Types are internal infinity-groupoids
Eric Finster, Antoine Allioux, Matthieu Sozeau

To cite this version:
Eric Finster, Antoine Allioux, Matthieu Sozeau. Types are internal infinity-groupoids. LICS 2021, Jun 2021, Rome, Italy. hal-03133144v2

HAL Id: hal-03133144
https://inria.hal.science/hal-03133144v2
Submitted on 5 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract—By extending type theory with a universe of definitionally associative and unital polynomial monads, we show how to arrive at a definition of opetopic type which is able to encode a number of fully coherent algebraic structures. In particular, our approach leads to a definition of $\infty$-groupoid internal to type theory and we prove that the type of such $\infty$-groupoids is equivalent to the universe of types. That is, every type admits the structure of an $\infty$-groupoid internally, and this structure is unique.

I. INTRODUCTION

Homotopy Type Theory has brought a new perspective to intensional Martin-Löf type theory: the higher identity types of a type endow it with the structure of an $\infty$-groupoid, and ideas from homotopy theory provide us with a means to predict and understand the resulting tower of identifications. While this perspective has been enormously clarifying with respect to our understanding of the notion of proof-relevant equality, leading, as it has, to a new class of models as well as new computational principles, a number of difficulties remain in order to complete the vision of type theory as a foundation for a new, structural mathematics based on homotopy-theoretic and higher categorical principles.

Foremost among these difficulties is the following: how does one describe a well behaved theory of algebraic structures on arbitrary types? The fundamental difficulty in setting up such a theory is that, in a proof relevant setting, nearly all of the familiar algebraic structures (monoids, groups, rings, and categories, to take a few) become infinitary. Of course, we will need to be somewhat more precise about what exactly we mean by algebraic structures. This leaves us in a strange position: absent a theory of algebraic structures, we have nothing to use to encode higher structures. Internally to type theory, however, we do not have recourse to such techniques. Indeed, without further hypotheses, we do not even expect that the most basic objects of the theory, types themselves, are presented by set-level structures. This leaves us in a strange position: absent a theory of algebraic structures, we have nothing to use to encode higher structures!

We suggest that a possible explanation for this phenomenon is the following: contrary to our experience with set-level mathematics, where an algebraic structure (i.e. a “structured set”) can itself be defined just in terms of sets: underlying sets, functions, sets of relations and so on, when we pass to the world of homotopy theoretic mathematics, the notion of type and structured type are simply no longer independent of each other in the same way. Consequently, some primitive notion of structured type must be defined at the same time as the notion of type itself. The present work is a first attempt at rendering this admittedly somewhat vague idea precise.

We begin by imagining a type theory which, in addition to defining a universe $\mathcal{U}$ of types, defines at the same time a universe $\mathcal{S}$ of structures. Of course, we will need to be somewhat more precise about what exactly we mean by structure. Category theory suggests that one way of representing a structure is by the monad on $\mathcal{U}$ which it defines, so we might think of $\mathcal{S}$ as a universe of monads. In practice, however, it will be useful to restrict to a particularly well behaved class of monads, having reasonable closure properties, and for which we have a good understanding of their higher dimensional counterparts. We submit that a reasonable candidate for such a well-behaved collection is the class of polynomial monads [1].

We feel that this is an appropriate class of structures for a number of reasons. A first reason is that this class of monads arises quite naturally in type theory already: indeed, a large literature exists on the interpretation of inductive and coinductive types as initial algebras and terminal coalgebras for polynomial monads, and we consider our work as deepening and extending this connection. Furthermore, this class of algebraic structures enjoys some pleasant properties...
which make them particularly amenable to “weakening”. For example, the very general approach to weakening algebraic structures developed by Baez and Dolan in [2] can be smoothly adapted to the polynomial case. While the cited work employs the language of symmetric operads, connections with the theory of polynomial functors were already described in [3], and moreover, recent work [4] has shown that, in type theory, we should expect symmetric operads to in fact be subsumed by the theory of polynomial monads.

The central intuition of Baez and Dolan’s approach, is that each polynomial monad $M$ determines its own higher dimensional collection of shapes (the $M$-opetopes) generated directly from the syntactic structure of its terms. They go on to introduce the notion of an $M$-opetopic type which is, roughly, a collection of well formed decorations of these shapes, and the notion of weak $M$-algebra is then defined as an $M$-opetopic type satisfying certain closure properties. In this sense, their approach differs from, say, approaches based on simplices, cubes or spheres in that the geometry is not fixed ahead of time, but adapted to the particular structure under consideration.

With these considerations in mind, our plan in the present work is to put the idea of a type theory with primitive structures to the test. What might it look like, and what might it be able to prove? In order to answer these questions, we will build a prototype of the theory using Agda, and exploit the recent addition of rewrite rules [5] which permits us to extend definitional equality by new well-typed reductions. The use of such rewrites is necessary to ensure that our primitive structures are not subject to the same infinite regress of coherence conditions which has so far obstructed more naive attempts and defining such objects.

Concretely, we will introduce a universe $\mathbb{M}$ whose elements we think of as codes for polynomial monads and describe the structures they decode to. Because we think of the objects of the universe $\mathbb{M}$ as primitives of our theory, on the same level as types, we allow ourselves the freedom to prescribe their computational behavior: in particular, we will equip them with definitional associativity and unit laws using the rewrite mechanism alluded to above. We emphasize that if structures are taken as defined in parallel with types, then this kind of definitional behavior should be no more surprising than, say, the definitional associativity of function composition.

We then show how the existence of our universe $\mathbb{M}$ has some strong consequences. In particular, it allows us to implement the Baez and Dolan definition of opetopic type alluded to above, and subsequently to define a number of weak higher dimensional structures. Among the structures which we are able to define using this technique are $\infty$-monoids and groups, $(\infty, 1)$-categories and presheaves over them (in particular, our setup leads to a definition of simplicial type), and as a special case, $\infty$-groupoids themselves.

There arises, then, the problem of justifying the correctness of our definitions. In order to do so, we will take up the example of $\infty$-groupoids in some detail. Indeed, since the homotopical interpretation of type theory asserts that types should “be” $\infty$-groupoids, it seems natural to compare these two objects. Our main result is the following:

**Theorem.** There is an equivalence

$$U \simeq \infty\text{-Grp}$$

In other words, every type admits the structure of an $\infty$-groupoid in our sense, and that structure is unique. This theorem, therefore, can be regarded as a (constructive) internalization of the intuition provided by the various meta-theoretic results to this effect [6], [7].

**A. Related Work**

The so-called coherence problem, which is the main motivation for the present work, has been considered by a number of authors. We briefly compare our approach with two prominent other strains of thought.

1) **Synthetic Structures:** One way to avoid some of the problems posed by the definition of higher dimensional structures is to simply enlarge the collection of basic objects to include them. Such approaches may be described as synthetic, in that they do not reduce higher structures to more primitive objects in exactly the same way that homotopy type theory itself does not define $\infty$-groupoids in terms of sets. This point of view is often adopted, for example, in the research into directed type theories, whether they be aimed at specific structures like higher categories as in [8], or allow for more general directed spaces as in [9].

While our theory does indeed add some new primitive structures to type theory, we collect these structures in a universe and decode them into collections of ordinary types and maps between them. Moreover, we go on to use these additional strict structures to give analytic, internal definitions of higher structures.

2) **Two-Level Type Theory:** Perhaps the closest related work to the current approach is that of Two-Level Type Theory [10]. There it is advocated to add a second “level” to type theory with a set-truncated equality type which one can then use to make meta-theoretic statements about the inner level, whose objects are typically taken to be the homotopically meaningful ones. The two-level approach provides, thus, a great deal of generality and flexibility at the cost of restricting the applicability of homotopical interpretation of types to the inner theory. It is likely, for example, that our theory could be developed inside a two-level system and many of the rewrites we employ proven as theorems in the outer level. By constrast, our approach is, we feel, somewhat more economical, extending the theory with a specific set of rewrite rules, and pointing towards the possibility of a useful theory of higher structures without the need to restrict homotopical principles like univalence.
B. Preliminaries

The basis of our metatheory is the type theory implemented in Agda [11] which is an extension of the predicative part of Martin-Löf type theory [12]. Among the particular types that Agda implements, we shall use inductive types, records and coinductive records.

As such, we adopt a style similar to Agda code, writing \((x : A) \rightarrow B x\) for the dependent product (although we occasionally also employ the \(\Pi_{x : A} B x\) notation if it improves readability). We also make use of the implicit counterpart of the dependent products, written \(\{ x : A \rightarrow B x \}\). This allows us to hide arguments which can be inferred from the context and hence clarify our notation. Non-dependent functions are denoted \(A \rightarrow B\), as usual. Functions enjoy the usual \(\eta\) conversion rule.

We shall make extensive use of coinductive record types, as well as copatterns for producing elements of these types. We write \(\top\) for the empty record with a constructor \(tt\) and projections \(\text{fst}\) and \(\text{snd}\). Pairs which are not dependent are denoted \(A \times B\).

We write \(\bot\) for the empty type, using absurd patterns where appropriate, and writing \(\bot\)-elim for the unique function for \(\bot\) to any type.

The identity type \(\_\equiv\_ : \{ A : \mathcal{U} \} \times \{ y : A \} \rightarrow \mathcal{U}\) is an inductive type with one constructor \(\text{refl}\) whose center of contraction will be referred to as \(\text{ctr}\).

Other notions defined in the HoTT book [13] will be employed including equivalence of types denoted \(X \simeq Y\), function extensionality denoted \(\text{funext}\), and the Univalence Axiom.

We write \(\mathcal{U}\) for the universe of small types, and \(\mathcal{U}_1\) for the next universe when necessary.

In order to unclutter and clarify the presentation, we occasionally take liberties with the formal definitions, for example, silently inserting applications to functional extensionality when necessary, or reassociating \(\Sigma\)-types in order to avoid a proliferation of commas. Our formal development in Agda allows no such informalities to remain.

II. A Universe of Polynomial Monads

As we have explained in the introduction, type theory appears to lack the ability to speak about infinitely coherent algebraic structures, and our strategy for addressing this problem will be to distinguish a collection of such structures which we consider as defined by the theory itself. We do so using a common technique in the type theory literature: that of introducing a universe. We write \(\\mathcal{M} : \mathcal{U}\) for our universe, and we think of its elements as codes for polynomial monads. Just as a typical type theoretic universe has some collection of base types and some collection of type constructors, so we will populate our universe with a collection of “base monads” and “monad constructors”.

In other words: we will have a syntax of structures which parallels the syntax for types.

Typically, a universe of types \(\mathcal{U}\) comes equipped with a decoding function \(El : \mathcal{U} \rightarrow \mathcal{U}\). In the case of our universe of monads \(\mathcal{M}\), each element \(M : \mathcal{M}\) will decode not to a single type, but to a collection of types and type families equipped with some structure. We will use rewrite rules to specify the computational behavior of this structure.

A. Polynomial Structure

To begin, we first equip each \(M : \mathcal{M}\) with an underlying polynomial or indexed container [13]. This is accomplished by postulating the following collection of decoding functions:

- \(\text{Idx} : \mathcal{M} \rightarrow \mathcal{U}\)
- \(\text{Cns} : (M : \mathcal{M}) \rightarrow \text{Idx} M \rightarrow \mathcal{U}\)
- \(\text{Pos} : (M : \mathcal{M}) \{ i : \text{Idx} M \} \rightarrow \text{Cns} M i \rightarrow \mathcal{U}\)
- \(\text{Typ} : (M : \mathcal{M}) \{ i : \text{Idx} M \} (c : \text{Cns} M i) \rightarrow \text{Pos} M c \rightarrow \text{Idx} M\)

Polynomials of this sort appear in the computer science literature as the “data of a datatype declaration”. They can equivalently be seen as a way to describe the signature of an algebraic theory: concretely, the elements of \(\text{Idx} M\), which we refer to as indices serve as the sorts of the theory, and for \(i : \text{Idx} M\), the type \(\text{Cns} M i\) is the collection of operation symbols whose “output” sort is \(i\). The type \(\text{Pos} M c\) then assigns to each operation a collection of “input positions” which are themselves assigned an index via the function \(\text{Typ}\).

It follows that every monad \(M\) induces a functor \(\square : (\text{Idx} M \rightarrow \mathcal{U}) \rightarrow (\text{Idx} M \rightarrow \mathcal{U})\) called its extension given by

\[
[M] X i = \sum_{(c : \text{Cns} M i)} (p : \text{Pos} M c) \rightarrow X (\text{Typ} M c p)
\]

We may think of the value of this functor at a type family \(X : \text{Idx} M \rightarrow \mathcal{U}\) as the type of constructors of \(M\) with inputs decorated by elements of \(X\). Indeed, we will frequently refer to a dependent function of the form

\[
(p : \text{Pos} M c) \rightarrow X (\text{Typ} M c p)
\]

where \(X\) is as above, as a decoration of \(c\) by elements of \(X\).

B. Monadic Structure

Next, for each monad \(M : \mathcal{M}\), we are going to equip the underlying polynomial of \(M\) with an algebraic structure: specifically, that structure required on the underlying polynomial so that the associated extension \([M]\) becomes a monad. In the case at hand, this takes the form of a pair of functions

- \(\eta : (M : \mathcal{M}) (i : \text{Idx} M) \rightarrow \text{Cns} M i\)
- \(\mu : (M : \mathcal{M}) \{ i : \text{Idx} M \} (c : \text{Cns} M i) \rightarrow (\delta : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)) \rightarrow \text{Cns} M i\)

which equip \(M\) with a multiplication and unit operation. We remark that the second argument \(\delta\) of the multiplication \(\mu\) is a decoration of \(c\) in the family \(\text{Cns} M\) of constructors, so that we can think of the input to this function as a “two-level” tree.

Crucial for what follows will be that the monads we consider are cartesian in the sense of [11]. Type theoretically,
the means we require each monad $M$ to come equipped with equivalences

$$\text{Pos} M (\eta M i) \simeq \top$$

$$\text{Pos} M (\mu M c \delta) \simeq \sum_{(p : \text{Pos} M c)} \text{Pos} M (\delta p)$$

Since we are already modifying the definitional equality of our type theory, it may be tempting to require these equivalences definitionally by asserting that the type of positions reduces when applied to constructors of the appropriate form. However, this will not work: as we will see below, when we come to populate our universe with concrete monads and monad constructors, the equivalences we find are often in fact not definitional, even if they remain provable. As an alternative, we will equip each monad with introduction, elimination and computation rules for its positions which will in effect guarantee that we always have the required equivalence. Each monad definition will then be required to implement these rules in a manner consistent with the various required typing laws.

In the case of $\eta$, for example, we postulate introduction and elimination rules of the form

$$\eta\text{-pos} : (M : \mathcal{M}) (i : \text{Idx} M) \rightarrow \text{Pos} M (\eta M i)$$

$$\eta\text{-pos-elim} : (M : \mathcal{M}) (i : \text{Idx} M)$$

$$\rightarrow (X : (p : \text{Pos} M (\eta M i)) \rightarrow \mathcal{U})$$

$$\rightarrow (u : X (\eta\text{-pos} M i))$$

$$\rightarrow (p : \text{Pos} M (\eta M i)) \rightarrow X p$$

with typing rule

$$\text{Typ} M (\eta M i) p \sim i \quad (\text{Typ}\eta)$$

and computation rule

$$\eta\text{-pos-elim} M i X u (\eta\text{-pos} M i) \sim u$$

Notice these are exactly the rules for an inductively defined indexed unit type$^{4}$ in particular, decorations of the constructor $\eta M i$ in a type family $X : \text{Pos} M (\eta M i) \rightarrow \mathcal{U}$ are completely determined by a single element $x : X i$, a fact which we record in the following definition to reduce clutter below:

$$\eta\text{-dec} M X \{i \} x = \eta\text{-pos-elim} M i (\lambda \_ \rightarrow X i) x$$

Next, for the multiplication $\mu$, our rules simply mimic the pairing and projections of the dependent sum. That is, we postulate an introduction rule

$$\mu\text{-pos} : (M : \mathcal{M}) \{i : \text{Idx} M\} \{c : \text{Cns} M i\}$$

$$\rightarrow \{\delta : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)\}$$

$$\rightarrow \{p : \text{Pos} M c\} \rightarrow (q : \text{Pos} M (\delta p))$$

$$\rightarrow \text{Pos} M (\mu M c \delta)$$

and elimination rules

$$\mu\text{-pos-fst} : (M : \mathcal{M}) \{i : \text{Idx} M\} \{c : \text{Cns} M i\}$$

$$\rightarrow \{\delta : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)\}$$

$$\rightarrow \text{Pos} M (\mu M c \delta) \rightarrow \text{Pos} M c$$

$$\mu\text{-pos-snd} : (M : \mathcal{M}) \{i : \text{Idx} M\} \{c : \text{Cns} M i\}$$

$$\rightarrow \{\delta : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)\}$$

$$\rightarrow \text{Pos} M (\delta (\mu\text{-pos-fst} M p))$$

with typing rule

$$\text{Typ} M (\mu M c \delta) p \sim \mu\text{-pos-fst} M p \text{ (Typ}\mu$$

$$\text{Typ} M (\delta (\mu\text{-pos-fst} M p)) \text{ (Typ}\eta$$

$$\text{Typ} M (\mu\text{-pos-snd} M p) \text{ (Typ}\eta$$

and computation rules

$$\mu\text{-pos-fst} M (\mu\text{-pos} M p q) \sim p \text{ (}\mu\text{-pos-fst}$$

$$\mu\text{-pos-snd} M (\mu\text{-pos} M p q) \sim q \text{ (}\mu\text{-pos-snd}$$

$$\mu\text{-pos} M (\mu\text{-pos-fst} M p) (\mu\text{-pos-snd} M p) \sim p \text{ (}\mu\text{-pos}\eta$$

With the handling of positions in place, we can now state the unitality and associativity axioms for the monads in our universe. These take the form of reductions:

$$\mu M c (\lambda p \rightarrow \eta M (\text{Typ} M c p)) \sim c \text{ (}\mu\eta\tau$$

$$\mu M (\eta M i) \delta \sim \delta (\eta\text{-pos} M i) \text{ (}\mu\eta\lambda$$

$$\mu M (\mu M c \delta) \epsilon \sim \epsilon \text{ (}\mu\mu$$

$$\mu M c (\lambda p \rightarrow \mu M (\delta p) (\lambda q \rightarrow \epsilon (\mu\text{-pos} M p q))) \text{ (}\mu\eta$$

Additionally, we must posit laws which assert that the constructors and eliminators for positions are compatible with these equations. We omit these for brevity, but the interested reader may consult the development for details.

While we will not undertake an extensive investigation of the meta-theoretic properties of our system in this article, we wish to pause briefly to make at least of few observations to justify its well-formedness. For example, there are critical pairs in the rewrite equations for the monad laws (between the first equation and the others) so we need to ensure confluence and termination.

**Lemma 1** (Strong confluence for $\eta$ and $\mu$). *The rewrite rules are strongly confluent [13], hence globally confluent.*

**Proof.** The rewrite system is strongly confluent using the rules $(\text{Typ}\eta)$ and $(\text{Typ}\mu)$ and the associated reduction rules for $\mu\text{-pos}$. We show the case for $\mu\eta\tau$ and $\mu\eta\lambda$. We omit $M$ which is fixed here.

$$\mu (\eta i) (\lambda p \rightarrow \eta (\text{Typ} (\eta i) p)) \sim (\eta\eta\eta) \eta i$$

$$\mu (\eta i) (\lambda p \rightarrow \eta (\text{Typ} (\eta i) p)) \sim (\eta\eta\eta) \eta i$$

$$(\lambda p \rightarrow \eta (\text{Typ} (\eta i) p)) (\eta\text{-pos}\ i) \sim (\eta\eta\eta) \eta i$$

$$\eta (\text{Typ} (\eta i) (\eta\text{-pos}\ i)) \sim (\text{Typ}\eta) \eta i$$

$^4$In principle, we would also like to have an $\eta$-rule for the unit $\eta$, that is, we would prefer the negative version as we have below for $\mu$ but unfortunately this is not possible with the current implementation of rewriting in Agda.
The resolution of the \(\mu\)-\(\eta\)-\(\mu\)-\(\mu\) pair can be found in the appendix of the extended version of this article [16] Lemma 3.

**Proposition 1** (Termination of rewriting). All of the above rules form a terminating rewrite system.

**Proof.** The \(\mu\)-pos, \(\eta\)-pos and \(\mathrm{Typ}\)-\(\eta\) rewrite rules are obviously terminating. For \(\mathrm{Typ}\)-\(\mu\), \(\mu\)-\(\eta\)-\(\mu\) and \(\mu\)-\(\mu\), we need to use a dependency-pairs path ordering as introduced by [17] to verify termination. In particular for associativity, a lexicographic lifting of the subterm relation is not enough to verify \(\mu\)-\(\mu\)’s termination as we are going under binders and applying the \(\delta\) and \(\epsilon\) functions to subterms. This is a variant of the ordinal type eliminator proven to terminate in [17] Example 14, p11, which requires to ensure that the constructor types of our monads are inductively generated. All the monads considered in this article satisfy this.

We omit the remaining structure, which has a similar flavor.

2) The pullback monad: Our first monad constructor starts from a monad \(M : M\) and a family \(X : \mathsf{Id}x M \to U\) and refines the indices of \(M\) by additionally decorating the inputs and output of each constructor by elements of \(X\). We refer to the resulting monad as the pullback of \(M\) along \(X\) (cf. [2] Section 2.4). We implement this construction by first postulating a function

\[
\mathsf{Pb} : (M : M) (X : \mathsf{Id}x M \to U) \to M
\]

which adds the necessary code to our universe. We next define the polynomial part of \(\mathsf{Pb} M X\) as follows:

\[
\begin{align*}
\mathsf{Id}x (\mathsf{Pb} M X) & = \sum_{(i : \mathsf{Id}x M)} X i \\
\mathsf{Cns} (\mathsf{Pb} M X) (i, x) & = \sum_{(c : \mathsf{Cns} M i)} \prod_{(p : \mathsf{Pos} M c)} X (\mathsf{Typ} M c p) \\
\mathsf{Pos} (\mathsf{Pb} M X) (c, \nu) & = \mathsf{Pos} M c \\
\mathsf{Typ} (\mathsf{Pb} M X) (c, \nu) p & = (\mathsf{Typ} M c p, \nu p)
\end{align*}
\]

The unit for the pullback monad simply calls the unit of the underlying monad and decorates its input with the same value as its output:

\[
\eta (\mathsf{Pb} M X) (i, x) = (\eta M i, \eta\text{-dec} M X x)
\]

As for the multiplication of the pullback monad, it again simply calls the multiplication of the underlying monad, this time decorating the result using the decorations of the second-level constructors, and forgetting the intermediate decoration. That is, we have

\[
\mu (\mathsf{Pb} M X) (c, \nu) \delta = (\mu M c \delta', \nu')
\]

where

\[
\begin{align*}
\delta' & = \text{fst} (\delta p) \\
\nu' & = \text{snd} (\delta (\mu\text{-pos-fst} p)) (\delta (\mu\text{-pos-snd} p))
\end{align*}
\]

The remaining structure is easily worked out from these definitions, and we omit the details.

3) The Slice Monad: The Baez-Dolan slice construction is at the heart of the opetopic approach: it is this construction which allows us to “raise the dimension” of the coherences in our algebraic structures. In our setting, it will take the form of a
monad constructor \( \text{Slice} : \mathcal{M} \rightarrow \mathcal{M} \). The basic intuition is that, for a monad \( M : \mathcal{M} \), the monad \( \text{Slice} M \) may be described as the monad of relations in \( M \). In order to realize this intuition, we have to find a way to encode the relations of \( M \) as some kind of data, just as the identity type encodes the relations in an ordinary type as data. This data will then serve as the constructors for the slice monad.

To begin, for a monad \( M : \mathcal{M} \), let us define
\[
\text{Idx}(\text{Slice} M) = \Sigma_{i : \text{Idx} M} \text{Cns} M i
\]
That is, the indices of the monad \( \text{Slice} M \) are exactly the constructors of the monad \( M \). Next, we are going to capture the notion of relation in \( M \) with the help of a certain inductive family, defined as follows:

\[
\text{data Tree} : \text{Idx}(\text{Slice} M) \rightarrow \mathcal{U} \quad \text{where}
\begin{align*}
\text{If} & : (i : \text{Idx} M) \rightarrow \text{Tree}(i, \eta M i) \\
\text{nd} & : (i : \text{Idx} M) \quad (c : \text{Cns} M i) \\
& \quad \rightarrow (\delta : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)) \\
& \quad \rightarrow (\epsilon : (p : \text{Pos} M c) \rightarrow \text{Tree}(\text{Typ} M c p, \delta p)) \\
& \quad \rightarrow \text{Tree}(i, \mu M c \delta)
\end{align*}
\]
And we define \( \text{Cns}(\text{Slice} M) = \text{Tree} \).

The reader familiar with the theory of inductive types may recognize this as a modified form of the indexed \( W \)-type associated to a polynomial or indexed container. Here, as in that case, the elements of this type are trees generated by the constructors of the polynomial in question (the underlying polynomial of \( M \), in the case at hand). The difference in the present setup is that our polynomial is equipped with a multiplication and unit, and we reflect this fact by indexing our trees not just over the indices (as is typically the case) but also over the constructors, applying the multiplication and unit as appropriate. The result is that we may view an element \( \eta : \text{Cns}(\text{Slice} M)(i, c) \) as “a tree generated by the constructors of \( M \) whose image under iterated multiplication is \( c \)”. It is in this sense that this definition captures the relations in the original monad \( M \).

We now turn to the rest of the structure required to complete the definition of \( \text{Slice} M \). Intuitively speaking, the positions of a tree \( \sigma \) will be its internal nodes. This can be accomplished by defining the positions by recursion on the constructors as follows:
\[
\text{Pos} (\text{Slice} M) (\text{If} i) = \bot
\]
\[
\text{Pos} (\text{Slice} M) (\text{nd} c \delta \epsilon) = \top \sqcup \sum_{p : \text{Pos} M c} \text{Pos} (\text{Slice} M) (\epsilon p)
\]
In other words, if our tree is a leaf, it has no positions, and if it is a node, its positions consist of either the unit type (to record the current node) or else the choice of a position of the base constructor and, recursively, a node of the tree attached to that position.

Finally, the typing function \( \text{Typ} (\text{Slice} M) \sigma p \) just projects out the constructor of \( M \) occurring at the node of \( \sigma \) specified by position \( p \):
\[
\begin{align*}
\text{Typ} (\text{Slice} M) (\text{If} i) () &= () \\
\text{Typ} (\text{Slice} M) (\text{nd} \{ i \} c \delta \epsilon) (\text{inl} \text{tt}) &= (i, c) \\
\text{Typ} (\text{Slice} M) (\text{nd} \{ i \} c \delta \epsilon) (\text{inl}(p, q)) &= \\
\text{Typ} (\text{Slice} M) (\epsilon p) q
\end{align*}
\]
It remains to describe the unit and multiplication of the slice monad. In accordance with the general laws for monads, the unit constructor needs to have a unique position, and since the positions of a given tree are given by occurrences of constructors, this implies that the unit at a given constructor \( c \) should be the corolla, that is, a tree with one node consisting of \( c \) itself. Therefore we set:
\[
\begin{align*}
\eta (\text{Slice} M)(i, c) &= \\
\text{nd} c (\lambda p \rightarrow \eta M (\text{Typ} M c p)) (\lambda p \rightarrow \text{If} (\text{Typ} M c p))
\end{align*}
\]
Note that this definition would not be type correct without the assumption that \( M \) is definitionally right unital. A similar remark applies to the rest of the definitions of the slice monad in this section. Indeed, it is exactly the problem of completing the definition of the slice monad without any assumptions of truncation which motives the introduction of our monadic universe in the first place.

Let us now sketch the definition of the multiplication in the slice monad. As hypotheses, we are given a tree \( \sigma : \text{Cns}(\text{Slice} M)(i, c) \) for some \( i : \text{Idx} M \) and \( c : \text{Cns} M i \), as well as a decoration
\[
\phi : (p : \text{Pos} (\text{Slice} M) \sigma) \rightarrow \text{Cns} (\text{Slice} M) (\text{Typ} (\text{Slice} M) \sigma p)
\]
In view of the preceding discussion, this means that \( \phi \) assigns to every position of \( \sigma \) a tree which multiplies to the constructor which inhabits that position. The multiplication of \( \text{Slice} M \) may intuitively be described as “substituting” each of these trees into the node it decorates.

The definition of \( \mu (\text{Slice} M) \) will require an auxiliary function \( \gamma \) with the following type:
\[
\gamma : (M : \mathcal{M}) \{ i : \text{Idx} M \} (c : \text{Cns} M i) \\
\rightarrow (\sigma : \text{Cns} (\text{Slice} M)(i, c)) \\
\rightarrow (\phi : (p : \text{Pos} M c) \rightarrow \text{Cns} M (\text{Typ} M c p)) \\
\rightarrow (\psi : (p : \text{Pos} M c) \rightarrow \text{Cns} (\text{Slice} M) (\text{Typ} M c p, \phi p)) \\
\rightarrow \text{Cns} (\text{Slice} M)(i, \mu M c \phi)
\]
The intuition of this function is that \( \gamma \) grafts the tree specified by \( \psi \) onto the appropriate leaf of the tree \( \sigma \) (indeed, \( \gamma \) may be seen as an incarnation of multiplication in the free monad generated by the underlying polynomial of \( M \)). This function simply operates by induction and may be defined as follows:
\[
\begin{align*}
\gamma M (\text{If} i) \delta \epsilon &= \epsilon (\eta \text{-pos} M i) \\
\gamma M (\text{nd} c \delta \epsilon) \phi \psi &= \text{nd} c \delta' \epsilon'
\end{align*}
\]
where we define
\[
\phi' p q = \phi(\mu \text{-pos} M c \delta p q)
\]
present article, however, we omit the details. As we will not have need of this construction in the same setup as the non-dependent case, simply repeating each extension \( M \) reason, when we are given a monad \( M \) using propositional equalities as it is encoded directly in the we do not need to axiomatize the notion of homomorphism \( M \). Aspects of Martin-Löf type theory, it is perhaps not surprising \( \psi \) a family of types dependent on a base type, a dependent monad \( M \). Rather, it is a monad \( M \) of our polynomial monads. We note there is a potential point of \( \psi \) for the dependent monad \( M \). We refer the reader to the formalization for details on the remaining constructions handling positions in the slice monad.

D. Dependent monads

Since the notion of dependent type is one of the primitive aspects of Martin-Löf type theory, it is perhaps not surprising that we quickly find ourselves in need of a dependent version of our polynomial monads. We note there is a potential point of confusion here: while a dependent type can be thought of as a family of types dependent on a base type, a dependent monad in our sense is not a family of monads. Rather, it is a monad structure on dependent families of indices and constructors indexed over the indices and constructors of the base monad \( M \). Put another way, under the equivalence between dependent types with domain \( A \) and functions with codomain \( A \), our dependent monads over a base monad \( M \) correspond to monads \( M' \) equipped with a cartesian homomorphism to \( M \). The advantage of working in a dependent style, however, is that we do not need to axiomatize the notion of homomorphism using propositional equalities as it is encoded directly in the typing of the multiplication operator.

To begin, let us postulate, for each monad \( M : \mathbb{M} \), a universe \( \mathbb{M}_\downarrow : \mathbb{M} \rightarrow \mathcal{U} \) of monads over \( M \).

\[
\mathbb{M}_\downarrow : \mathbb{M} \rightarrow \mathcal{U}
\]

That is, for \( M : \mathbb{M} \), the inhabitants of \( \mathbb{M}_\downarrow M \) are codes for monads equipped with a cartesian morphism to \( M \). For this reason, when we are given a monad \( M \) and a dependent monad \( M_\downarrow : \mathbb{M}_\downarrow M \), we often speak of the pair \( (M, M_\downarrow) \) as a monad extension.

The decoding functions for dependent monads follow the same setup as the non-dependent case, simply repeating each of the definitions fiberwise. And since the dependent case resembles so closely the non-dependent one, we have attempted to systematically name dependent versions of the the monadic structure introduced above by appending a “\( \downarrow \)” to the previously given name. For example, \( \text{Idx}_\downarrow \) for the dependent version of the family \( \text{Idx} \) of indices.

As a first step, a dependent monad will be equipped with a polynomial lying over the base polynomial. This corresponds to the following three dependent families:

\[
\begin{align*}
\text{Idx}_\downarrow & : \{M : \mathbb{M}\} \rightarrow \mathbb{M}_\downarrow M \rightarrow \text{Idx} M \rightarrow \mathcal{U} \\
\text{Cns}_\downarrow & : \{M : \mathbb{M}\} (\mathbb{M}_\downarrow M) \{i : \text{Idx} M\} \rightarrow \text{Idx}_\downarrow M_\downarrow i \rightarrow \text{Cns} M i \rightarrow \mathcal{U} \\
\text{Typ}_\downarrow & : \{M : \mathbb{M}\} (\mathbb{M}_\downarrow M) \rightarrow \{i : \text{Idx} M\} \{i_\downarrow : \text{Idx}_\downarrow M_\downarrow i\} \rightarrow \{c : \text{Cns} M i\} (c_\downarrow : \text{Cns}_\downarrow M_\downarrow i \downarrow c) \rightarrow \text{Pos} M c \rightarrow \text{Idx}_\downarrow M_\downarrow (\text{Typ} M c p)
\end{align*}
\]

The reader will notice, however, that there is no analog of dependent positions. This is because we are modelling cartesian morphisms of monads, and therefore positions of a dependent constructor \( c_\downarrow : \text{Cns}_\downarrow M_\downarrow i \downarrow c \) should be the same as those of the underlying constructor \( c \). By working fiberwise, we can reflect this requirement directly in the type signature.

The monadic structure of a dependent monad simply operates fiberwise, following the multiplication in the base monad:

\[
\begin{align*}
\eta_\downarrow & : \{M : \mathbb{M}\} (\mathbb{M}_\downarrow i) \rightarrow \{i : \text{Idx} M\} \\
\mu_\downarrow & : \{M : \mathbb{M}\} (\mathbb{M}_\downarrow i \downarrow c) \rightarrow \{c : \text{Cns} M i\}
\end{align*}
\]

The fact that we require the multiplication of a family of dependent constructors to live over the multiplication of the base constructors (and similarly for the unit) is what guarantees the homomorphism property alluded to above.

Our dependent monads must also be equipped with equalational laws making them compatible with the corresponding laws of the monads they live over. For example, the typing functions for \( \eta_\downarrow \) and \( \mu_\downarrow \) respect the indices of parameters, just as in the base case:

\[
\begin{align*}
\text{Typ}_\downarrow & M_\downarrow (\eta_\downarrow M \downarrow i) p \rightsquigarrow i_\downarrow \\
\text{Typ}_\downarrow & M_\downarrow (\mu_\downarrow M \downarrow c \downarrow i) p \rightsquigarrow \\
& \quad \text{Typ}_\downarrow (\mu_\text{-pos-fst} \downarrow \downarrow M p) \\
& \quad (\mu_\text{-pos-snd} \downarrow \downarrow M p)
\end{align*}
\]

\[\psi' \ p \ q = \psi (\mu \text{-pos} M c \psi' \ p \ q)\]

\[\delta' \ p = \mu M (\psi' \ p)\]

\[\epsilon' \ p = \gamma M (\psi' \ p)\]
There are similar laws asserting the definitional associativity and unitality of the multiplication, but as these all follow exactly the same pattern, we omit the details here and refer the curious reader to the implementation.

We remark that, because their positions are the same, decorations of the dependent constructor \( \eta \downarrow \) are essentially constant just as in the case of \( \eta \), and there is therefore an analogous function \( \eta \text{-dec}\downarrow \) generating such decorations from a single piece of data with a similar definition. This function occurs occasionally in the code below.

E. Populating the dependent universe

We now quickly describe dependent counterparts of the base monads and monad constructors of the previous section. As most of the definitions are routine and easily deduced from the absolute case, the presentation here is brief.

1) The identity monad: The dependent identity monad is parametrized by a type \( A : \mathcal{U} \) and indexed over the identity monad \( \text{Id} \). That is, we have a dependent monad constructor of the form

\[
\text{Id} \downarrow : \mathcal{U} \to \mathcal{M} \downarrow \text{Id}
\]

Its polynomial part is defined by

\[
\begin{align*}
\text{Idx} \downarrow (\text{Id} \downarrow A) \text{tt} &= A \\
\text{Cns} \downarrow (\text{Id} \downarrow A) \text{xt} &= T \\
\text{Pos} \downarrow (\text{Id} \downarrow A) \text{tt} &= T \\
\text{Typ} \downarrow (\text{Id} \downarrow A) \{ i \downarrow = x \} \text{tt} &= x
\end{align*}
\]

As in the base case, the multiplication and unit all take values in the unit type, making the structure essentially trivial.

2) The dependent pullback monad: Just as we can refine the indices of a base monad, so the dependent pullback monad allows us to refine the indices of a dependent monad. Its constructor takes the form

\[
\begin{align*}
Pb \downarrow : \{ M : \mathcal{M} \} (M \downarrow : \mathcal{M} \downarrow M) \{ X : \text{Idx} M \to \mathcal{U} \} \\
&\to (X \downarrow : \{ i : \text{Idx} M \to \text{Idx} M \downarrow i \to X \downarrow i \to \mathcal{U} \} \\
&\to \mathcal{M} \downarrow (Pb M X)
\end{align*}
\]

Note that the family \( X \downarrow \) may also depend on elements of the refining family \( X \) for the base monad. The underlying polynomial of the dependent pullback is then defined as follows:

\[
\begin{align*}
\text{Idx} \downarrow (Pb \downarrow M \downarrow X \downarrow) (i, x) &= \sum_{(i \downarrow : \text{Idx} M \downarrow i)} X \downarrow i \downarrow x \\
\text{Cns} \downarrow (Pb \downarrow M \downarrow X \downarrow) (i, x, c) &= \sum_{(c \downarrow : \text{Cns} M \downarrow i \downarrow c)} \prod_{(p : \text{Pos} M \downarrow c)} X \downarrow (\text{Typ} \downarrow M \downarrow c \downarrow p) (\nu p) \\
\text{Typ} \downarrow (Pb \downarrow M \downarrow X \downarrow) (c, \nu) &= \text{Typ} \downarrow M \downarrow c \downarrow p, \nu \downarrow p
\end{align*}
\]

3) The dependent slice monad: Finally, the dependent slice monad extends the Baez-Dolan slice construction to the dependent case. Its monad constructor is typed as follows:

\[
\text{Slice} \downarrow : \{ M : \mathcal{M} \} (M \downarrow : \mathcal{M} \downarrow M) \to \mathcal{M} \downarrow (\text{Slice} M)
\]

As for the absolute case, the indices are given by the dependent constructors. That is, we set

\[
\text{Idx} \downarrow (\text{Slice} \downarrow M \downarrow) (i, c) = \sum_{i \downarrow : \text{Idx} M \downarrow i} \text{Cns} \downarrow M \downarrow i \downarrow c
\]

Similarly, the type of constructors \( \text{Cns} \downarrow (\text{Slice} \downarrow M \downarrow) \) are trees lying over a tree in the base. This corresponds to the following (rather verbose) inductive type:

\[
\begin{align*}
data \text{Tree} \downarrow : \{ i : \text{Idx} (\text{Slice} M) \} \\
&\to (i \downarrow : \text{Idx} (\text{Slice} M \downarrow)) \\
&\to \text{Tree} i \to \mathcal{U} \text{ where} \\
\text{If} \downarrow : \{ i : \text{Idx} M \} (i \downarrow : \text{Idx} M \downarrow i) \\
&\to \text{Cns} \downarrow (\text{Slice} M \downarrow) (i \downarrow, \eta \downarrow M \downarrow i \downarrow) (\text{If} i) \\
\text{nd} \downarrow : \{ i : \text{Idx} M \} \{ c : \text{Cns} M \downarrow i \} \\
&\to \{ \delta : (p : \text{Pos} M \downarrow c) \to \text{Cns} M (\text{Typ} M \downarrow c \downarrow p) \} \\
&\to \{ \epsilon : (p : \text{Pos} M \downarrow c) \to \text{Cns} M (\text{Typ} M \downarrow c \downarrow p, \delta \downarrow p) \} \\
&\to \{ \delta : (p : \text{Pos} M \downarrow c) \to \text{Cns} M (\text{Typ} M \downarrow c \downarrow p, \delta \downarrow p) \} \\
&\to \text{Cns} \downarrow (\text{Slice} M \downarrow) (\text{Typ} M \downarrow c \downarrow p, \delta \downarrow p) \\
&\to \text{Cns} \downarrow (\text{Slice} M \downarrow) (i \downarrow, M \downarrow C \downarrow c \downarrow \delta \downarrow p) (\text{nd} c \delta \epsilon)
\end{align*}
\]

The rest of the description of the dependent slice follows exactly the same pattern: duplicating the definitions and laws of the base case routinely in each fiber.

III. Opetopic Types

In this section, we show how to use the universes introduced above in order to implement Baez and Dolan’s definition of opetopic type \([2]\). We go on to explain how to use this definition to capture the notion of weak \( M \)-algebra, and finish with some examples.

Definition 1. An opetopic type over a monad \( M \) is defined coninductively as follow:

\[
\begin{align*}
\text{record} & \quad \text{OpetopicType} (M : \mathcal{M}) : \mathcal{U}_1 \text{ where} \\
&\quad \text{C} : \text{Idx} M \to \mathcal{U} \\
&\quad \mathcal{R} : \text{OpetopicType} (\text{Slice} (Pb M C))
\end{align*}
\]

We see from the definition that an opetopic type consists of an infinite sequence of dependent families

\[
C X, C (\mathcal{R} X), C (\mathcal{R} (R X)), \ldots
\]

whose domain is the set of indices of a monad whose definition incorporates all the previous families in the sequence. Given an opetopic type \( X : \text{OpetopicType} M \), we will often denote this sequence of dependent types more succinctly as just \( X_0, X_1, X_2, \ldots \) since the destructor notation quickly becomes
quite heavy. We will use a similar convention for the series of monads \(M = M_0, M_1, M_2 \ldots\) generated by the definition. That is, we have:

\[
\begin{align*}
M_0 &= M \\
M_1 &= \text{Slice}(\text{Pb} M_0, X_0) \\
M_2 &= \text{Slice}(\text{Pb} M_1, X_1) \\
& \vdots \\
M_n &= \text{Slice}(\ldots) \\
\end{align*}
\]

Before describing the connection between opetopic types and weak \(M\)-algebras, let us give some examples of how to think of the resulting dependent families as “fillers” for a collection of “shapes” generated by the monad \(M\). For concreteness, we will fix \(M = \text{Id}\) in our examples. Given \(X : \text{OpetopicType} \text{Id}\), we can define the type of objects of \(X\) as simply

\[
\begin{align*}
\text{Obj} : U \\
\text{Obj} = C \times tt
\end{align*}
\]

Next, after a single slice, \(X\) provides us with a type of arrows between any two objects which can be defined as follows:

\[
\begin{align*}
\text{Arrow} : (x : \text{Obj}) \to U \\
\text{Arrow} xy &= C(RX) \\
& (\langle tt, y \rangle, (tt, n-\text{dec} \text{Id}(C X) x))
\end{align*}
\]

Furthermore, for a loop \(f\) in \(X\), that is, an arrow with the same domain and codomain, \(X\) includes a family whose elements can be thought of as “null-homotopies of \(f\)”, and which is defined by

\[
\begin{align*}
\text{Null} : (x : \text{Obj})(f : \text{Arrow} xx) \to U \\
\text{Null} xy &= C(R(RX)) \\
& (\langle \langle tt, x \rangle, (tt, n-\text{dec} \text{Id}(C X) x) \rangle, f), \\
& \text{If} (tt, x), \bot-\text{elim})
\end{align*}
\]

More examples of shapes and filling families may be found in the development.

**A. Weak Algebras and Fibrant Opetopic Types**

We now wish to describe how an opetopic type \(X : \text{OpetopicType} M\) encodes the structure of a weak \(M\)-algebra. Before we begin, it will be convenient to adopt the following convention: recall that \(X\) consists of an infinite sequence of dependent types following the form of Equation [1] In the discussion which follows, instead of working with a fixed opetopic type \(X\), we will rather just work with abstract type families \(X_0, X_1, \ldots\) over monads \(M = M_0, M_1, \ldots\) following the same pattern of dependences. We will then freely add new families of the form \(X_1\) to our hypotheses as they become necessary. The advantage of working this way is that our definitions are parameterized over just that portion of the opetopic type which is necessary, as opposed to depending on the entire opetopic type \(X\) itself, and consequently, we will be able to reuse our definitions and constructions starting at any point in the infinite sequence generated by \(X\).

We recall that for \(M\) a polynomial monad, an \(M\)-algebra consists of a carrier family \(C : \text{Id} X \to U\) together with a map

\[
\alpha : \{i : \text{Id} M\} \to [M] C i \to C i
\]

which satisfies some equations expressing the compatibility of \(\alpha\) with the multiplication of \(M\). Indeed, it is the need for a complete description of these equations in all dimensions which motivates the present work. Now, clearly the first dependent type \(X_0 : \text{Id} M \to U\) may serve as a carrier for an \(M\)-algebra structure. Let us now see what else this sequence of families provides us with.

After one iteration, we obtain a type family \(X_1 : \text{Id} M_1 \to U\), and unfolding the definition of the indices of the slice and pullback monads, we find that the domain of \(X_1\) takes the form

\[
\sum_{i : \text{Id} M} \sum_{x : X_0} \sum_{c : \text{Cns} M i} (p : \text{Pos} M c) \to X_0(\text{Typ} M c p)
\]

The elements of this type are 4-tuples \((i, x, c, \nu)\), and we now observe that the three elements \(i, c\) and \(\nu\) are typed such that they are exactly the arguments of the hypothetical algebra map \(\alpha\) introduced above. We may regard the family \(X_1\), therefore, as a relation between triples \((i, c, \nu)\) and elements \(x : X_0 i\), and in order to define a map \(\alpha\), we only need to impose that this relation is functional in the sense that there is a unique \(x\) determined by any such triple. When this is the case, we will say that the family \(X_1\) is multiplicative. That is, we define:

\[
\begin{align*}
\text{is-mult} : & \{ X_0 : \text{Id} M_0 \to U \} \times \{ X_1 : \text{Id} M_1 \to U \} \to U \\
\text{is-mult} \{ X_0 \} X_1 &= \{ i : \text{Id} M \} (c : \text{Cns} M i) \\
& \to (\nu : (p : \text{Pos} M c) \to X_0(\text{Typ} M c p)) \\
& \to \text{is-contr} (\sum_{x, X_0 i} X_1(i, x, c, \nu))
\end{align*}
\]

Supposing we are given a proof \(m_1 : \text{is-mult} X_1\), we can define an algebra map \(\alpha\) as above by

\[
\alpha(c, \nu) = \text{fst}(\text{ctr}(m_1(c, \nu)))
\]

Furthermore, we will write

\[
\alpha-\text{wit}(c, \nu) = \text{snd}(\text{ctr}(m_1(c, \nu)))
\]

for the associated element of the relation \(X_1(i, \alpha(c, \nu), c, \nu)\) which witnesses this multiplication.

Let us now suppose that our sequence extends one step further, that is, that we are given a type family \(X_2 : \text{Id} M_1 \to U\) and a proof \(m_2 : \text{is-mult} X_2\). We now show how to use this further structure to derive some of the expected laws for the algebra map \(\alpha\) we have just defined. As a first example, we expect \(\alpha\) to satisfy a unit law: decorating a unit constructor with some element \(x\) and then applying \(\alpha\) should return the element \(x\) itself. In other words, we expect to be able to prove

\[
\begin{align*}
\alpha^n-\text{coh} : & \{ i : \text{Id} M \} (x : X_0 i) \\
& \to \alpha(\eta M i, \eta-\text{dec} M X_0 x) \equiv x
\end{align*}
\]


To prove this equality, let us define the following function:
\[
\eta_{-\text{alg}_{m_2}} : \{ i : \text{ldx} M \} (x : X_0 i) \\
\to X_1 ((i, x), (\eta M i, \eta_{\text{dec} M} X_0 x))
\]
\[
\eta_{-\text{alg}_{m_2}} = \text{fst} (\text{ctr} (m_2 (l f (i, x)) -\text{elim}))
\]
Now we simply notice that the pairs
\[
\text{ctr} (m_1 (\eta M i) (\eta_{\text{dec} M} X_0 x)) \equiv (x, \eta_{-\text{alg}} x)
\]
must be equal as indicated, since they inhabit a contractible space. Projecting on the first factor gives exactly the desired equation.

We also expect our algebra map \(\alpha\) to satisfy an equation expressing its compatibility with multiplication of the following form:
\[
\alpha_{-\text{coh}} : \{ i' : \text{ldx} M \} (c' : \text{Cns} M i) \\
\to (\delta' : (p : \text{Pos} M c) \to \text{Cns} M (\text{Typ} M c p))
\]
\[
\to (\nu' : (p : \text{Pos} M c') (q : \text{Pos} M (\delta' q)) \\
\to X_0 (\text{Typ} M (\delta' p) q))
\]
\[
\to \alpha (\mu M c' \delta') (\lambda p \to \nu' (\mu_{\text{pos-fst}} p) (\mu_{\text{pos-snd}} p)) \equiv \alpha c' (\lambda p \to \alpha (\delta' p) (\nu' p))
\]

We note that this equation is simply the type theoretic translation of the familiar commutative diagram
\[
\begin{array}{ccc}
[M] X_0 & \xleftarrow{\mu X_0} & [M] X_0 \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
[M] X_0 & \xrightarrow{\alpha} & X_0
\end{array}
\]
To prove this axiom, we use \(m_2\) to define the following multiplication operation on elements of the family \(X_1\):
\[
\mu_{-\text{alg}_{m_2}} : \{ i : \text{ldx} M \} (c : \text{Cns} M i) \\
\to (\nu : (p : \text{Pos} M c) \to X_0 (\text{Typ} M c p))
\]
\[
\to (\delta : (p : \text{Pos} M c) \\
\to \text{Cns} (\text{Pb} M X_0) (\text{Typ} (\text{Pb} M X_0) (c, \nu) p))
\]
\[
\to (x_0 : X_0 i) (x_1 : X_1 (i, x_0, c, \nu))
\]
\[
\to (x : (p : \text{Pos} M c) \to X_1 (\text{Typ} (\text{Pb} M X_0) (c, \nu), \delta p))
\]
\[
\to X_1 (i, x_0, \mu (\text{Pb} M X_0) (c, \nu) \delta)
\]
\[
\mu_{-\text{alg}_{m_2}} = \text{fst} (\text{ctr} (m_2 \sigma \theta))
\]
where
\[
\sigma = \text{nd} (c, \nu) \delta (\lambda p \to \eta M_1 (\text{Typ} M c p, \nu p, \delta p))
\]
is the two-level tree consisting of a base node \((c, \nu)\), as well as a second level of constructors specified by the decoration \(\delta\), and \(\theta\) is the decoration of the nodes of \(\sigma\) by elements of \(X_1\) defined by:
\[
\theta (\text{inl} \text{tt}) = x_1
\]
\[
\theta (\text{inr} (p, \text{inl} \text{tt})) = \bar{x} p
\]
Now instantiating our function \(\mu_{-\text{alg}_{m_2}}\) with arguments
\[
c = c' \\
x_1 = \alpha_{-\text{wit}} (c, \nu)
\]
\[
\nu p = \alpha (\delta' p, \nu' p) \\
\delta p = (\delta' p, \nu' p)
\]
\[
x_0 = \alpha (c, \nu)
\]
\[
\bar{x} p = \alpha_{-\text{wit}} (\delta' p, \nu' p)
\]
we find that the pairs
\[
\text{ctr} (m_1 (\mu M c' \delta') (\lambda p \to \nu' (\mu_{\text{pos-fst}} p) (\mu_{\text{pos-snd}} p))) \equiv (\alpha (c, \nu), \mu_{-\text{alg}_{m_2}} c \nu x_0 \bar{x} \delta \bar{x})
\]
again inhabit a contractible space, whereby their first components are equal, giving the desired equation.

We may think of the functions \(\eta_{-\text{alg}_{m_2}}\) and \(\mu_{-\text{alg}_{m_2}}\) defined above as the nullary and binary cases of a multiplicative operation on the relations of our algebra structure. The key insight, as we have seen, is that this multiplicative structure encodes exactly the laws for the algebra map \(\alpha\) defined one level lower. Similarly, if we are able to extend our sequence on further step to a family \(X_3\) which is itself multiplicative, then we will be able to show that the operations \(\eta_{-\text{alg}_{m_2}}\) and \(\mu_{-\text{alg}_{m_2}}\) themselves satisfy unit and associativity laws, and this in turn encodes the “2-associativity” and “2-unitality” of the algebra map \(\alpha\). This motivates the following definition:

**Definition 2.** An opetopic type \(X\) over a monad \(M\) is said to be **fibrant** if we are given an element of the following coinductively defined property:

\[
\text{record} \text{ is-fibrant } \{ M : M \} \{ X : \text{OpetopicType} M \} : \text{U}
\]
where
\[
\text{car-is-mult} : \text{is-mult} (M (C (\text{R} X)))
\]
\[
\text{rel-is-fibrant} : \text{is-fibrant} (\text{R} X)
\]

Fibrant opetopic types, therefore, are our definition of infinitely coherent \(M\)-algebras, with the multiplicativity of the relations further in the sequence witnessing the higher dimensional laws satisfied by the structure earlier in the sequence.

**B. Higher structures**

We now use the preceding notions to define a number of coherent algebraic structures. A first example is that we obtain an internal definition of the notion of \(\infty\)-groupoid as follows:

**Definition 3.** An \(\infty\)-**groupoid** is a fibrant opetopic type over the identity monad. That is,

\[\infty\text{-Grp} = \Sigma (X : \text{OpetopicType Id}) \text{is-fibrant} X \]

We will attempt to justify the correctness of this definition in the sections which follow.

Next, it happens that the monad \(\text{Slice Id}\) is in fact the monad whose algebras are monoids, and consequently, our setup leads naturally to the definition of an \(\infty\)-**type**, that is, a type with a coherently associative binary operation.

**Definition 4.** An \(\infty\)-**type** is a fibrant opetopic type over the first slice of the identity monad.

\[\infty\text{-type} = \Sigma (X : \text{OpetopicType} (\text{Slice Id})) \text{is-fibrant} X \]
Furthermore, the notion of $A_\infty$-group can now be defined by imposing an invertibility axiom. A classical theorem of homotopy theory asserts that the type of $A_\infty$-groups is equivalent to the type of pointed, connected spaces via the loop-space construction. It would be interesting to see if the techniques of this article lead to a proof of this fact in type theory.

The notion of $\infty$-category can also be defined using this setup. Recall that an opetopic type over the identity monad $\Id$ has both a type of objects and a type of arrows (Equations 2 and 3). In the definition of $\infty$-groupoid above, the invertibility of the arrows in the underlying opetopic type is a consequence of the fact that the family of arrows is assumed to be multiplicative. Consequently, we obtain a reasonable notion of a pre-$\infty$-category by simply dropping this assumption, and only requiring fibrancy after one application of the destructor $R$:

$$\text{pre-}\infty\text{-Cat} = \sum_{(X: \text{OpType}_\Id)} \text{is-fibrant} (R X)$$

The prefix "pre" here refers to the fact that this definition is missing a completeness axiom asserting that the invertible arrows coincide with paths in the space of objects, that is, an axiom of univalence in the sense of \cite{18}. Such an axiom is easily worked out in the present setting, but as it would distract us slightly from the main objective of the present work, we will not pursue the matter here.

IV. THE $\infty$-GROUPOID ASSOCIATED TO A TYPE

In this section, we use the machinery we have set up to produce an $\infty$-groupoid associated to any type and eventually prove it is unique. As a first step, we will need a source of opetopic types. Here is where the notion of dependent monad becomes important: we now show that every dependent monad gives rise to an opetopic type. The reason for this phenomenon is simple: since our dependent monad constructors mirror the monad constructors of the absolute case, any monad extension $\langle M, M_\downarrow \rangle$ in fact gives rise to a new monad extension as follows:

$$M \quad \mapsto \quad \text{Slice} (Pb \downarrow M_\downarrow)$$

$$M_\downarrow \quad \mapsto \quad \text{Slice}_\downarrow (Pb_\downarrow M_\downarrow (\lambda j k \rightarrow j \equiv k))$$

Notice how by pulling back along $\Id X M_\downarrow$, the identity type gives us a canonical family along which to apply the $Pb_\downarrow$ constructor. Iterating this construction, then, we find that associated to every monad extension $\langle M, M_\downarrow \rangle$, is an infinite sequence

$$\langle M, M_\downarrow \rangle = (M_0, M_{\downarrow 0}), (M_1, M_{\downarrow 1}), (M_2, M_{\downarrow 2}), \ldots$$

where $(M_{i+1}, M_{\downarrow i+1})$ is obtained from $(M_i, M_{\downarrow i})$ by the above transformation.

The above construction provides us with our desired source of opetopic types. Formally, we define (using copattern notation)

$$\downarrow \text{OpType} M M_\downarrow : \text{OpType}_\text{Type} M$$

$$C (\downarrow \text{OpType} M M_\downarrow) = \Id X M_\downarrow$$

$$R (\downarrow \text{OpType} M M_\downarrow) = \downarrow \text{OpType} (\text{Slice} (Pb (\Id X M_\downarrow)))$$

Specializing to the case of the identity monad, we obtain the following:

**Definition 5.** For a type $A : \U$, the underlying opetopic type of $A$ is defined to be the opetopic type associated to the dependent identity monad determined by $A$. That is, the opetopic type

$$\downarrow \text{OpType} \Id (\Id X A)$$

in the notation of the previous paragraph.

In order to show that every type $A$ determines an $\infty$-groupoid in our sense, our next task is to show that this opetopic type is in fact fibrant.

A. Algebraic Extensions

Let $M : M_\llbracket$ and $M_\downarrow : M_{\downarrow \llbracket}$. We will say that the extension $\langle M, M_\downarrow \rangle$ is algebraic if we have a proof

$$\text{is-algebraic} : (M : M_\llbracket) \langle M_\downarrow : M_{\downarrow \llbracket} \rangle \rightarrow \U$$

$$\text{is-algebraic} = \{ i : \Id X M \} \{ c : \text{Cns} M i \}
\rightarrow (\nu : (p : \text{Pos} M c) \rightarrow \Id X M_\downarrow (\text{Typ} M c p))$$

$$\text{is-contr} \left( \sum_{(i : \Id X M) (c : \text{Cns} M i)} \sum_{\text{Typ} M \downarrow c \equiv \nu} \right)$$

An algebraic extension should be thought of as roughly analogous to a generalized kind of optification: if we think of the constructors as generalized arrows between their input indices and output, then the hypothesis says we know a family of lifts over the source of our constructor, and the conclusion is that there exists a unique "pushforward" consisting of a lift over the output as well as a constructor connecting the two whose typing function agrees with the provided input lifts. Such a hypothesis is one way of encoding an $M$-algebra, which motivates the name for this property. See \cite[Section 6.3]{19}.

The main use of the notion of algebraic extension is the following lemma, whose proof is entirely straightforward:

**Lemma 2.** Suppose the pair $\langle M, M_\downarrow \rangle$ is an algebraic extension. Then the relation $\Id X M_{\downarrow \llbracket}$ is multiplicative.

Consequently, just as dependent monads are a source of opetopic types, algebraic extensions can be thought of as a source of multiplicative relations. Hence if we want to prove fibrancy of the opetopic type associated to a monad extension, we will need to know which of the extensions in the generated sequence are algebraic. Our main theorem is that after a single iteration of the slice construction, every monad extension becomes algebraic. That is

**Theorem 1.** Let $\langle M, M_\downarrow \rangle$ be a monad extension. Then slice extension $\langle M_1, M_{\downarrow 1} \rangle$ is algebraic.
A proof can be found in the extended version of this article. The importance of the theorem is that it has the following immediate corollaries:

**Corollary 1.** Let \((M, M^\downarrow)\) be an algebraic extension. Then the opetopic type \(\downarrow\text{OpType} M M^\downarrow\) is fibrant.

**Proof.** The base case of the coinduction is Lemma 2 and the coinductive case is covered by Theorem 1.

**Corollary 2.** There is a map \(\Gamma : \mathcal{U} \to \infty\text{-Grp}\).

**Proof.** Let \(A : \mathcal{U}\) be a type. A short calculation shows that the monad extension \((\text{Id}, \text{Id}_A, A)\) is algebraic. The result therefore follows from Corollary 1.

### B. Uniqueness of the Groupoid Structure

We now turn to the task of showing the map \(\Gamma : \mathcal{U} \to \infty\text{-Grp}\) is an equivalence. Observe that there is a forgetful map \(\Upsilon : \infty\text{-Grp} \to \mathcal{U}\) which is given by extracting the type of objects (Equation 2) from the opetopic type underlying a monad extension \((M, M^\downarrow)\).

The base case of the coinduction is Lemma 2 and the coinductive case is covered by Theorem 1.

We have taken some liberties in the presentation of this theorem (strictly speaking, we have not stated precisely in what sense the second equivalence \(e_1\) is “over” the equivalence \(e_0\), nor precisely what equality is implied by symbol the \(\equiv_{e_0, e_1}\)) but these omissions can be made perfectly rigorous by standard techniques, and we feel the statement above conveys the essential ideas perhaps more clearly than a fully elaborated statement, which would require a great deal more preparation, not to mention space. See the appendix of the extended version of this article for a proof of Theorem 2.

We at last obtain our desired equivalence:

**Theorem 3.** The map

\[
\Gamma : \mathcal{U} \to \infty\text{-Grp}
\]

to the typing function of \(\mathcal{U}\), we will additionally need to add such an equivalence to our list of hypotheses.

To recap: assuming the equivalences \(e_0\) and \(e_1\) amounts to requiring that the first two stages of the opetopic type \(X\) are equivalent to the indices and constructors of the dependent monad \(M^\downarrow\), respectively. What structure remains? Well, the dependent constructors of \(M^\downarrow\) are equipped with the unit and multiplication operators \(\eta_1\) and \(\mu_1\). But now, recall from Section III-A that if the family of relations \(X_1\) extends further in the sequence to a family \(X_2\) and we have a proof \(m_2 : \text{is-mult} X_2\), then the family \(X_1\) can be equipped with a multiplicative structure given by the functions \(\eta_{\text{-alg}_{m_2}}\) and \(\mu_{\text{-alg}_{m_2}}\) defined there. This is the case in the current situation, if we assume that the opetopic type \(X\) is fibrant (in fact, we only need assume that \(\mathcal{R} X\) is fibrant to make this statement hold). Therefore, the last piece of information in order that \(X\) “completely agrees” with the dependent monad \(M^\downarrow\) is that the equivalence \(e_1\) is additionally a homomorphism, sending \(\eta_{\downarrow} \downarrow \to \eta_{\text{-alg}_{m_2}}\) and \(\mu_{\downarrow} \downarrow \to \mu_{\text{-alg}_{m_2}}\). Our theorem now is that this data suffices to prove an equivalence of opetopic types:

**Theorem 2.** Suppose \((M, M^\downarrow)\) is a monad extension and \(X : \text{OpetopicType} M\) an opetopic type such that \(\mathcal{R} X\) is fibrant. Moreover, suppose we are given the data of

- An equivalence \(e_0 : (i : \text{Id} X) \to \text{Id} X i \equiv X_0 i\)
- An equivalence \(e_1 : (i : \text{Id} M) \to \text{Id} M i \equiv_{e_0} X_1\)

over the previous equivalence \(e_0\) amounts to saying that the relations of the family \(X_1\) “are” just the dependent constructors of \(M^\downarrow\) (again, reindexed according to the typing of their input and output positions). As this is again part of the data already provided by the dependent monad \(M^\downarrow\), we will additionally need to add such an equivalence to our list of hypotheses.

Then there is an equivalence of opetopic types

\[X \equiv_{e_0, e_1} \eta_{\text{-alg}_{m_2}}\text{ and }\mu_{\text{-alg}_{m_2}}\]

over the previous equivalence \(e_0\) amounts to saying that the relations of the family \(X_1\) “are” just the dependent constructors of \(M^\downarrow\) (again, reindexed according to the typing of their input and output positions). As this is again part of the data already provided by the dependent monad \(M^\downarrow\), we will additionally need to add such an equivalence to our list of hypotheses.

Then there is an equivalence of opetopic types

\[X \equiv_{e_0, e_1} \eta_{\text{-alg}_{m_2}}\text{ and }\mu_{\text{-alg}_{m_2}}\]

We have taken some liberties in the presentation of this theorem (strictly speaking, we have not stated precisely in what sense the second equivalence \(e_1\) is “over” the equivalence \(e_0\), nor precisely what equality is implied by symbol the \(\equiv_{e_0, e_1}\)) but these omissions can be made perfectly rigorous by standard techniques, and we feel the statement above conveys the essential ideas perhaps more clearly than a fully elaborated statement, which would require a great deal more preparation, not to mention space. See the appendix of the extended version of this article for a proof of Theorem 2.

We at last obtain our desired equivalence:

**Theorem 3.** The map

\[
\Gamma : \mathcal{U} \to \infty\text{-Grp}
\]
is an equivalence.

Proof. Given $G : \infty\text{-Grp}$, we let $A : U$ be its type of objects. We now apply Theorem 1 with $M = \text{Id}$ and $M' = \text{Id} A$. We may take $c_0$ to be the identity. The equivalence $c_1$ is a consequence of [13 Theorem 5.8.2] and the required equalities are a straightforward calculation. □

V. CONCLUSION

We have presented an approach to defining higher coherent structures in homotopy type theory by equipping type theory with a primitive set of structures collected into a universe $\mathbb{M}$ of polynomial monads, and demonstrated that this approach can be used to prove non-trivial theorems about these structures. In this brief formal section, we compare some related approaches and survey some of the possible directions and applications.

A. Future Directions

1) Symmetric Structures: A natural class of structures which escapes the capabilities of our current approach is that of symmetric structures, that is, those which incorporate higher analogs of commutativity. Examples would include $E_\infty$ groups and monoids, symmetric monoidal categories, and general $\infty$-operads and their algebras.

2) Higher Category Theory: As we have seen, one higher structure which is amenable to treatment by our methods is that of an $\infty$-category. An obvious point to follow up on, then, is how much of the well developed theory of $\infty$-categories can be formalized in this manner.

3) A General Theory of Structures: As we have mentioned in the introduction, we see the present work as a first step towards a general theory of types and structures. And though we feel certain that at least some of the ideas of the present work will carry over to such a theory, a complete picture of the basic principles remains to be understood. Moreover, a careful investigation of the interaction of our techniques with univalent implementations of type theory (such as cubical type theory) also remains for future work.

Accompanying such a general theory, we anticipate a deeper investigation of the meta-theoretic properties of our proposed approach. For example, the Agda implementation is limited by the expressivity of rewrite rules, and complicated by the explicit universe construction, while a proper extension of MLTT would allow for the investigation of meta-theoretic properties like decidability of type checking and strong normalization using techniques like normalization-by-evaluation (and potentially settling the conjecture of [11-13]). Furthermore, we have not touched at all on the potential models of our system, topic which deserves we feel deserves careful attention.

REFERENCES


APPENDIX

Lemma 3 (Strong confluence for the $\mu c \eta \eta \eta \mu \mu$-pair). We fix a monad $M$ and show that the two rules can always be joined: if $t \leadsto M(M(M(t)))$ and $t \leadsto M(t')$ then $t'' \leadsto M(t'')$. 

Proof. In case of overlap, the first application reduces in one step:

$\mu (\mu c \delta) (\lambda (\mu c \delta) p) \leadsto M\mu\eta\mu c \delta$

We show that applying the $\mu c \eta \eta \mu \eta$ rule instead results in the same term, where only one rule applies on a given subterm at each step:

$\mu (\mu c \delta) (\lambda (\mu c \delta) p) \leadsto M\mu\eta\mu c \delta \leadsto M\mu\eta\mu c \delta$
\[ μ \circ (λ p \to μ (δ p)) (λ q \to η (Typ (δ p) q))) \]
\[ \sim _{\text{def}} μ \circ (λ p \to δ p) \]

\textbf{Theorem 1} Let \((M, M ↓)\) be a monad extension. Then slice extension \((M ↓, M ↓ ↓)\) is algebraic.

\textbf{Proof.} Unfolding the definitions, we have that \(\text{Idx} M \downarrow\) is the iterated sum:
\[
\sum (i: \text{Idx} M) \sum (j: M ↓ ↓ i) \sum (c: \text{Cns} M ↓) \sum (\nu: \text{Typ} M ↓ ↓ j)
\]
then a element
\[
(p: \text{Pos} M c) \to \text{Idx} M ↓ ↓ (\text{Typ} M c p)
\]
and for a given \((i, j, c, ν)\), the type \(\text{Idx} M ↓ ↓ i\) of dependent indices is itself an iterated sum of the form:
\[
\sum (j: \text{Idx} M ↓ ↓ i) \sum (r: j ↓) \sum (d: \text{Cns} M ↓) \sum (τ: \text{Typ} j ↓ ↓ d) \equiv ν
\]
In other words, if the tuple \((i, j, c, ν)\) is seen as a constructor in \(c\) with inputs and output decorated by elements of \(\text{Idx} M ↓ ↓ i\), then an element \((j' , r , d, τ)\) lying over it is simply a constructor of the dependent monad \(M ↓ ↓ c\) whose intrinsic typing information (that is, indices assigned by the underlying dependent polynomial of \(M ↓ ↓ c\)) matches the decoration of \(c\).

The constructors follow a similar pattern, but now assembled into trees: a constructor of the base monad \(M \downarrow\) is a tree \(σ\) whose internal nodes carry the additional information of a decoration of their incoming and outgoing edges by elements of \(\text{Idx} M ↓ ↓ i\). A constructor of \(M ↓ ↓ i\) lying over \(σ\) is itself a tree \(σ ↓\) built from dependent constructors of \(M ↓↓\), and carrying the additional information of equalities witnessing that the typing information of each node agrees with the decoration of the node it lies over in \(σ\), in the sense of the previous paragraph.

Now, the situation of the theorem is the following: we are given a 4-tuple \((i, j, c, ν) : \text{Idx} M ↓\) and a tree \(σ : \text{Cns} M ↓ i\) (\(i, j, c, ν\)) together with a decoration of this tree \(φ\) which assigns to each of the positions of \(M ↓ i\) then a element \(\mu i\) and \(ν = η \text{-dec} j\), and we may take
\[
ω = (j \text{-refl}, η ↓ ↓ j ∪ η \text{-dec} ↓ \text{-refl})
\]
\[
σ ↓ = (j \text{-refl})
\]
\[
ζ = \text{funext} ↓ \text{-elim}
\]
For the inductive case, that is, when \(σ = n \epsilon M ↓ c\), evaluating the decoration \(φ\) on the position corresponding to the base of our tree, that is \(φ(\text{inl} t t)\), we obtain a 4-tuple \((j', r, d, τ)\) where \(j' : \text{Idx} M ↓ ↓ i, r : j' ≡ j, d : \text{Cns} M ↓ ↓ c j'\) and \(τ : \text{Typ} ↓ ↓ d \equiv ν\). Then we define
\[
ω = (j', r, \mu i ↓ ↓ d \delta ↓ j', τ')
\]
\[
σ ↓ = n \epsilon ↓ ↓ i (d, τ) (λ p → δ ↓ j' p, τ' p q) \epsilon ↓ j'
\]
in which the decorations \(δ ↓ j'\) and \(ε ↓ j'\), as well as the typing compatibilities \(τ'\) and \(ζ'\) are obtained from the induction hypothesis, and \(β\) is a coherence asserting that the typing information of a constructor is unchanged when it is transported along an equality of indices.

In summary: the decoration \(φ\) assigns to each node of the tree \(σ\) a dependent constructor lying over the constructor occupying that node, together with compatibility information about the typing. By induction on \(σ\), we accumulate these dependent constructors into dependent tree \(σ ↓\) lying over \(σ\), and at the same time accumulate witnesses that the typing of the nodes in the resulting tree agrees with \(φ\), which it clearly does by construction.

To show uniqueness, we need to see that any other triple \((ω', σ', ζ')\) is equal to the one defined above. The proof again proceeds by induction on the given tree \(σ\), and the crucial point is that all the data of this 3-tuple can be eliminated away. To see this, let us expand \(ω' = (j', r, d, τ)\). Now, \(r : j' ≡ j\), so we may eliminate and assume \(j' = j\). Similarly, \(ζ' : \text{Typ} ↓ ↓ σ ↓ \equiv φ\) is just an equality between the typing function for \(σ ↓ j'\) and the original decoration \(φ\). Since \(φ\) was a free parameter to begin with, we can freely eliminate \(ζ'\) at this point and assume that \(φ\) actually is typing function \(σ ↓ j'\). We are left with just the dependent tree \(σ ↓ j'\) and the remaining two elements of \(ω', \text{namely} (d, τ)\). But now, as we proceed by induction on \(σ\), we can also proceed by induction on the dependent tree \(σ ↓ j'\). And in fact, this completely forces the values of \((d, τ)\) via the indexing of dependent trees: for example, when \(σ = \text{Id} \text{-dec} j\), the only valid possibility for \(σ ↓ j\) is \(\text{Id} ↓ (j, \text{refl})\), and in this case, we necessarily have \((d, τ) = (n ↓ ↓ j \text{-refl}, η \text{-dec} ↓ \text{-refl})\) as required. The nd case is similar, even if the path algebra is considerably more involved.

\textbf{Theorem 2} Suppose \((M, M ↓)\) is a monad extension and \(X : \text{OpetopicType} M\) an opetopic type such that \(R X\) is fibrant. Moreover, suppose we are given the data of

\begin{itemize}
  \item An equivalence \(e_0 : (i : \text{Idx} M) \to \text{Idx} M ↓ ↓ i ≃ X_0 i\)
  \item An equivalence \(e_1 : (i : \text{Idx} M ↓) \to \text{Idx} M ↓ ↓ i ≃ e_0 X_1\)
  \item Proofs that \(s : η ↓ ↓ M ↓ ↓ ≃ e_0 . e_1, η \text{-alg}_{m_2}\) and \(t : μ ↓ ↓ M ↓ ↓ ≃ e_0 . e_1, μ \text{-alg}_{m_2}\)
\end{itemize}

Then there is an equivalence of opetopic types
\[
X ≃_o ↓ \text{OpType} M M ↓
\]

\textbf{Proof.} The proof is by coinduction, and so it suffices to produce elements \(e'_0, e'_1, s', t'\) whose types are the same as
those of the corresponding elements in the hypotheses, but modified by the transformations
\[
(M, M \perp) \rightarrow (M_1, M^p_1) \quad X \rightarrow R X
\]
Next, we observe that it follows from univalence that the equivalences \(e_0\) and \(e_1\) can be eliminated from the hypotheses since the opetopic type \(X\) is abstract. In other words, we are free to assume that \(X_0 = \text{Idx}_\perp M_1\) and \(X_1 = \text{Idx}_\perp M^p_1\) so that the first remaining abstract family of \(X\) is
\[
X_2 : \text{ldx}_\perp M_2 \rightarrow U
\]
Moreover, after making these simplifications, the equalities \(s\) and \(t\) take the following types:
\[
s : (i : \text{ldx}_\perp PM) (j : \text{ldx}_\perp PM ↓_j i) \rightarrow (j, \eta ↓_j PM ↓_j j) \equiv \eta_\text{alg}_{m_2} (\text{fst} i) (\text{snd} i)\]
\[
t : (i : \text{ldx}_\perp PM) (c : \text{Cns} PM i) \rightarrow (\delta : \text{Pos} PM c) \rightarrow \text{Cns} PM (\text{Typ} PM c p) \rightarrow (j : \text{ldx}_\perp PM ↓_j i) (d : \text{Cns}_\perp PM ↓_j c) \rightarrow (\delta ↓_j (p : \text{Pos} PM c) \rightarrow \text{Cns}_\perp PM ↓_j (\text{Typ}_j PM ↓_j d p) (\delta p) \rightarrow (j, \mu ↓_j PM ↓_j d i) \equiv \mu_\text{alg}_{m_2} (\text{fst} i) (\text{snd} i) (\delta (\text{snd} i) (j, d) \rightarrow (\lambda p \rightarrow (\text{Typ}_j M ↓_j, (\text{fst} d) p, (\text{snd} d) p, (\delta ↓_j p))
\]
where we have set
\[
PM = \text{Pb}_\perp M (\text{ldx}_\perp M_\perp)\]
\[
PM ↓_j = \text{Pb}_\perp M_\perp (\lambda j k \rightarrow j \equiv k)
\]
to simplify the notation.
Now, we may take \(e_0\) to be the identity, since \(e_1\) is the identity. It therefore remains to construct the equivalence \(e_1'\) of type
\[
(i : \text{ldx}_\perp M_2) \rightarrow \text{ldx}_\perp M_\perp ↓_2 i \approx X_2 i
\]
and check the two required equations. We claim that it in fact suffices to construct a map
\[
e : (i : \text{ldx}_\perp M_2) \rightarrow \text{ldx}_\perp M^p_2 i \rightarrow X_2 i
\]
and that any such map is necessarily an equivalence. This follows from the fact that both \(\text{ldx}_\perp M_\perp \perp\) and \(X_2\) are multiplicative relations. The former by Theorem[1] and the latter by assumption.

Now, unfolding all the definitions and eliminating the redundancies, we find that the map \(e\) depends on the data of a tree \(\sigma : \text{Cns} M_1 (i, j, c, ν)\) where \((i, j, c, ν) : \text{ldx}_\perp M_1\), a decoration \(\phi\) of the nodes of this tree in the family \(\text{ldx}_\perp M^p_1\), a dependent tree \(\sigma^p\) living over \(\sigma\) and a proof \(\phi^p\) that the typing function of \(\sigma^p\) agrees with the data determined by \(\phi\) (the situation is analogous to that encountered in the proof of Theorem[1]). We may therefore proceed by simultaneous induction on \(\sigma\) and \(\sigma^p\). When \(\sigma = \text{lf} (i, j)\) and \(\sigma^p = \text{lf} (j, \text{refl})\), we must produce an element of \(X_2\) of type:
\[
X_2(i_s, j_s, \text{lf} (i, j), \phi)
\]
where
\[
i_s = (i, j, \eta M i, \eta \text{dec} j)\]
\[
j_s = (j, \text{refl}, \eta M \downarrow j, \eta \text{dec} \text{refl})
\]
on the other hand, we have an element
\[
x_2 : X_2(i_s, (\eta_\text{alg}_{m_2} i_s (\text{lf} (i, j) \phi), (i, j), \phi) \rightarrow \text{snd} (\text{ctr} (m_2 (\text{lf} (i, j) \phi)))
\]
since \(\eta_\text{alg}_{m_2}\) is defined using the multiplicativity of the relation \(X_2\). But the term \((i, j) (\text{lf} \phi)\) obtained from the hypothesis \(s\) now gives
\[
j_s \equiv \eta_\text{alg}_{m_2} i_s (\text{lf} (i, j) \phi)
\]
and so we obtain the desired result by transporting \(x_2\) along this equality.

Before moving on to the inductive case of a node, let us pause to anticipate how we intend to prove the equations \(s'\) and \(t'\), as this will have serious consequences for how we proceed. These equations will now be over the equivalence \(e_1'\) which we are in the process of constructing. Specifically, they take the form
\[
s' : (i : \text{ldx}_\perp PM i_1) (j : \text{ldx}_\perp PM ↓_j i) \rightarrow e_1'(i, \eta PM i_1) (j, \eta PM ↓_j j) \equiv \eta_\text{alg}_{m_3} (\text{fst} i) (\text{snd} i)
\]
\[
t' : (i : \text{ldx}_\perp PM i_1) (c : \text{Cns} PM i_1) \rightarrow (\delta : \text{Pos} PM c) \rightarrow \text{Cns} PM (\text{Typ} PM c p) \rightarrow (j : \text{ldx}_\perp PM ↓_j i) (d : \text{Cns}_\perp PM ↓_j c) \rightarrow (\delta ↓_j (p : \text{Pos} PM c) \rightarrow \text{Cns}_\perp PM ↓_j (\text{Typ}_j PM ↓_j d p) (\delta p) \rightarrow (j, \mu ↓_j PM ↓_j d i) \equiv \mu_\text{alg}_{m_3} (\text{fst} i) (\text{snd} i) (\delta (\text{snd} i) (j, d) \rightarrow (\lambda p \rightarrow (\text{Typ}_j M ↓_j, (\text{fst} d) p, (\text{snd} d) p, (\delta ↓_j p))
\]
where now each of monads have advanced by a single slice
\[
PM_1 = \text{Pb}_\perp M_1 (\text{ldx}_\perp M_\perp)
\]
\[
PM ↓_j = \text{Pb}_\perp M^p_\perp (\lambda j k \rightarrow j \equiv k)
\]
Furthermore, the left side of each equality now includes an application of the equivalence \(e_1'\), and the right side uses the unit and multiplicative operators \(\eta_\text{alg}_{m_3}\) and \(\mu_\text{alg}_{m_3}\) corresponding to the fact that the next family \(X_3\) is also multiplicative.

What this means is that, if we want to ensure that the equivalence \(e_1'\) sends the required elements to applications of the unit \(\eta_\text{alg}_{m_3}\) and multiplication \(\mu_\text{alg}_{m_3}\), we will need to use these functions in the definition of \(e_1'\) and furthermore, we need to use them both. But since the unit case
$\eta$-$\text{alg}_{m_3}$ in the next dimension corresponds to a corolla, that is, a tree with a single node in the present dimension, we will need to now make a case split depending on whether or not the node in the tree $\sigma$ we are recursing on has descendants or not. This is possible as soon as the monad $M$ we are considering is finitary in the sense that the type of positions is merely equivalent to finite type. In this case, the property of a tree being a corolla becomes decidable, and so we can make the required case split. This is not a problem for our intended application since it is easily checked that the identity monad has this property and, moreover, that it is inherited both by pulling back and slicing. Consequently, we will freely proceed now under this hypothesis.

Now, returning to the definition of our equivalence $e_1'$. We are in the inductive case so that we have $\sigma = \text{nd}(c,\nu) \delta \epsilon$ and $\sigma_\downarrow = \text{nd}(c_\downarrow,\nu_\downarrow) \delta_\downarrow \epsilon_\downarrow$. Our goal is to produce an element of type

$X_2(i_s,j_s,\text{nd}(c,\nu) \delta \epsilon,\phi)$

where now

$i_s = (i,j,\mu PM(c,\nu) \delta)$

$j_s = (j,\text{refl},\mu_\downarrow PM_\downarrow(c_\downarrow,\nu_\downarrow) \delta_\downarrow)$

We now make a case distinction based on whether the tree $\sigma$ is a corolla or not. In case that it is, one can prove equalities

$u : (i,j,\mu PM(c,\nu) \delta) \equiv (i,j,c,\nu)$

$v : (j,\text{refl},\mu_\downarrow PM_\downarrow(c_\downarrow,\nu_\downarrow) \delta_\downarrow) \equiv \phi(\text{inl tt})$

But then the result follows from transporting the term $\eta$-$\text{alg}_{m_3}(i,j,c,\nu)(\phi(\text{inl tt}))$ in the fibration $X_2$ along the equalities $u$ and $v$.

If $\sigma$ is not a corolla, but rather has proper descendants, then we proceed as follows: by appealing to the induction hypothesis, we obtain a family of elements of $X_2$ parameterized by the positions $p : \text{Pos}Mc$. These assemble, together with the witness for the binary multiplication of $c_\downarrow$ and $\delta_\downarrow$ under $\mu$-$\text{alg}_{x_2}$, into the arguments for $\mu$-$\text{alg}_{x_3}$, which has the correct type up to a transport along the equality given by our hypothesis $t$.

This completes the definition of $e_1'$ and, as the reader can see, we have thus achieved our goal of using the multiplicative operators $\eta$-$\text{alg}_{m_3}$ and $\mu$-$\text{alg}_{m_3}$ in the construction of the required equivalence. It remains to check the equations $s'$ and $t'$ above, a long calculation which we will not reproduce here. It is not hard to see that, up to some path algebra, $s'$ is by definition. Verifying $t'$ is slightly more involved: in this case, one must proceed by induction on the trees occurring in the arguments $c$ and $c_\downarrow$, again splitting into three cases: that of a leaf, a corolla, and a tree with at least 2 nodes. Additionally, during the course of the induction, one uses that the operators $\eta$-$\text{alg}_{m_3}$ and $\mu$-$\text{alg}_{m_3}$ are themselves associative and unital under the assumption that the opetopic type $X$ was fibrant. \qed