L_2 and BIBO stability of systems with variable delays

Catherine Bonnet, Jonathan R Partington

L_2 and BIBO stability of systems with variable delays

Catherine Bonnet* and Jonathan R. Partington†

March 11, 2020

Abstract

This paper considers L_2 and BIBO stability and stabilization issues for systems with time-varying delays which can be of retarded or neutral type. An important role is played by a nominal system with fixed delays which are close to the time-varying ones. Under stability or stabilizability conditions of this nominal system, sufficient conditions are given in order to ensure similar properties for the system with time-varying delays.

Keywords: neutral delay systems, time-varying delays, L_2-stability, $BIBO$-stability

In memory of Ruth Curtain.

1 Introduction and notation

There has been a growing interest in systems with time-varying delays in recent years, in particular due to the study of large-scale networked systems where delays may have to be considered to be time-varying for better accuracy of the model. The presence of time-varying delays induces complex behaviours which have been pointed out and studied by many authors (see e.g. [14, 15, 21] the monograph [7] and the references therein). Although the case of autonomous systems has been widely studied, much less work has been dedicated to such systems submitted to arbitrary external inputs. Several studies, e.g. [2], consider the L^2-input-output norm of the time-varying delay operator, however to the best of our knowledge, no study has been made of the input–output stability of a standard feedback scheme involving a system with time-varying delays. We can find in [19] an input-output setting for determining rate-based flow controllers for communication networks in the presence of time-varying delays; however, an H_∞ optimization problem is considered here which deals with several types of perturbations acting on the state-space system including time-varying coefficients and time-varying delays and this renders the comparison with our work inappropriate. Several papers (for example [8, 9, 12, 20]) mention an input–output approach for systems with time-varying delays; however the input–output framework is not considered for the given system but is introduced once an initially autonomous system with time-varying delays has been transformed into a system with external input by means of the introduction of additional variables given in terms of the states.

*Université Paris Saclay, Inria, CNRS, CentraleSupelec, Laboratoire des signaux et systèmes, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France Catherine.Bonnet@inria.fr

†School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K. J.R.Partington@leeds.ac.uk

In our setting, the classical condition on the boundedness of the derivative of the time-varying delay by a constant strictly lesser than one is not necessary. Note that this is also the case in [22] where H_∞ state-feedback control of systems time-varying input delays is considered. Physical considerations (such as the order of events being preserved in the presence of a time-varying delay) may impose that for a time-varying delay $\tau(t)$ the function $t \mapsto t - \tau(t)$ is strictly increasing, but since this only implies that the derivative of τ is bounded by one, a relaxation of the classical condition of boundedness is of use (*moderately varying delays* were first considered in [10]).

We consider two versions of input–output stability: so-called H_∞-stability (a finite gain when the signals are measured in terms of energy, that is, L_2 norm), and BIBO stability (bounded inputs give bounded outputs). Retarded and neutral type systems with pointwise or distributed delays are studied here. Note that in [8], the input-output approach to neutral systems has been presented, but the results were confined to slowly-varying delays.

The very general framework is introduced in Section 2, and the stability analysis is given in Section 3. Then, a discussion of feedback stabilization is given in Section 4. Finally, numerical examples appear in Section 5. The results we derive yield very explicit estimates for stability margins, as will be seen by considering the examples we present.

Notation

H_∞, or $H_\infty(\mathbb{C}_+)$, denotes the space of bounded analytic functions on the complex right half-plane $\mathbb{C}_+$, with the supremum norm $\|F\|_\infty = \sup_{s\in\mathbb{C}_+} |F(s)|$. The H^∞ norm gives an explicit expression for the L_2 input/output gain of a stable system.

In addition, $\mathcal{A}$ denotes the space of distributions of the form

$$g(t) = g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i),$$

where $g_a \in L_1(0,\infty)$ and $\sum_0^\infty |g_i| < \infty$; here, $\delta(t - t_i)$ denotes a delay Dirac distribution, $0 = t_0 < t_1, t_2, t_3 \dots$ the t_k being distinct. This is equipped with the norm

$$\|g\|_{\mathcal{A}} = \int_0^\infty |g_a(t)|\, dt + \sum_{i=0}^{\infty} |g_i|,$$

which gives an expression for the L_∞ input/output gain of a stable system. Then $\hat{\mathcal{A}}$ is the space of Laplace transforms of functions in $\mathcal{A}$, which is a subalgebra of H^∞.

2 The class of systems studied

In this paper we shall consider the variable-delay input–output system

$$\begin{aligned} \dot{x}(t) + \sum_{\ell=1}^{L} A_{-\ell}\dot{x}(t - \gamma_\ell(t)) &= Ax(t) + \sum_{j=1}^{J} A_j x(t - \tau_j(t)) + \int_0^{\delta(t)} h(\theta) I x(t-\theta)\, d\theta \\ &\quad + Bu(t) + \sum_{k=1}^{K} B_k u(t - \sigma_k(t))\, d\theta, \quad (t > 0), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{C}^n$ denotes the state and $u(t) \in \mathbb{C}^p$ the input, both assumed zero for $t \leq 0$, the matrices A, $A_{-\ell}$, A_j, B, B_k and I (identity matrix) have the appropriate sizes, the function $h(\theta)$ is of polynomial type and the delays $\gamma_\ell(t)$, $\tau_j(t)$, $\sigma_k(t)$ and $\delta(t)$ are positive (to ensure stability, further hypotheses will be required later).

Were the delays constant (say, $\gamma_\ell(t) = H_\ell$, $\tau_j(t) = h_j$, $\delta(t) = D$, and $\sigma_k(t) = T_k$ for each j, k and l), then this system would have a transfer function given by

$$G(s) = \left(sI + \sum_{\ell=1}^{L} A_{-\ell} s e^{-H_\ell s} - A - \sum_{j=1}^{J} A_j e^{-h_j s} - \int_0^D h(\theta) e^{-s\theta} \, d\theta \right)^{-1} \left(B + \sum_{k=1}^{K} B_k e^{-T_k s} \right),$$

and H_∞ stability (bounded L_2 gain) would correspond to G being analytic and bounded in the right half-plane $\mathbb{C}_+$.

Let us mention that the considered distributed delay will give rise only to polynomial and exponential terms in the transfer function allowing one to divide the considered class of systems into retarded and neutral type classes as in the generic (distributed delay free) case.

In [9], Fridman and Gil' considered an autonomous system with variable delays, given by

$$\dot{x}(t) = \sum_{j=1}^{\ell} A_j x(t - \tau_j(t)), \qquad (t > 0),$$

with initial conditions specified on an interval $[-\eta, 0]$. They analysed its asymptotic stability using a frequency-domain approach.

It is our aim to analyse the H_∞ and BIBO stability of the system (1), using methods similar to those of [9] in a far more general context, assuming stability of the nominal system where the delays are fixed.

Consider the *nominal system* (as always, with zero initial conditions),

$$\dot{v}(t) + \sum_{\ell=1}^{L} A_{-\ell} \dot{v}(t - H_\ell) = Av(t) + \sum_{j=1}^{J} A_j v(t - h_j) + \int_0^D h(\theta) I v(t - \theta) \, d\theta + Bu(t) + \sum_{k=1}^{K} B_k u(t - T_k), \tag{2}$$

for $t > 0$,

Throughout the rest of the paper, we make the following hypothesis.

Hypothesis (H): we assume that $\sum_{\ell=1}^{L} \|A_{-\ell}\| < 1$.

If the L_2 gain from u to v is finite we have that $\dot{v}(t) + \sum_{\ell=1}^{L} A_{-\ell} \dot{v}(t - H_\ell)$ is in L_∞ and there exists a constant K such that

$$\left\| \dot{v}(t) + \sum_{\ell=1}^{L} A_{-\ell} \dot{v}(t - H_\ell) \right\|_\infty \leq K \|u\|_\infty.$$

From this inequality and the fact that $\|\dot{v}(t-H)\|_\infty = \|\dot{v}(t)\|_\infty$, we get that

$$\left(1 - \sum_{\ell=1}^{L} \|A_{-\ell}\|\right) \|\dot{v}\| \leq K\|u\|_\infty,$$

which means, as $\sum_{\ell=1}^{L} \|A_{-\ell}\| < 1$, that we also have a finite L_∞ gain from u to $\dot{v}$.

If the L_2 gain from u to v is finite we have that the associated transfer function G_{nom} is in H_∞, its denominator being

$$\det(sI + \sum_{\ell=1}^{L} A_{-\ell} s e^{-H_\ell s} - A - \sum_{j=1}^{J} A_j e^{-h_j s} - \int_0^D h(\theta) I e^{-\theta s} d\theta)$$

We can easily deduce from this that $sG(s)$ is also in H_∞ as there is no problem with properness and no problem with boundedness on the imaginary axis. This means that there is a finite L_2 gain from u to $\dot{v}$.

We shall need to consider the system

$$\dot{z}(t) + \sum_{\ell=1}^{L} A_{-\ell}\dot{z}(t-H_\ell) = Az(t) + \sum_{j=1}^{J} A_j z(t-h_j) + \int_0^D h(\theta) I z(t-\theta)\, d\theta + w(t), \qquad (t > 0).$$

For the same reasons as above, if the L_2-gain (from w to z) is finite, we also have a L_2 finite gain from w to $\dot{z}$ and if the L_∞-gain (from w to z) is finite we also have a finite L_∞ gain from w to $\dot{z}$.

Let us denote :

M_2^{nom}	:	the L_2 gain from u to v
M_∞^{nom}	:	the L_∞ gain from u to v
M_∞^{nomd}	:	the L_∞ gain from u to $\dot{v}$
M_2^{nomd}	:	the L_2 gain from u to $\dot{v}$
M_2	:	the L_2 gain from w to z
M_∞	:	the L_∞ gains from w to z
M_2^d	:	the L_2 gain from w to $\dot{z}$
M_∞^d	:	the L_∞ gain from w to $\dot{z}$

3 Stability analysis

Our standing assumptions, together with Hypothesis (H) are that

$0 \leq H_\ell - \eta_\ell \leq \gamma_\ell(t) \leq H_\ell + \eta_\ell$
$0 \leq h_j - \mu_j \leq \tau_j(t) \leq h_j + \mu_j$
$0 \leq D - \epsilon \leq \delta(t) \leq D + \epsilon$
$0 \leq T_k - \nu_k \leq \sigma_k(t) \leq T_k + \nu_k$

for all t and for each j, k and ℓ, where $\eta_\ell, \mu_j, \epsilon, and \nu_k$ are positive constants.

Writing $y = x - v$ we have

$$\begin{aligned}\dot{y}(t) + \sum_{\ell=1}^{L} A_{-\ell}\dot{y}(t - H_\ell) &= Ay(t) \\ &\quad + \sum_{j=1}^{J} A_j y(t - h_j) + \sum_{j=1}^{J} A_j[x(t - \tau_j(t)) - x(t - h_j)] \\ &\quad + \int_0^D h(\theta) I y(t - \theta)\, d\theta + \int_D^{\delta(t)} h(\theta) I x(t - \theta)\, d\theta \\ &\quad + \sum_{\ell=1}^{L} A_{-\ell}[\dot{x}(t - H_\ell) - \dot{x}(t - \gamma_\ell(t))] \\ &\quad + \sum_{k=1}^{K} B_k[u(t - \sigma_k(t)) - u(t - T_k)], \qquad (t > 0).\end{aligned}$$

We may write this as

$$\begin{aligned}\dot{y}(t) + \sum_{\ell=1}^{L} A_{-\ell}\dot{y}(t - H_\ell) &= Ay(t) + \sum_{j=1}^{J} A_j y(t - h_j) + \int_0^D h(\theta) I y(t - \theta)\, d\theta \\ &\quad + \sum_{j=1}^{J} A_j f_j(t) + \sum_{\ell=1}^{L} A_{-\ell}\tilde{f}_\ell(t) + \int_D^{\delta(t)} h(\theta) I x(t - \theta)\, d\theta + \sum_{k=1}^{K} B_k g_k(t)\end{aligned} \tag{3}$$

with

$$\begin{aligned} f_j(t) &= x(t - \tau_j(t)) - x(t - h_j), \\ \tilde{f}_\ell(t) &= \dot{x}(t - H_\ell) - \dot{x}(t - \gamma_\ell(t)), \\ \text{and} \qquad g_k(t) &= u(t - \sigma_k(t)) - u(t - T_k). \end{aligned}$$

We now need supplementary conditions to ensure that the functions $\tilde{f}$, f_j and g_k lie in $L_2(0, \infty)$ or $L_\infty(0, \infty)$.

3.1 The special case of retarded type systems

We begin with the case that all $A_{-\ell} = 0$, since the system is then more robust to perturbations. The equations under consideration are

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{J} A_j x(t - \tau_j(t)) + \int_0^{\delta(t)} h(\theta) I x(t - \theta)\, d\theta + Bu(t) + \sum_{k=1}^{K} B_k u(t - \sigma_k(t)) \quad (t > 0), \tag{4}$$

$$\dot{v}(t) = Av(t) + \sum_{j=1}^{J} A_j v(t - h_j) + \int_0^D h(\theta) I v(t - \theta)\, d\theta + Bu(t) + \sum_{k=1}^{K} B_k u(t - T_k), \qquad (t > 0) \tag{5}$$

$$\dot{y}(t) = Ay(t)+\sum_{j=1}^{J} A_j y(t-h_j)+\int_0^D h(\theta)Iy(t-\theta)\,d\theta+\sum_{j=1}^{J} A_j f_j(t)+\int_D^{\delta(t)} h(\theta)Ix(t-\theta)\,d\theta+\sum_{k=1}^{K} B_k g_k(t), \tag{6}$$

and

$$\dot{z}(t) = Az(t) + \sum_{j=1}^{J} A_j z(t-h_j) + \int_0^D h(\theta)Iz(t-\theta)\,d\theta + w(t), \qquad (t>0). \tag{7}$$

Theorem 3.1. *1) Suppose that the system (5) is BIBO-stable.*
If $M_\infty^d \sum_{j=1}^{\ell} \mu_j \|A_j\| < 1$ *and* $M_\infty \epsilon \|h\|_\infty \left(\sum_{j=1}^{\ell} M_\infty^d \mu_j \|A_j\|(1-M)^{-1}+1\right) < 1$*, then the system (4) is BIBO-stable.*

2) Suppose that the system (5) is H_∞*-stable and that* $\dot{u} \in L_2$*.*
If $M = M_2^d \sum_{j=1}^{\ell} \mu_j \|A_j\| < 1$ *and* $M_2 \epsilon \|h\|_\infty (\sum_{j=1}^{\ell} \mu_j \|A_j\|(1-M')^{-1}+1) < 1$*, then the system (4) is* H_∞*-stable in the sense that there is a finite* L^2 *gain between* $(u,\dot{u})$ *and* x*.*

Proof. 1) The basic calculation is as follows:

$$\begin{aligned}\|\dot{x}\|_\infty &\le \|\dot{v}\|_\infty + \|\dot{y}\|_\infty \\ &\le \|\dot{v}\|_\infty + M_\infty^d(\sum_{j=1}^{\ell} \|A_j\|\|f_j\|_\infty + \epsilon\|h\|_\infty\|x\|_\infty + \sum_{k=1}^{m} \|B_k\|\|g_k\|_\infty)\end{aligned}$$

and then we shall bound $\|f_j\|_\infty$ in terms of $\|\dot{x}\|_\infty$.

We have

$$\|f_j\|_\infty = \sup_t \left\| \int_{t-\tau_j(t)}^{t-h_j} \dot{x}(s)\,ds \right\| \le \mu_j \|\dot{x}\|_\infty$$

and

$$\|g_k\|_\infty = \sup_t \|[u(t-\sigma_k(t)) - u(t-T_k)]| \le 2\|u\|_\infty$$

So that, provided that $M := M_\infty^d \sum_{j=1}^{\ell} \mu_j \|A_j\| < 1$, we have

$$\|\dot{x}\|_\infty \le \|\dot{v}\|_\infty + M\|\dot{x}\|_\infty + M_\infty^d \epsilon \|h\|_\infty \|x\|_\infty + M_\infty^d \sum_{k=1}^{m} 2\|B_k\|\|u\|_\infty,$$

or

$$\|\dot{x}\|_\infty \le (1-M)^{-1} \left(M_\infty^{nomd}\|u\|_\infty + M_\infty^d \epsilon\|h\|_\infty\|x\|_\infty + M_\infty^d \sum_{k=1}^{m} 2\|B_k\|\|u\|_\infty \right),$$

Now $x = v + y$ and so by (5) and (6) we have

$$\begin{aligned}\|x\|_\infty &\le \|v\|_\infty + \|y\|_\infty \\ &\le M_\infty^{nom}\|u\|_\infty + M_\infty(\sum_{j=1}^{\ell} \|A_j\|\|f_j\|_\infty + \epsilon\|h\|_\infty\|x\|_\infty + \sum_{k=1}^{m} \|B_k\|\|g_k\|_\infty) \\ &\le M_\infty^{nom}\|u\|_\infty + M_\infty(\sum_{j=1}^{\ell} \mu_j\|A_j\|\|\dot{x}\|_\infty + \sum_{k=1}^{m} 2\|B_k\|\|u\|_\infty + \epsilon\|h\|_\infty\|x\|_\infty)\end{aligned}$$

Let $\tilde{M} := M_\infty \epsilon \|h\|_\infty \left(1 + \sum_{j=1}^{\ell} \mu_j \|A_j\| \|M_\infty^d (1-M)^{-1} \right)$, we have

$$\begin{aligned}(1-\tilde{M})\|x\|_\infty \quad &\leq \quad M_\infty^{nom}\|u\|_\infty + \\ & M_\infty \left(\sum_{j=1}^{\ell} \mu_j \|A_j\|(1-M)^{-1}(M_\infty^{nomd} + M_\infty^d \sum_{k=1}^{m} 2\|B_k\|) + \sum_{k=1}^{m} 2\|B_k\| \right) \|u\|_\infty\end{aligned}$$

which under the condition $\tilde{M} < 1$ gives a finite L_∞ gain from u to x.

2) In the case of L_2-stability, we start with the same inequality as above. First, similarly to [9], we have, recalling that $x(t) = \dot{x}(t) = 0$ for $t \leq 0$,

$$\begin{aligned}\|f_j\|_2^2 \quad &= \quad \int_0^\infty \left\| \int_{t-\tau_j(t)}^{t-h_j} \dot{x}(s)\, ds \right\|^2 dt \leq \int_0^\infty \mu_j \int_{t-h_j-\mu_j}^{t-h_j} \|\dot{x}(s)\|^2 \, ds\, dt \\ &\leq \quad \mu_j^2 \int_0^\infty \|\dot{x}(r)\|^2 \, dr = \mu_j^2 \|\dot{x}\|_2^2.\end{aligned}$$

Then, to bound $\|g_k\|_2$ requires some restrictive conditions on u. With the condition that $\dot{u} \in L_2$, we get $\|g_k\|_2 \leq \nu_k \|\dot{u}\|_2$ and the stability result obtained in the case that there are input delays will take the form

$$\|x\|_2 \leq C_1 \|u\|_2 + C_2 \|\dot{u}\|_2.$$

Here again, provided that $M' := M_2^d \sum_{j=1}^{\ell} \mu_j \|A_j\| < 1$, we have

$$\|\dot{x}\|_2 \leq \|\dot{v}\|_2 + M'\|\dot{x}\|_2 + \epsilon \|h\|_\infty \|x\|_2 + M_2^d \sum_{k=1}^{m} \nu_k \|B_k\| \|\dot{u}\|_2,$$

or

$$\|\dot{x}\|_2 \leq (1-M')^{-1} \left(M_2^{nomd} \|u\|_2 + \epsilon \|h\|_\infty \|x\|_2 + M_2^d \sum_{k=1}^{m} \nu_k \|B_k\| \|\dot{u}\|_2 \right),$$

By (5) and (6) we have

$$\|x\|_2 \leq M_2^{nom} \|u\|_2 + M_2 (\sum_{j=1}^{\ell} \mu_j \|A_j\| \|\dot{x}\|_2 + \epsilon \|h\|_\infty \|x\|_2 + \sum_{k=1}^{m} \nu_k \|B_k\| \|\dot{u}\|_2),$$

that is,

$$\begin{aligned}&\left(1 - M_2 \epsilon \|h\|_\infty (\sum_{j=1}^{\ell} \mu_j \|A_j\| (1-M')^{-1} + 1) \right) \|x\|_2 \\ &\leq \left(M_2^{nom} + M_2 M_2^{nomd} \sum_{j=1}^{\ell} \mu_j \|A_j\| (1-M')^{-1} \right) \|u\|_2 + \left(M_2 \sum_{k=1}^{m} \nu_k \|B_k\| (M_2^d + 1) \right) \|\dot{u}\|_2,\end{aligned}$$

which gives, if $M_2\epsilon\|h\|_\infty(\sum_{j=1}^{\ell}\mu_j\|A_j\|(1-M')^{-1}+1)<1$, a finite L_2 gain from $(u,\dot u)$ to x if there are input delays.

□

Remark 3.1. *If $B_k=0$ for all k, or if the σ_j are constant functions for all j, we do not need to impose the condition that $\dot u$ lies in L_2 in order to get H_∞-stability.*

3.2 The general case of neutral systems

In general it is easier to destabilize a neutral delay system by a perturbation, and here we simply give a result for BIBO stability.

Theorem 3.2. *Suppose that the system (2) is BIBO-stable.
If $M_\infty^d(\sum_{j=1}^{\ell}\mu_j\|A_j\|+2\sum_{l=1}^{L}\|A_{-l}\|)<1$ then the system (1) is BIBO-stable.*

Proof. As for retarded systems we start with the inequality

$$\|\dot x\|_\infty \le \|\dot v\|_\infty + M_\infty^d(\sum_{j=1}^{\ell}\|A_j\|\|f_j\|_\infty + \sum_{k=1}^{m}\|B_k\|\|g_k\|_\infty + \sum_{l=1}^{L}\|A_{-l}\|\tilde f\|_\infty + \epsilon\|h\|_\infty\|x\|_\infty).$$

We have
$$\|\tilde f\|_\infty = \sup_t \|\dot x(t-\gamma(t)) - \dot x(t-H)\| \le 2\|\dot x\|_\infty,$$
whereas
$$\|f_j\|_\infty \le \mu_j\|\dot x\|_\infty$$
and
$$\|g_j\|_\infty \le 2\|u\|_\infty$$
as in the retarded case.

Provided that $M'':=M_\infty^d(\sum_{j=1}^{\ell}\mu_j\|A_j\|+2\sum_{l=1}^{L}\|A_{-l}\|)<1$, we have

$$\|\dot x\|_\infty \le (1-M'')^{-1}\left(M_\infty^{nomd}\|u\|_\infty + M_\infty^d\sum_{k=1}^{m}2\|B_k\|\|u\|_\infty + \epsilon\|h\|_\infty\|x\|_\infty\right),$$

From the relation $x=v+y$, we get a finite L_∞ gain from u to x.

□

4 Stabilization properties

4.1 The case of retarded systems

We consider here the stabilization of system (4) through the following standard feedback scheme where r and d are external input signals.

Retarded systems are H_∞-stabilizable (see [3]) and so strongly stabilizable (see [18]). They are also $BIBO$-stabilizable (see [3]) and those with commensurate delays are strongly $BIBO$-stabilizable (see [16]).

Therefore, we may consider here only strong stabilization of system (4) by a controller K, with convolution kernel $\mathcal{K}$. In this case, instead of studying the stability of the four matrix

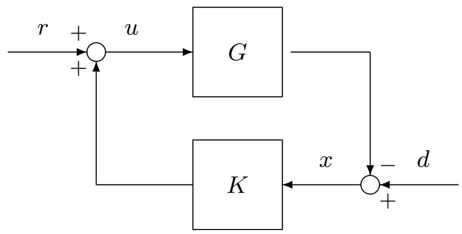

Figure 1: Standard Feedback Configuration

entries between $(r, d)^T$ and $(x, u)^T$, it becomes sufficient to only study the stability of that between r and x. In the following, the control law u is then taken of the type $u = \mathcal{K} * x + r$.

The closed-loop has the following equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{j=1}^{\ell} A_j x(t-\tau_j(t)) + \int_0^{\delta(t)} h(\theta)x(t-\theta)\,d\theta + B(\mathcal{K}*x)(t) + \sum_{k=1}^{m} B_k(\mathcal{K}*x)(t-\sigma_k(t)) \\ &\quad + Br(t) + \sum_{k=1}^{m} B_k r(t-\sigma_k(t)), \qquad (t>0),\end{aligned} \tag{8}$$

Consider the system

$$\begin{aligned}\dot{v}(t) &= Av(t) + \sum_{j=1}^{\ell} A_j v(t-h_j) + \int_0^{D} h(\theta)v(t-\theta)\,d\theta + B(\mathcal{K}*v)(t) + \sum_{k=1}^{m} B_k(\mathcal{K}*v)(t-T_k) \\ &\quad + Br(t) + \sum_{k=1}^{m} B_k r(t-T_k), \qquad (t>0),\end{aligned} \tag{9}$$

and let us denote:

M_∞^{cl}	:	the L_∞-gain of the closed-loop
M_∞^{cld}	:	L_∞-gain between input and derivative of the state of the closed-loop
M_2^{cl}	:	the L_2-gain of the closed-loop
M_∞^{clnomd}	:	L_∞-gain between input r and $\dot{v}$

Theorem 4.1. *1) Let us suppose that the nominal system (5) has commensurate delays. Let K be a stable controller which BIBO-stabilizes the nominal system (5), the L_∞-gain of K being denoted M_∞^K.*

If $M_\infty^{cl}(\sum_{j=1}^{\ell} \mu_j \|A_j\| + \sum_{k=1}^{m} \nu_k \|B_k\| M_\infty^K + \epsilon \|h\|_\infty) < 1$ then the controller K stabilizes system (4) in a BIBO sense.

2) Let K be a stable controller which H_∞-stabilizes the nominal system (5), the L_2-gain of K being denoted M_2^K. Let us suppose that $\dot{r}$ is in L_2.

If $M_2^{cl}(\sum_{j=1}^{\ell} \mu_j \|A_j\| + \sum_{k=1}^{m} \nu_k \|B_k\| M_2^K) < 1$ then the controller K stabilizes system (4) in the sense that there is a finite L_2 gain between r and x.

Proof. 1) Let $\tilde{K}$ be a stable BIBO-stabilizing controller of system (5) with convolution kernel $\mathcal{K}$.

Considering the stabilization of (4) and (5) we deal with the following equations, where as before $y = x - v$:

and

$$\begin{aligned}\dot{y}(t) &= Ay(t) + \sum_{j=1}^{\ell} A_j y(t-h_j) + \sum_{j=1}^{\ell} A_j f_j(t) \\ &+ B(\mathcal{K} * y)(t) + \int_D^{\delta(t)} h(\theta) x(t-\theta)\, d\theta \\ &+ \sum_{k=1}^{m} B_k(\mathcal{K} * \tilde{f}_k)(t) + \sum_{k=1}^{m} B_k(\mathcal{K} * \tilde{r})(t), \qquad (t>0),\end{aligned}$$

with $f_j(t) = x(t-\tau_j(t)) - x(t-h_j)$, $\tilde{f}_k(t) = x(t-\sigma_k(t)) - x(t-T_k)$ and $\tilde{r}(t) = r(t-\sigma_k(t)) - r(t-T_k)$.

Using the same arguments as in section 3, we get

$$\|\dot{x}\|_\infty \leq \|\dot{v}\|_\infty + M_\infty^{cld}(\sum_{j=1}^{\ell} \mu_j \|A_j\| \|\dot{x}\|_\infty + \sum_{k=1}^{m} \nu_k \|B_k\| M_\infty^K \|\dot{x}\| + 2\sum_{k=1}^{m} \|B_k\| M_\infty^K \|r\|_\infty + \epsilon \|h\|_\infty \|x\|_\infty).$$

If $1 - M_\infty^{cld}(\sum_{j=1}^{\ell} \mu_j \|A_j\| + \sum_{k=1}^{m} \nu_k \|B_k\| M_K) > 0$, the above inequality gives us:

$$(1 - M_\infty^{cld}(\sum_{j=1}^{\ell} \mu_j \|A_j\| + \sum_{k=1}^{m} \nu_k \|B_k\| M_K)) \|\dot{x}\| \leq (M_\infty^{clnomd} + 2M_\infty^{cld} \sum_{k=1}^{m} \|B_k\| M_\infty^K) \|r\|_\infty + \epsilon \|h\|_\infty \|x\|_\infty$$

As before writing $x = y + v$ we get a L_∞-bound between r and x.

2) Taking $\dot{r}$ in L_2 enables us, as in the Proof of Theorem 3.1, to bound $\|\dot{r}\|_2$, which proves the result.

□

4.2 The case of neutral systems

Proposition 4.1. *Let us suppose that there exits a stable controller K which BIBO-stabilizes the nominal system (2), the L_∞-gain of K being denoted M_∞^K, the L_∞-gain of the closed-loop being denoted M_∞^{cl} and the L_∞-gain between the input and the derivative of the state of the closed-loop being denoted M_∞^{cld}.*

If $M_\infty^{cld}(\sum_{j=1}^{\ell} \mu_j \|A_j\| + \sum_{k=1}^{m} \nu_k \|B_k\| M_\infty^K + 2 + \sum_{l=1}^{L} \|A_{-l}\|) < 1$ then K BIBO-stabilizes system (1).

Proof. For neutral systems, we deal with equations

$$\begin{aligned}\dot{v}(t) + + \sum_{l=1}^{L} A_{-l}\dot{v}(t-H_l) &= Av(t) + \sum_{j=1}^{\ell} A_j v(t-h_j) + B\mathcal{K} * v(t) + \sum_{k=1}^{m} B_k \mathcal{K} * v(t-T_k) \\ &+ + \int_0^D h(\theta) v(t-\theta)\, d\theta + Br(t) + \sum_{k=1}^{m} B_k r(t-T_k), \qquad (t>0),\end{aligned}$$

and

$$
\begin{aligned}
\dot{y}(t)+\sum_{l=1}^{L}A_{-l}\dot{v}(t-H_l) &= Av(yt)+\sum_{j=1}^{\ell}A_j y(t-h_j)+\sum_{j=1}^{\ell}A_j f_j(t)\\
&+\sum_{l=1}^{L}A_{-l}(x(t-H_l)-x(t-\gamma(t))++\int_0^D h(\theta)y(t-\theta)\,d\theta\\
&+\int_D^{\delta(t)} h(\theta)x(t-\theta)\,d\theta\\
&+B(K*y)(t)+\sum_{k=1}^{m}B_k(\mathcal{K}*y)(t-T_k)+Br(t)+\sum_{k=1}^{m}B_k r(t-T_k)\\
&+\sum_{k=1}^{m}B_k(\mathcal{K}*\tilde{f}_k)+\sum_{k=1}^{m}B_k\mathcal{K}*(r(t-\sigma_k(t))-r(t-T_k)), \qquad (t>0),
\end{aligned}
$$

from which we get

$$
\begin{aligned}
\|\dot{x}\|_\infty \leq \|\dot{v}\|_\infty + M_\infty^{cld}(2\sum_{l=1}^{L}\|A_{-l}\|\|\dot{x}\|_\infty \sum_{j=1}^{\ell}\|A_j\|\mu_j\|\dot{x}\|_\infty\\
+\sum_{k=1}^{m}\|B_k\|\nu_k M_\infty^K\|\dot{x}\| + 2\sum_{k=1}^{m}\|B_k\|M_\infty^K\|r\|_\infty+\epsilon\|h\|_\infty\|x\|_\infty).
\end{aligned}
$$

□

Although the existence of a stable controller cannot be guaranteed in the general case, let us mention that the existence of a stabilizing controller can be indeed guaranteed in the particular case of systems with commensurate delays.

It has been shown in [5] that neutral systems with commensurate delays and a finite number of poles in $\{\mathrm{Re}\, s > a\}$ with $a < 0$ are stabilizable in an H_∞-sense: coprime factorizations over H_∞ have been determined and the set of all stabilizing controllers was given. It is not difficult to see that the coprime factorizations are also in $\hat{\mathcal{A}}$, inducing BIBO-stabilizability as well.

The next Proposition shows that, under hypothesis (H) systems (2) with commensurate delays fall into the study of [5].

Proposition 4.2. *If the system (2) has commensurate delays, then there exists $a < 0$ such that system (2) only has a finite number of poles in $\{\mathrm{Re}\, s > a\}$.*

Proof. It is well-known [1] that the location of chains of poles of system (2) can be determined from the denominator of its transfer function, more precisely from the coefficient (containing exponential terms) of the term s^n.

Considering $\det(sI - A + \sum_{\ell=1}^{L} A_\ell s e^{-s\ell H} - \sum_{j=1}^{J} A_j e^{-sh_j} - \int_0^D h(\theta) I e^{-\theta s} d\theta)$ we notice that it is sufficient to look at $\det(sI + \sum_{\ell=1}^{L} A_{-\ell} s e^{-s\ell})$ as other terms do not contribute to the coefficient of the term s^n.

Letting $z = e^{-s\ell}$, we get $\det(sI + \sum_{\ell=1}^{L} A_{-\ell} s e^{-s\ell}) = s^n \det(I + \sum_{\ell=1}^{L} A_{-\ell} z^\ell)$.

Now, assume that there is $z \in \mathbb{C}$ such that $|z| \leq 1$ and $\det(I + \sum_{\ell=1}^{L} A_{-\ell} z^\ell) = 0$. Then there exits $x^* \in \mathbb{C}^n \setminus 0$ with $\|x^*\| = 1$ such that $x^* = -(\sum_{\ell=1}^{L} A_{-\ell} z^\ell) x^*$.

In this case, by hypothesis (H)

$$1 = \| - x^* \| = \| \sum_{\ell=1}^{L} A_{-\ell} z^\ell x^* \| \le \sum_{\ell=1}^{L} \|A_{-\ell}\| \|x^*\| |z| \le (\sum_{\ell=1}^{L} \|A_{-\ell}\|) |z| < 1$$

which is absurd so all roots of $\det(I + \sum_{\ell=1}^{L} A_{-\ell} z^\ell) = 0$ are of modulus strictly greater than one entailing that all roots in s are strictly in the left half plane. All chains of poles of neutral system (2) are then asymptotic to vertical axes located in the open right half-plane ensuring a finite number of pole in $\{\operatorname{Re} s > a\}$ with $a < 0$. □

5 Example

Consider the elementary delay system

$$\dot{x}(t) + x(t-h) = u(t),$$

with transfer function $G_h(s) = 1/(s + e^{-sh})$. This is H_∞ and BIBO stable provided that $0 \le h < \pi/2$ (see, for example, [17, Chap. 6]).

Now we consider the perturbed system

$$\dot{x}(t) + x(t - \tau(t)) = u(t), \tag{10}$$

with $0 \le h < \tau(t) \le h + \mu$.

By Theorem 3.1, we have H^∞ stability if $\mu < M_2^{d-1}$.

For $h = 0, 0.5, 1$ and 1.5 these values are 1, 0.63, 0.32 and 0.03 respectively; naturally $h + \mu < \pi/2$ in all cases.

For BIBO stability a similar result holds, except that we require the BIBO norm of $sG_h(s) = 1 - e^{-sh} G_h(s)$, which is not easy to calculate as we do not have an explicit form of the impulse response. One way round this is to use the Hardy–Littlewood inequality given in [11, p. 182] (see also [4]), namely that

$$\|G_h\|_{BIBO} \le \frac{1}{2} \|G_h'\|_{L^1(i\mathbb{R})}.$$

Using this bound, we find that for $h = 0, 0.5, 1$ and 1.5 the BIBO norm of G_h is at most 1, 1.01, 2.96 and 39.1 respectively, giving BIBO stability for μ at most 0.5, 0.50, 0.25 and 0.025 respectively.

Now for $h = 2$ the system G_h is not stable, but it is easily stabilized with the constant controller $K = -1$, giving a closed-loop transfer function of

$$G_{cl}(s) = \frac{s + e^{-2s}}{s + 1 + e^{-2s}}.$$

Calculations indicate that $\|G_{cl}\|_\infty = 1.54$ and $\|G_{cl}\|_{BIBO} \le 3.89$. We may therefore apply Theorem 4.1, and conclude that K stabilizes the system (10) provided that $2 \le \tau(t) \le 2 + \mu$, where $\mu = 0.26$ in the BIBO case and $\mu = 0.65$ in the H_∞ case.

Acknowledgements

We are grateful to Asmahan Alajyan for useful discussions of the numerical results, A. Fioravanti for recalling to us a crucial fact in the proof of Theorem 4.2. and M. Souza for his help in making our proof more direct.

References

[1] R. Bellman and K.L. Cooke, *Differential-difference equations.* Academic Press, New York–London, 1963.

[2] C. Briat, *Linear parameter-varying and time-delay systems, Analysis, Observation, Filtering & Control.* Advances in Delays and Dynamics, Springer 2015.

[3] C. Bonnet and J.R. Partington, Bézout factors and L^1-optimal controllers for delay systems using a two-parameter compensator scheme. *IEEE Trans. Automat. Control* 44 (1999), no. 8, 1512–1521.

[4] C. Bonnet and J.R. Partington, Analysis of fractional delay systems of retarded and neutral type, *Automatica J. IFAC* 38 (2002), no. 8, 1133–1138.

[5] C. Bonnet, A.R. Fioravanti, and J.R. Partington, Stability of neutral systems with commensurate delays and poles asymptotic to the imaginary axis. *SIAM J. Control Optim.* 49 (2011), no. 2, 498–516.

[6] A.R. Fioravanti, H_∞ analysis and control of time-delay systems by methods in frequency domain, *PhD Thesis*, Université Paris-Sud, Paris XI, 2011.

[7] E. Fridman, *Introduction to Time-Delay Systems, Analysis and Control*, Birkhaüser, Springer, 2014.

[8] E. Fridman, On robust stability of linear neutral systems with time-varying delays. *IMA J. Math. Control Inform.* 25 (2008), no. 4, 393–407.

[9] E. Fridman and M. Gil', Stability of linear systems with time-varying delays: a direct frequency domain approach. *J. Comput. Appl. Math.* 200 (2007), no. 1, 61–66.

[10] E. Fridman and U. Shaked, Input-Output approach to stability and L_2-gain analysis of systems with time-varying delays *Systems and Control Letters* 55 (2006) 1041–1053.

[11] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations.* Encyclopedia of Mathematics and its Applications, 34. Cambridge University Press, Cambridge, 1990.

[12] C.-Y. Kao and A. Rantzer, Stability analysis of systems with uncertain time-varying delays. *Automatica J. IFAC* 43 (2007), no. 6, 959–970.

[13] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equation* Springer Verlag, New York, 1993.

[14] J. Louisell, Delay-differential systems with time-varying delay: new directions for stability theory *Kybernetica* 37 (2001), no. 3, 239–251.

[15] W. Michiels, V. Van Assche and S.-I. Niculescu, Stabilization of Time-Delay Systems With a Controlled Time-Varying Delay and Applications, *IEEE Trans. on Automatic Control* 50 (2005), no. 4, 493–504.

[16] K.M. Mikkola and A.J. Sasane, Bass and topological stable ranks of complex and real algebras of measures, functions and sequences. *Complex Anal. Oper. Theory*, 4 (2010), no. 2, 401–448.

[17] J.R. Partington, *Linear operators and linear systems.* London Mathematical Society Student Texts, 60. Cambridge University Press, Cambridge, 2004.

[18] A. Quadrat, On a general structure of the stabilizing controllers based on stable range. *SIAM J. Control Optim.* 42 (2004), no. 6, 2264–2285.

[19] P.F. Quet, B. Ataslar, A. Iftar, H. Ozbay, S. Kalyanaranam and T. Kang, Rate-Based Flow Controllers for Communication Networks in the Presence of Uncertain Time-Varying Multiple Time-Delays *Automatica* 38 (2002), 917–928.

[20] E. Shustin and E. Fridman, On delay-derivative-dependent stability of systems with fast-varying delays. *Automatica J. IFAC* 43 (2007), no. 9, 1649–1655.

[21] E. Verriest, State Space for Time Varying Delay, Time Delay Systems: Methods, Applications and New Trends, Lecture Notes in Control and Information Sciences, Springer, (2012), 135–146.

[22] C. Yuan and F. Wu, H_∞ state-feedback control of linear systems with time-varying input delays. *Proc of IEEE 55th Conference on Decision and Control (CDC), Las Vegas, USA* (2016).