and Ad influence agent i, then i follows the opinion of De. We will consider two possibilities here: (i) the network g consists only of links created by De, and (ii) random interactions between agents can add links to the network. We begin with case (i) and first deal with the general cost function of De given in (3), and then illustrate the results with the linear cost function of De given in (4). Then, we relax the assumption that only the network built by De determines the social interactions of agents. More precisely, two agents may interact even if they have not been connected by De.

5.1 De Determines the Social Network

In this section we assume that links between agents occur only if De has formed them. Since we want to find the SPNE, let us start with the optimal strategy of Ad. First, since $C(\frac{n(n-1)}{2}, n) < 1$, and the tie-breaking rule favors De in equilibrium, De obtains a strictly positive payoff for all s^{Ad} and for any $\mathcal{N}$ and α. Ad influences even one agent, it results in a strictly negative payoff for Ad since $c_{Ad} > 0$. Therefore, in equilibrium, Ad must play $s^{Ad} = \emptyset$ to obtain a zero profit. The following result sums up this observation.

Lemma 1 *Suppose the payoff functions of players* De *and* Ad *are given by Equations* (5) *and* (6) *respectively. In an SPNE,* Ad *always chooses to influence no agent, and in an SPNE the strategy of* De *is such that for every agent* $i \in \mathcal{N}$, $\theta_i^{Fi} = 1$.

From the above lemma, we know the strategy played by Ad in an SPNE. Moreover, from the payoff function given in Equation (5), the best response for player De consists in minimizing the cost function $C(\sharp E(g), \sharp \mathcal{I}_{De})$ given that Ad will not influence any agent. Let $\Xi \subseteq G[\mathcal{N}] \times 2^{\mathcal{N}}$ be the set of pairs $(g, \mathcal{I}_{De})$ such that there does not exist a strategy for Ad that allows him to obtain a strictly positive payoff. With a slight abuse of notation we refer to as *winning strategy* of De any pair $(g, \mathcal{I}_{De})$ such that all agents vote 1 regardless of the strategy of Ad – given that Ad has to obtain a positive payoff with it. Moreover, we call a *minimal winning strategy* for De a strategy which is a winning network with the minimal number of links given $\mathcal{N}$, α and $\mathcal{I}_{De}$. Finally, an *optimal strategy* is a strategy minimizing the cost of De given that Ad plays a best response against it, that is a *strategy of* De *at the SPNE* – it is the cheapest strategy. Formally, $(g^\star, \mathcal{I}_{De}^\star)$ is an optimal strategy for De if and only if

$$(g^\star, \mathcal{I}_{De}^\star) \in \arg\min\{C(\sharp E(g), \sharp \mathcal{I}_{De}) : (g, \mathcal{I}_{De}) \in \Xi\}.$$

In the following, we denote by $(g^\star, \mathcal{I}_{De}^\star)$ a typical pair in Ξ which minimizes the cost function. In our first result, we provide a minimizing program whose solution is the optimal strategy for De at the SPNE. To simplify the presentation, let $k_{B1}(i, g) = \min\{k_{Ad}, \sharp \mathcal{N}_i(g) \setminus \mathcal{I}_{De}\}$ and $\kappa = 2\alpha - 1$. Note that $\kappa \in (0, 1]$. Clearly, for every agent $i \in \mathcal{I}_{De}$, we must have $\mathcal{N}_i^1(g) \geq \lceil \kappa \mathcal{N}_i^0(g) \rceil$ to obtain $\theta_i^{\text{Fi}} = 1$, that is the number of neighbors of i in $\mathcal{I}_{De}$ has to be greater than κ times the number of neighbors of i in $\mathcal{N} \setminus \mathcal{I}_{De}$. The proof of the following result is given in Appendix A.2.

Proposition 1 *Suppose that payoff functions of players De and Ad are respectively given by Equations (5) and (6). We have $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \leq \lceil \frac{n}{2} \rceil$ for every $i \in \mathcal{I}_{De}$. Moreover, The Designer's strategy, $(g, \mathcal{I}_{De})$, is an optimal strategy if and only if it is a solution of the following minimizing program:*

$$\underset{(g,\mathcal{I}_{De}) \in G[\mathcal{N}] \times 2^{\mathcal{N}}}{\arg\min} \quad C(\sharp E(g), \sharp \mathcal{I}_{De}) \tag{Prg}$$

$$\text{subject to} \quad \forall i \in \mathcal{I}_{De},\ \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \kappa k_{B1}(i,g), \tag{Cons. 1}$$

$$\forall i \in \mathcal{N} \setminus \mathcal{I}_{De},\ \exists j \in \mathcal{I}_{De},\ \mathcal{N}_i(g) = \{j\}. \tag{Cons. 2}$$

The above proposition allows us to rule out several architectures that cannot belong to the set of optimal strategies. More precisely, each agent, who is not influenced by De, has to be connected with exactly one agent that De influences. Similarly, agents influenced by De have to satisfy a ratio between their neighbors in $\mathcal{I}_{De}$ and their neighbors in $\mathcal{N} \setminus \mathcal{I}_{De}$; this ratio is given in (Cons. 1).

In the rest of this section, we provide results when payoff functions of players De and Ad are respectively given by Equations (5) and (6), and the cost function is given by (3). We begin by presenting the possible range of values for $\sharp\mathcal{I}_{De}$ when De plays an optimal strategy. Moreover, for each value of $\sharp\mathcal{I}_{De}$ inside this range, we provide the minimal number of links required in any optimal strategy. Finally, using these facts we establish that there exist only three possible optimal candidate strategies.

In the next proposition, we provide the minimal number of agents that De has to influence in order to ensure that all agents vote 1 in Stage 3. We need an additional definition for the presentation of this result. Let $\bar{x} = \bar{x}(\kappa, n)$ with

$$\bar{x} = \underset{x \in [\![1,n]\!]}{\arg\min} \left\{ x \left\lfloor \frac{x-1}{\kappa} \right\rfloor \geq n - x \right\}. \tag{7}$$

In Lemma 5 (see Appendix A.3), we establish that either $\bar{x} = \lceil \sqrt{\kappa n} \rceil$ or $\bar{x} = \lceil \sqrt{\kappa n} \rceil + 1$. Moreover, when $\alpha = 1$, i.e. $\kappa = 1$, $\bar{x} = \sqrt{n}$.

Let $\sharp\mathcal{I}_{De}{}^{\min}$ denote the minimal number of agents that De has to influence in an optimal strategy.

Proposition 2 *In any optimal strategy, the minimal number of agents De must influence is:*

$$\sharp\mathcal{I}_{De}{}^{\min} = \min\{\bar{x}, \lceil \kappa k_{Ad} \rceil + 1\}.$$

The intuition behind this result – the proof is in Appendix A.3 – is simple. There are two possible necessary conditions for obtaining a winning strategy (and therefore an optimal strategy for De).

1. Suppose that the maximal number of agents Ad can influence, k_{Ad}, is low. Then, it is necessary for one agent i_c to have a number of neighbors in $\mathcal{I}_{De}$ (weighted by κ) which is at least equal to the optimal number of attacks of Ad. When $\sharp\mathcal{N}_{i_c}(g) \cap \mathcal{I}_{De} \geq k_{Ad}$, it is possible for i_c to have a neighborhood that includes all agents that are not influenced by De without risking having $\theta_{i_c}^{\text{Fi}}(g) = 0$.
2. Suppose that the maximal number of agents Ad can influence, k_{Ad}, is large. Each agent influenced by De must have a number of neighbors who belong to $\mathcal{I}_{De}$ (weighted by κ) that is at least equal to the number of his neighbors who are not in $\mathcal{I}_{De}$. Morever, due to (Cons. 2) of Proposition 1, agents in $\mathcal{I}_{De}$ have $n - \sharp\mathcal{I}_{De}$ links with agents that are not influenced by De. Since an agent $i \in \mathcal{I}_{De}$ has at most $\sharp\mathcal{I}_{De} - 1$ neighbors in $\mathcal{I}_{De}$, the minimal number of agents influenced by De that leads to an optimal strategy is given by Inequality (7). More precisely, in Inequality (7) the lefthand side is the maximal number of agents in $\mathcal{N} \setminus \mathcal{I}_{De}$ that agents in $\mathcal{I}_{De}$ influence without being influenced by them (agents in $\mathcal{I}_{De}$ will not change their vote because of their links with agents in $\mathcal{N} \setminus \mathcal{I}_{De}$) and the righthand side is the total number of agents that are not influenced by De.

To illustrate these two points, consider the following example.

Example 4 Let $\mathcal{N} = [\![1, 28]\!]$, $k_{Ad} = 20$ and $\alpha = 0.75$. Moreover, we assume that the cost function of De is given by Equation (4). Let $i \in \mathcal{I}_{De}$. Since $\alpha = 0.75$, each neighbhor $j \in \mathcal{I}_{De}$ of i allows him to have two neighbors who do not belong to this set without voting 0. Consequently, we partition $\mathcal{N}$ into two subsets $\mathcal{I}_{De} = [\![1, 4]\!]$ and $\mathcal{N} \setminus \mathcal{I}_{De} = [\![5, 28]\!]$. Consider that each agent in $\mathcal{I}_{De}$ is linked with all other agents in $\mathcal{I}_{De}$, that is 3 agents in $\mathcal{I}_{De}$. Since each neighbors in $\mathcal{I}_{De}$ allows $i \in \mathcal{I}_{De}$ to have at most 2 neighbors in $\mathcal{N} \setminus \mathcal{I}_{De}$, i has at most 6 neighbors in $\mathcal{N} \setminus \mathcal{I}_{De}$. Because $\sharp\mathcal{I}_{De} = 4$, by (Cons 2) given in Proposition 1 there are at most 24 agents in $\mathcal{N} \setminus \mathcal{I}_{De}$. Obviously, if there are fewer than 4 agents in $\mathcal{I}_{De}$, it is not possible for De to get that all agents to vote 1. Consider the same example except that $k_{Ad} = 2$. A network where agent $i_c \in \mathcal{I}_{De}$ is linked to all the other agents, one of which is in $\mathcal{I}_{De}$ allows De to be sure that all agents vote 1.

We now present the minimal number of links that De has to form in an optimal strategy given the number of agents she influences, $\mathcal{I}_{De}$, i.e. the minimal winning strategies:

Proposition 3 *For $\sharp\mathcal{I}_{De} \geq \sharp\mathcal{I}_{De}^{\min}$, any minimal winning strategy has at least $L^{\min}(\mathcal{I}_{De})$ links with*

$$L^{\min}(\mathcal{I}_{De}) = \min\left(\left\lceil \frac{\kappa(n - \sharp\mathcal{I}_{De})}{2} \right\rceil, \ \lceil \kappa k_{Ad} \rceil\right) + n - \sharp\mathcal{I}_{De}. \tag{8}$$

Again, the intuition of Proposition 3 – the proof is in Appendix A.3 – can be divided into two cases.

1. Suppose that k_{Ad} is large. Then, the sum of degrees in the sub-network $g[\mathcal{I}_{De}]$ between agents in $\mathcal{I}_{De}$ has to be equal to $n - \sharp\mathcal{I}_{De}$ (weighted by κ), so there are $(n - \sharp\mathcal{I}_{De})/2$ links

(weighted by κ) between agents in $\mathcal{I}_{De}$. Moreover, there are $n - \sharp\mathcal{I}_{De}$ links between agents in $\mathcal{I}_{De}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{De}$ by (Cons. 2) given in Proposition 1.

2. Suppose that k_{Ad} is low. Then, one agent in $\mathcal{I}_{De}$ has to form links with at least k_{Ad} (weighted by κ) agents in $\mathcal{I}_{De}$. Again by (Cons. 2) we know that there are $n - \sharp\mathcal{I}_{De}$ links between agents in $\mathcal{I}_{De}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{De}$.

We now provide an example which establishes that there are situations where it is not possible to reach the bound $L^{\min}(\mathcal{I}_{De})$.

Example 5 Let $\mathcal{N} = [\![1, 27]\!]$, $k_{Ad} = 27$, and $\kappa = 3/7$. Clearly, we have $\sharp\mathcal{I}_{De}{}^{\min} = \left\lceil \sqrt{3/7 \times 27} \right\rceil = 4$. Similarly, we have $L^{\min}(\mathcal{I}_{De}{}^{\min}) = \lceil 3/14 \times (23) \rceil + 23 = 28$. Obviously, 23 links are required between agents in $\mathcal{I}_{De}{}^{\min}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{De}{}^{\min}$. Suppose now that there are only 5 links between agents in $\mathcal{I}_{De}{}^{\min}$. Then, two agents in $\mathcal{I}_{De}{}^{\min}$ have formed links with 3 agents in $\mathcal{I}_{De}{}^{\min}$, and two agents in $\mathcal{I}_{De}{}^{\min}$ have formed links with 2 agents in $\mathcal{I}_{De}{}^{\min}$. The former may form links with at most 14 agents in $\mathcal{N} \setminus \mathcal{I}_{De}{}^{\min}$ and the latter may form links with at most $2 \times \lfloor 2 \times 7/3 \rfloor = 8$ agents in $\mathcal{N} \setminus \mathcal{I}_{De}{}^{\min}$. Consequently one agent in $\mathcal{N} \setminus \mathcal{I}_{De}{}^{\min}$ does not satisfy (Cons. 2) in Proposition 1: $L^{\min}(\mathcal{I}_{De}{}^{\min})$ is not a sufficient number of links for obtaining a winning strategy.

Propositions 2 and 3 allow us to establish the main result of this section – the proof is in Appendix A.3 –.

Theorem 1 *The optimal strategy for De is:*

- *the influenced empty network strategy, infl-$\emptyset$, or*
- *a minimal core-periphery network where De influences at least $\sharp\mathcal{I}_{De}{}^{\min}$ agents, $(\kappa, \mathcal{I}_{De})$-imcp, with $\sharp\mathcal{I}_{De} \geq \sharp\mathcal{I}_{De}{}^{\min}$, or*
- *an influenced partial star network where De influences at least $\sharp\mathcal{I}_{De}{}^{\min}$ agents, $(\kappa, p, \mathcal{I}_{De})$-ips, with $\sharp\mathcal{I}_{De} \geq \sharp\mathcal{I}_{De}{}^{\min}$ and $p \geq \lceil \kappa k_{Ad} \rceil$.*

Strategy infl-$\emptyset$ is clear when the cost of forming links is high for De. The intuitions behind the two other strategies are examined successively. In both cases, we know that by (Cons. 2) De has to ensure that every agent she does not influence is linked with one agent she influences.

1. Suppose that k_{Ad} is large. More precisely, Ad is able to influence any agent who is not influenced by De. Then due to (Cons. 1) and (Prg), every agent i influenced by De has to satisfy $\mathcal{N}_i^1(g) = \lceil \kappa \mathcal{N}_i^0(g) \rceil$. When c_L/c_{De} is sufficiently low, min-cp networks where De influences at least $\sharp\mathcal{I}_{De}{}^{\min}$ agents is an optimal strategy for De. When c_L/c_{De} is sufficiently high, two strategies are candidates for being optimal: (i) the empty network where De influences all agents, and (ii) partial star networks where De influences at least $\sharp\mathcal{I}_{De}{}^{\min}$ agents.

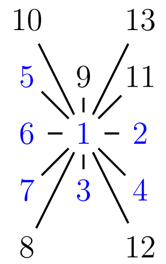

Figure 3: Network g

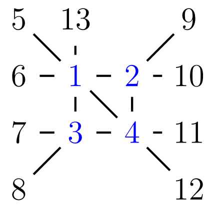

Figure 4: Network g'

2. Suppose that k_{Ad} is low. Then due to (Cons. 1) and (Prg), every agent influenced by De has to satisfy $\mathcal{N}_i^1(g) = \lceil \kappa k_{Ad} \rceil$. It follows that two strategies are candidates for being optimal: (i) the empty network where De influences all agents, and (ii) partial star networks where De influences at least $\sharp \mathcal{I}_{De}{}^{\min}$ agents.

It is worth noting that if De builds the partial star network when k_{Ad} is large, then she has to influence more agents than in the min-cp network. Consequently, as shown in the next example, there exist pairs (c_L, c_{De}) where it is not optimal for De to build the partial star network.

Example 6 Let $\mathcal{N} = [\![1, 13]\!]$, $\alpha = 1$, and $k_{Ad} = 13$. Moreover, payoff functions of players De and Ad are respectively given by Equation (4) and (6). Network g given in Figure 3 with $\mathcal{I}_{De} = [\![1, 7]\!]$ and network g' given in Figure 4 with $\mathcal{I}_{De} = [\![1, 4]\!]$ represent strategies where De builds a partial star and a min-cp network where she influences the minimal number of agents – colored blue. The partial star network requires 7 agents to influence and 12 links while the min-cp network requires 4 agents to influence and 14 links. Consequently, if $c_L/c_{De} < 3/2$, then strategies where De builds partial star networks cannot be optimal.

In Theorem 1, we provide strategies that are candidates for being optimal. We may ask whether there exist parameters where each of these strategies is an optimal one. In Example 7, we assume that the cost function of De is linear and show that each strategy presented in Theorem 1 is optimal under some conditions. More precisely, in this example parameters are chosen for obtaining indifference for De between these strategies. Hence, this example establishes the possibilities for situations where it may be that different winning strategies with different values of $\sharp \mathcal{I}_{De}$ are SPNE. Indeed although the value of the utilities is unique at the SPNE, the corresponding strategies may be different.

Example 7 Suppose the cost function of De is given by Equation (4). Moreover, suppose that

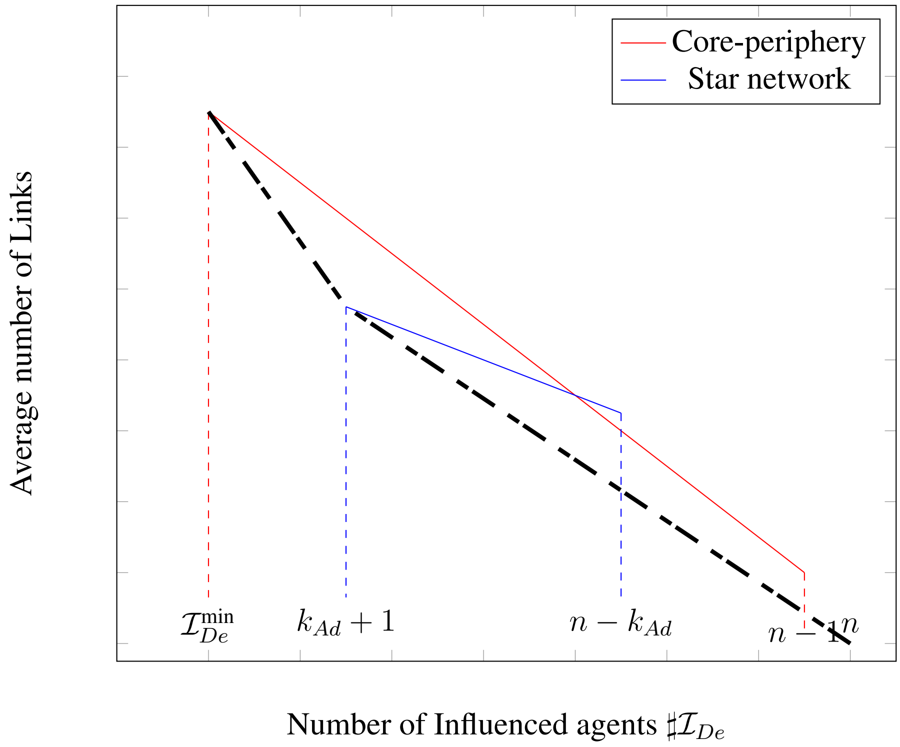

Figure 5: Minimal number of links in the network as a function of the number of influenced nodes $\mathcal{I}_{De}$.

$\alpha=\kappa=1$, $n=10$, $k_{Ad}=3$, $c_L=2$, $c_{DE}=3$, then we have the following results:[11]

$\sharp\mathcal{I}_{De}$	4	5	6	7	8	9	10
$\sharp$ Links in partial star	9	8	7	6	4	2	$\times$
$\sharp$ Links in min-cp Network	9	8	6	5	3	2	$\times$
Empty Network	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	0
Minimal cost for De	30	31	30	31	30	31	30

Thus depending on the values of the parameters n and c_{Ad} the three candidates provided in Theorem 1 may be optimal.

By way of illustration, we briefly discuss the case where (i) the cost function of player De is linear, i.e. the cost function of De is given by Equation (4), and (ii) $\alpha=1$, that is each agent imitates the average initial opinion of his neighbors and does not take into account his own initial opinion.[12] In order to simplify the presentation, we denote by $\mathcal{I}_{De}^{x}$ a typical set of agents influenced by De with $\sharp\mathcal{I}_{De}^{x}=x$. Recall that $\bar{x}=\lceil\sqrt{n}\,\rceil$ when $\alpha=1$.

Notice the following:

- Suppose $n\geq k_{Ad}+1$. For $p=k_{Ad}$ and $\sharp\mathcal{I}_{De}=k_{Ad}+1$, any $(1,p,\mathcal{I}_{De})$-ips strategy

[11] '$\times$' means that the architecture is not defined as an equilibrium.

[12] This situation will serve as a reference case for Section 5.2 where unwanted links by De occur with a positive probability.

contains exactly $p + n - \sharp\mathcal{I}_{De}$ links.[13]

- Consider the following strategy: Start with an empty network and add exactly $\lceil \frac{\sharp\mathcal{N}\setminus\mathcal{I}_{De}}{2} \rceil$ links randomly between pair of agents in $\mathcal{I}_{De}$ – this is possible since $\sharp\mathcal{N}\setminus\mathcal{I}_{De} \leq \sharp\mathcal{I}_{De}(\sharp\mathcal{I}_{De} - 1)$ by definition of ${\mathcal{I}_{De}}^{\min}$. Then, for any agent $i \in \mathcal{I}_{De}$, connect it to up to $\sharp\mathcal{N}_i^1(g)$ agents in $\mathcal{N} \setminus \mathcal{I}_{De}$, making sure that each agent of $\mathcal{N} \setminus \mathcal{I}_{De}$ is linked with exactly one agent in $\mathcal{I}_{De}$. One can check that it is a $(1, \mathcal{I}_{De})$-imcp strategy with $\sharp\mathcal{I}_{De} = \lceil \sqrt{n} \rceil$. With this strategy De forms exactly $n - \sharp\mathcal{I}_{De} + \left\lceil \frac{n - \sharp\mathcal{I}_{De}}{2} \right\rceil$ links.

Thus, when $\sharp\mathcal{I}_{De} \geq \max\{{\mathcal{I}_{De}}^{\min}, k_{Ad} + 1\}$, there exists a winning strategy where the network contains exactly $L^{\min}(\mathcal{I}_{De})$ links (see Proposition 3). As $L^{\min}$ is piecewise linear and the cost functions are linear the extremum values of $\sharp\mathcal{I}_{De}$ will correspond to optimal strategies. Note however that because of the ceiling function in the slope of $L^{\min}$, the optimal may be either reached at $\bar{x}$ or $\bar{x} + 1$. That is, depending on the values of the ratio c_L/c_{De}, $\sharp\mathcal{I}_{De} \in \{0, k_{Ad} + 1, \lceil \sqrt{n} \rceil, \lceil \sqrt{n} \rceil + 1\}$. Therefore, four strategies are candidates for being optimal: the influenced empty network, infl-$\emptyset$, the $(1, \mathcal{I}_{De})$-imcp strategies where $\sharp\mathcal{I}_{De} \in \{\lceil \sqrt{n} \rceil, \lceil \sqrt{n} \rceil + 1\}$, and the connected $(1, k_{Ad}, \mathcal{I}_{De}^{k_{Ad}+1})$-ips strategy, that is strategy where De influences $k_{Ad} + 1$ agents and builds a star network. Due to the linearity of the cost function of De, intuitions about the intervals where each of the previous candidate for being equilibrium are simple. In Figure 5, red line corresponds to that achieved at the mcp-network (in average), while the orange line corresponds to the one achieved at the influenced star network. The thick black line represents the convex envelope. It is sufficient to compare the slope of each line with the value of c_{De}/c_L to establish which strategy is optimal for De. In particular, if c_{De}/c_L is higher than the value of all the slopes of lines drawn in Figure 5, then the optimal strategy consists in forming a winning network with the lowest number of agents influenced by De. Conversely, if c_{De}/c_L is higher than the value of all the slopes of lines drawn in Figure 5, then the optimal strategy consists in forming a winning network with the highest number of agents influenced by De.[14]

5.2 Random Interactions among Non-Linked Agents

In this section, we assume that non-linked agents have a positive probability of interaction, altering the neighborhood of each agent created by De. Moreover, we assume that the cost function is linear, i.e. Equation (4) holds and $k_{Ad} = n$.

We consider the following timing of the game:

1. De chooses her strategy $(g, \mathcal{I}_{De})$;
2. Nature forms a link between every pair of agents (i, j) who are not linked in g with probability $\varpi \in [0, 1]$;

[13] In this presentation, we consider only the case where $k_{Ad} \leq \lfloor n/2 \rfloor$ and therefore $\sharp\mathcal{I}_{De} = k_{Ad} + 1$.

[14] The details of the linear case are available from the authors on request.

3. Ad chooses his strategy given $(g, \mathcal{I}_{De})$.

The timing of the game and $k_{Ad} = n$ together ensure that De obtains a non strictly positive payoff when the network (and the set of influenced agents) obtained after the move that allows random interactions is no longer a winning strategy.

First, for presenting expected payoffs of De and Ad, we need to define a realization g^{ϖ} of g. Network g is a subnetwork of g^{ϖ}. The probability that each link occurs between two non-linked agents is ϖ; the probability of occurrence of the links is i.i.d. In other words, meetings between agents are independent events. Network g^{ϖ} is obtained by adding links to g. Let $\lambda(g^{\varpi} \mid g, \varpi)$ be the probability that g^{ϖ} is realized given that De built network g and probability ϖ. We have:

$$\lambda(g^{\varpi} \mid g, \varpi) = \prod_{ij \in E(g^{\varpi}) \setminus E(g)} \varpi \prod_{i'j' \notin E(g^{\varpi})} (1-\varpi) = \varpi^{\sharp E(g^{\varpi}) \setminus E(g)} (1-\varpi)^{\frac{n(n-1)}{2} - \sharp E(g^{\varpi})}. \quad (9)$$

A winning realization is a pair $(g^{\varpi}, \mathcal{I}_{De})$ which is a winning strategy, i.e. all agents vote 1 in such a pair. Let $R(g)$ be the set of realizations associated with g. In a winning realization Ad does not have any strategy that allows him to ensure that at least one agent vote 0. Let $WR(g; \mathcal{I}_{De}) \subseteq R(g)$ be the set of winning realizations of g given $\mathcal{I}_{De}$. By assuming that the cost function of De is given by Equation (4) the expected payoff obtained by De is:

$$\mathbb{E}u^{De}(\boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi]) = \sum_{g^{\varpi} \in WR(g, \mathcal{I}_{De})} \lambda(g^{\varpi} \mid g, \varpi) - c_{De} \sharp \mathcal{I}_{De} - c_L \sharp E(g). \quad (10)$$

Similarly, the expected payoff of Ad is:

$$\mathbb{E}u^{Ad}(\boldsymbol{\theta}^{\text{Fi}}[s^{De}, s^{Ad}; \Psi, \Phi]) = 1 - \sum_{g^{\varpi} \in WR(g, \mathcal{I}_{De})} \lambda(g^{\varpi} \mid g, \varpi) - \sharp \mathcal{I}_{Ad} c_{Ad}. \quad (11)$$

At first glance it might seem that the possibility of occurence of such unwanted links (by De) is always harmful for De. Indeed, when the number of agents De influences is lower than the number of agents she does not influence, the probability that "bad" links (which involve agents non influenced by De) is higher than the probability that "good" links (which involve only agents influenced by De) occur. However, the possibility that unwanted links may occur is not always harmful for De. For instance, suppose that $\alpha = 1$ and the optimal strategy of De is the infl-$\emptyset$ strategy. If $\varpi^{\frac{n(n-1)}{2}}$ is sufficiently close to 1, then De has an incentive to influence only $\lfloor n/2 \rfloor + 1$ agents instead of influencing all the agents to obtain that all agents vote 1. In this case, the probability that the complete network occurs is sufficiently high and De will obtain a higher payoff with this strategy than with the infl-$\emptyset$ strategy. Let us now provide a lower bound for ϖ such that the strategies candidate for being optimal are the same as those given in Theorem 1 is not modified. The proof of this result is given in Appendix B.

Proposition 4 *Suppose that $\varpi \leq \frac{c_L}{4n}$. Then, the strategies candidate for being optimal are the same as those given in Theorem 1.*

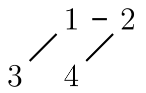

Figure 6: g

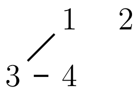

Figure 7: g^1

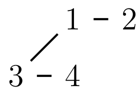

Figure 8: g^2

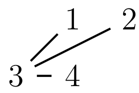

Figure 9: g^3

It is worth noting that in Proposition 4, ϖ depends on n. Let us illustrate this point when $\alpha = 1$. In a $(1, \mathcal{I}_{De}^{\lceil \sqrt{n} \rceil})$-imcp strategy, $\lceil \sqrt{n} \rceil$ agents belong to $\mathcal{I}_{De}$ and $n - \lceil \sqrt{n} \rceil$ agents does not belong to $\mathcal{I}_{De}$. When this strategy is played by De, if a link occurs, then the probability that this link is formed between any agent i and an agent $j \in \mathcal{N} \setminus \mathcal{I}_{De}$ is at least $1 - \frac{\binom{\lceil \sqrt{n} \rceil}{2}}{\binom{n}{2}} = 1 - \frac{\lceil \sqrt{n} \rceil!(n-2)!}{n!(\lceil \sqrt{n} \rceil - 2)!} = 1 - \frac{\lceil \sqrt{n} \rceil(\lceil \sqrt{n} \rceil - 1)}{n(n-1)} \geq 1 - \frac{\lceil \sqrt{n} \rceil}{n} \geq 1 - \frac{\sqrt{n}+1}{n}$. Notice that $\lim_{n \to +\infty} 1 - \frac{\sqrt{n}+1}{n} = 1$. In other words, when the number of agents is very large and a link occurs, the probability that it involves an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$ becomes very large when De uses a $(1, \mathcal{I}_{De}^{\lceil \sqrt{n} \rceil})$-imcp strategy. Obviously, this type of links makes the $(1, \mathcal{I}_{De}^{\lceil \sqrt{n} \rceil})$-imcp strategies inefficient.

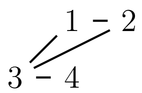

Figure 10: g^4

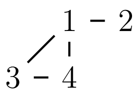

Figure 11: g^5

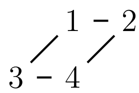

Figure 12: g^6

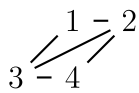

Figure 13: g^7

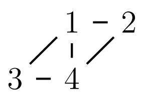

Figure 14: g^8

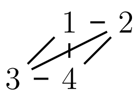

Figure 15: g^9

We now illustrate the probabilitic case in a specific situation where $\alpha = 1$, $\mathcal{N} = [\![1, 4]\!]$. Recall that $\sharp \mathcal{I}_{De}^{\min} = 2$ when $\alpha = 1$ and $\mathcal{N} = [\![1, 4]\!]$. It follows that there are two possibilities to examine according to the value of $\mathcal{I}_{De}$ – given that De always obtains $1 - 4c_{De}$ when she influences 4 agents: $\sharp \mathcal{I}_{De} = 2$ and $\sharp \mathcal{I}_{De} = 3$. Note that De does not form any link between agents who are not influenced by her. Let us explore successivley both cases:

1. $\sharp \mathcal{I}_{De} = 2$, say $\mathcal{I}_{De} = \{1, 2\}$. Note that there are only two network where De obtains all agents vote 1. These network called g and g^6 are drawn in Figures 6 and 12 – up to a relabeling of agents. It follows that if De forms no links, then the probability that g occurs is $2\varpi^3(1 - \varpi)^2$. When De forms one link, she has two possibilities: either she forms a link between two agents in $\mathcal{I}_{De}$ or she forms a link between an agent in $\mathcal{I}_{De}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$. The former leads to a probability of obtaining g equal to $2\varpi^2(1 - \varpi)^2$, and the latter leads to a probability of obtaining g or g^6 equal to $\varpi^2(1 - \varpi)^2$. Consequently, De has always an incentive to form a link between two agents she influences. When De forms two links she has two possibilities: either she forms a link between two agents in $\mathcal{I}_{De}$ and one link between an agent in $\mathcal{I}_{De}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$, or both links are between and

agent in $\mathcal{I}_{De}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$. In both cases the probability of obtaining g or g^6 is equal to $\varpi(1-\varpi)^2$. Finally, when De forms three links, the probability that g or g^6 occurs is $(1-\varpi)^2$.

2. $\sharp\mathcal{I}_{De} = 3$, say $\mathcal{I}_{De} = \{1, 2, 3\}$. We draw in Figures 7 to 15 the different networks where all agents vote 1 when $\mathcal{I}_{De} = \{1, 2, 3\}$ up to a relabeling of agents. By using similar arguments as in the previous point and the list of networks g^1 to g^7 we obtain the following results.[15] When De forms no links the probability that all agents vote 1 is $\varpi^6 + 6\varpi^2(1-\varpi)^2 + 6\varpi^4(1-\varpi)$. When De forms 1 link the probability that all agents vote 1 is $\varpi + \varpi(1-\varpi)$. When De forms 2 links the probability that all agents vote 1 is: $\max\{1-\varpi(1-\varpi)^2, 1-(1-\varpi)^3\}$. When De forms 3 links then she can ensure to obtain that all agents vote 1 with network g^2. It is also possible when she forms 4, 5 or 6 links.

Let us now provide the optimal strategies of De for some specific sets of parameters. More precisely, we assume that $N = [\![1, 4]\!]$. We define the following strategies for De: $S_1 : \mathcal{I}_{De} = \{1, 2\}, E(g) = \{12, 13, 24\}$, $S_2 : \mathcal{I}_{De} = \{1, 2, 3\}, E(g) = \{12, 13, 34\}$, and $S_3 : \mathcal{I}_{De} = \{1, 2, 3\}, E(g) = \{12, 13\}$, $S_4 : \mathcal{I}_{De} = \{1, 2, 3\}, E(g) = \emptyset$, and $S_5 : \mathcal{I}_{De} = \{1, 2, 3, 4\}, E(g) = \emptyset$. In the following table, we provide an optimal strategy for De up to a relabeling of agents for several value of ϖ and c_L given that $c_{De} = 0.07$.

ϖ \ c_L	1/1000	1/100	11/100
1/1000	S_1	S_1	S_5
87/100	S_2	S_3	S_4
99/100	S_4	S_4	S_4

Let us provide some observations through these examples.

1. When the probability of unwanted links is very low, then the optimal strategy is the same as in the benchmark model, S_1 – see Proposition 4.

2. When the probability of unwanted links is very high, then the optimal strategy consists in S_4: De forms no links, and influences a number of agents that allow her to obtain that each agent votes 1 in the complete network. Note that in this case De incurs costs lower than the ones she incurs in the benchmark model.

3. When the probability of unwanted links is moderate, then some intermediate strategies, where De influences a number of agents in $[\![{\mathcal{I}_{De}}^{\min} + 1, n-1]\!]$, become optimal. In particular, in S_3, De influences 3 agents, $n > 3 > \sharp{\mathcal{I}_{De}}^{\min}$. Moreover, the number of links and number of agents that De influences depend on the relative cost of c_L and c_{De}.

[15]Here we indicate the probability associated with the strategy of De which maximizes the probabililty to obtain that all agents vote 1.

6 Extensions

In the following we discuss briefly two assumptions made in the benchmark model. First, we relax the unanimity assumption. Specifically, we allow player De to win when a majority of the agents vote 1. Second, we relax the assumption that De is a better influencer than Ad. More precisely, when an agent is influenced both by De and Ad, then he follows the opinion of Ad.

6.1 Influencing only the Majority

In this section, we consider a situation where De obtains a strictly positive payoff if and only if $\sharp\mathcal{N}(1, \text{Fi}) \geq \lceil n/2 \rceil$.[16] The majority rule is incorporated into the model by modifying De's payoff function. Specifically,

$$u^{De}_{\text{Maj}}(\theta^{Fi}[s^{De}, s^{Ad}; \Psi, \Phi]) = \begin{cases} 1 - c_L \sharp E(g) - c_{De} \sharp \mathcal{I}_{De} & \text{if } \sharp\mathcal{N}(1, \text{Fi}) \geq \lceil \frac{n}{2} \rceil, \\ -c_L \sharp E(g) - c_{De} \sharp \mathcal{I}_{De} & \text{otherwise.} \end{cases} \tag{12}$$

Player Ad's payoff function is defined in the same way. In the majority case, the payoff of player Ad with

$$u^{Ad}_{\text{Maj}}(\theta^{Fi}[s^{De}, s^{Ad}; \Psi, \Phi]) = \begin{cases} 1 - c_{Ad} \sharp \mathcal{I}_{Ad} & \text{if } \sharp\mathcal{N}(1, \text{Fi}) < \lceil \frac{n}{2} \rceil, \\ -c_{Ad} \sharp \mathcal{I}_{Ad} & \text{otherwise.} \end{cases}$$

Following the arguments given in Lemma 1, at the SPNE Ad does not influence any agent. Let us provide some properties of the winning strategies of De according to the size of $\mathcal{I}_{De}$.

- If $\mathcal{I}_{De} = \emptyset$, then there exists no winning strategy for De.
- If $\mathcal{I}_{De} = \{i\}$, then consider a partial star where i is the central agent and which has $\lceil n/2 \rceil$ peripheral agents. All other agents are isolated. This is a winning strategy (regardless of α).
- If $\sharp\mathcal{I}_{De} \in [\![2, \lceil n/2 \rceil - 1]\!]$, then consider a partial star where i is the central agent and which has $\lceil n/2 \rceil - \sharp\mathcal{I}_{De} + 1$ peripheral agents. All other agents are isolated. This is a winning strategy (regardless of α).
- If $\sharp\mathcal{I}_{De} \geq \lceil n/2 \rceil$, then the empty network is a winning strategy.

A partial star with $\sharp\mathcal{I}_{De} - 1$ isolated agents influenced by De is less costly than a $(\kappa, \mathcal{I}_{De})$-imcp strategy since the number of links in the former is lower than in the latter. This relies on the fact that (i) there exists no link between agents in $\mathcal{I}_{De}$ in the partial star, (ii) and the number of links between agents in $\mathcal{I}_{De}$ and agents not belonging to $\mathcal{I}_{De}$ is the same in both strategies.
Thus the associated cost for De is:

$$\min\{\lceil n/2 \rceil c_{De}, \min_{x \in [\![1, \lceil n/2 \rceil - 1]\!]} (x c_{De} + (\lceil n/2 \rceil - x + 1) c_L)\}$$

This leads to the following proposition:

[16] Typically majority would require strictly greater than half, $\sharp\mathcal{N}(1, \text{Fi}) \geq \lceil n/2 \rceil + 1$. But in this case, the results presented in this section would be qualitatively the same.

Proposition 5 *Suppose the payoff functions of players De and Ad are given by Equations* (5) *and* (6) *respectively. An optimal strategy is independant of the value of α and of c_{Ad}. Further:*

1. *If $\frac{c_L}{c_{De}} > \left(1 - \frac{1}{\lceil n/2 \rceil}\right)$, then in her optimal strategy De forms no link and influences exactly $\lceil n/2 \rceil$ agents.*
2. *If $\frac{c_L}{c_{De}} < \left(1 - \frac{1}{\lceil n/2 \rceil}\right)$, then in her optimal strategy De forms $\lceil n/2 \rceil$ links and influences exactly 1 agent. The resulting network is a partial-star.*
3. *If $\frac{c_L}{c_{De}} = \left(1 - \frac{1}{\lceil n/2 \rceil}\right)$, then previous two strategies are equilibria.*

It is worth noting that De shapes the network according to her costs c_L and c_{De} without taking into account the cost for influencing agents that Ad incurs. One difficulty induced by the strategies described in Proposition 5 is that partial star networks where only the centre is influenced are possibly "unstable" in the following sense. If we provide the agents a second opportunity for revising their opinion, then all peripheral agents will vote 0. This occurs *only if* the cost for influencing agents is zero for Ad. We have not considered this possibility in the previous sections as it would not change the results except that Ad would be indifferent between all his strategies in equilibrium. Let us illustrate the fact that the star is unstable when $c_{Ad} = 0$ through the following example.

Example 8 Supppose $\mathcal{N} = [\![1, 12]\!]$, $\alpha = 1$, $\frac{c_L}{c_{De}} = \frac{1}{8}$ and $c_{Ad} = 0$. Then, an optimal strategy for De consists in forming a partial star and influencing agent 1. Similarly a *weakly* best response for Ad consists in influencing all agents. Network g^1 drawn in Figure 16 where agents with opinion 0 are colored red and agent 1 with opinion 1 is colored blue summarizes the strategies played by De and Ad. Due to the peer effect De obtains that agents in $[\![2, 7]\!]$ vote 1 see network g^2 drawn in Figure 17. However, if an opportunity to revise is given to each agent, then agents in $[\![2, 7]\!]$ vote 0 see network g^3 drawn in Figure 18. Consequently, votes made by agents in $[\![2, 7]\!]$ are not "stable".

Due to this problem, we introduce strategies, called *stable majority configurations*, that are stable in the following sense:
For every $i \in \mathcal{N}(1, \theta^{\text{Fi}})$,

$$\frac{\sharp(\mathcal{N}_i(g) \cap \mathcal{N}(1, \theta^{\text{Fi}}))}{\sharp\mathcal{N}_i(g)} \geq \frac{1}{2}. \tag{13}$$

Due to Inequality (13), every agent i who votes 1 has a majority of neighbors who vote 1. Consequently, i does not modify his vote when he obtains an opportunity to revise his opinion. This means that if an iterated process were introduced in the model, the set of agents who vote 1 would maintain their decision throughout the process.

We first present a lemma that provides the size of the set of isolated agents who are not influenced by De, its proof is given in Appendix C.

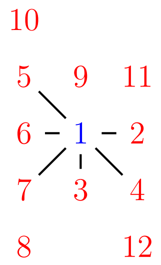

Figure 16: Network g^1

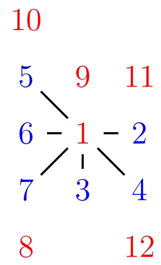

Figure 17: Network g^2

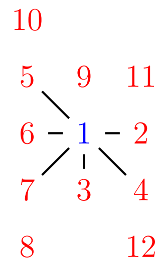

Figure 18: Network g^3

Lemma 2 *For obtaining a stable majority configuration, in optimal strategy, there are $n - \lceil n/2 \rceil$ isolated agents who are not influenced by De.*

By using similar arguments as in Lemma 1 Ad influences no agent in an SPNE – again De has an incentive to choose a winning strategy. Moreover, by Lemma 2, we know that De focuses her resources, i.e. links and influence, only on $\lceil n/2 \rceil$ agents. Consequently, the results obtained in Theorem 1 is preserved given that De focuses on $\lceil n/2 \rceil$ agents. We need to modify Equation (7) for taking into account this specific strategy of De:

$$\hat{x} = \arg\min_{x \in [\![1,n]\!]} \left\{ x \left\lfloor \frac{x-1}{\kappa} \right\rfloor \geq \left\lceil \frac{n}{2} \right\rceil - x \right\}. \tag{14}$$

Straighforward computations lead to $\hat{x} = \left\lceil \sqrt{\kappa \lceil n/2 \rceil} \right\rceil$ or $\hat{x} = \left\lceil \sqrt{\kappa \lceil n/2 \rceil} \right\rceil + 1$. We have ${\mathcal{I}_{De}}^{\min}[\text{maj}] = \min\{\hat{x}, \lceil \kappa k_{Ad} \rceil + 1\}$. By using the same arguments as in Theorem 1, we obtain the following result.

Proposition 6 *Optimal strategies, that always lead to a stable majority configuration, are the same as those identified in Theorem 1 but involve only $\lceil n/2 \rceil$ agents: The optimal strategy for De is:*

1. *the empty network where De influences $\lceil n/2 \rceil$ agents, or*
2. *a network g with two sub-networks g' and g'', g' is a star with $\lceil n/2 \rceil$ agents and where De influences at least $\sharp {\mathcal{I}_{De}}^{\min}[maj]$ agents, and g'' is the empty network, or*
3. *a network g with two sub-networks g' and g'', g' is a min-cp network with $\lceil n/2 \rceil$ agents where De influences at least $\sharp {\mathcal{I}_{De}}^{\min}[maj]$ agents, and g'' is the empty network.*

6.2 Strong Adversary

In this section, we assume that when both players De and Ad influence agent i, then $\theta_i^{\text{In}} = 0$. In order to simplify the presentation, we assume that n is even.[17] In the following, we say that Ad is

[17]The results obtained when n is odd are qualitatively the same, except that there exist conditions where in her optimal strategies player De chooses $\sharp \mathcal{I}_{De} = n - 1$.

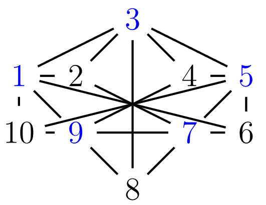

Figure 19: Network g^1

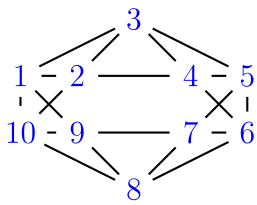

Figure 20: Network g^2

the best primary influencer.

In order to present the results, we define useful strategies for player De.

1. In the $(a, b, \mathcal{X})$-groups-regular strategy, $\mathcal{N}$ is partitionned into two subsets: $\mathcal{X}$ and $\mathcal{N} \setminus \mathcal{X}$, where $\mathcal{X} = \mathcal{I}_{De}$. Moreover, when $a \times \sharp\mathcal{X}$ is even, we have

$$\begin{cases} \sum_{j\in\mathcal{X}} \mathbb{A}_{i,j}(g) = a & \text{if } i \in \mathcal{X}, \\ \sum_{j\in\mathcal{X}} \mathbb{A}_{i,j}(g) = b, \sum_{j\notin\mathcal{X}} \mathbb{A}_{i,j}(g) = 0 & \text{otherwise.} \end{cases} \tag{15}$$

When $a \times \sharp\mathcal{X}$ is odd, conditions provided in (15) hold except that there is a unique agent $i \in \mathcal{X}$ for whom we have $\sum_{j\in\mathcal{X}} \mathbb{A}_{i,j}(g) = a + 1$.

2. In the $(a, \mathcal{N})$-regular strategy, we have $\mathcal{N} = \mathcal{I}_{De}$ and g is a regular network where all agents have a degree equal to a.

In Figure 19, g^1 represents a $(2, 3, \{1, 3, 5, 7, 9\})$-groups-regular strategy where every agent influenced by De is colored blue. In Figure 20, g^2 represents a $(4, [\![1, 10]\!])$-regular strategy where every agent influenced by De is colored blue.

Since Ad is the best primary influencer, he chooses a (minimal) number of agents to influence in order to obtain that for an agent, say i, $\theta_i^{\text{Fi}} = 0$. When k_{Ad} is sufficiently high, De has no strategy for which she can get all agents to vote 1. We begin our analysis by providing conditions under which De has the possibility of having a winning strategy. In the next proposition, we establish the minimal number of agents that player De has to influence in order to have winning strategies. Next proposition follows Lemma 6 given in Appendix D, it provides conditions under which De has strategies for obtaining $\theta_i^{\text{Fi}} = 1$ for every $i \in \mathcal{N}$.

Proposition 7 *Suppose that payoff functions of players De and Ad are respectively given by Equations* (5) *and* (6) *and n is even.*

1. *Suppose that $2(1-\alpha)k_{Ad} < 1$. If $n \geq \lceil 2\alpha k_{Ad} \rceil + 1$, then ${\mathcal{I}_{De}}^{\min} = \lceil 2\alpha k_{Ad} \rceil + 1$. Otherwise, there is no winning strategy.*

2. *Suppose that $2(1-\alpha)k_{Ad} > 1$. If $n \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, then ${\mathcal{I}_{De}}^{\min} = \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$. Otherwise, there is no winning strategy.*

3. *When $2(1-\alpha)k_{Ad} = 1$, then previous two results hold.*

We now provide an intuition for the previous result. First, Ad can target two types of agents: either he tries to obtain $\theta_i^{\text{Fi}} = 0$ for $i \in \mathcal{I}_{De}$, or $\theta_i^{\text{Fi}} = 0$ for $i \notin \mathcal{I}_{De}$. Second, Ad has two possible strategies: either he influences both agent i and some of his neighbors or he only influences some neighbors of i. The threshold given in Proposition 7 follows straighforward computations.
In the following proposition, we provide the strategies candidate for being optimal when the cost function is given by Equation (3). Obviously, we restrict our attention to cases where De has an incentive to choose a strategy where she has an incentive to influence some agents (and possibly form links).

Proposition 8 *Suppose that the cost function* (3) *is convex and* n *is even.*

1. *Suppose that* $(1-\alpha)k_{Ad} < \frac{1}{2}$. *If* De *has a winning strategy, then there are only two strategies candidate for being optimal:* $(\lceil 2\alpha k_{Ad} \rceil, 2k_{Ad}, \mathcal{I}_{De})$*-groups-regular strategy where* $\sharp \mathcal{I}_{De} \in [\![\sharp {\mathcal{I}_{De}}^{\min}, n-1]\!]$, *or* $(\lceil 2\alpha k_{Ad} \rceil, \mathcal{N})$*-regular strategy.*
2. *Suppose that* $\frac{1}{2} < (1-\alpha)k_{Ad} < \alpha$. *If* De *has a winning strategy, then there are only two strategies candidate for being optimal:* $\left(\left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, 2k_{Ad}, \mathcal{I}_{De}\right)$*-groups-regular strategy where* $\sharp \mathcal{I}_{De} \in [\![\sharp {\mathcal{I}_{De}}^{\min}, n-1]\!]$, *or* $\left(\left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, \mathcal{N}\right)$*-regular strategy.*
3. *Suppose that* $(1-\alpha)k_{Ad} > \alpha$. *If* De *has a winning strategy, then there are only two strategies candidate for being optimal:* $\left(\left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, \mathcal{I}_{De}\right)$*-groups-regular strategy where* $\sharp \mathcal{I}_{De} \in [\![\sharp {\mathcal{I}_{De}}^{\min}, n-1]\!]$, *or* $\left(\left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, \mathcal{N}\right)$*-regular strategy.*

Moreover, when $(1-\alpha)k_{Ad} = \frac{1}{2}$, *points* 1. *and* 2. *hold, and when* $(1-\alpha)k_{Ad} = \alpha$ *points* 2. *and* 3. *hold.*

Previous result highlights that there are two types of equilibrium strategies.

1. In the first strategy, De makes a distinction between agents in $\mathcal{I}_{De}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{De}$. Here De takes into account the strategy where Ad only targets the neighbors of agent i for whom Ad wants to obtain $\theta_i^{\text{Fi}} = 0$.
2. In the second strategy, De chooses to form a regular network and ensure that Ad cannot influence enough agents to obtain an agent $i \in \mathcal{N}$ such that $\theta_i^{\text{Fi}} = 0$. It takes into account the strategy where Ad targets both i and his neighbors for whom Ad wants to obtain $\theta_i^{\text{Fi}} = 0$.

We now illustrate this result in the case where $\alpha = 1$ and the cost function of De is linear. By Proposition 7, we know that $\sharp \mathcal{N}_i(g, \mathcal{I}_{De}) = 2k_{Ad}$ for each $i \in \mathcal{N}$. The cost associated with $(2k_{Ad}, 2k_{Ad}, \mathcal{I}_{De})$-groups-regular strategy is

$$\sharp \mathcal{I}_{De} c_{De} + \left(\left\lceil \frac{\sharp \mathcal{I}_{De} 2k_{Ad}}{2} \right\rceil + (n - \sharp \mathcal{I}_{De}) 2k_{Ad} \right) c_L,$$

when there is $i \notin \mathcal{I}_{De}$. When every $i \in \mathcal{I}_{De}$, the cost incurred by De is

$$nc_{De} + \left\lceil \frac{n 2k_{Ad}}{2} \right\rceil c_L.$$

Since $\sharp\mathcal{I}_{De}2k_{Ad}$ and $n2k_{Ad}$ are even, the cost incurred by De can be written

$$c(\sharp\mathcal{I}_{De}) = \sharp\mathcal{I}_{De}(c_{De} - c_L k_{Ad}) + n2k_{Ad}c_L$$

where $\sharp\mathcal{I}_{De} \in [\![2k_{Ad}+1, n]\!]$. Because $c(\sharp\mathcal{I}_{De})$ is affine, there are two possibilities to examine for establishing the minimal cost incurred by De. The minimal cost incurred by De occurs when $\sharp\mathcal{I}_{De} = \mathcal{I}_{De}{}^{\min} = 2k_{Ad}+1$ or when $\sharp\mathcal{I}_{De} = n$. Straightforward computations lead to $c(\sharp\mathcal{I}_{De}{}^{\min}) \geq c(n)$ if and only if $\frac{c_L}{c_{De}} \geq \frac{1}{k_{Ad}}$.

Proposition 9 *Suppose $\alpha = 1$ and the cost function of De is given by Equation* (4)*. Suppose also that De has a winning strategy. If $\frac{c_L}{c_{De}} \leq \frac{1}{k_{Ad}}$, then $(2k_{Ad}, 2k_{Ad}, \mathcal{I}_{De}{}^{\min})$-groups-regular strategy is optimal. Otherwise, $(2k_{Ad}, \mathcal{N})$-regular strategy is optimal.*

Let us provide an example that allows to capture the intuition behind the results given in this section.

Example 9 Suppose $\mathcal{N} = [\![1, 10]\!]$, $k_{Ad} = 2$, $\alpha = 1$, and the cost function of De is linear with $c_L = 1$, $c_{De} = 3$. Since $\alpha = 1$, Ad has always an incentive to influence only the neighbors of i and has no incentive to influence i himself. Indeed, when $\alpha = 1$, i does not take into account his own initial opinion to form his final one. Note that Ad can target (i) either the neighboorhood of an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$, (ii) or the neighboorhood of an agent in $\mathcal{I}_{De}$.
Let us deal with the former case where Ad wants to obtain that $i \in \mathcal{N} \setminus \mathcal{I}_{De}$ votes 0. In that case Ad has an incentive to influence neighbors of i if they are less than 4 to have been influenced by De – if De has influenced 4 neighbors of i, then i votes 1.
The latter case leads to the same type of result: every agent $i \in \mathcal{I}_{De}$ has to get at least 4 neighbors in $\mathcal{I}_{De}$ in order to ensure that i votes 1.
It follows that each agent $i \in \mathcal{N}$ must have 4 neighbors in their neighboorhood. Consequently, the minimal size of $\mathcal{I}_{De}$ is 5. Due to the linearity of the cost function of De, there are two strategies candidate for being an equilibrium: $(4, 4, [\![1, 5]\!])$-group regular – up to a relabeling of agents – where the size of $\mathcal{I}_{De}$ is minimal, and $(4, [\![1, 10]\!])$-regular strategy where the size of $\mathcal{I}_{De}$ is maximal. Straightforward computations leads to the cost incurred by De with the former strategy is: $(\frac{5\times4}{2} + 5 \times 4) \times 1 + 5 \times 3 = 45$ and the cost incurred by De with the latter strategy is: $\frac{10\times4}{2} \times 1 + 10 \times 3 = 50$. Consequently, $(4, 4, [\![1, 5]\!])$-group regular strategy is optimal for De.
We now assume that $c_L = 3$ and $c_{De} = 4$. Then the cost of strategies $(4, 4, [\![1, 5]\!])$-group regular and $(4, [\![1, 10]\!])$-regular are respectively $(\frac{5\times4}{2}+5\times4)\times3+5\times4 = 110$ and $\frac{10\times4}{2}\times3+10\times4 = 100$. Consequently, $(4, [\![1, 10]\!])$-regular strategy is optimal for De.

Step 1: × — × — × — × — × — 1 — 1 — × — × — × — × — ×

Step 2: × — × — × — × — 1 — 1 — 1 — 1 — × — × — × — ×

⋮ ⋮

Step 6: 1 — 1 — 1 — 1 — 1 — 1 — 1 — 1 — 1 — 1 — 1 — 1

Figure 21: Example 10 when $T \geq 6$

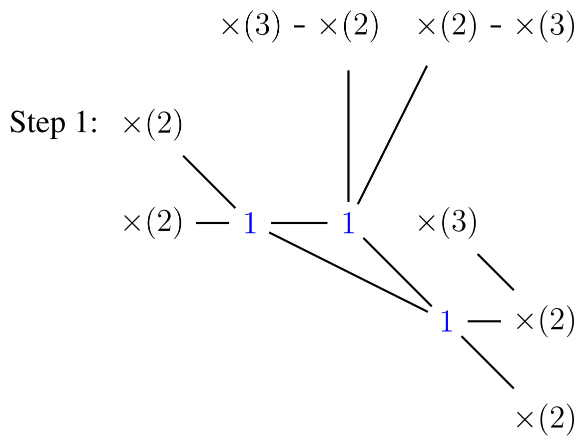

Figure 22: Example 10 when $T = 3$

7 Concluding Remarks

In this paper, we examine the optimal strategy of a designer who wants that at least a majority of the agents choose a specific action while there is another player who wants her to fail. An important component in this opinion war is the fact that each agent's peer group in the network affects their opinion. We have assumed that the agents form their opinion after fully interacting with their neighbors, i.e. they do not wish to change their minds after their interaction. An alternative way to think of this process is to say that they interact repeatedly and in each interaction they may have a different opinion. Finally at some future point in time they cast their vote.
We will now show through an example that allowing agents to interact for a large number of periods is not very interesting. Obviously having infinitely many periods implies that agents will not have the option to choose an action. When we restrict attention to finitely many periods of interaction between agents we show that the earlier results provide the upper bounds on the costs and therefore the payoffs of De.

Example 10 Let $\mathcal{N} = [\![1, 12]\!]$, $\alpha = 1$ and $k_{Ad} = 12$. Moreover, we assume that we are in the unanimity world, i.e. requires that all agents vote 1 after T steps. Finally, to make it more interesting, we only consider cases where the cost of influencing agents is high relative to the cost of forming links – De has an incentive to minimize the number of agents she influences. Let De chooses a strategy where she forms a line and influences two connected agents as shown in Figure 21 (Step 1). In Figure 21 each agent influenced by De is denoted by '1' and each agent who is not influenced by De is denoted by '$\times$'. Then due to the imitation process at Step $T = n/2 = 6$ all agents vote 1, regardless of the agents influenced by Ad. Consequently, the fact that De does not need to obtain that all agents choose 1 immediately allows her to reduce her cost in equilibrium. Note that the previous strategy is optimal for De since the cost of formation of links is sufficiently low – relative to the cost of influence. Indeed, it is clear that she cannot let agents she does not influence isolated. Moreover, it is obvious that De cannot influence only one agent, so De has to influence at least 2 agents.

Finally, it is worth noting that the number of agents that De has to influence depends on the number of periods. When De builds the network drawn in Figure 22, and influences 3 agents, all agents vote 1 after three periods – we indicate for every agent not influenced by De the step where she will choose 1 in bracket regardless of the agents Ad influences. This shows that the cost De incurs to get all agents to vote 1 decreases with the number of periods she has for obtaining unanimity among agents.

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Appendix A. Benchmark Model

A.1 Existence of $(q, \mathcal{X})$-*mcp* network

To show the existence of a $(q, \mathcal{X})$-mcp network for every $q \leq 1$, it is sufficient to establish that the set of $(q, \mathcal{X})$-cp networks is non-empty. Indeed, since this set is finite, it admits at least one minimal element with regard to the number of links. Since $q \leq 1$, a $(1, \mathcal{X})$-mcp is a $(q, \mathcal{X})$-cp network. The existence of $(1, \mathcal{X})$-mcp implies that the set of $(q, \mathcal{X})$-mcp networks is non-empty. Let us construct a process that leads to a $(1, \mathcal{X})$-mcp network g. We build network g as follows:

1. Start with the empty network.
2. While $\sharp E(g[\mathcal{X}])$ is such that $\sharp E(g[\mathcal{X}]) < \left\lceil \frac{n - \sharp \mathcal{X}}{2} \right\rceil$, take two unlinked agents $i, j \in \mathcal{X}$ such that $\sharp \mathcal{N}_i(g), \sharp \mathcal{N}_j(g) \in \min_{\ell \in \mathcal{X}} \{\sharp \mathcal{N}_\ell(g)\}$, do $ij \in E(g)$. When $\sharp E(g[\mathcal{X}]) = \left\lceil \frac{n - \sharp \mathcal{X}}{2} \right\rceil$ go to 3.
3. While there exists $j \in \mathcal{N} \setminus \mathcal{X}$, with $\mathcal{N}_j(g) = \emptyset$, take $i \in \mathcal{X}$ with $\sum_{\ell \in \mathcal{X}} \mathbb{A}_{i,\ell}(g) \geq \sum_{\ell \in \mathcal{N} \setminus \mathcal{X}} \mathbb{A}_{i,\ell}(g) + 1$, do $ij \in E(g)$. Stop.

A.2 Proof of Proposition 1

In order to present the proof, we present two lemmas.

Lemma 3 *Suppose that payoff functions of players De and Ad are respectively given by Equations* (5) *and* (6)*. Let $(g, \mathcal{I}_{De})$ be a winning strategy. For every $i \in \mathcal{I}_{De}$, we have $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \kappa k_{B1}$.*

Proof The condition is obvious for isolated agents in $\mathcal{I}_{De}$. Consider a non-isolated agent $i \in \mathcal{I}_{De}$. A winning strategy requires that $\theta_i^{\text{Fi}} = 1$. By Equation (1), this is true when for every $i \in \mathcal{N}$,

$$1/2 \leq (1-\alpha) + \alpha\overline{\Theta}_i = 1 - \alpha + \alpha\frac{\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De})}{\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) + k_{B1}}.$$

Thus

$$(\alpha - 1/2)\left(\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) + k_{B1}\right) \leq \alpha\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De})$$

Hence

$$\left(\alpha - \frac{1}{2}\right)k_{B1} \leq \frac{1}{2}\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De})$$

which leads to the conclusion.

□

Lemma 4 *Suppose that payoff functions of players De and Ad are respectively given by Equations* (5) *and* (6)*. Let $(g, \mathcal{I}_{De})$ belong to an minimal winning strategy. If $i \in \mathcal{N} \setminus \mathcal{I}_{De}$, then $\mathcal{N}_i(g) = \{j\}$, with $j \in \mathcal{I}_{De}$.*

Proof Suppose that $(g, \mathcal{I}_{De})$ is a minimal winning strategy. First, if $i \in \mathcal{N} \setminus \mathcal{I}_{De}$, then i is not isolated. Otherwise, Ad chooses to influence agent i, and the strategy is not a winning one, a contradiction. Second, we show that if $i, j \in \mathcal{N} \setminus \mathcal{I}_{De}$, then $ij \notin E(g)$. It is clear that links between agents $i, j \in \mathcal{N} \setminus \mathcal{I}_{De}$ cannot allow De to save links between agents in $\mathcal{I}_{De}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{De}$. Consequently, De has no incentive to form links between agents i and j in $\mathcal{N} \setminus \mathcal{I}_{De}$. Similarly, if $i \in \mathcal{N} \setminus \mathcal{I}_{De}$ has a unique neighbor who is influenced by De, then $\theta_i^{\text{Fi}} = 1$. Again, an additional link between i and another agent in $\mathcal{I}_{De}$ is useless and costly and thus not formed in a minimal winning strategy. □

Proof of Proposition 1 First, we establish that $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \leq \lceil \frac{n}{2} \rceil$ for every $i \in \mathcal{I}_{De}$. To introduce a contradiction, suppose that there exists $i \in \mathcal{N}$ such that $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \lceil \frac{n}{2} \rceil + 1$. Then, $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \left\lceil \frac{\sharp\mathcal{N}_i(g)}{2} \right\rceil + 1 \geq \frac{\sharp\mathcal{N}_i(g)+2}{2}$. We have $2\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \sharp\mathcal{N}_i(g) + 2 \Rightarrow \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \sharp\mathcal{N}_i(g) - \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) + 2 \Rightarrow \mathcal{N}_i^1(g) \geq \lceil \mathcal{N}_i^0(g) \rceil + 2 \Rightarrow \mathcal{N}_i^1(g) \geq \lceil \kappa\mathcal{N}_i^0(g) \rceil + 2$. Consequently, if $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{De}) \geq \lceil \frac{n}{2} \rceil + 1$, then it is possible for De to decrease $C(\sharp E(g), \sharp\mathcal{I}_{De})$ by removing a link and obtain a winning network, a contradiction.

Second, note that an optimal strategy for De is a minimal winning strategy. We divide the proof into two parts.

First, we establish that if $(g^\star, \mathcal{I}_{De}^\star)$ is an optimal strategy, then $(g^\star, \mathcal{I}_{De}^\star)$ is a solution of Program (1). We know that an optimal strategy $(g^\star, \mathcal{I}_{De}^\star)$ for De has to satisfy the two necessary conditions given in Lemmas 3 and 4, i.e. (Cons. 1) and (Cons. 2) given in Proposition 1. Moreover, an optimal strategy has to minimize the cost incurred by De. The result follows.

Second, we show that if $(g^\star, \mathcal{I}_{De}^\star)$ is a solution of the Program (1), then it is an optimal strategy. Suppose that the solution of program given in Proposition (1) is not an optimal strategy for De.

This means that there exists a winning strategy $(g, \mathcal{I}_{De})$ less costly than $(g^\star, \mathcal{I}^\star_{De})$. Such a pair $(g, \mathcal{I}_{De})$ has to violate one of the two constraints. Pair $(g, \mathcal{I}_{De})$ is not a winning strategy since it violates one of Lemmas 3 and 4.

□

A.3 Optimal Strategy of De Under General Cost Function

Lemma 5 *Let*

$$\bar{x} = \underset{x \in [\![1, \sharp\mathcal{N}]\!]}{\arg\min} \left\{ x \left\lfloor \frac{x-1}{\kappa} \right\rfloor \geq n - x \right\}. \tag{16}$$

Then, $\bar{x} = \lceil \sqrt{\kappa n} \rceil$ *or* $\bar{x} = \lceil \sqrt{\kappa n} \rceil + 1$.

Proof Consider the real valued function $f : x \mapsto x \lfloor \frac{x-1}{\kappa} \rfloor - n + x$. We seek $\bar{x}$ (integer) being the minimal value such that $f(\bar{x}) \geq 0$. Let $w = \sqrt{\kappa n}$. Then $w \lfloor \frac{w-1}{\kappa} \rfloor = \sqrt{\kappa n} \left\lfloor \frac{\sqrt{\kappa n}-1}{\kappa} \right\rfloor \leq \sqrt{\kappa n} \left(\frac{\sqrt{\kappa n}}{\kappa} - 1 \right) = n - \sqrt{\kappa n} = n - w$ and thus $f(w) \leq 0$. Let $y = \sqrt{\kappa n} + 1$. Then $y \left\lfloor \frac{y-1}{\kappa} \right\rfloor = (\sqrt{\kappa n} + 1) \left\lfloor \frac{\sqrt{\kappa n}}{\kappa} \right\rfloor > (\sqrt{\kappa n} + 1) \left(\frac{\sqrt{\kappa n}}{\kappa} - 1 \right) = n + \frac{\sqrt{\kappa n}}{\kappa} - \sqrt{\kappa n} - 1 = n + \frac{\sqrt{\kappa n}}{\kappa} - y > n - y$. Thus $f(y) > 0$. Since f is strictly increasing any value $x \geq y$ satisfies $f(x) \geq 0$. Similarly, any value $x < w$ satisfies $f(x) < 0$. The conclusions follows from the fact that $\bar{x}$ is the smallest integer for which f is non negative.

□

To illustrate the fact that $\bar{x}$ can either take the value $\lceil \sqrt{\kappa n} \rceil$ or $\lceil \sqrt{\kappa n} \rceil + 1$, we construct the following example:

Example 11 Let $\kappa = 0.6$. If $n = 20$ then we have $3 \times \lfloor 2/0.6 \rfloor = 9 < 17$ and $4 \times \lfloor 3/0.6 \rfloor = 20 > 16$ thus $\bar{x} = 4 = \lceil \sqrt{0.6 \times 20} \rceil$. On the other hand, if $n = 15$, then $3 \times \lfloor 2/0.6 \rfloor = 9 < 12$ and $4 \times \lfloor 3/0.6 \rfloor = 20 > 11$ thus $\bar{x} = 4 = \lceil \sqrt{0.6 \times 15} \rceil + 1$.

Proof of Proposition 2 Again, note that in a winning strategy – and so in an optimal strategy – there exists at least one agent influenced by the Designer, i.e. $\mathcal{I}_{De} \neq \emptyset$. By Proposition 1 an optimal strategy has to satisfy (Cons. 1) and (Cons. 2). Since for every $i \in \mathcal{N}$, $k_{B1}(i, g) = \min\{k_{Ad}, \sharp\mathcal{N}_i(g) \setminus \mathcal{I}_{De}\}$ two cases can occur:

1. For every $i \in \mathcal{I}_{De}$, $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g)) \leq k_{Ad}$. Let $i \in \mathcal{I}_{De}$, from (Cons. 1) given in Proposition 1, $\sharp\mathcal{N}^1_i(g) \geq \kappa k_{B_1}$. Thus, $\mathcal{I}_{De} - 1 \geq \sharp\mathcal{N}^1_i(g) \geq \kappa \min\left\{\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g)), k_{Ad}\right\} = \kappa\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g))$. Hence, $\frac{\sharp\mathcal{I}_{De}-1}{\kappa} \geq \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g))$ and finally $\left\lfloor \frac{\sharp\mathcal{I}_{De}-1}{\kappa} \right\rfloor \geq \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g))$ since the right hand side is an integer. Summing over all agents of $\mathcal{I}_{De}$, we obtain that $\sharp\mathcal{I}_{De} \left\lfloor \frac{\sharp\mathcal{I}_{De}-1}{\kappa} \right\rfloor \geq \sum_{i \in \mathcal{I}_{De}} \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g))$. Now, by (Cons. 2) given in Proposition 1, $n - \sharp\mathcal{I}_{De} = \sum_{i \in \mathcal{I}_{De}} \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}^1_i(g))$ and thus necessarily $\sharp\mathcal{I}_{De} \left\lfloor \frac{\sharp\mathcal{I}_{De}-1}{\kappa} \right\rfloor \geq n - \sharp\mathcal{I}_{De}$. By definition $\bar{x}$ is the minimum integer that satisfies the above inequality. A winning strategy with $\sharp\mathcal{I}_{De} = \bar{x}$ agents exists : it is a $(\kappa, \mathcal{I}_{De})$-imcp strategy – see Appendix A.1.

2. There exists $i \in \mathcal{I}_{De}$ such that $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) > k_{Ad}$. Since $\sharp\mathcal{N}_i^1(g) \geq \kappa k_{B1} = \kappa k_{Ad}$, necessarily $\sharp\mathcal{I}_{De} \geq \kappa k_{Ad} + 1$.

   Consider a star-network with the centre being influenced by De as well as $\lceil \kappa k_{Ad} \rceil$ peripheral agents. It is a winning strategy satisfying the minimal number of influenced agents. It is possible to construct it if and only if $n \geq \kappa \lceil k_{Ad} \rceil + 1$.

□

Proof of Proposition 3 First, we deal with the number of links between agents in $\mathcal{I}_{De}$. Two cases can occur:

1. For every $i \in \mathcal{I}_{De}$, $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) \leq k_{Ad}$. Let $i \in \mathcal{I}_{De}$, from (Cons. 1) given in Proposition 1, $\sharp\mathcal{N}_i^1(g) \geq \kappa k_{B1}$. Thus, $\sharp\mathcal{N}_i^1(g) \geq \kappa \min\left\{\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)), k_{Ad}\right\} = \kappa\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$. By summing over all agents of $\mathcal{I}_{De}$, we have $\sum_{i\in\mathcal{I}_{De}} \sharp\mathcal{N}_i^1(g) \geq \kappa \sum_{i\in\mathcal{I}_{De}}(\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$. Note that $\sum_{i\in\mathcal{I}_{De}}(\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$ represents the total number of links joining agents in $\mathcal{I}_{De}$ to agents in $\mathcal{N} \setminus \mathcal{I}_{De}$. Thus by (Cons. 2) given in Proposition 1, $\sum_{i\in\mathcal{I}_{De}}(\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))) = n - \sharp\mathcal{I}_{De}$. Therefore the number of links in the network $g[\mathcal{I}_{De}]$ is at least $\left\lceil \frac{\kappa(n-\sharp\mathcal{I}_{De})}{2} \right\rceil$.
2. There exists $i \in \mathcal{I}_{De}$ such that $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) > k_{Ad}$. First, by Proposition 1 $\sharp\mathcal{N}_i^1(g) \leq \left\lceil \frac{n}{2} \right\rceil$. Second since $\sharp\mathcal{N}_i^1(g) \geq \kappa k_{B1} = \kappa k_{Ad}$, necessarily network $g[\mathcal{I}_{De}]$ contains at least $\lceil \kappa k_{Ad} \rceil$ links.

Now the number of links joining an agent in $\mathcal{I}_{De}$ to an agent in $\mathcal{N} \setminus \mathcal{I}_{De}$ is exactly $n - \sharp\mathcal{I}_{De}$ in an optimal strategy from (Cons. 2) given in Proposition 1 which concludes the proof. □

Proof of Theorem 1 Recall that an optimal strategy is a winning strategy where De cannot remove a link – without influencing more agents, or reducing the size of the set of agents she influences – without forming additional links. Consider a winning strategy such that $\sharp\mathcal{I}_{De} = n$. Then, the optimal strategy is the influenced empty network. When $\sharp\mathcal{I}_{De} < n$, two cases can occur:

1. For every $i \in \mathcal{I}_{De}$, $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) \leq k_{Ad}$. The optimal strategy is a $(\kappa, \mathcal{I}_{De})$-imcp strategy with $\sharp\mathcal{I}_{De} \geq \sharp{\mathcal{I}_{De}}^{\min}$. Indeed, it is a winning strategy since it satisfies Lemmas 3 and 4 and Proposition 2. This strategy is optimal since the number of links in the $(\kappa, \mathcal{I}_{De})$-imcp strategy satisfies the bound given in Proposition 3.

2. There exists $i \in \mathcal{I}_{De}$ such that $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) > k_{Ad}$. Then, from Lemma 3, for that agent i, $\sharp\mathcal{N}_i^1(g) \geq \lceil \kappa k_{B1} \rceil = \lceil \kappa k_{Ad} \rceil$. Thus, the number of links between agents in $\mathcal{I}_{De}$ is at least $\lceil \kappa k_{Ad} \rceil$. Further, from Lemma 4, there are at least $\sharp\mathcal{N} \setminus \mathcal{I}_{De}$ links between agents of $\mathcal{I}_{De}$ and agents of $\mathcal{N} \setminus \mathcal{I}_{De}$. Thus the optimal strategy is obtained when there are exactly $\lceil \kappa k_{Ad} \rceil$ links in $g[\mathcal{I}_{De}]$, all connected to agent i. This strategy is an $(\kappa, p, \mathcal{I}_{De})$-ips strategy, with $\sharp\mathcal{I}_{De} \geq \sharp{\mathcal{I}_{De}}^{\min}$ and $p \geq \lceil \kappa k_{Ad} \rceil$.

□

Appendix B. Possibility of Interaction between Non-linked Agents

Proof of Proposition 4 Let $\mathbb{P}^W(g)$ be the probability to obtain a winning network from g, and $\mathbb{P}_i^W(g)$ be the probability that agent i satisfies (Cons. 1 & 2) after some links have been formed by Nature. Note that $\mathbb{P}^W(g) = \prod_{i\in\mathcal{N}} \mathbb{P}_i^W(g)$ since every agent has to satisfy (Cons. 1 & 2) in a winning network.

First we provide a lower bound for the expected payoff associated with a network, say g^{mw}, which is (i) a winning network before Nature forms links, and (ii) minimal, i.e. it is not possible to obtain a network with a lower number of links which is a winning one. We have $\mathbb{P}^W(g^{mw}) = \prod_{i\in\mathcal{N}} \mathbb{P}_i^W(g^{mw}) \geq \prod_{i\in\mathcal{N}}(1-\varpi)^n = n(1-\varpi)^n \geq (1-\varpi)^{n^2} = ((1-\varpi)^n)^2 \geq (1-n\varpi)^2 \geq (1-\frac{c_L}{4})^2$. The first inequality follows the fact that if Nature does not form any links, then the realization of g^{mw} is a winning network. The third and the last inequalities follow the assumption that $\varpi \leq \frac{c_L}{4n} < \frac{1}{n}$. We conclude that $\mathbb{P}^W(g^{mw}) \geq 1 - 2\frac{c_L}{4} = 1 - \frac{c_L}{2}$.

We now establish that De has no incentive to build a non-minimal winnning network, g^w, instead of g^{mw}. The difference between the expected payoff of g^w and g^{mw} is bounded by: $1 - \sharp E(g^w)c_L - (1 - \frac{c_L}{2} - c_L\sharp E(g^{mw}))) = (\frac{1}{2} - (\sharp E(g^w) - \sharp E(g^{mw})))c_L < 0$ since $\sharp E(g^w) - \sharp E(g^{mw}) \geq 1$.

Finally, we establish that De has no incentive to build a network, say g^ℓ, which is non-winning – before Nature forms links. For every agent $i \in \mathcal{N}$ for which (Cons. 1 & 2) does not hold in g^ℓ, Nature has to form at least b_i links in order to obtain agent i satisfies (Cons. 1 & 2) in a winning network g^w. Let $M(\ell)$ be the minimal set of links that allows g^ℓ to be a winning network, i.e. there is no set of links with lower cardinality that allows to obtain a winning network; $m(\ell) \geq 1$ is the cardinality of $M(\ell)$. Similarly, let $\mathcal{S}(\ell)$ be the set of agents that are involved in links in $M(\ell)$. We have $\mathbb{P}^W(g^\ell) \leq \sum_{\ell=b_i}^{B} \binom{B}{\ell}\varpi^\ell$, with $i \in \mathcal{S}_\ell$ and where $B = \sharp\mathcal{I}_{De} - \sharp\mathcal{N}_i^1(g^\ell)$. Note that $\binom{B}{\ell}\varpi^\ell = B\varpi\frac{(B-1)\varpi}{2}\dots\frac{(B-\ell)\varpi}{\ell} \leq (B\varpi)^\ell$. Moreover, $\sum_{\ell=b_i}^{B}(B\varpi)^\ell = (B\varpi)^{b_i}\frac{(1-(B\varpi)^{B+1-b_i})}{1-B\varpi} \leq \frac{(B\varpi)^{b_i}}{1-B\varpi} \leq \frac{B\varpi}{1-B\varpi}$ since $b_i \geq 1$ and $B\varpi < 1$. Since $\varpi < \frac{c_L}{4n}$ and $B < n$, $\frac{B\varpi}{1-B\varpi} < \frac{B\frac{c_L}{4n}}{1-B\frac{c_L}{4n}} \leq \frac{nc_L}{4n-nc_L} < \frac{c_L}{2}$. We conclude that $\mathbb{P}^W(g^\ell) \leq \frac{c_L}{2}$.

We now compute the difference between the minimal expected payoff of De with g^{mw} and the maximal one with g^ℓ: $1 - \frac{c_L}{2} - \sharp E(g^{mw})c_L - \left(\frac{c_L}{2} - (\sharp E(g^{mw}) - m(\ell))c_L\right) > 1 - c_L - \left(\frac{n(n-1)}{2} - 1\right)c_L = 1 - \frac{n(n-1)}{2}c_L \geq 0$. The first inequality follows the fact that the maximal number of links in g^{mw} is $\frac{\lceil\sqrt{n}\rceil(\lceil\sqrt{n}\rceil-1)}{2} + n - \lceil\sqrt{n}\rceil$ and the fact that $n \geq 4$.

□

Appendix C. The Majority Case

Proof of Lemma 2 Let $\mathcal{I}n\mathcal{I}$ be the set of agents in $\mathcal{N} \setminus \mathcal{I}_{De}$ who are isolated in g, i.e. it is the set of agents who are non-influenced by De and isolated. First, observe that $\sharp\mathcal{I}n\mathcal{I} \leq n - \lceil n/2 \rceil$, otherwise the payoff of De is negative. Second, by Equations (12), if every agent $i \notin \mathcal{I}n\mathcal{I}$ votes 1, then De obtains a strictly positive payoff when $\sharp\mathcal{I}n\mathcal{I} \leq n - \lceil n/2 \rceil$. In the following, we establish that strategies of player De where $\sharp\mathcal{I}n\mathcal{I} \leq n - \lceil n/2 \rceil - 1$ are not optimal.

1. Strategies where $\sharp\mathcal{I}n\mathcal{I} \leq n - \lceil n/2 \rceil - 1$ and $\theta_i^{\text{Fi}} = 1$, for every $i \notin \mathcal{I}n\mathcal{I}$, are not optimal. Obviously, only such strategies that lead to $\sharp\mathcal{N}(1, \text{Fi}) > \lceil \frac{n}{2} \rceil$ could be optimal. We show that they are not. Indeed, $\sharp\mathcal{N}(1, \text{Fi}) > \lceil \frac{n}{2} \rceil$ can be obtained by De in two ways: either by not creating any links and influencing all the agents involved in $\mathcal{N} \setminus \mathcal{I}_{De}$ or by forming links and influencing some agents. In the first case, De can improve his payoff by reducing $\sharp\mathcal{I}_{De}$ while preserving inequality $\sharp\mathcal{N}(1, \text{Fi}) \geq \lceil \frac{n}{2} \rceil$. In the second case, there exists at least one agent, say i, with $\theta_i^{\text{Fi}} = 1$, such that $\theta_i^{\text{In}} = 0$. This means that i is linked to at least one agent in $\mathcal{I}_{De}$. De can improve his payoff by removing this link and maintaining inequality $\sharp\mathcal{N}(1, \text{Fi}) \geq \lceil \frac{n}{2} \rceil$, a contradiction.

2. Strategies where $\sharp\mathcal{I}n\mathcal{I} \leq n - \lceil n/2 \rceil - 1$ and there exists $i_0 \in \mathcal{N}$ such that $i_0 \notin \mathcal{I}n\mathcal{I}$ with $\theta_{i_0}^{\text{Fi}} = 0$, are not optimal. There are two possibilities depending on whether agent i_0 belongs to $\mathcal{I}_{De}$. (i) Assume that $i_0 \notin \mathcal{I}_{De}$. Costly link(s) in which i_0 is involved does (do) not benefit player De, a contradiction with the optimality of the strategy. (ii) Assume that i_0 has been influenced by De. There are two possibilities.

   (a) There exists $j \in \mathcal{N}_{i_0}(g)$ with $j \notin \mathcal{N}(1, \text{Fi})$. This means that the strategy of player De is not optimal since the link between j and i_0 does not contribute to the presence of agents in $\mathcal{N}(1, \text{Fi})$, a contradiction.

   (b) Every agent $j \in \mathcal{N}_{i_0}(g)$ belongs to $j \in \mathcal{N}(1, \text{Fi})$. Recall that in an optimal strategy all agents in $\mathcal{N} \setminus \mathcal{I}_{De}$ and in $\mathcal{N}(1, \text{Fi})$ are linked with a unique agent in $\mathcal{I}_{De}$. Consequently, Inequality (13) does not hold for j, a contradiction.

□

Appendix D. Ad Is the Stronger Influencer

In this section, we establish Propositions 7 and 8. First we begin with a lemma. Let $\mathcal{N}_i(g, \mathcal{I}_{De})$ be the set of neighbors of agent i who are influenced by De.

Lemma 6 *Suppose that payoff functions of players De and Ad are respectively given by Equations* (5) *and* (6) *and n is even.*

- *Suppose that* $(1-\alpha)k_{Ad} \leq \frac{1}{2}$. *If* $\sharp\mathcal{I}_{De} \geq \lceil 2\alpha k_{Ad} \rceil + 1$, *then*

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \begin{cases} \lceil 2\alpha k_{Ad} \rceil & \text{if } i \in \mathcal{I}_{De}, \\ 2k_{Ad} & \text{otherwise}. \end{cases}$$

  If $\sharp\mathcal{I}_{De} < \lceil 2\alpha k_{Ad} \rceil + 1$, *then* $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \emptyset$.

- *Suppose that* $\frac{1}{2} < (1-\alpha)k_{Ad} \leq \alpha$. *If* $\sharp\mathcal{I}_{De} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, *then*

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \begin{cases} \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil & \text{if } i \in \mathcal{I}_{De}, \\ 2k_{Ad} & \text{otherwise}. \end{cases}$$

  If $\sharp\mathcal{I}_{De} < \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, *then* $\mathcal{N}_i(g,\mathcal{I}_{De}) = \emptyset$.

- *Suppose that* $(1-\alpha)k_{Ad} > \alpha$. *If* $\sharp\mathcal{I}_{De} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, *then*

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil, \text{ for every } i \in \mathcal{N}.$$

  If $\sharp\mathcal{I}_{De} < \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, *then* $\mathcal{N}_i(g,\mathcal{I}_{De}) = \emptyset$.

Proof Note that if g is non-empty and $\mathcal{I}_{De} \neq \emptyset$, then for every $i \in \mathcal{N}$, $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) > k_{Ad}$, otherwise De obtains a strictly negative payoff. Ad has two possibilities concerning the agents he influences when he wants to obtain $\theta_i^{\text{Fi}} = 0$: either he influences only the neighbors of agent i, or he influences both agent i and his neighbors.

First we deal with the case where Ad wants to obtain $\theta_i^{\text{Fi}} = 0$ for $i \in \mathcal{I}_{De}$.

1. When Ad influences only the neighbors of agent i, De has to ensure that the following inequality holds:

$$1-\alpha+\alpha\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})-k_{Ad}}{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})} \geq 1/2. \tag{17}$$

   It follows that $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2\alpha k_{Ad}$.

2. When Ad influences agent i and his neighbors, De has to ensure that the following inequality holds:

$$\alpha\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})-k_{Ad}+1}{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})} \geq 1/2, \tag{18}$$

   that is $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \frac{\alpha(k_{Ad}-1)}{\alpha-1/2}$.

Consequently, $\theta_i^{\text{Fi}} = 1 \Leftrightarrow \sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \max\{\frac{\alpha(k_{Ad}-1)}{\alpha-1/2}, 2\alpha k_{Ad}\}$. Let us compare $\frac{\alpha(k_{Ad}-1)}{\alpha-1/2}$ and $2\alpha k_{Ad}$. We obtain $k_{Ad} \geq \frac{1}{2(1-\alpha)}$ if and only if $\frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \geq 2\alpha k_{Ad}$. We get $\theta_i^{\text{Fi}} = 1$ if and only if

$$\left[k_{Ad} \geq \frac{1}{2(1-\alpha)} \text{ and } \sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \frac{\alpha(k_{Ad}-1)}{\alpha-1/2}\right] \text{ or } \left[k_{Ad} \leq \frac{1}{2(1-\alpha)} \text{ and } \sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2\alpha k_{Ad}\right]$$

Second we deal with the case where Ad wants to obtain $\theta_i^{\text{Fi}} = 0$ for $i \notin \mathcal{I}_{De}$. When Ad influences only the neighbors of agent i, De has to ensure that the following inequality holds:

$$\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})-k_{Ad}}{\sharp\mathcal{N}_i(g,\mathcal{I}_{De})} \geq 1/2, \tag{19}$$

that is $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2k_{Ad}$. When Ad influences both i and his neighbors inequality (18) holds. Let us compare $\frac{\alpha(k_{Ad}-1)}{\alpha-1/2}$ and $2k_{Ad}$. We obtain $k_{Ad} \geq \frac{\alpha}{1-\alpha}$ if and only if $\frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \geq 2k_{Ad}$. We get $\theta_i^{\text{Fi}} = 1$ if and only if

$$\left[k_{Ad} \geq \frac{\alpha}{1-\alpha} \text{ and } \sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \frac{\alpha(k_{Ad}-1)}{\alpha-1/2}\right] \text{ or } \left[k_{Ad} \leq \frac{\alpha}{1-\alpha} \text{ and } \mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2k_{Ad}\right].$$

Because $\alpha \in (1/2, 1]$, $\frac{1}{2(1-\alpha)} \leq \frac{\alpha}{1-\alpha}$. Consequently, we have to examine three intervals for completing the analysis.

- Suppose $(1-\alpha)k_{Ad} \leq \frac{1}{2}$. Then necessarily inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2\alpha k_{Ad}$ holds when Ad influences $i \in \mathcal{I}_{De}$ and inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2k_{Ad}$ holds when Ad influences $i \notin \mathcal{I}_{De}$. De has to choose the lowest number of neighbors of i which satisfies the previous inequalities:

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \begin{cases} \lceil 2\alpha k_{Ad} \rceil & \text{if } i \in \mathcal{I}_{De}, \\ 2k_{Ad} & \text{otherwise.} \end{cases} \tag{20}$$

  Note that when $(1-\alpha)k_{Ad} \leq \frac{1}{2}$, $\lceil 2\alpha k_{Ad} \rceil + 1 \geq 2k_{Ad}$. If $\sharp\mathcal{I}_{De} \geq \lceil 2\alpha k_{Ad} \rceil + 1$, then (20) holds. Otherwise, De cannot satisfy necessary condition: $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \lceil 2\alpha k_{Ad} \rceil$ for $i \in \mathcal{I}_{De}$.

- Suppose $\frac{1}{2} \leq (1-\alpha)k_{Ad} \leq \alpha$. Then necessarily inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \frac{\alpha(k_{Ad}-1)}{\alpha-1/2}$ holds when Ad influences $i \in \mathcal{I}_{De}$ and $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq 2k_{Ad}$ holds when Ad influences $i \notin \mathcal{I}_{De}$. De has to choose the lowest number of neighbors of i which satisfies the previous inequalities:

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \begin{cases} \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil & \text{if } i \in \mathcal{I}_{De}, \\ 2k_{Ad} & \text{otherwise.} \end{cases} \tag{21}$$

  Note that when $(1-\alpha)k_{Ad} \geq \frac{1}{2}$, $\left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1 \geq 2k_{Ad}$. If $\sharp\mathcal{I}_{De} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, then (21) holds. Otherwise, De cannot satisfy necessary condition: $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil$ for $i \in \mathcal{I}_{De}$.

- Suppose $(1-\alpha)k_{Ad} > \alpha$. Then necessarily inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil$ holds when Ad influences $i \in \mathcal{I}_{De}$ or when Ad influences $i \notin \mathcal{I}_{De}$. De has to choose the lowest number of neighbors which satisfies the previous inequality:

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil. \tag{22}$$

  If $\sharp\mathcal{I}_{De} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$, then (22) holds. Otherwise, De cannot satisfy $\sharp\mathcal{N}_i(g,\mathcal{I}_{De}) = \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil$ for $i \in \mathcal{I}_{De}$.

□

Proof of Proposition 7 The proof is straightforward from Lemma 6. Indeed for every value of k_{Ad}, Lemma 6 provides a necessary condition for the value of $\sharp\mathcal{I}_{De}{}^{\min}$:

- if $(1-\alpha)k_{Ad} \leq \frac{1}{2}$, then there is winning strategy if $\sharp\mathcal{I}_{De}{}^{\min} \geq \lceil 2\alpha k_{Ad} \rceil + 1$,
- if $\frac{1}{2} < (1-\alpha)k_{Ad} \leq \alpha$, then $\sharp\mathcal{I}_{De}{}^{\min} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$,
- If $(1-\alpha)k_{Ad} > \alpha$, then $\sharp\mathcal{I}_{De}{}^{\min} \geq \left\lceil \frac{\alpha(k_{Ad}-1)}{\alpha-1/2} \right\rceil + 1$.

□

Proof of Proposition 8 Due to Proposition 7, we know the minimal size of $\sharp\mathcal{I}_{De}$ for a winning strategy, $\sharp\mathcal{I}_{De}{}^{\min}$. That is in a winning strategy $\sharp\mathcal{I}_{De} \in [\![\sharp\mathcal{I}_{De}{}^{\min}, n]\!]$. Moreover, by Lemma 6, we know conditions that a winning strategy has to satisfy. Clearly, when $(1-\alpha)k_{Ad} \leq \frac{1}{2}$, $(\lceil 2\alpha k_{Ad} \rceil, 2k_{Ad}, \sharp\mathcal{I}_{De})$-groups-regular strategies and $(\lceil 2\alpha k_{Ad} \rceil, \mathcal{N})$-regular strategy, with $\sharp\mathcal{I}_{De} \in [\![\sharp\mathcal{I}_{De}{}^{\min}, n-1]\!]$, allow to satisfy conditions given in Lemma 6 and minimize the number of links given the size of $\mathcal{I}_{De}$. Similarly, by using Lemma 6, we obtain the two other parts of the proposition.

□