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# Sequential Predictors for Delay Compensation for Discrete Time Systems with Time-Varying Delays \*

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## Abstract

We study time-varying linear discrete time systems with uncertainties and time-varying **measurement** delays, whose outputs are perturbed by uncertainty. We build sequential predictors, which ensure input-to-state stability with respect to the uncertainties and which can be constructed using output values under arbitrarily long delays. The number of required sequential predictors is any upper bound for the delay **in our feedback stabilized closed loop systems**. We illustrate our work in a digital control problem for a continuous time system that is discretized through sampling.

*Key words:* Delay, robustness, discrete time, prediction

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## 1 Introduction

The compensation of delays is a central topic in systems and controls, because of the ubiquity of delays in engineering; see, e.g., [14] and [22]. Delay compensation usually involves constructing a feedback control that is calculated from time lagged state or output measurements and which ensures that a system is uniformly globally asymptotically stable to an equilibrium. While much of the delay compensation literature is on continuous time systems, there are notable applications leading to discrete time systems; see, e.g., [4, 9, 11, 15, 23, 26].

One method for delay compensation is emulation, which does not use information about the delay in the control design, and where one computes upper bounds on the delays under which the control still ensures global asymptotic stability, when delayed measurements are used to replace the current measurements in the control [16, 20]. Emulation is usually based on transforming a Lyapunov function for a closed loop undelayed system into a Lyapunov-Krasovskii functional for a delayed system, and has an advantage that it allows us to use more basic feedback control designs for undelayed systems. However, emulation may only provide conservative estimates of the maximum delays that the system can tolerate, and so is not always suited to applications in which the delays are long relative to the total response time of the dynamics.

This led to a literature on other delay compensation methods, where the control design uses information about the delays, such as [28]. One such method is the reduction model approach, which was first explored in [1]; see also [21] for time-varying systems. The reduction model approach shares the useful feature with the prediction approaches in [14] that it is able to compensate for arbitrarily long delays. However, a potential challenge for the implementation of standard prediction or reduction model approaches is that their controls normally require storing past control values over an interval of times, or, in the continuous time case, are only expressed implicitly as solutions of integral equations that do not have explicit solutions.

Sequential predictors (which were first discussed in [2]) are another delay compensation method, where the distributed terms in standard predictive controls are replaced by dynamic extensions. These extensions contain copies of the original system evolving on different time scales; see [3] and [18]. Since they do not use distributed terms in the controls, sequential predictors can be a useful alternative to addressing the computational challenges that can arise from using standard predictive controls. However, to the best of our knowledge, sequential predictors had not been developed for delay compensation in time-varying discrete time systems with outputs and time-varying delays.

Discrete time systems are useful for modeling digital control and discrete event systems, which are prone to delays that are analogous to the delays in continuous time. This motivates our work, which provides a discrete time analog of continuous time works such as [3], [18], and [25]. We cover discrete time linear time-varying systems with time-varying **measurement** delays, whose dynamics and output have uncertainties. We prove input-to-state stability (or

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ISS) with respect to the uncertainties, allowing arbitrarily large bounds on the delays without using distributed terms in the controls. This contrasts with notable works such as [8] whose sums are analogous to the integral terms in prediction-based delay compensation for continuous time systems. Our work is inspired by, and builds on, the notable work [4] by allowing time-varying systems and output feedback; these features were not allowed in [4]. See Remark 1 for more comparisons with [4]. The nondecreasing condition we will place on our delays naturally models degrading responsiveness of a control, and complements the requirement in [4] that there is a  $k_* > 0$  such that the delay is non-increasing on  $[k_*, +\infty)$ . Our delays are allowed to decrease on any finite length interval  $[0, k^*)$  as long as they are non-decreasing on  $[k^*, +\infty)$ . See also the notable work [10] on delay compensation for linear time invariant discrete time systems using linear matrix inequalities, under structural conditions on the dynamics for the uncertainty that we do not require here, and with no added uncertainties in the output or monotonicity assumptions on the delays.

We use this standard notation, in which the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. Let  $\|\cdot\|$  denote the usual Euclidean 2-norm and the corresponding matrix norm, and let  $\|\cdot\|_\infty$  denote the corresponding  $\mathcal{L}_\infty$  supremum norm. Let  $\sup_{\mathcal{I}}$  be the supremum over an interval  $\mathcal{I}$ . Let  $\mathcal{K}$ ,  $\mathcal{KL}$ , and  $\mathcal{K}_\infty$  be the usual classes of comparison functions as defined in [13, Chapter 4],  $I_n$  be the  $n$  dimensional identity matrix, and  $\mathbb{Z}$  denote the set of all integers. A time-varying discrete time system of the form

$$X_{k+1} = f(k, X_k, X_{k-h_k}, D_k) \quad (1)$$

with state space  $\mathbb{R}^n$  and a nonnegative integer valued bounded delay  $h_k$  is called input-to-state stable (which is also abbreviated as ISS [13, Chapter 4]) with respect to the sequence  $D_k \in \mathbb{R}^d$  provided there are  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that for each integer initial time  $k_0 \geq 0$  and each  $\mathbb{R}^n$ -valued initial function  $\phi$  for (1) (with  $\phi$  having domain  $\mathbb{Z} \cap [-\sup_k h_k + k_0, k_0]$ ), we have

$$\|X_k\| \leq \beta(\|\phi\|_\infty, k - k_0) + \gamma(\|D\|_{[k_0, k]}) \quad (2)$$

for all  $k \geq k_0$  and all choices of the sequence of  $D_k$ 's (which is equivalent to (2) with  $\|D\|_{[k_0, k]}$  replaced by  $\|D\|_\infty$ , by causality). However, for our state dynamics in our theorem, we choose the initial times  $k_0$  for the state to always be  $k_0 = 0$ , and we choose constant initial functions at  $k_0 = 0$ . For square matrices  $M_1$  and  $M_2$  of the same size, we use  $M_1 \leq M_2$  to mean that  $M_2 - M_1$  is nonnegative definite. We set  $\mathbb{Z}_i = \{j \in \mathbb{Z} : j \geq i\}$  for all  $i \in \mathbb{Z}$ , and  $\text{Ceiling}(r) = \min\{z \in \mathbb{Z} : z \geq r\}$  for all  $r \in \mathbb{R}$ , and we use 0 to denote the zero matrix of the appropriate dimensions.

## 2 Main Result

### 2.1 Assumptions and Statement of Theorem

We study linear time-varying systems of the form

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + d_k \\ y_k = C_k x_k + v_k \end{cases} \quad (3)$$

with a known output sequence  $y_k$  and known sequences  $A_k \in \mathbb{R}^{n \times n}$ ,  $B_k \in \mathbb{R}^{n \times m}$ , and  $C_k \in \mathbb{R}^{s \times n}$ , where the sequences  $d_k \in \mathbb{R}^n$  and  $v_k \in \mathbb{R}^s$  represent uncertainties and so are not assumed to be known, and where the control input value  $u_k$  will be computed from time lagged output measurements  $y_{k-\ell}$  for values  $\ell \geq h_k$ . The sequence  $h_k \in \mathbb{Z}_0$  represent **measurement** delays, and we assume that at each time  $k \in \mathbb{Z}_0$ , the values  $h_i$  are known for  $i \leq k$ . However, we do not require knowledge of future delay values. Later, we choose the  $u_k$ 's to ensure that the closed loop system satisfies an ISS property with respect to the disturbances  $D_k = (d_k, v_k)$ . We assume the following, where Assumption 1 is used to show that our closed loop system is causal (i.e., independent of future state values) and which is an analog of sufficient conditions for causality from [4]:

**Assumption 1** *The sequence  $h_k \in \mathbb{Z}_0$  is bounded. Also, there is a  $k^* \in \mathbb{Z}_0$  such that  $\max\{h_{k+1}, 1\} - 1 \leq \max\{h_k, 1\} \leq \max\{h_{k+1}, 1\}$  for all  $k \in \mathbb{Z}_{k^*}$ .  $\square$*

**Assumption 2** *The sequences of known matrices  $A_k \in \mathbb{R}^{n \times n}$ ,  $B_k \in \mathbb{R}^{n \times m}$ , and  $C_k \in \mathbb{R}^{s \times n}$  in (3) are bounded. Also, there exist known bounded sequences  $K_k \in \mathbb{R}^{m \times n}$  and  $L_k \in \mathbb{R}^{n \times s}$  such that the systems*

$$p_{k+1} = (A_k + B_k K_k) p_k + q_k \quad \text{and} \quad (4)$$

$$r_{k+1} = (A_k + L_k C_k) r_k + s_k \quad (5)$$

are ISS with respect to the sequences  $q_k$  and  $s_k$  on  $\mathbb{R}^n$ .  $\square$

Assumption 1 holds for all nondecreasing positive valued delays  $h_k$  with growth rates of at most 1, and other delays; see our illustration in Section 4. Note for later use that Assumption 1 implies that for all  $k \in \mathbb{Z}_{k^*}$  and  $i \in \{1, \dots, \max\{h_k, 1\}\}$ , we have  $i \leq \max\{h_{k+i}, 1\}$ . When the sequences  $A_k$ ,  $B_k$ , and  $C_k$  are constant and denoted by  $A$ ,  $B$ , and  $C$  respectively, Assumption 2 holds when  $(A, B)$  is controllable and  $(A, C)$  is observable; this can be checked by choosing constant  $K$  and  $L$  so that  $A + BK$  and  $A + LC$  are Schur stable and picking the constant sequences  $K_k = K$  and  $L_k = L$ ; see [11, Example 3]. See Sections 3-4 for time-varying cases where our assumptions hold. Our main result is as follows, where our requirement that  $z_k^i = 0$  for all  $k \leq 0$  and  $i$  ensures that  $u_k$  in (6) is defined even if  $k - \sigma(k) < 0$ :

**Theorem 1** *Let Assumptions 1-2 hold, and choose sequences  $K_k$  and  $L_k$  that satisfy the requirements of Assumption 2. Let  $\bar{h} \in \mathbb{Z}_1$  be an upper bound on the sequence  $h_k$ . Consider the system (3) in closed loop with the feedback*

$$u_k = K_k z_{k-\sigma(k)}^{\sigma(k)}, \quad \text{where } \sigma(k) = \max\{h_k, 1\}, \quad (6)$$

where  $z_k^r$  for each  $r \in \{1, 2, \dots, \bar{h}\}$  is the state of the  $r$ th subsystem of the  $\mathbb{R}^{n\bar{h}}$ -dimensional dynamical extension

$$\begin{cases} z_{k+1}^1 = A_{k+1} z_k^1 + B_{k+1} \mathcal{G}_1(k) + \alpha_{1,k} \\ z_{k+1}^2 = A_{k+2} z_k^2 + B_{k+2} \mathcal{G}_2(k) + \alpha_{2,k} \\ \vdots \\ z_{k+1}^{\bar{h}} = A_{k+\bar{h}} z_k^{\bar{h}} + B_{k+\bar{h}} \mathcal{G}_{\bar{h}}(k) + \alpha_{\bar{h},k}, \end{cases} \quad (7)$$

with the constant initial functions being defined by  $z_k^i = 0$  for  $i = 1, \dots, \bar{h}$  and integers  $k \leq 0$ , and where

$$\begin{aligned} \alpha_{1,k} &= L_{k+1}C_{k+1}z_k^1 - L_{k+1}y_{k+1}, \text{ and where} \\ \alpha_{2,k} &= L_{k+2}C_{k+2}[z_k^2 - A_{k+1}z_k^1 - B_{k+1}\mathcal{G}_1(k)] \\ &\vdots \\ \alpha_{\bar{h},k} &= L_{k+\bar{h}}C_{k+\bar{h}}[z_k^{\bar{h}} - A_{k+\bar{h}-1}z_k^{\bar{h}-1} \\ &\quad - B_{k+\bar{h}-1}\mathcal{G}_{\bar{h}-1}(k)] \text{ if } \bar{h} > 1 \end{aligned} \quad (8)$$

for all  $k \in \mathbb{Z}_0$ , where the  $\mathcal{G}_i$ 's are defined by

$$\mathcal{G}_i(k) = \begin{cases} u_{k+i}, & \text{if } k \in \mathbb{Z}_{k^*} \text{ and } i \leq \sigma(k+i) \\ K_{k+i}z_k^1, & \text{otherwise} \end{cases} \quad (9)$$

for all  $i \in \{1, \dots, \bar{h}\}$  and  $k$ , where  $k^*$  satisfies the requirements from Assumption 1. Then (3) in closed loop with the dynamic control given by (6)-(8) is ISS with respect to the disturbance  $D_k = (d_k, v_k)$  on its state space  $\mathbb{R}^n$ .  $\square$

## 2.2 Causality, Feasibility and Implementability of Method

Before proving our theorem, we provide remarks that explain in detail why our sequential predictor design is causal and discuss its implementability, and we explain in more detail why no knowledge of future delay values is required.

**Remark 1** The required values  $z_{k-\sigma(k)}^{\sigma(k)}$  in (6) can be computed from (7)-(8) using the values of  $\mathcal{G}_i$  for  $i = 1, \dots, \sigma(k)$ . Later we will set  $z_k^0 = x_k$  for all  $k \in \mathbb{Z}_0$ . The max in the  $\sigma(k)$  formula in Assumption 1 and in (6) is used because  $h_k$  could be 0, and because when  $h_k = 0$ , we cannot use the control value  $u_k = K_k x_k$  in (3) (because  $x_k$  may not be available for measurement). By (6) and (9), we have

$$\mathcal{G}_i(k) = K_{k+i}z_{k+i-\sigma(k+i)}^{\sigma(k+i)} \quad (10)$$

if  $i \in \{0, \dots, \sigma(k+i)\}$  and  $k \in \mathbb{Z}_{k^*}$ . Hence, at each  $k \in \mathbb{Z}_0$ , the right side of (7) does not depend on future  $z$  values.

While future delay values are needed to compute  $\mathcal{G}_i(k)$  in (10), the only  $\mathcal{G}_i$  values being used to compute the control value  $u_k$  at each time  $k$  are  $\mathcal{G}_i(q)$  for values  $q \leq k - \sigma(k) - 1$  and  $i \leq \sigma(k)$ . In fact, at each time  $k$ , the only  $z$  values we require in (6) are  $z_{k-\sigma(k)}^{\sigma(k)}$ , which can be computed using the values

$$\mathcal{R}(i, \ell, k) = z_{k-\sigma(k)-\ell}^i \text{ and } y_{k+1-\ell} \quad (11)$$

for  $i \leq \bar{h}$  and  $\ell \in \mathbb{Z}_1$ . Therefore, no future delay values are needed in the control. Also, while  $y_{k+1}$  is used in the  $\alpha_{1,k}$  formula, it is the case that at each time  $k$  when we compute the control value  $u_k$ , the only  $\alpha$  values we use are  $\alpha_{i,k-\ell}$  with  $\ell \geq 1$ , so the control does not require future output values, so our feedback control is causal. Our work contrasts with [4], which required a  $k_* \in \mathbb{Z}_1$  such that the nonincreasing condition  $h_{k+1} - h_k \leq 0$  holds for all  $k \geq k_*$ .  $\square$

**Remark 2** Each  $\mathbb{R}^n$ -valued  $z^i$  dynamic for  $i = 1, 2, \dots, \bar{h}$  in (7) is called a sequential predictor. Since the sequential predictors are interconnected in a chain, they are also called

chain predictors. At each time  $k$ , only the state of the  $\sigma(k)$ -th sequential predictor is used in the control (6).  $\square$

**Remark 3** Analogously to the continuous time case in [18], the sequential predictors in Theorem 1 consist of multiple copies (7) of the original system running on different time scales, with additional stabilizing terms (8). However, one key difference between the continuous and discrete time sequential prediction results is in the number of dynamical extensions. For instance, in [18], the number  $m_*$  of sequential predictors was required to satisfy

$$m_* > 11.4\bar{k}h \quad (12)$$

where  $\bar{k}$  was a global uniform Lipschitz constant for the right side of the dynamics (as a function of the state) and  $h$  was the constant delay, which gives the requirement  $\bar{k} \geq |A|_\infty$  for linear time-varying systems of the form  $\dot{x}(t) = A(t)x(t) + B(t)u(t-h)$  where  $A$  and  $B$  are bounded continuous matrix valued functions. On the other hand, Theorem 1 only needs  $\bar{h}$  sequential predictors, where  $\bar{h}$  is an upper bound on  $h_k$ . Our results are new, even when  $h_k$  is a constant delay, because we allow output feedback (which was not allowed in [4]). While the sequential predictors in the continuous time case used the state of the last of the sequential predictors to build the feedback  $u$ , the feedback control (6) in Theorem 1 calls for using a different sequential predictor for different possible delay values  $h_k$ .  $\square$

## 2.3 Proof of Theorem 1

While our control switches between sequential predictors, our dynamics do not lend themselves to the methods in previous switched systems stability proofs (e.g., because there are no common Lyapunov functions or other standard ingredients from switched systems arguments). This motivates the following novel proof method (which we believe has not been used in the literature on discrete-time observers with time-varying measurement delays or any other literature), which involves proving input-to-state stability estimates for paired states  $(e_k^i, \alpha_{i,k})$  for  $i' = 1, \dots, \bar{h}$  where

$$e_k^i = z_k^i - z_{k+1}^{i-1} \text{ for all } k \in \mathbb{Z}_0 \text{ and } i \in \{1, 2, \dots, \bar{h}\}, \quad (13)$$

and where  $z_k^0 = x_k$  for all  $k \in \mathbb{Z}_0$ .

Let

$$\bar{S} = \max\{\sigma(k) : k \in \mathbb{Z}_{k^*}\}. \quad (14)$$

Then for each  $i \in \{1, \dots, \bar{S}\}$ , we can find a  $k \in \mathbb{Z}_{k^*}$  such that  $i \leq \sigma(k+i)$ . For instance, if  $i = \sigma(\ell) - j$  for some  $\ell \in \mathbb{Z}_{k^*}$  and some  $j \in \{0, \dots, \sigma(\ell) - 1\}$ , then Assumption 1 implies that  $i \leq \sigma(\ell+i)$ . Hence, for each  $i \in \{1, \dots, \bar{S}\}$ , we can let  $\bar{k}_i$  denote the smallest  $k \in \mathbb{Z}_{k^*}$  such that  $i \leq \sigma(k+i)$ . Then  $i \leq \sigma(\ell+i)$  for all  $\ell \geq \bar{k}_i$ ,  $i \in \{1, \dots, \sigma(k)\}$ , and  $k \geq k^*$ , by Assumption 1. Next, note that with the choice

$$Z_k = (z_k^0, z_k^1, \dots, z_k^{\bar{h}}), \quad (15)$$

we can use the boundedness of the coefficient matrices in (3) to find a constant  $\bar{L} > 0$  such that

$$|Z_{k+1}| \leq \bar{L}(\sup\{|Z_\ell| : \ell = 0, \dots, k\} + |(d, v)|_\infty) \quad (16)$$

for all  $k \in \mathbb{Z}_0$  (using (3) to write  $y_{k+1}$  from (8) in terms

of earlier  $x_k$  and  $z_k$  and  $D_k$  values, and (6)-(7)). Hence, we can argue by induction on  $k$  to construct a function  $\underline{\gamma} \in \mathcal{K}_\infty$  such that

$$|x_k| \leq |Z_k| \leq \underline{\gamma}(|x_0|) + \underline{\gamma}(\|(d, v)\|_\infty), \quad 0 \leq k \leq \bar{h} + k^{\max} \quad (17)$$

holds along all solutions of the closed loop system from the statement of our theorem for all initial states  $x_0$  with the choice  $k^{\max} = \max_i \bar{k}_i$ , by using our assumption that the initial functions for the  $z^i$ 's for  $i = 1, \dots, \bar{h}$  are 0.

Set  $\alpha_{0,k} = d_k$  for all  $k \in \mathbb{Z}_0$ . These formulas and (13) give

$$\begin{aligned} e_{k+1}^i &= A_{k+i}z_k^i + B_{k+i}\mathcal{G}_i(k) + \alpha_{i,k} - z_{k+2}^{i-1} \\ &= A_{k+i}z_k^i + B_{k+i}\mathcal{G}_i(k) + \alpha_{i,k} \\ &\quad - [A_{k+i}z_{k+1}^{i-1} + B_{k+i}\mathcal{G}_{i-1}(k+1) + \alpha_{i-1,k+1}] \\ &= A_{k+i}[z_k^i - z_{k+1}^{i-1}] + \alpha_{i,k} - \alpha_{i-1,k+1} \end{aligned} \quad (18)$$

if  $k \in \mathbb{Z}_{k^*}$  and  $i \leq \sigma(k+i)$ , where  $\mathcal{G}_0$  is defined by the formula (9) with  $i = 0$ , because  $\mathcal{G}_i(k) = \mathcal{G}_{i-1}(k+1)$  when  $k \in \mathbb{Z}_{k^*}$  and  $i \leq \sigma(k+i)$ . Therefore,

$$\begin{aligned} e_{k+1}^1 &= A_{k+1}[z_k^1 - z_{k+1}^0] + \alpha_{1,k} - \alpha_{0,k+1} \\ &= A_{k+1}[z_k^1 - A_k x_k - B_k u_k] \\ &\quad + \alpha_{1,k} - A_{k+1}d_k - d_{k+1} \end{aligned} \quad (19)$$

and

$$\begin{aligned} e_{k+1}^i &= A_{k+i}[z_k^i - A_{k+i-1}z_k^{i-1} \\ &\quad - B_{k+i-1}\mathcal{G}_{i-1}(k) - \alpha_{i-1,k}] \\ &\quad + \alpha_{i,k} - \alpha_{i-1,k+1} \text{ for all } i \in \{2, \dots, \bar{h}\}. \end{aligned} \quad (20)$$

Here and in the sequel, all equalities and inequalities should be understood to hold for all integers  $k \geq k^{\max}$  and  $i \in \{1, 2, \dots, \sigma(k+i)\}$ , unless otherwise indicated.

Also, our choice of  $\alpha_{1,k}$  from (8) gives

$$\alpha_{1,k} = L_{k+1}C_{k+1}(z_k^1 - A_k x_k - B_k u_k - d_k) - L_{k+1}v_{k+1}. \quad (21)$$

From (8) and (19)-(21), it follows that

$$\begin{aligned} e_{k+1}^1 &= (A_{k+1} + L_{k+1}C_{k+1})e_k^1 - L_{k+1}v_{k+1} - d_{k+1} \\ e_{k+1}^i &= -A_{k+i}\alpha_{i-1,k} - \alpha_{i-1,k+1} \\ &\quad + (A_{k+i} + L_{k+i}C_{k+i})(z_k^i - A_{k+i-1}z_k^{i-1} - B_{k+i-1}\mathcal{G}_{i-1}(k)) \\ &= -A_{k+i}\alpha_{i-1,k} - \alpha_{i-1,k+1} \\ &\quad + (A_{k+i} + L_{k+i}C_{k+i})(z_k^i - z_{k+1}^{i-1} + \alpha_{i-1,k}) \\ &= L_{k+i}C_{k+i}\alpha_{i-1,k} - \alpha_{i-1,k+1} \\ &\quad + (A_{k+i} + L_{k+i}C_{k+i})e_k^i \text{ if } i \geq 2. \end{aligned} \quad (22)$$

Also, (8) and (21) give

$$\begin{aligned} \alpha_{1,k} &= L_{k+1}C_{k+1}e_k^1 - L_{k+1}v_{k+1} \text{ and} \\ \alpha_{i,k} &= L_{k+i}C_{k+i}[z_k^i - A_{k+i-1}z_k^{i-1} \\ &\quad - B_{k+i-1}\mathcal{G}_{i-1}(k) - \alpha_{i-1,k}] \\ &\quad + L_{k+i}C_{k+i}\alpha_{i-1,k} \\ &= L_{k+i}C_{k+i}[z_k^i - z_{k+1}^{i-1}] \\ &\quad + L_{k+i}C_{k+i}\alpha_{i-1,k} \\ &= L_{k+i}C_{k+i}[e_k^i + \alpha_{i-1,k}] \text{ if } i \geq 2. \end{aligned} \quad (23)$$

Since the  $e^1$  dynamics in (22) can be written in the form

$$r_{k+2} = (A_{k+1} + L_{k+1}C_{k+1})r_{k+1} + s_{k+1}, \quad (24)$$

where  $r_k = e_{k-1}^1$  and  $s_k = -L_k v_k - d_k$

for all  $k \in \mathbb{Z}_{k^*}$ , it follows from Assumption 2 that there are  $\beta_1^* \in \mathcal{KL}$  and  $\gamma_1^* \in \mathcal{K}_\infty$  such that

$$|r_\ell| \leq \beta_1^*(|r_{k^*+1}|, \ell - k^* - 1) + \gamma_1^*(L^\sharp \|(d, v)\|_\infty) \quad (25)$$

for all  $\ell \in \mathbb{Z}_{k^*+1}$ , where  $L^\sharp = |L|_\infty + 1$ . Also,

$$\begin{aligned} |r_{k^*+1}| &= |e_{k^*}^1| = |z_{k^*}^1 - z_{k^*+1}^0| \leq |z_{k^*}^1| + |z_{k^*+1}^0| \\ &\leq 2(\underline{\gamma}(|x_0|) + \underline{\gamma}(\|(d, v)\|_\infty)), \end{aligned} \quad (26)$$

where  $\underline{\gamma} \in \mathcal{K}_\infty$  is from the start of the proof. Combining (25) (with the choice  $\ell = k+1$ ) and (26) (to upper bound the  $|r_{k^*+1}|$  argument of  $\beta_1^*$  in (25)) gives

$$|e_k^1| \leq \beta_1^{**}(|x_0|, k) + \gamma_1^{**}(\|(d, v)\|_\infty) \quad (27)$$

for all  $k \in \mathbb{Z}_{k^*}$ , where

$$\beta_1^{**}(s, k) = \beta_1^*(4\underline{\gamma}(s), \max\{k - k^*, 0\}) \quad (28)$$

is of class  $\mathcal{KL}$  and

$$\gamma_1^{**}(s) = \gamma_1^*(L^\sharp s) + \beta_1^*(4\underline{\gamma}(s), 0), \quad (29)$$

where we used the fact that  $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$  holds for all functions  $\gamma \in \mathcal{K}$  and nonnegative  $a$  and  $b$ , and the fact that class  $\mathcal{KL}$  functions are of class  $\mathcal{K}$  in their first argument and nonincreasing in their second argument.

This and the first line of (23) provide functions  $\beta_1 \in \mathcal{KL}$  and  $\gamma_1 \in \mathcal{K}_\infty$  such that

$$\max\{|e_k^1|, |\alpha_{1,k}|\} \leq \beta_1(|x_0|, k) + \gamma_1(\|(d, v)\|_\infty) \quad (30)$$

for all  $k \in \mathbb{Z}_{k^*}$ . For instance, we can choose

$$\begin{aligned} \beta_1(s, k) &= \max\{1, |LC|_\infty\} \beta_1^{**}(s, k) \\ \text{and } \gamma_1(s) &= \max\{1, |LC|_\infty\} \gamma_1^{**}(s) + |L|_\infty s \end{aligned} \quad (31)$$

to satisfy our requirements.

Now we argue by induction, assuming that  $\bar{S} \geq 2$ ; later in the proof, we discuss the  $\bar{S} = 1$  case. Our induction hypothesis will be that  $N \in \{1, \dots, \bar{S} - 1\}$  is such that there are functions  $\beta_N \in \mathcal{KL}$  and  $\gamma_N \in \mathcal{K}_\infty$  such that

$$\max\{|e_k^i|, |\alpha_{i,k}|\} \leq \beta_N(|x_0|, k) + \gamma_N(\|(d, v)\|_\infty) \quad (32)$$

holds for all  $k \in \mathbb{Z}_{k^{\max}}$  and for all  $i \in \{1, 2, \dots, \min\{\sigma(k+i), N\}\}$ . We proceed by induction on  $N$ . The preceding condition is satisfied for  $N = 1$ , by the argument above.

To show that it holds for  $N + 1$  if it holds for a value  $N \in \mathbb{Z}_1 \cap [1, \bar{S})$ , we can assume that  $i = N + 1 \leq \sigma(k+i)$ . Then  $N + 1 \leq \sigma(k + N + 1)$  for all  $k \in \mathbb{Z}_{k^{\max}}$ , so (22) gives

$$\begin{aligned} r_{k+N+2} &= (A_{k+N+1} + L_{k+N+1}C_{k+N+1})r_{k+N+1} \\ &\quad + s_{k+N+1} \text{ where } r_{k+N+1} = e_k^{N+1} \text{ and} \\ s_{k+N+1} &= L_{k+N+1}C_{k+N+1}\alpha_{N,k} - \alpha_{N,k+1} \end{aligned} \quad (33)$$

if  $k \in \mathbb{Z}_{k^{\max}}$ . Since  $N + 1 \leq \sigma(k + N + 1)$ , Assumption 1 gives  $N \leq \sigma(k + N + 1) - 1 \leq \sigma(k + N)$ , so our induction hypothesis gives  $\max\{|\alpha_{N,k}|, |\alpha_{N,k+1}|\} \leq$

$\beta_N(|x_0|, k) + \gamma_N(|(d, v)|_\infty)$ , so we obtain

$$|s_{k+N+1}| \leq (|LC|_\infty + 1)(\beta_N(|x_0|, k) + \gamma_N(|(d, v)|_\infty)) \quad (34)$$

if  $k \in \mathbb{Z}_{k^{\max}}$ . Therefore, our ISS condition on the dynamics (5) from Assumption 2 implies that

$$\begin{aligned} |e_k^{N+1}| &= |r_{k+N+1}| \\ &\leq \beta_1^*(|r_{k^{\max}+N+1}|, 0) \\ &\quad + \gamma_1^*(|LC|_\infty + 1)\beta_N(|x_0|, 0) \\ &\quad + (|LC|_\infty + 1)\gamma_N(|(d, v)|_\infty) \\ &\leq \beta_1^*(|e_{k^{\max}}^{N+1}|, 0) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\gamma_N(|(d, v)|_\infty)) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\beta_N(|x_0|, 0)) \end{aligned} \quad (35)$$

if  $k \in \mathbb{Z}_{k^{\max}}$ , where  $\beta_1^* \in \mathcal{KL}$  and  $\gamma_1^* \in \mathcal{K}_\infty$  are from the previous part of the proof. Moreover,

$$\begin{aligned} |e_{k^{\max}}^{N+1}| &= |z_{k^{\max}}^{N+1} - z_{k^{\max}+1}^N| \leq |z_{k^{\max}}^{N+1}| + |z_{k^{\max}+1}^N| \\ &\leq 2(\underline{\gamma}(|x_0|) + \underline{\gamma}(|(d, v)|_\infty)). \end{aligned} \quad (36)$$

Hence, we can combine (35) with (36) to get

$$\begin{aligned} |e_k^{N+1}| &\leq \beta_1^*(4\underline{\gamma}(|x_0|), 0) + \beta_1^*(4\underline{\gamma}(|(d, v)|_\infty), 0) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\gamma_N(|(d, v)|_\infty)) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\beta_N(|x_0|, 0)) \end{aligned} \quad (37)$$

for all  $k \in \mathbb{Z}_{k^{\max}}$ .

The preceding bound provides a  $\gamma^{**} \in \mathcal{K}_\infty$  such that

$$|e_{\mathcal{M}(k)}^{N+1}| \leq \gamma^{**}(|x_0|) + \gamma^{**}(|(d, v)|_\infty) \quad (38)$$

for all  $k \in \mathbb{Z}_{k^{\max}}$ , where  $\mathcal{M}(k) = \text{Ceiling}(0.5(k + k^{\max}))$ . Therefore, our ISS condition on the dynamics (5) from Assumption 2 implies that

$$\begin{aligned} |e_k^{N+1}| &\leq \beta_1^*(|e_{\mathcal{M}(k)}^{N+1}|, k - \mathcal{M}(k)) \\ &\quad + \gamma_1^*(\sup\{|s_{\ell+N+1}| : \mathcal{M}(k) \leq \ell \leq k\}) \\ &\leq \beta_1^*(2\gamma^{**}(|x_0|), k - \mathcal{M}(k)) \\ &\quad + \beta_1^*(2\gamma^{**}(|(d, v)|_\infty), 0) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\beta_N(|x_0|, \mathcal{M}(k))) \\ &\quad + \gamma_1^*(2(|LC|_\infty + 1)\gamma_N(|(d, v)|_\infty)) \end{aligned} \quad (39)$$

for all  $k \in \mathbb{Z}_{k^{\max}}$ , by (34). This provides functions  $\hat{\beta}_{N+1}^a \in \mathcal{KL}$  and  $\hat{\gamma}_{N+1}^a \in \mathcal{K}_\infty$  such that

$$|e_k^{N+1}| \leq \hat{\beta}_{N+1}^a(|x_0|, k) + \hat{\gamma}_{N+1}^a(|(d, v)|_\infty) \quad (40)$$

for all  $k \geq k^{\max}$ .

Therefore, our formula for  $\alpha_{N+1, k}$  (which is the special case of last equality in (23) for  $i = N + 1$ ) and our inductive hypothesis give

$$\begin{aligned} \max\{|e_k^{N+1}|, |\alpha_{N+1, k}|\} &\leq \\ \beta_{N+1}^a(|x_0|, k) + \gamma_{N+1}^a(|(d, v)|_\infty) &\text{ for all } k \in \mathbb{Z}_{k^{\max}}, \end{aligned} \quad (41)$$

$$\text{where } \beta_{N+1}^a(s, k) = \max\{|LC|_\infty + 1, \hat{\beta}_{N+1}^a(s, k) + |LC|_\infty \beta_N(s, k)\} \quad (42)$$

is of class  $\mathcal{KL}$ , and where

$$\begin{aligned} \gamma_{N+1}^a(s) &= |LC|_\infty \gamma_N(s) \\ &\quad + \max\{|LC|_\infty, 1\} \hat{\gamma}_{N+1}^a(s) \end{aligned} \quad (43)$$

is of class  $\mathcal{K}_\infty$ . Therefore, by choosing

$$\beta_{N+1} = \max\{\beta_{N+1}^a, \beta_N\} \text{ and } \gamma_{N+1} = \max\{\gamma_{N+1}^a, \gamma_N\},$$

we can satisfy the requirements of the inductive step.

Choosing  $N = \bar{S} - 1$  in the preceding argument if  $\bar{S} \geq 2$  (or using (30) if  $\bar{S} = 1$ ), we obtain

$$|e_k^i| \leq \beta_{\bar{S}}(|x_0|, k) + \gamma_{\bar{S}}(|(d, v)|_\infty) \quad (44)$$

for all  $k \in \mathbb{Z}_{k^{\max}}$  and  $i \in \{1, \dots, \sigma(k + i)\}$ . Hence, can use a telescoping sum to get

$$\begin{aligned} |z_{k-\sigma(k)}^{\sigma(k)} - x_k| &= \left| \sum_{j=0}^{\sigma(k)-1} (z_{k-\sigma(k)+j}^{\sigma(k)-j} - z_{k-\sigma(k)+j+1}^{\sigma(k)-j-1}) \right| \\ &\leq \sum_{j=0}^{\sigma(k)-1} |e_{k-\sigma(k)+j}^{\sigma(k)-j}| \\ &\leq \bar{S} \beta_{\bar{S}}(|x_0|, k) + \bar{S} \gamma_{\bar{S}}(|(d, v)|_\infty) \end{aligned} \quad (45)$$

if  $k \geq \sigma(k) + k^{\max}$ , where the first inequality used the triangle inequality, and where the second inequality used the fact that

$$\sigma(k) - j \leq \sigma(k - \sigma(k) + j + (\sigma(k) - j)) = \sigma(k) \quad (46)$$

for all  $j \in \{0, \dots, \sigma(k) - 1\}$ . Therefore, for all  $k \geq \bar{h} + k^{\max}$ , we can use (6) to write the closed loop system from the statement of our theorem in the form

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k K_k z_{k-\sigma(k)}^{\sigma(k)} + d_k \\ &= (A_k + B_k K_k) x_k + d_k \end{aligned} \quad (47)$$

where for all  $k \geq \bar{h} + k^{\max}$ , the sequence

$$q_k = B_k K_k (z_{k-\sigma(k)}^{\sigma(k)} - x_k) + d_k \quad (48)$$

satisfies

$$|q_k| \leq \bar{b} \bar{S} (\beta_{\bar{S}}(|x_0|, k) + \gamma_{\bar{S}}(|(d, v)|_\infty)) + |d|_\infty, \quad (49)$$

where  $\bar{b} = |BK|_\infty$ .

Therefore, the ISS assumption on (4) in Assumption 1 ensures that there are  $\beta_1 \in \mathcal{KL}$  and  $\gamma_1 \in \mathcal{K}_\infty$  such that

$$\begin{aligned} |x_k| &\leq \beta_1(|x_{\bar{h}+k^{\max}}|, 0) \\ &\quad + \gamma_1(\bar{S} \bar{b} (\beta_{\bar{S}}(|x_0|, 0) + \gamma_{\bar{S}}(|(d, v)|_\infty)) + |d|_\infty) \\ &\leq \beta_1(|x_{\bar{h}+k^{\max}}|, 0) + \gamma_1(2\bar{S} \bar{b} \beta_{\bar{S}}(|x_0|, 0)) \\ &\quad + \gamma_1(2(\bar{S} \bar{b} \gamma_{\bar{S}}(|(d, v)|_\infty) + |d|_\infty)) \\ &\text{if } k \geq \bar{h} + k^{\max}, \end{aligned} \quad (50)$$

so by (17), we can obtain a function  $\bar{\gamma}_1 \in \mathcal{K}_\infty$  such that

$$|x_k| \leq \bar{\gamma}_1(|x_0|) + \bar{\gamma}_1(|(d, v)|_\infty) \quad (51)$$

for all  $k \geq \bar{h} + k^{\max}$ . Therefore, if we set

$$\bar{\mathcal{M}}(k) = \text{Ceiling}((k + \bar{h} + k^{\max})/2)$$

for all  $k \geq \bar{h} + k^{\max}$ , and if we evaluate (51) at  $\bar{\mathcal{M}}(k)$ , then we can again use the ISS assumption on the system (4) in

Assumption 1 to get

$$\begin{aligned}
|x_k| &\leq \beta_1(|x_{\mathcal{M}(k)}|, k - \bar{\mathcal{M}}(k)) + \gamma_1(\sup_{\ell \geq \mathcal{M}(k)} |q_\ell|) \\
&\leq \beta_1(2\bar{\gamma}_1(|x_0|), k - \bar{\mathcal{M}}(k)) + \beta_1(2\bar{\gamma}_1(|(d, v)|_\infty), 0) \\
&\quad + \gamma_1(2\bar{b}\bar{S}\bar{\beta}_{\bar{S}}(|x_0|, \bar{\mathcal{M}}(k))) \\
&\quad + \gamma_1(2(\bar{b}\bar{S}\bar{\gamma}_{\bar{S}}(|(d, v)|_\infty) + |d|_\infty))
\end{aligned} \tag{52}$$

if  $k \geq \bar{h} + k^{\max}$ . This allows us to find functions  $\hat{\beta} \in \mathcal{KL}$  and  $\hat{\gamma} \in \mathcal{K}_\infty$  such that

$$|x_k| \leq \hat{\beta}(|x_0|, k) + \hat{\gamma}(|(d, v)|_\infty) \tag{53}$$

for all  $k \geq \bar{h} + k^{\max}$ . We can then combine (53) with (17) to obtain the final ISS estimate

$$|x_k| \leq \beta_*(|x_0|, k) + \gamma_*(|(d, v)|_\infty) \text{ for all } k \in \mathbb{Z}_0, \tag{54}$$

where

$$\begin{aligned}
\beta_*(s, k) &= \hat{\beta}_1(s, k) + \underline{\gamma}(s) \max\{\bar{h} + k^{\max} + 1 - k, 0\} \\
\text{and } \gamma_*(s) &= \hat{\gamma}(s) + \underline{\gamma}(s)
\end{aligned} \tag{55}$$

for all  $s \geq 0$  and  $k \in \mathbb{Z}_0$ , which proves the theorem.

### 3 Checking Assumption 2

We believe that Theorem 1 is novel, even in the special case where the coefficient matrices or delays are constant ones. In this section, we present two methods for checking Assumption 2. For our first method, we show how our assumptions hold when the coefficient matrices have the form

$$\begin{aligned}
A_k &= A_* + \Delta_{A,k}, B_k = B_* + \Delta_{B,k} \\
\text{and } C_k &= C_* + \Delta_{C,k},
\end{aligned} \tag{56}$$

where  $(A_*, B_*)$  is a constant controllable pair and  $(A_*, C_*)$  is a constant observable pair, under suitable bounds on the allowable suprema of their time-varying parts  $\Delta_{A,k}$ ,  $\Delta_{B,k}$ , and  $\Delta_{C,k}$ . Second, we show how to apply our theorem to sampled-data time-varying continuous time systems.

#### 3.1 Time-Varying Coefficients

If the bounded sequences  $A_k$ ,  $B_k$ , and  $C_k$  are in the form (56) where  $(A_*, B_*)$  is controllable and  $(A_*, C_*)$  is observable, then we can find bounds on the allowable suprema

$$\begin{aligned}
|\Delta_A|_\infty &= \sup\{|\Delta_{A,k}| : k \geq 0\}, \\
|\Delta_B|_\infty &= \sup\{|\Delta_{B,k}| : k \geq 0\}, \text{ and} \\
|\Delta_C|_\infty &= \sup\{|\Delta_{C,k}| : k \geq 0\}
\end{aligned} \tag{57}$$

that ensure that Assumption 2 is satisfied (where we use  $\Delta_A$  to denote the function  $\Delta_A(k) = \Delta_{A,k}$  and similarly for the other coefficient matrices), as follows. Choose constant matrices  $K_*$  and  $L_*$  such that

$$M_1 = A_* + B_*K_* \text{ and } M_2 = A_* + L_*C_* \tag{58}$$

are Schur stable. Then we can use standard converse Lyapunov theory for discrete time systems (e.g., from [12]) to construct constant positive definite matrices  $P_1$  and  $P_2$  (which are symmetric) such that  $M_i^\top P_i M_i - P_i \leq -I_n$  for  $i = 1, 2$ . Then, using the definitions

$$N_{1,k} = A_k + B_k K_* \text{ and } N_{2,k} = A_k + L_* C_k, \tag{59}$$

we can construct a constant  $\epsilon_0 \in (0, 1)$  such that

$$N_{i,k}^\top P_i N_{i,k} - P_i \leq -\epsilon_0 I_n \tag{60}$$

holds for  $i = 1, 2$  and all  $k \in \mathbb{Z}_0$  for  $i = 1, 2$ , provided

$$\begin{aligned}
|\hat{\Delta}_1^\top P_1 [\hat{\Delta}_1 + 2(A_* + B_* K_*)]|_\infty &< 1 \text{ and} \\
|\hat{\Delta}_2^\top P_2 [\hat{\Delta}_2 + 2(A_* + L_* C_*)]|_\infty &< 1,
\end{aligned} \tag{61}$$

where  $\hat{\Delta}_1 = \Delta_A + \Delta_B K_*$  and  $\hat{\Delta}_2 = \Delta_A + L_* \Delta_C$ ; the sufficiency of the conditions (61) for the existence of the required constant  $\epsilon_0$  follows because (61) imply that

$$\begin{aligned}
&x^\top (N_{i,k}^\top P_i N_{i,k} - P_i) x \\
&= x^\top (M_i^\top P_i M_i - P_i) x + x^\top (\hat{\Delta}_i^\top P_i \hat{\Delta}_i + 2\hat{\Delta}_i^\top P_i M_i) x \\
&\leq -(1 - s_*) |x|^2 \text{ for all } x \in \mathbb{R}^n \text{ and } i = 1, 2,
\end{aligned} \tag{62}$$

where  $s_* \in (0, 1)$  is an upper bound for the left sides in (61), because  $N_{i,k} = M_i + \hat{\Delta}_{i,k}$  for  $i = 1, 2$  and  $k \in \mathbb{Z}_0$ . This gives time invariant Lyapunov functions  $V_1(x) = x^\top P_1 x$  and  $V_2(x) = x^\top P_2 x$  for

$$x_{k+1} = (A_k + B_k K_*) x_k \text{ and } x_{k+1} = (A_k + L_* C_k) x_k \tag{63}$$

respectively, which are ISS Lyapunov functions for the systems in Assumption 2 with the constant sequences  $K_k = K_*$  and  $L_k = L_*$ . It follows from well-known ISS results for discrete time systems (e.g., from [11]) that Assumption 2 is satisfied. We illustrate this in Section 4.1.

#### 3.2 Sampling in Continuous Time Systems

We can combine Theorem 1 with the method from Section 3.1 and the Kalman-Ho-Narendra criterion from [27, Section 3.4] to compensate for arbitrarily long delays and arbitrarily large sampling intervals in sampled-data time-varying continuous time systems. To see how, assume that we are given a continuous time system of the form

$$\begin{cases} \dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t - h(t)) + \delta_*(t) \\ y(t) = \mathcal{C}(t)x(t) + \delta_{**}(t) \end{cases} \tag{64}$$

with the output  $y$  and the piecewise continuous bounded uncertainties  $\delta_*$  and  $\delta_{**}$  and the bounded piecewise constant delay  $h : [0, \infty) \rightarrow \mathbb{Z}_0$ , where the bounded continuous matrix valued functions  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  have the forms

$$\begin{aligned}
\mathcal{A}(t) &= A_a + \delta_A(t), \mathcal{B}(t) = B_a + \delta_B(t), \\
\text{and } \mathcal{C}(t) &= C_* + \delta_C(t),
\end{aligned} \tag{65}$$

and where  $(A_a, B_a)$  is controllable and  $(A_a, C_*)$  is observable. We also assume that the constant  $\delta_s > 0$  (which will later serve as our sample rate) is such that the following condition holds for every pair  $(\lambda, \mu)$  of eigenvalues of  $A_a$  and so also for every pair  $(\lambda, \mu)$  of eigenvalues of  $A_a^\top$ : the constant  $\delta_s(\lambda - \mu)$  is not a nonzero integer multiple of  $2\pi i$ .

Now we assume that we are implementing a digital control, so the control  $u$  is constrained to be constant on each interval of the form  $[k\delta_s, (k+1)\delta_s)$  and that the perturbed output measurements  $y(t) = \mathcal{C}(t)x(t) + \delta_{**}(t)$  are only available at the discrete times  $k\delta_s$  for  $k \in \mathbb{Z}_0$ . Letting  $\Phi_A$  de-

note the fundamental solution associated with the system  $\dot{x}(t) = \mathcal{A}(t)x(t)$  (as defined, e.g., in [27, Appendix C.4]), we can then apply variation of parameters to the system (64) on each interval  $[k\delta_s, (k+1)\delta_s]$ , to obtain a time-varying discrete time system of the form (3), with

$$\begin{aligned} x_k &= x(k\delta_s), \quad A_k = \Phi_{\mathcal{A}}((k+1)\delta_s, k\delta_s), \\ C_k &= \mathcal{C}(\delta_s k), \quad d_k = \int_{k\delta_s}^{(k+1)\delta_s} \Phi_{\mathcal{A}}((k+1)\delta_s, s) \delta_*(s) ds, \\ B_k &= \int_{k\delta_s}^{(k+1)\delta_s} \Phi_{\mathcal{A}}((k+1)\delta_s, s) \mathcal{B}(s) ds, \\ \text{and } v_k &= \delta_{**}(k\delta_s) \end{aligned} \quad (66)$$

for all  $k \in \mathbb{Z}_0$ . This is a time-varying analog of the time invariant sampled-data system from [27, Section 3.4].

Also, our assumption on  $\delta_s$  implies that the pair

$$(A_*, B_*) = \left( e^{\delta_s A_a}, \int_0^{\delta_s} e^{(t-s)A_a} B_a ds \right) \quad (67)$$

is controllable, and that the pair  $(A_*, C_*)$  is observable. This follows by applying Theorem 4 and Lemma 3.4.1 from [27, Section 3.4] to the controllable pairs  $(A_a, B_a)$  and  $(A_a^T, C_a^T)$ , and is an application of the Kalman-Ho-Narendra criterion. Using the fact that

$$|\phi_{\mathcal{A}}(t, s) - e^{(t-s)A_a}| \leq e^{(t-s)|A_a|} (e^{(t-s)|\delta_A|} - 1) \quad (68)$$

holds for all  $s \geq 0$  and  $t \geq s$  (which follows, e.g., from [19, Lemma A.2]), and choosing the  $P_i$ 's and  $K_*$  and  $L_*$  as in Section 3.1 so that the matrices  $M_1$  and  $M_2$  in (58) (with the preceding choices of  $A_*$ ,  $B_*$ , and  $C_*$ ) are Schur stable, we can then find bounds on the allowable sup norms  $|\delta_A|_\infty$ ,  $|\delta_B|_\infty$ , and  $|\delta_C|_\infty$  of the time-varying parts of  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  such that the requirements of Section 3.1 are satisfied when we choose  $\Delta_{A,k} = A_k - A_*$ ,  $\Delta_{B,k} = B_k - B_*$ , and  $\Delta_{C,k} = C_k - C_*$ . This allows us to prove ISS for sampled data discrete time systems associated with the continuous time system (64).

While expressed in terms of the fundamental solution  $\Phi_{\mathcal{A}}$  (which is generally not available in closed form when  $\mathcal{A}$  is time varying), the  $\Phi_{\mathcal{A}}$  values in (66) can be computed by interconnecting our sequential predictors from Theorem 1 with **continuous time dynamic extensions**

$$\begin{cases} \lambda_1'(t) = \mathcal{A}(t)\lambda_1(t), & \lambda_2'(t) = -\lambda_2(t)\mathcal{A}(t) \\ \lambda_1(i\delta_s) = \lambda_2(i\delta_s) = I_n \end{cases} \quad (69)$$

indexed by  $i \in \mathbb{Z}_0$ , whose matrix valued solutions  $\lambda_1$  and  $\lambda_2$  satisfy

$$\Phi_{\mathcal{A}}(t, s) = \lambda_1(t)\lambda_2(s) \quad (70)$$

for all  $s$  and  $t$  in  $[i\delta_s, (i+1)\delta_s]$  and all  $i$ . To check (70), note that for each  $i \in \mathbb{Z}_0$ , the function  $\omega(t) = \lambda_1(t)\lambda_2(t)$  satisfies  $\dot{\omega}(t) = \mathcal{A}(t)\omega(t) - \omega(t)\mathcal{A}(t)$  for all  $t \in [i\delta_s, (i+1)\delta_s]$  and  $\omega(i\delta_s) = I_n$ , so  $\omega$  must be identically equal to  $I_n$  on  $[i\delta_s, (i+1)\delta_s]$  by standard results for uniqueness of solutions for linear differential equations (so  $\lambda_2 = \lambda_1^{-1}$ ), and then to note that  $\lambda_1(t) = \Phi_{\mathcal{A}}(t, i\delta_s)$  for all  $t \in [i\delta_s, (i+1)\delta_s]$  and use the semigroup property of fundamental solutions to get

$$\begin{aligned} \Phi_{\mathcal{A}}(t, s) &= \Phi_{\mathcal{A}}(t, i\delta_s)\Phi_{\mathcal{A}}(i\delta_s, s) = \lambda_1(t)\Phi_{\mathcal{A}}^{-1}(s, i\delta_s) \\ &= \lambda_1(t)\lambda_1^{-1}(s) = \lambda_1(t)\lambda_2(s) \end{aligned} \quad (71)$$

for each  $i \in \mathbb{Z}_0$ , which agrees with (70). Moreover, even though  $\lambda_2 = \lambda_1^{-1}$ , we do not have to invert  $\lambda_1$  because we can solve for the factors  $\lambda_1$  and  $\lambda_2$  directly using (69), and  $\lambda_1$  and  $\lambda_2$  are bounded by a constant over  $[0, +\infty)$ .

Hence, we can use Theorem 1 to compensate for arbitrarily infrequent periodic sampling in the control and output measurements, which corresponds to allowing arbitrarily large  $\delta_s$ 's. The number of required sequential predictors is any upper bound  $\bar{h} \geq 1$  on  $h_k$ . We illustrate this in Section 4.2.

## 4 Illustrations

### 4.1 Discrete Time Case

We study a generalization of the example from [6, Section V], which differs from the example in [6] because we allow one of the coefficient matrices to be time-varying and because we allow uncertainties. The dynamics have the form

$$\begin{aligned} x_{1,k+1} &= \frac{5}{4}x_{1,k} + x_{2,k} + \frac{1}{4}u_{1,k} + d_{1,k} \\ x_{2,k+1} &= -\frac{3}{8}x_{1,k} + \frac{1}{8}u_{2,k} + d_{2,k} \\ y_k &= C_k x_k + v_k \end{aligned} \quad (72)$$

where  $C_k = C_* + \Delta_{C,k}$ ,  $C_* = [1 \ 0]$ , and  $\Delta_{C,k} = [\delta_k \ 0]$  for a known function  $\delta_k$  of  $k$  for all  $k \in \mathbb{Z}_0$ . The work [6] studied the case where  $\delta_k = 0$ ,  $d_{1,k} = d_{2,k} = \sin(k)/9$ , and  $v_k = \sin(k^2)/9$  for all  $k$  and without the time-varying delay  $h_k$ , and it provided an observer design, which converged after a prescribed finite time when the perturbations  $d_{i,k}$  and  $v_k$  were set to zero.

By contrast, here we apply Theorem 1 to (72) to get a sequential predictor delay compensating control, by using the method from Section 3.1 with the constant choices

$$\begin{aligned} A_* &= \begin{bmatrix} \frac{5}{4} & 1 \\ -\frac{3}{8} & 0 \end{bmatrix}, \quad \Delta_{A,k} = 0, \quad K_* = \begin{bmatrix} -5 & -4 \\ 3 & 0 \end{bmatrix}, \\ B_* &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}, \quad \Delta_{B,k} = 0, \quad \text{and } L_* = \begin{bmatrix} -\frac{5}{4} \\ \frac{3}{8} \end{bmatrix}, \end{aligned} \quad (73)$$

but analogous reasoning applies under time-varying coefficients in the system, using nonzero  $\Delta_{A,k}$  and  $\Delta_{B,k}$  values. Our requirements from Section 3.1 can be satisfied using the positive definite matrices

$$P_1 = I_2 \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \quad (74)$$

In fact, since  $\Delta_A + \Delta_B K_* = 0$  and  $\Delta_{A,k} + L_* \Delta_{C,k} = L_* \Delta_{C,k} = L_* [\delta_k \ 0]$  for all  $k \in \mathbb{Z}_0$ , condition (61) holds if

$$|\Delta_C^T L_*^T P_2 (L_* \Delta_C + 2(A_* + L_* C_*))|_\infty < 1, \quad (75)$$

which can be expressed as

$$\sup_{k \geq 0} \left| \begin{bmatrix} \frac{59}{32} \delta_k^2 & -\frac{5}{2} \delta_k \\ 0 & 0 \end{bmatrix} \right| < 1, \quad (76)$$

or equivalently,

$$\sup_{k \geq 0} \delta_k^2 (3.399 \delta_k^2 + 6.25) < 1. \quad (77)$$



Then our result from Section 3.1 implies that we can compensate for any bounded time-varying delay  $h_k$  that satisfies Assumption 1, using the  $\bar{h}$  sequential predictors from Theorem 1 for any upper bound  $\bar{h} \geq 1$  on  $h_k$ .

#### 4.2 Sampled-Data Continuous Time System

We next apply the approach from Section 3.2 to a linearized version of a model of a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor that produces the continuous time tracking error dynamics

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = b_1 x_3(t) - a_1 \cos(x_{1r}(t))x_1(t) - a_2 x_2(t) \\ \dot{x}_3(t) = b_0 u(t - h(t)) - a_3 x_2(t) - a_4 x_3(t) \\ y(t) = x_1(t) + \delta_{**}(t), \end{cases} \quad (78)$$

where the  $a_i$ 's and  $b_i$ 's are positive constants,  $h : [0, \infty) \rightarrow \mathbb{Z}_0$  is a bounded piecewise constant delay, and

$$x_{1r}(t) = \frac{\pi}{2} \left(1 - e^{-0.1t^3}\right) \sin\left(\frac{8\pi}{5}t\right) \quad (79)$$

is the reference trajectory component from [5]; see [5] for the importance of this model for mechanical engineering applications. The work [5] derived the original model, [7] and [24] provide backstepping-based adaptive and sliding mode controls for the model, and [19] provided continuous-discrete observers for the unmeasured states in (78) and exponential ISS results using a continuous time controller.

However, these earlier works left open the problem of digital control of (78), i.e., cases where the values for the delay compensating controller are required to stay constant during the sampling intervals  $[k\delta_s, (k+1)\delta_s)$ . To see how Theorem 1 can address the preceding problem, we apply the method from Section 3.2, with the choices

$$A_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -a_2 & b_1 \\ 0 & -a_3 & -a_4 \end{bmatrix}, \quad B_a = [0 \ 0 \ b_0]^\top, \quad (80)$$

$$\delta_A(t) = \begin{bmatrix} 0 & 0 & 0 \\ -a_1 \cos(x_{1r}(t)) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and } C_* = [1 \ 0 \ 0],$$

and with  $\delta_B$  and  $\delta_C$  both being the zero functions, so  $\mathcal{A} = A_a + \delta_A$ ,  $\mathcal{B} = B_a$  and  $\mathcal{C} = C_*$ . Following [19], we choose  $a_2 = b_1 = a_3 = 0.5$ ,  $a_4 = 0.1$ , and  $b_0 = 1$ . Then we can satisfy the requirements from Section 3.2 under an appropriate bound on the constant  $a_1$  and for any positive constant sampling rate  $\delta_s$  such that  $\frac{1}{20\pi}\sqrt{84}\delta_s \notin \mathbb{Z}$ .

For instance, using the commands `StateFeedbackGains` and `DiscreteRiccatiSolve` in Mathematica [17] (to pick  $K_*$  and  $L_*$  to assign eigenvalues to  $M_1 = A_* + B_*K_*$  and  $M_2 = A_* + L_*C_*$  so that all of the eigenvalues of  $M_1$  and  $M_2$  are in the open unit disk centered at the origin in the complex plane, and then to construct the required matrices  $P_1$  and  $P_2$ , with the choices of  $A_*$  and  $B_*$  from

(67)) with the sampling rate  $\delta_s = 0.5$ , one can show that the requirements from Section 3 are met with

$$K_* = [-0.983972 \quad -1.3977 \quad -1.56992] \quad \text{and} \quad (81)$$

$$L_* = [-1.22643 \quad -0.592427 \quad -0.0774504]^\top$$

and for any value  $a_1 \in [0, 0.022]$  of the constant in the time-varying part  $\delta_A$  from (80). Then Theorem 1 provides the delay compensating controller for the corresponding sampled data discrete time system, where the number of sequential predictors depends on the delay bound.

We now express the closed loop system from our theorem for a discrete time sampled data system corresponding to (78), in a specific case using the preceding dynamics and parameter values. We can choose the sample rate  $\delta_s = 0.5$  and the on-off time-varying delay that is defined by  $h_k = \max\{0, (-1)^{k+1}\}$  for all  $k \in \mathbb{Z}_0$ , which satisfies our causality conditions from Assumption 1. Since the delay is bounded by 1, we can choose  $\bar{h} = 1$ , so we only require one sequential predictor in order to apply Theorem 1. Therefore, the closed loop tracking error system has the form

$$\begin{cases} X_{k+1} = A_k X_k + B_k K_* z_{k-1}^1 \\ z_{k+1}^1 = (A_{k+1} + L_* C_*) z_k^1 + B_{k+1} K_* z_k^1 - L_* Y_{k+1} \\ Y_k = C_* X_k + v_k \\ \lambda'_1(t) = \mathcal{A}(t)\lambda_1(t), \quad \lambda'_2(t) = -\lambda_2(t)\mathcal{A}(t) \end{cases} \quad (82)$$

on its state space  $\mathbb{R}^6 \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}$ , where  $\mathcal{A} = A_a + \delta_A$ , and where  $A_a$  and  $\delta_A$  and  $C_*$  are as defined by (80) with the  $a_i$  and  $b_i$  values as indicated above, and where  $x_{1r}$  is defined in (79),  $K_*$  and  $L_*$  are defined in (81), and the coefficient matrices are defined by

$$A_k = \Phi_{\mathcal{A}}((k+1)\delta_s, k\delta_s) = \lambda_1(0.5(k+1))\lambda_2(0.5k)$$

$$B_k = \int_{k\delta_s}^{(k+1)\delta_s} \Phi_{\mathcal{A}}((k+1)\delta_s, s)\mathcal{B}(s)ds \quad (83)$$

$$= \lambda_1(0.5(k+1)) \int_{0.5k}^{0.5(k+1)} \lambda_2(s)ds [0 \ 0 \ 1]^\top$$

where we used (70), and where we must use the required initial conditions  $z_k^1 = 0$  for all  $k \leq 0$  and  $\lambda_1(0) = \lambda_2(0) = I_3$ . The vector  $X_k$  represents the discretized state  $X_k = x(\delta_s k)$  where  $x$  is a solution of the original continuous time dynamics (78). In Figs. 1-2, we plot MATLAB simulations of the closed loop system (82) obtained using the preceding control design, which illustrate the ISS performance ensured by our theorem and so help validate our results in the special case of (82).

## 5 Conclusions

Our new sequential predictor delay compensation method for time-varying discrete time linear systems with outputs and time-varying **measurement** delays compensates for delays whose sup norms can be arbitrarily large, including nondecreasing delay sequences that were beyond the scope of earlier delay compensation methods for discrete time cases. Unlike earlier constructions of continuous time sequential predictors where the stabilizing feedback control

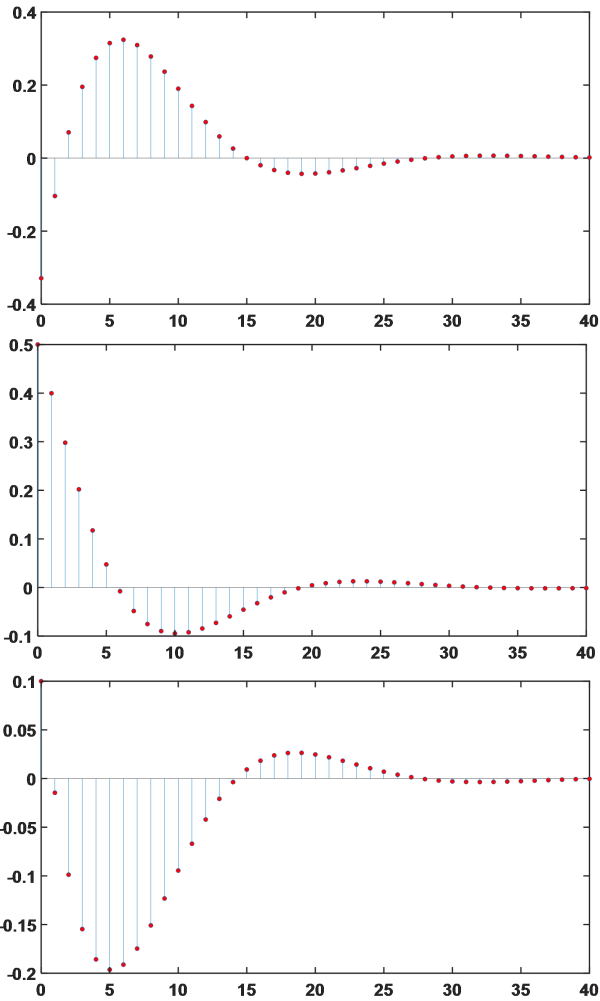


Fig. 1. Solution  $X_k = (X_{1k}, X_{2k}, X_{3k})$  of (82) with Disturbance  $v_k = 0$  for all  $k \in \mathbb{Z}_0$  Showing  $X_{1k}$  (Top),  $X_{2k}$  (Middle), and  $X_{3k}$  (Bottom) Illustrating Global Asymptotic Stability to 0

is constructed using the state of the last of the sequential predictors, our feedback control was expressed in terms of the sequential predictor whose index corresponds to the value of the delay at each time.

We used input-to-state stability to quantify the effects of uncertainties in the dynamics and in the outputs. Using discrete time systems, we modeled the effects of sampling in continuous time systems, which allowed the length of the sampling intervals to be arbitrarily large, in a digital control context where the control value is constrained to be kept constant between sampling times. We hope to merge our continuous and discrete time results to cover time-varying hybrid systems with **measurement delays and output functions, and to study the effects of delays in open loop unstable systems.**

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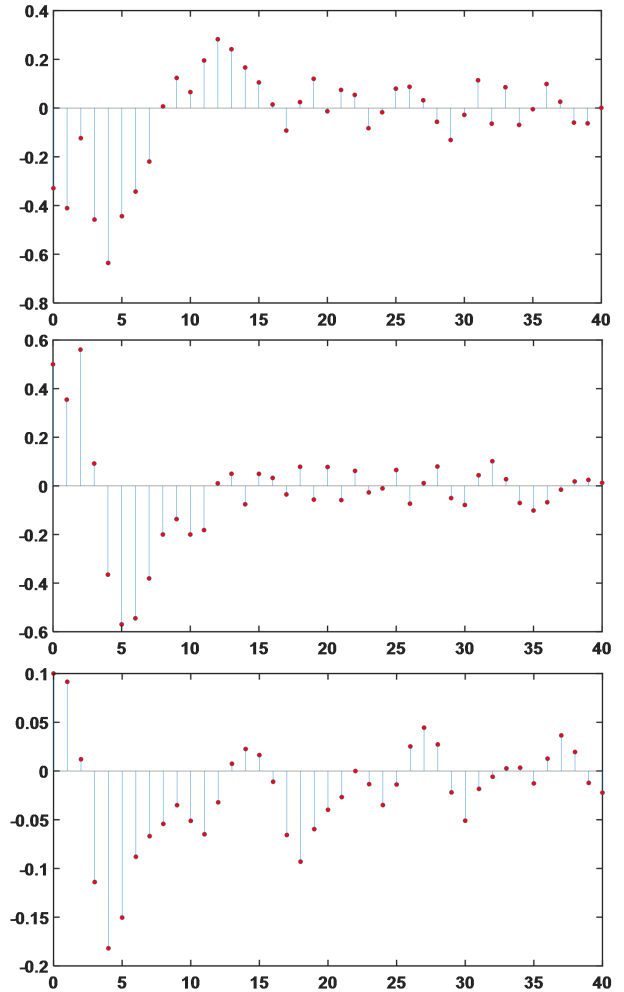


Fig. 2. Solution  $X_k = (X_{1k}, X_{2k}, X_{3k})$  of (82) with Disturbance  $v_k = 0.5 \sin(k^2/4)$ , Showing  $X_{1k}$  (Top),  $X_{2k}$  (Middle), and  $X_{3k}$  (Bottom) with Input-to-State Stability with Respect to  $v_k$

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