



## Security of the STARK-friendly hash functions

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# Security of the STARK-friendly hash functions

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## Motivation

ZK-STARK protocol is expected to be deployed on top of the Ethereum blockchain within the next year

→ its security and performance highly depend on the underlying **hash function**.

**Performance.** SFH are specified as sequences of low-degree polynomials or low-degree rational maps over a finite field.

### Security.

- algebraic attacks based on Gröbner basis [Albrecht et al. 19]...
- **all other cryptanalytic techniques.**

# MPC-friendly, Snark-friendly and Stark-friendly primitives

## Objectives:

- minimize the number of multiplications in large fields
- minimize the size of the polynomial relations representing the execution trace over a finite field.

## Examples:

- Cradic [Knudsen Nyberg 92], Misty [Matsui 97]
- MiMC [Albrecht et al. 16]

## SFH contenders

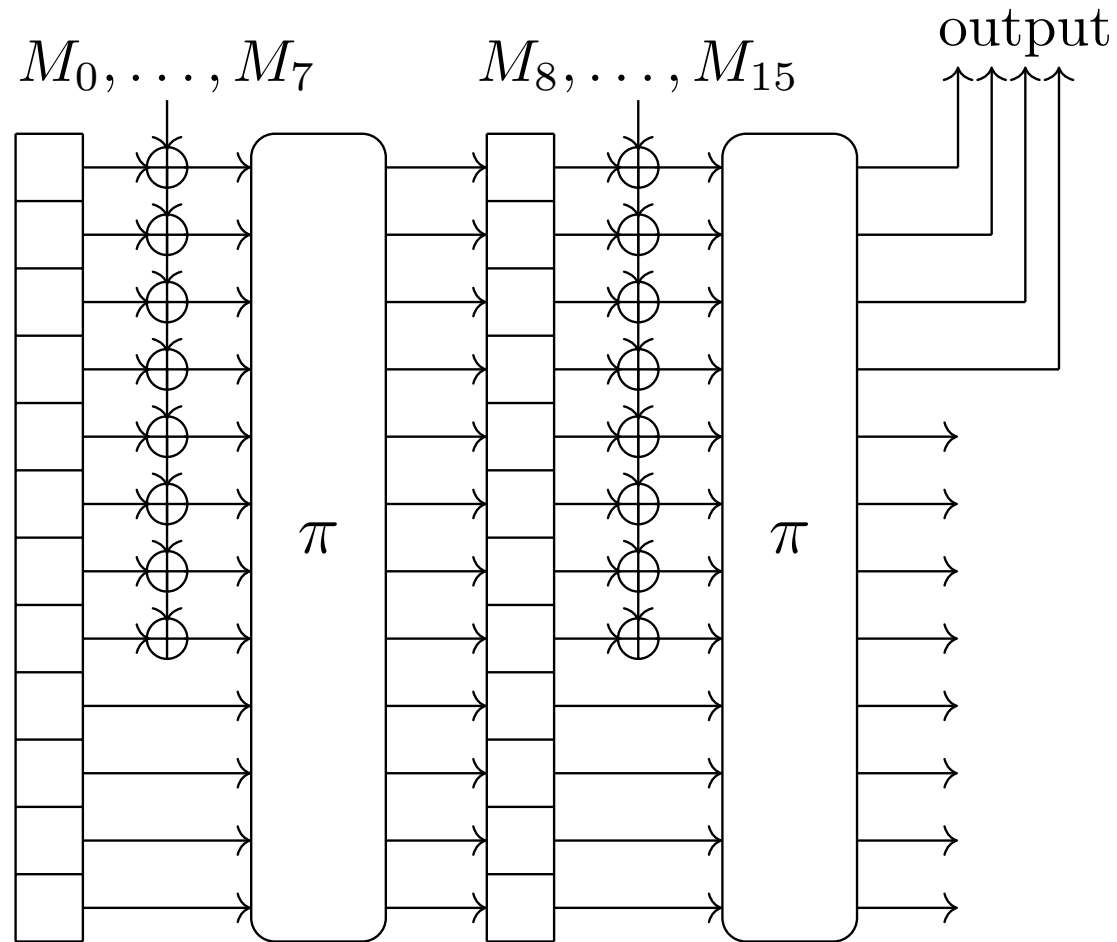
StarkWare challenges <https://starkware.co/hash-challenge/>

Three families of sponges with different permutations

- SPN with large blocks: **Vision** ( $\mathbb{F}_{2^n}$ ) and **Rescue** ( $\mathbb{F}_p$ ) [Aly et al. 19]
- **HadesMiMC** permutation: **Starkad** ( $\mathbb{F}_{2^n}$ ) and **Poseidon** ( $\mathbb{F}_p$ ) [Grassi et al. 19]
- **GMiMC** i.e.  $\text{GMiMC}_{\text{erf}}$  over  $\mathbb{F}_p$  [Albrecht et al. 19]

## Sponge construction

All candidates follow the same sponge construction with **blocksize  $t$**  and **capacity  $c$** .



## Parameters of the sponge

security level	$\log_2 q$	$c$	$t$	
128 bits	64	4	12	variant 128-d
	128	2	4	variant 128-a
	128	2	12	variant 128-c
	256	1	3	variant 128-b
	256	1	11	variant 128-e
256 bits	128	4	8	variant 256-a
	128	4	14	variant 256-b

## Performance for 128-bit security

Best candidate:

Variant 128-d:

$t = 12$  and  $c = 4$  over  $\mathbb{F}_q$

$$q = \begin{cases} 2^{63} \\ 2^{61} + 20 \times 2^{32} + 1 \end{cases}$$



## Compared performance for these parameters

prime fields are more STARK-friendly than binary fields

### Prime field:

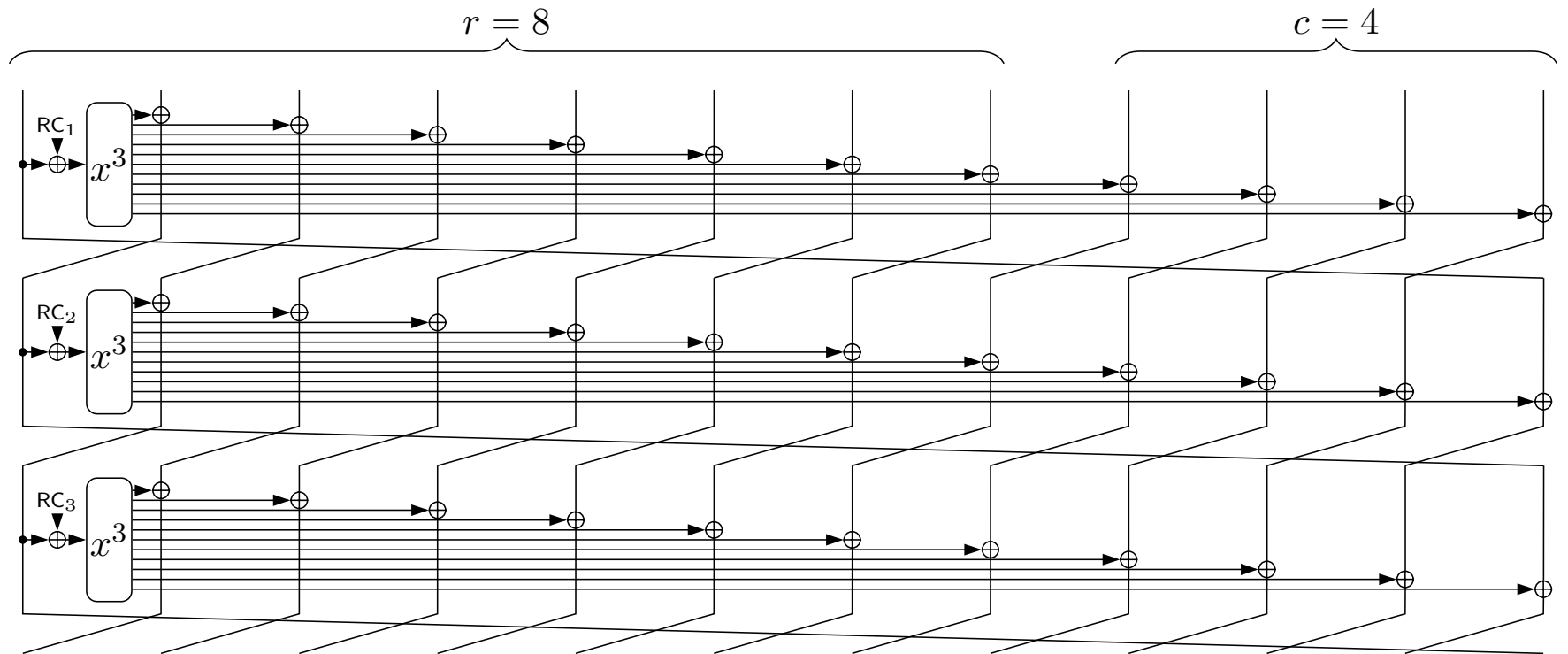
1. GMiMC
2. Rescue
3. Poseidon

### Binary field:

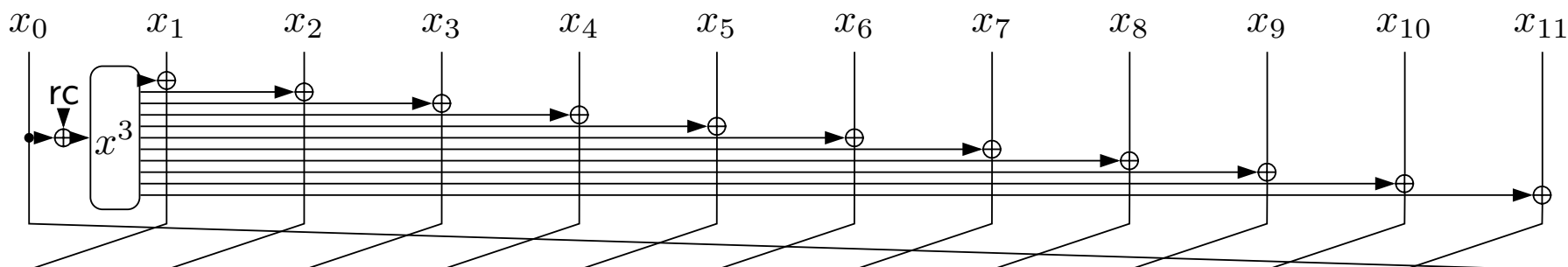
1. Vision
2. Starkad

**GMiMC**

## GMiMC with 101 rounds



## A differential distinguisher



### Original analysis:

best attack with a characteristic over  $(t + 1)$  rounds with probability  $(2q^{-1})^2$ .

### A better differential:

$$(0 \dots 0, \alpha, \alpha') \xrightarrow{\mathcal{R}^{t-2}} (\alpha, \alpha', 0 \dots 0) \xrightarrow{\mathcal{R}} (\alpha' + \beta, \beta \dots \beta, \alpha) \xrightarrow{\mathcal{R}} (\beta + \beta' \dots \beta + \beta', \alpha + \beta', \alpha' + \beta)$$

For  $\beta' = -\beta$ , we get an iterative differential

$$(0 \dots 0, \alpha, \alpha') \xrightarrow{\mathcal{R}^t} (0, \dots, 0, \alpha + \beta, \alpha' + \beta) \text{ with probability } 2q^{-1}$$

## A differential distinguisher

With this  $t$ -round differential with proba  $2q^{-1}$

- A differential characteristic over the **101** rounds with probability  $2^{-480}$  for a **732**-bit blocksize.
- With structures, we get valid pairs with **complexity  $2^{359}$**  (full permutation) and valid pairs with **complexity less than  $2^{128}$  for 58 rounds**.
- With a rebound-like technique, we expect to get valid pairs conforming with the differential over **58** rounds with complexity close to  $2^{64}$  (on-going work).

## Impossible differentials

**Original analysis:** best impossible differential over  $(2t - 2)$  rounds

**A better impossible differential over  $(3t - 4)$ :**

$$(0, \dots, 0, \alpha) \xrightarrow{\mathcal{R}^{3t-4}} (\beta, 0, \dots, 0)$$

for any nonzero  $\alpha \neq \beta$ .

## Integral attacks over $\mathbb{F}_q$

When  $q = 2^n$ .

For any (affine) subspace  $V \subset \mathbb{F}_2^n$  with  $\dim V > \deg F$ ,

$$\sum_{x \in V} F(x) = 0.$$

## Integral attacks over $\mathbb{F}_q$

When  $q = 2^n$ .

For any (affine) subspace  $V \subset \mathbb{F}_2^n$  with  $\dim V > \deg F$ ,

$$\sum_{x \in V} F(x) = 0.$$

Because, for  $V = b + \langle a_1, \dots, a_v \rangle$ ,

$$D_{a_1} D_{a_2} \dots D_{a_v} F(b) = \sum_{x \in V} F(x)$$

Not valid in odd characteristic.



But for any  $q$

For any exponent  $k$  with  $0 \leq k \leq q - 2$ ,

$$\sum_{x \in \mathbb{F}_q} x^k = 0$$

**General result.**

For any  $F : \mathbb{F}_q \rightarrow \mathbb{F}_q$  with  $\deg(F) \leq q - 2$ ,

$$\sum_{x \in \mathbb{F}_q} F(x) = 0 .$$

But for any  $q$

For any exponent  $k$  with  $0 \leq k \leq q - 2$ ,

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**General result.**

For any  $F : \mathbb{F}_q \rightarrow \mathbb{F}_q$  with  $\deg(F) \leq q - 2$ ,

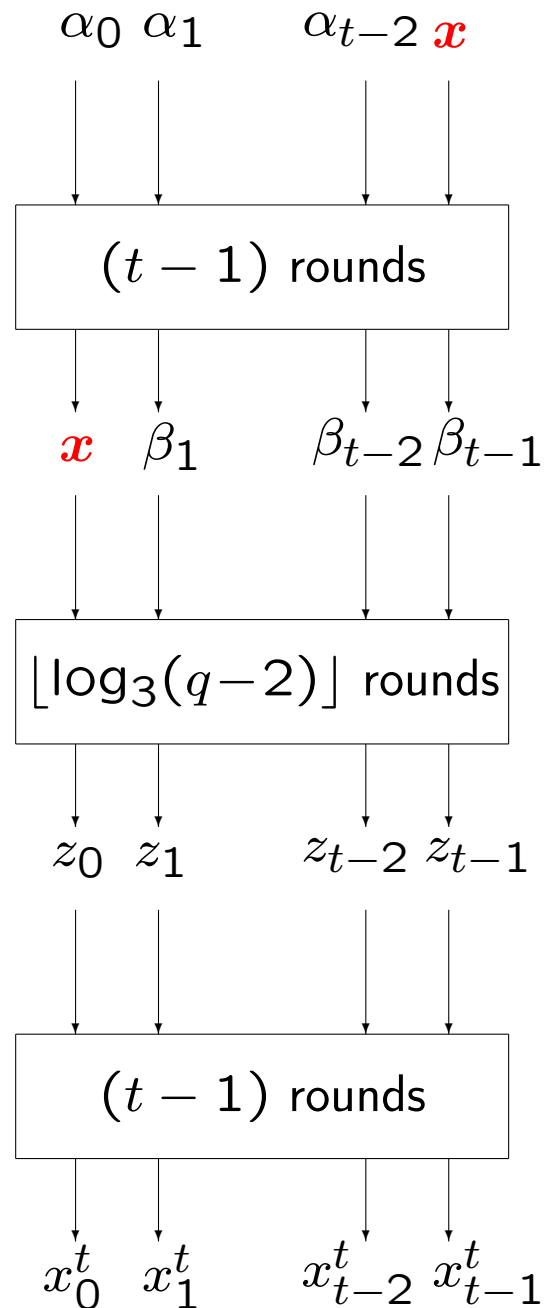
$$\sum_{x \in \mathbb{F}_q} F(x) = 0 .$$

**Less general than the property over  $\mathbb{F}_{2^n}$ :**

For any (affine) subspace  $V \subset \mathbb{F}_2^n$  with  $\dim V > \deg F$ ,

$$\sum_{x \in V} F(x) = 0$$

## Integral distinguisher on GMiMC



$$x \in \mathbb{F}_q$$

polynomial in  $x$   
of degree  $\leq q-2$

$$Q(x) = \sum_{i=1}^{t-1} x_i^t - (t-2)x_0^t$$

Until the degree does not exceed  $(q - 2)$

Input set.

$$\mathcal{X} = \{(\alpha_0, \dots, \alpha_{t-2}, \mathbf{x}), \mathbf{x} \in \mathbb{F}_q\}$$

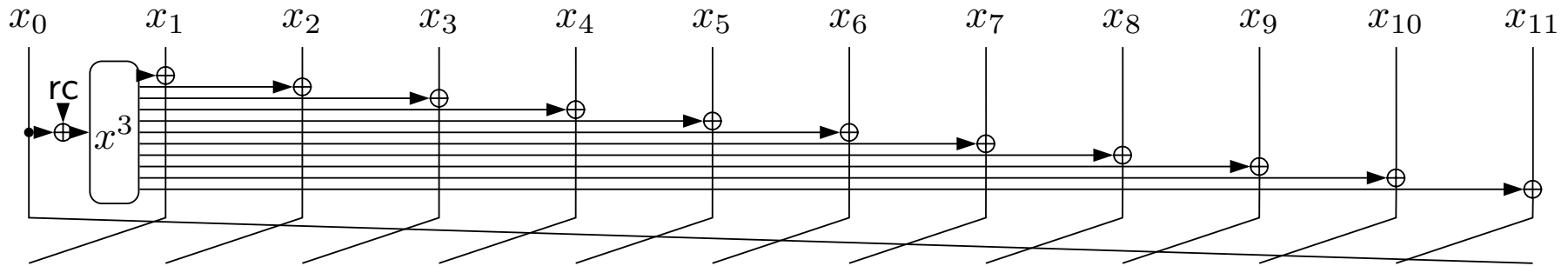
After  $(t - 1)$  rounds.

$$\mathcal{X}' = \{(\mathbf{x}, \beta_1, \dots, \beta_{t-1}), \mathbf{x} \in \mathbb{F}_q\}$$

After  $r$  rounds, the degree in  $\mathbf{x}$  of each branch is at most  $3^r$ .

$\Rightarrow$  all branches are balanced if  $3^r \leq q - 2$ .

## Adding $(t - 2)$ rounds



The inputs and outputs of Round  $\ell$  satisfy

$$x_i^\ell - x_{i+1}^\ell = x_{i+1}^{\ell-1} - x_{i+2}^{\ell-1}, \quad \forall i \leq t - 3$$

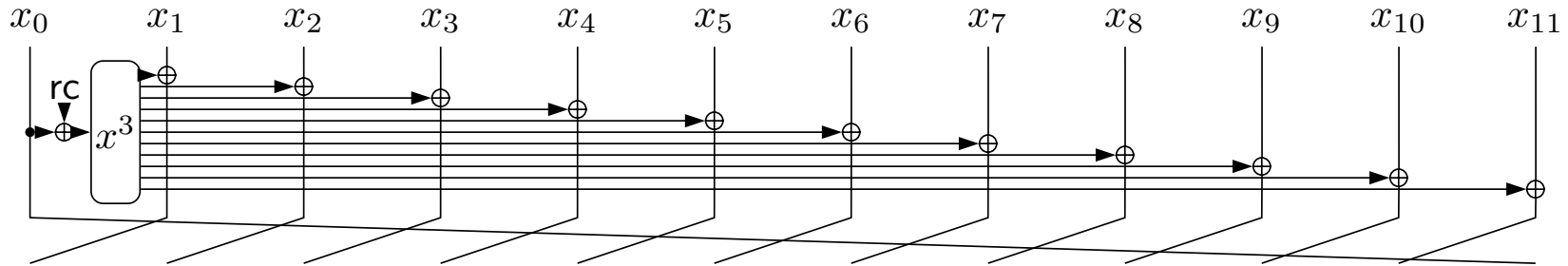
Over  $(t - 2)$  rounds,

$$x_0^{t-1} - x_1^{t-1} = x_{t-2}^1 - x_{t-1}^1$$

is a polynomial in  $\mathbf{x}$  of degree  $\leq (q - 2)$ .

$\Rightarrow$  Distinguisher with complexity  $q$  on  $(2t - 3 + \lfloor \log_3(q - 2) \rfloor)$  rounds (59 rounds)

## Adding one more round



The inputs and outputs of Round  $\ell$  satisfy

$$x_i^\ell = x_{i+1}^{\ell-1} + (x_j^\ell - x_{j+1}^{\ell-1}) \text{ and } x_{t-1}^\ell = x_0^{\ell-1}$$

$$\Rightarrow \sum_{i=0}^{t-1} x_i^{\ell-1} - (t-1)x_j^t = \sum_{i=0}^{t-1} x_i^\ell - (t-1)x_{j-1}^\ell$$

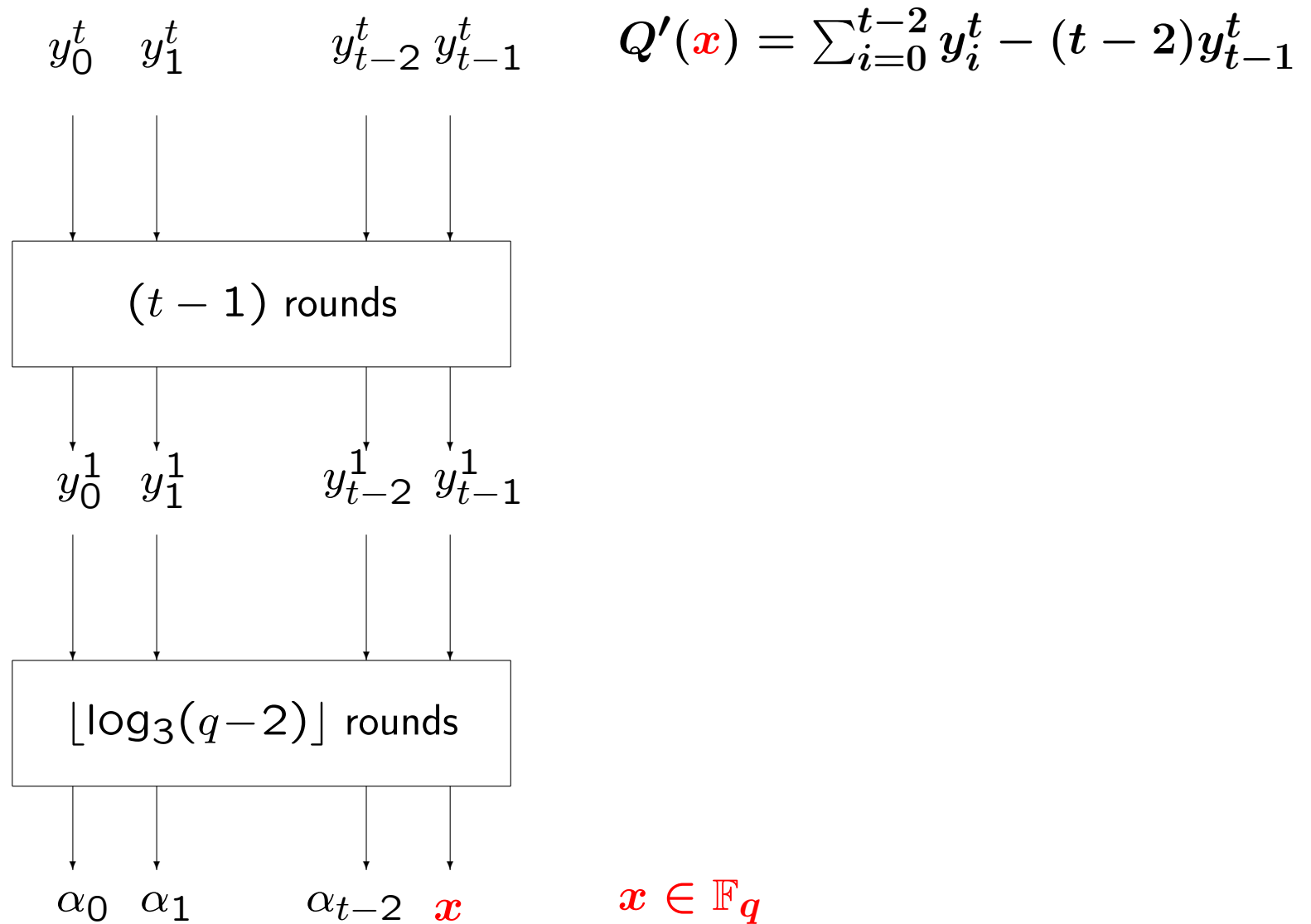
Over  $(t-1)$  rounds,

$$\sum_{i=0}^{t-1} x_i^1 - (t-1)x_{t-1}^1 = \sum_{i=0}^{t-1} x_i^t - (t-1)x_0^t$$

$\Rightarrow$  Distinguisher with complexity  $q$  on  $(2t-2 + \lfloor \log_3(q-2) \rfloor)$  rounds (60 rounds)

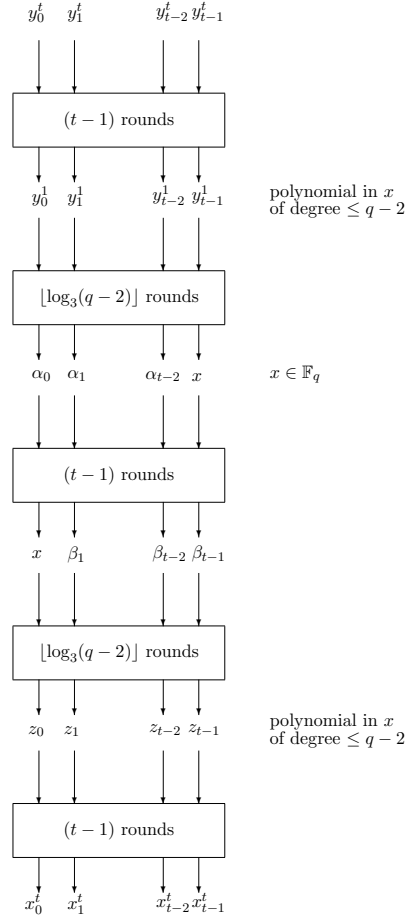
A few more rounds with two active branches (on-going work).

## Computing backwards



## Zero-sum partition on GMiMC on $(3t - 3 + 2\lfloor \log_3(q - 2) \rfloor)$ rounds (109)

$$\ell(x_0, \dots, x_{t-1}) = \sum_{i=1}^{t-1} x_i - (t - 2)x_0 \text{ sum to } 0$$

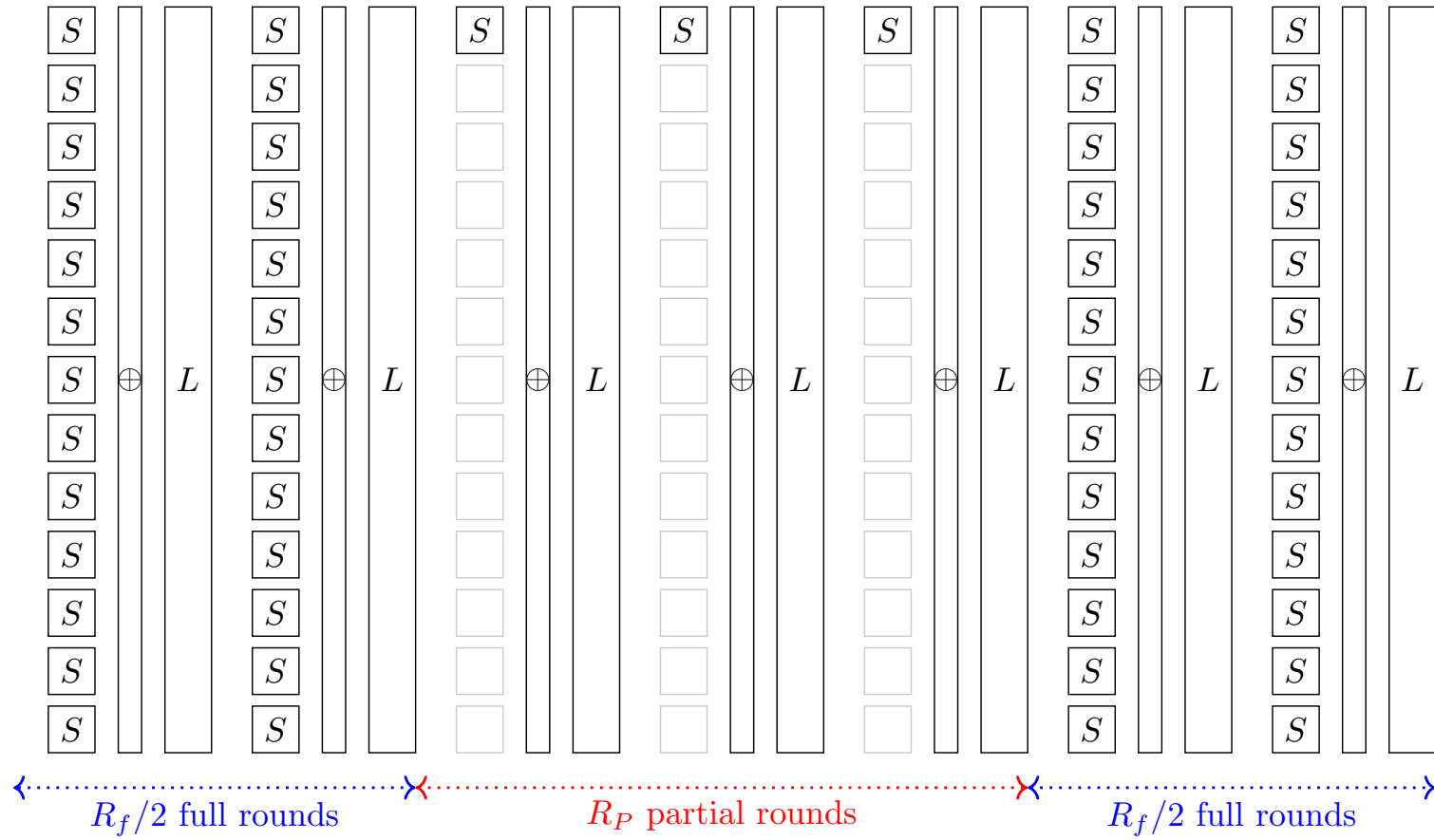


$$\ell'(y_0, \dots, y_{t-1}) = \sum_{i=0}^{t-2} y_i - (t - 2)y_{t-1} \text{ sum to } 0$$



**HadesMiMC**

# HadesMiMC



$R_f = 8$  full rounds and  $R_P = 43$  (binary) and  $R_P = 40$  (prime)

## Resistance against statistical attacks

Analysed without the partial rounds.

### Differential cryptanalysis:

$x^3$  has differential uniformity **2** over  $\mathbb{F}_q$ .

The best differential characteristic satisfies

$$\text{EDP} \leq \left( \frac{2}{q} \right)^{(t+1)R_f/2}$$

→  $R_f = 6$  are enough.

## Degree of the permutation over $\mathbb{F}_q$

Each coordinate is seen as a **multivariate polynomial over  $\mathbb{F}_q$**

**After  $r$  rounds:**

$$\sum_{u=(u_1, \dots, u_t)} \lambda_u \left( \prod_{i=1}^t x_i^{u_i} \right) \text{ where } u_i \leq 3^r$$

$\Rightarrow$  39 rounds are enough for Poseidon (40 for Starkad) to reach degree  $(q - 1)$  in each variable

$\Rightarrow \lceil \log_3(t) \rceil$  more rounds are enough to get total degree  $(q - 1)t$ .

**Remark:** StarkWare challenges with  $q \simeq 2^{256}$  and 96 rounds have degree at most  $2^{152}$  in each variable.

## Zero-sum partition over $\mathbb{F}_q$

State after the last full Sbox layer before the partial rounds.

$$\mathcal{X} = \{(\alpha_0, \dots, \alpha_{t-2}, \mathbf{x}), \mathbf{x} \in \mathbb{F}_q\}$$

After 38 rounds forwards.

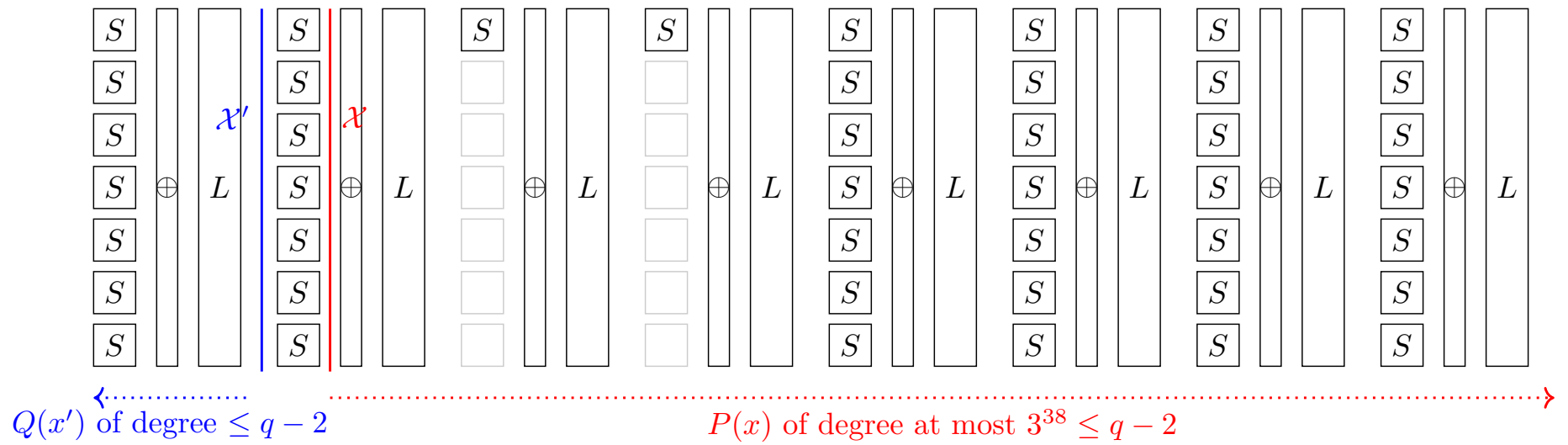
each coordinate has degree at most  $(q - 2)$ .

Computing backwards.

$$S^{-1} : x \mapsto x^s \text{ with } s = \frac{2q - 1}{3}$$

$\Rightarrow$  Zero-sum for  $R_f = 2 + 4$  and  $R_P = 34$  (35 for Starkad).

# Zero-sum partition over $\mathbb{F}_q$



## Improvement when $q = 2^n$

Each Boolean coordinate is seen as a **multivariate polynomial in  $nt$  variables over  $\mathbb{F}_2$**

**Degree over  $\mathbb{F}_{2^n}$  vs. binary degree.**

$$P(x) = \sum_{u \leq 2^n - 1} \lambda_u x^u$$

has binary degree

$$\max\{wt(u) : 0 \leq u < 2^n \text{ and } \lambda_u \neq 0\}$$

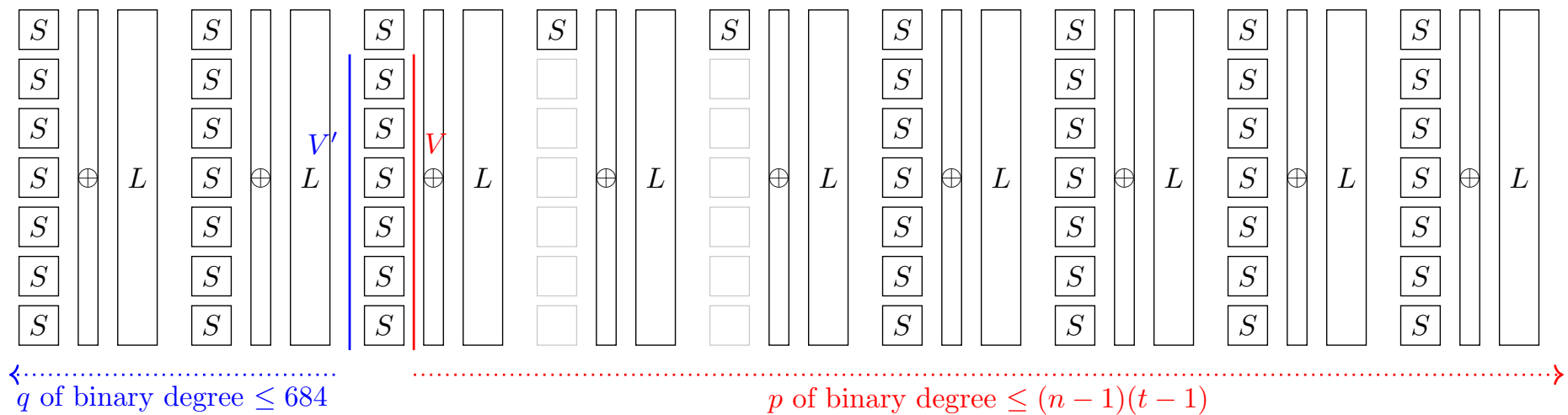
$\Rightarrow$  The inverse Sbox has binary degree  $\frac{n+1}{2}$ .

**Several rounds backwards [Boura, C. 13].**

- Two rounds backwards have binary degree  $\leq 684$
- Three rounds backwards have binary degree  $\leq 748$

# Zero-sum partition over $\mathbb{F}_2$ with $R_f = 3 + 4$ and $R_P = 35$

$$V = \{(0, x_1, \dots, x_{t-1}), x_i \in \mathbb{F}_{2^n}\}.$$





## When the MDS matrix has a small order

### How to propagate a subspace through all partial rounds?

Choose  $V$  such that all elements in each coset of  $L(V)$  have the same value on the first coordinate.

$$L(V) \subset H_0 = \{(0, x_1, \dots, x_{t-1}), x_i \in \mathbb{F}_q\}$$

or equivalently

$$V \subset \langle M_0 \rangle^\perp.$$

We can iterate this  $R_P$  times if

$$\mathcal{V} = H_0 \cap \bigcap_{r=0}^{R_P-1} L^r \left( \langle M_0 \rangle^\perp \right) \neq \{0\}$$

This holds if  $L^r = \text{Id}$  for some  $r \leq t - 2$ .

## When the MDS matrix is an involution

The internal states after each partial layer form a coset of  $V$  or of  $W = L(V)$ .

Special choice for  $V$ .

$$V = \{(\mathbf{x}v_0, \dots, \mathbf{x}v_{t-1}), \mathbf{x} \in \mathbb{F}_q\}$$

with  $v \in \mathcal{V}$ .

$\Rightarrow$  The outputs of the partial rounds vary in a coset of

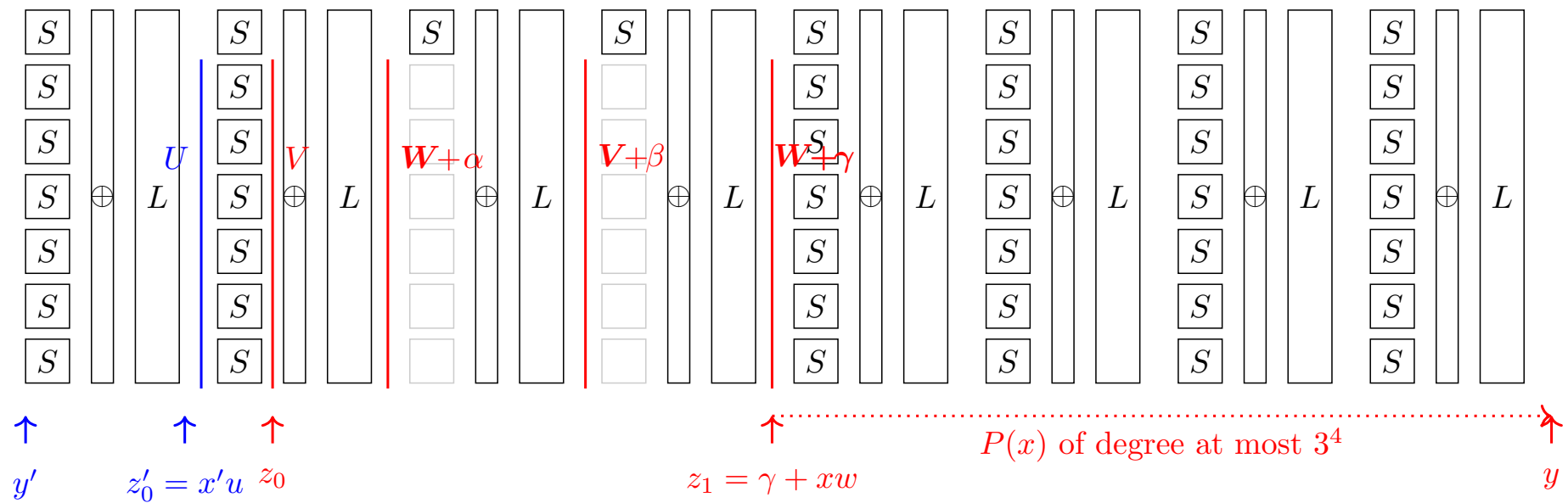
$$\{(\mathbf{x}w_0, \dots, \mathbf{x}w_{t-1}), \mathbf{x} \in \mathbb{F}_q\}$$

Forward direction.

Each output coordinate is a polynomial in  $\mathbf{x}$  of degree at most  $3^{R_f/2} \leq q - 2$ .

$\Rightarrow$  The output coordinates sum to zero.

# Zero-sum partition with $R_f = 2 + 4$ and any $R_P$ with complexity $q$



## Open question on the complexity of algebraic attacks

**Input:**  $(a_1, \dots, a_{t-k}) \in \mathbb{F}_q^{t-k}$  and  $(b_1, \dots, b_k) \in \mathbb{F}_q^k$

**Find**  $x_1, \dots, x_k \in \mathbb{F}_q^k$  such that

$$\pi(a_1, \dots, a_{t-k}, x_1, \dots, x_k) = (b_1, \dots, b_k, y_1, \dots, y_{t-k}) \text{ for some } y_1, \dots, y_{t-k}$$

**Degree of the univariate polynomial of the lexicographical Gröbner basis [Faugère-Perret].**

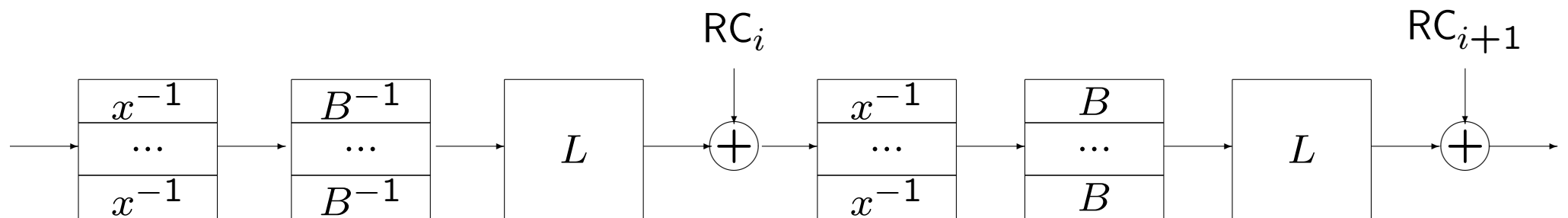
$$D = 3^{kR_f + R_P - 2k + 1}$$

Complexity for solving the system  $= D^2$ .

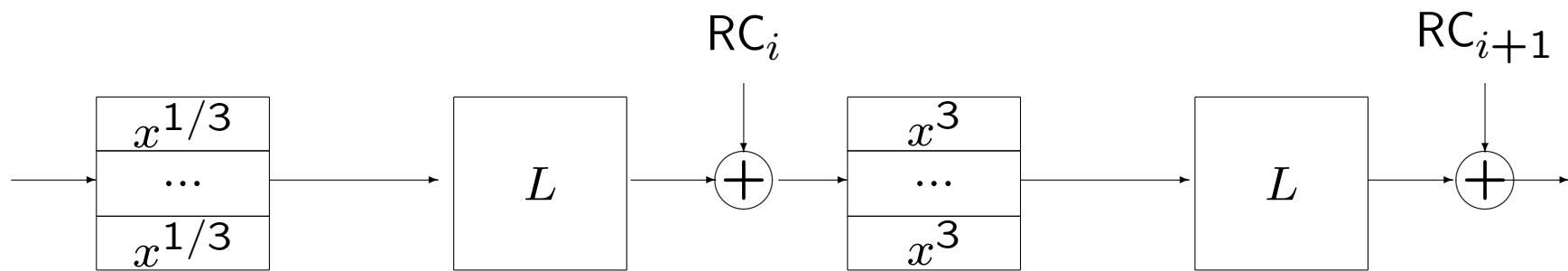
Variants aiming at 256-bit security have  $D \simeq 2^{170}$ .

# **Vision and Rescue**

## Vision (20 rounds)



## Rescue (20 rounds)



## Degree of Rescue

Activate one input coordinate  $x \in \mathbb{F}_p$

**After one round.**

$$\lambda x^{1/3} + \mu$$

$$\Rightarrow \text{degree } \frac{2p-1}{3}$$

**After the second Sbox layer.**

$$(\lambda x^{1/3} + \mu)^3$$

which contains only monomials  $x^{1/3}$ ,  $x^{2/3}$ ,  $x$  and a constant term.

$x^{2/3}$  has degree  $\frac{p+1}{3}$ .

$\Rightarrow$  The degree does not increase between the first and second round.

**But even by activating more inputs,** we cannot find an integral attack on more than 4 rounds.



## Conclusions

We need to **find the right tools** for analyzing symmetric primitives over non-binary fields:

- linear attacks and their variants?
- more general integral attacks?

### Open question:

Does the **form of  $q$**  affect the security?

For instance, if  $p = 2^{2^n} + 1$ ?