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Introduction to Quantum Computing

André Chailloux, Inria de Paris

WAIFI'20

Quantum physics and quantum computing

- Quantum physics (beginning. 20th century): physics at a very small scale. Planck, Heisenberg, Schrodinger, Einstein ...

- Particles behave differently
  - Can be in a superposition of physical states.
  - Are modified when observed and quantumness is destroyed (so very fragile)

- In the 1970's : idea to control quantum systems to simulate other quantum systems, and more generally for computing.

From quantum physics to quantum computing

Quantum physics

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

$$\frac{d}{dt}\hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}(t)] + \frac{\partial \hat{A}(t)}{\partial t},$$

$$\vdots$$

Benioff, Deutsch

Quantum computing

- Quantum bits
- Quantum operations
- Quantum gates and quantum computing model.

Goal of today's talk

Quantum computing

- Many promising applications.

- Quantum key distribution (BB84)/quantum cryptography.

- Quantum algorithms
  - Shor's algorithm (1994)
  - Grover's algorithm (1998)
  - ...

- Other things I'm not going to talk about: quantum simulation, quantum sensing.

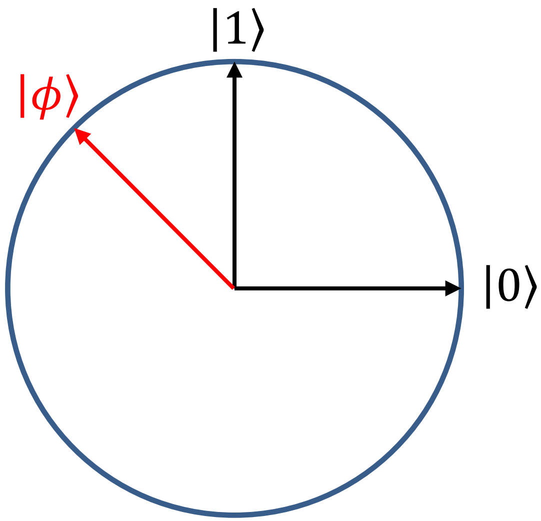
Outline of this talk

- Formalism of quantum computing

- Quantum algorithms

- General perspectives: quantum-secure cryptography, what can we do today from a hardware perspective.

- A quantum bit (or qubit) is a vector of norm 1 of the canonical $\mathbb{C}^2$ Hilbert space.
- Basis: $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$; $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
- $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$; $\alpha, \beta \in \mathbb{C}$
- Graphical representation of $|\phi\rangle$ when $\alpha, \beta \in \mathbb{R}$.



What does it mean to be in a superposition of states?

- Classical bits: 0;1. Unit of information. For eg. whether there is current in a wire (bit 1) ot not (bit 0)

- Qubits: basis = $\{|0\rangle, |1\rangle\}$. These basis elements also correspond to a physical state:
  - Spin of a particle (up = $|0\rangle$; down = $|1\rangle$).
  - Position of a particle (left = $|0\rangle$; right = $|1\rangle$).
  - Current in a wire (off = $|0\rangle$; on = $|1\rangle$).
  - Many other physical states can used to witness quantum effects (fortunately not the life of cats).

What does it mean to be in a superposition of states?

- So say I manage to construct $|\phi\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ that corresponds to the state of an electrical current ($|0\rangle =$ off, $|1\rangle =$ on)

- But say now I measure whether I have some electrical current or not.

- What ouptut do I get? Well 'on' or 'off', not both at the same time.
  - Probabilistic outcome: 'off' wp. 1/3 and 'on' wp. 2/3.

Quantum measurements

- More precisely, if I start from $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ and I measure:
  - I get outcome '0' wp. $|\alpha|^2$.
  - I get outcome '1' wp. $|\beta|^2$.

- After the measurement, the state collapses to the measured stated.
  - I get outcome '0' wp. $|\alpha|^2$. The state becomes $|\phi\rangle = |0\rangle$.
  - I get outcome '1' wp. $|\beta|^2$. The state becomes $|\phi\rangle = |1\rangle$.

- When a state is measured, it has to 'choose' in which state he really is.
  - Measurements destroy the 'quantumness' of the underlying physical particle
  - Measuring a quantum state changes it!

Quantum measurements: undetermination

- Measurements are the only way to get information about a quantum state.

- Suppose you are given $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$.
  - The only way to get some information about α and β is to perform a measurement on $|\phi\rangle$.
  - But this gives only 1 bit of information and then destroys all information about α and β (recall $\alpha, \beta \in \mathbb{C}$).

- Qubits can't be fully read. Actually, we can recover a very small amount of information about α, β.

- Measuring qubits destroys their quantum nature.

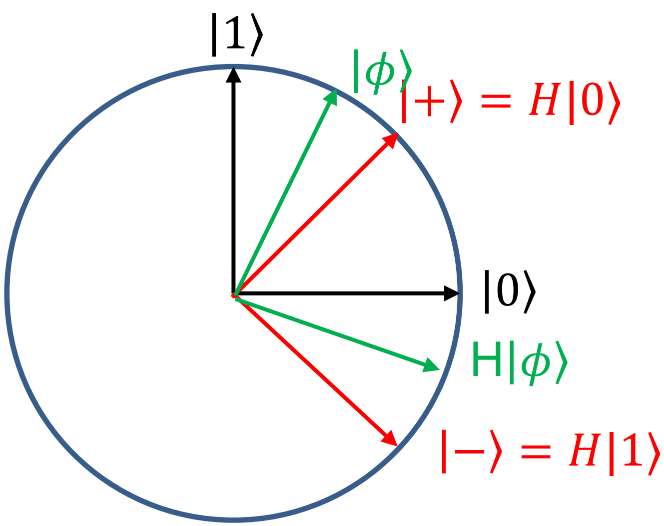
- However, we can perform quantum operations on qubits without destroying their quantum nature.

- Quantum operations are linear and norm preserving. They can be represented by unitary matrices U

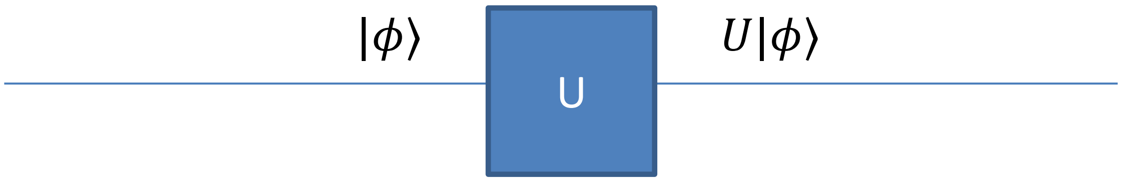
U= $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$: $\begin{pmatrix} a \\ b \end{pmatrix}$;$\begin{pmatrix} c \\ d \end{pmatrix}$ are each of norm 1 and orthogonal.

- Bit flip (BF): $U_{BF} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  - $U_{BF}|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \quad ; \quad U_{BF}|1\rangle = |0\rangle$
  - $U_{BF}\left(\alpha|0\rangle + \beta|1\rangle\right) = \beta|0\rangle + \alpha|1\rangle$.
- Hadamard gate: $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.
  - $H\,|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \coloneqq |+\rangle$.
  - $H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \coloneqq |-\rangle$.

- H gate.


$|1\rangle$
$|\phi\rangle$
$|+\rangle = H|0\rangle$
$|0\rangle$
$H|\phi\rangle$
$|-\rangle = H|1\rangle$

- Any 2 x 2 unitary matrix U is an admissible quantum operation on 1 qubit.



- A quantum unitary doesn't measure the quantum state, and thus preserves its quantumness.
- For eg. a photon going through a crystal, changing its polarity (which is the quantum state of the photon)
- Easy to do experimentally.

Recap of qubits

- A qubit $|\phi\rangle = \alpha|0\rangle + \beta|0\rangle$ represents a particle which is in a superposition of states (here '0' and '1').
- The laws of quantum mechanics allow to interact with $|\phi\rangle$ in 2 ways:
  - Via quantum unitary operations: preserves the quantumness.
  - Via measurments: destroys the quantumness. Only way to retrieve information about the state.

2 qubit states.

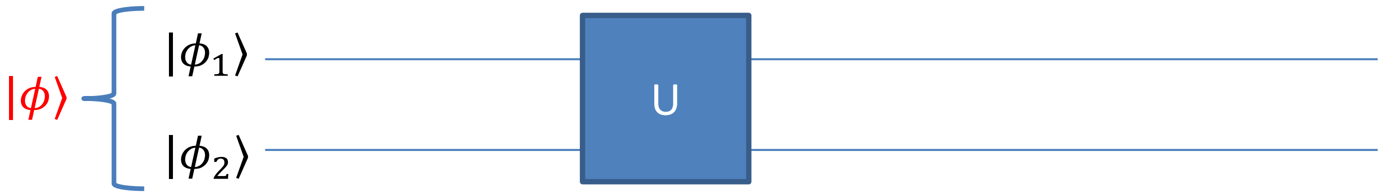
- 2 bits: 00;01;10;11.
- 2 qubit state: $|\phi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$.
  - $\alpha, \beta, \gamma, \delta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.
- $|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
- Again, 2 ways to interact:
  - Quantum operations on 2 qubits, represented by 4 x 4 unitary matrices. Eg.

$$U = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix} ; U|\phi\rangle = U \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 1/2 \begin{pmatrix} \alpha+\beta+\gamma+\delta \\ \alpha+\beta-\gamma-\delta \\ \alpha-\beta-\gamma+\delta \\ \alpha-\beta+\gamma-\delta \end{pmatrix}$$

  - Measurments: get outcome ‘00’ wp. $|\alpha|^2$. Similarly with ’01’, ’10’, ’11’.

2 qubits and entanglement

- Concatenation of 2 qubits



- $|\phi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$; $|\phi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$.
- What is the overall state $|\phi\rangle$ of the system?

$$|\phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

- Entanglement: not all 2 qubit states $|\phi\rangle$ are of the form $|\phi_1\rangle \otimes |\phi_2\rangle$.
  - Eg. $|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. This is an entangled state.

- [EinsteinPodolskiRosen’35] paradox: ‘spooky action at a distance’.

- Confirmed experimentally by Aspect.
  - Quantum mechanics is non-local as predicted by Bell (1964).
  - Non-locality has a very elegant explanation in terms of computer science concepts and information theory:

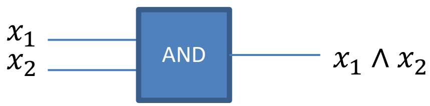
n qubit states

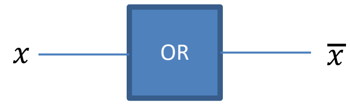
- n qubit state: $|\phi\rangle = \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle$.
  - Each $\alpha_i \in \mathbb{C}, \sum_i |\alpha_i|^2 = 1$.
  - Superposition of 2^n different states.

- Unitary operations: 2^n x 2^n matrices!
  - Even though these are quantum admissible operations: in general, we cannot make them efficiently.

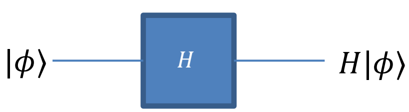
- Notions of complexity for quantum computing?

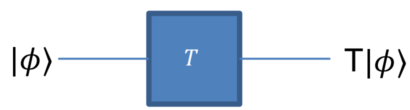
- Following gates are universal for (classical) computing:

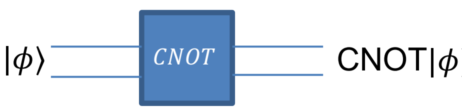




- Any function on n bits can be described using these 2 logical gates.
  - The minimum number of gates required to compute a function f is the circuit complexity of f (in this computational model).
- Quantum circuits, also an (approximate) universal set of gates.







$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{8}} \end{pmatrix} \qquad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Quantum circuit model and complexity

- Any quantum unitary U on n qubits can be approximated by performing only H,T,CNOT unitaires.
  - The minimum number of gates required to compute a unitary U is the quantum circuit complexity of U (in this computational model).

- All universal sets of 1 and 2 qubit gates are equivalent (up to a $\mathrm{polylog}(\frac{1}{\epsilon})$ factor where ϵ is the approximation factor).

Recap

- Quantum computing involves qubits which are in a superposition of bits.
  - 2 operations: unitaries and measurements.

- Unitaries on n qubits involve modifying 2^n complex amplitudes at the same time.
  - At the core of quantum speedups.
  - But operations can be very costly. Also recall that we can gather only n bits of information from the output.

- Computational model: the quantum circuit model.

Quantum error correction

Quantum noise

- As we said, a qubit looses its quantumness when observed.

- Actually, most physical interactions with the environment destroy this quantumness
  - Qubits are extremely fragile.

- This noise is the main problem when considering quantum computation.
  - Is there a way to solve this problem?

- Quantum error correction: we can embed our n working qubits (logical qubits) into n’ > n physical qubits.

- During the computation, we can ensure that our n’ physical qubits remain in our code subspace and correct them if needed.

- This introduces new qubits and new quantum operations, hence new errors.

Threshold theorem: If the noise level per gate is < C (absolute constant), then we can correct errors faster than they arrive.

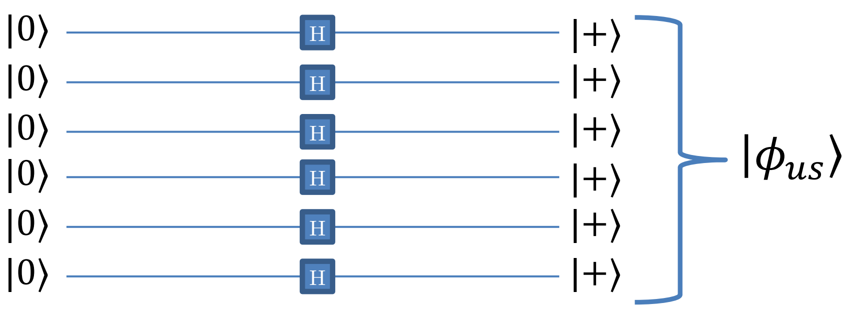
- So there is hope for quantum computing.

Quantum algorithms

- What can we do with quantum computing?

- Toy example 1: given n qubits, construct the uniform superposition $|\phi_{us}\rangle = \sum_{i\in\{0,1\}^n} \frac{1}{2^{n/2}} |i\rangle$.

- By performing basic computation, one can see that $|\phi_{us}\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle = H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle$



- $|\phi_{us}\rangle$ can be created in time n.

Transforming a classical circuit into a quantum circuit.

- Toy example 2: Quantum access to a classical function.

- We are given a function $f : \{0,1\}^n \rightarrow \{0,1\}^m$ with the description of an circuit computing f.
  - Goal: compute the unitary O_f such that $O_f(|x\rangle|0\rangle) = |x\rangle|f(x)\rangle$.

> Theorem: If the classical circuit for computing f has t gates then the unitary O_f can be constructed (approximately) using O(t) gates.

Shor's algorithm

Factoring problem

Input: a number N of n bits.
Goal: find the decomposition of N in prime numbers $N = \Pi_i p_i^{\alpha_i}$

- Actually, one can reduce this problem to period finding using a bit of number theory.

Period finding problem

Input: a function $f : \mathbb{N} \rightarrow \{0, \dots, N-1\}$ such that $\exists r \in \{0, \dots, N-1\}$ (unknown) st. $f(a) = f(b) \Leftrightarrow a = b \bmod r$
Goal: find r

- Classically, one needs $poly(N)$ time to solve this problem.
  - f can be computed in time $O(log^2(N)polylog(N))$.

Shor's algorithm

Period finding problem

Input: a function $f : \mathbb{N} \to \{0, \ldots, N-1\}$ such that $\exists r \in \{0, \ldots, N-1\}$ (unknown) st. $f(a) = f(b) \Leftrightarrow a = b \bmod r$
Goal: find r

- Shor's quantum algorithm: solves period finding in time $O(log^2(N)polylog(\log(N)))$ for this f.
  - $\Rightarrow$ solves factoring in the same time.

Shor's algorithm: 1 slide sketch

Period finding problem

Input: a function $f : \mathbb{N} \to \{0, \dots, N-1\}$ such that $\exists r \in \{0, \dots, N-1\}$ (unknown) st. $f(a) = f(b) \Leftrightarrow a = b \bmod r$
Goal: find r

- Take $m \approx 2\log(N)$. Construct $|\phi_{us}\rangle \otimes |0\rangle = \sum_{i\in\{0,1\}^m} \frac{1}{2^{m/2}} |i\rangle \otimes |0\rangle$.
- Apply O_f on $|\phi_{us}\rangle \otimes |0\rangle$ to get $|\psi\rangle := \sum_{i\in\{0,1\}^m} \frac{1}{2^{m/2}} |i\rangle \otimes |f(i)\rangle$.
  - Recall that $O_f(|i\rangle|0\rangle) = |i\rangle|f(i)\rangle$
- For a fixed second register ($|f(i_0)\rangle$), we get in the first register a uniform superposition proportional to $\sum_k |i_0 + kr\rangle$.
- The Quantum Fourier Transform allows us to find r in time $O(\log^2(N) polylog(\log(N)))$ for the function f needed for factoring.

Grover's algorithm

- Another iconic algorithm: Grover's algorithm

Search problem

Input: a function $f: \{0,1\}^n \rightarrow \{0,1\}$, with an efficient circuit description
Goal: find $x\ st.\ f(x) = 1$

- In the classical setting:
  - Worst case, a unique solution, need $O(2^n)$ calls to f.

- Quantum setting: Grover's algorithm, needs $O(2^{n/2})$ calls to O_f.

Perspectives

Quantum secure cryptography

- Shor's algorithm
  - Breaks factoring i.e. RSA
  - But also the Discrete Logarithm problem and the elliptic curve discrete log.

- Other problems (for public key crypto) for which we don't have an efficient quantum algorithm:
  - Lattice based crypto
  - Code based crypto
  - Isogeny based crypto
  - Multivariate based crypto
  - Hash based crypto

- Essential for retroactive security.

- Quantum Key Distribution (Quantum Cryptography)
  - Uses quantum communication to encode bits of a secret key.
  - An eavesdropper must observe the qubits and therefore destroy the quantumness: this can be detected.
  - Uses single qubits: we often use photons.
  - Can limit the effects of noise up to a few hundred kilometers: we don't use quantum error correction.

Cerberis^3 QKD System

Quantum Key Distribution for enterprise, government and telco production environments

- Complex network topologies (ring, hub and spoke)
- Interoperability with major Ethernet and OTN encryptors
- Easy integration in any data centre
- Centrally monitored solution
- Multiplexing of all channels on single fibre for metropolitan area

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What can we do today? Quantum Computing

- Many research teams try to construct qubits as robust as possible in order to be highly scalable.
  - At most a few qubits at the same time.

- With recent advances, companies such as IBM or Google construct quantum architecture with around 50 qubits
  - IBM: you can actually go online and run some quantum circuits on their quantum computer: quite noisy.
  - Google: used their computer to perform a task they call Quantum Supremacy: a computational problem that we don't know how to solve classically (but which is veeeery useless and can't be checked).
  - Also no error correction! Not very scalable.

Conclusion

- Quantum computing: a crazy idea.
- A long way to go: especially for Shor's algorithm: a few thousand logical qubits and million/billions physical qubits.
- Other applications that require much less qubits and that can tolerate some noise: constant/polynomial speedups, chemistry/simulation.
- Quantum key distribution works and can have some particular applications coupled with classical crypto.

Thank you for your attention