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Intersecting fractures in porous media: theoretical and numerical analysis

Laila Amir* Michel Kern^{†‡} Zoubida Mghazli[§] Jean E. Roberts

Abstract

This paper studies a model for flow in a fractured porous medium with intersecting fractures. The fractures are treated as lower-dimensional manifolds along which physical transmission conditions express the pressure jump and the continuity of the flux across the fractures. Specific attention is borne to the conditions when several fractures intersect. The resulting system is discretized with mixed finite element, and the well-posedness of both the continuous and the discrete problems are proved. Then a domain decomposition method is formulated, so that the problem is reduced to the set of fractures, and a simple preconditioner is proposed. Numerical results exemplify the performance of the method.

1 Introduction

Porous media are characterized by their heterogeneity across multiple scales. Fractures are extreme examples of heterogeneity, as they are structures within the porous medium with one dimension (their width) much smaller than the other two-dimensions. However they play an important role in the description of the flow as their permeability may differ greatly from that of the surrounding medium. The permeability in the fractures may be much larger than that of surrounding medium so that the fractures act as hydraulic conductors, providing easy pathways for fluid flow. On the other hand, if the permeability in the fractures is much smaller than that of the surrounding medium, then the fractures act as barriers that prevent flow across them. In this paper, we will be mostly interested in the former case, but the model presented is general.

The combination of the two characteristic features of fractures (small width and large influence on the flow) make the simulation of flow and transport in fractured porous media quite challenging. Accordingly, this topic has been the subject of numerous studies, with several numerical methods being proposed over the past few years.

Originally fractures were included in the flow model by means of so-called multi-continua methods, such as double porosity [23, 24], or stochastic models [21, 20]. These techniques are still in current use, as seen for example in [45, 47, 53, 42]. These methods made the study of large fracture networks feasible without the need for precise geometric description of the geometry of individual fractures.

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The development of advanced discretization techniques and meshing capabilities have enabled numerical methods that will describe the fracture network at the level of individual fractures. The size of the network, broadly characterized by the number of fractures that can be included in the model, is a defining feature of the works. We refer to the recent article [12] for a survey of different possible approaches, and will only quote a subset of the works with which we are most familiar.

One can roughly distinguish between Discrete Fracture Networks (DFNs), where only the fractures are included, and mixed-dimensional models where the interaction between the three-dimensional porous medium and the fractures is taken into account. The former class of models rests on the observation that fractures account for most of the flow, so that neglecting the flow in the porous medium may be justified. On the other hand, DNF models usually accommodate a much larger number of fractures, as the resulting model is intrinsically two-dimensional. [25, 55, 56, 30, 14, 41, 43, 9, 34, 40]

Following [5, 6, 48, 8], mixed-dimensional models treat the fractures as lower dimensional manifolds within the bulk medium. This usually results in a smaller number of fractures, as the whole three-dimensional volume has to be discretized. However, the simulation of such models remains challenging: one needs to choose between a conforming or a non-conforming mesh, and in the latter case, how to express the transmission conditions for pressure and flux across the fractures. A large number of numerical methods have been used [51, 52, 44, 1, 58, 32], among which one finds finite volume methods [4, 3, 27], extended finite elements (XFEM) [22, 29], and mixed finite elements [48, 7, 16, 13]. Models that specifically take into account the behavior of the flow at fracture intersection include [7, 16, 13, 59]. In most of the works cited above, the number of fractures in the domain remains limited, and is in any case much smaller than for DFNs. We note that a fairly general, and quite elegant, formulation for hierarchical problem has been proposed and analyzed in [16] (see also [51, 52]), including the use of non-matching grids and mortars.

The models have been extended to deal with non-linear flow in the fractures (Forchheimer model), see [33, 1] and multiphase flow, see [17, 37, 38, 42, 35, 57]. A posteriori error estimates were studied in [50, 39].

Domain decomposition (DD) methods appear naturally in the simulation of fractured media. The fractures define the subdomains between them, and the physical transmission conditions across the fractures give transmission conditions for the DD method, either of Dirichlet or Neumann type, or of Robin type. DD was first proposed as a possible formulation in [48, 5], was studied in more details in [28] and more recently in [2, 1]. The original mixed-dimensional problem is reduced to one posed only on the fractures, which is somewhat non-standard from the DD viewpoint, as it has to account both for the subdomain solutions, but also for the flow inside the fractures.

Finally, we note that a set of benchmarks for the simulation of flow in fractured media has been proposed in [11].

This paper deals with a model of fractures in a porous medium, and looks at the case where fractures can intersect. The model is concerned with so-called large fractures, where the number of fractures remains small, but their length are significant in comparison with the size of the overall domain. The model is an extension of the one described in [48] (and most of the papers cited above), and was already used in [7], as well as [13] or [16] to which this work is closely related. With respect to this last work, we felt it was still worthwhile to give a fairly direct analysis of the scheme, for conforming grids, as the machinery presented in [16] is quite

demanding.

The problem is discretized by means of mixed finite element. The present paper complements [7] by giving the details for the well posedness both for the continuous and for the discrete problem. In both cases, checking the hypotheses for the Babuska–Brezzi theory involves carefully constructing the solution to well-chosen auxiliary problems. Then a DD method is proposed along the lines of [1] or [2] (the basic idea for the formulation goes back to [5]). The reduction to an interface problem builds a Dirichlet to Neumann map. It was shown in [19] how to build this map for the case of mixed finite element, by using the hybrid formulation. One would then follow the same pattern for the interface problem (this is what is done in [25] for DFNs), however the Dirichlet to Neumann map is global, and the mixed problem can no longer be reduced to a positive definite problem for the multipliers. For this reason, the interface problem is discretized by means of cell centered finite volumes.

The outline for the rest of this paper is as follows: the physical problems, and the underlying geometric setting is described in Section 2. In Section 3, existence and uniqueness for the continuous problem is proved, while Section 4 does the same for the approximate problem. A DD method is described in Section 5 and numerical examples are presented in Section 6.

2 Flow in a porous medium: problem with fractures

We consider the flow of an incompressible fluid inside a domain containing a network of fractures that possibly intersect. We denote by Ω a convex domain in \mathbb{R}^d , $d = 2$ or 3 , and by $\partial\Omega$ its boundary.

Fractures are thin structures in the subsurface that can act either as channels with a higher hydraulic conductivity than that of the porous matrix, or on the contrary as impermeable barriers that prevent flow. As noted in Section 1 we adopt the point of view of representing the fractures as lower dimensional structures. Among the references listed there, see in particular [5, 6, 13, 16, 48, 51, 52, 59].

We assume that Ω is decomposed into a set of subdomains $\Omega_i, i = 1, \dots, N$, such that the boundary of each subdomain Ω_i is made up of the union of planar polygons, that may either be embedded in Ω or be part of the boundary $\partial\Omega$. We denote by $I = \{1, 2, \dots, N\}$ the set of (indexes of) subdomains, and for each $i \in I$, we let

$$S_i = \{j \in I \setminus \{i\} : \Omega_i \cap \Omega_j \neq \emptyset\}$$

be the set of its neighbors. Then,

$$\bar{\Omega} = \bigcup_{i \in I} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset.$$

We define a *fracture* as a face of a subdomain Ω_i that is embedded in Ω . For each fracture there exists exactly two subdomains, say Ω_i and Ω_j , that contain the fracture in their intersection, and the fracture can be denoted by $\bar{\gamma}_{ij}$:

$$\bar{\gamma}_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j \quad \text{for } i \in I, j \in S_i.$$

We note that our definition of fracture is in line with that of Boon et al. [16], but is different from that of Berrone et al. [13] where what we call a fracture is called a face, and a fracture is the union

of several faces. As a consequence of defining fractures as delimiting subdomain boundaries, fractures cannot end “in the middle” of a subdomain, that is a fracture boundary is either part of the domain boundary, or is part of the boundary of another fracture. This assumption may appear restrictive, but it could easily be relaxed by introducing additional planes that support “inactive fractures”, as is done in [13]. One would then have to distinguish between active and inactive fractures, and this only involves a change in the transmission conditions across the fractures.

The set \tilde{S} of all fractures can be identified with $\tilde{S} = \{(i, j) : i \in I, j \in S_i \text{ and } i < j\}$. It will occasionally be useful to count each fracture twice, and this is the purpose of the set $S = \{(i, j) : i \in I, j \in S_i\}$.

Fractures may intersect along edges belonging to two or more subdomains. If σ is such a segment, we denote by $T_\sigma = \{(i, j) \in \tilde{S}, \sigma \subset \bar{\gamma}_{ij}\}$ the set of all fractures (identified by the pair of indexes) containing σ as part of their boundary. We denote by T the set of all intersections:

$$T := \left\{ \sigma / \exists R \subset \tilde{S}, \sigma = \bigcap_{(i,j) \in R} (\partial \gamma_{ij}) \right\}$$

Last, we have for all intersections $\sigma \in T$,

$$\gamma_{ij} = (\bar{\gamma}_{ij} \cap \Omega) \setminus \sigma \quad \text{for } (i, j) \in T_\sigma,$$

and from now on, the fracture will be the open set γ_{ij} . Finally we let $\gamma = \cup_{(i,j) \in \tilde{S}} \gamma_{ij}$ be the set of all fractures.

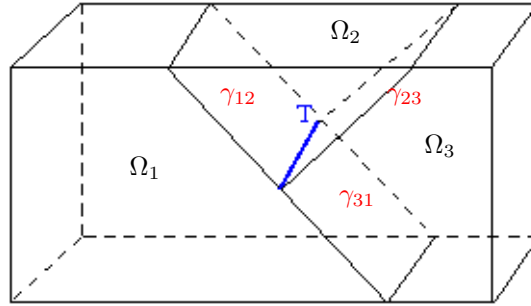


Figure 1: Geometry of domain

For the example depicted on Figure 1, we have three subdomains, so that $I = \{1, 2, 3\}$,

$$S_1 = \{2, 3\}, \quad S_2 = \{1, 3\}, \quad S_3 = \{2, 1\},$$

also

$$\tilde{S} = \{(1, 2), (1, 3), (2, 3)\}, \quad \text{while } S = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}.$$

There is only one intersection, $\sigma = \bar{\gamma}_{12} \cap \bar{\gamma}_{23} \cap \bar{\gamma}_{31}$, so that $T_\sigma = \tilde{S}$. We note that, in order for σ to be an edge of Ω_1 , this subdomain should be considered as a polyhedron with 7 faces (γ_{12} and γ_{31} are considered two different fractures).

We may now write the flow equations in the subdomains and in the fractures, as well as the transmission conditions across the fractures and in the intersections. We do not repeat the

derivation of the model, as this is by now well known. In this work, we rely on the model in [48] that takes into account both conductive and impervious fractures.

For each subdomain Ω_i , we find the usual flow equations (mass conservation and Darcy law):

$$\operatorname{div}(\vec{u}_i) = f_i \quad \text{on } \Omega_i \quad (1)$$

$$\vec{u}_i = -K_i \vec{\nabla} p_i \quad \text{on } \Omega_i \quad (2)$$

where p_i , \vec{u}_i , f_i and K_i are respectively the pressure, the Darcy velocity, the source term, and the permeability tensor.

The conditions on the exterior boundaries of Ω_i can be of either Dirichlet or Neumann types:

$$p_i = p_{Di} := p_d|_{\Gamma_{Di}} \quad \text{in } \Gamma_{Di} \quad (3)$$

$$\vec{u}_i \cdot \vec{\nu}_i = 0 \quad \text{in } \Gamma_{Ni}, \quad (4)$$

where $\Gamma_{Di} = \Gamma_i \cap \Gamma_D$ and $\Gamma_{Ni} = \Gamma_i \cap \Gamma_N$.

We suppose that K_i is a diagonal tensor and that there exists $0 < K_{\min}$ and $K_{\max} < \infty$ such that, for all $i \in I$

$$0 < K_{\min} \|\xi\|_0^2 \leq (K_i \xi, \xi) \leq K_{\max} \|\xi\|_0^2, \quad \forall \xi \in \mathbb{R}^d, \xi \neq 0. \quad (5)$$

At the interfaces $\gamma_{i,j}$, $(i, j) \in \tilde{S}$, instead of imposing the continuity of pressure as in a standard transmission problem, we impose as in [48] the condition

$$p_i = p_{i,j} + \frac{d_{i,j}}{4K_{i,j}^n} (\xi \vec{u}_i \cdot \vec{\nu}_i - (1 - \xi) \vec{u}_j \cdot \vec{\nu}_j) \quad \text{in } \gamma_{i,j}, \quad (6)$$

where $p_{i,j}$ is the pressure in the fracture $\gamma_{i,j}$, $K_{i,j}^n$ is the permeability of the fracture in the normal direction of that one, $d_{i,j}$ is the width of the fracture and ξ is a parameter that can be changed to obtain various alternatives. We suppose that $d_{i,j}$ is constant for all i and j and satisfies

$$0 < d_m \leq d_{i,j} \leq d_M < +\infty \quad (7)$$

Remark 1. Equation (6) takes into account a pressure jump across the fracture. It is obtained by integrating across the fracture in the direction normal to the fracture, cf [48]. Note that, as the width $d_{i,j}$ of the fracture is quite small, if in addition the fracture is highly permeable (in the direction normal to the fracture) i. e. if $K_{i,j}^n$ is sufficiently large, then the second term of the right-hand side of equation (6) is negligible, and equation (6) may be replaced by an equation expressing the continuity of the pressure across the fracture:

$$p_i = p_{i,j} = p_j \quad \text{on } \gamma_{i,j}. \quad (8)$$

This is the model which was studied in [6], and that we have used for the numerical examples in this article.

For the second transmission condition on the interfaces $\gamma_{i,j}$, instead of continuity of flow, one has, as in [6] and [48], a system of equations that models the flow in the fractures. The discontinuity of the flux across a fracture yields a source term in the equations describing the

flow in that fracture:

$$\operatorname{div}_{\gamma_{i,j}}(\vec{u}_{i,j}) = f_{i,j} + (\vec{u}_i \cdot \vec{\nu}_i + \vec{u}_j \cdot \vec{\nu}_j) \quad \text{in } \gamma_{i,j} \quad (9)$$

$$\vec{u}_{i,j} = -d_{i,j} K_{i,j}^\tau \vec{\nabla}_{\gamma_{i,j}} p_{i,j} \quad \text{in } \gamma_{i,j} \quad (10)$$

$$p_{i,j} = p_{d\gamma_{i,j}} \quad \text{on } \partial\gamma_{i,j} \cap \partial\Omega_D \quad (11)$$

$$\vec{u}_{i,j} \cdot \vec{\nu}_{i,j} = 0 \quad \text{on } \partial\gamma_{i,j} \cap \partial\Omega_N, \quad (12)$$

where

- $K_{i,j}^\tau$ is the tangential component of the permeability tensor of the fracture $\gamma_{i,j}$. We assume that K^τ satisfies a coercivity hypothesis similar to (5).
- $p_{d\gamma_{i,j}} \in H^{1/2}(\partial\gamma_{i,j} \cap \partial\Omega_D)$ (see (15) for the definition of that space) is the pressure given on its Dirichlet boundary,
- and $\vec{\nu}_{i,j}$ is the outward-pointing, unit, normal vector field on $\partial\gamma_{i,j}$.

The two operators $\operatorname{div}_\gamma$ and ∇_γ are, respectively, the surface divergence and the surface gradient on γ .

Finally, on each intersection $\sigma \in T$, the continuity of pressure and the continuity of flux are imposed:

$$p_{i,j} = p_\sigma \quad \text{in } \sigma, \quad \forall (i,j) \in T_\sigma \quad (13)$$

$$\sum_{(i,j) \in T_\sigma} \vec{u}_{i,j} \cdot \vec{\nu}_{i,j} = 0 \quad \text{in } \sigma, \quad (14)$$

where p_σ is (unknown) the pressure in σ .

3 The weak formulation of the problem and its well-posedness

Before giving the weak formulation of our problem, let us first recall some functional spaces. If $\mathcal{O} = \Omega$ is an open subset of \mathbb{R}^d ($d = 2, 3$) or $\mathcal{O} = \gamma$ is a $(d - \ell)$ -dimensional manifold in \mathbb{R}^d with $\ell = 1, 2$, we define

$$H^1(\mathcal{O}) := \{v \in L^2(\mathcal{O}) / \nabla_{\mathcal{O}} v \in (L^2(\mathcal{O}))^n, (n = d \text{ or } \ell)\}.$$

where $\nabla_{\mathcal{O}}$ is the gradient with respect to the derivative in Ω or in γ . The space of traces on the boundary of \mathcal{O} of functions in $H^1(\mathcal{O})$ is denoted by $H^{1/2}(\partial\mathcal{O})$, endowed with the norm

$$\|\phi\|_{\frac{1}{2}, \partial\mathcal{O}} = \inf_{\substack{r \in H^1(\mathcal{O}) \\ r|_{\partial\mathcal{O}} = \phi}} \|r\|_{1, \mathcal{O}}, \quad \forall \phi \in H^{1/2}(\partial\mathcal{O}).$$

If δ is a part of $\partial\mathcal{O}$, we denote

$$H^{1/2}(\delta) := \{\phi|_\delta / \phi \in H^{1/2}(\partial\mathcal{O})\}. \quad (15)$$

Let δ_D and δ_N be disjoint parts of $\partial\mathcal{O}$ such that $\partial\mathcal{O} = \bar{\delta}_D \cup \bar{\delta}_N$ and let $E_{N,0} : H^{1/2}(\delta_N) \rightarrow L^2(\partial\mathcal{O})$ be the extension operator defined by (see [36])

$$E_{N,0}(\eta) := \begin{cases} \eta & \text{on } \delta_N \\ 0 & \text{on } \delta_D \end{cases}, \quad \forall \eta \in H^{1/2}(\delta_N).$$

We define

$$H_{00}^{1/2}(\delta_N) := \{v|_{\delta_N} / v \in H^1(\mathcal{O}) \text{ and } v|_{\delta_D} = 0\}.$$

There holds (see [36])

$$H_{00}^{1/2}(\delta_N) = \{\eta \in H^{1/2}(\delta_N) / E_{N,0}(\eta) \in H^{1/2}(\partial\mathcal{O})\},$$

with the norm

$$\|\eta\|_{\frac{1}{2},00,\delta_N} := \|E_{N,0}(\eta)\|_{\frac{1}{2},\partial\mathcal{O}}.$$

We also need the space

$$\mathbf{H}(\text{div}, \mathcal{O}) := \{\vec{v} \in L^2(\mathcal{O})^n / \text{div}_{\mathcal{O}} \vec{v} \in L^2(\mathcal{O})\}, \text{ with } n = d \text{ or } n = d - \ell,$$

where $\text{div}_{\mathcal{O}}$ is the divergence operator with respect to the derivative on \mathcal{O} . The normal traces of functions in $\mathbf{H}(\text{div}, \mathcal{O})$ belong to the space $H^{-1/2}(\partial\mathcal{O})$ defined as the dual space of $H^{1/2}(\partial\mathcal{O})$, and we define $H_{00}^{-1/2}(\Gamma_N)$ as the dual of $H_{00}^{1/2}(\Gamma_N)$. We denote by $\langle \cdot, \cdot \rangle_{00, \frac{1}{2}, \Gamma_N}$ the duality between $H_{00}^{1/2}(\Gamma_N)$ and $H_{00}^{-1/2}(\Gamma_N)$.

3.1 The weak formulation

To give the weak formulation of the problem, we need to define two Hilbert spaces : \mathbf{W} , a space for the velocity and \mathbf{M} , a space for the pressure. Let

$$\begin{aligned} \mathbf{W} &= \prod_{i \in I} W_i \times \prod_{(i,j) \in \tilde{S}} W_{ij}, \\ \mathbf{M}_{\gamma} &= \prod_{i \in I} M_i \times \prod_{(i,j) \in \tilde{S}} M_{ij} \\ \mathbf{M}_{\mathbf{T}} &= \prod_{\sigma \in T} H_{00}^{1/2}(\sigma) \end{aligned} \tag{16}$$

and

$$\mathbf{M} = \mathbf{M}_{\gamma} \times \mathbf{M}_{\mathbf{T}}$$

where for $i \in I$ and $(i, j) \in \tilde{S}$

$$\begin{aligned} W_i &= \{\vec{v}_i \in \mathbf{H}(\text{div}, \Omega_i) / \vec{v}_i \cdot \vec{\nu}_i \in L^2(\gamma_{ij}), \forall j \in S_i \text{ and } \vec{v}_i \cdot \vec{\nu}_i|_{\Gamma_{N_i}} = 0\}, \\ W_{ij} &= \{\vec{v}_{ij} \in \mathbf{H}(\text{div}, \gamma_{ij}) / \vec{v}_{ij} \cdot \vec{\nu}_i|_{\partial\gamma_{ij} \cap \partial\Omega_N} = 0\}, \\ M_i &= L^2(\Omega_i), \\ M_{ij} &= L^2(\gamma_{ij}), \end{aligned}$$

equipped with the norms for $\mathbf{u} \in \mathbf{W}$ and $\mathbf{r} = (\mathbf{r}_{\gamma}, \boldsymbol{\mu}_T) \in \mathbf{M}$

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}}^2 &= \sum_{i \in I} (\|\vec{u}_i\|_{0, \Omega_i}^2 + \|\text{div } \vec{u}_i\|_{0, \Omega_i}^2) + \sum_{(i,j) \in \tilde{S}} (\|\vec{u}_{i,j}\|_{0, \gamma_{ij}}^2 + \|\text{div }_{\gamma_{ij}}(\vec{u}_{i,j})\|_{0, \gamma_{ij}}^2) \\ &\quad + \sum_{(i,j) \in \tilde{S}} \|\vec{u}_i \cdot \vec{\nu}_i + \vec{u}_j \cdot \vec{\nu}_j\|_{0, \gamma_{ij}}^2, \\ \|\mathbf{r}\|_{\mathbf{M}}^2 &= \sum_{i \in I} \|r_i\|_{0, \Omega_i}^2 + \sum_{(i,j) \in \tilde{S}} \|r_{i,j}\|_{0, \gamma_{ij}}^2 + \sum_{\sigma \in T} \|\mu_{\sigma}\|_{00, \frac{1}{2}, \sigma}^2. \end{aligned}$$

To obtain the mixed formulation we define two bilinear forms $\alpha_\xi : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$ and $\beta_T : \mathbf{W} \times \mathbf{M} \times \mathbf{M}_T \rightarrow \mathbb{R}$ by

$$\begin{aligned}\alpha_\xi(\mathbf{u}, \mathbf{v}) &= \sum_{i \in I} \int_{\Omega_i} K_i^{-1} \vec{u}_i \cdot \vec{v}_i + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} (d_{i,j} K_{i,j}^T)^{-1} \vec{u}_{i,j} \cdot \vec{v}_{i,j} \\ &+ \sum_{(i,j) \in S} \int_{\gamma_{i,j}} \frac{d_{i,j}}{4K_{i,j}^n} (\xi \vec{u}_i \cdot \vec{v}_i - (1-\xi) \vec{u}_j \cdot \vec{v}_j) \vec{v}_i \cdot \vec{v}_i \\ \beta_T(\mathbf{v}, \mathbf{r}) &= \sum_{i \in I} \int_{\Omega_i} \operatorname{div} \vec{v}_i r_i + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} (\operatorname{div}_{\gamma_{i,j}} \vec{v}_{i,j} - (\vec{v}_i \cdot \vec{v}_i + \vec{v}_j \cdot \vec{v}_j)) r_{i,j} \\ &- \sum_{\sigma \in T} \left\langle \mu_\sigma, \left(\sum_{(i,j) \in T_\sigma} \vec{v}_{ij} \cdot \nu_{\gamma_{i,j}} \right) \right\rangle_{00, \frac{1}{2}, \sigma},\end{aligned}$$

and let the linear forms $L_f : \mathbf{M} \rightarrow \mathbb{R}$ and $L_d : \mathbf{W} \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned}L_f(\mathbf{r}) &= \sum_{i \in I} \int_{\Omega_i} f_i r_i + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} f_{i,j} r_{i,j} \\ L_d(\mathbf{v}) &= \sum_{i \in I} \int_{\Gamma_{Di}} p_{di} \vec{v}_i \cdot \vec{\nu}_i + \sum_{(i,j) \in \tilde{S}} \int_{\partial \gamma_{i,j} \cap \partial \Omega_D} p_{d\gamma_{ij}} \vec{v}_{i,j} \cdot \vec{\nu}_{i,j}.\end{aligned}$$

With these definitions of spaces and forms, the weak formulation of the problem given by equations (1)–(4), (6), and (9)–(14) can be written as follows:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{W}, \mathbf{p} = (\mathbf{p}_\gamma, \boldsymbol{\lambda}_T) \in \mathbf{M} \text{ such that} \\ \alpha_\xi(\mathbf{u}, \mathbf{v}) - \beta_T(\mathbf{v}, \mathbf{p}) = L_d(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{W}, \\ \beta_T(\mathbf{u}, \mathbf{r}) = L_f(\mathbf{r}), \quad \forall \mathbf{r} \in \mathbf{M}, \end{cases} \quad (17)$$

3.2 Existence and uniqueness of the solution

Let \mathbf{V} be the space defined by

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{W} : \beta_T(\mathbf{v}, \mathbf{r}) = 0, \quad \forall \mathbf{r} \in \mathbf{M}, \}.$$

In the following lemmas we will show that the bilinear form $\alpha_\xi(\cdot, \cdot)$ is \mathbf{V} -elliptic and the bilinear form $\beta_T(\cdot, \cdot)$ satisfies the inf-sup condition.

Lemma 3.1. *If $\xi > \frac{1}{2}$ then the bilinear form $\alpha_\xi(\cdot, \cdot)$ is \mathbf{V} -elliptic.*

Proof. For $\mathbf{v} \in \mathbf{V}$, we have $\beta_T(\mathbf{v}, \mathbf{r}) = 0$ for all $\mathbf{r} \in \mathbf{M}$. By taking successively $\mu_\sigma = 0$, $r_{i,j} = 0$, and $r_i = 0$, for all σ and all i and j , we obtain in the distribution sense that

- $\operatorname{div} \vec{v}_i = 0$ in $L^2(\Omega_i)$, for $1 \leq i \leq 3$,
- $\operatorname{div}_{\gamma_{i,j}} \vec{v}_{i,j} = \vec{v}_i \cdot \vec{\nu}_i + \vec{v}_j \cdot \vec{\nu}_j$ in $L^2(\gamma_{i,j})$, for all $(i, j) \in \tilde{S}$,
- $\sum_{(i,j) \in \tilde{S}} \vec{u}_{i,j} \vec{\nu}_{i,j} = 0$ in $H_{00}^{-1/2}(\sigma)$, for all $\sigma \in T$.

The norm of $\mathbf{v} \in \mathbf{V}$ becomes

$$\|\mathbf{v}\|_{\mathbf{W}}^2 = \sum_{i \in I} \|\vec{v}_i\|_{0, \Omega_i}^2 + \sum_{(i,j) \in \tilde{S}} \left(\|\vec{v}_{i,j}\|_{0, \gamma_{ij}}^2 + 2\|v_i \cdot \nu_i + v_j \cdot \nu_j\|_{0, \gamma_{ij}}^2 \right)$$

and

$$\begin{aligned} \alpha_\xi(\mathbf{v}, \mathbf{v}) &= \sum_{i \in I} \int_{\Omega_i} K_i^{-1} \vec{v}_i \cdot \vec{v}_i + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{ij}} (d_{i,j} K_{i,j}^\tau)^{-1} \vec{v}_{i,j} \cdot \vec{v}_{i,j} \\ &\quad + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} \frac{d_{i,j}}{4K_{i,j}^n} \left[\xi(\vec{v}_i \cdot \vec{v}_i)^2 - 2(1-\xi)(\vec{v}_j \cdot \vec{v}_j)(\vec{v}_i \cdot \vec{v}_i) + \xi(\vec{v}_j \cdot \vec{v}_j)^2 \right]. \end{aligned}$$

By (5) and (7), the two first terms can be bounded from below. The last term define a quadratic form whose eigenvalues are 1 and $2\xi - 1$. We obtain

$$\begin{aligned} \alpha_\xi(\mathbf{v}, \mathbf{v}) &\geq K_{\max}^{-1} \left(\sum_{i \in I} \|\vec{v}_i\|_{0, \Omega_i}^2 + d_m \sum_{(i,j) \in \tilde{S}} \|\vec{v}_{i,j}\|_{0, \gamma_{ij}}^2 \right) \\ &\quad + \frac{d_m}{4} K_{\max}^{-1} \min(1, 2\xi - 1) \sum_{(i,j) \in S} (\|\vec{v}_i \cdot \vec{v}_i\|_{0, \gamma_{ij}}^2 + \|\vec{v}_j \cdot \vec{v}_j\|_{0, \gamma_{ij}}^2) \\ &\geq K_{\max}^{-1} \left[\min(1, d_m) \left(\sum_{i \in I} \|\vec{v}_i\|_{0, \Omega_i}^2 + \sum_{(i,j) \in \tilde{S}} \|\vec{v}_{i,j}\|_{0, \gamma_{ij}}^2 \right) \right. \\ &\quad \left. + \frac{d_m}{2} \min(1, 2\xi - 1) \left(\sum_{(i,j) \in \tilde{S}} \|\vec{v}_i \cdot \vec{v}_i + \vec{v}_j \cdot \vec{v}_j\|_{0, \gamma_{ij}}^2 \right) \right] \\ &\geq C \|\mathbf{v}\|_{\mathbf{W}}^2. \end{aligned}$$

where $C = K_{\max}^{-1} \min\{\min(1, d_m), \frac{d_m}{2} \min(1, 2\xi - 1)\}$ □

Lemma 3.2. *The bilinear form $\beta_T(\cdot, \cdot)$ satisfies the inf-sup condition.*

Proof. To show that β_T satisfies the inf-sup condition, it suffices to show that there is a positive constant C , such that given $\mathbf{r}^0 = (\mathbf{r}_\gamma^0, \boldsymbol{\mu}_T^0) \in \mathbf{M}$, there is an element $\mathbf{v}^0 \in \mathbf{W}$ such that

$$\beta_T(\mathbf{v}^0, \mathbf{r}^0) = \|\mathbf{r}^0\|_{\mathbf{M}}^2$$

$$\|\mathbf{v}^0\|_{\mathbf{W}} \leq C \|\mathbf{r}^0\|_{\mathbf{M}}.$$

For $i \in I$, the boundary of Ω_i is such that $\partial\Omega_i = \gamma_{ik} \cup \gamma_{ji} \cup \Gamma_{Di} \cup \Gamma_{Ni}$ for $k \neq i$ and $j \neq i$, in case we have an intersection of fractures. We will use the following notations: $\gamma_i^1 := \gamma_{ik}$ and $\gamma_i^2 := \gamma_{ji}$.

Given r_i^0 in $L^2(\Omega_i)$, for $i \in I$, let φ_i the variational solution of the auxiliary problem

$$\begin{cases} -\Delta \varphi_i &= r_i^0, & \text{in } \Omega_i \\ \frac{\partial \varphi_i}{\partial \nu_{\partial\Omega_i}} &= 0, & \text{on } \gamma_i^1 \cup \gamma_i^2 \cup \Gamma_{Ni} \\ \varphi_i &= 0, & \text{on } \Gamma_{Di}. \end{cases}$$

By the Lax-Milgram lemma there exists a unique solution $\varphi_i \in H_{0,\Gamma_{D_i}}^1$, and by Poincaré's inequality we have

$$|\varphi_i|_{1,\Omega_i} := \|\nabla\varphi_i\|_{0,\Omega_i} \leq C_i \|r_i\|_{0,\Omega_i}. \quad (18)$$

Let $\vec{v}_i^0 = \nabla\varphi_i$, then $\vec{v}_i^0 \in L^2(\Omega_i)^3$, $\operatorname{div}\vec{v}_i^0 = r_i^0 \in L^2(\Omega_i)$ and so $\vec{v}_i^0 \in \mathbf{H}(\operatorname{div}, \Omega_i)$ and moreover we have $\vec{v}_i^0 \cdot \vec{\nu}_i|_{\gamma_i^\ell} = 0$ for $\ell = 1, 2$ and $\vec{v}_i^0 \cdot \vec{\nu}_i|_{\Gamma_{N_i}} = 0$, which gives on the one hand that $\vec{v}_i^0 \cdot \vec{\nu}_i|_{\gamma_i^\ell} \in L^2(\gamma_i^\ell)$ for $\ell = 1, 2$, and on the other hand $\vec{v}_i^0 \cdot \vec{\nu}_i|_{\Gamma_{N_i}} = 0$, and so \vec{v}_i^0 is in W_i .

To define the remaining components $\vec{v}_{i,j}^0$, $(i, j) \in \tilde{S}$, we use the following results (see [36]). Since, for any $\sigma \in T$, $H_{00}^{1/2}(\sigma)$ is a Hilbert space with the inner product (where γ is one of $\gamma_{i,j}$, for $(i, j) \in T_\sigma$)

$$\langle \xi, \eta \rangle_{00, \frac{1}{2}, \sigma} := \langle E_{N,0}(\xi), E_{N,0}(\eta) \rangle_{\frac{1}{2}, \gamma}, \quad \forall \xi, \eta \in H_{00}^{1/2}(\sigma)$$

where $\langle \cdot, \cdot \rangle_{\frac{1}{2}, \gamma}$ is the inner product in $H^{1/2}(\gamma)$, we can consider the Riesz mapping $\mathcal{R}_{00, \sigma} : H_{00}^{-1/2}(\sigma) \rightarrow H_{00}^{1/2}(\sigma)$.

For each $\sigma \in T$ and $(i, j) \in T_\sigma$, let $\varphi_{i,j} \in H_{0, \partial\gamma_{i,j} \setminus \sigma}^1$ be the solution of

$$\int_{\gamma_{i,j}} \nabla\varphi_{i,j} \cdot \nabla\psi \, dx = \int_{\gamma_{i,j}} r_{i,j}^0 \psi \, dx - \left\langle \frac{1}{|T_\sigma|} \mathcal{R}_{00, \sigma}^{-1}(\mu_\sigma^0); \psi|_\sigma \right\rangle_\sigma, \quad \forall \psi \in H_{0, \partial\gamma_{i,j} \setminus \sigma}^1(\gamma_{i,j}), \quad (19)$$

where $|T_\sigma|$ is the number of elements of T_σ . By the Lax-Milgram lemma, this problem has a unique solution and according to the respective continuous dependence result and the duality between $H_{00}^{-1/2}(\sigma)$ and $H_{00}^{1/2}(\sigma)$ we have

$$|\varphi_{i,j}|_{1, \gamma_{i,j}} \leq C_{ij} \left\{ \|r_{i,j}^0\|_{0, \gamma_{i,j}} + \|\mu_\sigma^0\|_{00, \frac{1}{2}, \sigma} \right\}. \quad (20)$$

Now, let $\vec{v}_{i,j}^0 = -\nabla\varphi_{i,j}$. Then $\vec{v}_{i,j}^0 \in \mathbf{H}(\operatorname{div}_\gamma, \gamma_{i,j})$ since $\operatorname{div}\vec{v}_{i,j}^0 = r_{i,j}^0$, and we have

$$\vec{v}_{i,j}^0 \cdot \vec{\nu}|_\sigma = -\frac{1}{|T_\sigma|} \mathcal{R}_{00, \sigma}^{-1}(\mu_\sigma^0) \quad \text{and} \quad \vec{v}_{i,j}^0 \cdot \vec{\nu}|_{\partial\gamma_{i,j} \cap \Gamma_N} = 0$$

(equivalently $\mathcal{R}_{00, \sigma}(\vec{v}_{i,j}^0 \cdot \vec{\nu})|_\sigma = \frac{1}{|T_\sigma|} \mu_\sigma^0$) so that $\vec{v}_{i,j}^0 \in W_{i,j}$.

$$\begin{aligned} \beta_T(\mathbf{v}^0, \mathbf{r}^0), &= \sum_{i \in I} \int_{\Omega_i} \operatorname{div}\vec{v}_i^0 r_i^0 \, dx + \sum_{(i,j) \in \tilde{S}} \left[\int_{\gamma_{i,j}} (\operatorname{div}_\gamma \vec{v}_{i,j}^0 - (\vec{v}_i^0 \cdot \vec{\nu}_i + \vec{v}_j^0 \cdot \vec{\nu}_j)) r_{i,j}^0 \right] \\ &\quad - \sum_{\sigma \in T} \left\langle \mu_\sigma^0, \left(\sum_{(i,j) \in T_\sigma} \vec{v}_{i,j}^0 \cdot \vec{\nu}_{\gamma_{i,j}} \right) \right\rangle_{00, \frac{1}{2}, \sigma} \\ &= \sum_{i \in I} \int_{\Omega_i} |r_i^0|^2 + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} |r_{i,j}^0|^2 + \sum_{\sigma \in T} \langle \mu_\sigma^0; \mathcal{R}_{00, \sigma}^{-1}(\mu_\sigma^0) \rangle_{00, \frac{1}{2}, \sigma} \\ &= \sum_{i \in I} \|r_i^0\|_{0, \Omega_i}^2 + \sum_{(i,j) \in \tilde{S}} \|r_{i,j}^0\|_{0, \gamma_{i,j}}^2 + \sum_{\sigma \in T} \|\mu_\sigma^0\|_{00, \frac{1}{2}, \sigma}^2 \\ &= \|\mathbf{r}_\gamma^0\|_{\mathbf{M}_\gamma}^2 + \|\boldsymbol{\mu}_T^0\|_{\mathbf{M}_T}^2 \\ &= \|\mathbf{r}^0\|_{\mathbf{M}}^2. \end{aligned}$$

Also,

$$\begin{aligned}
\|\mathbf{v}^0\|_{\mathbf{W}}^2 &= \sum_{i \in I} (\|\bar{v}_i^0\|_{0,\Omega_i}^2 + \|\operatorname{div} \bar{v}_i^0\|_{0,\Omega_i}^2) + \sum_{(i,j) \in \tilde{S}} \|(\bar{v}_i^0 \cdot \bar{v}_i + \bar{v}_j^0 \cdot \bar{v}_j)\|_{0,\gamma_{ij}}^2 \\
&\quad + \sum_{(i,j) \in \tilde{S}} \left(\|\bar{v}_{i,j}^0\|_{0,\gamma_{ij}}^2 + \|\operatorname{div}_{\gamma_{ij}}(\bar{v}_{i,j}^0)\|_{0,\gamma_{ij}}^2 \right) \\
&= \sum_{i \in I} (|\varphi_i|_{1,\Omega_i}^2 + \|r_i^0\|_{0,\Omega_i}^2) + \sum_{(i,j) \in \tilde{S}} (|\varphi_{ij}|_{1,\gamma_{ij}}^2 + \|r_{i,j}^0\|_{0,\gamma_{ij}}^2).
\end{aligned}$$

By inequalities (18) and (20) we obtain

$$\begin{aligned}
\|\mathbf{v}^0\|_{\mathbf{W}}^2 &\leq (1 + 3C_0^2) \sum_{i \in I} \|r_i^0\|_{0,\Omega_i}^2 + \sum_{(i,j) \in \tilde{S}} (2 + C_{ij}^2) \|r_{i,j}^0\|_{0,\gamma_{ij}}^2 + \left(\sum_{(i,j) \in \tilde{S}} C_{ij}^2 \right) \sum_{\sigma \in T} \|\mu_\sigma\|_{00,\frac{1}{2},\sigma}^2 \\
&\leq \mathbf{C} \left(\|\mathbf{r}_\gamma^0\|_{\mathbf{M}_\gamma}^2 + \|\boldsymbol{\mu}_T^0\|_{\mathbf{M}_T}^2 \right), \\
&\leq \mathbf{C} \|\mathbf{r}^0\|_{\mathbf{M}}^2.
\end{aligned}$$

where $\mathbf{C} = \max \left\{ (1 + 3C_0^2), (2 + C_{ij}^2), \left(\sum_{(i,j) \in \tilde{S}} C_{ij}^2 \right) \right\}$. We deduce

$$\sup_{\mathbf{r} \in \mathbf{M}} \frac{\beta_T(\mathbf{v}, \mathbf{r})}{\|\mathbf{v}\|_{\mathbf{W}}} \geq \frac{\beta_T(\mathbf{v}^0, \mathbf{r}^0)}{\|\mathbf{v}^0\|_{\mathbf{W}}} = \frac{\|\mathbf{r}^0\|_{\mathbf{M}}^2}{\|\mathbf{v}^0\|_{\mathbf{W}}} \geq \frac{1}{\sqrt{\mathbf{C}}} \|\mathbf{r}^0\|_{\mathbf{M}}$$

which is the inf-sup condition. \square

From Lemmas 3.1 and 3.2, we immediately obtain the existence and uniqueness result.

Theorem 3.3. *Assume that the permeabilities K_i on the subdomains Ω_i , the permeabilities $K_{i,j}^\tau$ in the fractures $\gamma_{i,j}$ and the thickness $d_{i,j}$ are bounded above and below by a positive constants K_{\min} and K_{\max} , and assume also that $\xi > \frac{1}{2}$. Then the model problem (\mathcal{P}) has a unique solution.*

To conclude this section, we briefly discuss a possible alternative formulation. To keep the discussion simple, we only consider the case of a single intersection σ .

Remark 2. *The continuity on σ of the normal traces condition (14) was imposed in the equations through a Lagrange multiplier λ_σ that represents a pressure on σ . As a minimum of regularity ($H_{00}^{-1/2}$) for these normal traces has been considered, the multiplier, which is in the dual space, must be more regular ($H_{00}^{1/2}$). If we now want, for the approximation reasons for example, less regularity on λ_σ , we need to impose more regularity on the normal traces. We can then suppose that the sum of the normal traces is in $L^2(\sigma)$, and introduce this condition in the functional space. We then replace the space W_{ij} by*

$$W_{ij} = \left\{ \bar{v}_{ij} \in \mathbf{H}(\operatorname{div}, \gamma_{ij}) / (\bar{v}_{12} \cdot \bar{v}_{12} + \bar{v}_{23} \cdot \bar{v}_{23} + \bar{v}_{13} \cdot \bar{v}_{13})|_{\sigma} \in L^2(\sigma) / \bar{v}_i \cdot \bar{v}_i|_{\partial\gamma_{ij} \cap \partial\Omega_N} = 0 \right\} \quad (21)$$

and require the pressure to be only in $L^2(\sigma)$

$$M_T := \prod_{\sigma \in T} L^2(\sigma). \quad (22)$$

In the definition of $\beta_T(\cdot, \cdot)$, the duality bracket becomes an integral:

$$\left\langle \mu_\sigma ; \left(\sum_{(i,j) \in T_\sigma} \vec{v}_{ij} \cdot \vec{\nu}_{\gamma_{i,j}} \right) \right\rangle_{00, \frac{1}{2}, \sigma} = \int_\sigma \mu_\sigma \left(\sum_{(i,j) \in T_\sigma} \vec{v}_{ij} \cdot \vec{\nu}_{\gamma_{i,j}} \right) ds.$$

The proof of the existence theorem 3.3 goes through with the L^2 -norm instead of $H_{00}^{1/2}$ or $H_{00}^{-1/2}$ norms.

4 Approximation by mixed finite element

In this section we present the discrete setting for approximating problem (17).

We start with meshes on the fractures, and we consider meshes on the subdomains that are conforming across the fractures, that is, given subdomains Ω_i and Ω_j with a common fracture γ_{ij} , the trace of the meshes of Ω_i and Ω_j on γ_{ij} coincide, and they coincide with the mesh on γ_{ij} . Additionally, the meshes of the fractures are assumed to match along the intersections. We denote the mesh on subdomain Ω_i by \mathcal{T}_{ih} and by \mathcal{E}_{ih} the set of faces of \mathcal{T}_{ih} . Similarly, the mesh on a fracture γ_{ij} is denoted by \mathcal{T}_{ijh} and the set of edges of \mathcal{T}_{ijh} is \mathcal{E}_{ijh} . Each intersection σ inherits a mesh $\mathcal{T}_{\sigma h}$ from the fractures that contain it.

To approximate the problem, the Raviart-Thomas elements of lowest order are used. For numerical purposes, it will be more convenient to use the formulation given in Remark 2, so that we only work with functions, and not dual spaces. The approximation spaces $M_{h,i}$, $W_{h,i}$, $M_{h,i,j}$ and $W_{h,i,j}$ are subspaces respectively of $L^2(\Omega_i)$, $\mathbf{H}(\text{div}, \Omega_i)$, $L^2(\gamma_{i,j})$ and $\mathbf{H}(\text{div}_{\gamma_{i,j}}, \gamma_{i,j})$ such that $(M_{h,i}, W_{h,i})$ is the lowest order Raviart-Thomas-Nédélec space on Ω_i , and $(M_{h,i,j}, W_{h,i,j})$ is the lowest order Raviart-Thomas space on $\gamma_{i,j}$ (see [36, Chap. 3] or [15, Sec. 2.3] for the precise definitions). The subspace $M_{h,\sigma}$ of $L^2(\sigma)$ is composed of functions that are constant on each cell of the mesh. The approximation spaces \mathbf{W}_h and \mathbf{M}_h , are the finite dimension subspaces of \mathbf{W} and \mathbf{M} defined by

$$\begin{aligned} \mathbf{W}_h &= \prod_{i \in I} W_{h,i} \times \prod_{(i,j) \in \tilde{S}} W_{h,i,j} \subset \mathbf{W}, \\ \mathbf{M}_h &= \prod_{i \in I} M_{h,i} \times \prod_{(i,j) \in \tilde{S}} M_{h,i,j} \times \prod_{\sigma \in T} M_{h,\sigma} \subset \mathbf{M}. \end{aligned}$$

The global discrete problem is written as:

$$\begin{aligned} &\text{Find } (\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{W}_h \times \mathbf{M}_h \text{ such that} \\ (\mathcal{P}_h) \quad &\alpha_\xi(\mathbf{u}_h, \mathbf{v}) - \beta_T(\mathbf{v}, \mathbf{p}_h) = -L_d(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{W}_h \\ &\beta_T(\mathbf{u}_h, \mathbf{r}) = L_f(\mathbf{r}), \quad \forall \mathbf{r} \in \mathbf{M}_h. \end{aligned}$$

To show that the discrete problem (\mathcal{P}_h) is well posed, and to obtain error estimates, it suffices to show the analogs of Lemma 3.1 and 3.2, and to show that the constants in the inequalities can be chosen to be independent of h .

First we show that $\alpha_\xi : \mathbf{W}_h \times \mathbf{W}_h \rightarrow \mathbb{R}$ is uniformly \mathbf{V}_h -elliptic, where

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{W}_h : \beta_T(\mathbf{v}_h, \mathbf{r}_h) = 0, \quad \forall \mathbf{r}_h \in \mathbf{M}_h\}, \quad (23)$$

and that the ellipticity constant can be chosen independently of h .

Generally $V_h \not\subset V$, but if the inclusion holds, then uniform ellipticity of $\alpha_\xi(\cdot, \cdot)$ follows. We can prove this inclusion by building an operator $P_h = (P_{\gamma h}, P_{Th}) : \mathbf{M}_\gamma \times M_T \longrightarrow \mathbf{M}_{\gamma h} \times M_{Th}$ such that, for all $\mathbf{r} = (\mathbf{r}_\gamma, \mu_T) \in \mathbf{M}$ (cf [31, Lemma 4.2], or [15, Prop. 5.4.2])

$$\beta_T(\mathbf{v}_h, \mathbf{r} - P_h \mathbf{r}) = 0, \quad \forall \mathbf{v}_h \in \mathbf{W}_h. \quad (24)$$

We take for $P_{\gamma h}$ the projection operator $P_{\gamma h} = \left((P_{ih})_{i \in I}, (P_{ijh})_{(i,j) \in \tilde{S}} \right) : \mathbf{M}_\gamma \longrightarrow \mathbf{M}_{\gamma h}$ defined by the following degrees of freedom

for $i \in I$

$$\int_K (r_i - P_{ih}(r_i)) dx_K = 0, \quad \forall K \in \mathcal{T}_{ih}; \quad (25)$$

for $(i, j) \in \tilde{S}$

$$\int_E (r_{ij} - P_{ijh}(r_{ij})) dx_E = 0, \quad \forall E \in \mathcal{T}_{ijh}. \quad (26)$$

Similarly, the projection operator $P_{Th} = (P_{\sigma h})_{\sigma \in T} : M_T \longrightarrow M_{Th}$ is defined by

for $\sigma \in T$

$$\int_e (\mu_\sigma - P_{\sigma h}(\mu_\sigma)) ds_e = 0, \quad \forall e \in \mathcal{T}_{\sigma h}. \quad (27)$$

Let $\mathbf{r} = (\mathbf{r}_\gamma, \mu_T)$ be given in \mathbf{M} . We have, for all $\mathbf{v}_h \in \mathbf{W}_h$

$$\begin{aligned} \beta_T(\mathbf{v}_h, \mathbf{r} - P_h \mathbf{r}) &= \sum_{i \in I} \int_{\Omega_i} \operatorname{div} \vec{v}_i (r_i - P_{ih}(r_i)) \\ &\quad + \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} (\operatorname{div}_{\gamma_{i,j}} \vec{v}_{i,j} - (\vec{v}_i \cdot \vec{\nu}_i + \vec{v}_j \cdot \vec{\nu}_j)) (r_{i,j} - P_{ijh}(r_{i,j})) \\ &\quad - \sum_{\sigma \in T} \int_\sigma (\mu_\sigma - P_{\sigma h}(\mu_\sigma)) \left(\sum_{(i,j) \in T_\sigma} v_{ij} \cdot \nu_{\gamma_{i,j}} \right) \\ &= \sum_{i \in I} \sum_{K \in \mathcal{T}_{ih}} \int_K \operatorname{div} \vec{v}_i (r_i - P_{ih}(r_i)) \\ &\quad + \sum_{(i,j) \in \tilde{S}} \sum_{E \in \mathcal{T}_{ijh}} \int_E (\operatorname{div}_{\gamma_{i,j}} \vec{v}_{i,j} - (\vec{v}_i \cdot \vec{\nu}_i + \vec{v}_j \cdot \vec{\nu}_j)) (r_{i,j} - P_{ijh}(r_{i,j})) \\ &\quad - \sum_{e \in \mathcal{T}_{\sigma h}} \int_e (\mu_\sigma - P_{\sigma h}(\mu_\sigma)) \left(\sum_{(i,j) \in T_\sigma} \vec{v}_{ij} \cdot \vec{\nu}_{\gamma_{i,j}} \right). \end{aligned}$$

Now, for $\mathbf{v} \in \mathbf{W}_h$, we know that the Raviart-Thomas elements satisfy $\operatorname{div} \vec{v}_i$ is a constant in each $K \in \mathcal{T}_{ih}$; $\vec{v}_i \cdot \vec{\nu}_i$ is a constant in each $E \in \mathcal{E}_{ih}$, in particular on the elements of \mathcal{T}_{ijh} for some j ; $\operatorname{div}_{\gamma_{i,j}} \vec{v}_{i,j}$ is a constant in each $E \in \mathcal{T}_{ijh}$; $\vec{v}_{ij} \cdot \vec{\nu}_{\gamma_{i,j}}$ is a constant in each $e \in \mathcal{E}_{ijh}$ in particular on the elements of \mathcal{T}_{Th} . Then (24) is an immediate consequence of (25), (26) and (27), and we obtain the uniform \mathbf{V} -ellipticity of $\alpha_\xi(\cdot, \cdot)$.

To prove that the discrete inf-sup condition holds uniformly we use a characterization analogous to the last property. If there exists an operator Π_h such that, for all $\mathbf{v} \in \mathbf{W}$

$$\beta_T(\mathbf{v} - \Pi_h(\mathbf{v}), \mathbf{r}) = 0, \quad \forall \mathbf{r} \in \mathbf{M}_h \quad (28)$$

and if there exists a constant $C > 0$ independent of h such that

$$\|\Pi_h(\mathbf{v})\|_{\mathbf{W}} \leq C \|\mathbf{v}\|_{\mathbf{W}}, \quad \forall \mathbf{v} \in \mathbf{W} \quad (29)$$

then the discrete inf-sup condition is satisfied (cf. [31, Prop. 4.2], or [15, Prop. 5.4.2]).

Additional regularity assumptions are required in order to define the interpolation operator associated with \mathbf{W} . If $s > 2$ and we denote $\mathbf{W}^s := \mathbf{W} \cap \prod_{i \in I} L^s(\Omega_i) \cap \prod_{(i,j) \in \tilde{S}} L^s(\gamma_{ij})$, then one may define (see [18, 15])

$$\Pi_h := \left((\Pi_{ih})_{i \in I}, (\Pi_{ijh})_{(i,j) \in \tilde{S}} \right) : \mathbf{W}^s \longrightarrow \mathbf{W}_h.$$

by the following degrees of freedom

for $i \in I$

$$\int_E (\vec{v}_i - \Pi_{ih}(\vec{v}_i)) \cdot \vec{\nu}_E ds_E = 0, \quad \forall E \in \mathcal{E}_{ih} \quad (30)$$

for $(i, j) \in \tilde{S}$

$$\int_e (\vec{v}_{ij} - \Pi_{ijh}(\vec{v}_{ij})) \cdot \vec{\nu}_e ds_e = 0, \quad \forall e \in \mathcal{E}_{ijh}. \quad (31)$$

and the stability condition (29) holds for $\mathbf{v} \in \mathbf{W}^s$.

Let $\mathbf{v} \in \mathbf{W}$ be given. For all $\mathbf{r} \in \mathbf{M}_h$, we have

$$\begin{aligned} \beta_T(\mathbf{v} - \Pi_h(\mathbf{v}), \mathbf{r}) &= \sum_{i \in I} \int_{\Omega_i} \operatorname{div}(\vec{v}_i - \Pi_{ih}(\vec{v}_i)) r_i \\ &+ \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} (\operatorname{div}_{\gamma_{i,j}}(\vec{v}_{i,j} - \Pi_{ijh}(\vec{v}_{ij}))) r_{i,j} \\ &- \sum_{(i,j) \in \tilde{S}} \int_{\gamma_{i,j}} [(\vec{v}_i - \Pi_{ih}(\vec{v}_i)) \cdot \vec{\nu}_i + (\vec{v}_j - \Pi_{jh}(\vec{v}_j)) \cdot \vec{\nu}_j] r_{i,j} \\ &- \sum_{\sigma \in T} \int_{\sigma} \mu_{\sigma} \left(\sum_{(i,j) \in T_{\sigma}} (\vec{v}_{ij} - \Pi_{ijh}(\vec{v}_{ij})) \cdot \vec{\nu}_{\gamma_{i,j}} \right). \end{aligned}$$

By decomposing the integrals over the meshes, we obtain

$$\begin{aligned} \beta_T(\mathbf{v} - \Pi_h(\mathbf{v}), \mathbf{r}) &= \sum_{i \in I} \sum_{K \in \mathcal{T}_{ih}} \int_K \operatorname{div}(\vec{v}_i - \Pi_{ih}(\vec{v}_i)) r_i \\ &+ \sum_{(i,j) \in \tilde{S}} \sum_{E \in \mathcal{T}_{ijh}} \int_E (\operatorname{div}_{\gamma_{i,j}}(\vec{v}_{i,j} - \Pi_{ijh}(\vec{v}_{ij}))) r_{i,j} \\ &- \sum_{(i,j) \in \tilde{S}} \sum_{E \in \mathcal{T}_{ijh}} \int_E [(\vec{v}_i - \Pi_{ih}(\vec{v}_i)) \cdot \vec{\nu}_i + (\vec{v}_j - \Pi_{jh}(\vec{v}_j)) \cdot \vec{\nu}_j] r_{i,j} \\ &- \sum_{\sigma \in T} \sum_{e \in \mathcal{T}_{\sigma h}} \int_e \mu_{\sigma} \left(\sum_{(i,j) \in T_{\sigma}} (\vec{v}_{ij} - \Pi_{ijh}(\vec{v}_{ij})) \cdot \vec{\nu}_{\gamma_{i,j}} \right). \end{aligned}$$

Notice that the last two terms in the expression of $\beta_T(\mathbf{v} - \Pi_h(\mathbf{v}), \mathbf{r})$ vanish since r_{ij} is constant in E and μ_σ is constant over e . On the other hand, since r_i is constant in each $K \in \mathcal{T}_{ih}$ an integration by part and (30) give

$$\int_K \operatorname{div}(\vec{v}_i - \Pi_{ih}(\vec{v}_i)) r_i = \int_{\partial K} (\vec{v}_i - \Pi_{ih}(\vec{v}_i)) \cdot \nu_{ij} r_i = \sum_{E \subset \partial K} \int_E (\vec{v}_i - \Pi_{ih}(\vec{v}_i)) \cdot \vec{\nu}_{ij} r_i = 0.$$

Similarly, since $r_{i,j}$ is constant in each $E \in \mathcal{T}_{ijh}$ we have by integration by part and (31)

$$\int_E (\operatorname{div}_{\gamma_{i,j}}(\vec{v}_{i,j} - \Pi_{ijh}(\vec{v}_{ij}))) r_{i,j} = \sum_{e \in \partial E} \int_e (\vec{v}_{i,j} - \Pi_{ijh}(\vec{v}_{ij})) \cdot \vec{\nu}_{ij} r_{i,j} = 0.$$

This shows that (28) holds together with (29) establishes the discrete inf-sup.

The discrete inf-sup condition and the \mathbf{V}_h -ellipticity give the existence and uniqueness of the solution and we have an optimal error estimate (see [18])

5 Domain decomposition and reduction to an interface problem

In this section we show that we can use domain decomposition techniques to eliminate all volume unknowns and to reduce the problem to one set on the fractures.

We will only consider a simplified setup, with only one intersection, so that $T = \sigma$. Also, we consider only the model of very permeable fractures with continuity of the pressure, and thus replace equation (6) by (8). This hypothesis had already been used in [48] for introducing a domain decomposition method in a medium with a single fracture. Formally, it corresponds to the limit $K_{i,j}^n \rightarrow \infty$. We also assume that all fractures have only Dirichlet boundary conditions

A Neumann-Neumann domain decomposition method for mixed finite element has been presented and analyzed in [19] (see also [54]). In that work it was shown that using the mixed hybrid formulation led to a positive definite Dirichlet to Neumann map (or Schur complement at the discrete level). We follow this approach to eliminate the volume unknowns in the subdomains and reduce the problem to the fractures. In order to obtain a positive definite system, we then solve the resulting problem by a cell-centered finite volume scheme as already hinted at in [48] and detailed in [2]. By doing this, we see that the pressure traces on the boundary for the problem in the subdomains coincide with the fracture cell pressure unknowns.

To define the Dirichlet to Neumann map, we use linearity to decompose, for each subdomain Ω_i , $i \in I$, the solution of problem (1), (2) with boundary conditions (3), (4) and (8) as $\vec{u}_i = \vec{u}_i^0 + \vec{u}_i^*$ and $p_i = p_i^0 + p_i^*$, with (\vec{u}_i^0, p_i^0) and (\vec{u}_i^*, p_i^*) the respective solutions of

$$\begin{array}{llll} \operatorname{div} \vec{u}_i^0 = 0 & \text{on } \Omega_i & \operatorname{div} \vec{u}_i^* = f_i & \text{on } \Omega_i \\ \vec{u}_i^0 = -K_i \vec{\nabla} p_i^0 & \text{on } \Omega_i & \vec{u}_i^* = -K_i \vec{\nabla} p_i^* & \text{on } \Omega_i \\ p_i^0 = 0 & \text{in } \Gamma_{Di} & p_i^* = p_{di} & \text{in } \Gamma_{Di} \\ \vec{u}_i^0 \cdot \vec{\nu}_i = 0 & \text{in } \Gamma_{Ni} & \vec{u}_i^* \cdot \vec{\nu}_i = 0 & \text{in } \Gamma_{Ni} \\ p_i^0 = p_{i,j} & \text{in } \gamma_{i,j} & p_i^* = 0 & \text{in } \gamma_{i,j}. \end{array} \quad (\mathcal{P}_i^0) \qquad (\mathcal{P}_i^*)$$

and we let $\chi_i = (\vec{u}_i^* \cdot \vec{\nu}_i)|_{\gamma_i}$.

We now define the Dirichlet to Neumann map (aka Steklov–Poincaré operator) by

$$S_i : \quad \bigcup_{j \in S_i} M_{hij} \rightarrow \bigcup_{j \in S_i} \mathbf{M}_{hij} \\ \lambda_i := (\lambda_{ij})_{j \in S_i} \rightarrow (\vec{u}_i^0 \cdot \vec{\nu}_{ij})|_{\gamma_{ij}}$$

where \bar{u}_i^0 is the solution of problem (\mathcal{P}_i^0) with boundary data $p_{ij} = \lambda_{ij}$.

Then for each fracture γ_{ij} , $(i, j) \in \tilde{S}$, we can rewrite equation (9) as

$$S_i(\lambda_{ij}) + S_j(\lambda_{ij}) - \operatorname{div}_{\gamma_{i,j}} \vec{u}_{i,j} = f_{i,j} + \chi_i + \chi_j \quad \text{on } \gamma_{i,j} \quad (32)$$

together with Darcy's law (10), the boundary conditions

$$\begin{aligned} \lambda_{ij} &= p_{d\gamma} & \text{in } \partial\gamma \cap \partial\Omega \\ \lambda_{i,j} &= \lambda_\sigma & \text{in } \sigma, \quad i \in I, j \in S_i \end{aligned}$$

and condition (14) over the intersection σ .

We define the global Steklov–Poincaré operator by extending λ_i by 0 to γ γ_i , and we do the same for χ_i :

$$\mathcal{S}(\lambda) = \sum_I \mathcal{S}_i(\lambda) \quad \text{and} \quad \chi = \sum_I \chi_i.$$

We note that because the Steklov–Poincaré operators are global, it is not possible to use a hybrid formulation to reduce the problem to the set of edges as was done in [25, 55]. Indeed, one can no longer eliminate locally the cell unknowns in Darcy's law. For this reason, we return to the primal formulation of the problem over the fractures. One could keep the pressure–velocity formulation with mixed finite element, as is done in [1], at the cost of losing the positive definite character of the problem, and then the resulting system can be solved by GMRES.

We write the system over the global interface:

$$\begin{aligned} S(\lambda) - \operatorname{div}_\gamma \left(d_\gamma K_\gamma^\tau \vec{\nabla}_\gamma \lambda \right) &= f_\gamma + \chi & \text{in } \gamma \\ \lambda &= p_{d\gamma} & \text{in } \partial\gamma \cap \partial\Omega \\ \lambda_{i,j} - \lambda_\sigma &= 0 & \text{in } \sigma, \quad (i, j) \in \tilde{S} \\ \sum_{(i,j) \in \tilde{S}} \left(-d_{i,j} K_{i,j}^\tau \vec{\nabla}_\gamma \lambda_{i,j} \right)_{|\sigma} \cdot \nu_{i,j} &= 0 & \text{in } \sigma, \end{aligned} \quad (\mathcal{P}_\gamma^*)$$

where d_γ , K_γ^τ and f_γ are defined in an obvious way.

System (\mathcal{P}_γ^*) is equivalent to the original problem (17), but involves only the unknowns on the fractures. In comparison with domain decomposition problem, system (\mathcal{P}_γ^*) features the additional term $-\operatorname{div}_\gamma \left(d_\gamma K_\gamma^\tau \vec{\nabla}_\gamma \lambda \right)$ that takes into account the flow in the fractures. The Flow in the intersection is represented by the last equation in (\mathcal{P}_γ^*) .

We discretize the reduced system by a cell-centered finite volume method (see [2] or [26]).

For each fracture γ_{ij} , $(i, j) \in \tilde{S}$, we write the balance equation for each cell $K \in \mathcal{T}_{ijh}$

$$S(\lambda)|_K + \sum_{L \in \mathcal{N}(K)} \tau_{KL} (\lambda_K - \lambda_L) + \sum_{\substack{e \subset K \\ e \in \mathcal{E}_{ijh} \cap \partial\gamma_{ij}}} \tau_{K,e} (\lambda_K - p_{d\gamma_{i,j}}) + \sum_{e \in \mathcal{T}_{\sigma h}} \tau_{K,e} (\lambda_K - p_{\sigma,e}) = f_K, \quad (33)$$

where τ_{KL} and $\tau_{K,e}$ the transmissivities across the edge between cells K and L and at the boundary, and where it is understood that the second sum or third sums may be empty for interior cells. We emphasize that there is only *one* unknown on the intersection, expressing the pressure continuity in equation (13).

We also write an equation for the continuity of the flux across the intersection σ :

$$\sum_{e \in \mathcal{T}_{\sigma h}} \sum_{K \supset e} \tau_{K,\sigma} (\lambda_K - \lambda_{\sigma,e}) = 0, \quad (34)$$

where the inner sum is over all the fractures that contain the intersection.

The system formed by equations (33) and (34) is a square system where the unknowns are one pressure unknown for each cell, and one for each edge in the intersection. The resulting matrix is symmetric and positive definite. Indeed, the Steklov–Poincaré operator is positive semi-definite, and this is also the case for the part coming from the finite volume, as one can be by the usual discrete integration by part technique. Last, the Dirichlet boundary conditions make the whole system definite.

The system can be solved by a conjugate gradient method. At each iteration, evaluation of the operator on a vector λ requires the solution of a Dirichlet problem in each subdomain, given the values on the fractures. The pressure unknowns on the fracture are the values of the pressure traces at the center of the faces of the volume mesh for each subdomain.

A preconditioner is required to accelerate the convergence of the conjugate gradient algorithm. The Neumann–Neumann preconditioner and its two–level variants (see [46, 60, 49] for non-overlapping domain decomposition methods. However in this case, it is likely that a simpler preconditioner could be more effective: if we compare the differential order of the terms in (\mathcal{P}_γ^*) , we see that the divergence term is second order, while the Steklov–Poincaré operator are first order operators¹. So, unless the value of dK_γ^τ is very small, we should expect that a preconditioner based on inverting the differential part of the operator should be efficient. This is confirmed experimentally in Section 6.2.

6 Numerical results

6.1 Description the test cases

In this part we present some numerical results to illustrate the properties of our model. We consider a set of tests that highlight the effect of the fractures on the performance of the domain decomposition algorithm. In all the numerical experiments, we consider a domain Ω represented in Figure 2 that contains in addition of the 3 fractures of the problem model (figure 1) a single isolated fracture.

Three tests cases are considered. In all three cases, we impose a pressure equal to 1 on top of the fractures and a pressure equal to 0 on the bottom of the fractures. The permeability of the bulk will be taken equal to 1, while that of the fractures is larger, as we only consider the model with permeable fractures.

6.1.1 Horizontal flow

In the first experiment, a pressure gradient is imposed from left to right ($p = 1$ on the left boundary, $p = 0$ on the right, and no flow on the other boundaries). We show the influence of the fractures by comparing the results obtained without any fractures, and then with the fractures as in Figure 2. In the first case, the domain decomposition method becomes the usual

¹The authors are indebted to Y. Achdou for this remark.

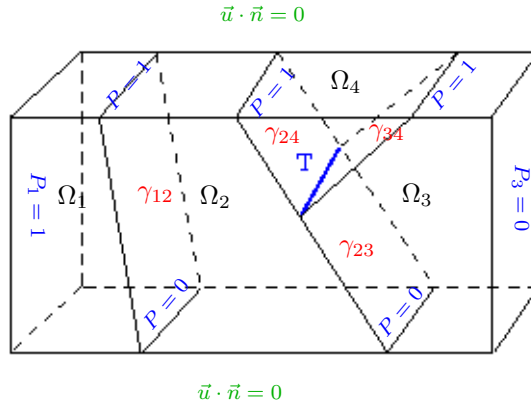


Figure 2: Geometry and boundary conditions for the first case test

one. Then boundary conditions on the fractures, are Dirichlet conditions, with $p = 1$ at the top and $p = 0$ at the bottom.

The permeability in the fracture multiplied by its thickness is equal to 100, which is greater than 1 the permeability in subdomains. This means that the fluid tends to flow rapidly along the fracture.

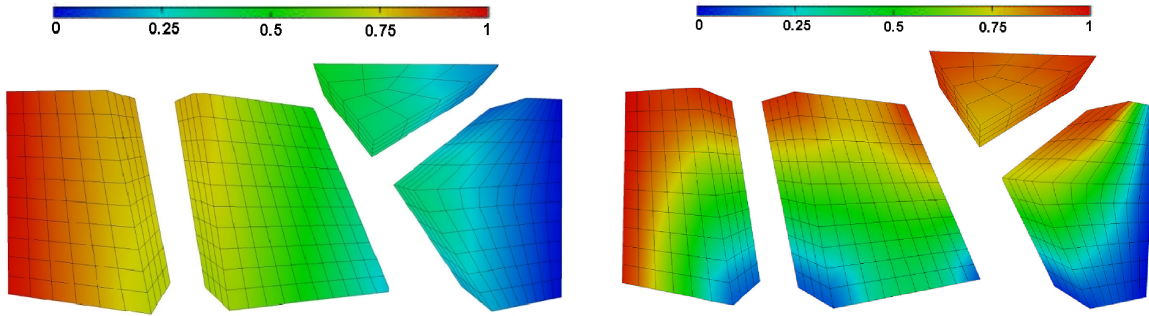


Figure 3: Horizontal flow. Pressure field obtained without (left) and with (right) the fractures

Figure 3 on the left represents the pressure obtained when no fractures are present (we represented each subdomains separately). The flow is only horizontal, the pressure varies linearly from the left to the right. If we take into account the fractures, the situation change, as shown on Figure 3 on the right. In addition to the horizontal flow, we now see a visible downward effect for the flow in the fractures.

6.1.2 Flow due to the fractures

In the second case test, we impose a pressure equal to 1 on the both sides of Ω . Without fractures the pressure would be constant, but Figure 4 shows a significant effect due to the fractures. By comparing this figure and the figure to the right of Figure 3 in Section 6.1.1, one also sees the difference due to the boundary condition imposed on the right.

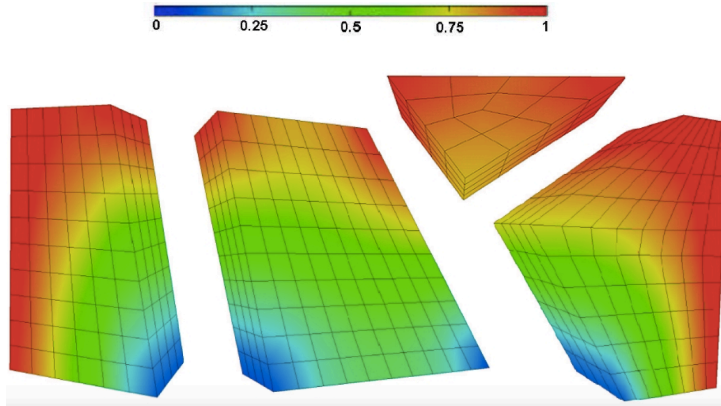


Figure 4: The flow is only due to the pressure difference between the top and bottom of the fractures

6.1.3 Vertical flow

For the third test we impose a vertical gradient of pressure. Figure 5 represents the geometry

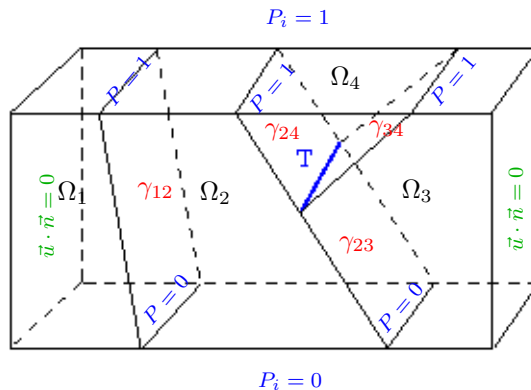


Figure 5: Boundary conditions for the third test case

and the boundary conditions for this test case: we impose an vertical gradient of the pressure, $p = 1$ on the top boundary, $p = 0$ on the bottom boundary and a zero flux condition on the other boundaries.

Figure 6 compares the pressure fields obtained without the fractures (on the left), and with the fracture (on the right). One sees a linear gradient from top to bottom on the left figure, while the presence of the three intersecting fractures leads to a pressure concentration near the intersection, on all three subdomains that contain the intersection.

6.2 Performance of the preconditioner

We briefly discuss the performance of the preconditioner introduced at the end of Section 5.

The left part of Figure 7 compares the convergence of the conjugate gradient without and with the preconditioner. As can be seen, the simple preconditioning by the flow in the fracture alone is quite efficient. The right part of Figure 7 shows how the behavior of the preconditioner varies

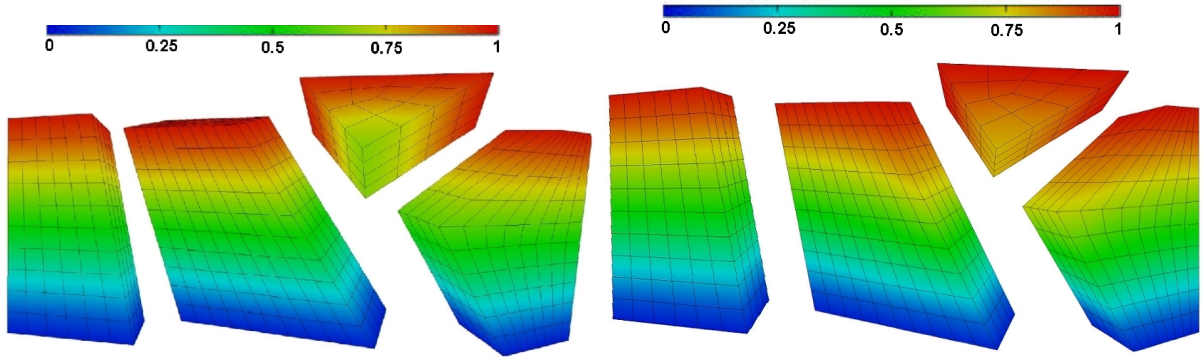


Figure 6: Vertical flow. Pressure field obtained without (left) and with (right) the fractures

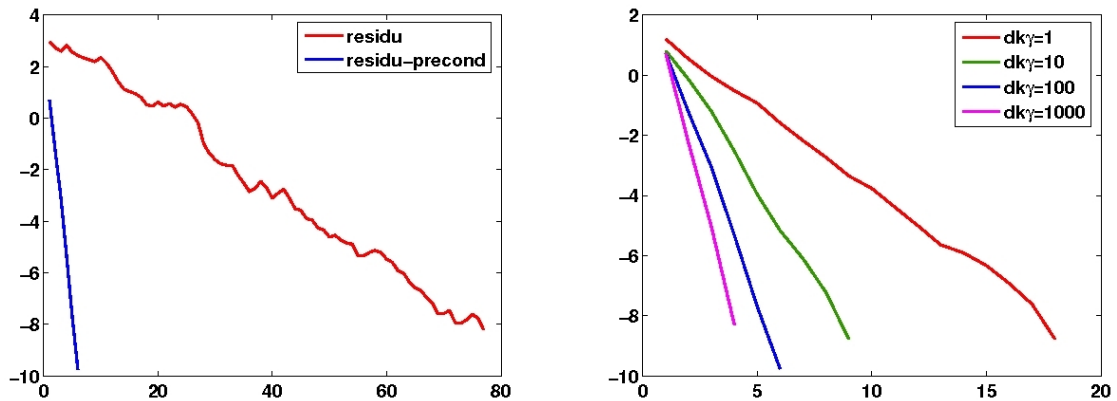


Figure 7: Performance of the preconditioner. Left: log of residual vs number of iterations, in red without preconditioning, in blue, with preconditioning. Right: performance for different values of the fracture permeability

with the permeability of the fracture. As expected from the motivations for the preconditioner given in Section 5, the preconditioner becomes more effective for larger values of the fracture permeability.

7 Conclusion

In this paper we present a model of the flow in a porous medium in 3 dimension that takes into account the intersecting fractures. In this model, we assimilate the fractures, that is assumed to be more permeable than the surrounding medium, to interfaces through which the pressure is continuous but flux is discontinuous. We reformulate the model by a method of domain decomposition in order to have a problem in the whole of the interfaces. The use of an iterative method to solve the system in the global interface require the study and the test of a preconditioner. We will extend this work in case of the transport, and the case of nonconforming mesh by using the method of mortar elements [10, 56, 55, 54].

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