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## ► To cite this version:

George Krait, Sylvain Lazard, Guillaume Moroz, Marc Pouget. Certified numerical algorithm for isolating the singularities of the plane projection of generic smooth space curves. *Journal of Computational and Applied Mathematics*, 2021, 394, pp.113553. 10.1016/j.cam.2021.113553 . hal-03161393

**HAL Id: hal-03161393**

<https://inria.hal.science/hal-03161393v1>

Submitted on 6 Mar 2021

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# Certified numerical algorithm for isolating the singularities of the plane projection of generic smooth space curves

George Krait<sup>1</sup>, Sylvain Lazard<sup>1</sup>, Guillaume Moroz<sup>1</sup>, and Marc Pouget<sup>1</sup>

*Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy*

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## Abstract

Isolating the singularities of a plane curve is the first step towards computing its topology. For this, numerical methods are efficient but not certified in general. We are interested in developing certified numerical algorithms for isolating the singularities. In order to do so, we restrict our attention to the special case of plane curves that are projections of smooth curves in higher dimensions. This type of curve appears naturally in robotics applications and scientific visualization. In this setting, we show that the singularities can be encoded by a regular square system whose solutions can be isolated with certified numerical methods. Our analysis is conditioned by assumptions that we prove to be generic using transversality theory. We also provide a semi-algorithm to check their validity. Finally, we present experiments, some of which are not reachable by other methods, and discuss the efficiency of our method.

*Keywords:* Transversality, Generic Singularities, Certified Numerical Algorithms, Interval Analysis, Singular Curve Topology

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## 1. Introduction

2 The problem of computing the topology of a real plane curve consists of computing a piecewise-linear plane  
3 graph that can be deformed continuously into that curve. Such a problem is critical for drawing plane curves with  
4 the correct topology. One of the main challenges is to isolate the singular points efficiently and correctly. The aim  
5 of this paper is to do so with certified numerical methods and we show that this could be achieved for the specific  
6 class of plane curves that are projections of  $C^\infty$  smooth curves in higher dimension.

7 Although this class of curves seems specific, it appears naturally in visualization and robotic applications and  
8 the curves from this class often contain singularities. When visualizing a curve given by  $n - 1$  implicit equations  
9 in  $n$  dimensions for instance, we compute its projection in 2D to display it on a screen. This class of curves also  
10 appears in robotic applications. For instance given a robot with two degrees of freedom that moves in the plane,  
11 the set of points it can reach is bounded by a curve. In this case, computing the correct topology of this curve is  
12 often needed for deciding if a specific position is reachable. This curve is usually the projection of a smooth curve  
13 embedded in a space of higher dimension, and it often contains singular points.

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*Email address:* 1 (Firstname.Name@inria.fr)

14 By certified algorithms, we refer to algorithms that always output mathematically correct results in a given  
15 model of computation; for instance, randomized Las-Vegas algorithms are (usually) certified, but randomized  
16 Monte-Carlo algorithms are not; numerical methods that may miss solutions or output spurious solutions are not  
17 certified. We consider in this paper the RAM model of computation. Recall that the singular points of a plane  
18 curve, defined by the equation  $f(x, y) = 0$ , are the solutions of the system defined by  $f(x, y) = \frac{\partial f}{\partial x}(x, y) =$   
19  $\frac{\partial f}{\partial y}(x, y) = 0$ ; it should be stressed that this system is over-determined, i.e., it has more equations than variables,  
20 which prevents the use of certified numerical methods such as interval Newton methods [MKC09]. On the other  
21 hand, symbolic methods can solve such over-determined systems but they are restricted to algebraic systems and  
22 their complexity is high with respect to the degree of the equations.

23 *Main contributions.* In this paper, we present a square and regular system that encodes the singularities of the  
24 plane projection of a  $C^\infty$  smooth curve in  $\mathbb{R}^n$  (Theorems 11 & 27). Our approach does not use elimination  
25 theory to compute the equation of the projected curve and it is not restricted to the algebraic case: it applies to the  
26 larger class of  $C^\infty$  smooth curves. Being square and regular, this system can thus be solved with state-of-the-art  
27 certified numerical methods based on interval arithmetic or certified homotopy tracking. However it encodes the  
28 singularities of the plane projection only if some assumptions, defined in Section 2.4, are satisfied. Our second main  
29 result is that those assumptions are satisfied generically, which we prove using transversality theory (Section 7).  
30 We also present Semi-algorithm 4 that checks whether a given curve satisfies our assumptions, that is, an algorithm  
31 that stops if and only if the assumptions are satisfied. The combination of these results provides a method that is  
32 both numerical and certified for isolating the singularities of the plane projection of a generic curve. Finally,  
33 we present several experiments and discuss the efficiency of our algorithm in Section 6. Our contribution is a  
34 generalization of [IMP16b] that only considers the 3-dimensional case and is in the same spirit as the work of  
35 Delanoue et al. [DL14].

36 We also address the case of curves that are the silhouettes of smooth surfaces in  $\mathbb{R}^n$  (the silhouette being the  
37 set of points on the surface where the tangent plane projects on the plane of projection in a line or a point). Such  
38 curves naturally appear in parametric systems since they partition the parametric space with respect to the number  
39 of solutions of the system. For such curves, we were only able to prove some partial results on their genericity  
40 (see Section 2.4, Proposition 61 and Conjecture 62), but our other main results hold (Theorems 11 & 27 and our  
41 semi-algorithms).

42 *State of the art.* The problem of isolating the singularities of a plane curve is a special case of the problem of  
43 isolating the solutions of a zero-dimensional system in  $\mathbb{R}^2$ . We give a concise summary of the state of the art of  
44 certified methods for these two problems, organized in two main classes.

45 *Symbolic methods.* Symbolic methods are widely used for solving in a certified way zero-dimensional alge-  
46 braic systems. Classical such methods are based on Gröbner bases, resultant theory and univariate representations  
47 (see e.g., [CLO92, BPR06]). In this context, methods dedicated to the bivariate case have also been designed (see

[Hon96, GVK96, BLM<sup>+</sup>16, vdHL18] and references therein). Compared to numerical methods, these methods are adapted to over-constrained or non-regular systems. On the other hand, they suffer several drawbacks. They are not adaptive in the sense that solving in a small region is not easier than solving for all solutions. They are limited to algebraic systems and their complexity is high with respect to the degree of the system.

*Certified numerical methods.* When a zero-dimensional system is regular (Definition 1), its solutions can be isolated in a certified way using interval-arithmetic subdivision methods [Neu91, MKC09] or homotopy approaches with certified path tracking (see [BL13] and references therein). However, these methods do not directly work for isolating the singularities of a plane curve given by the equation  $f(x, y) = 0$  because the system  $f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$  that encodes the singularities is neither square nor regular. On the other hand, the curve may not be given by its implicit equation and computing this representation may not be required nor desirable. There are only few contributions designing certified numerical algorithms, even with additional restrictions on the curve and its singularities. When the curve is algebraic, Burr et al. [BCGY12] use separation bounds to isolate the singularities via a subdivision algorithm but the worst-case values of such bounds make it inefficient in practice. Lien et al. [LSVY14, LSVY20] study the special case of a singular curve that is a union of non-singular ones such that the singularities are only transverse intersections between them. They propose a subdivision algorithm using the Moore-Kioustelidis interval test for isolating the square and regular system defined by two curves. No implementation is available but such a subdivision scheme in two dimensions is expected to be efficient. In the case where the plane curve is defined as a projection, only two contributions present certified numerical approaches for isolating the singularities: Delanoue and Lagrange [DL14] consider the apparent contours of smooth surfaces in  $\mathbb{R}^4$  and Imbach et al. [IMP16b] handle the plane projections of smooth curves in  $\mathbb{R}^3$  using a subdivision scheme locally in four dimensions. Even though subdivision approaches may suffer in practice from the curse of dimensionality, Imbach et al. observe experimentally that, for algebraic curves, their approach is more efficient than computing the implicit equation of the projected plane curves and its singularities using symbolic methods.

The rest of the paper is organized as follows: In Section 2, we introduce notation and the assumptions we consider in our approach. In Section 3, we introduce the so-called *Ball system* that characterizes the singularities of the plane projection and we prove, in Section 4, that it is regular at its solutions. In Section 5, we provide a semi-algorithm to check the assumptions introduced in Section 2. Experiments are presented in Section 6. Finally, in Section 7, we prove the genericity of our assumptions, with a focus on the case of silhouette curves in Section 7.3.

## 2. Notation and assumptions

The main technical notation is summarized in Table 3 at the end of the paper. For a positive integer  $n$ , a closed (resp. open)  $n$ -box is the Cartesian product of  $n$  closed (resp. open) intervals. Assume that  $n \geq 3$  and let  $B$  be an open  $n$ -box and  $\bar{B}$  be the topological closure of  $B$  with respect to the usual topology in  $\mathbb{R}^n$ . Let  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  denote the set of smooth functions (i.e., differentiable infinitely many times) from  $\mathbb{R}^n$  to  $\mathbb{R}^{n-1}$ . Consider the function  $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ . We denote by  $\mathfrak{C}$  (resp.  $\bar{\mathfrak{C}}$ ) the solution set of the

82 system  $\{P_1(x) = \dots = P_{n-1}(x) = 0\}$ , with  $x = (x_1, \dots, x_n) \in B$  (resp. with  $x \in \overline{B}$ ). Also, consider the  
83 projection  $\pi_{\mathfrak{C}}$  (resp.  $\pi_{\overline{\mathfrak{C}}}$ ) from  $\mathfrak{C}$  (resp.  $\overline{\mathfrak{C}}$ ) to the  $(x_1, x_2)$ -plane. Unless otherwise stated, the plane projection of  
84 a point  $x \in \mathbb{R}^n$  is  $(x_1, x_2)$ . Our main goal is to compute the cusp points and nodes of  $\pi_{\mathfrak{C}}$ . If  $\overline{\mathfrak{C}}$  is a smooth curve  
85 (see the definition below), define  $\mathfrak{L}_c$  (resp.  $\overline{\mathfrak{L}}_c$ ) as the set of points  $q$  in  $\mathfrak{C}$  (resp.  $\overline{\mathfrak{C}}$ ) where the tangent line, denoted  
86 by  $T_q\mathfrak{C}$ , (resp.  $T_q\overline{\mathfrak{C}}$ ) is orthogonal to the  $(x_1, x_2)$ -plane. We also define the set  $\mathfrak{L}_n$  (resp.  $\overline{\mathfrak{L}}_n$ ) to be the set of points  
87  $q$  in  $\mathfrak{C}$  (resp.  $\overline{\mathfrak{C}}$ ) such that the cardinality of the pre-image of  $\pi_{\mathfrak{C}}(q)$  under  $\pi_{\mathfrak{C}}$  (resp.  $\pi_{\overline{\mathfrak{C}}}$ ) is at least two without  
88 counting multiplicities. We will see later that, under some generic assumption,  $\mathfrak{L}_c$  (resp.  $\mathfrak{L}_n$ ) is the set of points in  
89  $\mathfrak{C}$  that project to cusps (resp. nodes), which explains the subscript c (resp. n).

### 90 2.1. Regular and singular points

91 Let  $m \geq 1$  be an integer,  $V$  be a subset of  $\mathbb{R}^m$  and  $p \in V$ . We call  $p$  a regular (or smooth) point of  $V$  if  $V$  is  
92 a sub-manifold at  $p$ , that is, there exist a neighborhood  $W$  of  $p$  in  $\mathbb{R}^m$ , an integer  $k > 0$  and  $k$  smooth functions  
93  $\varphi_1, \dots, \varphi_k$  defined over  $W$ , such that  $V \cap W$  is the set of solutions of  $\{\varphi_1(x) = \dots = \varphi_k(x) = 0\}$  in  $W$  and  
94 the rank of the matrix  $\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \dots & \frac{\partial \varphi_1}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \dots & \frac{\partial \varphi_k}{\partial x_m} \end{pmatrix}$  evaluated at  $q$  is  $k$  [Dem00, Definition 2.2.2]. We call this matrix the  
95 Jacobian matrix of the system  $\{\varphi_1(x) = \dots = \varphi_k(x) = 0\}$  and we denote it by  $J_{(\varphi_1, \dots, \varphi_k)}$ . If  $q$  is not a regular  
96 point of  $V$ , we call it a singular point. If all points in  $V$  are regular, then  $V$  is called regular or smooth. Otherwise,  
97  $V$  is called singular.

98 For  $\varphi = (\varphi_1, \dots, \varphi_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ , we denote by  $T_q\varphi$  its derivative (also known as the tangent map) at the  
99 point  $q$ . Note that the Jacobian matrix  $J_\varphi = J_{(\varphi_1, \dots, \varphi_k)}$  is the expression of the derivative in the canonical bases  
100 of  $\mathbb{R}^n$  and  $\mathbb{R}^k$ .

101 **Definition 1.** *let  $F = (f_1, \dots, f_n)$  be a vector of smooth real-valued functions that are defined in  $\mathbb{R}^n$  and let*  
102  *$a \in \mathbb{R}^n$  be a solution of the system  $\{F = 0\}$ . We say that the latter system is regular at  $a \in \mathbb{R}^n$  if the determinant*  
103 *of its Jacobian matrix, evaluated at  $a$ , does not vanish. We call  $\{F = 0\}$  regular if it is regular at all of its*  
104 *solutions.*

### 105 2.2. Multiplicity in zero-dimensional systems

106 We recall the definition of multiplicity in the univariate case before generalizing it to higher dimensions.

107 **Definition 2.** *Let  $f$  be a real smooth function at  $a \in \mathbb{R}$ . The multiplicity of  $f$  at  $a$  is the integer  $\text{mult}_a(f(x)) =$   
108  $\min\{k \in \mathbb{N} \mid \frac{\partial^k f}{\partial x^k}(a) \neq 0\}$  if it exists, otherwise  $\text{mult}_a(f(x)) = \infty$ . For the case  $a = 0$ , we write  $\text{mult}(f) =$   
109  $\text{mult}_a(f)$  for simplicity.*

110 **Definition 3** ([CLO05, Definition 4.2.1]). *For integers  $m \geq n \geq 1$ , let  $G = (g_1(x), \dots, g_m(x))$  be a polynomial*  
111 *function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  and  $q$  be a solution of the system  $\{G = 0\}$ . Let  $\mathbb{R}[x]$  be the ring of polynomials with  $n$*   
112 *variables and define  $\mathbb{R}[x]_q = \{\frac{h_1}{h_2} \mid h_1, h_2 \in \mathbb{R}[x], h_2(q) \neq 0\}$  to be the localization of  $\mathbb{R}[x]$  at  $q$ . Define the*

113 intersection multiplicity of  $q$  in the system  $\{G = 0\}$  (or equivalently the multiplicity of the system  $\{G = 0\}$  at  $q$ ) to  
 114 be the dimension of the real vector space  $\frac{\mathbb{R}[x]_q}{I_G}$ , where  $I_G$  is the ideal generated by the set  $\{\frac{g_1}{1}, \dots, \frac{g_m}{1}\}$  in  $\mathbb{R}[x]_q$ .

115 The previous definition is classical for the algebraic case. However, in our paper, we are interested in curves  
 116 defined as the zero locus of smooth functions. For this goal, we consider a more general definition for a system  
 117  $S = \{f_1(x) = \dots = f_m(x) = 0\}$  with  $f_i \in C^\infty(\mathbb{R}^n, \mathbb{R})$ . Let  $a$  be a solution of  $S$  and  $k$  be a non-negative  
 118 integer, we define the dual space of rank  $k$ , denoted by  $D_a^k[S]$ , to be the vector space of all linear combinations  $c$   
 119 of differential functionals  $\frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$  with  $k_1 + \dots + k_n \leq k$  such that:

- 120 (a)  $D_a^0[S] = \text{span}(\{\frac{\partial^0}{\partial x_1^0 \dots \partial x_n^0}\})$ ,
- 121 (b)  $c$  in  $D_a^k[S]$  applied to  $f_i$ , evaluated at  $a$  is zero for all integers  $1 \leq i \leq m$ , and
- 122 (c) for all  $i \in \{1, \dots, n\}$ , the anti-differentiation transformation  $\phi_j$  applied to  $c$  in  $D_a^k[S]$  is in  $D_a^{k-1}[S]$ .

123 The anti-differentiation transformation  $\phi_j$  is the linear operator mapping the order  $h$  differential functional  
 124  $\frac{\partial^h}{\partial x_1^{h_1} \dots \partial x_j^{h_j} \dots \partial x_n^{h_n}}$  to the order  $(h-1)$  differential functional  $\frac{\partial^{h-1}}{\partial x_1^{h_1} \dots \partial x_j^{h_j-1} \dots \partial x_n^{h_n}}$  if  $h_j > 0$  or to the order 0  
 125 differential functional  $\frac{\partial^0}{x_j^0}$  otherwise, where  $h = \sum_{i=1}^n h_i$ .

126 **Definition 4** ([DLZ11, Definition 1]). Let  $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$  such that  $F^{-1}(0)$  is a finite set and let  $a \in \mathbb{R}^n$   
 127 be a solution of the system  $S = \{F = 0\}$ . Consider the ascending chain of dual spaces  $D_a^0[F] \subseteq D_a^1[F] \subseteq$   
 128  $\dots \subseteq D_a^h[F] \subseteq \dots$ . If there exists an integer  $\alpha$  such that  $D_a^\alpha[F] = D_a^{\alpha+1}[F]$ , then the dimension of the vector space  
 129  $D_a^\alpha[F]$  is called the multiplicity of  $a$  in the system  $S$ . If such an  $\alpha$  does not exist, the multiplicity is, by convention,  
 130 infinite.

131 For polynomial systems, the two definitions are equivalent [DLZ11, Theorem 2] and in addition the following  
 132 proposition shows that algebraic tools can be used in the smooth case.

133 **Proposition 5** ([DLZ11, Corollary 3]). For an integer  $k \geq n$ , let  $F = (f_1, \dots, f_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$  and let  
 134  $a \in \mathbb{R}^n$  be a solution of the system  $\{F = 0\}$ . Suppose that the multiplicity of  $a$  in  $\{F = 0\}$  is  $m < \infty$ , then the  
 135 intersection multiplicity at  $a$  of the polynomial system  $\{G = (g_1, \dots, g_k) = 0\}$  is also  $m$ , where  $g_i$  is equal to the  
 136 Taylor expansion of  $f_i$  at  $a$  up to degree at least  $m$ .

### 137 2.3. Singularities of plane curves, nodes and ordinary cusps

138 **Definition 6** ([AGZV12, §17.1]). For  $i \in \{1, 2\}$ , let  $C_i$  be a plane curve defined in a neighborhood  $U_i \subset \mathbb{R}^2$  of  
 139  $p_i$  by the 0-set of a smooth function  $f_i$ . The pairs  $(p_1, C_1)$  and  $(p_2, C_2)$  are equivalent, and thus define the same  
 140 plane curve singularity, if there exists a diffeomorphism  $\varphi$  from  $U_1$  to  $U_2$  such that  $f_1 = f_2 \circ \varphi$  and  $\varphi(p_1) = p_2$ .

141 In particular, a singularity is of type  $A_k$  if the curve is locally defined at the origin by the 0-set of the function  
 142  $x^2 - y^{k+1}$ . As important special cases,  $A_1$  is called a node singularity and  $A_2$  is called an ordinary cusp singularity,  
 143 see Figure 1.



Figure 1: Left: At an  $A_1$  singularity, two branches of the curve intersect transversally. Right: At an  $A_{2k+1}$  singularity with  $k > 1$ , the tangent lines of the two branches at the intersection point coincide.

144 **Remark 7.** *It is worthwhile to notice that a curve  $C$  is an ordinary cusp at a point  $p$  if  $C$  can be locally parame-*  
 145 *terized with  $(z^2, z^3)$  and  $p$  corresponds to the value  $z = 0$ . This remark is helpful to characterize ordinary cusps*  
 146 *in Section 3.*

#### 147 2.4. Assumptions

148 Our goal is to design a numerical algorithm for isolating the singularities that appear in the plane projection  
 149 of a curve  $\mathcal{C}$  in  $\mathbb{R}^n$ . Numerical algorithms usually cannot handle degenerate cases, that is, singularities in our  
 150 context. However, under some assumptions on  $\mathcal{C}$ , we succeed to isolate in a certified way some singularities of the  
 151 projection. Namely, we require that the singularities are “generic”, that is, only nodes can appear in the projection  
 152 (Assumption  $\mathcal{A}_5$ ). Our other assumptions on  $\mathcal{C}$  are, roughly speaking, that it is smooth ( $\mathcal{A}_1$ ), that its projection  
 153 only has a discrete set of singularities ( $\mathcal{A}_4$ ), and that at most two points of  $\mathcal{C}$  project on each singularity ( $\mathcal{A}_3$ );  
 154 see Figure 2. We will prove in Section 5 that our numerical algorithm is certified and terminates under these  
 155 assumptions and in Section 7 that these assumptions are generically satisfied.

156  $\mathcal{A}_1$  – For all  $q \in \overline{\mathcal{C}}$ ,  $\text{rank}(J_P(q)) = n - 1$ . In particular,  $\overline{\mathcal{C}}$  is a smooth curve.<sup>1</sup>

157  $\mathcal{A}_2$  – The set  $\overline{\Sigma_c}$  is discrete and does not intersect the boundary of  $B$ .

158  $\mathcal{A}_3$  – For all points  $p = (\alpha, \beta) \in \pi_{\overline{\mathcal{C}}}(\overline{\mathcal{C}})$ , the pre-image of  $p$  under  $\pi_{\overline{\mathcal{C}}}$  consists of at most two points in  $\overline{B}$  counted  
 159 with multiplicities in the system  $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ .

160  $\mathcal{A}_4$  – The set  $\overline{\Sigma_n}$  is discrete and does not intersect the boundary of  $B$ .

161  $\mathcal{A}_5$  – The singular points of  $\pi_{\mathcal{C}}(\mathcal{C})$  are only nodes (see Definition 6).

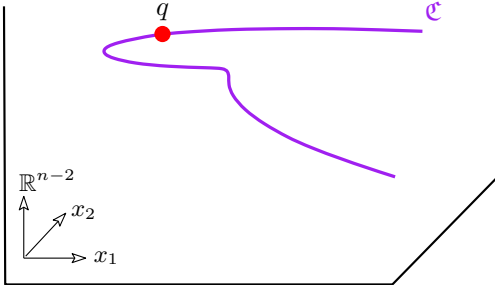
162 We also consider a weaker version of Assumption  $\mathcal{A}_5$  in which ordinary cusps can also appear in the projection:

163  $\mathcal{A}_5^-$  – The singular points of  $\pi_{\mathcal{C}}(\mathcal{C})$  are only ordinary cusps or nodes (see Definition 6).

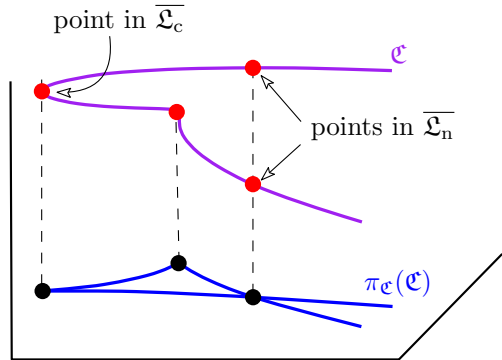
164 **Definition 8.** *Assumptions  $\mathcal{A}_{1-5}$  are called the strong assumptions and Assumptions  $\mathcal{A}_{1-4}$  and  $\mathcal{A}_5^-$  are called the*  
 165 *weak assumptions.*

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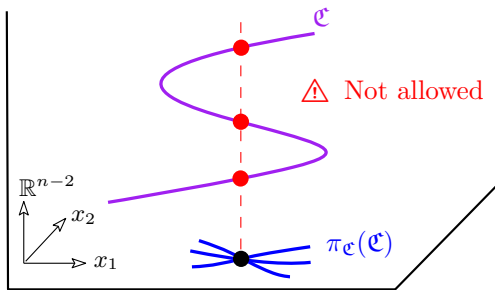
<sup>1</sup>Note that the converse is not true as the vertical (double) line defined by  $x_1^2 = x_2 = 0$  in  $\mathbb{R}^3$  is smooth but the rank of its Jacobian is never full.



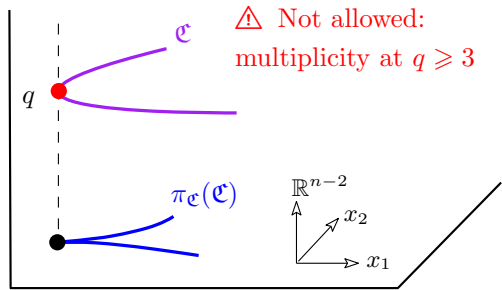
(a) Assumption  $\mathcal{A}_1$ :  $\bar{\mathcal{C}}$  is a smooth; the rank of the Jacobian of  $P$  at  $q$  is full.



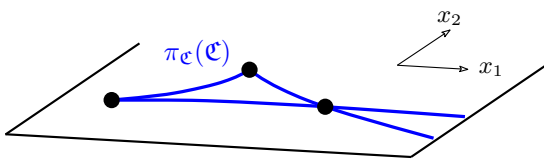
(b) Assumptions  $\mathcal{A}_2$  and  $\mathcal{A}_4$ : the sets  $\bar{\mathcal{L}}_c$  and  $\bar{\mathcal{L}}_n$  are finite and do not intersect the boundary of  $B$ .



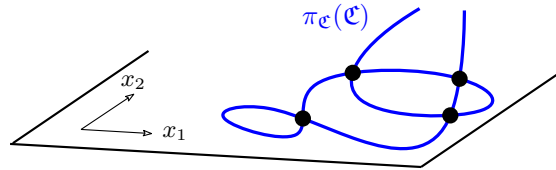
(c) Assumption  $\mathcal{A}_3$ : No three points (counted with multiplicity) have the same projection.



(d) Assumption  $\mathcal{A}_3$ : No three points (counted with multiplicity) have the same projection.



(e) Assumption  $\mathcal{A}_5^-$ : points in  $\pi_{\mathcal{C}}(\mathcal{C})$  are either smooth, nodes or ordinary cusps.



(f) Assumption  $\mathcal{A}_5$ : points in  $\pi_{\mathcal{C}}(\mathcal{C})$  are only smooth or nodes.

Figure 2: Illustration of the assumptions.



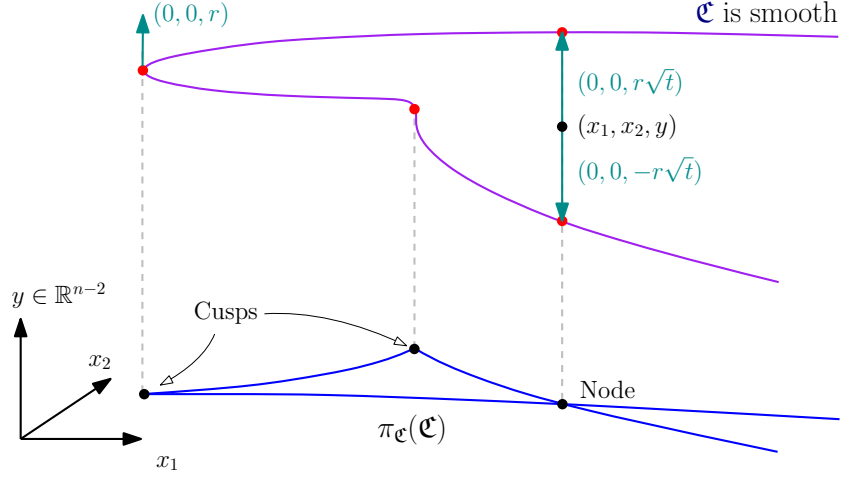


Figure 3: Illustration of a node and cusps in the plane projection of a smooth curve.

166 Our motivation for also considering these weak assumptions is dual. First, our certified algorithm for isolating  
 167 the singularities of the projection of curves satisfying the strong assumptions also works, to some extent, if only  
 168 the weak assumptions hold: namely, it outputs a *superset* of the isolating boxes of the singularities. Second, we  
 169 conjecture that our weak assumptions are satisfied by silhouette curves of generic surfaces (see Proposition 61 and  
 170 Conjecture 62).

### 171 3. Modelling system

172 Our goal in this section is, under the weak assumptions, to encode the singularities of the projected curve  
 173  $\pi_{\mathcal{C}}(\mathcal{C})$  by a square and regular (see Definition 1) system so that it is solvable with certified numerical methods. In  
 174 Section 3.1, we first define this system  $\text{Ball}(P)$  and state the first main result of this section, Theorem 11, showing  
 175 that the Ball system exactly encodes the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$ . In Sections 3.2 and 3.3, we study the singularities  
 176 obtained as the projections of the points in  $\mathcal{L}_n$  and  $\mathcal{L}_c$ , respectively. Theorem 11 is proved in Section 3.4. In  
 177 Section 4, we will prove our second main result, Theorem 27, stating necessary and sufficient conditions for the  
 178 Ball system to be regular.

#### 179 3.1. Encoding the singular points of the plane projection

180 By Assumption  $\mathcal{A}_5^-$ , the singularities of the projected curve  $\pi_{\mathcal{C}}(\mathcal{C})$  are only nodes and cusps. Intuitively, a  
 181 node appears when two points of  $\mathcal{C}$  project to the same point and a cusp appears when projecting a point with a  
 182 tangent line orthogonal to the projection plane (see Figure 3). The idea to encode the nodes is to design a system  
 183 whose variables are the coordinates of two different points in  $\mathbb{R}^n$  constrained to be on  $\mathcal{C}$  and so that they have the

184 same plane projection. To encode a cusp, we design a system whose variables are the coordinates of one point  
185 in  $\mathbb{R}^n$  constrained to be on  $\mathfrak{C}$  with a tangent orthogonal to the projection plane. Furthermore, in order to apply  
186 certified numerical methods we need systems that are square and regular (Definition 1). To solve this issue and to  
187 gather the two systems into a single one, we first parameterize two different points of  $\mathfrak{C}$  with the same projection by  
188  $(x_1, x_2, y + r\sqrt{t})$  and  $(x_1, x_2, y - r\sqrt{t})$ , with  $x_1, x_2, t \in \mathbb{R}$ ,  $y, r$  in  $\mathbb{R}^{n-2}$  and  $\|r\| = 1$ . Then, given any function  
189  $f$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  so that  $f = 0$  is one of the  $n - 1$  hypersurfaces that define  $\mathfrak{C}$ , we introduce in Definition 9 two  
190 smooth functions  $S \cdot f$  and  $D \cdot f$ . When  $t > 0$ , they return, roughly speaking, the arithmetic mean and difference  
191 of  $f$  at the above two points, hence they both vanish if and only if the two points are on the hypersurface  $f = 0$ .  
192 When  $t = 0$ , the two points coincide and  $S \cdot f$  and  $D \cdot f$  return, roughly speaking,  $f$  evaluated at this point and the  
193 gradient of  $f$  (at that point) scalar the “vertical” vector  $(0, 0, r)$ ; hence, they both vanish if and only if the point is  
194 on the hypersurface  $f = 0$  and its tangent hyperplane is normal to the plane of projection. It follows that given a  
195 curve defined by  $P_1 = \dots = P_{n-1} = 0$ , the solutions of the so-called Ball system of all  $S \cdot P_i = D \cdot P_i = 0$  is  
196 the set of points on the curve that project to nodes and cusps (Theorem 11). Note that we consider  $\sqrt{t}$  instead of  $t$   
197 in the parameterization  $(x_1, x_2, y \pm r\sqrt{t})$  for ensuring the regularity of the Ball system when  $t = 0$  (because this  
198 ensures that the linear term of the Taylor expansion of  $D \cdot f$  does not vanish).

199 **Definition 9.** Let  $x_1, x_2, t$  be variables in  $\mathbb{R}$  and  $y, r$  in  $\mathbb{R}^{n-2}$ . For a smooth function  $f : \overline{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , we define  
200 the functions  $S \cdot f$  and  $D \cdot f$  from  $\mathbb{R}^{2n-1}$  to  $\mathbb{R}$ .

$$201 \quad S \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0 \end{cases}$$

202 and

$$203 \quad D \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ \nabla f(x_1, x_2, y) \cdot (0, 0, r), & \text{for } t = 0. \end{cases}$$

**Lemma 10.** If  $f$  is a smooth function defined on  $\overline{B} \subseteq \mathbb{R}^n$ , then both  $S \cdot f$  and  $D \cdot f$  are smooth functions on the  
subset

$$\overline{B}_{\text{Ball}} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geq 0, (x_1, x_2, y \pm r\sqrt{t}) \in \overline{B}, \|r\|^2 = 1\}$$

204 of  $\mathbb{R}^{2n-1}$ , where  $\|r\|$  denotes the Euclidean norm of  $r$ .

205 *Proof.* On the subset  $\overline{B}_{\text{Ball}}$  with  $t > 0$ , both  $S \cdot f(x_1, x_2, y, r, t)$  and  $D \cdot f(x_1, x_2, y, r, t)$  are the compositions of  
206 smooth functions, hence they are smooth functions.

207 For a point  $X = (x_1, x_2, y, r, t)$  in  $B_{\text{Ball}}$  with  $t = 0$ , we will prove that  $S \cdot f$  (resp.  $D \cdot f$ ) is a  $C^s$  function for  
208 an arbitrarily  $s$  which implies that  $S \cdot f$  (resp.  $D \cdot f$ ) is smooth. First define the function

$$209 \quad S_0 \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + rt) + f(x_1, x_2, y - rt)], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0. \end{cases}$$

210 Since  $S_0 \cdot f(x_1, x_2, y, r, t)$  is an even smooth function with respect to  $t$ , the partial derivatives of  $S_0 \cdot f$  with respect  
211 to  $t$  of odd orders, evaluated at  $X$ , are zero. For an integer  $s > 0$ , by the parameterized Taylor formula without  
212 remainder [Dem00, Proposition 4.2.2], there exist smooth functions  $a_i(x_1, x_2, y, r)$ , with integers  $0 \leq i < s$  such

213 that  $S_0 \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^{2i} + t^{2s} \cdot \phi(x_1, x_2, y, t)$ , where  $\phi(x_1, x_2, y, t)$  is a smooth function.

214 Notice that  $S \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^i + t^s \cdot \phi(x_1, x_2, y, \sqrt{t})$ , so that a partial derivative exists  
 215 up to order  $s$  at  $t = 0$ . Thus,  $S \cdot f(x_1, x_2, y, r, t)$  is a  $C^{s-1}$  function. This holds for any arbitrarily large  $s$ , hence

216  $S \cdot f(x_1, x_2, y, r, t)$  is a  $C^\infty$  function.

Now, we prove that  $D \cdot f$  is continuous at  $X = (x_1, x_2, y, r, 0)$ . Let  $X_i$  be a sequence that converges to  $X$ . To prove that  $D \cdot f(X_i)$  converges to  $D \cdot f(X)$ , it is enough to show that for a sequence  $t_i$  that converges to 0, then we have that  $D \cdot f(x_1, x_2, y, r, t_n)$  converges to  $D \cdot f(X)$ . We can assume that  $t_i \neq 0$  for all  $i$ , so that

$$\begin{aligned} \lim_{t_i \rightarrow 0} D \cdot f(x_1, x_2, y, r, t_i) &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - (f(x_1, x_2, y) - f(x_1, x_2, y)) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y)] \\ &\quad + \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \frac{1}{2} \nabla f \cdot (0, 0, r) - \frac{1}{2} \nabla f \cdot (0, 0, -r) \\ &= \nabla f \cdot (0, 0, r). \end{aligned}$$

217 We now prove that  $D \cdot f$  is smooth at  $X$ . Similarly to the proof of the case of  $S \cdot f$ , since the function  $\frac{1}{2}[f(x_1, x_2, y +$   
 218  $rt) - f(x_1, x_2, y - rt)]$  is odd with respect to  $t$ , there exist smooth functions  $b_i(x_1, x_2, y, r)$ , for  $1 \leq i < s$  and  
 219  $\psi(x_1, x_2, y, r, t)$  such that  $\frac{1}{2}[f(x_1, x_2, y + rt) - f(x_1, x_2, y - rt)] = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r) t^{2i+1} + t^{2s+1} \cdot \psi(x_1, x_2, y, t)$ .

220 Notice that  $D \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r) t^i + t^s \cdot \psi(x_1, x_2, y, \sqrt{t})$ , so that a partial derivative exists  
 221 up to order  $s$  at  $t = 0$ . Thus,  $D \cdot f(x_1, x_2, y, r, t)$  is a  $C^{s-1}$  function. This holds for any arbitrarily large  $s$ , hence  
 222  $D \cdot f(x_1, x_2, y, r, t)$  is a  $C^\infty$  function.  $\square$

**Theorem 11.** Consider  $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  that satisfies Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ .

Then,  $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$  is a solution of the Ball system

$$\text{Ball}(P) = \begin{cases} S \cdot P_1(X) = \dots = S \cdot P_{n-1}(X) = 0 \\ D \cdot P_1(X) = \dots = D \cdot P_{n-1}(X) = 0 \\ \|r\|^2 - 1 = 0 \end{cases} \quad (3.1)$$

223 if and only if  $(x_1, x_2)$  is a singular point of  $\pi_{\mathcal{C}}(\mathcal{C})$  (see Definition 9 for the notation  $S \cdot P_i$  and  $D \cdot P_i$ ).

224 We postpone the proof of Theorem 11 to the end of Section 3.3. As a first step, we study a mapping from the  
 225 solutions of the Ball system to pairs of points on the curve  $\mathcal{C}$ .

226 **Definition 12.** Let  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ . Define  $\widehat{\mathfrak{L}}_n$  to be the set of pairs  $(q_1, q_2)$  with  $q_1, q_2 \in \mathfrak{C}$ ,  $q_1 \neq q_2$  and  
 227  $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ , also define  $\widehat{\mathfrak{L}}_c$  to be the set of pairs  $(q_1, q_1)$  with  $q_1 \in \mathfrak{L}_c$ , and let  $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}_n \cup \widehat{\mathfrak{L}}_c$ .

228 **Lemma 13.** Consider  $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  and let  $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times$   
 229  $\mathbb{R}^{n-2} \times \mathbb{R}$ , with  $\|r\| = 1$ . Assume that  $P$  satisfies Assumption  $\mathcal{A}_1$ . Then  $X$  is a solution of  $\text{Ball}(P)$  if and only  
 230 if for the points  $q_1 = (x_1, x_2, y + r\sqrt{t})$  and  $q_2 = (x_1, x_2, y - r\sqrt{t})$ , the pair  $(q_1, q_2)$  is in  $\widehat{\mathfrak{L}}_n$ , or in  $\widehat{\mathfrak{L}}_c$  with  
 231  $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$  in  $T_{q_1}\mathfrak{C}$ .

232 *Proof.* Note that, by Assumption  $\mathcal{A}_1$ , the tangent space to the curve at any of its points is well defined and is a  
 233 line. First, assume that  $X$  is a solution of  $\text{Ball}(P)$ . We consider two cases:

- 234 (a) If  $t > 0$ , then since  $r \neq 0 \in \mathbb{R}^{n-2}$  we have that  $q_1 \neq q_2$ . Moreover, since  $S \cdot P_i(X) = D \cdot P_i(X) = 0$   
 235 for all  $i \in \{1, \dots, n-1\}$ , we deduce that  $P_i(q_1) = P_i(q_2) = 0$ , thus  $q_1, q_2 \in \mathfrak{C}$ . Moreover, since  
 236  $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2) = (x_1, x_2)$  we have  $q_1, q_2 \in \mathfrak{L}_n$ . Thus,  $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$ .
- 237 (b) If  $t = 0$ , then  $q_1 = q_2$ . First,  $P_i(q_1) = S \cdot P_i(X) = 0$ , for all indices  $i \in \{1, \dots, n-1\}$ , hence  $q_1 \in \mathfrak{C}$ .  
 238 Moreover, we have that  $0 = D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r)$ , for all  $i \in \{1, \dots, n-1\}$ , equivalently,  
 239  $J_P(q_1) \cdot (0, 0, r)^T = 0 \in \mathbb{R}^{n-1}$ , i.e., we have  $(0, 0, r) \in T_{q_1}\mathfrak{C}$ . Thus,  $q_1 \in \mathfrak{L}_c$  and hence,  $(q_1, q_1) \in \widehat{\mathfrak{L}}_c$ .

240 Now, let us prove the other direction:

- 241 (a) If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$ , then  $q_1 \neq q_2$  and  $t \neq 0$ . Also, since  $q_1, q_2 \in \mathfrak{C}$ , we can write that  $S \cdot P_i(X) =$   
 242  $\frac{1}{2}(P_i(q_1) + P_i(q_2)) = 0$ , and  $D \cdot P_i(X) = \frac{1}{2\sqrt{t}}(P_i(q_1) - P_i(q_2)) = 0$ , for all  $i \in \{1, \dots, n-1\}$ . Thus,  $X$   
 243 is a solution of  $\text{Ball}(P)$ .
- 244 (b) If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$  and  $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$  is in  $T_{q_1}\mathfrak{C}$ , one has  $q_1 = q_2 \in \mathfrak{L}_c \subseteq \mathfrak{C}$ , and  $t = 0$ . Moreover,  
 245 for all  $i \in \{1, \dots, n-1\}$  we have  $S \cdot P_i(X) = P_i(q_1) = 0$  and since  $(0, 0, r) \in T_{q_1}\mathfrak{C}$ , we can equivalently  
 246 write  $D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r) = 0$ . Thus,  $X$  is a solution of  $\text{Ball}(P)$ .  $\square$

247 **Definition 14.** Let  $\text{Sol}_{\text{Ball}(P)}$  be the solution set of  $\text{Ball}(P)$ . Define the function  $\Omega_P$  from  $\text{Sol}_{\text{Ball}(P)}$  to  $\widehat{\mathfrak{L}}$  that  
 248 sends  $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}^+$  to the ordered pair  $q_1 = (x_1, x_2, y + r\sqrt{t})$  and  
 249  $q_2 = (x_1, x_2, y - r\sqrt{t})$ . Notice that the function  $\Omega_P$  is well-defined by Lemma 13.

250 **Lemma 15.** If  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  satisfies Assumption  $\mathcal{A}_1$ , then  $\Omega_P$  is surjective.

251 *Proof.* For any pair  $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$  we have that the point  $X = (\frac{1}{2}(q_1 + q_2), \frac{\Pi_{\mathfrak{C}}(q_1 - q_2)}{\|q_1 - q_2\|}, \frac{1}{4}\|q_1 - q_2\|^2) \in \mathbb{R}^n \times$   
 252  $\mathbb{R}^{n-1} \times \mathbb{R}^+$  is a solution of  $\text{Ball}(P)$ , where  $\Pi_{\mathfrak{C}}(q_1 - q_2)$  is the vector in  $\mathbb{R}^{n-2}$  obtained by omitting the first two  
 253 coordinates (which are zeros) from  $q_1 - q_2$ . Note that  $\Omega_P(X) = (q_1, q_2)$ . If the pair  $(q_1, q_1)$  is in  $\widehat{\mathfrak{L}}_c$ , we define  
 254  $r$  in the following way: we take a unit vector  $v \in T_{q_1}\mathfrak{C}$  (the first two coordinates of  $v$  are zeros since  $q_1 \in \mathfrak{L}_c$ ).  
 255 We set  $r$  to be  $\Pi_{\mathfrak{C}}(v)$ . Again  $X = (q_1, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$  is a solution of  $\text{Ball}(P)$ , with  $\Omega_P(X) = (q_1, q_1)$ .  
 256 Thus,  $\Omega_P$  is surjective.  $\square$

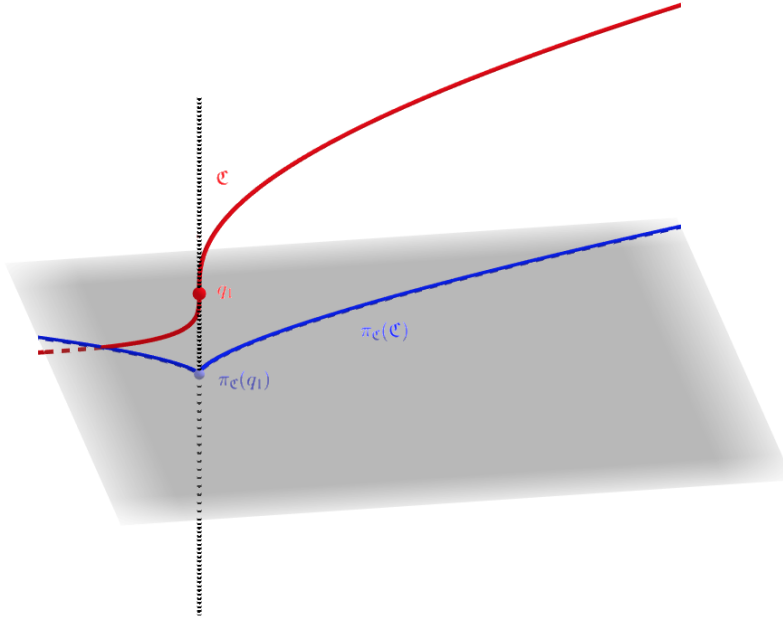


Figure 4: The curve  $\mathfrak{C}$  (red) and its plane projection  $\pi_{\mathfrak{C}}(\mathfrak{C})$  (blue) of Example 18 displaying a cusp singularity.

257 **Remark 16.** Notice that if  $X = (x_1, x_2, y, r, t)$  is in  $\text{Sol}_{\text{Ball}(P)}$ , then  $\Omega_P(X) \in \widehat{\mathfrak{L}}_n$  (resp.  $\Omega_P(X) \in \widehat{\mathfrak{L}}_c$ ) if and  
 258 only if  $t \neq 0$  (resp.  $t = 0$ ).

259 **Remark 17.** Preserving the notation in Lemma 13, notice that if  $X = (x_1, x_2, y, r, t)$  is a solution of  $\text{Ball}(P)$ ,  
 260 then  $X' = (x_1, x_2, y, -r, t)$  is another solution. Moreover, both solutions characterize the same unordered pair  
 261  $\Omega_P(X) = \Omega_P(X') = (q_1, q_2)$ . We call  $X$  and  $X'$  twin solutions. An alternative choice would have been to take  $r$   
 262 in a projective space instead of the sphere to identify these twin solutions.

**Example 18.** Refer to Figure 4. Let  $n = 3$  and  $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \in (-2, 2)\}$ . Define  
 $P_1(x_1, x_2, x_3) = x_1 - (x_3 - 1)^3$ ,  $P_2(x_1, x_2, x_3) = x_2 - (x_3 - 1)^2$  and  $P = (P_1, P_2)$ . The Jacobian matrix  
of  $P$  has full rank over  $\mathfrak{C}$ , thus Assumption  $\mathcal{A}_1$  is satisfied. The set  $\mathfrak{L}_n$  is empty since  $\pi_{\mathfrak{C}}$  is injective over  $\mathfrak{C}$ ,  
hence Assumption  $\mathcal{A}_4$  is satisfied. The only point of  $\mathfrak{C}$  with a tangent line orthogonal to the  $(x_1, x_2)$ -plane is  
 $q_1 = (0, 0, 1)$ , thus  $\mathfrak{L}_c = \{q_1\}$  and Assumption  $\mathcal{A}_2$  is satisfied. By Lemma 23, the multiplicity of the system  
 $\{P = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$  at its unique solution  $q_1$  is  $\min\{\text{mult}_1((x_3 - 1)^3), \text{mult}_1((x_3 - 1)^2)\} = \min\{3, 2\} =$   
2 (mult is defined in Definition 2). Moreover, for any point  $q_0 \in \mathfrak{C}$  different from  $q_1$ , the multiplicity of the

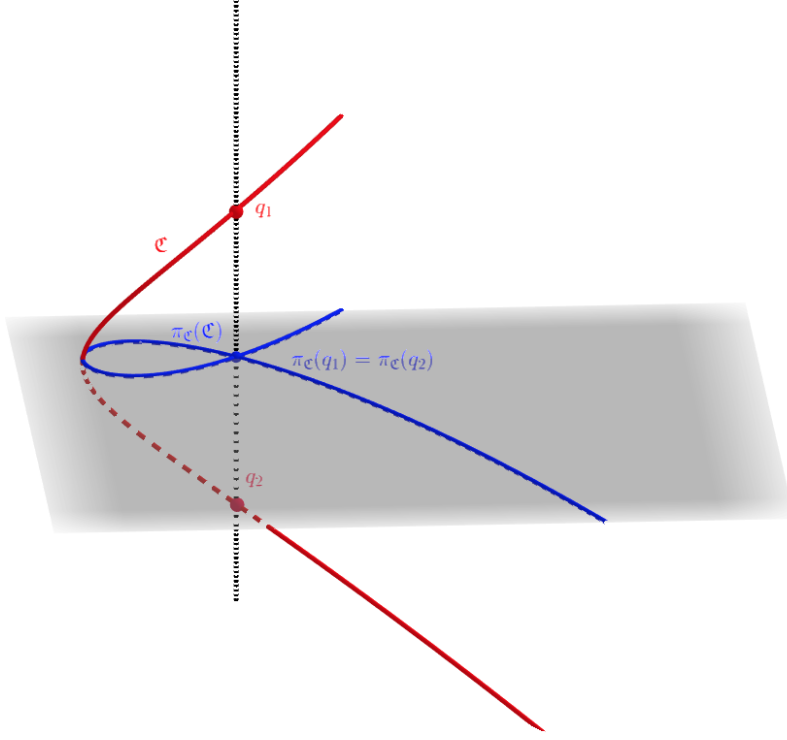


Figure 5: The curve  $\mathcal{C}$  (red) and its plane projection  $\pi_{\mathcal{C}}(\mathcal{C})$  (blue) of Example 19 displaying a node singularity.

corresponding system at its unique solution  $q_0$  is one, thus  $P$  satisfies Assumption  $\mathcal{A}_3$ . The system  $\text{Ball}(P)$ :

$$\begin{cases} x_1 - 3r^2ty + 3r^2t - y^3 + 3y^2 - 3y + 1 = 0 \\ x_2 - r^2t - y^2 + 2y - 1 = 0 \\ -r^3t - 3ry^2 + 6ry - 3r = 0 \\ -2ry + 2r = 0 \\ r^2 - 1 = 0 \end{cases} \quad (3.2)$$

263 has two twin solutions  $X = (0, 0, 1, 1, 0)$  and  $X' = (0, 0, 1, -1, 0)$  in  $B_{\text{Ball}(P)} \subset \mathbb{R}^{2 \cdot 3-1} = \mathbb{R}^5$  such that  
 264  $\Omega_P(X) = \Omega_P(X') = (q_1, q_1) \in \widehat{\mathcal{L}}_{\mathcal{C}}$ .

**Example 19.** Refer to Figure 5. Let  $B$  be defined as in Example 18. Define the functions  $P_1(x_1, x_2, x_3) = x_1 - (x_3^2 - 1)$ ,  $P_2(x_1, x_2, x_3) = x_2 - (x_3^3 - x_3)$  and  $P = (P_1, P_2)$ . The Jacobian matrix of  $P$  has full rank over  $\mathcal{C}$ , thus Assumption  $\mathcal{A}_1$  is satisfied. Moreover, the set  $\mathcal{L}_{\mathcal{C}}$  is empty and  $\mathcal{L}_{\mathfrak{n}} = \{q_1, q_2\}$ , with  $q_1 = (0, 0, 1)$ ,  $q_2 = (0, 0, -1)$ , i.e., Assumptions  $\mathcal{A}_2$  and  $\mathcal{A}_4$  are satisfied. The multiplicity of the system  $\{P = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$  at both

$q_1, q_2$  is equal to one, thus Assumption  $\mathcal{A}_3$  is also satisfied. The system  $\text{Ball}(P)$ :

$$\begin{cases} x_1 - r^2t - y^2 + 1 = 0 \\ x_2 - r^2ty - y^3 + y = 0 \\ -2ry = 0 \\ -r^3t - 3ry^2 + r = 0 \\ r^2 - 1 = 0 \end{cases} \quad (3.3)$$

265 has two twin solutions  $X = (0, 0, 0, 1, 1)$  and  $X' = (0, 0, 0, -1, 1)$  in  $\mathbb{R}^5$  such that  $\Omega_P(X) = \Omega_P(X') =$   
 266  $(q_1, q_2) \in \widehat{\mathcal{L}}_n$ .

### 267 3.2. Singularities induced by $\mathcal{L}_n$

268 We now study the types of singularities of the plane curve  $\pi_{\mathcal{C}}(\mathcal{C})$  obtained by projecting points in  $\mathcal{L}_n$ , that is  
 269 when several points of  $\mathcal{C}$  project to the same point. We begin by showing that the geometric property that the curve  
 270  $\mathcal{C}$  has a tangent orthogonal to the projection plane has an algebraic equivalent in terms of multiplicity.

271 **Lemma 20.** *Let  $P = (P_1 \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  satisfy Assumption  $\mathcal{A}_1$ . Let  $q$  be in  $\overline{\mathcal{C}}$  such that the*  
 272 *multiplicity of the system  $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$  at  $q$  is finite, where  $(\alpha, \beta) = \pi_{\overline{\mathcal{C}}}(q) \in \mathbb{R}^2$ .*  
 273 *Then,  $q \in \overline{\mathcal{L}}_c$  if and only if the multiplicity of the system  $S$  at  $q$  is at least two.*

274 *Proof.* Without loss of generality assume that  $q = 0 \in \mathbb{R}^n$ .

275 *Sufficiency:* Assume that  $q \in \overline{\mathcal{L}}_c$ . Let  $v = (v_1, \dots, v_n)$  be a non-trivial vector of the tangent line of  $\overline{\mathcal{C}}$  at  $q$ .  
 276 Thus,  $J_P(q) \cdot v^T = 0$ . By the definition of  $\overline{\mathcal{L}}_c$  we have  $v_1 = v_2 = 0$ . Define the differential operator  $c = \sum_{i=3}^n v_i \frac{\partial}{\partial x_i}$ .  
 277 Notice that  $c \cdot P_j = \sum_{i=3}^n v_i \frac{\partial P_j}{\partial x_i}(q) = 0$  for all integers  $1 \leq j \leq n-1$  (see [DLZ11, 2.1] for the definition of  $c \cdot P_j$ ).  
 278 Moreover, by the definition of  $c$  and since  $v_1 = v_2 = 0$ , we have  $c \cdot (x_1) = c \cdot (x_2) = 0$ . Hence,  $c \in D_q^1[S] \setminus D_q^0[S]$ .  
 279 Thus,  $\dim(D_q^1) > 1$ . Hence, the multiplicity of  $S$  at  $q$  is at least two.

280 *Necessity:* Assume that the multiplicity of  $S$  at  $q$  is at least two, then  $D_q^0[S] \subsetneq D_q^1[S]$ . This implies that there  
 281 exists a non-trivial differential operator  $c = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} \in D_q^1[S] \setminus D_q^0[S]$  such that:

282 (a) We have that  $c \cdot P_j = 0$  for all integers  $1 \leq j \leq n-1$  which implies that if we write  $v_i = c_i$ , with  $1 \leq i \leq n$ ,  
 283 the non-trivial vector  $v$  is in the tangent space of  $\overline{\mathcal{C}}$  at  $q$ .

284 (b) We have that  $c \cdot (x_1) = c \cdot (x_2) = 0$ , equivalently,  $c_1 = c_2 = 0$ . Thus,  $v_1 = v_2 = 0$ .

285 The tangent line to the curve at  $q$  is thus orthogonal to the  $(x_1, x_2)$ -plane. Thus,  $q \in \overline{\mathcal{L}}_c$ . □

286 **Lemma 21.** *Under Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ , if  $q \in \mathcal{L}_n$ , then  $\pi_{\mathcal{C}}(q)$  is a singular point of the plane curve*  
 287  *$\pi_{\mathcal{C}}(\mathcal{C})$ . More precisely, either  $\pi_{\mathcal{C}}(q)$  is of type  $A_{2k+1}^-$  with  $k \geq 0$ , or there exists a non-null smooth function  $g$*   
 288 *defined in a neighborhood of  $0 \in \mathbb{R}$  with  $\text{mult}(g) = \infty$  such that  $(\pi_{\mathcal{C}}(q), \pi_{\mathcal{C}}(\mathcal{C}))$  is equivalent (according to*  
 289 *Definition 6) to the curve defined by  $x^2 - g(y^2) = 0$  at the origin.*

290 *Proof.* Let  $p = \pi_{\mathcal{C}}(q)$ , according to  $\mathcal{A}_3$ ,  $\pi_{\mathcal{C}}^{-1}(p)$  has at most two points, and since  $q$  is in  $\mathfrak{L}_n$ , it also has at least two  
291 points. Define  $q'$  such that  $\pi_{\mathcal{C}}^{-1}(p) = \{q, q'\}$  and denote the plane curve  $\pi_{\mathcal{C}}(\mathcal{C})$  by  $C$ . Without loss of generality,  
292 one can assume that  $p = (0, 0)$ . In addition,  $\mathcal{A}_3$  also implies that the multiplicities of  $q$  and  $q'$  in the system  
293  $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$  are one. With Assumption  $\mathcal{A}_1$ , Lemma 20 then implies that the tangents to  $\mathcal{C}$   
294 at  $q$  and  $q'$  are not orthogonal to the  $(x_1, x_2)$ -plane. Thus there exists two neighborhoods  $N_q$  and  $N_{q'}$  of  $q$  and  $q'$  in  
295  $\mathbb{R}^n$  such that  $\pi$  restricted to  $\mathcal{C} \cap N_q$  (resp.  $\mathcal{C} \cap N_{q'}$ ) is an embedding. Let  $D_k$  be a sequence of open disks centered  
296 at  $p$  and of radius  $\frac{1}{k}$ . By contradiction, if for all  $k$ , there exists points  $q_k \in \mathcal{C}$  such that  $q_k$  is not in  $N_q \cup N_{q'}$   
297 and  $\pi_{\mathcal{C}}(q_k) \in D_k$ , then the limit  $q_{\infty}$  is a point of  $\overline{\mathcal{C}}$  distinct from  $q$  and  $q'$ , and  $\pi_{\mathcal{C}}(q_{\infty}) = p$ . If  $q_{\infty}$  is in  $B$ , it  
298 contradicts  $\mathcal{A}_3$  and if it is in  $\overline{B}$ , it contradicts  $\mathcal{A}_4$ . Thus for a small enough neighborhood of  $p$ , the projection of  
299 the curve is restricted to the projection of the two branches around  $q$  and  $q'$ . Finally, if for all  $D_k$ , the pre-image of  
300  $\pi^{-1}(D_k)$  contains a point in  $\overline{\mathfrak{L}_n} \setminus \{q, q'\}$ , then this contradicts the discreteness assumption  $\mathcal{A}_4$ . Thus there exists a  
301 neighborhood  $N \subseteq \mathbb{R}^2$  of  $p$  such that  $\pi_{\mathcal{C}}^{-1}(N)$  is a union of two smooth (Assumption  $\mathcal{A}_1$ ) open subsets of  $\mathcal{C}$  such  
302 that  $q$  is on one branch and  $q'$  on the other, and  $\pi_{\mathcal{C}}$  restricted to  $\pi_{\mathcal{C}}^{-1}(N) \setminus \{p, q'\}$  is an embedding. The projection  
303 of these two smooth branches are thus two smooth curves in the plane. Let these two smooth plane branches be  
304 defined by the zero sets of the smooth functions  $f_1$  and  $f_2$  in  $C^\infty(\mathbb{R}^2, \mathbb{R})$ . Let  $u$  (resp.  $u'$ ) be a non-zero tangent  
305 vector of  $\mathcal{C}$  at  $q$  (resp.  $q'$ ) and  $v$  (resp.  $v'$ ) be its projection in  $\mathbb{R}^2$ . We distinguish two cases:

306 (a) The vectors  $v$  and  $v'$  are independent in  $\mathbb{R}^2$ . Thus,  $v$  and  $v'$  give rise to a local coordinate system  $(x, y)$   
307 in a neighborhood of  $p$  in  $\mathbb{R}^2$ . The vector  $v$  being tangent to the zero set of  $f_1$ , one has  $\frac{\partial f_1}{\partial x}(p) = 0$  and  
308  $\frac{\partial f_1}{\partial y}(p) \neq 0$ . By the implicit function theorem [Dem00, Corollary 2.7.3.], we deduce that there exists a real  
309 smooth function  $h_1$  such that  $y = x^2 \cdot h_1(x)$  is a local parameterization of the zero set of  $f_1$ . Similarly,  
310 there exists a smooth function  $h_2$  such that  $x = y^2 \cdot h_2(y)$  is a local parameterization of the zero set of  $f_2$ .  
311 Thus  $(x, y) \in N$  iff  $f(x, y) = f_1(x, y)f_2(x, y) = 0$  iff  $(y - x^2 \cdot h_1(x))(x - y^2 \cdot h_2(y)) = 0$ , equivalently,  
312  $[y - x - x^2 \cdot h_1(x) + y^2 \cdot h_2(y)]^2 - [y + x - x^2 \cdot h_1(x) - y^2 \cdot h_2(y)]^2 = 0$ . The change of coordinates  
313  $X = y - x + x^2 \cdot h_1(x) + y^2 \cdot h_2(y)$  and  $Y = y + x + x^2 \cdot h_1(x) - y^2 \cdot h_2(y)$  is a diffeomorphism since  
314  $\det(J_{x,y}(X, Y))_p \neq 0$ . Then, the local equation of the curve  $C$  at  $p$  is of the form  $X^2 - Y^2$  with these new  
315 coordinates, which means that  $p$  is a  $A_1^-$  or node singularity.

316 (b) The vectors  $v$  and  $v'$  are co-linear. Then, choose  $v'' \in T_p \mathbb{R}^2$  linearly independent from  $v$ , the vectors  $v, v''$   
317 give rise to a coordinate system  $(x, y)$  at  $p$ . In this coordinate system, we thus have  $\frac{\partial f_1}{\partial x}(p) = \frac{\partial f_2}{\partial x}(p) = 0$ ,  
318  $\frac{\partial f_1}{\partial y}(p) \neq 0$  and  $\frac{\partial f_2}{\partial y}(p) \neq 0$ . By the implicit function theorem, there exist smooth functions  $h_1$  and  $h_2$  such  
319 that locally  $f(x, y) = 0$  if and only if  $(y - x^2 \cdot h_1(x))(y - x^2 \cdot h_2(x)) = 0$ . The last equality is equivalent  
320 to  $(2y - x^2(h_1(x) + h_2(x)))^2 - x^4(h_1(x) - h_2(x))^2 = 0$ . Assumption  $\mathcal{A}_4$  ensures that the projections of  
321 the two branches have only one common point, such that  $h_1(x) - h_2(x)$  does not vanish identically. We  
322 distinguish two cases:

323 (i)  $\text{mult}(h_1(x) - h_2(x)) = k \leq \infty$ , then  $h_1(x) - h_2(x) = x^k \cdot u$  with  $u(p) \neq 0$  and without loss



324 of generality, assume that  $u(p) > 0$ . The change of coordinates  $X = 2y - x^2(h_1(x) + h_2(x))$  and  
 325  $Y = x \cdot u^{\frac{1}{2+k}}$  is a diffeomorphism (notice that indeed  $u^{\frac{1}{2+k}}$  is a smooth function around  $p$ ). Then, the  
 326 local equation of the curve  $C$  at  $p$  is of the form  $X^2 - Y^{(2k+3)+1}$  with these new coordinates, which  
 327 means that  $p$  is a singularity of type  $A_{2k+3}^-$ .

328 (ii)  $\text{mult}(h_1(x) - h_2(x)) = \infty$ . Since the function  $x^4(h_1(x) - h_2(x))^2$  is even, by Theorem 49, there  
 329 exists a smooth function  $g$  such that  $x^4(h_1(x) - h_2(x))^2 = g(x^2)$ . Thus, taking the diffeomorphism  
 330  $X = 2y - x^2(h_1(x) + h_2(x))$  and  $Y = x$ , we get the second case of the claim.

331 □

### 332 3.3. Singularities induced by $\mathfrak{L}_c$

333 We now study the types of singularities of the plane curve  $\pi_c(\mathfrak{C})$  obtained by projecting points in  $\mathfrak{L}_c$ , that is  
 334 when the tangent to  $\mathfrak{C}$  is orthogonal to the projection plane. We start by locally parametrizing  $\mathfrak{C}$  around a point  
 335 in  $\mathfrak{L}_c$ . This parameterization will ease the computation of  $\text{Ball}(P)$  and its Jacobian in Section 4. In the rest  
 336 of this section and Section 4, the analysis is simplified by translating relevant points or assuming the curve  $\mathfrak{C}$  is  
 337 parametrizable by a specific variable. On the other hand, in our algorithmic Section 5, the input is not modifiable  
 338 at all but every computation uses interval arithmetic. This implies that the exact coordinates of a point may not be  
 339 known, instead we only compute with a box containing it and isolating it from other relevant points. The idea of  
 340 our semi-algorithms is to check that some function does not vanish on such a box. This then implies that such a  
 341 function does not vanish at the point this box contains. The *theoretical analysis* of this section can then be applied  
 342 to the point to deduce the appropriate property without the knowledge of the exact location of that point.

**Lemma 22.** *Let  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ . Let  $q \in \mathfrak{L}_c$  such that Assumption  $\mathcal{A}_1$  is satisfied in a neighborhood of  $q$  in  $B$ . Without loss of generality one can assume  $q = 0 \in \mathbb{R}^n$ . Then there exist an invertible matrix  $M$  of size  $(n-1) \times (n-1)$  of smooth functions in a neighborhood of  $q$  and smooth functions  $f_1, f_2, f_3, \dots, f_{n-1}$  defined in a neighborhood of  $0 \in \mathbb{R}$ , such that:*

$$\begin{pmatrix} x_1 - f_1(x_n) \\ x_2 - f_2(x_n) \\ x_3 - f_3(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix}, \quad (3.4)$$

343 with  $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\} > 1$  (mult is defined in Definition 2).

*Proof.* Since  $\text{rank}(J_P(q)) = n-1$  (Assumption  $\mathcal{A}_1$ ), there exists  $k \in \{1, \dots, n\}$  such that  $\det(M_k(q)) \neq 0$ , where  $M_k$  is the minor of  $J_P$  obtained by removing the  $k$ -th column. Notice that  $k \notin \{1, 2\}$ , since  $q \in \mathfrak{L}_c$  implies that  $\det(M_1(q)) = \det(M_2(q)) = 0$ . Without loss of generality, we assume that  $k = n$ . Using the implicit

function theorem [Corollary 2.7.3][Dem00], there exist smooth functions  $f_1, \dots, f_{n-1}$  of one variable such that we have that

$$P_j(f_1(x_n), \dots, f_{n-1}(x_n), x_n) = 0, j \in \{1, \dots, n-1\}. \quad (3.5)$$

344 Define the function  $\varphi$  that maps  $x_i$  to  $z_i = x_i - f_i(x_n)$ , for all  $i \in \{1, \dots, n-1\}$  and  $x_n$  to  $z_n = x_n$ .  
 345 We can see that  $\varphi$  is a diffeomorphism and  $z = (z_1, \dots, z_n)$  is a local coordinate system around  $q$ . Hence, we  
 346 can define the function  $G_j(z) = P_j \circ \varphi^{-1}(z) = P_j(x)$  for all integers  $1 \leq j \leq n-1$ . Using Hadamard's  
 347 Lemma [Dem00, Proposition 4.2.3] for the first  $n-1$  variables of  $z$ , we can write  $G_j(z) - G_j(0, \dots, 0, z_n) =$   
 348  $\sum_{i=1}^{n-1} z_i \cdot h_{ji}(z)$  for some smooth functions  $h_{ji}$ . Note that  $\varphi^{-1}(z) = (z_1 + f_1(z_n), \dots, z_{n-1} + f_{n-1}(z_n), z_n)$ . Hence,  
 349  $G_j(0, \dots, 0, z_n) = P_j \circ \varphi^{-1}(0, \dots, 0, z_n) = P_j(f_1(z_n), \dots, f_{n-1}(z_n), z_n) = P_j(f_1(x_n), \dots, f_{n-1}(x_n), x_n)$ .  
 350 The latter function is equal to zero by (3.5). Thus,  $P_j(x) = G_j(z) = \sum_{i=1}^{n-1} z_i \cdot h_{ji}(z) = \sum_{i=1}^{n-1} (x_i - f_i(x_n)) \cdot H_{ji}(x)$ ,  
 351 with  $H_{ji}(x) = h_{ji} \circ \varphi(x)$ .

352 Defining  $M_0 = \left( H_{ji} \right)_{1 \leq j, i \leq n-1}$  we get:

$$\begin{pmatrix} P_1 \\ \dots \\ P_{n-1} \end{pmatrix} = M_0 \cdot \begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix}.$$

353 Notice that  $M_0$  evaluated at  $q$  is the invertible matrix  $M_n(q)$ . Hence, by continuity of the determinant function,  
 354 there is a neighborhood of  $q$  in which  $M_0$  is invertible. Thus, writing  $M$  as the inverse of  $M_0$  we get:

$$Q_0 = \begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ \dots \\ P_{n-1} \end{pmatrix}. \quad (3.6)$$

355 To prove that  $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\} > 1$ , we take the Jacobian matrices of both sides of (3.6) and we  
 356 evaluate them at  $q = 0$ . We get the equation  $J_{Q_0}(q) = M(q) \cdot J_P(q)$ . By invertibility of  $M(q)$  we deduce that the  
 357  $k$ -th minors (obtained by removing the  $k$ -th column) of  $J_{Q_0}(q)$  and  $J_P(q)$  have the same rank. Computing  $J_{Q_0}(q)$   
 358 and considering the fact that  $\det(M_1(q)) = \det(M_2(q)) = 0$  implies that  $f_1'(0) = f_2'(0) = 0$ , we thus have that  
 359  $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$  is at least two.  $\square$

360 **Lemma 23.** *Preserving the notation and the assumptions in Lemma 22, the multiplicity  $m$  of the system  $S =$*   
 361  *$\{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$  at  $q$  is equal to  $d = \min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$ .*

362 *Proof.* First, we start with the case  $m < \infty$ . By Proposition 5, we can assume without loss of generality, that  
 363  $f_1, \dots, f_{n-1}$  are polynomials. Following the notation in Definition 3, let  $\mathbb{R}[x]$  (resp.  $\mathbb{R}[x_n]$ ) be the ring of poly-  
 364 nomials with  $n$  variables (resp. one variable) and  $\mathbb{R}[x]_q$  (resp.  $\mathbb{R}[x_n]_0$ ) be its localization at  $q$  (resp.  $0 \in \mathbb{R}$ ).  
 365 Also, define  $I_S$  to be the ideal generated by the polynomials of  $S$  in  $\mathbb{R}[x]_q$  (as  $I_G$  is defined in Definition 3),

366 i.e.,  $I_S = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_1, x_2 \rangle = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} -$   
367  $f_{n-1}(x_n), f_1(x_n), f_2(x_n) \rangle$ . If  $f_1(x_n) = f_2(x_n) = 0$ , then the ideal  $I_S$  is of dimension one, hence,  $S$  has an  
368 infinite number of solutions which contradicts the assumption that  $m < \infty$ . Thus,  $d < \infty$  which means that  
369 there exist  $h_1, h_2 \in \mathbb{R}[x_n]_0$  such that  $h_1(x_n)f_1(x_n) + h_2(x_n)f_2(x_n) = x_n^d$ . Thus,  $I_S = \langle x_1 - f_1(x_n), x_2 -$   
370  $f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_n^d \rangle$ . Note that the set  $\{x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_n^d\}$  is a  
371 Gröbner basis of  $I_S$  with respect to Local Lexicographical ordering  $x_1 > \dots > x_n$ . Hence, By [CLO05, Theorem  
372 4.4.3] we have  $\dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x]_q}{LT(I_S)}) = \dim(\frac{\mathbb{R}[x]_q}{\langle x_1, x_2, \dots, x_{n-1}, x_n^d \rangle})$ , where  $LT(I_S)$  is the ideal generated by the  
373 leading terms of  $I_S$ . Consequently,  $m = \dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x_n]_0}{\langle x_n^d \rangle}) = d$ .

374 Second, assume that  $m = \infty$ . We prove that  $d = \infty$ , that is,  $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$  for any positive integer  
375  $k$ . Preserving the notation in Definition 4, consider the dual space  $D_q^k[S]$ . We are going to show that for any  
376 positive integer  $k$  and any element  $c \in D_q^k[S] \setminus D_q^{k-1}[S]$  (which always exists since  $m = \infty$ ), the coefficient  $c_{x_n^k}$   
377 corresponding to  $\frac{\partial^k}{\partial x_n^k}$ , for  $c$ , is non-zero. We consequentially show that  $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$ . We prove the  
378 previous statements by induction on  $k$ .

379 For  $k = 1$ , since  $q \in \mathcal{L}_c$ , we already showed in the proof of Lemma 20 that a non-trivial element  $c = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$   
380 is in  $D_q^1[S] \setminus D_q^0[S]$  if and only if  $v = (v_1, \dots, v_n)$  is in  $T_q \mathcal{C}$ . On the other hand,  $T_q \mathcal{C}$  is generated by the vector  
381  $(f_1'(0), \dots, f_{n-1}'(0), 1)$ , thus  $c_{x_n^1} = v_n \neq 0$ . The function  $f_1(x_n)$  is in the set of functions generated by  $S$  thus  
382  $0 = c \cdot (f_1(x_n)) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \cdot (f_1(x_n)) = c_{x_n^1} \frac{\partial f_1}{\partial x_n}(0)$ , and thus  $\frac{\partial f_1}{\partial x_n}(0) = 0$ . Thus, the induction hypothesis  
383 holds for  $k = 1$ .

384 Define  $c' = \phi_n(c)$  and consider two cases:

385 (a)  $c' \in D_q^{k-1}[S] \setminus D_q^{k-2}[S]$ : By the induction hypothesis, the coefficient  $c'_{x_n^{k-1}}$  corresponding to  $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$  for  
386  $c'$  is non-zero and  $\frac{\partial^{k'} f_1}{\partial x_n^{k'}}(0) = \frac{\partial^{k'} f_2}{\partial x_n^{k'}}(0) = 0$ , for all  $k' < k$ . Notice that by the definition of  $\phi_n$ , we have  
387  $c_{x_n^k} = c'_{x_n^{k-1}} \neq 0$ . Hence,  $0 = c \cdot f_1(x_n) = \sum_{i=1}^k c_{x_n^i} \frac{\partial^i f_1}{\partial x_n^i}(0) = c_{x_n^k} \frac{\partial^k f_1}{\partial x_n^k}(0)$ . Hence,  $\frac{\partial^k f_1}{\partial x_n^k}(0) = 0$ . Similarly,  
388 we prove that  $\frac{\partial^k f_2}{\partial x_n^k}(0) = 0$ . Thus in Case (a), the lemma is proved.

389 (b)  $c' \in D_q^{k-2}[S]$ : Since  $c \in D_q^k[S] \setminus D_q^{k-1}[S]$ , there exists  $j \in \{1, \dots, n-1\}$  such that the element  $c'' = \phi_j(c)$   
390 is in  $D_q^{k-1}[S] \setminus D_q^{k-2}[S]$ . By the induction hypothesis, the coefficient  $c''_{x_n^{k-1}}$  corresponding to  $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$  for  $c''$ ,  
391 is non-zero. On the other hand,  $c_{x_j x_n^{k-1}} = c''_{x_n^{k-1}} \neq 0$ . Hence, since  $\phi_n(c_{x_j x_n^{k-1}} \frac{\partial^{k-1}}{\partial x_j \partial x_n^{k-1}}) \in D_q^{k-1}[S] \setminus$   
392  $D_q^{k-2}[S]$ , then so is  $\phi_n(c) = c'$  which contradicts the assumption. Thus, Case (b) is impossible.

393 □

394 With the additional Assumptions  $\mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ , one can give a more precise form of  $f_1$  and  $f_2$  in Equa-  
395 tion (3.4).

**Lemma 24.** *Let  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ . Let  $q \in \mathcal{L}_c$  such that Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$  hold in a neigh-  
borhood of  $q$  in  $B$ , then there exist an invertible matrix  $\widetilde{M}$  of size  $(n-1) \times (n-1)$  of smooth functions in a*

neighborhood of  $q$ , a smooth diffeomorphism  $\varphi$  defined in an open subset of  $\mathbb{R}^n$ , with  $z = (z_1, \dots, z_n) = \varphi^{-1}(x)$  and smooth functions  $f_3, \dots, f_{n-1}, g$  defined in a neighborhood of  $0 \in \mathbb{R}$ , such that

$$Q = \begin{pmatrix} z_1 - z_n \cdot g(z_n^2) \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} = \widetilde{M} \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} \circ \varphi, \quad (3.7)$$

396 on a neighborhood of  $q$ . Moreover, either  $\text{mult}(g(z_n)) = \infty$  or there exists an integer  $k > 0$  with  $g(z_n) = z_n^k$ .

397 *Proof.* Step 1: Equation (3.6) implies that  $Q_0$  and  $P$  define the same curve  $\mathfrak{C}$  in a neighborhood of  $q$  and that the  
398 function  $Q_0$  satisfies the same assumptions as  $P$  around  $q$ . By Lemma 23,  $d = \min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$   
399 is the multiplicity of the system  $\{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = 0, x_2 = 0\}$  at  $q$ . By Assumption  $\mathcal{A}_3$ , we have that  
400  $d = 2$ .

401 Without loss of generality, assume that  $\text{mult}(f_2(x_n)) = 2$  and  $\frac{\partial^2 f_2}{\partial x_n^2}(0) = 2$ . Hence, there is a smooth  
402 function  $v$  such that  $f_2(x_n) = x_n^2(1 + x_n \cdot v(x_n))$ . Now, consider the diffeomorphism  $\phi_n$  that sends  $x_n$   
403 to  $z_n = x_n \sqrt{1 + x_n \cdot v(x_n)}$ . We have that  $x_2 - f_2(x_n) = x_2 - z_n^2$ . Define  $\tilde{f}_1(z_n) = f_1(\phi_n^{-1}(z_n))$  and  
404  $\tilde{f}_2(z_n) = f_2(\phi_n^{-1}(z_n)) = z_n^2$ . Since  $\text{mult}(\tilde{f}_1(z_n)) = \text{mult}(f_1(x_n)) \geq d = 2$ , there exists a smooth function  
405  $h$  such that  $\tilde{f}_1(z_n) = z_n^2 h(z_n)$ . Write  $\tilde{f}_1(z_n) = z_n^2 [\frac{h(z_n) + h(-z_n)}{2} + \frac{h(z_n) - h(-z_n)}{2}]$ . Since  $\frac{h(z_n) + h(-z_n)}{2}$  (resp.  
406  $\frac{h(z_n) - h(-z_n)}{2}$ ) is even (resp. odd), then by Theorem 49 there exists a smooth function  $\xi_1$  (resp.  $\xi_2$ ) such that  
407  $\frac{h(z_n) + h(-z_n)}{2} = \xi_1(z_n^2)$  (resp.  $\frac{h(z_n) - h(-z_n)}{2} = z_n \xi_2(z_n^2)$ ). Thus,  $\tilde{f}_1(z_n) = z_n^2 (\xi_1(z_n^2) + z_n \xi_2(z_n^2))$ . Notice that  
408  $\xi_2(z_n^2)$  cannot be the zero function, otherwise  $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$  and  $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$  for all small enough  $\epsilon > 0$ ,  
409 which contradicts Assumption  $\mathcal{A}_4$ .

410 Step 2: We have two cases:

411 **Case 1:**  $\text{mult}(\xi_2(z_n)) = \infty$ , then define the diffeomorphism  $\phi$  which sends  $x_1$  to  $z_1 = x_1 - x_2 \xi_1(x_2)$ ,  $x_i$  to  
412  $z_i = x_i$  for all integers  $i \in \{2, \dots, n-1\}$  and  $x_n$  to  $z_n = x_n \sqrt{1 + x_n \cdot v(x_n)}$ . Taking  $g(z_n) = z_n \xi_2(z_n)$  and  
413  $\varphi = \phi^{-1}$  we prove the claim for the first case.

414 **Case 2:**  $\text{mult}(\xi_2(z_n)) = k < \infty$ , that is,  $\xi_2(z_n) = z_n^k u(z_n)$ , for some smooth function  $u$ , with  $u(0) \neq 0$  and  
415 an integer  $k \geq 0$ . Hence, we can write  $x_1 - \tilde{f}_1(z_n) = x_1 - z_n^2 \xi_1(z_n^2) - z_n^{2k+3} u(z_n^2) = x_1 - x_2 \xi_1(x_2) - z_n^{2k+3} u(x_2)$ .

416 So, defining the diffeomorphism  $\phi$  which sends  $x_i$  to  $z_i = x_i$  for all integers  $i \in \{2, \dots, n-1\}$ ,  $x_n$  to  $z_n =$   
417  $x_n \sqrt{1 + x_n \cdot v(x_n)}$  and  $x_1$  to  $z_1 = (x_1 - x_2 \xi_1(x_2)) u^{-1}(x_2)$  (which means that  $x_1 - f_1(x_n) = u(x_2)[z_1 - z_n^{2k+3}]$ ),  
418 we get that:

$$\begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi,$$

419 for a small enough neighborhood of  $q$ , where  $I_{n-2}$  is the identity matrix of size  $n - 2$ . Comparing with (3.4), we  
420 get:

$$M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi.$$

421 Hence, taking  $\widetilde{M} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix}^{-1} \cdot M$  and  $\varphi = \phi^{-1}$  we recover (3.7).  $\square$

422 Following the conclusion of Lemma 24, the reader may wonder whether the projection of  $q$  in  $\pi_{\mathcal{C}}$  is always  
423 singular. This is clear when  $g(x_n) = x_n^k$  for  $0 < k < \infty$  since this implies  $z_1^2 - z_2^{k+1} = 0$  and thus  $\pi_{\mathcal{C}}(q)$  is a  
424 singularity of the type  $A_{2k}$ . We next prove that the projection is also singular if  $\text{mult}(g(z_n)) = \infty$ .

425 **Lemma 25.** *Preserving the notation and the assumptions in Lemma 24, consider the function  $g$  defined in (3.7), if*  
426  *$\text{mult}(g(z_n)) = \infty$ , then  $\pi_{\mathcal{C}}(q)$  is singular in  $\pi_{\mathcal{C}}(\mathcal{C})$ .*

427 *Proof.* Since  $\text{mult}(g(z_n)) = \infty$ , then **Case 1** in the proof of Lemma 24 holds. Moreover, we saw in the same  
428 proof that  $\xi_2(z_n^2)$  (restricted to an open neighborhood of  $0 \in \mathbb{R}$ ) cannot be the zero function. This implies that  
429 neither is the function  $g(z_n^2) = z_n^2 \xi(z_n^2)$ , i.e.,  $g(z_n^2)$ , restricted to an open neighborhood of  $0 \in \mathbb{R}$ , is not the zero  
430 function. Assume for the sake of contradiction that  $\pi_{\mathcal{C}}(q)$  is smooth in  $\pi_{\mathcal{C}}(\mathcal{C})$ , then using the implicit function  
431 theorem, there exists a  $C^\infty$ -function defined in a neighborhood of  $0$  in  $\mathbb{R}$ , with  $f(0) = 0$  such that for a small  
432 neighborhood of  $\pi_{\mathcal{C}}(q)$  in  $\mathbb{R}^2$ , one of the following cases is satisfied:

433 (a)  $f(z_1) = z_2 \iff (z_1, z_2) \in \pi_{\mathcal{C}}(\mathcal{C})$ . Then, by (3.7), we have  $f(z_n g(z_n^2)) = z_n^2$ . Taking the second  
434 derivative of both sides with respect to  $z_n$  and then evaluating at  $0$  (recall that  $\text{mult}(g(z_n)) = \infty$ ), we get  
435 the contradiction  $0 = 2$ .

436 (b)  $f(z_2) = z_1 \iff (z_1, z_2) \in \pi_{\mathcal{C}}(\mathcal{C})$ . Then  $f(z_n^2) = z_n g(z_n^2)$ . The function  $z_n g(z_n^2)$  is an odd function but  
437 not the zero function, and on the other hand  $f(z_2)$  is an even function, which leads to a contradiction.

438 Thus, in both cases we have a contradiction, that is,  $f$  does not exist and  $\pi_{\mathcal{C}}(q)$  cannot be smooth in  $\pi_{\mathcal{C}}(\mathcal{C})$ .  $\square$

439 Returning to (3.7), notice that  $\varphi$  is defined in such a way that it preserves the singularity class of  $\pi_{\mathcal{C}}(\mathcal{C})$  at the  
 440 point  $\pi_{\mathcal{C}}(q)$ . In other words, if  $C$  is the plane projection of the curve defined by  $Q$  then  $(\pi_{\mathcal{C}}(\mathcal{C}), 0)$  and  $(C, 0)$  are  
 441 equivalent.

#### 442 3.4. Proof of Theorem 11

443 We first characterize the singularities of the projected curve  $\pi_{\mathcal{C}}(\mathcal{C})$  by the points in  $\mathcal{L}_n$  and  $\mathcal{L}_c$ . The proof of  
 444 Theorem 11 is then obtained via the bijection  $\Omega_P$  (Definition 14) between  $\widehat{\mathcal{L}}$  and the solutions of the Ball system.

445 **Lemma 26.** *If  $P$  satisfies Assumptions  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{A}_3$  and  $\mathcal{A}_4$ , then a point  $q \in \mathcal{C}$  projects to a singular point in*  
 446  *$\pi_{\mathcal{C}}(\mathcal{C})$  if and only if  $q \in \mathcal{L}_c \cup \mathcal{L}_n$ .*

447 *Proof.* If  $q \in \mathcal{L}_c \cup \mathcal{L}_n$ , then by Lemmas 21, 24, and 25,  $\pi_{\mathcal{C}}(q)$  is singular in  $\pi_{\mathcal{C}}(\mathcal{C})$ . Conversely, if  $q \notin \mathcal{L}_c \cup \mathcal{L}_n$ ,  
 448 we prove that  $\pi_{\mathcal{C}}(q)$  is smooth in  $\pi_{\mathcal{C}}(\mathcal{C})$ .

449 Since  $q \notin \mathcal{L}_c$ , the plane projection of  $T_q\mathcal{C}$  is a line, or equivalently, the derivative  $T_q\pi_{\mathcal{C}}$  of  $\pi_{\mathcal{C}}$  at  $q$  is injective.  
 450 Thus,  $\pi_{\mathcal{C}}$  is an immersion at  $q$  ([Dem00, Definition 2.9.3]). Hence, for a small enough neighborhood  $U_0$  of  $q$  in  
 451  $\mathbb{R}^n$ , we have that  $\pi_{\mathcal{C}}$  restricted to  $V = U_0 \cap \mathcal{C}$  is an embedding (see [Dem00, Proposition 2.9.6]). We are going to  
 452 prove that, assuming that  $U_0$  is small enough, the curve  $\pi_{\mathcal{C}}(\mathcal{C})$  has exactly one branch around  $\pi_{\mathcal{C}}(q)$  which implies  
 453 that  $\pi_{\mathcal{C}}(\mathcal{C})$  is smooth at  $\pi_{\mathcal{C}}(q)$  since  $\mathcal{C}$  is smooth at  $q$  by Assumption  $\mathcal{A}_1$ .

454 To prove this claim, assume that there exists an open subset  $U'_0$  in  $\mathbb{R}^n$  such that the set  $V' = U'_0 \cap \mathcal{C}$  and  $V$  are  
 455 disjoint, but  $\pi_{\mathcal{C}}(q)$  is in the closure of  $\pi_{\overline{\mathcal{C}}}(V')$ . Let  $q_k$  be a sequence of points in  $V'$  such that  $\pi_{\mathcal{C}}(q_k)$  converges  
 456 to  $\pi_{\mathcal{C}}(q)$ . Since  $\overline{B}$  is compact, there exists a convergent sub-sequence of  $q_k$  that has a limit  $q'$  in  $\overline{B}$ . Notice that  
 457  $\pi_{\overline{\mathcal{C}}}(q') = \pi_{\mathcal{C}}(q)$  by the continuity of  $\pi_{\overline{\mathcal{C}}}$ . Hence,  $q, q'$  are both in  $\overline{\mathcal{L}_n}$ . However, since  $q \notin \mathcal{L}_n$ , we must have that  
 458  $q' \notin B$ . Hence,  $q'$  is in the boundary of  $B$  which contradicts Assumption  $\mathcal{A}_4$ . Hence, the curve  $\pi_{\mathcal{C}}(\mathcal{C})$  has exactly  
 459 one smooth branch around  $\pi_{\mathcal{C}}(q)$  which concludes the proof.  $\square$

460 Finally, we prove that the solutions of the Ball system project to the singular points of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

461 *Proof of Theorem 11:* By Lemma 26, if  $(x_1, x_2)$  is singular in  $\pi_{\mathcal{C}}(\mathcal{C})$ , then there exists a point  $q_1 \in \mathcal{L}_c \cup \mathcal{L}_n$ , with  
 462  $\pi_{\mathcal{C}}(q_1) = (x_1, x_2)$ . If  $q_1 \in \mathcal{L}_c$ , let  $q_2 = q_1$  and otherwise let  $q_2$  be the unique (by Assumption  $\mathcal{A}_3$ ) point in  $\mathcal{L}_n$ ,  
 463 distinct from  $q_1$ , that projects onto  $(x_1, x_2)$ , i.e.  $\pi_{\mathcal{C}}(q_1) = \pi_{\mathcal{C}}(q_2) = (x_1, x_2)$ . Hence,  $(q_1, q_2)$  is in  $\widehat{\mathcal{L}}$ . Since  $\Omega_P$  is  
 464 surjective (Lemma 15), there exists  $X = (x_1, x_2, y, r, t) \in \text{Sol}_{\text{Ball}(P)}$  with  $\Omega_P(X) = (q_1, q_2)$ .

465 On the other hand, if  $X$  is a solution of  $\text{Ball}(P)$ , then by Lemma 13 the pair  $(q_1, q_2) = \Omega_P(X)$  is in  $\widehat{\mathcal{L}}$ . Hence,  
 466  $q_1 = (x_1, x_2, y + r\sqrt{t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$  is in  $\mathcal{L}_c \cup \mathcal{L}_n$ . Hence, by Lemma 26 the point  $(x_1, x_2)$  is singular in  
 467  $\pi_{\mathcal{C}}(\mathcal{C})$ .  $\square$

468 **4. Regularity of the Ball system**

469 In this section, our goal is to prove Theorem 27 determining necessary and sufficient conditions for  $\text{Ball}(P)$  to  
470 be regular.

471 **Theorem 27.** *Let  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  that satisfies Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ , then  $P$  satisfies Assumption  
472  $\mathcal{A}_5^-$  if and only if  $\text{Ball}(P)$  is regular in  $B_{\text{Ball}}$ .*

473 In order to prove Theorem 27, we are going to show that the Jacobian matrices of  $\text{Ball}(P)$  and  $\text{Ball}(Q)$   
474 evaluated at  $X$  have the same rank, where  $Q$  is defined in Equation (3.7). Recall that Equation (3.7) implies that  
475  $P$  and  $Q$  define the same curve around  $q$ . Notice also that if  $X = (q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$  is in  $\Omega_P^{-1}((q, q))$ ,  
476 then  $X \in \Omega_Q^{-1}((q, q))$ .

477 **Lemma 28.** *Let  $P$  and  $Q$  be as defined in (3.7). Under Assumption  $\mathcal{A}_1$ , let  $(q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$  be a  
478 solution of the system  $\text{Ball}(P)$  in  $B_{\text{Ball}}$ , then  $\text{Ball}(P)$  is regular at  $(q, r, 0)$  if and only if  $\text{Ball}(Q)$  is regular at the  
479 point  $(0, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$  (recall that for simplicity, we assume in Lemma 24 that  $q = 0 \in \mathbb{R}^n$ ).*

480 *Proof.* Let us write  $X = (q, r, 0)$ . We are going to prove that the Jacobian matrices of  $\text{Ball}(P)$  and  $\text{Ball}(Q)$   
481 evaluated at  $X$  have the same rank. By Remark 16 we have that  $\Omega_P(X) = (q, q) \in \widehat{\mathcal{L}}_c$  (see Definitions 14 and  
482 12), and hence,  $q \in \mathcal{L}_c$ . By Lemma 13 we have that  $(0, 0, r) \in T_q \mathcal{C}$ . We prove the claim in three steps:

483 **Step 1:** Let  $\widetilde{M} = (f_{ij})_{1 \leq i, j \leq n-1}$  be as defined in the Equality (3.7). We define  $S \cdot \widetilde{M}$  (resp.  $D \cdot \widetilde{M}$ ) to be the  
484 matrix  $(S \cdot f_{ij})_{1 \leq i, j \leq n-1}$  (resp.  $(D \cdot f_{ij})_{1 \leq i, j \leq n-1}$ ). Using the identity  $\frac{1}{2}(ab + cd) = \frac{1}{4}(a + c)(b + d) + \frac{1}{4}(a -$   
485  $c)(b - d)$ , one deduces the properties for any  $f, g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ :

$$S \cdot fg = (S \cdot f)(S \cdot g) + t(D \cdot f)(D \cdot g) \quad (4.1)$$

$$D \cdot fg = (D \cdot f)(S \cdot g) + (S \cdot f)(D \cdot g) \quad (4.2)$$

These identities applied to Equation (3.7) yield

$$\begin{pmatrix} S \cdot Q_1 \\ \cdots \\ \cdots \\ S \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

and

$$\begin{pmatrix} D \cdot Q_1 \\ \cdots \\ \cdots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

Combining the last two equations:

$$\begin{pmatrix} S \cdot Q_1 \\ \cdots \\ \cdots \\ S \cdot Q_{n-1} \\ D \cdot Q_1 \\ \cdots \\ \cdots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix} \quad (4.3)$$

486 Notice that  $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}_X = \begin{pmatrix} \widetilde{M}(q) & 0 \\ D \cdot \widetilde{M}(X) & \widetilde{M}(q) \end{pmatrix}$  (recall that in our case we have  $S \cdot \widetilde{M}(X) = \widetilde{M}(q)$ )

487 and that the latter matrix has an inverse (by Lemma 24,  $\widetilde{M}(q)$  is an invertible matrix of size  $n - 1$ ), namely,

488  $\begin{pmatrix} \widetilde{M}(q)^{-1} & 0 \\ -\widetilde{M}(q)^{-1} \cdot (D \cdot \widetilde{M})(X) \cdot \widetilde{M}(q)^{-1} & \widetilde{M}(q)^{-1} \end{pmatrix}$  which implies (by continuity of the determinant function)

489 that  $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}$  is invertible in a neighborhood of  $X$ .

**Step 2:** Writing  $y = (y_3, \dots, y_n)$  and  $r = (r_3, \dots, r_n)$ , consider the diffeomorphism  $\varphi$  defined in Lemma 24 and define the smooth function  $\psi$  over an open subset of  $\mathbb{R}^{2n-1}$  containing  $X$  which maps the point  $(x_1, x_2, y, r, t)$  to  $(\varphi_1, \varphi_2, S \cdot \varphi_3, \dots, S \cdot \varphi_n, D \cdot \varphi_3, \dots, D \cdot \varphi_n, t)$ . Notice that we have:

$$S \cdot (P_j \circ \varphi) = (S \cdot P) \circ \psi \text{ and } D \cdot (P_j \circ \varphi) = (D \cdot P) \circ \psi, \text{ for } 1 \leq j \leq n - 1, \quad (4.4)$$

490 since  $\varphi_i(x_1, x_2, y \pm r\sqrt{t}) = \psi_i \pm \psi_{n+i-2} \sqrt{\psi_{2n-1}}$  for all  $i \in \{3, \dots, n\}$ . In fact, using the last two equations we  
491 can also see that  $\psi^{-1}$  exists and is smooth. Thus,  $\psi$  is a diffeomorphism.

492 **Step 3:** Now, comparing (4.3) with (4.4) we get:



$$SD \cdot Q := \begin{pmatrix} S \cdot Q_1 \\ \dots \\ \dots \\ S \cdot Q_{n-1} \\ D \cdot Q_1 \\ \dots \\ \dots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot P_1 \\ \dots \\ \dots \\ S \cdot P_{n-1} \\ D \cdot P_1 \\ \dots \\ \dots \\ D \cdot P_{n-1} \end{pmatrix} \circ \psi.$$

Consider the vector  $SD \cdot P = (S \cdot P_1, \dots, S \cdot P_{n-1}, D \cdot P_1, \dots, D \cdot P_{n-1})^T$  and let  $J_{SD \cdot P}$ ,  $J_{SD \cdot Q}$  and  $J_\psi$  be the Jacobian matrices of  $SD \cdot P$ ,  $SD \cdot Q$  and  $\psi$ , respectively. Taking the Jacobian matrix of both sides of the last equality:

$$J_{SD \cdot Q} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot J_{SD \cdot P} \cdot J_\psi + \text{Jacobian} \left( \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \right) \cdot \begin{pmatrix} S \cdot P_1 \\ \dots \\ \dots \\ S \cdot P_{n-1} \\ D \cdot P_1 \\ \dots \\ \dots \\ D \cdot P_{n-1} \end{pmatrix} \circ \psi.$$

Evaluating the last equality at  $X = (0, r, 0)$  and using the fact that  $\psi(X) = \psi(0, r, 0) = (0, r, 0) = X$ , we note that the second term of the right-hand side is zero. One has:

$$J_{SD \cdot Q}(X) = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}_X \cdot J_{SD \cdot P}(X) \cdot J_\psi(X). \quad (4.5)$$

493 Computing  $J_\psi(X)$ , we get  $J_\psi(X) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial z_1}(0) & \frac{\partial \varphi_1}{\partial z_2}(0) & 0_{1 \times (2n-3)} \\ \dots & \dots & \dots \\ 0_{(2n-2) \times 1} & I_{2n-2} & \dots \end{pmatrix}$ , with  $\frac{\partial \varphi_1}{\partial z_1}(0) \neq 0$  according to the  
494 formula in Lemma 24.

495 Hence by Equation (4.5), it is straightforward to check that:

$$\begin{aligned} J_{\text{Ball}(Q)} &= \begin{pmatrix} J_{SD \cdot Q}(X) \\ 2X \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X \cdot \begin{pmatrix} J_{SD \cdot P}(X) \\ 2X \end{pmatrix} \cdot J_\psi(X) \\ &= \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X \cdot J_{\text{Ball}(P)}(X) \cdot J_\psi(X). \end{aligned}$$

496 Recalling that  $J_\psi(X)$  and  $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X$  are invertible matrices, the proof of the lemma fol-  
497 lows. □

498 Now, we are ready to prove Theorem 27, which characterizes the regularity of the solutions of  $\text{Ball}(P)$  under  
499 generic assumptions. We split the proof in the two Lemmas 32 and 33. Before that, we introduce a new assumption  
500 that helps to simplify the proof.

501 **Definition 29.** Let  $(q_1, q_2) \in \widehat{\mathfrak{L}}$ . We say that  $(q_1, q_2)$  satisfies Assumption  $\mathcal{A}_5^{-'}$  if  $q_1$  and  $q_2$  are isolated in  $\mathfrak{L}_n \cup \mathfrak{L}_c$   
502 and the following conditions are satisfied:

- 503 (a) If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$ , then the plane projections of the tangent lines of  $q_1$  and  $q_2$  to  $\mathfrak{C}$  are linearly independent.  
504 (b) If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$ , then the plane projection of a small enough neighborhood of  $q_1$  in  $\mathfrak{C}$  is an ordinary cusp at  
505  $\pi_{\mathfrak{C}}(q_1)$  and the multiplicity of the system  $\{P(x) = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$  at  $q_1$  is two.

506 Assumption  $\mathcal{A}_5^{-'}$  can be seen as a "local version" of Assumption  $\mathcal{A}_5^-$ . We are going to prove that if Assumptions  
507  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$  are satisfied, then Assumption  $\mathcal{A}_5^-$  is equivalent to the condition that Assumption  $\mathcal{A}_5^{-'}$  is  
508 satisfied for all  $\widehat{\mathfrak{L}}$ .

509 The main reason behind introducing Assumption  $\mathcal{A}_5^{-'}$ , is that we are going to prove in Lemma 32 that, under  
510 Assumption  $\mathcal{A}_1$ , a pair  $(q_1, q_2) \in \widehat{\mathfrak{L}}$  satisfies Assumption  $\mathcal{A}_5^{-'}$  if and only if every  $X$  in  $\Omega_P^{-1}(q_1, q_2)$  is a regular  
511 solution of  $\text{Ball}(P)$ , whereas Assumption  $\mathcal{A}_5^-$  is, in general, not sufficient for the regularity of the solutions of  
512  $\text{Ball}(P)$ . For example, take  $n = 3$  and  $P = (x_1 - x_3^6, x_2 - x_3^9)$ . We can see that  $P$  satisfies Assumption  $\mathcal{A}_1$ ,  
513 the set  $\mathfrak{L}_c$  consists of a unique point  $q = (0, 0, 0)$  and the set  $\mathfrak{L}_n$  is empty. The plane projection of  $\mathfrak{C}$  is the curve  
514 given by the equation  $x_1^3 - x_2^2 = 0$ . Hence, Assumption  $\mathcal{A}_5^-$  is satisfied. However, the multiplicity of the system  
515  $\{P(x_1, x_2, x_3) = 0 \in \mathbb{R}^2, x_1 = x_2 = 0\}$  at the point  $q$  equals 6 (Lemma 23). Hence, Assumption  $\mathcal{A}_5^{-'}$  is not  
516 satisfied and one can also check that  $\text{Ball}(P)$  is not regular.

517 The next definition and lemma are technical tools to handle the case of nodes in Lemma 32, and later in  
518 Lemma 54.

519 **Definition 30.** Consider  $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  satisfying Assumption  $\mathcal{A}_1$  and recall that we  
520 denote the Jacobian matrix of  $P$  at the point  $q$  by  $J_P(q)$ . We define the  $(n-1) \times (n-2)$  sub-matrix  $M_P(q)$  obtained  
521 by removing the first two columns of  $J_P(q)$  and the  $(n-1) \times 2$  sub-matrix  $N_P(q)$  formed by the first two columns  
522 of  $J_P(q)$ . Let  $q_1, q_2 \in \mathfrak{C}$ , we define the  $2n-2$  square matrix  $M(q_1, q_2) = \begin{pmatrix} N_P(q_1) & 0 & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & 0 \end{pmatrix}$ .

523 **Lemma 31.** Using the same assumption and notation as in Definition 30, let  $q_1$  and  $q_2$  be distinct points of  $\mathfrak{C}$  with  
524  $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ , then  $M(q_1, q_2)$  is invertible if and only if neither  $q_1$  nor  $q_2$  is in  $\mathfrak{L}_c$  and the plane projections of  
525 the tangent lines of  $\mathfrak{C}$  at  $q_1$  and  $q_2$  do not coincide.

526 *Proof.* We prove the converse statement using

$$\det(M(q_1, q_2)) = 0 \iff \text{There exist } \alpha \in \mathbb{R}^2 \text{ and } \beta, \gamma \in \mathbb{R}^{n-2} \text{ such that the vector}$$

$$x = (\alpha, \beta, \gamma) \text{ is not trivial and } M(q_1, q_2) \cdot x^T = 0.$$

527

$$\iff (\alpha, \beta) \text{ and } (\alpha, \gamma) \text{ are in the tangent lines } T_{q_1} \mathcal{C} \text{ and } T_{q_2} \mathcal{C}, \text{ respectively,}$$

$$\text{and at least one of them is not trivial.}$$

528

The last statement can be split in two cases:

529

•  $\alpha$  is not trivial, which is equivalent to saying that the plane projections of  $T_{q_1} \mathcal{C}$  and  $T_{q_2} \mathcal{C}$  are both generated by  $\alpha$  and coincide.

530

531

•  $\alpha = (0, 0)$ , which is equivalent to  $\beta$  or  $\gamma$  are not trivial, which is equivalent to  $T_{q_2} \mathcal{C}$  or  $T_{q_1} \mathcal{C}$  projects to a point in the plane, which is equivalent to  $q_1$  or  $q_2$  is in  $\mathcal{L}_c$ .  $\square$

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533

**Lemma 32.** Let  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  satisfy Assumption  $\mathcal{A}_1$ . Let  $X$  be a solution of  $\text{Ball}(P)$  and  $(q_1, q_2) = \Omega_P(X)$  (Definition 14), then  $X$  is a regular solution of  $\text{Ball}(P)$  if and only if  $(q_1, q_2)$  satisfies Assumption  $\mathcal{A}_5'$ .

534

535

*Proof.* Let  $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$  be a solution of  $\text{Ball}(P)$ . We consider two cases:

536

**Case (a):**  $t \neq 0$ , i.e.,  $q_1 \neq q_2$ .

537

It is easy to see that  $\frac{\partial(S \cdot P_i)}{\partial x_j}, \frac{\partial(D \cdot P_i)}{\partial x_j}, \frac{\partial(S \cdot P_i)}{\partial r_k}, \frac{\partial(D \cdot P_i)}{\partial r_k}, \frac{\partial(S \cdot P_i)}{\partial t}, \frac{\partial(D \cdot P_i)}{\partial t}$  are, respectively, equal to:  $S \cdot \frac{\partial(P_i)}{\partial x_j}, D \cdot$

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$\frac{\partial(P_i)}{\partial x_j}, t \cdot D \cdot \frac{\partial(P_i)}{\partial x_k}, S \cdot \frac{\partial(P_i)}{\partial x_k}, \frac{1}{2} \sum_{m=3}^n D \cdot \left( \frac{\partial P_i}{\partial x_m} \right) \cdot r_m, \frac{1}{2t} \left[ \sum_{m=3}^n S \cdot \left( \frac{\partial P_i}{\partial x_m} \right) \cdot r_m - D \cdot P_i \right]$ . Hence, by computing the

539

Jacobian matrix of the  $\text{Ball}(P)$  we get the matrix:

$$\begin{pmatrix} S \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & S \cdot \frac{\partial P_1}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2} \sum_{m=3}^n D \cdot \left( \frac{\partial P_1}{\partial x_m} \right) \cdot r_m \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2} \sum_{m=3}^n D \cdot \left( \frac{\partial P_{n-1}}{\partial x_m} \right) \cdot r_m \\ D \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & D \cdot \frac{\partial(P_1)}{\partial x_n} & S \cdot \frac{\partial(P_1)}{\partial x_3} \dots & S \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2t} \left[ \sum_{m=3}^n S \cdot \left( \frac{\partial P_1}{\partial x_m} \right) \cdot r_m - D \cdot P_1 \right] \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ D \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & S \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2t} \left[ \sum_{m=3}^n S \cdot \left( \frac{\partial P_{n-1}}{\partial x_m} \right) \cdot r_m - D \cdot P_{n-1} \right] \\ 0 & \dots & 0 & 2r_3 \dots & 2r_n & 0 \end{pmatrix}.$$

We denote by  $C_i$  (resp.  $L_i$ ) the  $i$ -th column (resp. line) of the latter matrix. Replace the last column  $C_{2n-1}$  with  $\sum_{m=1}^{n-2} \frac{r_{m+2}}{2t} C_{n+m} + C_{2n-1}$ , also for all integers  $1 \leq k \leq n-1$  we replace the line  $L_k$  with  $L_k + \sqrt{t} \cdot L_{k+n-1}$  and

then the line  $L_{k+n-1}$  with  $L_k - 2\sqrt{t}L_{k+n-1}$ . The resulting matrix is:

$$\begin{pmatrix} \frac{\partial(P_1)}{\partial x_1}(q_1) & \dots & \frac{\partial P_1}{\partial x_n}(q_1) & \sqrt{t} \cdot \frac{\partial(P_1)}{\partial x_3}(q_1) & \dots & \sqrt{t} \frac{\partial(P_1)}{\partial x_n}(q_1) & 0 \\ & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial(P_{n-1})}{\partial x_1}(q_1) & \dots & \frac{\partial P_{n-1}}{\partial x_n}(q_1) & \sqrt{t} \cdot \frac{\partial(P_{n-1})}{\partial x_3}(q_1) & \dots & \sqrt{t} \frac{\partial(P_{n-1})}{\partial x_n}(q_1) & 0 \\ \frac{\partial(P_1)}{\partial x_1}(q_2) & \dots & \frac{\partial(P_1)}{\partial x_n}(q_2) & -\sqrt{t} \frac{\partial(P_1)}{\partial x_3}(q_2) & \dots & -\sqrt{t} \frac{\partial(P_1)}{\partial x_n}(q_2) & 0 \\ \dots & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial(P_{n-1})}{\partial x_1}(q_2) & \dots & \frac{\partial(P_{n-1})}{\partial x_n}(q_2) & -\sqrt{t} \frac{\partial(P_{n-1})}{\partial x_3}(q_2) & \dots & -\sqrt{t} \frac{\partial(P_{n-1})}{\partial x_n}(q_2) & 0 \\ 0 \dots & & 0 & 2r_3 & \dots & 2r_n & \frac{1}{2t} \end{pmatrix}.$$

The determinant of the latter matrix is zero if and only if the determinant of the following matrix is zero:

$$M_0 = \begin{pmatrix} N_P(q_1) & M_P(q_1) & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & -M_P(q_2) \end{pmatrix},$$

where  $M_P(q_1), M_P(q_2)$  are the minors that are obtained, respectively, by removing the first two columns from  $J_P(q_1), J_P(q_2)$  and  $N_P(q_1), N_P(q_2)$  are the matrices formed by the first two columns of  $J_P(q_1), J_P(q_2)$ , respectively. By linear operations on  $M_0$ , we can see that  $M_0$  has same rank as the matrix  $M(q_1, q_2)$  (see Definition 30). Thus,  $X$  is regular for  $\text{Ball}(P)$  if and only if  $M(q_1, q_2)$  is invertible. By Lemma 31 we have that  $M(q_1, q_2)$  is invertible if and only if none of  $q_1, q_2$  is in  $\mathfrak{L}_c$  (and hence none of the plane projections of  $T_{q_1}\mathfrak{C}, T_{q_2}\mathfrak{C}$  is trivial) and the plane projection of their tangent spaces are different. Equivalently, the pair  $(q_1, q_2)$  is in  $\widehat{\mathfrak{L}}_n$  and satisfies Assumption  $\mathcal{A}_5^{-'}$ .

**Case (b):**  $t = 0$ , i.e.,  $q_1 = q_2$ .

Let us write  $q = q_1$ . We prove the claim in three steps:

Step 1: We first simplify  $P$ . Without loss of generality and by Lemma 22 we can assume that  $q = 0$  and  $P_1, \dots, P_{n-1}$  are, respectively, equal to  $x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n)$  with the property that  $\min\{\text{mult}(f_1), \text{mult}(f_2)\} \geq 2$ . For all  $i \in \{3, \dots, n-1\}$ , using Taylor's theorem, we can write  $f_i(x_n) = \sum_{j=1}^3 a_{i,j} x_n^j + x_n^4 h_i(x_n)$ , for some  $a_{i,j} \in \mathbb{R}$  and smooth functions  $h_i(x_n)$ . Since  $\min\{\text{mult}(f_1), \text{mult}(f_2)\} \geq 2$ , we can write  $f_1(x_n) = \sum_{j=2}^3 \alpha_j x_n^j + x_n^4 h_1(x_n)$  and  $f_2(x_n) = \sum_{j=2}^3 \beta_j x_n^j + x_n^4 h_2(x_n)$ . Notice that

$$(f_1(x_n), f_2(x_n), f_3(x_n), \dots, f_{n-1}(x_n), x_n)$$

is a local parameterization system of  $\mathfrak{C}$  around  $q$ . Since  $\dim(T_q\mathfrak{C}) = 1$  (Assumption  $\mathcal{A}_1$ ), there exists  $\lambda \in \mathbb{R}^*$  with  $(a_{3,1}, \dots, a_{n-1,1}, 1) = \lambda r$  (because the vectors  $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$  and  $(0, 0, a_{1,3}, \dots, a_{1,n-1}, 1)$  are in  $T_q\mathfrak{C} \setminus \{0\}$ ). In particular,  $r_n \neq 0$ .

Step 2: Now, we compute  $J_{\text{Ball}(P)}$  at  $X = (x_1, x_2, y, r, 0)$  by first computing it for  $X_t = (x_1, x_2, y, r, t)$  with  $t \neq 0$ , and then taking the limit when  $t$  goes to 0. Since the operator  $S$  is linear, we write  $S(x_i - f_i(x_n)) = S(x_i - \sum_{j=1}^3 a_{i,j} x_n^j) - S(x_n^4 h_i(x_n))$ . On the other hand, using the identity (4.1) we deduce that  $S(x_n^4 h_i(x_n)) =$

556  $S(x_n^4) \cdot S(h_i(x_n)) + tD(x_n^4) \cdot D(h_i(x_n))$ , for all  $i \in \{1, \dots, n-1\}$ . It is straightforward to see that  $S(x_n^4) =$   
557  $r_n^4 t^2 + 6r_n^2 t x_n^2 + x_n^4$  and  $tD(x_n^4) = 4r_n^3 x_n t^2 + 4r_n x_n^3 t$  with  $r = (r_3, \dots, r_n)$ . Hence, all of the first-order partial  
558 derivatives of  $S(x_n^4 h_i(x_n))$ , evaluated at  $X_t$ , converge to zero when  $t$  goes to 0. Hence, the partial derivatives of  
559 the functions  $S(x_i - f_i(x_n))$  and  $S(x_i - \sum_{j=1}^3 a_{i,j} x_n^j)$  evaluated at  $X$  are equal. Using an analogous argument, we  
560 deduce that the evaluation of the partial derivatives of the functions  $D(x_i - f_i(x_n))$  and  $D(x_i - \sum_{j=1}^3 a_{i,j} x_n^j)$ , at  
561  $X$  are also equal. Thus,  $J_{\text{Ball}(P)}(X_t)$  and  $J_{\text{Ball}(\bar{P})}(X_t)$  converge to the same limit  $J_{\text{Ball}(P)}(X)$ , where  $\bar{P}$  is the  
562 function obtained by truncating  $P$  beyond degree 3 with respect to the variable  $x_n$ .

Computing  $J_{\text{Ball}(P)}(X) = \lim_{t \rightarrow 0} J_{\text{Ball}(\bar{P})}(X_t)$ , we get:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -\alpha_2 r_n^2 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & -\beta_2 r_n^2 \\ 0 & 0 & \dots & \dots & -a_{3,1} & 0 & \dots & 0 & -a_{3,2} r_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1,1} & 0 & \dots & 0 & -a_{n-1,2} r_n^2 \\ 0 & 0 & \dots & \dots & -2\alpha_2 r_n & 0 & \dots & 0 & -\alpha_3 r_n^3 \\ 0 & 0 & \dots & \dots & -2\beta_2 r_n & 0 & \dots & 0 & -\beta_3 r_n^3 \\ 0 & 0 & \dots & \dots & -2a_{3,2} r_n & 1 & \dots & -a_{3,1} & -a_{3,3} r_n^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & -2a_{n-1,2} r_n & 0 & \dots & 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\ 0 & 0 & \dots & \dots & 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n & 0 \end{pmatrix}.$$

Hence, observing that the matrix is block diagonal, its determinant is zero if and only if the determinant of the following one is:

$$\begin{pmatrix} -2\alpha_2 r_n & 0 & \dots & 0 & 0 & -\alpha_3 r_n^3 \\ -2\beta_2 r_n & 0 & \dots & 0 & 0 & -\beta_3 r_n^3 \\ -2a_{3,2} r_n & 1 & 0 \dots & 0 & -a_{3,1} & -a_{3,3} r_n^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2a_{n-1,2} r_n & 0 & 0 \dots & 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\ 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n & 0 \end{pmatrix}.$$

Shifting the columns of the last matrix we get:

$$\begin{pmatrix} -\alpha_3 r_n^3 & -2\alpha_2 r_n & 0 & \dots & 0 & 0 \\ -\beta_3 r_n^3 & -2\beta_2 r_n & 0 & \dots & 0 & 0 \\ -a_{3,3} r_n^3 & -2a_{3,2} r_n & 1 & 0 \dots & 0 & -a_{3,1} \\ & \dots & \dots & & & \\ -a_{n-1,3} r_n^3 & -2a_{n-1,2} r_n & 0 & 0 \dots & 1 & -a_{n-1,1} \\ 0 & 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n \end{pmatrix}.$$

563 To compute the determinant of the second block, we expand it about the last row. Hence, the determinant of  
 564 the last matrix is zero if and only if  $r_n(\alpha_2\beta_3 - \alpha_3\beta_2)(r_n + \sum_{i=3}^{n-1} a_{i,1}r_i) = 0$ . Notice that, by Step 1, we have that  
 565  $r_n \neq 0$  and the third factor  $(r_n + \sum_{i=3}^{n-1} a_{i,1}r_i)$  is never zero since it is equal to  $\lambda$ . Thus,  $J_{\text{Ball}(P)}(X)$  is invertible iff

566  $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$ , equivalently, the matrix  $A = \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}$  is invertible.

567 Step 3: We now show that the invertibility of  $A$  is equivalent to the condition that  $(q, q)$  satisfies Assump-  
 568 tion  $\mathcal{A}_5^{-'}$ .

569 First assume that  $A$  is invertible. It follows that either  $\alpha_2 \neq 0$  or  $\beta_2 \neq 0$  and this yields that the minimum  
 570 of the multiplicities of  $f_1$  and  $f_2$  is 2. By Lemma 23, the multiplicity of the system  $\{P(x_1, x_2, y) = 0 \in$   
 571  $\mathbb{R}^{n-1}, (x_1, x_2) = \pi_{\mathcal{E}}(q)\}$  at  $q$  is equal to 2, thus Assumption  $\mathcal{A}_5^{-'}$  (b) is satisfied. Using the same notation as  
 572 in the proof of Lemma 24, one can write  $\tilde{f}_1(z_n) = z_n^2(\xi_1(z_n^2) + z_n\xi_2(z_n^2))$ . Notice that  $\xi_2(z_n^2)$  cannot be the zero  
 573 function, otherwise  $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$  and  $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$  for all small enough  $\epsilon > 0$ , which means that  $X$  would  
 574 be the limit of solutions  $X_\epsilon$  of  $\text{Ball}(P)$  with  $\Omega_P(X_\epsilon) \in \widehat{\mathcal{L}}_n$ .  $X$  would then be a non-isolated solution and thus a  
 575 non-regular solution of  $\text{Ball}(P)$  which contradicts the assumption. We then have two cases as in Lemma 24. The  
 576 first one is when  $\text{mult}(\xi_2(z_n)) = \infty$ , that would imply that  $\alpha_2 = \alpha_3 = 0$  and contradicts the invertibility of  $A$ .  
 577 We then must satisfy the second case  $\text{mult}(\xi_2(z_n)) = k < \infty$  and, after a change of variables, the first equation  
 578 of the system becomes equivalent to  $z_1 - z_n^{2k+3} = 0$ . The invertibility of  $A$  implies that  $k = 0$ . The projection  
 579 of the curve in the plane is thus locally parameterized by  $(z_n^3, z_n^2)$  and is an ordinary cusp, Assumption  $\mathcal{A}_5^{-'}$  (a) is  
 580 satisfied.

581 Second, assume that Assumption  $\mathcal{A}_5^{-'}$  is satisfied. By Lemma 23 and Assumption  $\mathcal{A}_5^{-'}$  (b), the minimum of  
 582 the multiplicities of  $f_1$  and  $f_2$  is 2. Using the proof of Lemma 24 once again, one can assume that  $f_2(z_n) = z_n^2$   
 583 and  $f_1(z_n) = z_n g(z_n^2)$  or  $f_1(z_n) = z_n^{2k+3}$ . By Assumption  $\mathcal{A}_5^{-'}$  (a), the projection is an ordinary cusp and thus  
 584 has a parameterization of the form  $(z_n^2, z_n^3)$ , that is  $f_1(z_n) = z_n^3$ . This implies that  $A$  is equivalent to  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  
 585 hence is invertible.  $\square$

586 **Lemma 33.** *If Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$  are satisfied, then Assumption  $\mathcal{A}_5^-$  is satisfied if and only if*  
 587 *Assumption  $\mathcal{A}_5^{-'}$  is satisfied for all  $(q_1, q_2) \in \widehat{\mathcal{L}} \subset B \times B$ .*

588 *Proof.* Assume that Assumption  $\mathcal{A}_5^-$  is satisfied and  $(q_1, q_2) \in \widehat{\mathfrak{L}}$ . If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$ , then by Lemma 24 and  
589 Assumption  $\mathcal{A}_5^-$  we must have that the plane projection of a small enough neighborhood of  $q_1$  in  $\mathfrak{C}$  is an ordinary  
590 cusp at  $\pi_{\mathfrak{C}}(q_1)$ . By Assumption  $\mathcal{A}_3$  and Lemma 20, the multiplicity of the mentioned system at  $q_1 = q_2$  is two.  
591 Thus,  $(q_1, q_2)$  satisfies Assumption  $\mathcal{A}_5^{-'}$ . If  $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$ , then by Lemma 21 and Assumption  $\mathcal{A}_5^-$ , we have that  
592  $\pi_{\mathfrak{C}}(q_1)$  is a node in  $\pi_{\mathfrak{C}}(\mathfrak{C})$ . Thus, we have that  $\pi_{\mathfrak{C}}(q_1)$  is a transverse intersection of two smooth branches of  
593  $\pi_{\mathfrak{C}}(\mathfrak{C})$ . Those branches are the plane projections of two disjoint branches of  $\mathfrak{C}$  each of which contains either  $q_1$  or  
594  $q_2$ . Hence, the plane projections of the tangent spaces of  $q_1$  and  $q_2$  to  $\mathfrak{C}$  are linearly independent. Thus,  $(q_1, q_2)$   
595 satisfies Assumption  $\mathcal{A}_5^{-'}$ .

596 Assume conversely that  $\mathcal{A}_5^{-'}$  is satisfied for all  $(q_1, q_2) \in \widehat{\mathfrak{L}}$ . By Lemma 26, any singular point of  $\pi_{\mathfrak{C}}(\mathfrak{C})$  is the  
597 plane projection of a point  $q_1 \in \mathfrak{L}_c \cup \mathfrak{L}_n$ . For some  $q_2 \in \mathfrak{C}$ , the pair  $(q_1, q_2)$  is in  $\widehat{\mathfrak{L}}$  (which satisfies Assumption  
598  $\mathcal{A}_5^{-'}$ ). Hence, if  $(q_1, q_2)$  is in  $\widehat{\mathfrak{L}}_n$  (resp. in  $\widehat{\mathfrak{L}}_c$ ) the plane projection of  $q_1$  is a node (resp. an ordinary cusp) by  
599 Lemma 21 (resp. Lemma 24).  $\square$

600 Lemmas 32 and 33 then imply Theorem 27.

## 601 5. Semi-algorithms to check assumptions and isolate singularities

602 In this section we present Semi-algorithm 3 that checks the weak assumptions of Section 2.4 and, if it termi-  
603 nates, outputs a superset of isolating boxes of the singularities of the plane projection  $\pi_{\mathfrak{C}}(\mathfrak{C})$  of  $\mathfrak{C}$ . We also present  
604 Semi-algorithm 4 that checks the strong assumptions of Section 2.4 and, if it terminates, outputs a set of isolating  
605 boxes of the singularities of  $\pi_{\mathfrak{C}}(\mathfrak{C})$ .

606 The main idea of these semi-algorithms comes from Theorems 11 and 27: Theorem 11 states that, under  
607 Assumptions  $\mathcal{A}_{1-4}$ , the singularities of  $\pi_{\mathfrak{C}}(\mathfrak{C})$  are the plane projections of the solutions of  $\text{Ball}(P)$ . Theorem 27  
608 states that, under the further Assumption  $\mathcal{A}_5^-$ ,  $\text{Ball}(P)$  is regular, so we can use certified numerical methods such  
609 as interval Newton methods [MKC09] to solve  $\text{Ball}(P)$ . In addition, in order to verify these assumptions, we use  
610 subdivision approaches based on interval arithmetic in a semi-algorithm framework.

611 We present in Section 5.1 the basics of interval arithmetic with the notation and definitions by Lin and  
612 Yap [LY11] and our semi-algorithms in Section 5.2.

### 613 5.1. Interval arithmetic

614 Recall that for some positive integer  $k$ , by a closed (resp. open)  $k$ -box  $\mathfrak{B}$ , we mean the Cartesian product of  
615  $k$  closed (resp. open) intervals. The width of a box  $\mathfrak{B}$ , denoted by  $w(\mathfrak{B})$ , is the maximal length of the intervals  
616 of that product. For a subset  $A \subset \mathbb{R}^k$ , the set  $IA$  is the set of all closed  $k$ -boxes that are contained in  $A$ . For  
617 the positive integer  $m$  and a function  $f : A \rightarrow \mathbb{R}^m$ , the function  $\square f : IA \rightarrow I\mathbb{R}^m$  is called an inclusion of  $f$  if  
618 the set  $f(\mathfrak{B}) = \{f(x) \mid x \in \mathfrak{B}\}$  is contained in  $\square f(\mathfrak{B})$ , for all  $\mathfrak{B} \in IA$ . An inclusion  $\square f$  of  $f$  is called a box  
619 function, if for any descending sequence of closed  $k$ -boxes  $\mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \dots$  that converges to a point  $q \in \mathbb{R}^k$ , the  
620 sequence  $\square f(\mathfrak{B}_1) \supset \square f(\mathfrak{B}_2) \supset \dots$  converges to  $f(q)$ . In the rest of this section, we assume that we are given a

621 box function  $\square f$  for any function  $f$  we consider. The command *subdivide* is applied to a closed  $k$ -box  $\overline{\mathfrak{B}}$ , and it  
 622 returns the set of boxes obtained by bisecting  $\overline{\mathfrak{B}}$  in all dimensions.

623 An interval matrix  $\square M$  is a matrix whose coefficients are intervals. It can also be seen as the set of all  
 624 matrices whose  $(i, j)$ -th coefficients belong to the  $(i, j)$ -th interval. The rank of an interval matrix  $\square M$ , denoted  
 625 by  $\text{rank}(\square M)$ , is the minimum of the ranks of all the matrices in this set.

## 626 5.2. Semi-algorithms

627 This section is dedicated to the proof of the following theorem. Recall that the weak and strong assumptions  
 628 are defined in Definition 8.

629 **Theorem 34.** *For an open  $n$ -box  $B$  and a smooth function  $P$  from  $\overline{B}$  to  $\mathbb{R}^{n-1}$ , Semi-algorithm 3 stops if and only  
 630 if  $P$  satisfies the weak assumptions in  $\overline{B}$  and then it returns a set of isolating boxes of all the singularities of  $\pi_{\mathfrak{C}}(\mathfrak{C})$ ,  
 631 plus possibly other spurious disjoint boxes. Semi-algorithm 4 stops if and only if  $P$  satisfies the strong assumptions  
 632 in  $\overline{B}$  and then it returns a set of isolating boxes of all the singularities of  $\pi_{\mathfrak{C}}(\mathfrak{C})$ .*

633 To check whether a given function  $P$  satisfies the weak assumptions ( $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$ ) in  $\overline{B}$ , we use  
 634 their relation to the solutions of  $\text{Ball}(P)$  studied in the previous sections. Recall that for any subset  $A \subseteq \mathbb{R}^n$ , we  
 635 defined  $A_{\text{Ball}} = \{(x_1, x_2, y, r, t) \mid t \geq 0, (x_1, x_2, y + r\sqrt{t}), (x_1, x_2, y - r\sqrt{t}) \in A, \|r\|^2 = 1\}$ . Let  $B$  be an open  
 636  $n$ -box and  $P$  be a smooth function from  $\overline{B}$  to  $\mathbb{R}^{n-1}$  that satisfies Assumption  $\mathcal{A}_1$  in  $\overline{B}$ . Consider the following  
 637 assumptions:

638  $\aleph_1$  - All solutions of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$  are regular.

639  $\aleph_2$  - For every solution  $X$  of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$ , none of the points of the pair  $\Omega_P(X)$  (Definition 14) is in the  
 640 boundary of  $B$ .

641  $\aleph_3$  - No two distinct solutions of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$ , except the twin solutions (Remark 17), have the same plane  
 642 projection.

643 The next lemma shows the relation between these new assumptions and those of Section 2.4. The motivation  
 644 of these alternative assumptions is that they are stated in terms of  $\text{Ball}(P)$ , which makes them easier to verify in  
 645 our semi-algorithms.

646 **Lemma 35.** *Let  $B$  be an open  $n$ -box and  $P$  be a smooth function from  $\overline{B}$  to  $\mathbb{R}^{n-1}$  that satisfies Assumption  $\mathcal{A}_1$   
 647 in  $\overline{B}$ . Then, Assumptions  $\aleph_1, \aleph_2$  and  $\aleph_3$  are satisfied if and only if Assumptions  $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$  are satisfied  
 648 in  $\overline{B}$ .*

649 *Proof.* If Assumptions  $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$  are satisfied in  $\overline{B}$ , then by Theorem 27 we have Assumption  $\aleph_1$  is  
 650 satisfied. Moreover, by Assumptions  $\mathcal{A}_2$  and  $\mathcal{A}_4$  we have that none of  $\overline{\mathfrak{L}}_n, \overline{\mathfrak{L}}_c$  intersects the boundary of  $B$ . By  
 651 Definition 14, for any solution  $X$  of  $\text{Ball}(P)$ , we have that the points of the pair  $\Omega_P(X)$  are in  $\overline{\mathfrak{L}}_n \cup \overline{\mathfrak{L}}_c$  and hence



652 are not on the boundary of  $B$ , which implies that Assumption  $\aleph_2$  is satisfied. Assume that Assumption  $\aleph_3$  is not  
653 satisfied, that is, there exist two distinct non-twin solutions  $X, X'$  that have the same plane projection  $p \in \mathbb{R}^2$ . Let  
654  $(q_1, q_2) = \Omega_P(X)$  and  $(q'_1, q'_2) = \Omega_P(X')$ . By Lemma 13, the pairs  $(q_1, q_2), (q'_1, q'_2)$  are distinct and the points  
655  $q_1, q_2, q'_1, q'_2$  have the same plane projection  $p$ . By Assumption  $\mathcal{A}_3$ , we cannot have three pairwise distinct points  
656 among  $q_1, q_2, q'_1, q'_2$ . Moreover, if the multiplicity at all of the points  $q_1, q_2, q'_1, q'_2$  is one, then  $(q_1, q_2), (q'_1, q'_2)$   
657 are in  $\widehat{\mathfrak{L}}_n$  and not distinct. Hence, at least a point say  $q_1$  has multiplicity larger than one, i.e.,  $q_1 \in \mathfrak{L}_c$  (Lemma  
658 20). Hence, the number of solutions counted with multiplicity is at least three which contradicts Assumption  $\mathcal{A}_3$ .  
659 Hence, Assumption  $\aleph_3$  is satisfied.

660 Now, assume that Assumptions  $\aleph_1, \aleph_2$  and  $\aleph_3$  are satisfied. Since, by Assumption  $\aleph_1$ ,  $\text{Ball}(P)$  is a regular  
661 square system, its solution set is a zero-dimensional manifold in the compact set  $\overline{B}_{\text{Ball}(P)}$  (regular value theorem).  
662 Hence,  $\text{Ball}(P)$  has a finite number of solutions in  $\overline{B}_{\text{Ball}}$ . Since  $\Omega_P$  (Definition 14) is surjective (Lemma 15),  
663 the set  $\widehat{\mathfrak{L}}$  (Definition 12) is also finite. Hence, the set  $\mathfrak{L}_c \cup \mathfrak{L}_n$  is finite (since  $\mathfrak{L}_c \cup \mathfrak{L}_n$  is the image of  $\widehat{\mathfrak{L}}$  under  
664 the surjective function  $(q_1, q_2) \rightarrow q_1$ ). Moreover, by Assumption  $\aleph_2$ , the set  $\overline{\mathfrak{L}}_n \cup \overline{\mathfrak{L}}_c$  does not intersect the  
665 boundary of  $B$ . Hence, Assumptions  $\mathcal{A}_2$  and  $\mathcal{A}_4$  are satisfied in  $\overline{B}$ . To prove that Assumption  $\mathcal{A}_3$  is satisfied, let  
666  $p = (\alpha, \beta) \in \pi_{\mathfrak{C}}(\mathfrak{C})$  and  $|\pi^{-1}(p)| \geq 3$ . For pairwise distinct points  $q_1, q_2, q_3 \in \pi^{-1}(p)$ , by Lemma 13, we have  
667 that there exist two distinct non-twin solutions  $X, X'$  of  $\text{Ball}(P)$ , with  $\Omega_P(X) = (q_1, q_2)$  and  $\Omega_P(X') = (q_1, q_3)$   
668 such that we have the same plane projection  $p$  which contradicts Assumption  $\aleph_3$ . Hence,  $\pi_{\mathfrak{C}}^{-1}(p)$  consists of at  
669 most two distinct points. We consider two cases:

- 670 (a)  $\pi_{\mathfrak{C}}^{-1}(p)$  has two distinct elements, say  $q_1, q_2$ . By Lemma 13, the pair  $(q_1, q_2)$  is in  $\widehat{\mathfrak{L}}_n$ , and hence, there exists  
671 a solution  $X = (\alpha, \beta, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$  of  $\text{Ball}(P)$ , with  $t \neq 0$  and  $\Omega_P(X) = (q_1, q_2)$ .  
672 Since  $X$  is a regular solution (Assumption  $\aleph_1$ ), by Lemma 32 we have that none of  $q_1, q_2$  is in  $\mathfrak{L}_c$ . Hence,  
673 by Lemma 20, the multiplicity of  $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$  at  $q_1$  (resp.  $q_2$ ) is one.  
674 Thus, the number of solutions counted with multiplicity is two.
- 675 (b)  $\pi_{\mathfrak{C}}^{-1}(p)$  has a unique point  $q$ . Let  $m$  denote the multiplicity of the system  $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 -$   
676  $\alpha = x_2 - \beta = 0\}$  at  $q$ . If  $m = 1$ , then we are done. If  $m > 1$ , then by Lemma 20 we have that  $q \in \mathfrak{L}_c$ .  
677 Hence, there exists a solution of  $\text{Ball}(P)$  of the form  $X = (\alpha, \beta, y, r, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$  such  
678 that  $\Omega_P(X) = (q, q)$  (Lemma 15). Since  $X$  is regular (Assumption  $\aleph_1$ ), by Lemma 32 we have that  $(q, q)$   
679 satisfies assumption  $\mathcal{A}'_5$ . In particular, the multiplicity  $m$  is equal to two.

680 Thus, for all  $p \in \pi_{\mathfrak{C}}(\mathfrak{C})$  the sum of the multiplicities of the solutions in the system  $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha =$   
681  $x_2 - \beta = 0\}$  is at most two, i.e., Assumption  $\mathcal{A}_3$  is satisfied. Now, since Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$   
682 are satisfied and since all solutions of  $\text{Ball}(P)$  are regular, by Theorem 27, we have that Assumption  $\mathcal{A}_5^-$  is also  
683 satisfied.  $\square$

684 Using Lemma 35, we are ready to check Assumptions  $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$  using  $\aleph_1, \aleph_2$  and  $\aleph_3$ . Since  
685 Lemma 35 requires Assumption  $\mathcal{A}_1$ , we start by checking that assumption with Semi-algorithm 1 that is based on

686 subdivision.

---

687 **Semi-algorithm 1** Checking Assumption  $\mathcal{A}_1$

---

688 **Input:** An open  $n$ -box  $B$  and a function  $P$  from  $\overline{B}$  to  $\mathbb{R}^{n-1}$ .

689 **Termination:** If and only if  $P$  satisfies Assumption  $\mathcal{A}_1$  in  $\overline{B}$ .

690 **Output:** True.

691 1:  $L := \{\overline{B}\}$   
692 2: **while**  $L \neq \emptyset$  **do**  
693 3:    $\mathfrak{B} := \text{pop}(L)$   
694 4:   **if**  $0 \in \square P(\mathfrak{B})$  and  $\text{rank}(\square J_P(\mathfrak{B})) < n - 1$  **then**  
695 5:     Subdivide  $\mathfrak{B}$  and add its children to  $L$ .  
696 6: **return** True.

---

697 **Lemma 36.** *Semi-algorithm 1 stops if and only if  $P$  satisfies Assumption  $\mathcal{A}_1$  in  $\overline{B}$ .*

699 *Proof.* If Semi-algorithm 1 stops, by the conditions in Step (4), the box  $\overline{B}$  is partitioned into two sets of boxes. A  
700 set of boxes that are disjoint with  $\overline{\mathcal{C}}$  and the other one is a set of boxes that contain parts of  $\overline{\mathcal{C}}$  that satisfy Assumption  
701  $\mathcal{A}_1$ . Thus, Assumption  $\mathcal{A}_1$  is satisfied in  $\overline{B}$ . On the other hand, assume that  $P$  satisfies Assumption  $\mathcal{A}_1$  in  $\overline{B}$  and  
702 Semi-algorithm 1 does not stop, then, for every positive real  $\epsilon$  there exists a closed box  $\overline{\mathfrak{B}}_\epsilon \subset \overline{B}$ , with  $w(\overline{\mathfrak{B}}_\epsilon) < \epsilon$   
703 such that the conditions in Step (4) are satisfied in  $\overline{\mathfrak{B}}_\epsilon$ . Consider the infinite chain  $\overline{\mathfrak{B}}_{\frac{1}{1}}, \overline{\mathfrak{B}}_{\frac{1}{2}}, \overline{\mathfrak{B}}_{\frac{1}{3}} \dots$  and take  
704  $q_k \in \overline{\mathfrak{B}}_{\frac{1}{k}}$ , with  $q_k \neq q_{k'}$  for  $k \neq k'$ . Since  $\overline{B}$  is compact, then there exists a subsequence of  $q_k$  that converges to  
705 a point on  $\overline{B}$  say  $q$ . Since  $\square P$  and  $\square J_P$  are box function we must have that  $P(q) = 0$  and  $\text{rank}(J_P(q)) < n - 1$ .  
706 Thus,  $q$  is a point in  $\overline{\mathcal{C}}$  that does not satisfy Assumption  $\mathcal{A}_1$  which is a contradiction. Hence, Semi-algorithm 1  
707 stops.  $\square$

708 The next step is to check Assumptions  $\aleph_1$  and  $\aleph_2$ . For this goal, we want to find a finite set of pairwise disjoint  
709 boxes in  $\overline{B}_{\text{Ball}}$  such that every box contains at most one solution of  $\text{Ball}(P)$  and the union of these boxes contains  
710 all solutions of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$ . Notice that, by the definition of box functions, for a closed  $(2n - 1)$ -box  $\overline{\mathfrak{U}}$ , if  
711  $0 \notin \square \text{Ball}(P)(\overline{\mathfrak{U}})$ , then  $\overline{\mathfrak{U}}$  does not contain a solution of  $\text{Ball}(P)$ , whereas the condition  $0 \in \square \text{Ball}(P)(\overline{\mathfrak{U}})$  does  
712 not necessarily imply that a solution is in  $\overline{\mathfrak{U}}$ . This is why the set we are going to find might have unnecessary boxes.  
713 However, we will see later that this is enough for our purpose. Before introducing Semi-algorithm 2, we define the  
714 following functions.

**Definition 37.** *Consider the set  $\mathbb{R}_{t \geq 0}^{2n-1} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geq 0\}$  and define*

$$f_{\text{Ball}}^\pm : \mathbb{R}_{t \geq 0}^{2n-1} \rightarrow \mathbb{R}^n$$

$$(x_1, x_2, y, r, t) \mapsto (x_1, x_2, y \pm r\sqrt{t})$$

715 Define the function  $f_{\text{Ball}} : \mathbb{R}_{t \geq 0}^{2n-1} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  that maps  $X$  to  $(f_{\text{Ball}}^+(X), f_{\text{Ball}}^-(X))$ . Notice that  $f_{\text{Ball}}$  is an  
716 extension of  $\Omega_P$  (Definition 14). By abuse of notation, for a set  $S \subset \mathbb{R}^{2n-1}$ , we define  $f_{\text{Ball}}(S)$  as  $f_{\text{Ball}}(S \cap \mathbb{R}_{t \geq 0}^{2n-1})$ .

---

717 **Semi-algorithm 2** Isolating the solutions of  $\text{Ball}(P)$  (under Assumption  $\mathcal{A}_1$ )

---

718 **Input:** An open  $n$ -box  $B$ , a function  $P$  from  $\overline{B}$  to  $\mathbb{R}^{n-1}$  such that  $P$  satisfies Assumption  $\mathcal{A}_1$  in  $\overline{B}$  and a  $(2n-1)$ -  
719 open box  $\mathcal{U}_0$  that contains  $\overline{B}_{\text{Ball}}$  (see Remark 40).

720 **Termination:** If and only if  $\text{Ball}(P)$  satisfies Assumptions  $\aleph_1$  and  $\aleph_2$  in  $\overline{B}_{\text{Ball}}$ .

721 **Output:** A list of pairwise disjoint isolating boxes of the solutions of  $\text{Ball}(P)$  in  $\mathcal{U}_0$  such that their images by  
722  $f_{\text{Ball}}$  lies in  $B \times B$ .

723 1:  $Solutions = \emptyset$ .

724 2:  $L := \{\mathcal{U}_0\}$ .

725 3: **while**  $L \neq \emptyset$  **do**

726 4:    $\mathcal{U} := pop(L)$ .

727 5:   **if**  $0 \notin \square \text{Ball}(P)(\overline{\mathcal{U}})$  or  $(\square f_{\text{Ball}}(\overline{\mathcal{U}})) \cap (\overline{B} \times \overline{B}) = \emptyset$  **then**

728 6:     Do nothing ( $\mathcal{U}$  is simply removed from  $L$ ).

729 7:   **else if**  $\text{rank}(\square J_{\text{Ball}(P)}(\overline{\mathcal{U}})) = 2n - 1$  and  $\square f_{\text{Ball}}(\epsilon\text{-inflation}(\overline{\mathcal{U}}))^2 \subset B \times B$  **then**

730 8:     **if**  $\epsilon\text{-inflation}(\mathcal{U})$  contains a solution of  $\text{Ball}(P)$  (see Remark 38) **then**

731 9:       Add  $\epsilon\text{-inflation}(\mathcal{U})$  to  $Solutions$ .

732 10:   **else**

733 11:     Subdivide  $\mathcal{U}$  and add its children to  $L$ .

734 12: Remove duplicates in  $Solutions$  (see Remark 38).

735 13: **return**  $Solutions$

---

736 **Remark 38.** Steps (8) and (12) are not detailed because they are standard in subdivision algorithms to handle  
737 the issue of solutions on or near box boundaries and ensuring that solution boxes are pairwise disjoint. We only  
738 sketch below how these steps are done and refer to Sta95, §5.9.1; Kea97; XY19] for details. In Step (8), an  
739 existence test is performed by evaluating an interval Newton operator on an  $\epsilon$ -inflation of the box  $\mathcal{U}$ . The inflated  
740 box  $\epsilon\text{-inflation}(\mathcal{U})$  is certified to contain a solution if its image by the interval Newton operator is contained in the  
741 interior of  $\epsilon\text{-inflation}(\mathcal{U})$ . When the existence test is positive, the solution may be on the boundary or even outside  
742  $\mathcal{U}$ , but still in the interior of  $\epsilon\text{-inflation}(\mathcal{U})$ . The side effect is that the same solution may be reported several times  
743 when it is on or near a boundary of the subdivision. This issue is then solved in Step (12) as follows. When two  
744 boxes in the set  $Solutions$  intersect, they must report the same solution, and in addition, this solution is in the  
745 intersection of the two boxes. In Step (12), we thus compute intersections between boxes and replace intersecting  
746 ones by their intersection box. The boxes in the output set  $Solutions$  are thus pairwise disjoint.

748 **Lemma 39.** Under Assumption  $\mathcal{A}_1$  in  $\overline{B}$ , if Semi-algorithm 2 stops, it returns a list of pairwise disjoint isolating  
749 boxes of the solutions of  $\text{Ball}(P)$  in  $\mathcal{U}_0$  such that their images by  $f_{\text{Ball}}$  lies in  $B \times B$ . Moreover, Semi-algorithm 2

---

<sup>2</sup>For a box  $\mathcal{U}$  and  $\epsilon > 0$ ,  $\epsilon\text{-inflation}(\mathcal{U})$  is the box that has the same center as  $\mathcal{U}$  and its width is that of  $\mathcal{U}$  multiplied by  $(1 + \epsilon)$ . The box  $\epsilon\text{-inflation}(\mathcal{U})$  thus contains  $\mathcal{U}$ , the exact value of  $\epsilon$  is not important for the algorithm and it is usually set to 0.1 in subdivision algorithms [Rum10].

750 stops if and only if  $\text{Ball}(P)$  satisfies Assumptions  $\aleph_1, \aleph_2$  in  $\overline{B}_{\text{Ball}}$ .

751 *Proof.* We first prove the correctness of the Semi-algorithm 2 assuming that it terminates. Since Step (5) is the  
 752 only time the algorithm discards boxes, it never discards a box that contains a solution of  $\text{Ball}(P)$  in  $\mathfrak{U}_0$  such that  
 753 its image by  $f_{\text{Ball}}$  lies in  $B \times B$ . Hence, all such solutions of  $\text{Ball}(P)$  lie in output boxes. The rank condition in  
 754 Step (7) guarantees that each output box contains at most one solution of  $\text{Ball}(P)$  [Sny92b, Theorem A.1]. The  
 755 fact that every output box contains at least one solution is ensured by a standard algorithm in Step (8) (see e.g.,  
 756 Neu91, Theorem 5.6.2; XY19] and Remark 38). Finally, by Step (12), the output boxes are pairwise disjoint, hence  
 757 the algorithm outputs isolating boxes of the solutions of  $\text{Ball}(P)$  in  $\mathfrak{U}_0$  such that their images by  $f_{\text{Ball}}$  lie in  $B \times B$ .

758 To prove the equivalence for the termination, first assume that Semi-algorithm 2 stops and returns *Solutions*.  
 759 According to the correctness proof, every solution  $X$  of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$  is regular and satisfies  $\Omega_P(X) \in B \times B$ .  
 760 Thus, Assumptions  $\aleph_1$  and  $\aleph_2$  are satisfied in  $\overline{B}_{\text{Ball}}$ .

761 On the other hand, assume that  $\aleph_1$  and  $\aleph_2$  hold in  $\overline{B}_{\text{Ball}}$ . We prove that Semi-algorithm 2 terminates. By As-  
 762 sumption  $\aleph_1$  all solutions in  $\overline{B}_{\text{Ball}}$  of the square system  $\text{Ball}(P)$  are regular. Hence, they form a zero-dimensional  
 763 manifold in the compact space  $\overline{B}_{\text{Ball}}$ . Thus, the solution set is finite. We now prove that for any box  $\overline{\mathfrak{U}} \in L$  with  
 764 a small enough width, one of the conditions in Step (5) or the conditions in Steps (7-8) are satisfied. Thus, in both  
 765 cases  $\overline{\mathfrak{U}}$  will be removed from  $L$ , and hence, Semi-algorithm 2 stops after a finite number of iterations. Due to  
 766 Assumption  $\aleph_2$ , after a finite number of iterations, no box  $\mathfrak{U}$  in  $L$  intersects the boundary of  $B \times B$ . Moreover, due  
 767 to the convergence of the box evaluations, we can also assume that either  $\square f_{\text{Ball}}(\epsilon\text{-inflation}(\overline{\mathfrak{U}})) \subset B \times B$ , which  
 768 is the second condition of Step (7), or  $(\square f_{\text{Ball}}(\overline{\mathfrak{U}})) \cap (\overline{B} \times \overline{B}) = \emptyset$  which is the second condition of Step (5).

769 If  $\mathfrak{U}$  does not contain a solution of  $\text{Ball}(P)$ , then due to convergence of the box function evaluation of  $\text{Ball}(P)$ ,  
 770 after a finite number of iterations, every children  $\mathfrak{U}'$  of  $\mathfrak{U}$  satisfies  $0 \notin \square \text{Ball}(P)(\overline{\mathfrak{U}'})$ , that is, it is discarded in  
 771 Step (5).

772 If  $\mathfrak{U}$  contains a solution of  $\text{Ball}(P)$  in  $\overline{B}_{\text{Ball}}$ , according to Assumption  $\aleph_1$ , it is a regular solution. Due to  
 773 the convergence of the box evaluation  $\det(J_{\text{Ball}(P)}(\overline{\mathfrak{U}}))$  will eventually be non zero and thus  $\text{rank}(\square J_{\text{Ball}(P)}(\overline{\mathfrak{U}}))$   
 774 will eventually be  $2n - 1$  after a finite number of iterations, which is the first condition of Step (7). Due to the  
 775 convergence of the interval Newton existence test, the condition of Step (8) will also be eventually satisfied (see  
 776 Remark 38). The refined box will then eventually be added in the Solutions list.

777 Thus, for any box in  $L$  with a small enough width, one of the conditions of Step (5) is satisfied or all of the  
 778 conditions in Step (7-8) are satisfied, thus it is either discarded or added to the output. Hence, Semi-algorithm 2  
 779 terminates.  $\square$

780 **Remark 40.** *Semi-algorithm 2 requires a closed  $(2n - 1)$ -box  $\overline{\mathfrak{U}}_0$  that contains  $\overline{B}_{\text{Ball}}$ . For instance the following*  
 781 *set could be used:  $\{(q, r, t) \in \mathbb{R}^{2n-1} \mid q \in \overline{B}, -1 \leq r_i \leq 1 \text{ for } i \in \{3, \dots, n\}, 0 \leq t \leq \frac{\xi^2}{4}\}$  with  $\xi =$*   
 782  *$\max\{\|q - q'\| \mid q, q' \in \overline{B}\}$ .*

783 Finally, using Lemma 35, Semi-algorithm 3 checks whether  $P$  satisfies Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$

784 in  $\overline{B}$  and outputs a superset of isolating boxes of the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

---

785 **Semi-algorithm 3** Checking the weak assumptions and computing a superset of the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$

---

786 **Input:** An open  $n$ -box  $B$  and a smooth function  $P$  from  $\overline{B}$  to  $\mathbb{R}^{n-1}$ .

787 **Termination:** If and only if  $P$  satisfies Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$  and  $\mathcal{A}_5^-$  in  $\overline{B}$ .

788 **Output:**  $N$ , a list of certified node singularities: a list of boxes in  $\mathbb{R}^{2n-1}$  whose projections in  $\mathbb{R}^2$  contain each a  
789 single node of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

790  $U$ , a list of uncertified singularities: a list of boxes in  $\mathbb{R}^{2n-1}$  whose projections in  $\mathbb{R}^2$  contain each at most one  
791 node or one cusp of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

792 The union of all these projected 2D boxes contains all the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

793 1: Check Assumption  $\mathcal{A}_1$  (Semi-algorithm 1).

794 2: Compute a closed  $(2n - 1)$ -box  $\overline{\mathfrak{M}}_0$  that contains  $\overline{B}_{\text{Ball}}$  (Remark 40).

795 3:  $L :=$  output of Semi-algorithm 2.

796 4: Keep refining all boxes  $\overline{\mathfrak{M}} \in L$  (see Remark 41), until no triplets of boxes overlap in projection. Then remove  
797 from  $L$  one box from every pair (see Remark 17). This ensures Assumption  $\mathfrak{N}_3$ .

798 5:  $N :=$  boxes of  $L$  that lie in the halfspace  $t > 0$ .

799 6:  $U :=$  boxes of  $L$  that intersect the hyperplane  $t = 0$ .

800 7: **return**  $N$  and  $U$ .

---

801 **Remark 41.** *The refinement of an isolating box of a solution is performed by iterative evaluation of an interval*  
802 *Newton operator; we refer to [Neu91, Theorem 5.6.2] for details.*  
803

804 To identify the possible cusp points in the set  $U$  returned by Semi-algorithm 3, one may wish to solve indepen-  
805 dently the Ball system with the additional constraint  $t = 0$  (by Remark 16). Unfortunately, in this case we have  
806 an over-determined system and thus we cannot certify its solutions numerically. In the special case of a silhouette  
807 curve, it is possible to identify cusp points with numerical algorithms in the case  $n = 3$  [IMP16a, Lemmas 9 &  
808 10], but we leave as a conjecture its generalization for  $n > 3$ .

809 On the other hand, for curves that satisfy the strong assumptions,  $\mathcal{A}_5$  ensures that there are no cusps in the  
810 projection, which is equivalent to  $\widehat{\mathcal{L}}_{\mathcal{C}}$  being empty and  $\text{Ball}(P)$  having no solutions on the hyperplane  $t = 0$  (by  
811 Remark 16). Hence, if Assumptions  $\mathcal{A}_{1-5}$  hold, we can refine the boxes output by Semi-Algorithm 3 until no box  
812 intersects  $t = 0$ . Boxes in the half-space  $t < 0$  correspond to imaginary points in  $\mathbb{C}^n$  (Definition 14). Then the  
813 boxes satisfying  $t > 0$  are all the isolating boxes of the nodes of  $\pi_{\mathcal{C}}(\mathcal{C})$  by Lemmas 39 and 13, Remark 16 and  
814 Lemma 26.

---

815 **Semi-algorithm 4** Checking Assumptions  $\mathcal{A}_{1-5}$  and isolating the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$

---

816 **Input:** An open  $n$ -box  $B$  and a smooth function  $P$  from  $\overline{B}$  to  $\mathbb{R}^{n-1}$ .

817 **Termination:** If and only if  $P$  satisfies Assumptions  $\mathcal{A}_{1-5}$  in  $\overline{B}$ .

818 **Output:** A list of boxes in  $\mathbb{R}^{2n-1}$  whose projections in  $\mathbb{R}^2$  are isolating boxes of the singularities  $\pi_{\mathcal{C}}(\mathcal{C})$  (all  
819 singularities are in some boxes and each box contains a unique singularity).  
820 1:  $N, U :=$  output of Semi-Algorithm 3.  
821 2: **for** all  $\bar{U} \in U$  **do**  
822 3: Keep refining  $\bar{U}$  (see Remark 41) until it does not intersect the hyperplane  $t = 0$  and discard it if it lies in  
823 the half-space  $t < 0$ .  
824 4: **return**  $N \cup U$ .

---

825

## 826 6. Implementation and experiments

827 We first describe, in Section 6.1, the algorithms we implemented in our software *Isolating\_singularities*<sup>3</sup> with,  
828 in particular, the refinements we considered for improving the running time of those presented in Section 5. We  
829 present in Section 6.2 the third-party libraries we use. In Section 6.3, we present our experiments on several  
830 analytic curves in 3 and 4 dimensions and discuss the efficiency of our implementation.

### 831 6.1. Algorithms

832 Semi-algorithms 3 and 4 of Section 5 take as input an open  $n$ -box  $B$  and a curve  $\mathcal{C}$  defined by a smooth  
833 function  $P$  from  $\bar{B}$  to  $\mathbb{R}^{n-1}$ . They terminate if and only if  $P$  satisfies our weak, respectively strong, assumptions  
834 of Definition 8. Upon termination, they output a superset of the singularities, respectively the singularities, of the  
835 curve  $\pi_{\mathcal{C}}(\mathcal{C})$ .

836 The algorithms we implemented are variants of Semi-algorithms 3 and 4. First, for visualization purposes we  
837 modified their output as follows. Instead of returning boxes in  $\mathbb{R}^{2n-1}$  isolating solutions of the Ball system, they  
838 now return their projections in  $\mathbb{R}^2$  that contain singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$ . In addition, they also return the projection  
839 in  $\mathbb{R}^2$  of the boxes enclosing the curve  $\mathcal{C}$ , that thus enclose the curve  $\pi_{\mathcal{C}}(\mathcal{C})$ .

840 The two main improvements of our implementations are described below. The first one solves a stability issue  
841 when singular points of the projection are induced by very close branches of the curve  $\mathcal{C}$  or by a close to vertical  
842 part of the curve. The second one is a generalization of an idea used in the three-dimensional case in [IMP18], it  
843 aims at reducing the domain where the Ball system is to be solved. It is critical for the efficiency since solving in  
844 this high-dimensional space is costly.

845 *Evaluating the operators  $S$  and  $D$  of Definition 9.* To solve the Ball system we need box functions for the operators  
846  $S$  and  $D$ . We first note that if  $P(x_1, x_2, y) \in C^\infty(\mathbb{R}^n, \mathbb{R})$  is a polynomial function,  $S \cdot P$  (resp.  $D \cdot P$ ) is also a  
847 polynomial function and can be computed from the terms of  $P(x_1, x_2, y + rt)$  that have even (resp. odd) exponents  
848 in the variable  $t$ , see [IMP16b, Lemma 6] for details.

---

<sup>3</sup>[https://github.com/gkrait/Isolating\\_singularities](https://github.com/gkrait/Isolating_singularities)

849 If  $P$  is a more general  $C^\infty$  function for which we have a box function, using Definition 9, computing  $S \cdot P$   
850 on any  $(2n - 1)$ -box  $\bar{\mathcal{U}}$  or computing  $D \cdot P$  on  $\bar{\mathcal{U}}$  such that its  $t$ -interval does not contain 0, is implemented from  
851 the box function of  $P$  and interval arithmetic. On the other hand, when the  $t$ -interval contains 0 or is close to 0,  
852 the division by  $\sqrt{t}$  in the formula for  $D \cdot P$  makes the computation undefined or unstable. In such non-polynomial  
853 cases, we use a Taylor form at order 3 [Ral83], that is, we compute a Taylor expansion with remainder at  $t = 0$  of  
854  $D \cdot P$  and evaluate by interval the third order derivative. We define a threshold  $\delta_{\text{Taylor}}$  (which we set to  $10^{-2}$  in  
855 our experiments) such that this Taylor form is used when the  $t$ -interval has values smaller than  $\delta_{\text{Taylor}}$ .

856 *Improvement of Semi-algorithm 3.* The domain  $B_{\text{Ball}}$  where the Ball system is solved is refined to reduce costly  
857 computations in the high-dimensional space  $\mathbb{R}^{2n-1}$  by first enclosing the curve  $\mathcal{C}$  in a union of boxes in the smaller  
858 space  $\mathbb{R}^n$ . We denote this set of boxes by *enclosing\_curve*. Our approach follows the observation that every cusp  
859 of  $\pi_{\mathcal{C}}(\mathcal{C})$  lies in the projection of a box of *enclosing\_curve* containing a point  $p$  of  $\mathcal{C}$  with a tangent orthogonal  
860 to the  $(x_1, x_2)$ -plane. Such a point  $p$  is both  $x_1$  and  $x_2$ -critical for  $\mathcal{C}$ , that is, both  $\det(M_1(p))$  and  $\det(M_2(p))$   
861 vanish ( $M_i$  denotes the minor of  $J_P$  obtained by removing the  $i$ -th column). In addition, every node in  $\pi_{\mathcal{C}}(\mathcal{C})$  is  
862 contained in:

- 863 (a) the projection of a box  $\mathfrak{B}$  in *enclosing\_curve* such that  $0 \in \det(\square M_1)(\mathfrak{B})$  and  $0 \in \det(\square M_2)(\mathfrak{B})$ , or
- 864 (b) the intersection of the plane projections of two boxes in *enclosing\_curve*.

865 To understand this observation, we say that  $\mathcal{C}$  is parameterizable by  $x_i$  in a box  $\mathfrak{B}$ , if for any particular value  
866  $\alpha$  of  $x_i$  in  $\mathfrak{B}$ , the hyperplane  $x_i = \alpha$  intersects the curve  $\mathcal{C}$  at most once in  $\mathfrak{B}$ . The interval implicit function  
867 theorem [Sny92a, Thm C5 p.291] states that a sufficient condition for  $\mathcal{C}$  to be parameterizable by  $x_i$  in  $\mathfrak{B}$ , is  
868 that  $0 \notin \det(\square M_i)(\mathfrak{B})$ . In a box  $\mathfrak{B}$  such that  $0 \notin \det(\square M_i)(\mathfrak{B})$  for  $i = 1$  or  $2$ ,  $\mathcal{C}$  is parameterizable in  $x_i$ ,  
869 thus  $\mathcal{C}$  cannot contain two points with the same  $x_i$  value, which implies that the projection of  $\mathcal{C} \cap \mathfrak{B}$  does not  
870 contain a node. It follows that such a box can only induce a node in the projection when it overlaps in projection  
871 with another box, this case being covered by criterion (b). All nodes or cusps are thus in the projection of boxes  
872 satisfying criteria (a) or (b). Using the mapping  $f_{\text{Ball}}$  of Definition 37 from the Ball system space  $\mathbb{R}^{2n-1}$  to pairs  
873 of points in  $\mathbb{R}^n$ , we only have to solve the Ball system on the pre-image of these particular boxes or pairs of boxes  
874 in the set *enclosing\_curve*. More precisely, the Ball system domains associated to boxes in *enclosing\_curve* are  
875 defined as follows:

- 876 (i) for a single box  $\mathfrak{B}$ , cross product of boxes  $\mathfrak{B}_{(x_1, x_2)}$  in  $\mathbb{R}^2$  and  $\mathfrak{B}_y$  in  $\mathbb{R}^{n-2}$ , the domain is

$$877 \quad \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}, (x_1, x_2, y) \in \mathfrak{B}, r \in [-1, 1]^{n-2}, 0 \leq t \leq \max(\frac{\|\mathfrak{B}_y - \mathfrak{B}_y\|^2}{4})\},$$

- 878 (ii) for a pair of boxes  $\mathfrak{B} = \mathfrak{B}_{(x_1, x_2)} \times \mathfrak{B}_y$  and  $\mathfrak{B}' = \mathfrak{B}'_{(x_1, x_2)} \times \mathfrak{B}'_y$ , the domain is

$$879 \quad \{(x_1, x_2, y, r, t), (x_1, x_2) \in \mathfrak{B}_{(x_1, x_2)} \cap \mathfrak{B}'_{(x_1, x_2)}, y \in \frac{1}{2}(\mathfrak{B}_y + \mathfrak{B}'_y), r \in [-1, 1]^{n-2} \text{ if } \mathfrak{B} \cap \mathfrak{B}' \neq \emptyset, \text{ otherwise, } r \in$$

$$880 \quad \frac{\mathfrak{B}_y - \mathfrak{B}'_y}{\|\mathfrak{B}_y - \mathfrak{B}'_y\|}, t \in \frac{\|\mathfrak{B}_y - \mathfrak{B}'_y\|^2}{4}\}.$$

881 To sum up, our improved versions of Semi-algorithm 3 and 4 consists of three steps: (i) computing a set  
882 *enclosing\_curve* of  $n$ -boxes that enclose the curve  $\mathcal{C}$ , (ii) finding the boxes in *enclosing\_curve* that satisfy the  
883 above criteria (a) or (b), and (iii) solving  $\text{Ball}(P)$  over the pre-image of these boxes under  $f_{\text{Ball}}$ . When the semi-  
884 algorithms terminate, they return the 2D projections of the boxes in the set *enclosing\_curve* that cover  $\pi_{\mathcal{C}}(\mathcal{C})$   
885 together with the boxes that isolate the singularities of  $\pi_{\mathcal{C}}(\mathcal{C})$ .

## 886 6.2. Third-party libraries

887 Our software is based on interval arithmetic, interval evaluations of analytic functions and an interval solver.  
888 We use the following libraries, Python-FLINT and Ibexsolve, for these tasks.

889 Python-FLINT is a Python extension module wrapping FLINT (Fast Library for Number Theory) and Arb  
890 (arbitrary-precision ball arithmetic), which offers a toolbox for interval arithmetic and evaluation of analytic func-  
891 tions.

892 Ibexsolve is a C++ end-user program that solves systems of non-linear equations rigorously, that is, it does not  
893 lose any solution and return each solution under the form of a small box enclosing the true value. It implements  
894 a classical branch-and-prune algorithm that interleaves contractions and branching (bisections) to enclose the so-  
895 lutions of a system at any given desired precision. However, as opposed to Arb, Ibexsolve has a fixed precision,  
896 hence when several solutions are closer to each other than this precision, it will correctly return an enclosing box  
897 for these solutions but it will fail at isolating them. In our software, we use the default precision which is  $10^{-7}$ .  
898 Ibexsolve, and thus also our software *Isolating\_singularities*, use a parameter *eps\_max* that defines a maximum  
899 width for the isolating boxes output by the solver (the box bisections are forced until all output boxes are not  
900 larger than *eps\_max*). We use Ibexsolve for solving the Ball system (Semi-algorithm 2) and also in a variant of  
901 Semi-algorithm 1 to check the smoothness of the curve  $\mathcal{C}$  and at the same time enclosing  $\mathcal{C}$  in a set of boxes of  $\mathbb{R}^n$ .

## 902 6.3. Experiments

903 In this section, we present four experiments performed with our software *Isolating\_singularities*. More pre-  
904 cisely, we applied our improved Semi-algorithm 3 on Experiment 1 and Semi-algorithm 4 on all other experiments.  
905 The first example is pedagogical and considers a simple analytic curve in  $\mathbb{R}^3$  that induces only one node and one  
906 ordinary cusp in  $\mathbb{R}^2$ . The second example considers a smooth analytic curve in  $\mathbb{R}^4$  that induces many nodes in  $\mathbb{R}^2$ .  
907 The third one considers sparse but reasonably-high-degree algebraic equations in  $\mathbb{R}^4$ . It should be stressed that, up  
908 to our knowledge, the two latter examples are out of reach by other methods: indeed, no other certified algorithm  
909 can handle non-algebraic curves in dimension higher than 3 and, for reasonably-high-degree algebraic equations in  
910  $\mathbb{R}^4$ , the bivariate equation defining their 2D projection often has a very high degree (see Section 6.3.3 for details).

911 Finally, in the fourth example, we exhibit the behavior of our software when a node in  $\mathbb{R}^2$  is induced by a pair  
912 of points (on the space curve) that are very close. Indeed, when the equations defining the space curve are not  
913 algebraic, the Ball system contains a division by  $\sqrt{t}$  (due to the formula of  $D \cdot P$ ), which may cause instability



		Enclosing curve $\mathfrak{C}$			Solving Ball system		Output	
Experiments	Boxes max. width	Tree size	Output boxes	Time	Tree size	Time	Total time	Singularity boxes
Experiment 1	0.1	535	134	0.1	70	3.6	3.7	2
	0.03	1835	456	0.3	90	3.8	4.1	
	0.01	5427	1354	0.7	188	4.4	5.1	
Experiment 2	0.1	2243	520	1.1	6098	52	53.1	43
	0.03	6759	1639	3.4	1078	35.4	38.8	
	0.01	19583	4847	10.2	372	35.8	45.6	
Experiment 3	0.1	1151	203	1.0	655	4.2	5.2	7
	0.03	2503	523	1.8	272	3.5	5.3	
	0.01	6347	1482	4.5	163	5.7	10.2	

Table 1: Running times (in seconds) and numbers of boxes in Experiments 1 to 3.

914 since  $t$  tends to zero when the distance between the pair of points tend to zero. For that purpose, we consider two  
915 very close skew lines defined analytically (and not algebraically).

916 We report the running times and other relevant parameters in Tables 1 and 2. Running times are in seconds on a  
917 sequential Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux. We emphasize that the experiments  
918 are done with a prototype implementation that is under ongoing development. The *tree size* columns reports the  
919 total number of boxes created during the subdivision algorithm either for enclosing the curve  $\mathfrak{C}$  in  $\mathbb{R}^n$  or for  
920 solving the Ball system in  $\mathbb{R}^{2n-1}$ . For the enclosing part, the column *output boxes* is the number of boxes in the  
921 set *enclosing\_curve*. For each experiment, we provide a visualization of the plane projected curve  $\pi_{\mathfrak{C}}(\mathfrak{C})$  with its  
922 singularities. On each figure, the green boxes are the plane projections of the boxes in *enclosing\_curve* that enclose  
923  $\mathfrak{C}$ , hence these green boxes enclose  $\pi_{\mathfrak{C}}(\mathfrak{C})$ . The black boxes are the projections of the Ball system solution boxes  
924 identifying nodes of the plane curve  $\pi_{\mathfrak{C}}(\mathfrak{C})$ .

925 For each experiment, we consider three values of *eps\_max* and it can be observed (see Table 1) that the smaller  
926 the value of *eps\_max*, the larger the set *enclosing\_curve*, and the longer it is to compute. As expected, even with  
927 the improvement to reduce the Ball system domains to be solved in, the subdivision in the high-dimensional space  
928  $\mathbb{R}^{2n-1}$  is the dominant step of the algorithm.

### 929 6.3.1. Experiment 1: Analytic curve in $\mathbb{R}^3$ generating one node and one ordinary cusp

We start with a pedagogical example pictured in Figure 6. Running times are given in Table 1. The curve  $\mathfrak{C}$  is defined in the box  $B = (-1, 4) \times (-1, 4) \times (-4.8, -1.4)$  by

$$P(x_1, x_2, x_3) = [x_1 - \cos(x_3)(3 + \sin^4(x_3)) + 3, \quad x_2 - \sin^2(x_3)(3 + \sin(2x_3))].$$

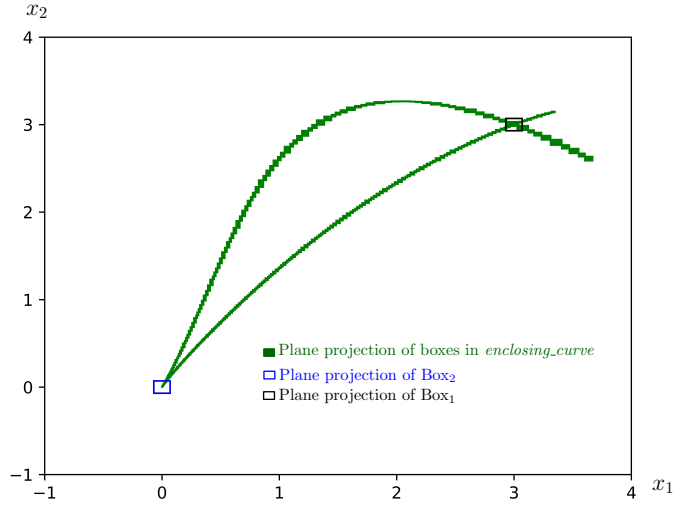


Figure 6: Experiment 1: Plane projection of an analytic curve in  $\mathbb{R}^3$  with one node and one ordinary cusp.

Our improved Semi-algorithm 3 outputs the following solutions for the Ball system in  $\mathbb{R}^5$ :

$$\begin{cases} N = \{\text{Box}_1 = [3, 3] \times [3, 3] \times [-3.15, -3.14] \times [1, 1] \times [2.4673, 2.4675]\} \\ U = \{\text{Box}_2 = [-0.06, 0.04] \times [-0.04, 0.07] \times [-3.15, -3.14] \times [1, 1] \times [-0.01, 0.01]\}. \end{cases}$$

930  $\text{Box}_1$  in the set  $N$  thus projects to a node of  $\pi_{\mathcal{C}}(\mathcal{C})$ .  $\text{Box}_2$  being in the set  $U$ , one cannot decide whether its  
 931 projection in the plane contains a node, a cusp or no singularity at all. On the other hand, one can notice on the  
 932 equation  $P = 0$  that the curve is parametrizable by the variable  $x_3$ . It is thus an easy computation to check that for  
 933 the value  $x_3 = -\pi$ , the point  $q = (0, 0, -\pi)$  is on the curve  $\mathcal{C}$  and its tangent line at  $q$  is generated by the vector  
 934  $(0, 0, 1)$  which is orthogonal to the projection plane. It is then clear that the projection of  $q$  generates a cusp that is  
 935 witnessed by  $\text{Box}_2$ .

### 936 6.3.2. Experiment 2: Analytic curve in $\mathbb{R}^4$ with many nodes

Figure 7 illustrates the output of our improved Semi-algorithm 4 for the curve defined by

$$\begin{aligned} P = [x_1 + 2 \sin(x_1) - \cos(x_4) - (3 \cos(x_3) - \cos(2.8571x_3)), \\ x_2 + 0.2 \cos(x_2) + (3 \sin(x_3) - \sin(2.8571x_3)) + \sin(x_4), \\ x_4^2 - \sin(x_2)] \end{aligned}$$

937 over the box  $B = (-1, 0) \times (-0.1, 3.5) \times (-20, 20) \times (-10, 10)$ . This curve has many nodes, some of them very  
 938 close to each other. Running times are given in Table 1.

### 939 6.3.3. Experiment 3: High degree algebraic curve in $\mathbb{R}^4$

The goal of this experiment is to emphasize the genericity of the assumptions and the efficiency of our software in the sparse polynomial case. The curve  $\mathcal{C}$  is defined in the 4-box  $B = (-1, 0.2) \times (-0.2, 1.4) \times (-10, 10)^2$  and

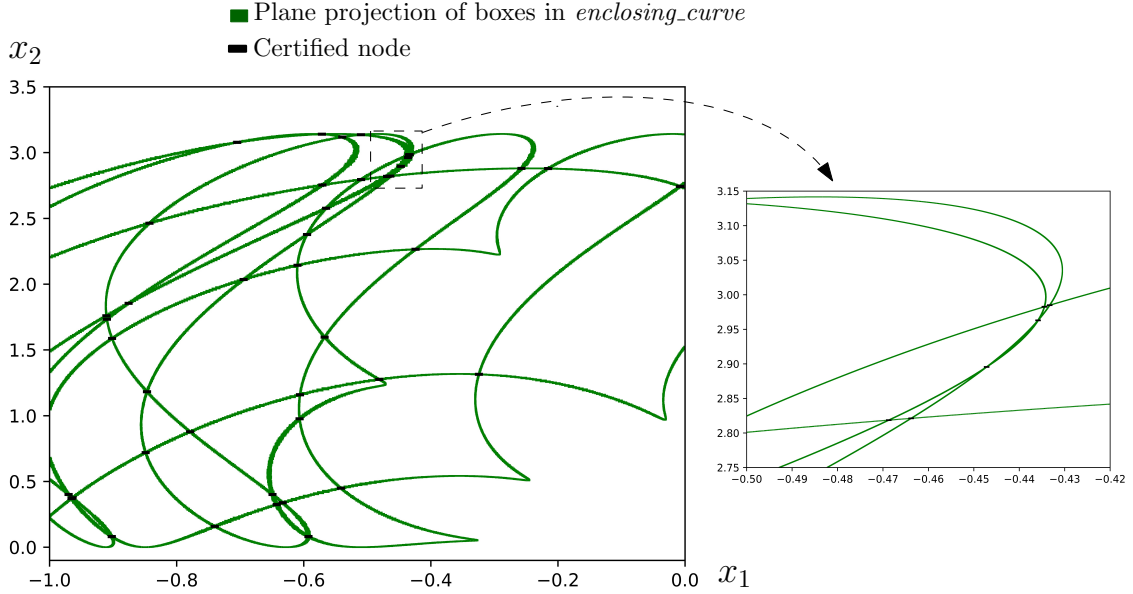


Figure 7: Experiment 2: Plane projection of an analytical curve  $\mathcal{C}$  in  $\mathbb{R}^4$ . Each of the 43 black boxes contains a node of  $\pi_{\mathcal{C}}(\mathcal{C})$  and is the projection of a box in  $\mathbb{R}^7$  containing one solution of  $\text{Ball}(P)$ .

is the zero set of three polynomials of degrees 17, 15 and 13, respectively that have a unique monomial of highest degree (which is monic) and 9 other random monomials of degrees at most 2 with integer coefficients in  $(-25, 25)$ .

$$\begin{aligned}
 P = & [x_1^{17} - 14x_1^2 - 7x_1x_3 - 7x_2^2 - 22x_2x_4 - x_3x_4 - 19x_4^2 + 8x_1 - 14x_3 + 9, \\
 & x_2^{15} + 8x_1x_3 - 14x_1x_4 - 15x_2^2 + 16x_2x_3 + 8x_2x_4 + 2x_3^2 + 13x_4^2 + 11x_1 + 11x_2, \\
 & x_3^{13} + 17x_1^2 - 15x_1x_2 + 4x_1x_3 - 20x_1x_4 + 2x_2^2 - 10x_2x_3 + 4x_2x_4 + 20x_4^2 - 23x_2].
 \end{aligned}$$

940 Figure 8 illustrates the 7 nodes of the projection of  $\mathcal{C}$  and running times are given in Table 1.

941 Note that since  $P$  is polynomial, the implicit equation of  $\pi_{\mathcal{C}}(\mathcal{C})$  can be computed using elimination theory and  
 942 its singularities can then be isolated using algebraic solvers. However, the implicit equation we obtained for  $\pi_{\mathcal{C}}(\mathcal{C})$   
 943 is defined by an irreducible bivariate polynomial of degree 442 with 51074 monomials. Isolating the singularities  
 944 of such a high-degree polynomial is then a real challenge.

945 Note also that our class of examples is rather specific and our software does not work that well if our defining  
 946 polynomials are dense or even if they have non-monic high-degree monomials. However, it should be stressed  
 947 that our software is a prototype and that it would be more efficient to use an interval solver specialized for the  
 948 algebraic case than the versatile Ibex solver we used. Such a specialized approach has been proved successful in  
 949 the 3-dimensional case by Imbach et al. [IMP18].

#### 950 6.3.4. Experiment 4: Two close lines in $\mathbb{R}^3$ generating a node

951 As mentioned in the preamble of Section 6.3, the purpose of this experiment { 13 is to study the behavior of our

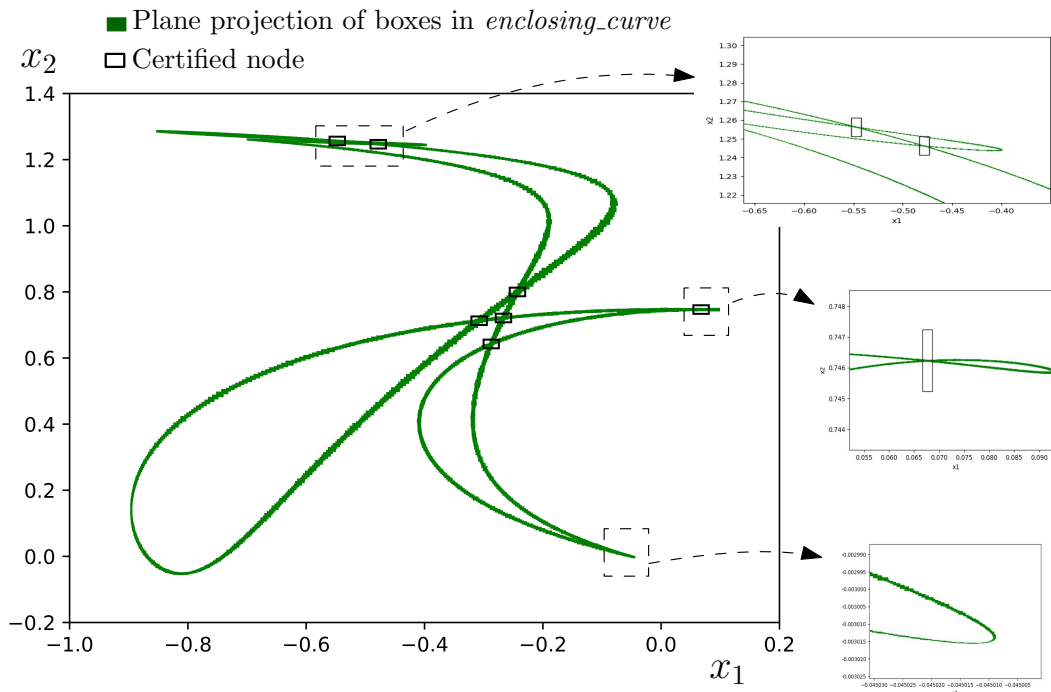


Figure 8: Experiment 3: High degree algebraic curve in  $\mathbb{R}^3$  generating 7 nodes.

952 software when a node in  $\mathbb{R}^2$  is induced by a pair of points (on the space curve) that are very close; namely when  
 953 a node  $(x_1, x_2) \in \mathbb{R}^2$  is induced by the pair of points  $(x_1, x_2, y \pm r\sqrt{t}) \in \mathbb{R}^n$  with  $t$  that tends to zero. Indeed,  
 954 when the equations defining the space curve are not algebraic, the Ball system contains a division by  $\sqrt{t}$  (due to  
 955 the formula of  $D \cdot P$ ), which may cause instability since  $t$  tends to zero when the distance between the pair of  
 956 points tend to zero.

957 The simplest example to consider is the two skew lines  $x_2 = x_1$  in the plane  $x_3 = \epsilon$  and  $x_2 = -x_1$  in the plane  
 958  $x_3 = -\epsilon$ , whose projection in the  $(x_1, x_2)$ -plane has a node at the origin, and to make  $\epsilon$  vary towards 0. The pair  
 959 of lines is thus defined by  $[\epsilon x_2 - x_1 x_3, (x_3 - \epsilon)(x_3 + \epsilon)]$  but, in order to have non-algebraic equations, we replace  
 960  $x_3$  by  $\sin x_3$  and define our two lines by  $P = [\epsilon x_2 - x_1 \sin x_3, \sin^2 x_3 - \epsilon^2]$  in the box  $B = (-1, 1)^3$ .

961 The goal of this experiment is to illustrate the stability of our software when  $\epsilon$  varies towards 0. Recall from  
 962 Section 6.1 that the  $D$  operator is evaluated on a box in two different ways depending on how close to zero is the  
 963  $t$ -interval of that box. The Ball system is thus solved either with Equation (3.1) (involving a division by  $\sqrt{t}$ ) when  
 964 the values of the  $t$ -interval are larger than a parameter  $\delta_{\text{Taylor}}$  (set to  $10^{-2}$ ) or using a Taylor expansion otherwise.

965 To illustrate the stability of our software, we compared in Table 2 its running times when  $\epsilon$  varies towards 0  
 966 with what it would be without using Taylor expansions. It shows that if we do not use Taylor expansions, the  
 967 solution is not certified (by the Ibexsolver; see Section 6.2) when  $\epsilon \leq 10^{-5}$ . On the other hand, our software  
 968 is stable, although its running time increases from 0.1 to 0.8 seconds when  $\epsilon$  gets smaller than or equal to  $10^{-2}$ ,  
 969 which is when the  $D$  operator starts to be evaluated using a Taylor expansion.

$\epsilon$ -values	> 1	1	0.99	0.9	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$	$10^{-6}$	$10^{-7}$
Time ( $\delta_{\text{Taylor}} = 10^{-2}$ )	$\mathfrak{C}$ is empty	$\mathfrak{C}$ is singular	Taylor forms are not triggered			0.8					
Time without Taylor forms			0.1			0.1		0.1		0.3	

Table 2: Experiment 4: Performances for different values of  $\epsilon$ .

## 7. Genericity of the assumptions

The key to prove the genericity of our assumptions is Thom's Transversality Theorem. We thus first recall, in Section 7.1, the basics of transversality theory using the notation of Demazure's book [Dem00]. We then prove, in Section 7.2, that all assumptions of Section 2 are satisfied for a generic curve. Finally, in Section 7.3, we consider the special case where the curve is the silhouette of a surface and prove that Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$  are generically satisfied in this case.

### 7.1. Preliminaries

We work with the set of smooth functions  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  with the weak (or compact-open) topology [Dem00, §3.9.2], that is convergence is understood as uniform on compact subsets and for any derivative. A subset of  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  is called residual if it contains the intersection of a countable family of dense open subsets. The space  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  is a Baire space [Dem00, Proposition 3.9.3], that is, every residual subset of  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  is dense. A property is generic in  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  if it is satisfied by a residual subset.

**Definition 42** ([Dem00, §3.8.3]). *Let  $E \simeq \mathbb{R}^n$  and  $F$  be two finite-dimensional real vector spaces and let  $r \geq 0$  be an integer. Let  $P^r(E, F)$  be the vector space of polynomial functions of degree at most  $r$  from  $E$  to  $F$ . For an open subset  $U$  of  $E$  (with respect to the usual topology on  $E$ ), let  $J^r(U, F) = U \times P^r(E, F)$  be the space of jets of order  $r$  of functions from  $U$  to  $F$ . Notice that  $J^r(U, F)$  can be identified with an open subset of  $\mathbb{R}^N$  for some positive integer  $N$ . Let  $f : U \rightarrow F$  be a smooth function, the jet of order  $r$  of  $f$  is the function*

$$j^r f : U \subset \mathbb{R}^n \rightarrow J^r(U, F) \subseteq \mathbb{R}^N$$

$$x \mapsto \left( x, f(x), \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x), \frac{\partial^2 f}{\partial x_1 \partial x_2}(x), \dots, \frac{\partial^r f}{\partial x_n^r}(x) \right).$$

Let  $W$  be a sub-manifold of  $J^r(U, F)$ . We say that  $j^r f$  is transverse to  $W$  if for all  $a \in U$  either  $j^r f(a) \notin W$  or every vector of  $\mathbb{R}^N$  can be written as a sum of a vector of  $T_{j^r f(a)}W$  and a vector in the image of the function  $T_a j^r f$ , where  $T_{j^r f(a)}W$  is the tangent space of  $W$  at  $j^r f(a)$  and  $T_a j^r f$  is the derivative function of  $j^r f$  at  $a$ .

**Theorem 43** (Thom's Transversality Theorem [Dem00, Theorem 3.9.4]). *Let  $E$  and  $F$  be two finite-dimensional vector spaces with  $U$  an open set in  $E$ . Let  $r \geq 0$  be an integer and  $W$  be a sub-manifold of  $J^r(U, F)$ . Then, the set of functions  $f \in C^\infty(U, F)$  such that  $j^r f$  is transverse to  $W$  is a dense residual subset of  $C^\infty(U, F)$ .*

988 **Proposition 44** ([Dem00, Corollary 3.7.3]). *Let  $U$  be an open subset of  $\mathbb{R}^n$ ,  $N \geq 1$  be an integer and  $W$  be a*  
989 *sub-manifold of the vector space  $\mathbb{R}^N$  of pure co-dimension  $m$ . Assume that the smooth function  $g : U \rightarrow \mathbb{R}^N$  is*  
990 *transverse to  $W$ , then  $g^{-1}(W)$  is a (possibly empty) sub-manifold of dimension  $n - m$ .*

991 The idea of the proofs of genericity of our assumptions is to express each assumption as a system of equations  
992 in the jet space. When this system defines a manifold  $W$ , Proposition 44 directly applies to pull back the manifold  
993 from the jet space to the ambient space of the curve. This pull back is a sub-manifold of the same co-dimension  
994 as  $W$ . A difficulty occurs when the system does not define a manifold. The following corollary overcomes this  
995 difficulty in the special case where the system is defined by analytic functions, in other words, the system defines  
996 an analytic variety. Such a variety does not need to be a manifold, but, using the Whitney stratification theorem  
997 [Whi65], the variety is written as a union of manifolds on which Thom's theorem is then applied.

998 **Corollary 45.** *Let  $E$  and  $F$  be two finite-dimensional vector spaces with  $E$  of dimension  $n$  and  $U$  an open set in*  
999  *$E$ . Let  $r \geq 0$  be an integer and  $W$  be an analytic variety of  $J^r(U, F)$  with co-dimension larger than  $n$ , then for a*  
1000 *generic  $P \in C^\infty(U, F)$ , the pre-image of  $W$  under  $j^r P$  is empty.*

1001 *Proof.* Let  $W = \bigcup_{i=1}^m W_i$  be a Whitney stratification of  $W$ , where the  $W_i$ 's are sub-manifolds. Since  $\text{codim}(W) >$   
1002  $n$ , we have that  $\text{codim}(W_i) > n$  for any integer  $1 \leq i \leq m$ . Let  $\Gamma_i = \{P \in C^\infty(U, F) \mid j^r P \text{ is transverse to } W_i\}$   
1003 and  $\Gamma = \bigcap_{i=1}^m \Gamma_i$ . By Theorem 43,  $\Gamma_i$  is residual and so is  $\Gamma$ . Moreover, by Proposition 44, for  $P \in \Gamma_i$  the pre-image  
1004 of  $W_i$  under  $j^r P$  is empty. Hence,  $(j^r P)^{-1}(W) = \bigcup_{i=1}^m (j^r P)^{-1}(W_i) = \emptyset$ . □

1005 We will also need a refined version of Thom's theorem in a multijet setting, that is for several points in the  
1006 source space simultaneously. We give the formal definitions of the multijet space and function but we do not  
1007 restate versions of Theorem 43, Proposition 44 and Corollary 45 that also hold for multijets.

**Definition 46** ([Dem00, §3.9.6]). *Let  $U$  be an open subset of  $\mathbb{R}^n$  and  $k \geq 1$  be an integer. We denote  $\Delta_{(k)}(U)$*   
*the subset of  $U^k$  consisting of sequences  $(a_1, \dots, a_k)$  of pairwise distinct points of  $U$ . For an integer  $r \geq 0$*   
*and a finite-dimensional space  $F$ , the  $k$ -multijet space of order  $r$ ,  $J_{(k)}^r(U, F)$ , is the subset of  $J^r(U, F)^k =$*   
 *$(U \times P^r(E, F))^k$  consisting of the  $k$ -tuples  $((a_1, p_1), \dots, (a_k, p_k))$ , with  $(a_1, \dots, a_k) \in \Delta_{(k)}(U)$ . Let  $f : U \rightarrow F$*   
*be a smooth function, the  $k$ -multijet of order  $r$  of  $f$  is the function*

$$j_{(k)}^r f : \Delta_{(k)}(U) \rightarrow J_{(k)}^r(U, F)$$

$$(a_1, \dots, a_k) \mapsto (j^r f(a_1), \dots, j^r f(a_k)).$$

1008 Finally, we gather several technical tools from algebra and analysis.

1009 **Proposition 47** ([BV88, Proposition 1.A.1.1]). *Let  $M(m, n)$  be the vector space of real matrices of size  $m \times n$*   
1010 *and  $r$  be a positive integer such that  $r < \min\{n, m\}$ . The determinantal variety,  $M_r$ , is the set of matrices in*  
1011  *$M(m, n)$  that have rank less than  $r + 1$ . Then, the following statements hold:*

1012 (a)  $M_r$  is an irreducible variety in  $M(m, n)$ .

1013 (b)  $M_r$  is of dimension  $r(n + m - r)$ .

1014 (c) The singular locus of  $M_r$  is  $M_{r-1}$ .

1015 **Lemma 48** ([Bôc64, §XIV.61 Theorem 1]). Let  $n \geq 2$  be an integer,  $\{x_{ij}\}_{1 \leq j, i \leq n}$  be a set of  $n^2$  variables  
 1016 and  $\mathbb{C}[x_{ij}]_{1 \leq j, i \leq n}$  be the ring of complex polynomials with variables  $\{x_{ij}\}$ . Then, the determinant of the matrix  
 1017  $(x_{ij})_{1 \leq i, j \leq n}$  is an irreducible polynomial in  $\mathbb{C}[x_{ij}]_{1 \leq j, i \leq n}$ .

1018 **Theorem 49** ([Whi43, Theorem 1 & 2]). Let  $f$  be an even (resp. odd) smooth function, then there exists a smooth  
 1019 function  $g$  such that  $f(x) = g(x^2)$  (resp.  $f(x) = x \cdot g(x^2)$ ).

## 1020 7.2. Genericity of the assumptions for a curve in $\mathbb{R}^n$

1021 We are going to prove that each assumption in Section 2 is generic. Hence, the combination of these assump-  
 1022 tions is also generic since a countable intersection of residual subsets in  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  is residual.

1023 **Lemma 50.** Assumption  $\mathcal{A}_1$  is generic.

*Proof.* Consider the jet of order 1 of the function  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ :

$$j^1 P : \mathbb{R}^n \rightarrow J^1(\mathbb{R}^n, \mathbb{R}^{n-1}) = \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n}$$

$$x \mapsto (x, P(x), J_P(x)) = (x, y, z).$$

1024 We represent the jet space by the variables  $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$  and  $z \in \mathbb{R}^{(n-1) \times n}$ . Abusing notation, we can see  
 1025 the variable  $z$  as an  $(n-1) \times n$ -matrix. Define the variety  $W = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n} \mid y =$   
 1026  $0, \text{rank}(z) \leq n-2\}$ . The variety  $W$  is a product of a determinantal variety in  $\mathbb{R}^{(n-1) \times n}$  of dimension  $n^2 - n - 2$   
 1027 (by Proposition 47) and a linear space of dimension  $n$  in  $\mathbb{R}^n \times \mathbb{R}^{n-1}$ . Thus,  $W$  is a variety of co-dimension  $n+1$   
 1028 in  $\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n}$ . Hence, by Corollary 45, there exists a residual subset  $\Gamma_1 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ , such that  
 1029 for  $P \in \Gamma_1$  the pre-image of  $W$  under  $j^1 P$  is empty. Consequently, for a generic  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  and any  
 1030  $q \in \overline{\mathcal{C}}$ , we have that  $q \notin (j^1 P)^{-1}(W) = \emptyset$ , thus  $\text{rank}(J_P(q)) = n-1$ , which is Assumption  $\mathcal{A}_1$ .  $\square$

1031 **Lemma 51.** Assumption  $\mathcal{A}_2$  is generic. Moreover, generically, the set  $\overline{\mathcal{L}}_c$  is empty.

1032 *Proof.* We consider the jet of order 1 of the function  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  as in the proof of Lemma 50 with the  
 1033 same notation. Define the matrix  $T_1(z)$  (resp.  $T_2(z)$ ) to be the sub-matrix of  $z$  obtained by removing the first  
 1034 (resp. second) column. Consider the variety  $W \subset J^1(\mathbb{R}^n, \mathbb{R}^{n-1})$  defined by  $\{y = 0 \in \mathbb{R}^{n-1}, \det(T_1(z)) =$   
 1035  $\det(T_2(z)) = 0\}$ . Notice that  $\overline{\mathcal{L}}_c$  is included in the pre-image of  $W$  under  $j^1 P$  since  $\overline{\mathcal{L}}_c$  is the set of points of  
 1036 the curve  $\overline{\mathcal{C}}$  that are both  $x_1$  and  $x_2$ -critical. By Lemma 48, we have that both  $\det(T_1(z))$  and  $\det(T_2(z))$  are  
 1037 irreducible polynomials. By [CLO92, §9.4 Prop 10], a proper sub-variety of an irreducible variety is of lower  
 1038 dimension, we deduce that the common zero locus of  $\det(T_1(z))$  and  $\det(T_2(z))$  is of co-dimension at least two.

1039 We deduce that  $\text{codim}(W) > n$ . By Corollary 45, there exists a residual subset  $\Gamma_2 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ , such that  
 1040 for  $P \in \Gamma_2 \cap \Gamma_1$ , the pre-image of  $W$  under  $j^1P$  is empty and hence  $\overline{\mathfrak{L}_c}$  is empty, which implies Assumption  
 1041  $\mathcal{A}_2$ . □

1042 **Lemma 52.** *Assumption  $\mathcal{A}_3$  is generic.*

*Proof.* Let us consider the 3-multijet of order 0:

$$j_{(3)}^0 P : \Delta_{(3)}(\mathbb{R}^n) \rightarrow J_{(3)}^0(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^3$$

$$(x, x', x'') \mapsto ((x, P(x)), (x', P(x')), (x'', P(x''))) = ((x, y), (x', y'), (x'', y''))$$

1043 where every element in the jet space  $J_{(3)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$  is of the form  $((x, y), (x', y'), (x'', y''))$ , where  $x =$   
 1044  $(x_1, \dots, x_n)$ ,  $x', x'' \in \mathbb{R}^n$  and  $y, y', y'' \in \mathbb{R}^{n-1}$ . Consider the linear sub-manifold  $W = \{x_1 = x'_1 = x''_1, x_2 =$   
 1045  $x'_2 = x''_2, y = y' = y'' = 0\}$ , the co-dimension of  $W$  is thus  $3n + 1$  which is larger than the dimension of the  
 1046 source space  $\Delta_{(3)}(\mathbb{R}^n)$  which is  $3n$ . Thus, by Corollary 45, there exists a residual subset  $\Gamma_3 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ ,  
 1047 such that for  $P \in \Gamma_3$ , the pre-image of  $W$  by  $j_{(3)}^0$  is empty, which translates to the fact that there are no pairwise  
 1048 distinct points  $q, q', q''$  in  $\overline{\mathfrak{C}}$  such that  $\pi_{\overline{\mathfrak{C}}}(q) = \pi_{\overline{\mathfrak{C}}}(q') = \pi_{\overline{\mathfrak{C}}}(q'')$ . This is also equivalent to saying that the system  
 1049  $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$  has at most two distinct solutions (without counting multiplicities)  
 1050 for any  $(\alpha, \beta) \in \mathbb{R}^2$ .

1051 Using  $\Gamma_1, \Gamma_2$  as defined in the proofs of Lemmas 50 & 51 and  $\Gamma_3$  defined above, we define  $\Gamma_4 = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$   
 1052 which is thus a residual set and let  $P$  be in  $\Gamma_4$ . Since  $P$  is in  $\Gamma_3$ , the system  $S$  has at most two distinct solutions. In  
 1053 addition, since  $P$  is in  $\Gamma_2 \cap \Gamma_1$ , one has that  $\overline{\mathfrak{L}_c}$  is empty and finally together with Lemma 20, since  $P$  is in  $\Gamma_1$ , this  
 1054 implies that these solutions have multiplicity exactly 1 in  $S$ . For  $P$  in the residual set  $\Gamma_4$ , the number of solutions  
 1055 counted with multiplicities of  $S$  is thus at most 2, which is Assumption  $\mathcal{A}_3$ . □

1056 **Lemma 53.** *Assumption  $\mathcal{A}_4$  is generic.*

*Proof.* Let us consider the 2-multijet of order 0 of  $P$ :

$$j_{(2)}^0 P : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^2$$

$$(x, x') \mapsto ((x, P(x)), (x', P(x'))) = ((x, y), (x', y'))$$

1057 where every element in the jet space  $J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$  is of the form  $((x, y), (x', y'))$ , where  $x = (x_1, \dots, x_n)$ ,  
 1058  $x' \in \mathbb{R}^n$  and  $y, y' \in \mathbb{R}^{n-1}$ . Consider the linear sub-manifold  $W = \{x_1 = x'_1, x_2 = x'_2, y = y' = 0\}$  of the jet  
 1059 space  $J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$ . Notice that,  $(j_{(2)}^0 P)^{-1}(W)$  contains the set  $\widehat{\mathfrak{L}}'_n = \{(q_1, q_2) \in \Delta_{(2)}(\mathbb{R}^n) \cap \overline{\mathfrak{C}} \times \overline{\mathfrak{C}} \mid \pi_{\overline{\mathfrak{C}}}(q_1) =$   
 1060  $\pi_{\overline{\mathfrak{C}}}(q_2)\}$  and  $\overline{\mathfrak{L}_n}$  is the image of  $\widehat{\mathfrak{L}}'_n$  by the projection  $(q_1, q_2) \rightarrow q_1$ . We have  $\dim(\Delta_{(2)}(\mathbb{R}^n)) = 2n$  and, since  $W$   
 1061 is linear, its co-dimension is easily computed  $\text{codim}(W) = 2(2n - 1) - (2 + 2(n - 1)) = 2n$ . Proposition 44  
 1062 thus yields that generically  $(j_{(2)}^0 P)^{-1}(W)$  is a sub-manifold of dimension zero that is a discrete set in  $\mathbb{R}^n$ , and so  
 1063 is  $\overline{\mathfrak{L}_n}$ .



1064 Now, we prove that, generically,  $\overline{\mathfrak{L}}_n$  does not intersect the boundary of  $B$ . The boundary  $\partial B$  of the box  $B$  is  
1065 included in the union of the supporting hyperplanes  $H_i$  of its  $2^n$  faces of dimension  $n - 1$ , that is  $\partial B = \bigcup_{i=1}^{2^n} H_i$ .  
1066 Define the linear sub-manifold  $W_i = \{((x, y), (x', y')) \in W \mid x \in H_i \text{ or } x' \in H_i\}$ , notice that this adds one  
1067 equation to  $W$  and thus increases the co-dimension of  $W$  by one, thus  $\text{codim}(W_i) = 2n + 1$ . By Corollary 45, we  
1068 have that generically, the pre-image of  $W_i$  under  $j_{(2)}^0 P$  is empty, which translates to the fact that there is no point  
1069 of  $\overline{\mathfrak{L}}_n$  on  $\partial B \cap H_i$ . This is also true for any  $i$  and thus, generically,  $\overline{\mathfrak{L}}_n$  does not intersect the boundary of  $B$ .  $\square$

1070 **Corollary 54.** *Assumption  $\mathcal{A}_5$  is generic.*

1071 *Proof.* Let  $B$  be an open  $n$ -box. We prove that for a generic  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ , the singular points of  $\pi_{\mathfrak{C}}(\mathfrak{C})$  are  
1072 only nodes (recall that by Lemma 26, under the generic assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ , the points in  $\mathfrak{C} \setminus (\mathfrak{L}_c \cup \mathfrak{L}_n)$   
1073 project to smooth points).

Let  $\Gamma_0$  be the set of  $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  such that  $P$  satisfies Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_4$ . The previous lemmas of this section show that  $\Gamma_0$  is residual in  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ . Let us consider the 2-multijet of order 1 of  $P$ :

$$j_{(2)}^1 P : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^1(\mathbb{R}^n, \mathbb{R}^{n-1}) \subseteq (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n})^2$$

$$(x, x') \mapsto ((x, P(x), J_P(x)), (x', P(x'), J_P(x'))) = ((x, y, z), (x', y', z'))$$

Let  $s, s'$  (resp.  $r, r'$ ) be the sub-matrices of  $z, z'$ , respectively, obtained by removing the first two columns (resp. obtained by the first two columns). Define the matrix  $M = \begin{pmatrix} r & 0 & s \\ r' & s' & 0 \end{pmatrix}$  and the variety

$$W = \{((x, y, z), (x', y', z')) \in (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n})^2 \mid y = y' = 0, x_1 = x'_1, x_2 = x'_2, \det(M) = 0\}.$$

1074 The variety  $W$  is a product of a determinantal variety and a linear space, thus its co-dimension is  $\text{codim}(W) \geq$   
1075  $2n + 1 > 2n = \dim(\Delta_{(2)}(\mathbb{R}^n))$ . Hence, by Corollary 45, there exists a residual subset  $\Gamma'_0$  in  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$  such  
1076 that for all  $P \in \Gamma'_0$ , the pre-image of  $W$  under  $j_{(2)}^1 P$  is empty.

1077 Let  $P$  be in the residual set  $\Gamma_0 \cap \Gamma'_0$ . By Lemma 31 and since  $\mathfrak{L}_c$  is empty, we deduce that for distinct  
1078  $q_1, q_2 \in \mathfrak{C}$  with  $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ , the plane projections of the lines  $T_{q_1} \mathfrak{C}$  and  $T_{q_2} \mathfrak{C}$  intersect transversely if and  
1079 only if  $j_{(2)}^1((q_1, q_2)) \notin W$ . Finally, by Lemma 21 (Step (a) of the proof), we deduce that  $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$  is a  
1080 node in  $\pi_{\mathfrak{C}}(\mathfrak{C})$ .  $\square$

### 1081 7.3. Genericity of the assumptions for the silhouette of a surface in $\mathbb{R}^n$

1082 In this section, we focus on the special case of silhouette curves of surfaces in  $\mathbb{R}^n$ . For an open  $n$ -box  $B$  and  $\tilde{P}$   
1083 in  $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$  such that  $S = \tilde{P}^{-1}(0)$  is a smooth 2-sub-manifold in  $\mathbb{R}^n$ , the silhouette of  $\tilde{P}$  is the set of points  
1084  $q$  of this surface  $S$  such that the projection (with respect to a fixed direction) of the tangent plane  $T_q S$  to  $\mathbb{R}^2$  is not  
1085 surjective. We prove that Assumptions  $\mathcal{A}_1, \mathcal{A}_2$  &  $\mathcal{A}_4$  are satisfied for a generic silhouette, and we only conjecture  
1086 that Assumptions  $\mathcal{A}_3$  &  $\mathcal{A}_5^-$  also hold generically. We start by formalizing the definition of the silhouette curve  
1087 algebraically.

1088 **Definition 55.** For an integer  $n \geq 3$ , let  $\tilde{P} = (P_1, \dots, P_{n-2}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ . Define the smooth function  
 1089  $P_{n-1} = \det \left( \left( \frac{\partial P_i}{\partial x_j} \right)_{\substack{1 \leq i \leq n-2 \\ 3 \leq j \leq n}} \right)$  and  $P = (P_1, \dots, P_{n-1})$ . We define the curve  $\mathfrak{C}$  (and  $\bar{\mathfrak{C}}$ ) as in Section 2 and call  
 1090 it the silhouette of  $\tilde{P}$ .

1091 **Proposition 56.** For a generic  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , the function  $P$  satisfies Assumption  $\mathcal{A}_1$ .

*Proof.* Consider the jet of order 1 of  $\tilde{P}$ :

$$j^1 \tilde{P} : \mathbb{R}^n \rightarrow J^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \simeq \mathbb{R}^{n^2-2} = \mathbb{R}^N$$

$$x \mapsto (x, \tilde{P}(x), J_{\tilde{P}}(x)) = (x, y, z).$$

We represent the jet space by the vectors  $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-2}$  and the  $((n-2) \times n)$ -matrix  $z \in \mathbb{R}^{(n-2) \times n}$ . Let  $T(z)$  denote the sub-matrix obtained by removing the first two columns of  $z$ . Define the variety  $W = \{y = 0, \det(T(z)) = 0\} = \{y = 0, \text{rank}(T(z)) \leq n-3\}$  in  $\mathbb{R}^N$ . According to Proposition 47,  $W = \text{Reg}(W) \cup \text{Sing}(W)$  where  $\text{Reg}(W)$  (resp.  $\text{Sing}(W)$ ) is the set of smooth (resp. singular) points in  $W$  and

$$\text{Reg}(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \text{rank}(T(z)) = n-3\}$$

$$\text{Sing}(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \text{rank}(T(z)) < n-3\}.$$

1092 In addition, Proposition 47 yields that  $\text{Reg}(W)$  is a manifold of co-dimension  $n-1$  and  $\text{Sing}(W)$  is a variety  
 1093 of co-dimension  $n+2$ . Since the co-dimension of  $\text{Sing}(W)$  is larger than that of the source space, Corollary 45  
 1094 implies that, generically,  $(j^1 \tilde{P})^{-1}(\text{Sing}(W)) = \emptyset$ . One thus has  $(j^1 \tilde{P})^{-1}(W) = (j^1 \tilde{P})^{-1}(\text{Reg}(W))$ .

Consider the function

$$\varphi : \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \rightarrow \mathbb{R}^{n-2} \times \mathbb{R}$$

$$\chi = (x, y, z) \mapsto (y, \det(T(z))),$$

1095 such that  $\varphi^{-1}(0) = W$ . Its Jacobian matrix is  $J_\varphi = \begin{pmatrix} 0_{(n-2) \times n} & I_{(n-2) \times (n-2)} & 0_{(n-2) \times (n-2)n} \\ 0_{1 \times (n)} & 0_{1 \times (n-2)} & v(z) \end{pmatrix}$ , where  $0_{k_1 \times k_2}$   
 1096 (resp.  $I_{k_1 \times k_2}$ ) is the zero (resp. identity) matrix of size  $k_1 \times k_2$  and the vector  $v(z)$  is the adjugate matrix of  $T(z)$   
 1097 written as the concatenation of its lines:  $v(z) = (\text{Adj}^{ij}(T(z)))_{\substack{1 \leq i \leq n-2 \\ 3 \leq j \leq n}} \in \mathbb{R}^{(n-2)^2}$ . Let  $\chi = (x, y, z) \in \text{Reg}(W)$ ,  
 1098 then  $\text{rank}(T(z)) = n-3$ , thus there exists a pair  $(i, j)$  such that  $\text{Adj}^{ij}(T(z)) \neq 0$ . Hence, the vector  $v(z)$  is  
 1099 non-trivial and  $J_\varphi(\chi)$  has full rank  $n-1$ . The function  $\varphi$  is thus a submersion on  $\text{Reg}(W)$ .

1100 Theorem 43 yields that, generically,  $j^1 \tilde{P}$  is transverse to the manifold  $\text{Reg}(W)$ . Together with the fact that  
 1101  $\varphi$  is a submersion on  $\text{Reg}(W)$ , [GG73, Lemma II.4.3 (p.52)] implies that  $P = \varphi \circ j^1 \tilde{P}$  is a submersion on  
 1102  $(j^1 \tilde{P})^{-1}(\text{Reg}(W)) = (j^1 \tilde{P})^{-1}(W) = (j^1 \tilde{P})^{-1}(\varphi^{-1}(0)) = (\varphi \circ j^1 \tilde{P})^{-1}(0) = P^{-1}(0) = \mathfrak{C}$ . In other words,  
 1103  $J_P$  has full rank  $n-1$  on  $\mathfrak{C}$ , which is Assumption  $\mathcal{A}_1$ .  $\square$

1104 **Proposition 57.** For a generic  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , the function  $P$  satisfies Assumption  $\mathcal{A}_2$ .

1105 *Proof.* First we prove that, generically,  $\overline{\mathfrak{L}}_c$  is discrete. For any  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$  consider  $j^2\tilde{P} : \mathbb{R}^n \rightarrow$   
1106  $J^2(\mathbb{R}^n, \mathbb{R}^{n-2}) \subset \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)} = \mathbb{R}^N$ . Assume that every element in  $\mathbb{R}^N$  is represented  
1107 as  $(x, y, z, h)$ , where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^{n-2}$ ,  $z \in \mathbb{R}^{(n-2) \times n}$  and  $h \in \mathbb{R}^{n^2(n-2)}$ . Abusing notation, we consider  
1108  $z$  as a  $((n-2) \times n)$ -matrix. Let  $T(z)$  denote the matrix obtained by removing the first two columns of  $z$ .  
1109 The Jacobian matrix  $J_P$  is a function of the derivatives  $(\frac{\partial P_i}{\partial x_j}, \frac{\partial^2 P_i}{\partial x_k \partial x_s})_{\substack{1 \leq i, l \leq n-2 \\ 1 \leq j, k, s \leq n}}$ , it can thus be seen in the jet  
1110 space as a function of  $z$  and  $h$ ,  $J_P(z, h)$ . Define the matrix  $T_1(z, h)$  (resp.  $T_2(z, h)$ ) to be the sub-matrix of  
1111  $J_P(z, h)$  obtained by removing the first (resp. second) column. Define the variety  $W = \{(x, y, z, h) \mid y = 0 \in$   
1112  $\mathbb{R}^{n-2}, \det(T(z)) = \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$ , so that  $\overline{\mathfrak{L}}_c$  is included in the pre-image of  $W$  under  
1113  $j^2\tilde{P}$ . Let  $W_1 = \{(x, y, z, h) \mid y = 0 \in \mathbb{R}^{n-2}, \det(T(z)) = 0\}$ , we already showed in the proof of Proposition 56  
1114 that  $W_1$  is an irreducible variety of co-dimension  $n - 1$ . In addition,  $\det(T_1(z, h))$  does not identically vanish  
1115 on  $W_1$ , thus  $W$  is a proper sub-variety of the irreducible variety  $W_1$  and [CLO92, §9.4 Prop 10] implies that  
1116  $\text{codim}(W) > \text{codim}(W_1) = n - 1$ .

1117 Now, write  $W = \text{Reg}(W) \cup \text{Sing}(W)$ , where  $\text{Reg}(W)$  (resp.  $\text{Sing}(W)$ ) is the set of smooth (resp. sin-  
1118 gular) points in  $W$ . Recall that  $\text{codim}(\text{Sing}(W)) > n$  since  $\text{Sing}(W)$  is a proper closed sub-variety of  $W$   
1119 [BCR98, Proposition 3.3.14]. By Corollary 45, there exists a residual set  $\Gamma' \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$  such that  
1120 if  $\tilde{P} \in \Gamma'$ , then the pre-image of  $\text{Sing}(W)$  under  $j^2\tilde{P}$  is empty. Define  $\Gamma = \{\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2}) \mid$   
1121  $j^2\tilde{P}$  is transverse to  $\text{Reg}(W)\} \cap \Gamma'$ . Notice that if  $\tilde{P} \in \Gamma$ , then  $\overline{\mathfrak{L}}_c$  is included in the pre-image of  $\text{Reg}(W)$  under  
1122  $j^2\tilde{P}$ . Hence, since  $\text{codim}(\text{Reg}(W)) = \text{codim}(W) \geq n$ , we have by Proposition 44 that  $\overline{\mathfrak{L}}_c$  is a sub-manifold of  
1123 dimension, at most, zero. Thus,  $\overline{\mathfrak{L}}_c$  is discrete for all  $\tilde{P} \in \Gamma$ . Using Theorem 43 we deduce that  $\Gamma$  is residual.

1124 The proof that  $\overline{\mathfrak{L}}_c$  does not intersect the boundary of  $B$  can be done analogously as in the proof of Lemma 53.  
1125 □

1126 **Proposition 58.** *For a generic  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , the function  $P$  satisfies Assumption  $\mathcal{A}_4$ .*

1127 *Proof.* Consider the 2-multijet  $j_{(2)}^1\tilde{P} : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^2$  of the function  
1128  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , where  $(\mathbb{R}^n \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{(n-2) \times n})^2$  is described by the coordinates  $x, x' \in \mathbb{R}^n, y, y' \in$   
1129  $\mathbb{R}^{n-2}$  and  $z, z' \in \mathbb{R}^{(n-2) \times n}$ . Abusing notation, we consider  $z$  and  $z'$  as  $((n-2) \times n)$ -matrices. Let  $T(z)$  (resp.  
1130  $T(z')$ ) denote the matrix obtained by removing the first two columns of  $z$  (resp.  $z'$ ). Define the variety  $W$  to be  
1131 the solution set of the system  $\{y = y' = 0, x_1 - x'_1 = x_2 - x'_2 = \det(T(z)) = \det(T(z')) = 0\}$ . Denote the  
1132 regular part of  $W$  by  $\text{Reg}(W)$ . By Proposition 47 (a) we deduce that  $W$  is of co-dimension  $2n$ . Using the same  
1133 argument in the proof of Proposition 56, we deduce that there exists a residual set  $\Gamma \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$  such that  
1134 if  $\tilde{P} \in \Gamma$ , then the image of  $\Delta_2(\mathbb{R}^n)$  under  $j_{(2)}^1\tilde{P}$  is contained in  $\text{Reg}(W)$ . Moreover, by Proposition 44, we have  
1135 that  $M_P = (j_{(2)}^1\tilde{P})^{-1}(\text{Reg}(W)) = (j_{(2)}^1\tilde{P})^{-1}(W)$  is a sub-manifold of dimension zero in  $\Delta_2(\mathbb{R}^n)$ . Notice that  
1136  $\overline{\mathfrak{L}}_n$  is the image of  $M_P$  under the projection  $(x, x') \rightarrow x$ . Since  $M_P$  is of dimension zero, then so is  $\overline{\mathfrak{L}}_n$ . Thus we  
1137 have just proven that, if  $\tilde{P} \in \Gamma$ , then  $\overline{\mathfrak{L}}_n$  is a sub-manifold of dimension zero. Hence,  $\overline{\mathfrak{L}}_n$  is discrete.

1138 The proof that  $\overline{\mathfrak{L}}_n$  does not intersect the boundary of  $B$  can be done analogously as in the proof of Lemma 53.  
1139 □

1140 Assumption  $\mathcal{A}_3$  can be rephrased by the three following assumptions:

1141  $\mathcal{A}_{3(a)}$  - There are no pairwise distinct  $q, q', q'' \in \bar{\mathcal{C}}$  such that  $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q') = \pi_{\mathcal{C}}(q'')$ .

1142  $\mathcal{A}_{3(b)}$  -  $\bar{\mathcal{L}}_{\mathcal{C}} \cap \bar{\mathcal{L}}_n = \emptyset$ .

1143  $\mathcal{A}_{3(c)}$  - For  $q \in \bar{\mathcal{L}}_{\mathcal{C}}$ , the multiplicity of the system  $\{P(x) = 0 \in \mathbb{R}^{n-1}, (x_1, x_2) = \pi_{\mathcal{C}}(q)\}$  at  $q$  is exactly two.

1144 Using this rephrasing, we next show that Assumptions  $\mathcal{A}_{3(a)}$  &  $\mathcal{A}_{3(b)}$  generically hold and we leave the genericity  
1145 of Assumption  $\mathcal{A}_{3(c)}$  as a conjecture.

1146 **Proposition 59.** *For a generic function  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , Assumption  $\mathcal{A}_{3(a)}$  holds.*

1147 *Proof.* Consider the 3-multijet  $j_{(3)}^1 \tilde{P} : \Delta_{(3)}(\mathbb{R}^n) \rightarrow J_{(3)}^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^3$ . Assume that  
1148 every element in  $(\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^3$  is of the form  $((x, y, z), (x', y', z'), (x'', y'', z''))$ , where  $x, x', x'' \in \mathbb{R}^n$ ,  
1149  $y, y', y'' \in \mathbb{R}^{n-2}$  and  $z, z', z'' \in \mathbb{R}^{(n-2) \times n}$ . Abusing notation, we consider  $z, z'$  and  $z''$  as  $((n-2) \times n)$ -matrices.  
1150 Let  $T(z), T(z'), T(z'')$  denote the matrices obtained by removing the first two columns of  $z, z', z''$ , respectively.  
1151 Consider the variety  $W$  defined by the equations:  $\{x_1 = x'_1 = x''_1, x_2 = x'_2 = x''_2, y = y' = y'' = 0 \in$   
1152  $\mathbb{R}^{n-2}, \det(T(z)) = \det(T(z')) = \det(T(z'')) = 0\}$ .

1153 Notice that  $\dim(\Delta_{(3)}(\mathbb{R}^n)) = 3n < 3n + 1 = \text{codim}(W)$ . Hence, by Corollary 45, we have that, generically,  
1154 the pre-image of  $W$  under  $j_{(3)}^1 \tilde{P}$  is empty. Hence, there are no pairwise different  $q, q', q'' \in \bar{\mathcal{C}}$  such that  $\pi_{\mathcal{C}}(q) =$   
1155  $\pi_{\mathcal{C}}(q') = \pi_{\mathcal{C}}(q'')$ .  $\square$

1156 **Proposition 60.** *For a generic function  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , Assumption  $\mathcal{A}_{3(b)}$  holds.*

1157 *Proof.* Consider the 2-multijet  $j_{(2)}^2 \tilde{P} : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^2(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)})^2$   
1158 of the function  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , where  $(\mathbb{R}^n \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)})^2$  is described by the  
1159 coordinates  $x, x' \in \mathbb{R}^n, y, y' \in \mathbb{R}^{n-2}, z, z' \in \mathbb{R}^{(n-2) \times n}$  and  $h, h' \in \mathbb{R}^{n^2(n-2)}$ . With abuse of notation we can  
1160 consider  $z$  and  $z'$  as  $((n-2) \times n)$ -matrices. Let  $T(z)$  (resp.  $T(z')$ ) denote the matrix obtained by removing the  
1161 first two columns of  $z$  (resp.  $z'$ ). Define the matrices  $T_1(z, h), T_2(z, h)$  as in the proof of Lemma 57 and the variety  
1162  $W$  to be the solution set of the system  $\{y = y' = 0 \in \mathbb{R}^{n-2}, x_1 - x'_1 = x_2 - x'_2 = \det(T(z)) = \det(T(z')) = 0,$   
1163  $\det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$ .

1164 Define varieties  $W' = \{(x, y, z, h) \mid y = y' = 0, \det(T(z)) = \det(T(z')) = 0, x_1 = x'_1, x_2 = x'_2\}$  and  
1165  $W'' = \{(x, y, z, h) \mid y = y' = 0, \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$ . Notice that  $W = W' \cap W''$ . Moreover,  
1166 we can find a smooth silhouette curve  $C$  that is not an orthogonal line to the  $(x_1, x_2)$ -plane and that contains two  
1167 distinct points  $q, q'$ , with  $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q')$  such that the projection of  $T_q C$  (resp.  $T_{q'} C$ ) onto  $\mathbb{R}^2$  is injective. Notice  
1168 that  $j_{(2)}^2 \tilde{P}(q, q') \in W' \setminus W''$ . Hence,  $W' \not\subseteq W''$ . Moreover, since  $W'$  is the Cartesian product of determinant  
1169 varieties (which are irreducible by Proposition 47(a)) with linear spaces, we have that  $W'$  is also irreducible  
1170 [BCR98, Theorem 2.8.3 (iii)]. In other words,  $W = W' \cap W''$  is a proper sub-variety of the irreducible variety  
1171  $W'$ . Hence,  $\dim(W) = \dim(W' \cap W'') < \dim(W')$ , equivalently,  $\text{codim}(W) > \text{codim}(W') = 2n$ . Hence,

1172 by Corollary 45 we have that, generically, the pre-image of  $W$  under  $j_{(2)}^2 \tilde{P}$  is empty. Since, by Proposition 56,  
 1173 Assumption  $\mathcal{A}_1$  (which is necessary to guarantee that  $\mathcal{L}'_c$  is well-defined) is also generic, we imply that, generically,  
 1174 there is no distinct pair  $q, q' \in \mathcal{C}$  such that  $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q')$  and  $q \in \overline{\mathcal{L}}_c$ , equivalently,  $\mathcal{L}'_c \cap \mathcal{L}'_n = \emptyset$  which proves  
 1175 the proposition.  $\square$

1176 We thus proved the following proposition that the silhouette of a generic surface in  $\mathbb{R}^n$  satisfies all assumptions  
 1177 except for Assumptions  $\mathcal{A}_{3(c)}$  and  $\mathcal{A}_{5-}$ . However, based on previous results with three variables [IMP16b], we  
 1178 formulate the following conjecture.

1179 **Proposition 61.** *For a generic function  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , Assumptions  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_{3(a)}, \mathcal{A}_{3(b)}, \mathcal{A}_4$ , hold.*

1180 **Conjecture 62.** *For a generic function  $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ , Assumptions  $\mathcal{A}_{3(c)}$  and  $\mathcal{A}_{5-}$  hold.*

## 1181 8. Conclusion

1182 We proposed a regular square system that encodes the singularities of the plane projection of a curve in  $\mathbb{R}^n$   
 1183 under some assumptions, which we proved to be generic via transversality theory. For the case of plane projections  
 1184 of silhouette curves, we proved the genericity of only some of the assumptions and we conjecture the genericity  
 1185 of the others (Proposition 61 and Conjecture 62). We provided semi-algorithms that check whether a given curve  
 1186 satisfies our assumptions and, if they terminate, output isolating boxes of the singularities or the plane projection  
 1187 (possibly with spurious boxes under some weak assumptions). The drawback of our approach is that the number  
 1188 of variables is doubled, which is an issue for subdivision methods that are exponential in the dimension. We  
 1189 partially overcame this issue by applying subdivision schemes in the space of doubled dimension only locally in  
 1190 the neighborhood of the points that project onto singularities (Section 6.1).

1191 A natural open question is the complexity of our semi-algorithms. It is worth noticing that our semi-algorithms  
 1192 use a subdivision approach with *diameter distance tests* as defined by Burr et al. [BGT20]. As such, it should be  
 1193 possible to study our complexities using the method of *continuous amortization*. This should yield explicit bounds  
 1194 for the case of polynomial input either in the worst case (as in [BGT20]), or in a smoothed complexity setting (as  
 1195 in [CETC19]).

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Symbol	Description (see Section 2 unless specified otherwise)
$C^\infty(\mathbb{R}^n, \mathbb{R}^k)$	Class of functions from $\mathbb{R}^n$ to $\mathbb{R}^k$ that are differentiable infinitely many times
$P = (P_1, \dots, P_{n-1})$	A function in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$
$\mathfrak{C}$	Curve defined by $P = 0$ in an open box $B$ in $\mathbb{R}^n$
$\pi_{\mathfrak{C}}$	Projection of $\mathfrak{C}$ onto the $(x_1, x_2)$ -plane
$\mathfrak{L}_{\mathfrak{C}}$	Set of points in $\mathfrak{C}$ where the tangent line is orthogonal to the $(x_1, x_2)$ -plane
$\mathfrak{L}_n$	Set of points $q$ in $\mathfrak{C}$ such that the cardinality of the pre-image of $\pi_{\mathfrak{C}}(q)$ is at least two without counting multiplicities
$\widehat{\mathfrak{L}}_n$	Set of $(q_1, q_2) \in \mathfrak{L}_n^2$ such that $q_1 \neq q_2$ and $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ (Def. 12)
$\widehat{\mathfrak{L}}_{\mathfrak{C}}$	Set of $(q_1, q_1)$ with $q_1 \in \mathfrak{L}_{\mathfrak{C}}$ (Def. 12)
$\widehat{\mathfrak{L}}$	$\widehat{\mathfrak{L}}_n \cup \widehat{\mathfrak{L}}_{\mathfrak{C}}$ (Def. 12)
$T_q \mathfrak{C}$	Line tangent to $\mathfrak{C}$ at $q$
$\overline{\mathfrak{C}}$	Topological closure of $\mathfrak{C}$
$\pi_{\overline{\mathfrak{C}}}, \overline{\mathfrak{L}}_{\mathfrak{C}}, \overline{\mathfrak{L}}_n, T_q \overline{\mathfrak{C}}$	Analogous as $\pi_{\mathfrak{C}}, \mathfrak{L}_{\mathfrak{C}}, \mathfrak{L}_n, T_q \mathfrak{C}$ with respect to $\overline{\mathfrak{C}}$ instead of $\mathfrak{C}$
$S \cdot f(x_1, x_2, y, r, t)$	Operator on $f$ : $\frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})]$ for $t \geq 0$ (Def. 9)
$D \cdot f(x_1, x_2, y, r, t)$	Operator on $f$ : $\frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})]$ for $t > 0$ , and $\nabla f(x_1, x_2, y) \cdot (0, 0, r)$ , otherwise (Def. 9)
Ball( $P$ )	Ball system (Thm. 11)
$\Omega_P$	Maps a solution $(x_1, x_2, y, r, t)$ of Ball( $P$ ) to the pair of points $(x_1, x_2, y \pm r\sqrt{t})$ in $\widehat{\mathfrak{L}}$ (Def. 14 & Fig. 3)
$j^r f, J^r(\mathbb{R}^n, \mathbb{R}^k)$	Jet of order $r$ of $f$ and space of jets of order $r$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^k$ (Def. 42)
$j_{(k)}^r f, J_{(k)}^r(\mathbb{R}^n, \mathbb{R}^k)$	Multijet and space of multijets (Def. 46)

Table 3: Table of the main symbols used throughout this paper.

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