## Saddlepoint Approximations of Cumulative Distribution Functions of Sums of Random Vectors

Dadja Anade, Jean-Marie Gorce, Philippe Mary, and Samir M. Perlaza

## RESEARCH

## REPORT

$\mathbf{N}^{\circ} 9388$
February 2021
Project-Teams MARACAS and

# Saddlepoint Approximations of Cumulative Distribution Functions of Sums of Random Vectors 

Dadja Anade, Jean-Marie Gorce, Philippe Mary, and Samir M. Perlaza<br>Project-Teams MARACAS and NEO<br>Research Report n ${ }^{\circ} 9388$ - version 1 - initial version February 2021 -<br>revised version Mai 2021 - 32 pages


#### Abstract

In this report, a real-valued function that approximates the cumulative distribution function (CDF) of a finite sum of real-valued independent and identically distributed random vectors is presented. The approximation error is upper bounded and thus, as a byproduct, an upper bound and a lower bound on the CDF are obtained. Finally, it is observed that in the case of lattice and absolutely continuous random variables, the proposed approximation is identical to the saddlepoint approximation of the CDF.


Key-words: Sums of independent random vectors, Saddlepoint approximations, Gaussian approximations

Dadja Anade and Jean-Marie Gorce are with the Laboratoire CITI, a joint laboratory between the Institut National de Recherche en Informatique et en Automatique (INRIA), the Universite de Lyon and the Institut National de Sciences Apliquées (INSA) de Lyon. 6 Av. des Arts 69621 Villeurbanne, France. (\{dadja.anade-akpo, jean-marie.gorce\} @inria.fr)

Philippe Mary is with the Laboratoire IETR and the Institut National de Sciences Apliquées (INSA) de Rennes. (philippe.mary@insa-rennes.fr)

Samir M. Perlaza is also with the Electrical and Computer Engineering Department, Princeton University, Princeton, 08544 NJ, USA; and the Laboratoire de Mathématiques GAATI, Université de la Polynésie Française, BP 6570, 98702 Faaa, French Polynesia.

This work was partially funded by the French National Agency for Research (ANR) under grant ANR-16-CE25-0001.

Parts of this work were presented at the IEEE International Symposium on Information Theory (ISIT), Melbourne, Australia, Jul. 2021 1].

```
RESEARCH CENTRE
GRENOBLE - RHÔNE-ALPES
Inovallée
6 5 5 \text { avenue de l'Europe Montbonnot}
3 8 3 3 4 \text { Saint Ismier Cedex}
```

Résumé : Dans ce rapport, une fonction qui approxime la fonction de répartition d'une somme de vecteurs aléatoires indépendants et identiquement distribués est présentée. L'erreur d'approximation est majorée, et par consequent, une borne supérieure et une borne inférieure sur la fonction de répartition sont obtenues. Finalement, pour des vecteurs aléatoires absolument continues ou lattices, l'approximation proposée est identique à l'approximation du point de selle de la fonction de répartition.

Mots-clés : Approximation du point de selle, approximation Gaussienne

## Contents

1 Notation 4
2 Introduction 4
3 Preliminaries 4
4 Gaussian Approximation 6
5 Exponentially Tilted Gaussian Approximation 7
5.1 Approximation of the Measure . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
5.2 Approximation of the CDF . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
5.3 Connexion with the Saddlepoint Approximation. . . . . . . . . . . . . . . . . . . 14

6 Examples 15
7 Final Remarks and Discussion 19

| Proof of Lemma 2 | 20 |
| :--- | :--- | :--- |

A. 1 Explicit Expression of (78) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
A. 2 Explicit Expression of (79) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23
A. 3 Upper Bound on (77) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24

B Proof of Theorem 3 25

| C Proof of Lemma 4 |
| :--- | :--- |
| 28 |

D Proof of Lemma $5 \quad 29$

## 1 Notation

The real numbers are denoted by $\mathbb{R}$, and the natural numbers are denoted by $\mathbb{N}$. In particular, $0 \notin \mathbb{N}$. The Borel sigma field on $\mathbb{R}^{k}$, with $k \in \mathbb{N}$, is denoted by $\mathscr{B}\left(\mathbb{R}^{k}\right)$. The Lebesgue measure on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right.$ ) is denoted by $\nu_{k}$. Given a discrete set $\mathcal{K}$, the biggest sigma field, i.e., the set of all its subsets, is denoted by $2^{\mathcal{K}}$. The Euclidian norm in $\mathbb{R}^{k}$ is denoted by $\|\cdot\|$. Given a set $\mathcal{A} \subseteq \mathbb{R}^{k}$, the closure of the set $\mathcal{A}$, denoted by clo $\mathcal{A}$, is defined by $\operatorname{clo} \mathcal{A} \triangleq\left\{\boldsymbol{x} \in \mathbb{R}^{k}: \forall r>0, \exists \boldsymbol{y} \in \mathcal{A},\|\boldsymbol{x}-\boldsymbol{y}\|<r\right\}$. A diagonal matrix whose diagonal is the vector $\boldsymbol{x} \in \mathbb{R}^{k}$ is denoted by diag $(\boldsymbol{x})$.

## 2 Introduction

The calculation of cumulative distribution functions (CDFs) of sums of random vectors is omnipresent in the realm of information theory. For instance, the joint decoding error probability in multi-user channels often boils down to the calculation of CDFs of random vectors, c.f., [2]. In the case of the memoryless Gaussian multiple access channel, under certain conditions on the channel inputs, the dependence testing bound corresponds to the CDF of a sum of independent and identically distributed (IID) random vectors [3, Theorem 2].
Unfortunately, the calculation of CDFs of random vectors requires elaborated numerical methods which often lead to errors that are difficult to quantify. From this perspective, approximations to these CDFs, e.g., Gaussian approximations and saddlepoint approximations [4]-8 have gained remarkable popularity $[9-15$. In the case of Gaussian approximations, multi-dimensional Berry-Esseen-type theorems provide upper bounds on the approximation errors, c.f., 16]. These bounds are particularly precise around the mean. Alternatively, saddlepoint approximations are known to be more precise than Gaussian approximations far apart from the mean. Unfortunately, this claim is often justified only by numerical analysis as formal upper bounds on the error induced by saddlepoint approximations are rather inexistent, c.f., 6 and $[7$. This paper contributes in this direction by introducing a real-valued function that approximates the CDF of a finite sum of real-valued IID random vectors. Both Gaussian and saddlepoint approximations are shown to be special cases of the proposed approximation, which is referred to as the exponentially tilted Gaussian approximation. The approximation error is upper bounded and both upper and lower bounds on the CDF are obtained.

## 3 Preliminaries

Let $n$ be a finite integer, with $n>1$, and let $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$ be independent random vectors such that each of them induces the probability measure $P_{\boldsymbol{Y}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$, with $k \in \mathbb{N}$. Denote by $K_{\boldsymbol{Y}}: \mathbb{R}^{k} \rightarrow \mathbb{R}$ the cumulant generating function (CGF) of each of these random variables. That is, for all $t \in \mathbb{R}^{k}$,

$$
\begin{equation*}
K_{\boldsymbol{Y}}(\boldsymbol{t})=\ln \left(\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\exp \left(\boldsymbol{t}^{\top} \boldsymbol{Y}\right)\right]\right) \tag{1}
\end{equation*}
$$

The gradient of the CGF $K_{\boldsymbol{Y}}$ is a function denoted by $K_{\boldsymbol{Y}}^{(1)}: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$. More specifically, for all $t \in \mathbb{R}^{k}$,

$$
\begin{equation*}
K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})=\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\boldsymbol{Y} \exp \left(\boldsymbol{t}^{\boldsymbol{\top}} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{t})\right)\right] . \tag{2}
\end{equation*}
$$

The Hessian of the CGF $K_{\boldsymbol{Y}}$ is a function denoted by $K_{\boldsymbol{Y}}^{(2)}: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k \times k}$. That is, for all $\boldsymbol{t} \in \mathbb{R}^{k}$,

$$
\begin{equation*}
K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{t})=\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})\right)\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})\right)^{\boldsymbol{\top}} \exp \left(\boldsymbol{t}^{\boldsymbol{\top}} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{t})\right)\right], \tag{3}
\end{equation*}
$$

Note that $K_{\boldsymbol{Y}}^{(1)}(\mathbf{0})$ and $K_{\boldsymbol{Y}}^{(2)}(\mathbf{0})$ are respectively the mean vector and the covariance matrix of each of the random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$. In the following, $K_{\boldsymbol{Y}}^{(2)}(\mathbf{0})$ is assumed to be positive definite.

Let also

$$
\begin{equation*}
\boldsymbol{X}_{n} \triangleq \sum_{t=1}^{n} \boldsymbol{Y}_{t} \tag{4}
\end{equation*}
$$

be a random vector that induces the probability measure $P_{X_{n}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$, with cumulative distribution function (CDF) denoted by $F_{\boldsymbol{X}_{n}}$. When for all $i \in\{1,2, \ldots, n\}$ the random vector $\boldsymbol{Y}_{i}$ in (4) is absolutely continuous and its corresponding CGF $K_{\boldsymbol{Y}}$ is such that the set

$$
\begin{equation*}
\left.\mathcal{C}_{\boldsymbol{Y}} \triangleq\left\{\boldsymbol{\theta} \in \mathbb{R}^{k}: K_{\boldsymbol{Y}}(\boldsymbol{\theta})<\infty\right\} \cap\right]-\infty, 0\left[{ }^{k}\right. \tag{5}
\end{equation*}
$$

is not empty, the CDF $F_{\boldsymbol{X}_{n}}$ can be written as a complex integral 4 . In particular, for all $\boldsymbol{x} \in \mathbb{R}^{k}$,

$$
\begin{equation*}
F_{\boldsymbol{X}_{n}}(\boldsymbol{x})=\int_{\boldsymbol{c}-\mathrm{i} \infty}^{\boldsymbol{c}+\mathrm{i} \infty} \frac{\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\tau})-\boldsymbol{\tau}^{\boldsymbol{\top}} \boldsymbol{x}\right)}{(2 \pi \mathrm{i})^{k} \prod_{t=1}^{k} \tau_{t}} \mathrm{~d} \boldsymbol{\tau} \tag{6}
\end{equation*}
$$

where i is the imaginary unit; $\boldsymbol{\tau}=\left(\tau_{1}, \tau_{2}, \ldots, \tau_{k}\right)$; and the constant $\boldsymbol{c}$ is arbitrarily chosen to satisfy $\boldsymbol{c} \in \mathcal{C}_{\boldsymbol{Y}}$. The complex integral in (6) is a multivariate Laplace inverse transform (10, 17, and can be approximated with high precision, as shown hereunder. Denote by $\mathcal{D}$ the following set

$$
\begin{equation*}
\mathcal{D} \triangleq\left\{\boldsymbol{u} \in \mathbb{R}^{k}: \exists \boldsymbol{t} \in\right]-\infty, 0\left[^{k}, n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})=\boldsymbol{u}\right\} \tag{7}
\end{equation*}
$$

and denote by $\boldsymbol{\tau}_{0} \in \mathbb{R}^{k}$ the unique solution in $\boldsymbol{\tau}$ to

$$
\begin{equation*}
K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\tau})=\frac{1}{n} \boldsymbol{x} \tag{8}
\end{equation*}
$$

For all $\boldsymbol{x} \in \mathcal{D}$, a Taylor series expansion of $n K_{\boldsymbol{Y}}(\boldsymbol{\tau})-\boldsymbol{\tau}^{\boldsymbol{\top}} \boldsymbol{x}$ in the neighborhood of $\boldsymbol{\tau}_{0}$, leads to the following asymptotic expansion of the integral in (6):

$$
\begin{equation*}
F_{\boldsymbol{X}_{n}}(\boldsymbol{x})=\hat{F}_{\boldsymbol{X}_{n}}(\boldsymbol{x})+O\left(\frac{\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\boldsymbol{\top}} \boldsymbol{x}\right)}{\sqrt{n}}\right) \tag{9}
\end{equation*}
$$

where the function $\hat{F}_{\boldsymbol{X}_{n}}: \mathcal{D} \rightarrow \mathbb{R}$ is

$$
\begin{equation*}
\hat{F}_{\boldsymbol{X}_{n}}(\boldsymbol{x})=\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\top} \boldsymbol{x}+\frac{n \boldsymbol{\tau}_{0}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right) F_{\boldsymbol{G}_{n}^{\left(\tau_{0}\right)}}\left(n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right) \tag{10}
\end{equation*}
$$

and the function $F_{G_{n}^{\left(\tau_{0}\right)}}: \mathbb{R}^{k} \rightarrow[0,1]$ is the CDF of a Gaussian random vector with mean vector $(0,0, \ldots, 0)$ and covariance matrix $n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right)$.
The vector $\boldsymbol{\tau}_{0}$ and the function $\hat{F}_{\boldsymbol{X}_{n}}$ in 10 are respectively referred to as the saddlepoint and the saddlepoint approximation of the $\mathrm{CDF}{F_{\boldsymbol{X}_{n}} \text {. In [4], it is shown that the approximation }}^{\text {. }}$ $\hat{F}_{\boldsymbol{X}_{n}}$ in (10) also holds for the case in which for all $i \in\{1,2, \ldots, n\}$ the vector $\boldsymbol{Y}_{i}$ in (4) is a lattice random vector. Moreover, when for all $i \in\{1,2, \ldots, n\}$ the random vector $\boldsymbol{Y}_{i}$ in (4) is a Gaussian random vector, then the saddlepoint approximation is exact. That is, $\hat{F}_{\boldsymbol{X}_{n}}$ and $F_{\boldsymbol{X}_{n}}$ are identical. Using the elements above, the main contributions of this work can be described as follows:
(a) A real-valued function that approximates the CDF $F_{\boldsymbol{X}_{n}}$ is presented. This approximation is shown to be identical to the saddlepoint approximation $\hat{F}_{\boldsymbol{X}_{n}}$ in $(9)$ when $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots$, $\boldsymbol{Y}_{n}$ are either absolutely continuous or lattices random vectors; and
(b) an upper bound on the error induced by the proposed approximation is also presented. The asymptotic behaviour with $n$ of the proposed upper bound is consistent with the one suggested by (9).

## 4 Gaussian Approximation

Let $\boldsymbol{\mu}_{\boldsymbol{X}_{n}} \in \mathbb{R}^{k}$ and $\underline{\boldsymbol{v}}_{\boldsymbol{X}_{n}} \in \mathbb{R}^{k \times k}$ be the mean vector and covariance matrix of the random vector $\boldsymbol{X}_{n}$ in (4). The Gaussian approximation of the measure $P_{\boldsymbol{X}_{n}}$ induced by $\boldsymbol{X}_{n}$ is the probability measure induced by a Gaussian vector with mean vector $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ and covariance matrix $\underline{\boldsymbol{v}}_{\boldsymbol{X}_{n}}$. The following theorem, known as the multivariate Berry-Esseen theorem [16], introduces an upper bound on the approximation error.

Theorem 1 ( [16, Theorem 1.1]) Assume that the measure $P_{\boldsymbol{Y}}$ induced by each of the random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$ in (4) satisfies,

$$
\begin{align*}
& \mathbb{E}_{P_{\boldsymbol{Y}}}[\boldsymbol{Y}]=(0,0, \ldots, 0), \text { and }  \tag{11}\\
& \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\boldsymbol{Y} \boldsymbol{Y}^{T}\right]=\frac{1}{n} \operatorname{diag}(1,1, \ldots, 1) \tag{12}
\end{align*}
$$

Let $P_{\boldsymbol{Z}_{n}}$ be the probability measure induced on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$ by a Gaussian random vector $\boldsymbol{Z}_{n}$ with mean vector $(0,0, \ldots, 0)$ and covariance matrix diag $(1,1, \ldots, 1)$. Then,

$$
\begin{equation*}
\sup _{\mathcal{A} \in \mathcal{C}_{k}}\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-P_{\boldsymbol{Z}_{n}}(\mathcal{A})\right| \leqslant \min \left(1, c(k) n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\|\boldsymbol{Y}\|^{3}\right]\right) \tag{13}
\end{equation*}
$$

where $\mathcal{C}_{k}$ is the set of all convex sets in $\mathscr{B}\left(\mathbb{R}^{k}\right)$; and the function $c: \mathbb{N} \rightarrow \mathbb{R}$ satisfies for all $k \in \mathbb{N}$,

$$
\begin{equation*}
c(k)=42 k^{\frac{1}{4}}+16 . \tag{14}
\end{equation*}
$$

The measure $P_{\boldsymbol{Z}_{n}}$ in (13) is often referred to as the Gaussian approximation of $P_{\boldsymbol{X}_{n}}$. Similarly, $F_{\boldsymbol{Z}_{n}}$, the CDF of the measure $P_{\boldsymbol{Z}_{n}}$, is referred to as the Gaussian approximation of $F_{\boldsymbol{X}_{n}}$. Theorem 1 leads to the following inequalities for all $\boldsymbol{x} \in \mathbb{R}^{k}$,

$$
\begin{equation*}
\underline{\Sigma}(n, \boldsymbol{x}) \leqslant F_{\boldsymbol{X}_{n}}(\boldsymbol{x}) \leqslant \bar{\Sigma}(n, \boldsymbol{x}) \tag{15}
\end{equation*}
$$

where,

$$
\begin{align*}
& \bar{\Sigma}(n, \boldsymbol{x}) \triangleq F_{\boldsymbol{Z}_{n}}(\boldsymbol{x})+\min \left(1, c(k) n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\|\boldsymbol{Y}\|^{3}\right]\right), \text { and }  \tag{16a}\\
& \underline{\Sigma}(n, \boldsymbol{x}) \triangleq F_{\boldsymbol{Z}_{n}}(\boldsymbol{x})-\min \left(1, c(k) n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\|\boldsymbol{Y}\|^{3}\right]\right) . \tag{16b}
\end{align*}
$$

That is, the functions $\underline{\Sigma}(n, \cdot)$ and $\bar{\Sigma}(n, \cdot)$ are respectively a lower and an upper bound on the $\operatorname{CDF} F_{\boldsymbol{X}_{n}}$.

## 5 Exponentially Tilted Gaussian Approximation

This section introduces two central results. First, given a convex set $\mathcal{A}$ in $\mathscr{B}\left(\mathbb{R}^{k}\right)$, the probability $P_{\boldsymbol{X}_{n}}(\mathcal{A})$, with $P_{\boldsymbol{X}_{n}}$ the probability measure induced by the random vector $\boldsymbol{X}_{n}$ in (4), is approximated by a function that is a measure but not necessary a probability measure. This function, which is parametrized by a vector in $\mathbb{R}^{k}$ that can be arbitrarily chosen, is often referred to as the exponentially tilted Gaussian approximation of $P_{\boldsymbol{X}_{n}}$. Second, using the first result, the CDF of $\boldsymbol{X}_{n}$ is approximated by a function that is not necessarily a CDF. Nonetheless, this function is parametrized by a vector in $\mathbb{R}^{k}$ that can be arbitrarily chosen to locally minimize the approximation error. Additionally, an upper bound on the approximation error induced by both functions is provided. As a by product, an upper bound and a lower bound are provided for both the measure $P_{\boldsymbol{X}_{n}}$ and the CDF $F_{\boldsymbol{X}_{n}}$.

### 5.1 Approximation of the Measure

Given $\boldsymbol{\theta} \in \Theta_{\boldsymbol{Y}}$, with

$$
\begin{equation*}
\Theta_{\boldsymbol{Y}} \triangleq\left\{\boldsymbol{t} \in \mathbb{R}^{k}: K_{\boldsymbol{Y}}(\boldsymbol{t})<\infty\right\} \tag{17}
\end{equation*}
$$

let $\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})}, \ldots, \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}$ be independent random vectors such that each of them induces the probability measure $P_{\boldsymbol{Y}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$ that satisfies for all $\boldsymbol{y} \in \mathbb{R}^{k}$,

$$
\begin{equation*}
\frac{\mathrm{d} P_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}}{\mathrm{d} P_{\boldsymbol{Y}}}(\boldsymbol{y})=\exp \left(\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) . \tag{18}
\end{equation*}
$$

That is, the probability measure $P_{\boldsymbol{Y}^{(\theta)}}$ is an exponentially tilted measure with respect to $P_{\boldsymbol{Y}}$. Denote by $P_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta}} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}$ and $P_{\boldsymbol{Y}_{1} \boldsymbol{Y}_{2} \ldots \boldsymbol{Y}_{n}}$ the joint probability measures respectively induced by the independent random vectors $\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})}, \ldots, \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}$ and $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$ in (4) on the measurable space $\left(\mathbb{R}^{k \times n}, \mathscr{B}\left(\mathbb{R}^{k \times n}\right)\right)$. Then, for all $j \in\{1,2, \ldots, n\}$ and for all $\boldsymbol{y}_{j} \in \mathbb{R}^{k}$, it holds that

$$
\begin{align*}
&\left.\frac{\mathrm{d} P_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}^{\mathrm{d} P_{\boldsymbol{Y}_{1}} \boldsymbol{Y}_{2} \ldots \boldsymbol{Y}_{n}}\left(\boldsymbol{y}_{1}, \boldsymbol{y}_{2}, \boldsymbol{y}_{2} \ldots \boldsymbol{y}_{n}\right)}{}\right)=\prod_{j=1}^{n} \frac{\mathrm{~d} P_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}}{\mathrm{d} P_{\boldsymbol{Y}}}\left(\boldsymbol{y}_{j}\right)  \tag{19}\\
&=\exp \left(\sum_{j=1}^{n}\left(\boldsymbol{\theta}^{\top} \boldsymbol{y}_{j}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right) . \tag{20}
\end{align*}
$$

Using this notation, for all $\mathcal{A} \subseteq \mathbb{R}^{k}$ and for all $\boldsymbol{\theta} \in \Theta_{\boldsymbol{Y}}$, with $\Theta_{\boldsymbol{Y}}$ defined in 17 , it holds that

$$
\begin{align*}
& P_{\boldsymbol{X}_{n}}(\mathcal{A})=\mathbb{E}_{P_{\boldsymbol{X}_{n}}}\left[\mathbb{1}_{\left\{\boldsymbol{X}_{n} \in \mathcal{A}\right\}}\right]  \tag{21a}\\
& =\mathbb{E}_{P_{\boldsymbol{Y}_{1} Y_{2} \ldots \boldsymbol{Y}_{n}}}\left[\mathbb{1}_{\left\{\sum_{j=1}^{n} \boldsymbol{Y}_{j} \in \mathcal{A}\right\}}\right]  \tag{21b}\\
& =\mathbb{E}_{P_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta}}} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}\left[\mathbb{1}_{\left\{\sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}} \frac{\mathrm{d} P_{\boldsymbol{Y}_{1} \boldsymbol{Y}_{2} \ldots \boldsymbol{Y}_{n}}}{}{\mathrm{~d} P_{\left.\boldsymbol{Y}_{1}^{(\boldsymbol{\theta}}\right)} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}\left(\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})}, \ldots, \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}\right)\right]  \tag{21c}\\
& =\mathbb{E}_{\boldsymbol{P}_{1}^{(\boldsymbol{\theta})} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}\left[\mathbb{1}_{\left\{\sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\left(\frac{\mathrm{d} \boldsymbol{P}_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta}}} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}{\mathrm{d} P_{\boldsymbol{Y}_{1} \boldsymbol{Y}_{2} \ldots \boldsymbol{Y}_{n}}}\left(\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})}, \ldots, \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}\right)\right)^{-1}\right]  \tag{21d}\\
& =\mathbb{E}_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})} \mathbf{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}\left[\mathbb{1}_{\left\{\sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}} \exp \left(\sum_{j=1}^{n}\left(K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})}\right)\right)\right] \tag{21e}
\end{align*}
$$

RR n ${ }^{\circ} 9388$

$$
\begin{equation*}
=\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}} \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})} \ldots \boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}}\left[\mathbb{1}_{\left\{\sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})}\right)\right] . \tag{21f}
\end{equation*}
$$

To ease the notation, consider the random vector

$$
\begin{equation*}
\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}=\sum_{j=1}^{n} \boldsymbol{Y}_{j}^{(\boldsymbol{\theta})} \tag{22}
\end{equation*}
$$

which induces the probability measure $P_{\boldsymbol{S}_{n}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right.$ ). Hence, plugging 22 in 21 f yields,

$$
\begin{equation*}
P_{\boldsymbol{X}_{n}}(\mathcal{A})=\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right] . \tag{23}
\end{equation*}
$$

The equality in (23) is central as it expresses the probability $P_{\boldsymbol{X}_{n}}(\mathcal{A})$ in terms of another measure $P_{\boldsymbol{S}_{n}^{(\theta)}}$, which is the sum of $n$ independent random vectors. From this observation, it follows that an approximation on $P_{\boldsymbol{X}_{n}}(\mathcal{A})$ can be obtained by arbitrarily replacing $P_{\boldsymbol{S}_{n}^{(\theta)}}$ by its Gaussian approximation, i.e., a probability measure $P_{Z_{n}^{(\theta)}}$ induced on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$ by a Gaussian random vector $\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}$ with the same mean vector and covariance matrix as the random vector $\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}$ in 22 . Denote by $\boldsymbol{\mu}_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}} \in \mathbb{R}^{k}$ and $\underline{\boldsymbol{v}}_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta}}} \in \mathbb{R}^{k \times k}$ the mean vector and the covariance matrix of the random vector $\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}$, respectively. Hence,

$$
\begin{align*}
\boldsymbol{\mu}_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta}}} & \triangleq \mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}}\left[\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right]  \tag{24a}\\
& =n \mathbb{E}_{P_{\left.\boldsymbol{Y}^{( }\right)}}\left[\boldsymbol{Y}^{(\boldsymbol{\theta})}\right]  \tag{24b}\\
& =n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\boldsymbol{Y} \exp \left(\boldsymbol{\theta}^{\top} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]  \tag{24c}\\
& =n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta}), \tag{24d}
\end{align*}
$$

where $K_{\boldsymbol{Y}}^{(1)}$ is the gradient vector of the CGF $K_{\boldsymbol{Y}}$ defined in (2). Alternatively,

$$
\begin{align*}
\underline{\boldsymbol{v}}_{\boldsymbol{Z}_{n}^{(\theta)}} & \triangleq \mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}}\left[\left(\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\left(\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\top}\right]  \tag{25a}\\
& =n \mathbb{E}_{P_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}}\left[\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\top}\right]  \tag{25b}\\
& =n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\boldsymbol{\top}} \exp \left(\boldsymbol{\theta}^{\top} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]  \tag{25c}\\
& =n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta}), \tag{25~d}
\end{align*}
$$

where $K_{\boldsymbol{Y}}^{(2)}$ is the Hessian matrix of the CGF $K_{\boldsymbol{Y}}$ defined in (3). The equality (25b) is a consequence of the random vector $S_{n}^{(\boldsymbol{\theta})}$ in (22) being a sum of independent random vectors. Hence, the central idea for providing an approximation to $P_{\boldsymbol{X}_{n}}$ is to approximate the RHS of (23) by the function $\eta_{\boldsymbol{Y}}: \mathbb{R}^{k} \times \mathscr{B}\left(\mathbb{R}^{k}\right) \times \mathbb{N} \rightarrow \mathbb{R}$, which is such that

$$
\begin{equation*}
\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n) \triangleq \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\boldsymbol{z}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right] \tag{26}
\end{equation*}
$$

Note that $\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$ can also be expressed as follows:
$\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$

$$
\begin{align*}
&= \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{z}\right) \mathbb{1}_{\{\boldsymbol{z} \in \mathcal{A}\}} \mathrm{d} P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{z})  \tag{27a}\\
&= \int_{\mathcal{A}} \frac{\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{z}\right)}{\sqrt{(2 \pi)^{k} \operatorname{det}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)}} \exp \left(-\frac{\left(\boldsymbol{z}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\top}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1}\left(\boldsymbol{z}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)}{2}\right) \mathrm{d} \nu_{k}(\boldsymbol{z}) \\
&= \frac{\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)}{\sqrt{(2 \pi)^{k} \operatorname{det}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)}}  \tag{27b}\\
& \int_{\mathcal{A}}^{\exp }\left(-\frac{\left(\boldsymbol{z}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\top}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1}\left(\boldsymbol{z}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)}{2}\right) \exp \left(-\frac{2 \boldsymbol{\theta}^{\top}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1} z}{2}\right) \mathrm{d} \nu_{k}(\boldsymbol{z})  \tag{27c}\\
&= \frac{\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})+\frac{\boldsymbol{\theta}^{\top} n K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2}-\boldsymbol{\theta}^{\boldsymbol{\top}} n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)}{\sqrt{(2 \pi)^{k} \operatorname{det}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)}} \\
& \int_{\mathcal{A}}\left(-\frac{\left(\boldsymbol{z}-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})+\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{\top} \boldsymbol{\theta}\right)^{\top}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1}\left(z-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})+\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{\top} \boldsymbol{\theta}\right)}{2}\right) \mathrm{d} \nu_{k}(\boldsymbol{z})
\end{align*}
$$

$$
\begin{equation*}
=\exp \left(n\left(K_{\boldsymbol{Y}}(\boldsymbol{\theta}) \frac{\boldsymbol{\theta}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2}-\boldsymbol{\theta}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\right) \frac{1}{\sqrt{(2 \pi)^{k} \operatorname{det}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)}} \tag{27d}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\mathcal{A}} \exp \left(-\frac{\left(z-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})+\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{\boldsymbol{\top}} \boldsymbol{\theta}\right)^{\top}\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1}\left(z-n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})+\left(n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{\top} \boldsymbol{\theta}\right)}{2}\right) \mathrm{d} \nu_{k}(\boldsymbol{z}) \tag{27e}
\end{equation*}
$$

$=\exp \left(n\left(K_{\boldsymbol{Y}}(\boldsymbol{\theta})+\frac{\boldsymbol{\theta}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2}-\boldsymbol{\theta}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\right) P_{\boldsymbol{H}_{n}^{(\boldsymbol{\theta})}}(\mathcal{A})$
where the probability measure $P_{\boldsymbol{H}_{n}^{(\theta)}}$ is the probability measure induced by a Gaussian random vector $\boldsymbol{H}_{n}^{(\boldsymbol{\theta})}$ with mean vector $n\left(K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})-K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}\right)$ and covariance matrix $n K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$.

Hence, the approximation of the probability $P_{\boldsymbol{X}_{n}}(\mathcal{A})$ by $\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$, follows from the arbitrary assumption that the probability measure $P_{S^{(\theta)}}$ can be approximated by the probability measure $P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}$. Given that $P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}$ is an exponentially tilted measure with respect to $P_{\boldsymbol{X}_{n}}$ and $P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}$ is the Gaussian approximation of $P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}$, the function $\eta_{\boldsymbol{Y}}$ in 26 is referred to as the exponentially tilted Gaussian approximation of $P_{\boldsymbol{X}_{n}}$.

The rest of this section follows by upper bounding the error induced by replacing $P_{\boldsymbol{S}_{n}^{(\theta)}}$ by its Gaussian approximation $P_{\boldsymbol{Z}_{n}^{(\theta)}}$ in 23 . That is, establishing an upper-bound on

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right|, \tag{28}
\end{equation*}
$$

which is the purpose of the following lemma.
Lemma 2 Given $\boldsymbol{\theta}=\left(\theta_{1}, \theta_{2}, \ldots, \theta_{k}\right) \in \Theta_{\boldsymbol{Y}}$, with $\Theta_{\boldsymbol{Y}}$ in 17 , and a convex set $\mathcal{A} \in \mathscr{B}\left(\mathbb{R}^{k}\right)$, it holds that

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\boldsymbol{T}} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}, P_{\boldsymbol{Z}_{n}^{(\theta)}}\right) \tag{29}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta\left(P_{\boldsymbol{S}_{n}^{(\theta)}}, P_{\boldsymbol{Z}_{n}^{(\theta)}}\right) \triangleq \sup _{\mathcal{B} \in \mathcal{C}_{k}}\left|P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\mathcal{B})-P_{\boldsymbol{Z}_{n}^{(\theta)}}(\mathcal{B})\right| \tag{30}
\end{equation*}
$$

the set $\mathcal{C}_{k}$ contains all convex sets in $\mathscr{B}\left(\mathbb{R}^{k}\right)$; and the vector $\boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})=\left(a_{1}(\mathcal{A}, \boldsymbol{\theta}), a_{2}(\mathcal{A}, \boldsymbol{\theta}), \ldots\right.$, $\left.a_{k}(\mathcal{A}, \boldsymbol{\theta})\right)$ is such that for all $i \in\{1,2, \ldots, k\}$,

$$
a_{i}(\mathcal{A}, \boldsymbol{\theta}) \triangleq\left\{\begin{array}{cl}
0 & \text { if } \theta_{i}=0  \tag{31}\\
\inf _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} & \text { if } \theta_{i}>0 \\
\sup _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} & \text { if } \theta_{i}<0
\end{array}\right.
$$

Proof: The proof of Lemma 2 is presented in Appendix A.
Note that the term $\Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta}}}\right)$ in (30) can be upper bounded by using Theorem 1 . For doing so, consider the function $\xi_{\boldsymbol{Y}}: \mathbb{R}^{k} \rightarrow \mathbb{R}$, such that for all $t \in \mathbb{R}^{k}$,

$$
\begin{equation*}
\xi_{\boldsymbol{Y}}(\boldsymbol{t}) \triangleq \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\left(\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})\right)^{\boldsymbol{\top}}\left(K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{t})\right)^{-1}\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})\right)\right)^{3 / 2} \exp \left(\boldsymbol{t}^{\top} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{t})\right)\right] . \tag{32}
\end{equation*}
$$

Using this notation, the following theorem introduces an upper bound on the error induced by the approximation of the probability $P_{\boldsymbol{X}_{n}}(\mathcal{A})$ by $\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$, where $\mathcal{A} \subseteq \mathbb{R}^{k}$ is a convex Borel measurable set and $\boldsymbol{\theta} \in \mathbb{R}^{k}$ is a fixed parameter.

Theorem 3 For all $\mathcal{A} \in \mathcal{C}_{k}$, with $\mathcal{C}_{k}$ the set of all convex sets in $\mathscr{B}\left(\mathbb{R}^{k}\right)$, and for all $\boldsymbol{\theta}=$ $\left(\theta_{1}, \theta_{2}, \ldots, \theta_{k}\right) \in \Theta_{\boldsymbol{Y}}$, with $\Theta_{\boldsymbol{Y}}$ in (17), it holds that

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{T} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right) \tag{33}
\end{equation*}
$$

where the functions c and $\eta_{\boldsymbol{Y}}$ are respectively defined in (14) and (27f); the vector $\boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta}) \triangleq$ $\left(a_{1}(\mathcal{A}, \boldsymbol{\theta}), a_{2}(\mathcal{A}, \boldsymbol{\theta}), \ldots, a_{k}(\mathcal{A}, \boldsymbol{\theta})\right)$ is defined in (31); and the function $\xi_{\boldsymbol{Y}}$ is defined in 32).

Proof: The proof of Theorem 3 is presented in Appendix $B$

### 5.2 Approximation of the CDF

The CDF $F_{\boldsymbol{X}_{n}}$ can be written in the form of the probability of a convex set in $\mathscr{B}\left(\mathbb{R}^{k}\right)$. That is, for all $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{k}\right) \in \mathbb{R}^{k}$, let the set $\mathcal{A}_{\boldsymbol{x}}$ be such that

$$
\begin{equation*}
\mathcal{A}_{\boldsymbol{x}}=\left\{\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \mathbb{R}^{k}: \forall i \in\{1,2, \ldots, k\}, t_{i} \leq x_{i}\right\} . \tag{34}
\end{equation*}
$$

Then, for all $\boldsymbol{x} \in \mathbb{R}^{k}$, it holds that

$$
\begin{equation*}
F_{\boldsymbol{X}_{n}}(\boldsymbol{x})=P_{\boldsymbol{X}_{n}}\left(\mathcal{A}_{\boldsymbol{x}}\right) . \tag{35}
\end{equation*}
$$

This observation allows to use Theorem 3 to approximate the CDF $F_{\boldsymbol{X}_{n}}$ of the random vector $\boldsymbol{X}_{n}$ in (4). Explicitly, for all $\boldsymbol{x} \in \mathbb{R}^{k}$ and for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\boldsymbol{Y}}$, with $\boldsymbol{\Theta}_{\boldsymbol{Y}}$ in (17), it holds that

$$
\begin{equation*}
\left|F_{\boldsymbol{X}_{n}}(\boldsymbol{x})-\eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}, \mathcal{A}_{\boldsymbol{x}}, n\right)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}\left(\mathcal{A}_{\boldsymbol{x}}, \boldsymbol{\theta}\right)\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right) . \tag{36}
\end{equation*}
$$

The approximation of the CDF $F_{\boldsymbol{X}_{n}}$ in (36) can be enhanced by choosing the parameter $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\boldsymbol{Y}}$ that minimizes the right-hand side (RHS) of (36). From this standpoint, the parameter $\boldsymbol{\theta}$ must be searched within a subset of $\Theta_{Y}$ in which

$$
\begin{equation*}
\boldsymbol{\theta}^{\top} \boldsymbol{a}\left(\mathcal{A}_{\boldsymbol{x}}, \boldsymbol{\theta}\right)=\boldsymbol{\theta}^{\top} \boldsymbol{x}<+\infty \tag{37}
\end{equation*}
$$

More specifically, given $\boldsymbol{A}_{\boldsymbol{x}}$ in (34), it follows from (31) that the minimization must be restricted to the set

$$
\begin{equation*}
\boldsymbol{\Theta}_{\boldsymbol{Y}}^{-} \triangleq\left\{\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}, \forall i \in\{1,2, \ldots, k\}, t_{i} \leq 0\right\} \tag{38}
\end{equation*}
$$

An arbitrary choice of $\boldsymbol{\theta}$ is the one that minimizes the exponential term $\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}\left(\mathcal{A}_{\boldsymbol{x}}, \boldsymbol{\theta}\right)\right)$, which depends on $\boldsymbol{x}$. Denote such a choice by $\boldsymbol{\theta}(\boldsymbol{x})$, which is defined in terms of the following quantity:

$$
\begin{equation*}
\boldsymbol{\tau}(\boldsymbol{x}) \triangleq \underset{\boldsymbol{t} \in \operatorname{clo} \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}}{\arg \min }\left(n K_{\boldsymbol{Y}}(\boldsymbol{t})-\boldsymbol{t}^{\boldsymbol{\top}} \boldsymbol{x}\right) \tag{39}
\end{equation*}
$$

where $\operatorname{clo} \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$is the closure of $\boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$. The uniqueness of $\boldsymbol{\tau}(\boldsymbol{x})$ in (39), for a given $\boldsymbol{x}$, follows from the fact that the set $\boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$in $(\sqrt[38)]{ }$ is convex and the function $n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}\left(\mathcal{A}_{\boldsymbol{x}}, \boldsymbol{\theta}\right)$ is strictly convex with respect to $\boldsymbol{\theta}$. More specifically, the difference between a strictly convex function, i.e., $n K_{\boldsymbol{Y}}(\boldsymbol{\theta})$ and a linear function, i.e., $\boldsymbol{\theta}^{\top} \boldsymbol{a}\left(\mathcal{A}_{\boldsymbol{x}}, \boldsymbol{\theta}\right)$ is strictly convex. The former is strictly convex due to the fact that the covariance matrix $K_{\boldsymbol{Y}}^{(2)}(\mathbf{0})$ is a positive definite matrix, c.f., 6 , Section 1.2] and [18, Theorem 7.1].

Given $\boldsymbol{\tau}(\overline{\boldsymbol{x}})$ in (39), the choice of the vector $\boldsymbol{\theta}$ to reduce the RHS of (36) is

$$
\boldsymbol{\theta}(\boldsymbol{x})=\left\{\begin{array}{cl}
\boldsymbol{\tau}(\boldsymbol{x}) & \text { if } \boldsymbol{\tau}(\boldsymbol{x}) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}  \tag{40}\\
\boldsymbol{\tau}(\boldsymbol{x})+\boldsymbol{\epsilon} & \text { otherwise }
\end{array}\right.
$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^{k}$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\|<r$, with $r>0$ arbitrary small; and second, $\boldsymbol{\theta}(\boldsymbol{x}) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$.

The following lemma, presents some of the properties of $\boldsymbol{\theta}(\boldsymbol{x})$.
Lemma 4 For all $\boldsymbol{x} \in \mathbb{R}^{k}, \boldsymbol{\theta}(\boldsymbol{x})$ in 40 satisfies

$$
\begin{equation*}
\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)^{\boldsymbol{T}} \boldsymbol{\theta}(\boldsymbol{x}) \geq 0 \tag{41}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{\theta}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)=\mathbf{0} \tag{42}
\end{equation*}
$$

where,

$$
\begin{equation*}
\boldsymbol{\mu}_{\boldsymbol{X}_{n}}=\left(\mu_{\boldsymbol{X}_{n, 1}}, \mu_{\boldsymbol{X}_{n, 2}}, \ldots, \mu_{\boldsymbol{X}_{n, k}}\right)^{\boldsymbol{T}} \tag{43}
\end{equation*}
$$

is the mean of the random vector $\boldsymbol{X}_{n}$ in (4).
Proof: The proof of Lemma 4 is presented in Appendix C.
Let the set $\mathcal{E}_{\boldsymbol{X}_{n}}$ be defined by

$$
\begin{equation*}
\mathcal{E}_{\boldsymbol{X}_{n}} \triangleq\left\{\left(x_{1}, x_{2}, \ldots, x_{k}\right) \in \mathbb{R}^{k}: \forall i \in\{1,2, \ldots, k\}, x_{i}>\mu_{X_{n, i}}\right\}, \tag{44}
\end{equation*}
$$

where for all $i \in\{1,2, \ldots, k\}, \mu_{X_{n, i}}$ is defined in (43). From (44), it holds that for all $\boldsymbol{x} \in \mathcal{E}_{\boldsymbol{X}_{n}}$, the vector $\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ is such that all components are strictly positive. Similarly, from (38), it follows that $\boldsymbol{\theta}(\boldsymbol{x})$ is a vector whose components are all nonpositive. Hence, from (41), it follows that,

$$
\begin{equation*}
\boldsymbol{\theta}(\boldsymbol{x})=\mathbf{0} \tag{45}
\end{equation*}
$$

which leads to the Gaussian approximation of the $\operatorname{CDF} F_{\boldsymbol{X}_{n}}$ at the point $\boldsymbol{x}$. Hence, for all $\boldsymbol{x} \in \mathcal{E}_{\boldsymbol{X}_{n}}$ the choice of $\boldsymbol{\theta}$ in (40) can still be improved. In this case, the objective is to focus on $1-F_{\boldsymbol{X}_{n}}(\boldsymbol{x})$ and write it as a sum of probability measures of convex sets with respect to $P_{\boldsymbol{X}_{n}}$. The following lemma provides such a result.

Lemma 5 For all $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{k}\right)^{T} \in \mathbb{R}^{k}$, with $k \in \mathbb{N}$, it holds that

$$
\begin{equation*}
1-F_{\boldsymbol{X}_{n}}(\boldsymbol{x})=\sum_{i=1}^{k} P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i)), \tag{46}
\end{equation*}
$$

where the set $\mathcal{B}(\boldsymbol{x}, i)$, with $i \in\{1,2, \ldots, k\}$, is

$$
\begin{equation*}
\mathcal{B}(\boldsymbol{x}, i)=\left\{\boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \mathbb{R}^{k}: \forall j \in\{1,2, \ldots, k\}, t_{j} \leqslant x_{j} \text { if } j<i, \text { and } t_{j}>x_{j} \text { if } j=i\right\} . \tag{47}
\end{equation*}
$$

Proof: The proof of Lemma 5 is presented in Appendix D.
Note that the choice of the sets $\mathcal{B}(\boldsymbol{x}, 1), \mathcal{B}(\boldsymbol{x}, 2), \ldots, \mathcal{B}(\boldsymbol{x}, k)$ in 46 is not unique, c.f., the inclusion-exclusion principle. There exist $k$ ! possible ways of choosing these sets. As shown later, each choice induces a different approximation on $F_{\boldsymbol{X}_{n}}$ and different approximation errors. For all $i \in\{1,2, \ldots, k\}$, the probability $P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))$ in (46) can be approximated by using Theorem 3. More specifically, for all $i \in\{1,2, \ldots, k\}$ and for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\boldsymbol{Y}}$,

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{B}(\boldsymbol{x}, i), n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{a}(\mathcal{B}(\boldsymbol{x}, i), \boldsymbol{\theta})\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right) . \tag{48}
\end{equation*}
$$

Similar to the previous discussion on the minimization of the RHS of (36), the minimization of the RHS of (48) is focused only on the term $\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{B}(\boldsymbol{x}, i), \boldsymbol{\theta})\right)$, and thus, the choice of $\boldsymbol{\theta}$ must be constrained to a subset of $\Theta_{\boldsymbol{Y}}$ in which $\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{B}(\boldsymbol{x}, i), \boldsymbol{\theta})$ is finite. That is, for all $\boldsymbol{x} \in \mathbb{R}^{k}$, the choice of $\boldsymbol{\theta}$ must satisfy that

$$
\begin{equation*}
\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{B}(\boldsymbol{x}, i), \boldsymbol{\theta})=\boldsymbol{\theta}^{\top} \boldsymbol{x}<+\infty \tag{49}
\end{equation*}
$$

More specifically, given a set $\mathcal{B}(\boldsymbol{x}, i)$, it follows from (31), that the the choice of $\boldsymbol{\theta}$ must be restricted to the set

$$
\begin{align*}
\boldsymbol{\Theta}_{\boldsymbol{Y}}^{i} \triangleq & \left\{\left(\theta_{1}, \theta_{2}, \ldots, \theta_{k}\right) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}: \forall j \in\{1,2, \ldots, k\}, \theta_{j} \leqslant 0 \text { if } j<i, \theta_{j} \geqslant 0 \text { if } j=i,\right. \\
& \text { and } \left.\theta_{j}=0 \text { otherwise }\right\} . \tag{50}
\end{align*}
$$

Denote such a choice by $\boldsymbol{\theta}_{i}(\boldsymbol{x})$, which is defined in terms of the following quantity:

$$
\begin{equation*}
\boldsymbol{\tau}_{i}(\boldsymbol{x})=\underset{\boldsymbol{t} \in \operatorname{clo} \boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}}{\arg \min }\left(n K_{\boldsymbol{Y}}(\boldsymbol{t})-\boldsymbol{t}^{\boldsymbol{\top}} \boldsymbol{x}\right) \tag{51}
\end{equation*}
$$

with $\operatorname{clo} \boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}$ the closure of $\boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}$. The uniqueness of $\boldsymbol{\tau}_{i}(\boldsymbol{x})$ in (51), for a given $\boldsymbol{x}$, follows from the fact that the set $\boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}$ in 50 is convex and the function $n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{B}(\boldsymbol{x}, i), \boldsymbol{\theta})$ is strictly convex with respect to $\boldsymbol{\theta}$, as discussed above.
Given $\boldsymbol{\tau}_{i}(\boldsymbol{x})$ in (51), the choice of the vector $\boldsymbol{\theta}$ to reduce the RHS of (48) is

$$
\boldsymbol{\theta}_{i}(\boldsymbol{x})=\left\{\begin{array}{cl}
\boldsymbol{\tau}_{i}(\boldsymbol{x}) & \text { if } \boldsymbol{\tau}_{i}(\boldsymbol{x}) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}  \tag{52}\\
\boldsymbol{\tau}_{i}(\boldsymbol{x})+\boldsymbol{\epsilon} & \text { otherwise }
\end{array}\right.
$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^{k}$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\|<r$, with $r>0$ arbitrary small; and second, $\boldsymbol{\theta}_{i}(\boldsymbol{x}) \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{i}$.
Finally, for all $i \in\{1,2, \ldots, k\}$, it holds from (48) that the probability $P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))$ can be approximated by $\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{B}(\boldsymbol{x}, i), n)$. Using these approximations in 46), the approximation error can be upper bounded as follows:

$$
\begin{align*}
& \left|1-F_{\boldsymbol{X}_{n}}(\boldsymbol{x})-\sum_{i=1}^{k} \eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{\mathcal{J}}(\boldsymbol{x}), \mathcal{B}(\boldsymbol{x}, i), n\right)\right| \\
& =\left|\sum_{i=1}^{k} P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))-\sum_{i=1}^{k} \eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{\mathcal{J}}(\boldsymbol{x}), \mathcal{B}(\boldsymbol{x}, i), n\right)\right|  \tag{53}\\
& \leq \sum_{i=1}^{k}\left|P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))-\eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x}), \mathcal{B}(\boldsymbol{x}, i), n\right)\right|  \tag{54}\\
& \leq \sum_{i=1}^{k} \exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x})\right)-\boldsymbol{\theta}_{i}^{\top}(\boldsymbol{x}) \boldsymbol{x}\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x})\right)}{\sqrt{n}}\right), \tag{55}
\end{align*}
$$

where the inequality in (54) follows from the triangular inequality; and the inequality in 55 follows from 48.

In order to ease the notation, let the functions $\zeta_{\boldsymbol{Y}}: \mathbb{N} \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ and $\delta_{\boldsymbol{Y}}: \mathbb{N} \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ be such that for all $(n, \boldsymbol{x}) \in \mathbb{N} \times \mathbb{R}^{k}$,

$$
\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x}) \triangleq\left\{\begin{array}{cc}
\eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}(\boldsymbol{x}), \mathcal{A}_{\boldsymbol{x}}, n\right) & \text { if } \boldsymbol{x} \notin \mathcal{E}_{\boldsymbol{X}_{n}}  \tag{56}\\
1-\sum_{i=1}^{k} \eta_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x}), \mathcal{B}(\boldsymbol{x}, i), n\right) & \text { if } \boldsymbol{x} \in \mathcal{E}_{\boldsymbol{X}_{n}}
\end{array}\right.
$$

RR n ${ }^{\circ} 9388$

$$
\delta_{\boldsymbol{Y}}(n, \boldsymbol{x}) \triangleq\left\{\begin{array}{cl}
\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta}(\boldsymbol{x}))-\boldsymbol{\theta}(\boldsymbol{x})^{\boldsymbol{\top}} \boldsymbol{x}\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta}(\boldsymbol{x}))}{\sqrt{n}}\right) & \text { if } \boldsymbol{x} \notin \mathcal{E}_{\boldsymbol{X}_{n}}  \tag{57}\\
\sum_{i=1}^{k} \exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x})\right)-\boldsymbol{\theta}_{i}^{\boldsymbol{\top}}(\boldsymbol{x}) \boldsymbol{x}\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}\left(\boldsymbol{\theta}_{i}(\boldsymbol{x})\right)}{\sqrt{n}}\right) & \text { if } \boldsymbol{x} \in \mathcal{E}_{\boldsymbol{X}_{n}} .
\end{array}\right.
$$

Using this notation, the following theorem summarizes the discussion above.
Theorem 6 For all $\boldsymbol{x} \in \mathbb{R}^{k}$, it holds that

$$
\begin{equation*}
\left|F_{\boldsymbol{X}_{n}}(\boldsymbol{x})-\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x})\right| \leqslant \delta_{\boldsymbol{Y}}(n, \boldsymbol{x}), \tag{58}
\end{equation*}
$$

where the functions $\zeta_{\boldsymbol{Y}}$ and $\delta_{\boldsymbol{Y}}$ are respectively defined in (56) and (57).
An immediate result from Theorem 6 is the following upper and lower bounds on $F_{\boldsymbol{X}_{n}}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{R}^{k}$,

$$
\begin{equation*}
\underline{\Omega}(n, \boldsymbol{x}) \leqslant F_{\boldsymbol{X}_{n}}(\boldsymbol{x}) \leqslant \bar{\Omega}(n, \boldsymbol{x}), \tag{59}
\end{equation*}
$$

where,

$$
\begin{align*}
& \bar{\Omega}(n, \boldsymbol{x}) \triangleq \zeta_{\boldsymbol{Y}}(n, \boldsymbol{x})+\delta_{\boldsymbol{Y}}(n, \boldsymbol{x}), \text { and }  \tag{60}\\
& \underline{\Omega}(n, \boldsymbol{x}) \triangleq \zeta_{\boldsymbol{Y}}(n, \boldsymbol{x})-\delta_{\boldsymbol{Y}}(n, \boldsymbol{x}) . \tag{61}
\end{align*}
$$

### 5.3 Connexion with the Saddlepoint Approximation

The following claim underlines the relation between the saddlepoint approximation $\hat{F}_{\boldsymbol{X}_{n}}$ in (10) and the function $\zeta_{\boldsymbol{Y}}$ in (56).

Claim 7 For all $\boldsymbol{x} \in \mathcal{D}$, with $\mathcal{D}$ in (7), it holds that

$$
\begin{equation*}
\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x})=\hat{F}_{\boldsymbol{X}_{n}}(\boldsymbol{x}), \tag{62}
\end{equation*}
$$

where the function $\hat{F}_{\boldsymbol{X}_{n}}$ and $\zeta_{\boldsymbol{Y}}$ are respectively defined in (10) and (56).
Proof: From (7), note that for all $\boldsymbol{x} \in \mathcal{D}$ the solution in $\boldsymbol{t}$ to

$$
\begin{equation*}
n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})=\boldsymbol{x} \tag{63}
\end{equation*}
$$

denoted by $\boldsymbol{\tau}_{0}$ exists and the components of $\boldsymbol{\tau}_{0}$ are all strictly negative. Thus, $\boldsymbol{\tau}_{0} \in \Theta_{\boldsymbol{Y}}^{-}$, with $\Theta_{Y}^{-}$in (38).
Note that the vector $\boldsymbol{\tau}_{0}$ is also a solution to 33 for all $\boldsymbol{x} \in \mathcal{D}$. This follows from the fact that the CGF $K_{\boldsymbol{Y}}$ is strictly convex and $K_{\boldsymbol{Y}}^{(1)}$ is the gradient vector of the CGF $K_{\boldsymbol{Y}}$. Thus, the vector $\boldsymbol{\theta}(\boldsymbol{x})$ in 40 satisfies

$$
\begin{equation*}
\boldsymbol{\theta}(\boldsymbol{x})=\boldsymbol{\tau}_{0} \tag{64}
\end{equation*}
$$

Then, for all $\boldsymbol{x} \in \mathcal{D}$, from (64), all the components of $\boldsymbol{\theta}(\boldsymbol{x})$ are strictly negative and thus, $\boldsymbol{x} \notin \mathcal{E}_{\boldsymbol{X}_{n}}$. Then, plugging (64) in (56) yields

$$
\begin{align*}
\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x}) & =\eta_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}, \mathcal{A}_{\boldsymbol{x}}, n\right),  \tag{65}\\
& =\exp \left(n\left(K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(1)}\left(\boldsymbol{\tau}_{0}\right)+\frac{\boldsymbol{\tau}_{0}^{\top} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right)\right) P_{\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}}\left(\mathcal{A}_{\boldsymbol{x}}\right)  \tag{66}\\
& =\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-n \boldsymbol{\tau}_{0}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(1)}\left(\boldsymbol{\tau}_{0}\right)+\frac{n \boldsymbol{\tau}_{0}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right) P_{\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}}\left(\mathcal{A}_{\boldsymbol{x}}\right)  \tag{67}\\
& =\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\top} \boldsymbol{x}+\frac{n \boldsymbol{\tau}_{0}^{\boldsymbol{\top}} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right) P_{\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}}\left(\mathcal{A}_{\boldsymbol{x}}\right) \tag{68}
\end{align*}
$$

$$
\begin{equation*}
=\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\boldsymbol{\top}} \boldsymbol{x}+\frac{n \boldsymbol{\tau}_{0}^{\top} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right) F_{\boldsymbol{H}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}}(\boldsymbol{x}), \tag{69}
\end{equation*}
$$

where $P_{\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}}$ is the probability measure induced by a Gaussian random vector $\boldsymbol{H}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}$ with mean vector $n\left(K_{\boldsymbol{Y}}^{(1)}\left(\boldsymbol{\tau}_{0}\right)-K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right)$ and covariance matrix $n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right)$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$; and $F_{\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}}$ is the CDF of the random vector $\boldsymbol{H}_{n}^{\left(\tau_{0}\right)}$. The equality in (66) follows from (27f). The equality in (68) follows from $\boldsymbol{\tau}_{0}$ being the solution in $\boldsymbol{t}$ to $n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})=\boldsymbol{x}$.
Let the random vector $\boldsymbol{Z}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}$ be such that

$$
\begin{equation*}
\boldsymbol{Z}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}=\boldsymbol{H}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}-n\left(K_{\boldsymbol{Y}}^{(1)}\left(\boldsymbol{\tau}_{0}\right)-K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right) \tag{70}
\end{equation*}
$$

which induces the probability measure $P_{Z_{2}^{\left(\tau_{0}\right)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$ and the corresponding CDF is denoted by $F_{\boldsymbol{Z}_{n}^{\left(\tau_{0}\right)}}$. Then, the mean vector and the covariance matrix of $\boldsymbol{Z}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}$ are respectively $(0,0, \ldots, 0)$ and $n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right)$. Thus, for all $\boldsymbol{x} \in \mathbb{R}^{k}$, it holds that

$$
\begin{align*}
F_{\boldsymbol{H}_{n}^{\left(\boldsymbol{\tau}_{0}\right)}}(\boldsymbol{x}) & =F_{\boldsymbol{Z}_{n}^{\left(\tau_{0}\right)}}\left(\boldsymbol{x}-n\left(K_{\boldsymbol{Y}}^{(1)}\left(\boldsymbol{\tau}_{0}\right)-K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right)\right)  \tag{71}\\
& =F_{\boldsymbol{Z}_{n}^{\left(\tau_{0}\right)}}\left(n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right) \tag{72}
\end{align*}
$$

where the equality in $(72)$ follows from $\boldsymbol{\tau}_{0}$ being the solution in $\boldsymbol{t}$ to $n K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{t})=\boldsymbol{x}$.
Plugging (72) in 69) yields

$$
\begin{align*}
\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x}) & =\exp \left(n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}_{0}\right)-\boldsymbol{\tau}_{0}^{\boldsymbol{\top}} \boldsymbol{x}+\frac{n \boldsymbol{\tau}_{0}^{\top} K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}}{2}\right) F_{\boldsymbol{Z}_{n}^{\left(\tau_{0}\right)}}\left(n K_{\boldsymbol{Y}}^{(2)}\left(\boldsymbol{\tau}_{0}\right) \boldsymbol{\tau}_{0}\right)  \tag{73}\\
& =\hat{F}_{\boldsymbol{X}_{n}}(\boldsymbol{x}) \tag{74}
\end{align*}
$$

where the equality in 74 follows from 10 . This concludes the proof.

## 6 Examples

Consider the case in which the independent random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$ in (4), with $n$ fixed, are such that for all $i \in\{1,2, \ldots, n\}$,

$$
\boldsymbol{Y}_{i} \triangleq\left(\begin{array}{cc}
1 & 0  \tag{75}\\
\rho & \sqrt{1-\rho^{2}}
\end{array}\right)\binom{B_{1}}{B_{2}}
$$

where $\rho \in[0,1)$ is the Pearson correlation coefficient between the components of $\boldsymbol{Y}_{i}$; and both $B_{1}$ and $B_{2}$ are independent Bernoulli random variables with parameter $p=0.25$. The mean of $\boldsymbol{X}_{n}$ in (4) is $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}=n p\left(1, \rho+\sqrt{1-\rho^{2}}\right)^{\top}$.
Given a vector $\boldsymbol{x} \in \mathbb{R}^{2}$, Figure 1 depicts the set $\mathcal{A}_{\boldsymbol{x}}$ in (34) (blue rectangle); the set $\mathcal{B}(\boldsymbol{x}, 1)$ (grey rectangle), and the set $\mathcal{B}(\boldsymbol{x}, 2)$ (yellow rectangle) in 47). Three cases are distinguished with respect to the given vector $\boldsymbol{x} \in \mathbb{R}^{2}$ and the mean vector $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ in (43). In Sub-figure 1 a the mean vector $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ belongs to the set $\mathcal{B}(\boldsymbol{x}, 1)$. In Sub-figure 1 b , the mean vector $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ belongs to the set $\mathcal{B}(\boldsymbol{x}, 2)$. In these two cases, the approximation of the CDF $F_{\boldsymbol{X}_{n}}$ is done using the set $\mathcal{A}_{\boldsymbol{x}}$, i.e., using the equality in (35). In Sub-figure 1c the mean vector $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$ belongs to the set $\mathcal{A}_{\boldsymbol{x}}$. In this case, the approximation of the CDF $F_{\boldsymbol{X}_{n}}$ is done using the sets $\mathcal{B}(\boldsymbol{x}, 1)$ and $\mathcal{B}(\boldsymbol{x}, 2)$. That is, using the equality in 46).

Figures $2 \sqrt{4}$ depict the $\mathrm{CDF} F_{\boldsymbol{X}_{n}}$ of $\boldsymbol{X}_{n}$ in (4); the Gaussian approximation $F_{\boldsymbol{Z}_{n}}$ in $\sqrt{16}$; the saddlepoint approximation $\zeta_{\boldsymbol{Y}}$ in (56); and the saddlepoint upper and lower bounds $\bar{\Omega}$ in (60) and $\underline{\Omega}$ in (61); through the line $a \boldsymbol{d}+\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$. The plots on the left and the center in Figures $2 \| 4$ are respectively for fixed vectors $\boldsymbol{d}=(1,1)^{\top}$ and $\boldsymbol{d}=(1,-1)^{\mathrm{T}}$, as a function of $a$. The pots on the right in Figures $2 \cdot 4$ are in function of $\rho$ for a fixed point in the line $a \boldsymbol{d}+\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$, with $a \in\{-6,-12,-24\}$ and $\boldsymbol{d}=(1,1)^{\top}$, i.e., the tail of the distribution in the direction of the vector $\boldsymbol{d}=(1,1)^{\mathrm{T}}$. Note that Gaussian and saddlepoint approximations are particularly precise near to the mean $\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$. That is, when $a=0$. Nonetheless, away from the mean, i.e., $a<-4$ when $n=25$, or $a<-10$ when $n \in\{50,100\}$, the Gaussian approximation induces a large approximation error. Note that this is in sharp contrast with the saddlepoint approximation.

For the value of $n=50$, Figure 3 , the lower bound $\underline{\Omega}$ is negative, except when $a>5$. Alternatively, the Gaussian upper and lower bounds $\bar{\Sigma}$ in 16 a and $\underline{\Sigma}$ in 16 b are trivial. That is, the lower bound is negative and the upper bound is bigger than one, which highlights the lack of formal mathematical arguments to evaluate the Gaussian approximation. For instance, note that when $a<-10$, the Gaussian approximation is bigger than the upper bound due to the saddle point approximation. In particular, note that Figure 3 (Right) highlights the fact that the same observation holds for all values of $\rho$.


Figure 1: Example of the set $\mathcal{A}_{\boldsymbol{x}}$ in (34) (blue rectangle); the set $\mathcal{B}(\boldsymbol{x}, 1)$ (grey rectangle), and the set $\mathcal{B}(\boldsymbol{x}, 2)$ (yellow rectangle) in (47).




Figure 2: Sum of the independent random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$, with $n=25$, such that for all $i \in\{1,2, \ldots, n\}, \boldsymbol{Y}_{i}$ satisfies 75 .


Figure 3: Sum of the independent random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$, with $n=50$, such that for all $i \in\{1,2, \ldots, n\}, \boldsymbol{Y}_{i}$ satisfies 75 .


Figure 4: Sum of the independent random vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$, with $n=100$, such that for all $i \in\{1,2, \ldots, n\}, \boldsymbol{Y}_{i}$ satisfies 75 .

## 7 Final Remarks and Discussion

A final remark is the fact that for all $\boldsymbol{x} \in \mathcal{D}$, the saddlepoint approximation $\hat{F}_{\boldsymbol{X}_{n}}(\boldsymbol{x})$ in (10) is identical to $\zeta_{\boldsymbol{Y}}(n, \boldsymbol{x})$ in (56). That is, the saddlepoint approximation $\hat{F}_{\boldsymbol{X}_{n}}$ can be obtained from Theorem 6 in the special case in which the vectors $\boldsymbol{Y}_{1}, \boldsymbol{Y}_{2}, \ldots, \boldsymbol{Y}_{n}$ in (4) are either lattice or absolutely continuous random vectors. Additionally, it is worth to highlight that Theorem 6 holds for all random variables whose CGF exists. Under this condition, the Multivariate BerryEssen theorem in [16, Theorem 1.1], Theorem 1] is a special case of Theorem 3 for the choice $\boldsymbol{\theta}=\mathbf{0}$.

The advantages of approximating the probability of a convex set in $\mathscr{B}\left(\mathbb{R}^{k}\right)$ by using Theorem 3 instead of Theorem 1 are twofold. First, the proposed upper bound on the approximation error depends on the set to be approximated. Second, both the approximation and the upper bound on the approximation error are parametrized by $\boldsymbol{\theta} \in \Theta_{\boldsymbol{Y}}$, with $\Theta_{\boldsymbol{Y}}$ in (17). Thus, the vector $\boldsymbol{\theta}$ in (33) can be tuned to minimize the upper bound on the approximation error. Nonetheless, such optimization is not trivial. In this work, a non necessarily optimal choice has been made for obtaining an approximation of the CDF $F_{\boldsymbol{X}_{n}}$ in Theorem 6. That being said, the possibility to obtain tighter upper bounds on the approximation error on the measure $P_{\boldsymbol{X}_{n}}$ from Theorem 3, and on the approximation error on the CDF $F_{\boldsymbol{X}_{n}}$ from Theorem 6 is not negligible.
In the single dimension case, i.e., $k=1$, Theorem 3 leads to the same approximation on the measure $P_{\boldsymbol{X}_{n}}$ in [19, Theorem 2]. Nonetheless, the upper bound provided in [19, Theorem 2] on the approximation error is better than the one provided by Theorem 3 .

## A Proof of Lemma 2

The proof relies on noticing that:

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right|=\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\boldsymbol{z}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right]\right| \tag{76}
\end{equation*}
$$

and thus, plugging (23) in the RHS of (76) yields

$$
\begin{align*}
& \left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \\
& =\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\left|\mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}}\left[\exp \left(-\boldsymbol{\theta}^{\mathrm{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right]-\mathbb{E}_{P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}}\left[\exp \left(-\boldsymbol{\theta}^{\mathrm{\top}} \boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right]\right| . \tag{77}
\end{align*}
$$

In the following, explicit expressions for the terms

$$
\begin{align*}
& \mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right], \text { and }  \tag{78}\\
& \mathbb{E}_{P_{\boldsymbol{z}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right], \tag{79}
\end{align*}
$$

in (77) are obtained.

## A. 1 Explicit Expression of 78

From (78), the following holds

$$
\begin{equation*}
\mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right]=\int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{x}\right) \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{x}) \tag{80}
\end{equation*}
$$

The next step consists in writing the function $\exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{x}\right)$ in the right hand-side of 80) as a Lebesgue integral. For doing so, consider the set $\mathcal{K}=\{1,2, \ldots, k\}$ and let the set-valued functions $\mathcal{I}^{-}: \Theta_{\boldsymbol{Y}} \rightarrow 2^{\mathcal{K}}, \mathcal{I}: \Theta_{\boldsymbol{Y}} \rightarrow 2^{\mathcal{K}}$, and $\mathcal{I}^{+}: \Theta_{\boldsymbol{Y}} \rightarrow 2^{\mathcal{K}}$ be respectively defined such that for all $\boldsymbol{u}=\left(u_{1}, u_{2}, \ldots, u_{k}\right) \in \Theta_{\boldsymbol{Y}}$

$$
\begin{align*}
& \mathcal{I}^{-}(\boldsymbol{u})=\left\{i \in\{1,2, \ldots, k\}: u_{i}<0\right\}  \tag{81}\\
& \mathcal{I}(\boldsymbol{u})=\left\{i \in\{1,2, \ldots, k\}: u_{i}=0\right\}, \text { and }  \tag{82}\\
& \mathcal{I}^{+}(\boldsymbol{u})=\left\{i \in\{1,2, \ldots, k\}: u_{i}>0\right\} \tag{83}
\end{align*}
$$

Then, for all $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{k}\right) \in \mathbb{R}^{k}$, the following holds

$$
\begin{align*}
& \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{x}\right) \\
& =\exp \left(-\theta_{1} x_{1}-\theta_{2} x_{2}-\ldots-\theta_{k} x_{k}\right)  \tag{84}\\
& =\exp \left(-\theta_{1} x_{1}\right) \exp \left(-\theta_{2} x_{2}\right) \ldots \exp \left(-\theta_{k} x_{k}\right)  \tag{85}\\
& =\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \exp \left(-\theta_{i} x_{i}\right)\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \exp \left(-\theta_{j} x_{j}\right)\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \exp \left(-\theta_{s} x_{s}\right)\right)  \tag{86}\\
& =\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \int_{-\infty}^{x_{i}}-\theta_{i} \exp \left(-\theta_{i} t_{i}\right) \mathrm{d} t_{i}\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_{0}^{1} \mathrm{~d} t_{j}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \int_{x_{s}}^{\infty} \theta_{s} \exp \left(-\theta_{s} t_{s}\right) \mathrm{d} t_{s}\right) \tag{87}
\end{align*}
$$

$$
\begin{equation*}
=(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \int_{-\infty}^{x_{i}} \theta_{i} \exp \left(-\theta_{i} t_{i}\right) \mathrm{d} t_{i}\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_{0}^{1} \mathrm{~d} t_{j}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \int_{x_{s}}^{\infty} \theta_{s} \exp \left(-\theta_{s} t_{s}\right) \mathrm{d} t_{s}\right) \tag{88}
\end{equation*}
$$

To ease the notation, for all $u \in \mathbb{R}$ and for all $a \in \mathbb{R}$, let the set $\mathcal{B}_{u, a}$ be:

$$
\mathcal{B}_{u, a} \triangleq \begin{cases}{[0,1]} & \text { if } u=0  \tag{89}\\ (-\infty, a] & \text { if } u<0 \\ {[a, \infty)} & \text { if } u>0\end{cases}
$$

Then, using the notation in (89), the equality in (88) can be written as follow

$$
\begin{align*}
& \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{x}\right) \\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_{i}, x_{i}}} \theta_{i} \exp \left(-\theta_{i} t_{i}\right) \mathrm{d} t_{i}\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_{j}, x_{j}}} \mathrm{~d} t_{j}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_{s}, x_{s}}} \theta_{s} \exp \left(-\theta_{s} t_{s}\right) \mathrm{d} t_{s}\right)(90) \\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{B}_{\theta_{1}, x_{1}}} \int_{\mathcal{B}_{\theta_{2}, x_{2}}} \ldots \int_{\mathcal{B}_{\theta_{k}, x_{k}}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) \mathrm{d} t_{k} \ldots \mathrm{~d} t_{2} \mathrm{~d} t_{1}, \tag{91}
\end{align*}
$$

where the equality in follows from the linearity of the integration. To ease the notation, consider the set defined as

$$
\begin{equation*}
\mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}=\mathcal{B}_{\theta_{1}, x_{1}} \times \mathcal{B}_{\theta_{2}, x_{2}} \times \ldots \times \mathcal{B}_{\theta_{k}, x_{k}}, \tag{92}
\end{equation*}
$$

where for all $i \in\{1,2, \ldots, k\}$, the set $\mathcal{B}_{\theta_{i}, x_{i}}$ is defined in 89). Then, plugging (92) in (88) yields

$$
\begin{align*}
\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{x}\right) & =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}} \exp \left(-\boldsymbol{\theta}^{\mathrm{\top}} \boldsymbol{t}\right) \mathrm{d} \nu_{k}(\boldsymbol{t})  \tag{93}\\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}\}}\right.} \mathrm{d} \nu_{k}(\boldsymbol{t}), \tag{94}
\end{align*}
$$

where $\nu_{k}$ is the Lebesgue measure on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$.
Then, plugging (94) in 80) yields

$$
\begin{align*}
& \mathbb{E}_{P_{S_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right] \\
& =\int_{\mathbb{R}^{k}}(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) \mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}\}}\right.} \mathrm{d} \nu_{k}(\boldsymbol{t}) \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{x})  \tag{95}\\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}\right\}} \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} \nu_{k}(\boldsymbol{t}) \mathrm{d} P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{x})  \tag{96}\\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k \times 2}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) \mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}\}} \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} \nu_{k} \cdot P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{t}, \boldsymbol{x}),\right.} \tag{97}
\end{align*}
$$

RR n ${ }^{\circ} 9388$
where the Lebesgue integral in (97) is with respect to the product measure $\nu_{k} \cdot P_{\boldsymbol{S}_{n}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k \times 2}, \mathscr{B}\left(\mathbb{R}^{k \times 2}\right)\right)$. Note that the integral in (97) is absolutely integrable. Thus, using the Fubini's Theorem 20 , the right hand-side of (97) can be written as follows
$\mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right]$
$=(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) \mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}\}}\right.} \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} P_{\left.\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}\right)}(\boldsymbol{x}) \mathrm{d} \nu_{k}(\boldsymbol{t})$.

From (92\}, the indicator $\mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}\right\}}$ in (98) can be written as follows

$$
\begin{equation*}
\mathbb{1}_{\left\{\boldsymbol{t} \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}\right\}}=\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}\right)\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{i} \leqslant x_{i}\right\}}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{s} \geqslant x_{s}\right\}}\right) . \tag{99}
\end{equation*}
$$

To ease the notation, let the set

$$
\begin{equation*}
\overline{\mathcal{B}}_{\theta, t}=\overline{\mathcal{B}}_{\theta_{1}, t_{1}} \times \overline{\mathcal{B}}_{\theta_{2}, t_{2}} \times \ldots \times \overline{\mathcal{B}}_{\theta_{k}, t_{k}} \tag{100}
\end{equation*}
$$

where, for all $i \in\{1,2, \ldots, k\}$, the set $\overline{\mathcal{B}}_{\theta_{i}, t_{i}}$ is defined by:

$$
\overline{\mathcal{B}}_{\theta_{i}, t_{i}}=\left\{\begin{array}{cl}
\mathbb{R} & \text { if } \theta_{i}=0  \tag{101}\\
\left(-\infty, t_{i}\right] & \text { if } \theta_{i}>0 \\
{\left[t_{i}, \infty\right)} & \text { if } \theta_{i}<0
\end{array}\right.
$$

Then, plugging (100) in (99) yields

$$
\begin{equation*}
\mathbb{1}_{\left\{t \in \mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{x}}\right\}}=\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}\right) \mathbb{1}_{\left\{\boldsymbol{x} \in \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}\right\}} . \tag{102}
\end{equation*}
$$

Hence, plugging 102 in (98) yields

$$
\begin{align*}
& \mathbb{E}_{P_{S_{n}^{(\boldsymbol{\theta}}}}\left[\exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}) \in \mathcal{A}\}}\right.}\right] \\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}\right) \\
& \left.\mathbb{1}_{\left\{\boldsymbol{x} \in \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}\right\}} \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}} \boldsymbol{x}\right) \mathrm{d} \nu_{k}(\boldsymbol{t})  \tag{103}\\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right)\left(\prod_{j \in \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}\right) \\
& \int_{\mathbb{R}^{k}}^{\mathbb{1}_{\left\{\boldsymbol{x} \in \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}\right\}} \mathbb{1}_{\{\boldsymbol{x} \in \mathcal{A}\}} \mathrm{d} P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}(\boldsymbol{x}) \mathrm{d} \nu_{k}(\boldsymbol{t})}  \tag{104}\\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathbb{R}^{k}} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right)\left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}\right) P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right) \mathrm{d} \nu_{k}(\boldsymbol{t})(105)
\end{align*}
$$

Note that the support of the integrand in is a subset of $\mathbb{R}^{k}$. Hence, the objective of the following lines is to characterize a subset of $\mathbb{R}^{k}$, denoted by $\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})$, that contains the support of the integrand in 105). The integrand is different from zero if $P_{S_{n}^{(\theta)}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right)$ and $\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}$ are simultaneously strictly positive. On the first hand, given a vector $\boldsymbol{t}=$ $\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \mathbb{R}^{k}$, the product $\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\left\{t_{j} \in[0,1]\right\}}$ is strictly positive if and only if for all $i \in\{1,2, \ldots, k\}$, it holds that

$$
\begin{equation*}
t_{i} \in[0,1] \text { if } \theta_{i}=0 \tag{106}
\end{equation*}
$$

On the other hand, given $\boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \mathbb{R}^{k}$, a necessary condition for $P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta}}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right)$ to be strictly positive is that the set $\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}$ is not empty. Now a necessary condition for the non-emptiness of $\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}$ is that the set $\overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}$ contains at least one element $\boldsymbol{e}=\left(e_{1}, e_{2}, \ldots, e_{k}\right)$ such that for all $i \in\{1,2, \ldots, k\}$,

$$
\begin{equation*}
\inf _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} \leqslant e_{i} \leqslant \sup _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} . \tag{107}
\end{equation*}
$$

The inequalities in 107 impose some conditions on the given vector $\boldsymbol{t}$. More specifically, from the definition of the set $\overline{\mathcal{B}_{\boldsymbol{\theta}, \boldsymbol{t}}}$ in 100 , the vector $\boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{k}\right)$ must satisfy for all $i \in\{1,2, \ldots, k\}$,

$$
\begin{cases}t_{i} \geqslant \underset{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}}{\inf } b_{i} & \text { if } \theta_{i}>0 ; \text { and }  \tag{108}\\ t_{i} \leqslant \sup _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} & \text { if } \theta_{i}<0\end{cases}
$$

Hence, from 106 and 108$)$ the set $\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})$ can be defined as follows

$$
\begin{align*}
\mathcal{D}(\mathcal{A}, \boldsymbol{\theta}) \triangleq & \triangleq\left\{\left(t_{1}, t_{2}, \ldots, t_{k}\right) \in \mathbb{R}^{k}: \forall i \in\{1,2, \ldots, k\}, t_{i} \in[0,1] \text { if } \theta_{i}=0\right. \\
& \left.t_{i} \geqslant \inf _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} \text { if } \theta_{i}>0, \text { and } t_{i} \leqslant \sup _{\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in \mathcal{A}} b_{i} \text { if } \theta_{i}<0\right\} \tag{109}
\end{align*}
$$

Then, the equality in can be written as follows

$$
\begin{align*}
& \mathbb{E}_{P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{S}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right] \\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right) \mathrm{d} \nu_{k}(\boldsymbol{t}) . \tag{110}
\end{align*}
$$

## A. 2 Explicit Expression of 79

Following similar steps as in Subsection A.1 the following holds with the random vector $\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}$ :

$$
\begin{align*}
& \mathbb{E}_{P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}}\left[\exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\left\{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})} \in \mathcal{A}\right\}}\right] \\
& =(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right) \mathrm{d} \nu_{k}(\boldsymbol{t}) . \tag{111}
\end{align*}
$$

## A. 3 Upper Bound on 77

The proof ends by plugging (110) and (111) in the right hand-side of (77). This yields,

$$
\begin{align*}
& \left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \\
& =\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta}) \mid(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right) \mathrm{d} \nu_{k}(\boldsymbol{t})\right. \\
& -(-1)^{\left|\mathcal{I}^{-}(\boldsymbol{\theta})\right|}\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right) P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right) \mathrm{d} \nu_{k}(\boldsymbol{t}) \mid  \tag{112}\\
& =\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta}) \mid\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right)\right. \\
& \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{t}\right)\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right)-P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}}\right)\right) \mathrm{d} \nu_{k}(\boldsymbol{t}) \mid \tag{113}
\end{align*}
$$

The set $\overline{\mathcal{B}}_{\boldsymbol{\theta}, \boldsymbol{t}} \cap \mathcal{A}$ in 113 is convex Borel measurable, given that $\mathcal{A}$ is convex Borel measurable from the assumptions of the lemma. From (30), it holds that

$$
\begin{equation*}
\left|P_{\boldsymbol{S}_{n}^{(\theta)}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}\right)-P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\left(\mathcal{A} \cap \overline{\mathcal{B}}_{\boldsymbol{\theta}, t}\right)\right| \leqslant \Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\right) . \tag{114}
\end{equation*}
$$

Then, plugging (114) in (113) yields

$$
\begin{align*}
& \left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \\
& \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})}-\theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\right) \nu_{k}(\mathrm{~d} \boldsymbol{t})(\overline{\text { a }}  \tag{115}\\
& =\exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right) \Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\right)\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})}-\theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \nu_{k}(\mathrm{~d} \boldsymbol{t})(. \tag{.116}
\end{align*}
$$

The expression $\int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \mathrm{d} \nu_{k}(\boldsymbol{t})$ in (116) using the notation in (89) and (92) can be written in the form

$$
\begin{align*}
& \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{t}\right) \mathrm{d} \nu_{k}(\boldsymbol{t}) \\
& =\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})} \int_{-\infty}^{a_{i}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\theta_{i} t_{i}\right) \mathrm{d} t_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \int_{a_{s}(\mathcal{A}, \boldsymbol{\theta})}^{\infty} \exp \left(-\theta_{s} t_{s}\right) \mathrm{d} t_{s}\right)  \tag{117}\\
& =\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})}-\frac{\exp \left(-\theta_{i} a_{i}(\mathcal{A}, \boldsymbol{\theta})\right)}{\theta_{i}}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \frac{\exp \left(-\theta_{s} a_{s}(\mathcal{A}, \boldsymbol{\theta})\right)}{\theta_{s}}\right)  \tag{118}\\
& =\frac{\exp \left(-\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right)}{\left(\prod_{i \in \mathcal{I}^{-}(\boldsymbol{\theta})}-\theta_{i}\right)\left(\prod_{s \in \mathcal{I}^{+}(\boldsymbol{\theta})} \theta_{s}\right)}, \tag{119}
\end{align*}
$$

where the vector $\boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})=\left(a_{1}(\mathcal{A}, \boldsymbol{\theta}), a_{2}(\mathcal{A}, \boldsymbol{\theta}), \ldots, a_{k}(\mathcal{A}, \boldsymbol{\theta})\right)$ is defined in 31). Hence, plugging (119) in 116) yields

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\right) \tag{120}
\end{equation*}
$$

which completes the proof.

## B Proof of Theorem 3

From Lemma 2 it holds that

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta\left(P_{\boldsymbol{S}_{n}^{(\theta)}}, P_{\boldsymbol{Z}_{n}^{(\theta)}}\right) . \tag{121}
\end{equation*}
$$

Hence, the objective is to provide an upper bound on $\Delta\left(P_{\boldsymbol{S}_{n}^{(\theta)}}, P_{\boldsymbol{Z}_{n}^{(\theta)}}\right)$. An upper bound would be obtained immediately from Theorem 1, except for the fact that the vectors $\boldsymbol{Y}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{Y}_{2}^{(\boldsymbol{\theta})}, \ldots$, $\boldsymbol{Y}_{n}^{(\boldsymbol{\theta})}$ in (22) do not have means $(0,0, \ldots, 0)$ and variances $\frac{1}{n} \operatorname{diag}(1,1, \ldots, 1)$, as required by Theorem 1. Denote by $\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} \in \mathbb{R}^{k}$ and $\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} \in \mathbb{R}^{k \times k}$, respectively, the mean vector and the covariance matrix of these random vectors, for some $\boldsymbol{\theta} \in \Theta_{\boldsymbol{Y}}$, with $\Theta_{\boldsymbol{Y}}$ in (17). Then, the following holds,

$$
\begin{align*}
\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} & \triangleq \mathbb{E}_{P_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}}\left[\boldsymbol{Y}^{(\boldsymbol{\theta})}\right]  \tag{122a}\\
& =\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\boldsymbol{Y} \exp \left(\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]  \tag{122b}\\
& =K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta}), \tag{122c}
\end{align*}
$$

where $K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})$ is the gradient vector of the CGF $K_{\boldsymbol{Y}}$ defined in (2). Alternatively,

$$
\begin{align*}
\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} & \triangleq \mathbb{E}_{P_{\boldsymbol{Y}^{(\theta)}}}\left[\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\mathrm{\top}}\right]  \tag{122~d}\\
& =\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\mathrm{T}} \exp \left(\boldsymbol{\theta}^{\mathrm{\top}} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]  \tag{122e}\\
& =K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta}) \tag{122f}
\end{align*}
$$

where $K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})$ is the Hessian matrix of the CGF $K_{\boldsymbol{Y}}$ defined in (3). Let the Cholesky decomposition of the matrix $\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}$ be

$$
\begin{equation*}
\underline{v}_{Y^{(\theta)}}=\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}} \underline{\underline{Y}}_{\boldsymbol{Y}^{(\theta)}}^{\mathrm{\top}}, \tag{123}
\end{equation*}
$$

where $\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}$ is a real lower triangular matrix. Note that the matrix $\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}$ is nonsingular. This follows from the assumption that the covariance matrix $K_{\boldsymbol{Y}}^{(2)}(\mathbf{0})$ is positive definite, which implies that the CGF $K_{\boldsymbol{Y}}$ is strictly convex and thus, its Hessian matrix $K_{\boldsymbol{Y}}^{(2)}$ is positive definite. Let the random vector $\boldsymbol{R}_{n}^{(\boldsymbol{\theta})}$ be such that

$$
\begin{equation*}
\boldsymbol{R}_{n}^{(\boldsymbol{\theta})} \triangleq \frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}^{-1}\left(\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}-n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right) \tag{124}
\end{equation*}
$$

which induces the probability measure $P_{\boldsymbol{R}_{n}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$. Plugging (22) in (124) yields:

$$
\begin{align*}
\boldsymbol{R}_{n}^{(\boldsymbol{\theta})} & =\frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}^{-1} \sum_{j=1}^{n}\left(\boldsymbol{Y}_{j}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)  \tag{125}\\
& =\sum_{j=1}^{n} \frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}^{-1}\left(\boldsymbol{Y}_{j}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right) \tag{126}
\end{align*}
$$

RR n ${ }^{\circ} 9388$

$$
\begin{equation*}
=\sum_{j=1}^{n} \boldsymbol{U}_{j}^{(\boldsymbol{\theta})} \tag{127}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{U}_{j}^{(\boldsymbol{\theta})} \triangleq \frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}^{-1}\left(\boldsymbol{Y}_{j}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right) \tag{128}
\end{equation*}
$$

is a random vector that induces the probability measure $P_{\boldsymbol{U}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$. Thus, the random vector $\boldsymbol{R}_{n}^{(\boldsymbol{\theta})}$ in 124 is the sum of $n$ IID random vectors $\boldsymbol{U}_{1}^{(\boldsymbol{\theta})}, \boldsymbol{U}_{2}^{(\boldsymbol{\theta})}, \ldots$, $\boldsymbol{U}_{n}^{(\boldsymbol{\theta})}$ such that each of them induces the probability measure $P_{\boldsymbol{U}^{(\boldsymbol{\theta})}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right.$ ), which satisfies,

$$
\begin{align*}
\mathbb{E}_{P_{\boldsymbol{U}^{(\theta)}}}\left[\boldsymbol{U}^{(\boldsymbol{\theta})}\right] & =\frac{1}{\sqrt{n}} \mathbb{E}_{P_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}}\left[\underline{\underline{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}\right)\right]  \tag{129}\\
& =\frac{1}{\sqrt{n}} \underline{l}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}^{-1} \mathbb{E}_{P_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}}\left[\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}\right)\right]  \tag{130}\\
& =0 \tag{131}
\end{align*}
$$

and

$$
\begin{align*}
& \mathbb{E}_{P_{\boldsymbol{U}^{(\boldsymbol{\theta}}}}\left[\boldsymbol{U}^{(\boldsymbol{\theta})} \boldsymbol{U}^{(\boldsymbol{\theta}) \boldsymbol{\top}}\right]=\frac{1}{n} \mathbb{E}_{P_{\boldsymbol{Y}^{(\theta)}}}\left[\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\left.\boldsymbol{Y}^{(\boldsymbol{\theta}}\right)}\right)\right)^{\mathrm{T}}\right]  \tag{132}\\
& =\frac{1}{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1} \mathbb{E}_{P_{\boldsymbol{Y}^{(\theta)}}}\left[\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\theta)}}\right)\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{\mathrm{\top}}\right]\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1}\right)^{\top}  \tag{133}\\
& =\frac{1}{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1} \underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\theta)}}\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1}\right)^{\boldsymbol{\top}}  \tag{134}\\
& =\frac{1}{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{\top}\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}}^{-1}\right)^{\top}  \tag{135}\\
& =\frac{1}{n} \operatorname{diag}(1,1, \ldots, 1) \text {. } \tag{136}
\end{align*}
$$

Similarly, let the random vector $\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}$ be such that

$$
\begin{equation*}
\boldsymbol{W}_{n}^{(\boldsymbol{\theta})} \triangleq \frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}^{-1}\left(\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}-n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right), \tag{137}
\end{equation*}
$$

which induce the probability measure $P_{W_{n}^{(\theta)}}$ on the measurable space $\left(\mathbb{R}^{k}, \mathscr{B}\left(\mathbb{R}^{k}\right)\right)$. The mean vector and the covariance matrix of the random vector $\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}$ are identical to those of the random vector $\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}$. See for instance (24) and (25). Then, from $(124)$ and (137), it holds that the mean vector and the covariance matrix of the random vector $\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}$ are identical to those of the random vector $\boldsymbol{R}_{n}^{(\boldsymbol{\theta})}$.
The rest of the proof follows by noticing that for all $\mathcal{B} \in \mathscr{B}\left(\mathbb{R}^{k}\right)$, the set $\mathcal{S}(\mathcal{B})$ defined by

$$
\begin{align*}
\mathcal{S}(\mathcal{B}) & \triangleq\left\{\boldsymbol{y} \in \mathbb{R}^{k}: \exists \boldsymbol{x} \in \mathcal{B}, \boldsymbol{y}=\frac{1}{\sqrt{n}} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}^{-1}\left(\boldsymbol{x}-n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right\}  \tag{138}\\
& =\left\{\boldsymbol{y} \in \mathbb{R}^{k}:\left(\sqrt{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} \boldsymbol{y}+n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right) \in \mathcal{B}\right\} . \tag{139}
\end{align*}
$$

allows writing that

$$
\begin{equation*}
P_{\boldsymbol{S}_{n}^{(\theta)}}(\mathcal{B})=\mathbb{E}_{P_{S_{n}^{(\theta)}}}\left[\mathbb{1}_{\left\{\boldsymbol{S}_{n}^{(\theta)} \in \mathcal{B}\right\}}\right] \tag{140}
\end{equation*}
$$

$$
\begin{align*}
& =\mathbb{E}_{P_{\boldsymbol{R}_{n}^{(\boldsymbol{\theta}}}}\left[\mathbb{1}_{\left.\left\{\left(\sqrt{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}} \boldsymbol{R}_{n}^{(\boldsymbol{\theta})}+n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}\right) \in \mathcal{B}\right\}\right]}\right]  \tag{141}\\
& =P_{\boldsymbol{R}_{n}^{(\theta)}}\left(\left\{\boldsymbol{y} \in \mathbb{R}^{k}:\left(\sqrt{n} \underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\theta)}} \boldsymbol{y}+n \boldsymbol{\mu}_{\boldsymbol{Y}^{(\theta)}}\right) \in \mathcal{B}\right\}\right)  \tag{142}\\
& =P_{\boldsymbol{R}_{n}^{(\theta)}}(\mathcal{S}(\mathcal{B})) . \tag{143}
\end{align*}
$$

Similarly, from 137) and 138, it holds that

$$
\begin{equation*}
P_{\boldsymbol{Z}_{n}^{(\theta)}}(\mathcal{B})=P_{W_{n}^{(\theta)}}(\mathcal{S}(\mathcal{B})) . \tag{144}
\end{equation*}
$$

This implies that,

$$
\begin{align*}
\Delta\left(P_{\boldsymbol{S}_{n}^{(\theta)}}, P_{\boldsymbol{Z}_{n}^{(\theta)}}\right) & =\sup _{\mathcal{B} \in \mathcal{C}_{k}}\left|P_{\boldsymbol{R}_{n}^{(\theta)}}(\mathcal{S}(\mathcal{B}))-P_{\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}}(\mathcal{S}(\mathcal{B}))\right|  \tag{145a}\\
& \leq \sup _{\mathcal{S} \in \mathcal{C}_{k}}\left|P_{\boldsymbol{R}_{n}^{(\boldsymbol{\theta}}}(\mathcal{S})-P_{\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}}(\mathcal{S})\right|  \tag{145b}\\
& =\Delta\left(P_{\boldsymbol{R}_{n}^{(\boldsymbol{\theta}}}, P_{\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}}\right) \tag{145c}
\end{align*}
$$

where the inequality in 145 b is a consequence of the fact that the set $\mathcal{C}_{k}$ of all convex sets in $\mathscr{B}\left(\mathbb{R}^{k}\right)$ is stable under linear transformations. Then, from the multivariate Berry-Essen Theorem (Theorem 1), it holds that

$$
\begin{align*}
& \Delta\left(P_{\boldsymbol{R}_{n}^{(\boldsymbol{\theta}}}, P_{\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}}\right) \\
& \leq \min \left(1, c(k) n \mathbb{E}_{P_{\boldsymbol{U}^{(\boldsymbol{\theta}}}}\left[\left\|\boldsymbol{U}^{(\boldsymbol{\theta})}\right\|^{3}\right]\right)  \tag{146a}\\
& =\min \left(1, c(k) n \mathbb{E}_{P_{\boldsymbol{U}^{(\boldsymbol{\theta}}}}\left[\left(\boldsymbol{U}^{(\boldsymbol{\theta}) \mathrm{T}} \boldsymbol{U}^{(\boldsymbol{\theta})}\right)^{\frac{3}{2}}\right]\right)  \tag{146b}\\
& =\min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\boldsymbol{Y}^{(\theta)}}}\left[\left(\left(\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right)^{\top}\left(\underline{\boldsymbol{l}}_{\left.\boldsymbol{Y}^{(\boldsymbol{\theta}}\right)}\right)^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right)^{\frac{3}{2}}\right]\right)  \tag{146c}\\
& =\min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}}\left[\left(\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{\boldsymbol{\top}}\left(\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}\right)^{-1}\right)^{\boldsymbol{\top}}\left(\underline{\boldsymbol{l}}_{\boldsymbol{Y}^{(\boldsymbol{\theta}}}\right)^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right)^{\frac{3}{2}}\right]\right)  \tag{146d}\\
& =\min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\boldsymbol{Y}^{(\theta)}}}\left[\left(\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{\boldsymbol{\top}}\left(\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{-1}\left(\boldsymbol{Y}^{(\boldsymbol{\theta})}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right)^{\frac{3}{2}}\right]\right)  \tag{146e}\\
& =\min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\left(\left(\boldsymbol{Y}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{\boldsymbol{\top}}\left(\underline{\boldsymbol{v}}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)^{-1}\left(\boldsymbol{Y}-\boldsymbol{\mu}_{\boldsymbol{Y}^{(\boldsymbol{\theta})}}\right)\right)^{3 / 2} \exp \left(\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]\right),( \tag{146f}
\end{align*}
$$

where $c$ is the function defined in (14).
Finally, plugging (122c) and (122f in (146f) yields

$$
\begin{align*}
& \Delta\left(P_{\boldsymbol{R}_{n}^{(\theta)}}, P_{\boldsymbol{W}_{n}^{(\boldsymbol{\theta})}}\right) \\
& \leq \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{Y}}\left[\left(\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)^{\top}\left(K_{\boldsymbol{Y}}^{(2)}(\boldsymbol{\theta})\right)^{-1}\left(\boldsymbol{Y}-K_{\boldsymbol{Y}}^{(1)}(\boldsymbol{\theta})\right)\right)^{3 / 2} \exp \left(\boldsymbol{\theta}^{\top} \boldsymbol{Y}-K_{\boldsymbol{Y}}(\boldsymbol{\theta})\right)\right]\right)  \tag{147a}\\
& =\min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right), \tag{147b}
\end{align*}
$$

RR n ${ }^{\circ} 9388$
where the function $\xi_{Y}$ is defined in (32). Plugging (147b) in 145 c yields

$$
\begin{equation*}
\Delta\left(P_{\boldsymbol{S}_{n}^{(\boldsymbol{\theta})}}, P_{\boldsymbol{Z}_{n}^{(\boldsymbol{\theta})}}\right) \leq \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right) \tag{148}
\end{equation*}
$$

Finally, plugging (148) in (121) yields

$$
\begin{equation*}
\left|P_{\boldsymbol{X}_{n}}(\mathcal{A})-\eta_{\boldsymbol{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)\right| \leqslant \exp \left(n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{a}(\mathcal{A}, \boldsymbol{\theta})\right) \min \left(1, \frac{c(k) \xi_{\boldsymbol{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right) \tag{149}
\end{equation*}
$$

which completes the proof.

## C Proof of Lemma 4

Note that for all $\boldsymbol{x} \in \mathbb{R}^{k}$, it holds from (40) and from the fact that $\mathbf{0} \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$, that

$$
\begin{equation*}
n K_{\boldsymbol{Y}}(\boldsymbol{\theta}(\boldsymbol{x}))-\boldsymbol{\theta}^{\boldsymbol{\top}}(\boldsymbol{x}) \boldsymbol{x} \leq\left(n K_{\boldsymbol{Y}}(\mathbf{0})-\mathbf{0}^{\boldsymbol{\top}} \boldsymbol{x}\right)=0 \tag{150}
\end{equation*}
$$

Moreover, for all $\boldsymbol{x} \in \mathbb{R}^{k}$ and for all $\boldsymbol{\theta} \in \operatorname{clo} \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$, it holds that

$$
\begin{align*}
n K_{\boldsymbol{Y}}(\boldsymbol{\theta})-\boldsymbol{\theta}^{\top} \boldsymbol{x} & =n \log \left(\mathbb{E}_{P_{\boldsymbol{Y}}}\left[\exp \left(\boldsymbol{\theta}^{\top} \boldsymbol{Y}\right)\right]\right)-\boldsymbol{\theta}^{\top} \boldsymbol{x}  \tag{151}\\
& \geq n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\log \left(\exp \left(\boldsymbol{\theta}^{\top} \boldsymbol{Y}\right)\right)\right]-\boldsymbol{\theta}^{\top} \boldsymbol{x}  \tag{152}\\
& =n \mathbb{E}_{P_{\boldsymbol{Y}}}\left[\boldsymbol{\theta}^{\top} \boldsymbol{Y}\right]-\boldsymbol{\theta}^{\boldsymbol{\top}} \boldsymbol{x}  \tag{153}\\
& =\boldsymbol{\theta}^{\top} n \mathbb{E}_{P_{\boldsymbol{Y}}}[\boldsymbol{Y}]-\boldsymbol{\theta}^{\top} \boldsymbol{x}  \tag{154}\\
& =\boldsymbol{\theta}^{\top}\left(n \mathbb{E}_{P_{\boldsymbol{Y}}}[\boldsymbol{Y}]-\boldsymbol{x}\right)  \tag{155}\\
& =\boldsymbol{\theta}^{\top}\left(\mathbb{E}_{P_{\boldsymbol{X}_{n}}}\left[\boldsymbol{X}_{n}\right]-\boldsymbol{x}\right)  \tag{156}\\
& =\boldsymbol{\theta}^{\top}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}-\boldsymbol{x}\right), \tag{157}
\end{align*}
$$

where the inequality in 152 follows from Jensen's inequality [21, Section 2.6]; the equality in 156 follows from (4); and the equality in 157 follows from (43).
From (150) and (157), it holds that for all $\boldsymbol{x} \in \mathbb{R}^{k}, \boldsymbol{\theta}(\boldsymbol{x})$ in (40) satisfies

$$
\begin{equation*}
\boldsymbol{\theta}^{\boldsymbol{\top}}(\boldsymbol{x})\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}-\boldsymbol{x}\right) \leq 0, \tag{158}
\end{equation*}
$$

which implies that $\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)^{\mathrm{T}} \boldsymbol{\theta}(\boldsymbol{x}) \geq 0$ and proves the inequality in 41.
From (150) and (157), it holds that for $\boldsymbol{x}=\boldsymbol{\mu}_{\boldsymbol{X}_{n}}$,

$$
\begin{equation*}
n K_{\boldsymbol{Y}}\left(\boldsymbol{\tau}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)\right)-\boldsymbol{\tau}^{\boldsymbol{\top}}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right) \boldsymbol{\mu}_{\boldsymbol{X}_{n}}=\left(n K_{\boldsymbol{Y}}(\mathbf{0})-\mathbf{0}^{\boldsymbol{\top}} \boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)=0 \tag{159}
\end{equation*}
$$

Thus, the uniqueness of $\boldsymbol{\tau}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)$ implies from 159) that

$$
\begin{equation*}
\boldsymbol{\tau}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)=\mathbf{0} \tag{160}
\end{equation*}
$$

Finally, note that $\mathbf{0} \in \boldsymbol{\Theta}_{\boldsymbol{Y}}^{-}$and thus, from 40 , it holds that

$$
\begin{equation*}
\boldsymbol{\theta}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)=\boldsymbol{\tau}\left(\boldsymbol{\mu}_{\boldsymbol{X}_{n}}\right)=\mathbf{0} \tag{161}
\end{equation*}
$$

which concludes the proof.

## D Proof of Lemma 5

For all $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{k}\right) \in \mathbb{R}^{k}$, it holds that

$$
\begin{align*}
1-F_{\boldsymbol{X}_{n}}(\boldsymbol{x}) & =1-\mathbb{E}_{\boldsymbol{X}_{n}}\left[\prod_{t=1}^{k} \mathbb{1}_{\left\{X_{n, t} \leq x_{t}\right\}}\right]  \tag{162}\\
& =\mathbb{E}_{\boldsymbol{X}_{n}}\left[1-\prod_{t=1}^{k} \mathbb{1}_{\left\{X_{n, t} \leq x_{t}\right\}}\right]  \tag{163}\\
& =\mathbb{E}_{\boldsymbol{X}_{n}}\left[\max \left\{\mathbb{1}_{\left\{X_{n, t}>x_{t}\right\}}: t \in\{1,2, \ldots, k\}\right\}\right] . \tag{164}
\end{align*}
$$

The proof continues by using a property of the max function provided by the following lemma.
Lemma 8 For all $k \in \mathbb{N}$, and for all $\left(a_{1}, a_{2}, \ldots, a_{k}\right) \in\{0,1\}^{k}$, it holds that

$$
\begin{equation*}
\max \left\{a_{1}, a_{2}, \ldots, a_{k}\right\}=a_{1}+\sum_{i=2}^{k} a_{i} \prod_{j=1}^{i-1}\left(1-a_{j}\right) \tag{165}
\end{equation*}
$$

Proof: The proof is made by recurrence. For $k=1$, the results is trivial. For $k=2$, it holds that

$$
\begin{equation*}
\max \left\{a_{1}, a_{2}\right\}=a_{1}+a_{2}\left(1-a_{1}\right) \tag{166}
\end{equation*}
$$

Assume that for all $k \in \mathbb{N}$ and for all $\left(b_{1}, b_{2}, \ldots, b_{k}\right) \in\{0,1\}^{k}$, it holds that

$$
\begin{equation*}
\max \left\{b_{1}, b_{2}, \ldots, b_{k}\right\}=b_{1}+\sum_{i=2}^{k} b_{i} \prod_{j=1}^{i-1}\left(1-b_{j}\right) \tag{167}
\end{equation*}
$$

Let $b_{k+1}$ a real be such that $b_{k+1} \in\{0,1\}$. Then, it holds that

$$
\begin{equation*}
\max \left\{b_{1}, b_{2}, \ldots, b_{k}, b_{k+1}\right\}=\max \left\{b_{k+1}, \max \left\{b_{1}, b_{2}, \ldots, b_{k}\right\}\right\} \tag{168}
\end{equation*}
$$

Note that $\max \left\{b_{1}, b_{2}, \ldots, b_{k}\right\} \in\{0,1\}$. Then, plugging (166) in 168) yields

$$
\begin{equation*}
\max \left\{b_{1}, b_{2}, \ldots, b_{k}, b_{k+1}\right\}=b_{k+1}+\left(1-b_{k+1}\right) \max \left\{b_{1}, b_{2}, \ldots, b_{k}\right\} . \tag{169}
\end{equation*}
$$

Then, plugging (167) in 169 yields

$$
\begin{align*}
\max \left\{b_{1}, b_{2}, \ldots, b_{k}, b_{k+1}\right\} & =b_{k+1}+\left(1-b_{k+1}\right)\left(b_{1}+\sum_{i=2}^{k} b_{i} \prod_{j=1}^{i-1}\left(1-b_{j}\right)\right)  \tag{170}\\
& =b_{k+1}+\left(b_{1}\left(1-b_{k+1}\right)+\sum_{i=2}^{k} b_{i}\left(1-b_{k+1}\right) \prod_{j=1}^{i-1}\left(1-b_{j}\right)\right) . \tag{171}
\end{align*}
$$

Let $a_{1}, a_{2}, \ldots, a_{k+1}$ be $(k+1)$ numbers such that

$$
\begin{equation*}
a_{1}=b_{k+1}, \text { and } \forall i \in\{1,2, \ldots, k\}, a_{i+1}=b_{i} . \tag{172}
\end{equation*}
$$

Then, it holds that

$$
\begin{equation*}
\max \left\{a_{1}, a_{2}, \ldots, a_{k}, a_{k+1}\right\}=\max \left\{b_{1}, b_{2}, \ldots, b_{k}, b_{k+1}\right\} \tag{173}
\end{equation*}
$$

RR $n^{\circ} 9388$

$$
\begin{align*}
& =a_{1}+\left(a_{2}\left(1-a_{1}\right)+\sum_{i=2}^{k} a_{i+1}\left(1-a_{1}\right) \prod_{j=1}^{i-1}\left(1-a_{j+1}\right)\right)  \tag{174}\\
& =a_{1}+\left(a_{2}\left(1-a_{1}\right)+\sum_{i=2}^{k} a_{i+1}\left(1-a_{1}\right) \prod_{j=2}^{i}\left(1-a_{j}\right)\right)  \tag{175}\\
& =a_{1}+\left(a_{2}\left(1-a_{1}\right)+\sum_{i=2}^{k} a_{i+1} \prod_{j=1}^{i}\left(1-a_{j}\right)\right)  \tag{176}\\
& =a_{1}+\left(a_{2}\left(1-a_{1}\right)+\sum_{i=3}^{k+1} a_{i} \prod_{j=1}^{i-1}\left(1-a_{j}\right)\right)  \tag{177}\\
& =a_{1}+\sum_{i=2}^{k+1} a_{i+1} \prod_{j=1}^{i-1}\left(1-a_{j}\right), \tag{178}
\end{align*}
$$

which concludes the proof by recurrence.
Using Lemma 8 in 164 , it follows that

$$
\begin{align*}
1-F_{\boldsymbol{X}_{n}}(\boldsymbol{x}) & =\mathbb{E}_{\boldsymbol{X}_{n}}\left[\mathbb{1}_{\left\{X_{n, 1}>x_{1}\right\}}+\sum_{i=2}^{k} \mathbb{1}_{\left\{X_{n, i}>x_{i}\right\}} \prod_{j=1}^{i-1} \mathbb{1}_{\left\{X_{n, j} \leqslant x_{j}\right\}}\right]  \tag{180}\\
& =\mathbb{E}_{\boldsymbol{X}_{n}}\left[\mathbb{1}_{\left\{X_{n, 1}>x_{1}\right\}}\right]+\sum_{i=2}^{k} \mathbb{E}_{\boldsymbol{X}_{n}}\left[\mathbb{1}_{\left\{X_{n, i}>x_{i}\right\}} \prod_{j=1}^{i-1} \mathbb{1}_{\left\{X_{n, j} \leqslant x_{j}\right\}}\right]  \tag{181}\\
& =P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, 1))+\sum_{i=2}^{k} P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i))  \tag{182}\\
& =\sum_{i=1}^{k} P_{\boldsymbol{X}_{n}}(\mathcal{B}(\boldsymbol{x}, i)) \tag{183}
\end{align*}
$$

where the set $\mathcal{B}(\boldsymbol{x}, i)$ is defined in (47). The equality in (181) follows from the linearity of the expectation. This concludes the proof.

## References

[1] D. Anade, Jean-Marie Gorce, P. Mary, and S. M. Perlaza, "Saddlepoint approximation of cumulative distribution functions of sums of random vectors," in Proc. of the IEEE International Symposium on Information Theory (ISIT), Melbourne, Victoria, Australia, Jul. 2021.
[2] V. Y. F. Tan and O. Kosut, "On the dispersions of three network information theory problems," IEEE Transactions on Information Theory, vol. 60, no. 2, pp. 881-903, Feb. 2014.
[3] E. MolavianJazi and J. N. Laneman, "A random coding approach to Gaussian multiple access channels with finite blocklength," in Proc. of the 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Illinois, USA, Oct. 2012, pp. 286-293.
[4] J. E. Kolassa, "Multivariate saddlepoint tail probability approximations," The Annals of Statistics, vol. 31, no. 1, pp. 274-286, Feb. 2003.
[5] J. Kolassa and J. Li, "Multivariate saddlepoint approximations in tail probability and conditional inference," Bernoulli, vol. 16, no. 4, pp. 1191-1207, Nov. 2010.
[6] J. L. Jensen, Saddlepoint Approximations. New York, NY, USA: Clarendon press - Oxford, 1995.
[7] R. Butler, Saddlepoint approximations with applications. Cambridge, NY, USA: Cambridge University Press, 2007.
[8] W. Feller, An Introduction to Probability Theory and Its Applications, 2nd ed. New York, NY, USA: John Wiley and Sons, 1971, vol. 2.
[9] A. Martinez, J. Scarlett, M. Dalai, and A. G. i Fàbregas, "A complex-integration approach to the saddlepoint approximation for random-coding bounds," in Proc. of the 11th International Symposium on Wireless Communications Systems (ISWCS), Barcelona, Spain, Aug. 2014, pp. 618-621.
[10] T. Erseghe, "Coding in the finite-blocklength regime: Bounds based on Laplace integrals and their asymptotic approximations," IEEE Transactions on Information Theory, vol. 62, no. 12, pp. 6854-6883, Dec. 2016.
[11] J. Font-Segura, G. Vazquez-Vilar, A. Martinez, A. G. i Fàbregas, and A. Lancho, "Saddlepoint approximations of lower and upper bounds to the error probability in channel coding," in Proc. of the 52nd Annual Conference on Information Sciences and Systems (CISS), Princeton, NJ, USA, Mar. 2018, pp. 1-6.
[12] G. Vazquez-Vilar, A. G. i Fàbregas, T. Koch, and A. Lancho, "Saddlepoint approximation of the error probability of binary hypothesis testing," in Proc. of the IEEE International Symposium on Information Theory (ISIT), Vail, CO, USA, Jun. 2018, pp. 2306-2310.
[13] J. Font-Segura, A. Martinez, and A. G. i Fàbregas, "Saddlepoint approximation of the costconstrained random coding error probability," in Proc. of the IEEE Information Theory Workshop (ITW), Guangzhou, China, Jan. 2018, pp. 1-5.
[14] J. Font-Segura, A. Martinez, and A. G. i Fàbregas, "Asymptotics of the random coding error probability for constant-composition codes," in Proc. of the IEEE International Symposium on Information Theory (ISIT), Paris, France, Jul 2019, pp. 2947-2951.
[15] A. Lancho, J. Ostman, G. Durisi, T. Koch, and G. Vazquez-Vilar, "Saddlepoint approximations for short-packet wireless communications," IEEE Transactions on Wireless Communications, vol. 19, no. 7, pp. 4831-4846, Jul. 2020.
[16] M. Raič, "A multivariate BerryâĂŞEsseen theorem with explicit constants," Bernoulli, vol. 25, no. 4A, pp. 2824-2853, Nov. 2019.
[17] R. J. Beerends, Fourier and Laplace transforms. Cambridge, NY, USA: Cambridge University Press, 2003.
[18] O. E. Nielsen, Information and exponential families : in statistical theory. Chichester U.K., NY, USA: John Wiley \& Sons, 2014.
[19] D. Anade, J.-M. Gorce, P. Mary, and S. M. Perlaza, "An upper bound on the error induced by saddlepoint approximations - Applications to information theory," Entropy, vol. 22, no. 6, p. 690, Jun. 2020.
[20] R. Ash, Probability and measure theory. San Diego, CA, USA: Harcourt/Academic Press, 2000.
[21] T. M. Cover, Elements of information theory. Hoboken, N.J, USA: Wiley-Interscience, 2006.

RESEARCH CENTRE
GRENOBLE - RHÔNE-ALPES
Inovallée
655 avenue de l'Europe Montbonnot
38334 Saint Ismier Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105-78153 Le Chesnay Cedex
inria.fr

