Edward Sang's computation of sines

Denis Roegel

11 January 2021

Contents

Edward Sang (1805-1890) was probably the greatest calculator of logarithms of the 19th century [1, 8, 9, 10, 23, 31]. Sang spent 40 years computing tables of logarithms and trigonometric functions, with the assistance from his daughters Flora (1838-1925) and Jane (1834-1878). The result fills about 50 manuscript volumes, plus a number of transfer duplicates.

Sang's purpose was in particular to provide fundamental tables, including for the decimal division of the quadrant. In 1890 [1, p. 189], he wrote that

> In addition to the results being accurate to a degree far beyond what can ever be needed in practical matters, [the collection of computations] contains what no work of the kind has contained before, a complete and clear record of all the steps by which those results were reached. Thus we are enabled at once to verify, or if necessary, to correct the record, so making it a standard for all time.
>
> For these reasons it is proposed that the entire collection be acquired by, and preserved in, some official library, so as to be accessible to all interested in such matters; so that future computers may be enabled to extend the work without the need of recomputing what has been already done; and also so that those extracts which are judged to be expedient may be published.

I have analyzed and reconstructed the main tables computed by Sang, and a summary of these reconstructions is given in a separate guide [51].

The present document gives an overview of the techniques used by Sang in order to complete his canon of sines. The first section summarizes the articles and notes transcribed in section 3.

1 Summary of the volumes of tables of sines

The sines are scattered over three volumes of tables:

- K40/1: sines from $25'$ to $25'$
- K40/2: sines from $5'$ to $5'$
- K41-K42: sines from $1'$ to $1'$

There is also a more specialized volume of sines, K44, which gives the sines but with an additional multiplicative factor. This table was constructed to help in the making of a table of circular segments.

I have reconstructed all these tables in four separate volumes.

2 Sang's computation of sines

The canon of sines was computed using bisections by square root extractions and quinquisections (or quinquesections). In other words, given some known sines, intermediate values were obtained half way, or by dividing the intervals in five. This process was repeated until a sufficiently small sine value was found. Then, the sine of the smallest angle was used to compute the sines of its multiples using finite differences. The present section summarizes this procedure and draws in particular on an unpublished notice written by Sang in 1882 and transcribed in section 3.7.

2.1 Dividing the quadrant in 80 parts

First, Sang obtained the values of $\sin 20^{\mathrm{c}}$, $\sin 50^{\mathrm{c}}$ and $\sin 60^{\mathrm{c}}$:

$$\begin{aligned}
\sin 20^{\mathrm{c}} &= \frac{\sqrt{5}-1}{4} \\
\sin 50^{\mathrm{c}} &= \sqrt{\frac{1}{2}} \\
\sin 60^{\mathrm{c}} &= \frac{\sqrt{5}+1}{4}
\end{aligned}$$

Sang possibly also computed

$$\cos 20^{\mathrm{c}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

and then

$$\begin{aligned}
\sin 40^{\mathrm{c}} &= 2\sin 20^{\mathrm{c}}\cos 20^{\mathrm{c}} \\
\sin 80^{\mathrm{c}} &= 2\sin 40^{\mathrm{c}}\cos 40^{\mathrm{c}} = 2\sin 40^{\mathrm{c}}\sin 60^{\mathrm{c}}
\end{aligned}$$

Now Sang had the sines and cosines of 20^{c}, 40^{c}, 50^{c}, 60^{c}, and 80^{c}.

Then Sang computed $\sin 10^{\mathrm{c}}$ by bisection, perhaps using

$$\sin^2 10^{\mathrm{c}} = \frac{1-\cos 20^{\mathrm{c}}}{2}$$

Then $\sin 30^{\mathrm{c}}$ was computed from $\cos 60^{\mathrm{c}}$, then $\sin 70^{\mathrm{c}} = \cos 30^{\mathrm{c}}$ from $\cos 60^{\mathrm{c}}$, and $\sin 90^{\mathrm{c}} = \cos 10^{\mathrm{c}}$ from $\cos 20^{\mathrm{c}}$.

Now Sang had the sines and cosines of all multiples of 10^{c}.

Further, each 10^{c} interval was bisected three times. Indeed, if $\sin a$, $\cos a$, $\sin b$ and $\cos b$ are known for two angles a and b, we have

$$\sin\left(\frac{a+b}{2}\right) = \sqrt{\frac{1-\cos(a+b)}{2}}$$

Eventually, Sang obtained the sines and cosines for all multiples of $1^{\mathrm{c}}.25$, that is the entire quadrant was divided in 80 parts.

The sines were computed on 33 places and copied to 30 places in a two pages list which is given in volume K40.

2.2 Dividing the quadrant in 400 parts

2.2.1 Computing $\sin 0^{c}.25$

Now, Sang divides every $1^{c}.25$ interval in five parts, using a quinquisection. Sang considers the equation

$$\sin 5a = 16\sin^5 a - 20\sin^3 a + 5\sin a$$

which is easy to check by developing $\sin(3a + 2a)$ and so forth. This equation is used to find the value of $\sin a$ from that of $\sin 5a$. In order to solve this equation, Sang uses an iterative method which he had published in 1829 [67] and which is based on Taylor's theorem. He considers the function

$$\mathcal{E}(x) = 16x^5 - 20x^3 + 5x - sin5a$$

and his objective is to find x such that $\mathcal{E}(x) = 0$.

Sang starts with some value of x, and then modifies x to obtain a new value of x, and so on. This is basically done using Newton-Raphson's method in which

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with $f(x) = \mathcal{E}(x)$.

Sang's procedure is a little bit more complicated, because he does more along the way. It is best to first see his procedure in action. The first computation is with $a = 25'$. Sang starts with $\sin 1^{c}25' = 0.01963\ldots$ found earlier and with the first approximation $x_1 = 0.003926$. He then computes the five quantities which are the values of $\mathcal{E}(x)$ and its derivatives:

$$\begin{aligned}
\mathcal{E} &= 16x_1^5 - 20x_1^3 + 5x_1 - sin5a \\
{}_1\mathcal{E} &= 80x_1^4 - 60x_1^2 + 5 \\
{}_2\mathcal{E} &= 320x_1^3 - 120x_1 \\
{}_3\mathcal{E} &= 960x_1^2 - 120 \\
{}_4\mathcal{E} &= 1920x_1 \\
{}_5\mathcal{E} &= 1920
\end{aligned}$$

Sang finds in particular

$$\begin{aligned}
\mathcal{E} &= -0.0000049027\ldots \\
{}_1\mathcal{E} &= \quad 4.99907521044\ldots
\end{aligned}$$

From this Sang sets $\delta x = x_2 - x_1 = -\frac{\mathcal{E}}{{}_1\mathcal{E}} \approx 0.00000098072\ldots$ and obtains $x_2 = 0.003926 + 0.00000098072\ldots = 0.00392698072\ldots$

By repeating this process two more times, Sang obtains the value of $\sin 25'$ correct to 30 decimal places. Sang also computed $\cos 25'$, probably with

$$\cos 25' = \sqrt{1 - (\sin 25')^2}$$

Although obtaining the successive values of x only requires the computation of $\mathcal{E}$ and its first derivative, Sang framed his procedure by computing all the expressions $\mathcal{E}$, ${}_1\mathcal{E}$, ${}_2\mathcal{E}$, etc., for each new approximation, and this was summarized in the following table:

${}_5\mathcal{E}$	${}_4\mathcal{E}$		${}_3\mathcal{E}$		${}_2\mathcal{E}$		${}_1\mathcal{E}$		$\mathcal{E}$		x	
	${}_5\mathcal{E}$	$\frac{\delta x}{1}$	${}_4\mathcal{E}$	$\frac{\delta x}{1}$	${}_3\mathcal{E}$	$\frac{\delta x}{1}$	${}_2\mathcal{E}$	$\frac{\delta x}{1}$	${}_1\mathcal{E}$	$\frac{\delta x}{1}$		δx
			${}_5\mathcal{E}$	$\frac{\delta x^2}{1.2}$	${}_4\mathcal{E}$	$\frac{\delta x^2}{1.2}$	${}_3\mathcal{E}$	$\frac{\delta x^2}{1.2}$	${}_2\mathcal{E}$	$\frac{\delta x^2}{1.2}$		
					${}_5\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$	${}_4\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$	${}_3\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$		
							${}_5\mathcal{E}$	$\frac{\delta x^3 4}{1.2.3.4}$	${}_4\mathcal{E}$	$\frac{\delta x^4}{1.2.3.4}$		
									${}_5\mathcal{E}$	$\frac{\delta x^5}{1.2.3.4.5}$		
${}_5\mathcal{E}$	${}_4\mathcal{E}'$		${}_3\mathcal{E}'$		${}_2\mathcal{E}'$		${}_1\mathcal{E}'$		$\mathcal{E}'$		x'	

Assuming values of ${}_5\mathcal{E}$, ${}_4\mathcal{E}$, etc., have been found for some value of x, and a value of δx was computed as shown above, this table shows how to compute the new values of ${}_5\mathcal{E}$, ${}_4\mathcal{E}$, etc., from the old ones and from δ. For instance, the third column means that

$$ {}_3\mathcal{E}' = {}_3\mathcal{E} + {}_4\mathcal{E} \times \frac{\delta x}{1} + {}_5\mathcal{E} \times \frac{\delta x^2}{1.2} $$

where ${}_3\mathcal{E}'$ is the new value of ${}_3\mathcal{E}$. What Sang actually does, and he says so explicitely in his 1829 memoir [67], is to use Taylor's theorem. Given a function f such as the sine function, we have

$$ f(x + \delta x) = f(x) + f'(x)\delta x + f''(x)\frac{\delta x^2}{2!} + \cdots $$

Replacing f by f', f'', etc., we also have

$$ \begin{aligned} f'(x + \delta x) &= f'(x) + f''(x)\delta x + f'''(x)\frac{\delta x^2}{2!} + \cdots \\ f''(x + \delta x) &= f''(x) + f''(x)\delta x + f''''(x)\frac{\delta x^2}{2!} + \cdots \\ &\cdots\cdots \end{aligned} $$

Sang then merely writes the new values of ${}_4\mathcal{E}$, ${}_3\mathcal{E}$, etc., using Taylor's theorem.

The expressions in the above table stop when the derivative is zero, which is the case beyond ${}_5\mathcal{E}$.

In addition, these derivatives are useful because we can express the expressions $\cos^2 a$, $\cos 2a$, $\sin^2(2a)$, $\sin 3a$ and $\cos 4a$ as combinations of $\mathcal{E}$, ${}_1\mathcal{E}$, ${}_2\mathcal{E}$ and ${}_3\mathcal{E}$, and this can serve as checks. Replacing x by $\sin a$ in the expressions for ${}_1\mathcal{E}$, ${}_2\mathcal{E}$, and ${}_3\mathcal{E}$, it is easy to check that:

$$\cos^2 a = \frac{7}{8} - \frac{1}{960}{}_3\mathcal{E}$$
$$\cos 2a = \frac{3}{4} - \frac{1}{480}{}_3\mathcal{E}$$
$$(\sin 2a)^2 = \frac{3}{8} + \frac{1}{960}{}_3\mathcal{E} - \frac{1}{20}{}_1\mathcal{E}$$
$$\sin 3a = \frac{3}{2}x - \frac{1}{80}{}_2\mathcal{E}$$
$$\cos 4a = \frac{1}{4} - \frac{1}{480}{}_3\mathcal{E} + \frac{1}{10}{}_1\mathcal{E}$$

Although the computation of $\sin 25'$ is somewhat tedious, it only had to be done once.

2.2.2 Computing $\sin(n \times 0^{\mathfrak{c}}25')$

Once $\sin 25'$ was computed, the values of $\sin 50'$, $\sin(3 \cdot 25')$, etc., were obtained using second differences. Consider the scheme

sine	Δ^1	Δ^2
$\sin(P-2a)$		$\sin(P-a) - 2\sin(P-2a) + \sin(P-3a)$
	$\sin(P-a) - \sin(P-2a)$	
$\sin(P-a)$		$\sin(P) - 2\sin(P-a) + \sin(P-2a)$
	$\sin(P) - \sin(P-a)$	
$\sin(P)$		$\sin(P+a) - 2\sin P + \sin(P-a)$
	$\sin(P+a) - \sin(P)$	
$\sin(P+a)$		$\sin(P+2a) - 2\sin(P+a) + \sin(P)$
$\cdots$	$\cdots$	$\cdots$

Now since

$$\sin(P+a) - 2\sin P + \sin(P-a) = -2\operatorname{ver} a \cdot \sin P$$

where $\operatorname{ver} a = 1 - \cos a$ is the versine, we can rewrite the previous scheme of differences as:

sine	Δ^1	Δ^2
$\sin(P-2a)$		$-2\operatorname{ver} a \cdot \sin(P-2a)$
	$\sin(P-a) - \sin(P-2a)$	
$\sin(P-a)$		$-2\operatorname{ver} a \cdot \sin(P-a)$
	$\sin(P) - \sin(P-a)$	
$\sin(P)$		$-2\operatorname{ver} a \cdot \sin P$
	$\sin(P+a) - \sin(P)$	
$\sin(P+a)$		$-2\operatorname{ver} a \cdot \sin(P+a)$
$\cdots$	$\cdots$	$\cdots$

Assuming $\ldots, \sin(P-2a), \sin(P-a), \sin P$ have been found, we can see that $\sin(P+a)$ can be obtained by computing $2\operatorname{ver} a \cdot \sin P$.

It is in order to facilitate these computations that Sang computed a table of values of $2 \operatorname{ver} 25'$.[1]

The values computed by the method of differences can be checked every 5 values with those already computed in the previous step.

2.3 Dividing the quadrant in 2000 parts

The same procedure was used to compute $\sin 5'$ and the sines of multiples of $5'$ were again obtained using second differences. In this case, Sang computed an auxiliary table of values of $2 \operatorname{ver} 5'$.[2] The previously computed values of $\sin 25'$ could be used as checks.

2.4 Dividing the quadrant in 10000 parts

Finally, the quinquisection was used one last time to compute $\sin 1'$, and the sines of multiples of $1'$ were again obtained using second differences. In this case, Sang computed an auxiliary table of values of $2 \operatorname{ver} 1'$.[3] The previously computed values of $\sin 5'$ could be used as checks.

2.5 Accuracy

The sines in volume K40/1 ($25'$) seem to be correctly rounded to 30 places and are occasionally off by one unit of the 30th place. The same is true for volume K40/2 ($5'$). The sines in volumes K41-K42 ($1'$) seem to be correct to 15 places, except for a few very rare cases. Details are given in the individual descriptions.

[1] National Library of Scotland, Edinburgh, Acc.10780/74. (not seen)

[2] National Library of Scotland, Edinburgh, Acc.10780/74 (not seen) and volume K40 (Acc.10780/50).

[3] National Library of Scotland, Edinburgh, Acc.10780/52 (volume K42).

3 Transcriptions of Sang's articles

In this section, I reproduce transcriptions of several articles or notes relevant to Sang's construction of the canon of sines. For several of his notes, manuscript versions are extant, but it is not certain that they were written by Sang's hand. For instance the notice written in 1882 (section 3.7) contains the incorrect expressions

$$\sin a = \sqrt{\left\{\frac{1}{2} - \frac{1}{2}\sin 2a\right\}}$$

$$\cos a = \sqrt{\left\{\frac{1}{2} + \frac{1}{2}\sin 2a\right\}}$$

and such errors are very surprising. Some of the notes may have been copied by Sang's daughters, and they may have escaped a close scrutiny. In any case, I have tried to check all the values and all the computations and corrected them where necessary.

All the footnotes in this section are my own, and none are by Sang.

3.1 Introduction to the $25'$ canon of sines (1876)

Volume K40 contains a short seven page introduction to the canon of sines at intervals of $25'$. I am not reproducing this introduction here as its scope is similar to that of the 1882 notice transcribed in section 3.7. A summary will suffice.

In this introduction Sang gives the sines of the first division of the quadrant into 80 parts, probably in the order of their computation. So, we find 20^c, 40^c, 60^c, 80^c, 10^c, 30^c, 50^c, 70^c, 90^c, 5^c, 15^c, etc. Then, Sang gives practically the same procedure as that summarized in 1882 and computes $\sin 25'$.

Sang also gives the values of $\cos^2 a$, $\cos 2a$, $\sin^2 2a$, etc., as well as $2 \operatorname{ver} a$ for $a = 25'$. Sang observes that a table of the first hundred multiples of $2 \operatorname{ver} 25'$ was formed[4] in order to ease the computation of the table.

The resulting table of sines and first and second differences for every $25'$ is given in volume K40 and in my reconstruction K40/1 [59].

A final note in this table is dated 23 July 1876, which suggests that this table of sines was completed in 1876.

3.2 Introduction to the $5'$ canon of sines (1877)

Volume K40 also contains an introduction to the canon of sines at $5'$ intervals. Sang computed $\sin 5'$ using the same procedure as that outlined for the computation of $\sin 25'$. Sang took as a first approximation[5]

$$\sin a \approx \frac{1}{5}\sin 5a + \frac{4}{125}\sin^3 5a$$

[4]National Library of Scotland, Edinburgh, Acc.10780/74. (not seen)

[5]Sang's manuscript incorrectly states that he took

$$\sin a \approx \frac{1}{5}\sin 5a + \frac{1}{500}\sin^3 5a$$

but this does not correspond to the approximation of $\sin 5'$ he takes.

Eventually, Sang obtained $\sin 5'$ correctly to 30 places and also computed several derived sines and cosines.

Sang then gave several checks extracted from his unpublished notes on the "Numerical values of recurring functions."[6]

The resulting table of sines for every $5'$ is given in volume K40 and in my reconstruction K40/2 [60].

The final date in the table is 30 December 1877.

3.3 On the construction of the Canon of Sines (1878)

On the construction of the Canon of Sines, for the decimal division of the Quadrant.[7]

The convenience of having only one system of numeration is so well recognized that there is no need for any discussion. Already the numerical nomenclatures of all nations having any culture have been converted to one, namely the decimal, system, and traces of the ancient use of dozens, scores,[8] fifteens or sixties can be found in only a very few of them. Although we count our hours in sixty minutes, we do not date the present as the year *thirty-one sixties and seventeen.*[9] Yet, in the matters of measure, weight and value the old and irksome divisions continue to be used; nay, even in those departments of science which most need laborious calculation we continue to employ the ancient scale of division.

It is indeed remarkable that, while men of business are striving for uniformity in the modes of counting and of measuring, trigonometers and astronomers should remain unconcerned as to the subdivision of arcs and of time. We are rapidly approaching the anomalous position of using the decimal division of the Earth's quadrant as the source of our standards of weights and measures, and of yet rejecting that division in our notation of angles.

The want of Trigonometrical Tables suitable to the new division is the real cause of this backwardness; the construction of these tables is, essentially, the first step to the universal employment of the decimal system. This has been long and well recognised; in the end of the last century Borda computed the decimal Canon; this computation was superseded by that which the French Government caused to be made under the superintendence of Prony,[10] but neither of these has been put to press.[11] The only centesimal table available to the Trigonometer is that given, for each minute of the quadrant, in Callet's

[6] National Library of Scotland, Edinburgh, Acc.10780/59 (not examined). These notes seem to comprise several hundred pages of tables which I have unfortunately not had the time to consult.

[7] Volume K40, National Library of Scotland, Edinburgh, Acc 10780/50. Published in 1878 [74]. The present transcription is that of the manuscript, which has minor differences with the published version.

[8] A score is 20.

[9] Hence 1877.

[10] See [39].

[11] Sang was apparently unaware of the publication of Borda and Delambre's table in 1801 [15].

Tables Portatives;[12] it was collated with the manuscripts of Borda and of Prony, but is presented in a most inconvenient form.

The eminent astronomer LaPlace, adopted the decimal division of arcs, of distance and of time in his Mécanique Celeste published in the last year of the century.[13] The force of this illustrious example might, long since, have gained universal acceptance for the system, had not the non-publication of the requisite canon prevented all progress in this direction.

Notwithstanding many solicitations and even the offer by the English Government to defray a share of the expense, the tables computed in the Bureau du Cadastre remain unpublished, and the fact of their existence remains a discouragement to other computers.

Now fifty years ago, having fallen upon a method of approximating very rapidly to the roots of numerical equations, published in 1829 under the title "Solution of Algebraic Equations of All Orders",[14] I applied it to the quinquisection of an arc, and thus obtained directly the sine of any proposed decimal division of the quadrant. After proceeding a short way in the construction of the Canon by this method, I laid it aside from the conviction that the labour could, at best, only produce a repetition of what had already been accomplished.

Many years ago, after having felt for long the want of a table of Logarithms more extensive than any existing, I designed a nine-place table up to one million; and having carried the manuscript fifteen-place table up to 300 000, laid it before this Society, whose Council did me the exceedingly great favour of presenting to Government a memorial soliciting aid in the completion and publication.

One of our scientific periodicals, in noticing this memorial, supplied to Government a most potent reason for not acceding to the request, in this, that the labours of Prony in the Bureau du Cadastre had already anticipated and surpassed all that can be done in future in this department of calculation.

Forced thus into a critical examination of the Cadastre Tables, so far as that can be accomplished by help of published documents, I arrived at most unexpected conclusions.

In the first place, the fundamental table, that of the Logarithms of the Prime Numbers, as given by Legendre in his work on Elliptic Functions,[15] was found to be correct up to 2000; that is as far as Abraham Sharp[16] and other ancient computers had gone. But, of the primes between 2000 and 10000 computed in the Bureau, only five have their logarithms correctly given while almost all of the other logarithms err on one side.

[12] See [7].
[13] See [16].
[14] See [67].
[15] See [24].
[16] See [84].

In the second place, Henry Briggs' original table had been collated by Prony's assistants, and errors in the tenth and higher places had been found; yet no notice had been taken of the very many errors in the fourteenth and even in the thirteenth place.

Thirdly, the system of computation adopted was so imperfect that although differences of the sixth order were extended to the thirty-sixth place, the results were liable to errors up to the twelfth figure.

Lastly; the Cadastre calculations were used by M. Lefort for correcting Adrian Vlacq's ten-place table,[17] that table of which all the subsequently published seven-place tables are abridgements; with the result of sometimes putting Vlacq in the wrong when he is right. That is to say the Cadastre calculations cannot be trusted for the compilation of a ten-place logarithmic table.

Such being the case with that part of Prony's great work which was comparable with previously published tables, we are unable to place confidence in the Trigonometrical portion which necessarily is almost entirely new; and we are forced, when contemplating the compilation of the Canon of Sines, to hold the Cadastre Tables as non existent or, at best, as useful for controlling palpable errors of press.

In actual Trigonometrical calculations we very seldom use the sines and tangents themselves, but employ their logarithms instead; wherefore both the canon of Sines and the Logarithmic Canon are needed as the joint foundation of our working tables. Having carried the table of Logarithms as far as to 370 000, and being satisfied of the insufficiency of the work done in the Bureau du Cadastre, I resumed the computation of the Sines, and have now proceeded to such a length as to be able to submit the methods employed, to the Society and through it to the mathematical public.

The decimal division of the quadrant is effected by bisections and quinquisections, and the first thing to be determined is the order in which these operations should be taken. Now we obtain the sine of the half arc not from the sine of the whole arc but from its cosine,[18] or rather from its co-versed-sine;[19] when the arc is small the co-versed-sine, or defect of the cosine from the radius, is represented by a very small decimal fraction, the number of whose effective figures is less than the total number of figures in the calculations; we have to extract the square root of its half, which square root can only be true to as many effective figures, and thus cannot be extended to the full number. If then we desire to obtain accuracy to a specific number of decimal places we must extend our first calculations far beyond. Whereas the sine of an odd submultiple is deduced directly from the sine of the whole arc, and the computation is such as to retain the full number of effective figures. Hence we must proceed by first making all the requisite bisections, and thereafter the quinquisections.

[17] Vlacq's tables [86, 87] were published in 1628 and 1633.

[18] *i.e.*, $\sin x = \sqrt{(1-\cos 2x)/2}$.

[19] coversine$(x) = 1 - \sin x$, so Sang rather means the versine $\operatorname{ver} x = 1 - \cos x$.

There is, however, an obvious exception. The sines of 20^c, 40^c, 60^c and 80^c are obtained by the solution of a quadratic equation, so that this quinquisection naturally comes first; the bisections afterwards and then the remaining quinquisections.

Having prescribed unit in the thirtieth decimal place as the limit of inexactitude, and taken three figures more to obviate the accumulation of the minute errors inevitable in all approximate work, I computed the sines of these four arcs to thirty-three places.

The bisection of the quadrant gave the sine of 50^c; that of 60^c gave for 30^c and for its complement 70^c; that of 20^c gave for 10^c and 90^c, thus completing the table for each tenth part of the quadrant.

In the same way the bisections of 10^c, 30^c, 50^c, 70^c and 90^c gave the sines of the intermediate fifth degrees.

The bisections were continued in this way until the quadrant was divided into 80 equal parts.

In performing this work each of the roots

$$\sin a = \sqrt{\left\{\frac{1}{2} - \frac{1}{2}\cos 2a\right\}}, \qquad \cos a = \sqrt{\left\{\frac{1}{2} + \frac{1}{2}\cos 2a\right\}}$$

was extracted twice, and $\sin a$ was also found from the division

$$\sin a = \frac{\sin 2a}{2\cos a},$$

by which means the values of $\sin a$, and $\cos a$ were securely got, and the previous computations of $\sin 2a$, $\cos 2a$ again verified. There were, as a matter of course, small errors in the last places, but the results, to the thirtieth place were made sure.

The quinquisections were obtained by help of the equation[20]

$$16.\sin^5 a - 20.\sin^3 a + 5.\sin a - \sin 5a = 0 = \varepsilon.$$

Regarding ε as a function of $\sin a$, its successive derivatives are

$$\begin{aligned}
{}_1\varepsilon &= 80.\sin^4 a - 60.\sin^2 a + 5\\
{}_2\varepsilon &= 320.\sin^3 a - 120.\sin a\\
{}_3\varepsilon &= 960.\sin^2 a - 120\\
{}_4\varepsilon &= 1920.\sin a\\
{}_5\varepsilon &= 1920 \qquad\qquad ;
\end{aligned}$$

[20]The exponents have been moved to make the formulæ more understandable.

now, while resolving the equation, we get the values of all the derivatives; so, taking advantage of these, we have

$$\begin{aligned}
\cos a^2 &= \frac{7}{8} - \frac{1}{960}{}_3\varepsilon \\
\cos 2a &= \frac{3}{4} - \frac{1}{480}{}_3\varepsilon \\
\sin 3a &= \frac{3}{2}\sin a - \frac{1}{80}{}_2\varepsilon \\
\cos 4a &= \frac{1}{4} - \frac{1}{480}{}_3\varepsilon + \frac{1}{10}{}_1\varepsilon \qquad ;
\end{aligned}$$

the first of these gives us, by extracting the square root, $\cos a$.

Applying this method to the arc $1^c 25'$, the eightieth part of the quadrant, I obtained the sine and cosine of $25'$, and proceeded to compute the functions of its multiples by help of second differences, according to the well known formula

$$\sin(n+1)a - 2\sin na + \sin(n-1)a = -\sin na . 2\operatorname{ver} a;$$

and, because the multiplier $2\operatorname{ver} a$ is to be repeatedly used, a table of its multiples was constructed, in the case of $a = 25'$, up to the hundredth, in the cases of $a = 5'$ and $a = 1'$, up to the thousandth multiple.

The sines of these multiples being computed continuously, an error in any part of the work, propagated subsequent errors which could not possibly be overlooked in comparing the results at each fifth step with the previously computed test values. In this way immunity from error was obtained excepting in the last places, where small errors are inevitable.

In performing the multiplication $\sin na . 2\operatorname{ver} a$ stopping at the 33^{rd} place, the last figure of each partial product may err in excess or in defect by $\frac{1}{2}$; now it is possible, though not likely, that all of these errors may be on one side, and therefore there is a *possible* error in the last place of the total product, of half as many units as there are lines in the multiplication. Now by using the table of multiples up to one thousand, we reduce the number of lines to one third and therefore the possible amount of residual error in the same ratio; so that the auxiliary table both saves labour and augment the exactitude of the result.

These final-place errors were corrected at each fifth step by altering the last figures of the second differences, and thus the accumulation of those errors was prevented. In the whole calculation of the sines of the quarter degrees it was not found necessary to alter any one second difference to so much as the limit of possible error, and therefore we may hold that the manuscript table of the sines of these arcs is absolutely correct to within the prescribed degree of precision, namely unit in the thirtieth decimal place.

The next quinquisection, conducted in the same way, gave the sine and cosine of $5'$; the functions of whose multiples were obtained as before and compared, at each fifth step, with the previous work. The table of sines to every fifth minute is already well advanced.

The third quinquisection gave the sine and cosine of the single minute of the decimal division. A table of the multiples of $2\,\mathrm{ver}\,1'$ has been constructed[21] up to 1000, and has been used in forming a good beginning of the Canon of Sines to each single minute.

For the purpose of preventing all error in the record of these calculations the second differences only were copied from the duplicate scroll calculations, and the successive first differences and sines were thence recomputed on the record sheet. Since any error in copying, or even in the original computation, was necessarily continued and extended into the afterpart of the record, its detection was rendered certain, so that the recorded results may be implicitely relied on. To make the record more secure each page was copied on thin transfer-paper on which no alteration can be made without being obvious.

The method of computing might have been extended to differences of the fourth order, in which case the common multiplier, $4(\mathrm{ver}\,a)^2$, would have been much smaller;[22] and the terms of the products fewer. But, on the other hand, the entries in the record would have been more, and the effects of the residual errors would have been much greater and more troublesome in correction. For the interpolation of each quarter degree the saving obtained by the use of fourth differences would have been unappreciable; for those of each fifth and of each single minute, it would have been hardly such as to compensate for the inconveniences just mentioned; but for the future interpolations of each tenth second and of each single second, the fourth differences may be advantageously used.

When the canon of sines shall have been completed, the computation of the working table, that of Logarithmic Sines will be easy; particularly by help of my fifteen-place table of logarithms.

Beginning, as John Nepair did,[23] at the sine of the whole quadrant and proceeding downwards, the computed logarithmic sines may be verified by their differences which are small. When the work has been brought down to the log sin of the half-quadrant, the farther progress is easy and rapid, for the formula

$$2.\sin a.\cos a = \sin 2a$$

gives $\log\sin a = \log\frac{1}{2} + \log\sin 2a - \log\cos a$, so that the differences of any order may be got at once from the previously tabulated differences of that

[21] National Library of Scotland, Edinburgh, Acc.10780/52 (volume K42).

[22] The second differences are the values of the sines multiplied by $2\,\mathrm{ver}\,a$, the fourth differences are the values of the second differences multiplied by $2\,\mathrm{ver}\,a$, hence the values of the sines multiplied by $4(\mathrm{ver}\,a)^2$, and so on.

[23] See [25, 27, 37].

order, and, what is most worthy of remark, may be used without the fear of an accumulation of residual error.

The table of logarithmic tangents follows as a matter of course.

3.4 Recording original computations (1878)

On the precautions to be taken in recording and using the records of, original computations.[24]

The real utility of Tables of numerical results is only secured by making them accessible to those computers who may require them; and the essence of their utility lies in this that the labour of a single computer saves that of many others.

It is indispens[a]ble that those who use the tables be able to rely implicitely on the accuracy of the tabulated numbers, and that they have a ready means of detecting any error should the existence of one by suspected.

I do not mean, at present, to say a word on the mechanical arrangements of setting up the types, of stereotyping, revising the proofs, and printing; these have already often been discussed, but I shall take the matter up at this critical point: — the investigator has in his hands a set of printed or of manuscript tables, which he means to use in his researches, and he wishes to know whether the individual book be or be not to be trusted. This confidence must necessarily be influenced by the history of the book and by its correlatives. Thus if it be a stereotype copy of a work in extensive circulation, he may accept the general opinion as to its accuracy; but if the table be one seldom used, such as those which serve as the foundation of working tables, this source of confidence fails him.

The nature of the case may be most clearly seen from an example:

We propose to extend the logarithmic canon beyond the limits to which it has been already printed; this extension must be founded on the logarithms of the prime numbers; now Abraham Sharp computed, to 61 places, the logarithm of every prime number up to 1097; these were printed in Sherwin's collection,[25] and thence re-printed by Callet in his Tables Portatives;[26] shall we then build our more extensive tables on the computation by Sharp? Sharp was known as a most zealous and careful computer; both Sherwin and Callet would take care that the numbers be correctly copied, yet for all that, we cannot venture to found on Sharp's work because there is an essential omission.

If we were to proceed to compute, by help of these, the logarithms of larger primes, and if, after a lengthened series of operations, we were to find a disagreement; we should be left in doubt as to which of the many logarithms

[24] Volume K40, National Library of Scotland, Edinburgh, Acc 10780/50. Published in 1878 [75].
[25] See [84].
[26] See [7].

that had been used may be in fault; we should have to re-compute such of Sharp's logarithms as might be implicated, while the labour and irksomeness of the search would be intolerable.

In all such calculations we seek to arrive at the result by two independent processes. All the use, therefore, that could be made of Sharp's table, was to hold his work as constituting one of these processes; a great use certainly, yet, at best, only half of what it might have been.

Now in the computation, to twenty-eight places, of the logarithms of the prime numbers, no error whatever was discovered among those given by Callet; so here we have an instance of records, in themselves quite exact, and yet insufficient to obviate subsequent re-computation.

The means of readily verifying the record are awanting; these means must necessarily vary with the nature of the tabulated functions.

In the volumes containing the computation of the logarithms to 28 places of all primes below ten thousand, which was laid before the Society, the articulate steps of every calculation are recorded and indexed, so that if an error be suspected in any one logarithm we have the means of instantly verifying the table or of detecting the source of the error. Had such a record accompanied Sharp's admirable table, the need for subsequent re-calculation would have been entirely obviated.

The vast majority of tables have their arguments arranged with equal differences and consequently the functions progressing gradually; and, for the most part, these tables have been constructed by help of differences. It is then sufficient to record the differences along with the values of the functions. For the canon of sines I have found it convenient to place the first difference, with the sign +, and then the second difference, with the sign −, below the preceding sine, as shown below

$1^{c}\ 12'$	.01759	20113	41099	72108	93607	04490
	+ 15	70551	06706	95090	77046	38118
	−	4	37940	65992	07711	07444
1..13	.01774	90664	47806	67199	70653	42608
	+ 15	70546	68766	29098	69335	30674
	−	4	41815	82853	79716	53726
1..14	.01790	61211	16572	96298	39988	73281

This arrangement enables the computer to examine any sine which he wishes to extract, so as to guard against any typographical error; and, if the table of the multiples of 2 ver $1'$ were appended, to check readily the computation itself. When, however, the differences have only a few figures, the ordinary method of placing them in separate columns, is to be preferred; it saves room while the practical calculator has no difficulty in performing the requisite additions or subtractions. The utility of this arrangement has been long recognized.

In the case of tables for common use, which are, in general, abbreviations of original calculations to a greater number of places, it is enough to give the first differences when these vary much.

When such means of verification have been provided, the user of the Table can make sure that the number which he extracts contains no error; and if all users were to make this examination habitually, printed tables, and above all those printed from stereotype, would be gradually freed from errors of press.

3.5 The series for the sine and cosine (1878)

On a new investigation of the series for the sine and cosine of an arc.[27]

The sines of the successive equidifferent arcs form a progression having for its general character the relation

$$\phi_{n-1} - 2\phi_n + \phi_{n+1} = \phi_n \cdot v,$$

and the properties of sines may be deduced from this general formula. Viewed in this light, the angular functions become cases only of more general ones.

If we suppose A, B, C to be three consecutive terms of such a progression we must have

$$A - 2B + C = vB,$$

from which, when three of the four quantities, A, B, C, v are given, the fourth may be found. Let then A and B, the first and second terms of the progression, and v the common coefficient, be given; the succeeding terms may be computed thus:—[28]

$$\begin{array}{lcll}
\phi 0 & = & A & \\
 & & -A & +B \\
 & & & +Bv \\
\hline
\phi 1 & = & & \;\;B \\
 & & -A & +B(1+v) \\
 & & -Av & +B(\;\;+2v+v^2) \\
\hline
\phi 2 & = & -A & +B(2+v) \\
 & & -A(1+v) & +B(1+3v+v^2) \\
 & & -A(\;\;+2v+v^2) & +B(\;\;+3v+4v^2+v^3) \\
\hline
\phi 3 & = & -A(2+v) & +B(3+4v+v^2) \\
 & & -A(1+3v+v^2) & +B(1+6v+5v^2+v^3) \\
\hline
\phi 4 & = & -A(3+4v+v^2) & +B(4+10v+6v^2+v^3),
\end{array}$$

[27] Published in 1878 [73]. The manuscript is at the National Library of Scotland, Edinburgh, Acc.10780/50.

[28] In this scheme, the first lines ($\phi 0$, $\phi 1$, etc.) are the values of the function. The second lines are the first differences, and the third lines are the second differences.

from which it is obvious that the coefficient of $-A$ in the expression for ϕn, is a transcript of that of B in the preceding expression for $\phi(n-1)$. Hence, for the present, we may confine our attention to the latter.

The coefficients of B form the following progression:—

in $\phi 0$ $\quad 0$
in $\phi 1$ $\quad 1$
in $\phi 2$ $\quad 2+v$
in $\phi 3$ $\quad 3+4v+v^2$
in $\phi 4$ $\quad 4+10v+6v^2+v^3$
in $\phi 5$ $\quad 5+20v+21v^2+8v^3+v^4$
in $\phi 6$ $\quad 6+35v+56v^2+36v^3+10v^4+v^5$
in $\phi 7$ $\quad 7+56v+126v^2+120v^3+55v^4+12v^5+v^6$
in $\phi 8$ $\quad 8+84v+252v^2+330v^3+220v^4+78v^5+14v^6+v^7$
in $\phi 9$ $\quad 9+120v+462v^2+792v^3+715v^4+364v^5+105v^6+16v^7+\&c.$

and in general

in ϕn $\quad \dfrac{n}{1}+\dfrac{n-1}{1}\dfrac{n}{2}\dfrac{n+1}{3}v+\dfrac{n-2}{1}\dfrac{n-1}{2}\dfrac{n}{3}\dfrac{n+1}{4}\dfrac{n+2}{5}v^2+\&c.$

When v is positive the formulæ belong to the class of catenarian functions; when v is negative, to the circular ones.

If we put $\sin pa$ for $\phi 0$, $\sin(p+1)a$ for $\phi 1$, and $-\operatorname{chord}^2 a$ for v, we obtain[29]

$$\sin(p+n)a=-\sin pa\left\{\frac{n-1}{1}-\frac{n-2}{1}\frac{n-1}{2}\frac{n}{3}\operatorname{cho}^2 a+\&c.\right\}$$
$$+\sin(p+1)a\left\{\frac{n}{1}-\frac{n-1}{1}\frac{n}{2}\frac{n+1}{3}\operatorname{cho}^2 a+\&c.\right\}$$

and in thus putting $p=0$

$$\sin na=n\sin a\left\{1-\frac{n^2-1}{1.2}\frac{\operatorname{cho}^2 a}{3}+\frac{n^2-1}{1.2}\frac{n^2-4}{3.4}\frac{\operatorname{cho}^4 a}{5}-\&c.\right\}$$

And again writing $na=A$, $a=\frac{A}{n}$, this takes the form

$$\sin A=n\sin\frac{A}{n}\left\{1-\frac{n^2-1}{1.2}\frac{1}{3}\left(\operatorname{cho}\frac{A}{n}\right)^2+\frac{n^2-1}{1.2}\frac{n^2-4}{3.4}\frac{1}{5}\left(\operatorname{cho}\frac{A}{n}\right)^4-\&c.\right\}$$

Now when n becomes indefinitely great, $n\sin\frac{A}{n}$ becomes A, so also $n\operatorname{cho}\frac{A}{n}$, wherefore

$$\sin A=\frac{A}{1}-\frac{A^3}{1.2.3}+\frac{A^5}{1.2.3.4.5}-\&c.$$

[29] The manuscript, as well as the published article, mistakenly wrote $v=-\operatorname{chord} a$. I also corrected a few other related typos. We have $\operatorname{chord} a=2\sin(a/2)$.

In order to obtain the series for the cosine we must put $pa = \frac{\pi}{2}$, and therefore $\sin(p+1)a = \cos a$, which gives

$$\cos na = -1\left\{\frac{n-1}{1} - \frac{n-2}{1}\frac{n-1}{2}\frac{n}{3}\,\text{cho}^2 a + \&\text{c.}\right\} + \cos a\left\{\frac{n}{1} - \frac{n-1}{1}\frac{n}{2}\frac{n+1}{3}\,\text{cho}^2 a + \&\text{c.}\right\}$$

and if, in this, we substitute for $\cos a$, its value $1 - \frac{1}{2}\,\text{cho}^2 a$,

$$\cos na = 1 - \frac{n^2}{2}\,\text{cho}^2 a + \frac{(n-1)n^2(n+1)}{24}\,\text{cho}^4 a - \&\text{c.}$$

whence,[30] proceeding as before,

$$\cos A = 1 - \frac{A^2}{1.2} + \frac{A^4}{1\ldots 4} - \frac{A^6}{1\ldots 6} + \&\text{c.}$$

3.6 Introduction to the $1'$ canon of sines (1881)

Canon of Sines.[31]

Narrative.

On January 1878, the canon of sines to thirty-three places for each two-thousandth part of the quadrant was laid on the table of the Royal Society of Edinburgh. The present work is the sequel thereof, being the interpolation of four termes between each contiguous pair of the previous results.

The intention was to have carried the interpolation to thirty-three places so as to have the sines true to the thirtieth place; and for this the sine and cosine of one minute were computed and a table of one thousand multiples of $2\,\text{ver}\,1'$ was written out;[32] the computation of the sines was also begun, all as mentioned in the manuscript "Record of the computation of the table of sines and cosines to the decimal division of the quadrant";[33] in which record, all the essentiel steps of the process are detailed.

---

[30] The previous development was given incorrectly by Sang, although he was led to the correct conclusion. The correct development is

$$\cos na = 1 - \sum_{i=0}^{\infty} \frac{\text{cho}^{2i+2} a}{(2i+2)!} \prod_{j=0}^{i} (n^2 - j^2).$$

[31] Volume K41, National Library of Scotland, Edinburgh, Acc 10780/51.

[32] National Library of Scotland, Edinburgh, Acc.10780/52 (volume K42).

[33] This must refer to "On the precautions…", reproduced here in section 3.4.

Meanwhile, during the preparation, for a special purpose, of a table of the equation of time at intervals of one day from the mean equinox, an exceedingly simple solution of Kepler's problem presented itself. An account of this solution was submitted to the Royal Academy of Science of Turin, and is printed in their Atti for 1879, under the title "Nouveau Calcul des Mouvements Elliptiques", along with some specimen tables.[34]

For the utilisation of this method, the values of circular segments measured in degrees of surface for each of the 40 000 minutes of the circumference, and tables of true-anomalies for each degree in orbits of each degree of ellipticity were computed and laid on the table of the Royal Society of Edinburgh in July 1879.

In order to render these tables still more convenient for the use of astronomers, the corresponding mean anomalies and radii vectores are wanting; but the computation of these requires the use of the logarithmic canon to the centesimal division. The only accessible table of this kind is that given to seven places for each minute in Callet's Tables Portatives.[35]

On proceeding to make use of this table, an obvious discrepancy in the column of differences led to an examination which revealed, on the single page for 31^c no fewer than twenty-three errors of this kind.[36]

The absolute need of a new calculation of the logarithmic canon even to seven places being thus apparent, the construction of the canon of sines was resumed, and the question arose 'to how many decimal places should it be carried?'.

In scarcely any department of applied science does the degree of exactitude attainable by observation exceed what is indicated by nine decimal places; for the great mass of calculation, seven places suffice; in ordinary surveying and in navigation five places are enough. In the construction, then, of fundamental tables, to serve, by abbreviation, as the basis of all others, fifteen decimal places are amply sufficient even in those critical cases where the figures to be rejected are close to the limit 5000.... These considerations led to the adoption of fifteen places in my table of logarithm to one million, and have again determined the same number for the canon of snes. The previous calculation affords an example, a very rare one certainly, of what may be needed; the sine of $65^c35'$ to thirty places is

$$.85549\ 98500\ 00003\ 84200\ 36673\ 55275$$

so that in shortening to the usual seven places, we can discover whether to write .855 4998 or .855 4999 only by carrying the calculation to the fifteenth place.

[34] See [77].

[35] See [7].

[36] I am not sure what Sang had in mind. I have for instance checked the values of the logarithms of the sines from $31^c00'$ to $31^c50'$ in Callet's tables [7], and I found no error in the last four places. There may however be errors for other values on that page, as the page gives the logarithms of the sines, cosines and tangents for every decimal minute between 31^c and 32^c, to seven places, together with their differences.

We may form some idea of the meaning of accuracy to fifteen decimal places by taking for the radious of our circle, the Earth's distance from the Sun, and noting the value of unit in the last place. The radius of the Earth is about 252 000 000 of English inches, and, taking the parallax at $8''.95$, our distance from the Sun is 23 000 times this radius;[37] hence our assumed unit (the unit of astronomers) is, in inches, represented by the figures

5 796 000 000 000;

and unit, in the last place of our table means .005796 or about the 173^{d} part of an inch, the seventh part of a millimetre.[38]

To assert, then, that the results given in the table are true to the last place, is to assert that the estimated distances from point to point in the Earth's supposed circular orbit do not err by the one hundredth part of an inch.

The computation was made in the same way as for the preceding table; that is, to say, the second difference was got by multiplying the last obtained sine by $2\operatorname{ver} 1'$, the multiplication being performed by help of the table of products. Eighteen places were kept on the scroll papers, and the products were tested by their first and second differences before being used in computing the next sine. The following is an example of the multiplication[39] extending from $34^{\mathrm{c}}..45'$ to $34^{\mathrm{c}}..50'$.

[37] The parallax of the Sun is the angle of the radius of the Earth as seen from the Sun. We have $r/l = \sin 8''.95$ where r is the radius of the Earth and l the distance of the Sun. $1/\sin 8''.95 \approx 23000$.

[38] The distance of the Sun being taken as the unit, 10^{15} less is about 0.005796 inches.

[39] The value above the $(34^{\mathrm{c}})46'$ line is the second difference. These second differences are computed by multiplying $2\operatorname{ver} 1' = 0.000000024674010952\ldots$ with the sines. For instance, $2\operatorname{ver} 1' \times \sin 34^{\mathrm{c}}45' = 0.00\ldots12709900716$. The three lines above the second differences show how the multiplication was decomposed. In the first case, $\sin 34^{\mathrm{c}}45' = 0.515|1128|749|\ldots$ and we have $515 \times 2467010952 =$ **1270707115640**280, $1128 \times 2467401 =$ **2783228**328, $749 \times 2467 =$ **1847**783. The values of the sines were however not always split uniformly. For instance, for the next second difference, we have $\sin 34^{\mathrm{c}}46' = 0.515|2475|0501|\ldots$, later we have $\sin 34^{\mathrm{c}}47' = 0.515|382|1224|\ldots$. Similarly, the value of $2\operatorname{ver} 1'$ was not always taken with the same number of decimals, even with identical places in the sines. And the value 6106819 in the second second difference should be 6106818. The figures on the right margin, 3322 178, etc., are the tabulated third differences: for instance 127 09900 716+3321 867 = 127 13222 583, and the figures in between (311, 318, etc.) are the tabulated fourth differences.

				3322	178
45′	127	07115	640		
		2783	228		
		1	848		311
	127	09900	716	3321	867
46′	127	07115	640		
		6106	819		
			124		318
	127	13222	583	3321	549
47′	127	07115	640		
		9425	472		
		3	020		311
	127	16544	132	3321	238
48′	127	07115	640		
		12731	790		
		17	940		315
	127	19865	370	3320	923
49′	127	07115	640		
		16062	781		
		7	872		
	127	23186	293		

The second differences so obtained were then applied as in the subjoined specimen[40]

	13	46428	18592	547	−3
45′	.51511	28749	07028	080	−8
		127	09900	716	
	13	46301	08691	828	
46′	.51524	75050	15719	900	
		127	13222	583	
	13	46173	95469	245	
47′	.51538	21224	11189	145	
		127	16544	132	
	13	46046	78925	113	
48′	.51551	67270	90114	258	
		127	19865	370	
	13	45919	59059	743	
49′	.51565	13190	49174	001	
		127	23186	293	
	13	45792	35873	450	−1
50′	.51578	58982	85047	451	−4

[40] In this specimen, −8 in the right margin means that the correct value computed in the 5′ canon is 8 units smaller than the value obtained by adding differences. It is this correct value which was used as the starting point for adding the next first differences (072 + 828 = 900).

Each fifth sine having been compared with that previously computed to 33 places, the requisite correction was marked on the margin of the scroll and a correction averaging the third part thereof was set opposite to the preceding first differences. These corrections were taken into account in the subsequent work.

The scroll calculations having been, in this way, completed for (illegible), the transcription into the pages of the present volume was effected, not by copying the computed sines, but by copying only the second differences and by applying these as had been done in the scroll work itself. By this means any error in copying was made sure of detection, as was also any error in the additions or subtractions.

Only, since the last three figures are omitted, it became necessary to adjust the last figure of the second difference so as that the thence resulting sine should be true to the nearest figure.

The table before us is not properly an interpolation; it is a new and independent work, the sole use of the preceding table having been to present the accumulation of the errors caused by the omission of the nineteenth and following figures. It furnishes, indeed, a severe check upon the former table and has been the means of discovering therein one false figure, namely in the sine of $80^{c}45'$.

No other error has been detected in the long preceding calculations and the author ventures to hope that there is not a single error in the present volume.

6 Molends Terrace
Edinburgh
12 September 1881

3.7 French notice to the canon of sines (1882)

The following text was written in French but does not seem to have been published. It was possibly a draft for a communication in a French journal, in order to obtain some support for a publication of the canon of sines.

Although the text is intelligible, the French is often faulty. Instead of polishing the writing, I have decided to keep it as Sang left it, but I added a few corrections in square brackets where I felt they are really necessary. (I have left some obviously incorrect spellings.) Some of the contents of this notice is similar to that found in other texts transcribed here, so that the general understanding should not be difficult.

Notice du Canon des Sinus à quinze places, pour chaque minute. Contribution au Système Décimal.[41]

Tandis que dans tous les pays, les négociants s'empressent à adopter le système décimal pour les poids et pour les monnaies, et que le [les] mécaniciens replacent [remplacent] leurs mesures locales par celles fondues [fondées] sur la division du meridien terrestre, il est à remarquer que les géomètres et les astronomes ne font aucun pas pour l'introduction de cet système dans leurs travails [travaux] ; et au moment nous sommes dans la position étrange que le système métrique va devenir universel sauf dans l'endroit de sa naissance.

Il n'est pas difficile de découvrir la cause de cet phénomène. Appercevant les avantages de l'uniformité, je divise mon cercle astronomique en quatre-cent degrés :— mais à quoi bon cette division ? Il n'y a pas des tables centésimales. Même pour la trigonométrie ordinaire, nous n'avons que la table très incommode de Callet.[42] Les déclinaisons des etoiles, leurs ascensions, les réfractions astronomiques et une foule des autres quantités doivent être exprimées en décimales pour que je puis mettre en usage mon cercle ainsi divisé. Nous ne pouvons pas faire un pas isolé dans la transition de l'un système à l'autre ; néanmoins il n'y a pas de doute que la division centésimale va-t-être universellement adoptée ; ce n'est qu'une affaire de temps.

Il y a long temps que Archimède le Syracusan a montré au roi Gelon les pouvoirs transcendents de la numération ordonnée. Quatre siècles sont écoulés depuis que Johann Müller de Königsberg a introduit la division décimale du rayon ; trois cents depuis que Reinholdt a calculé les sinus d'après la même division ; et puis il y a deux siècles et demi que John Nepair a couronnée le système par la notation des fractions decimales, et par l'invention des Logarithmes. En 1658 John Newton a publié la Trigonometria Britannica[43] dans laquelle, d'après l'avis de Henry Briggs, le degré est divisée en cent minutes ; et au commencement de cet dix-neuvième siècle, les savans français ont proposé, comme le dit M. Callet, de renouveller l'idée de Briggs en le portant un

[41]National Library of Scotland, Edinburgh, Acc 10780/81 [79].
[42]See [7].
[43]See [30].

pas en arrière ;[44] c'est à dire ils ont proposé la division centésimale du quart de cercle. Voila donc que les pas de progrès quoique lents sont sûrs. Il est à espérer que l'advent [avènement] du vingtième siècle ne trouve que des traces des anciens systèmes.

Le gouvernement français [a] ordonné la construction de tables décimales tres étendues ; et la publication en avait fait quelques progrès quand les événements politiques y faisaient [mirent] fin. Néanmoins les tables calculées au Bureau du Cadastre sont devenues celèbres.[45] En 1818 le gouvernement d'Angleterre faisait à celui de la France, l'offre de partager les frais de la publication, mais cette offre ne portait pas de fruit, et les tables de Prony restent inedites.

Chez nous, les étudiants des mathématiques à l'université d'Edimbourg, le bruit courrait que cet manque de succès résultat [résultait] de la découverte des inexactitudes dans les calculs ; et quoique je ne pouvais jamais tracer d'autorisation pour cet bruit, l'impression m'en restait toujours.

La formation des tables trigonométriques fondementales, qui peuvent être abrégées pour la pratique exige des logarithmes étendus à plusieurs places additionnelles ; c'est pourquoi j'ai projeté,[46] vers l'an 1825 le calcul d'une nouvelle table logarithmique jusqu'à un million. Les calculs preparatifs ont été portés à 28 places pour être surs à 25. Pour les nombres au dela de 100 000 le calcul a été restreint à quinze places. À propos d'exemplaires raccourcis à neuf places et soumis aux mathématiciens, M. Govi a rendu en 1873 un rapport[47] à l'Académie Royale de Turin. Le manuscrit à quinze places a été continué jusqu'à 370 000.

Le journal anglais « Nature » opposait à la publication de cette table, l'existence et la perfection de celles du Cadastre, qui, selon lui, ne peuvent jamais être surpassées. Ainsi forcé malgré moi à l'examen de ces tables fameuses, j'ai y trouvé des grandes imperfections. Des logarithmes des nombres premiers entre deux milles et vingt milles, calculés au Cadastre et publiés par Legendre dans son traité des fonctions elliptiques,[48] quatre seulement sont exacts, et presque toutes les erreurs sont dans un même sens. Parmi les corrections sur Vlacq signalées par M. Lefort, à l'aide du manuscrit de Prony, j'ai trouvé une en tort ; et ainsi l'opinion de M. Lefort est bien établie que l'exactitude des tables du Cadastre ne depasse pas l'unité dans la douzième place ; c'est à dire que ces tables cèdent aux calculs anciens de Henry Briggs.

On a suivi, au Cadastre, la méthode d'interpolation dite « méthode Mouton »[49] en portant les differences du sixième ordre jusqu'à vingt six places, et en insérant 199 intermédiares. En appliquant les différences on a rejeté deux

[44] Sang means that Briggs had divided the degree in 100 parts, but that the French endeavour went further up (back) in that it also divided the quadrant in 100 parts.

[45] See [39].

[46] Indeed, Sang seems to have had the idea of constructing tables shortly after the joint project of publishing the Cadastre tables was abandoned [1, p. 185].

[47] See [20].

[48] See [24].

[49] See [49].

chiffre pour chaque ordre. L'incertitude provenant de cette méthode peut être evalué en representant une erreur d'unité dans la dernière place par un millimètre : alors l'erreur en resultant à la fin du procedé peut ou être nulle ou peut entourer l'orbite lunaire.

Cette même méthode a été employée pour les canons trigonométriques, et notre confiance est tout-à-fait détruite.

Un nouveau calcul des fonctions angulaires étant ainsi de rigeur, j'ai laissé à part la table logarithmique pour entreprendre le Canon des Sinus ; et en janvier 1878 j'ai placé devant la Société Royale d'Edimbourg un manuscrit contenant, à trente trois places, les sinus des deux milles parties du quadrant, avec leurs différences premières et secondes ; et j'avais commencé le calcul pour chaque minute quand le progré fut interrompu par la découverte d'une solution très simple au problème de Kepler. L'Académie Royale de Turin m'a fait la grande faveur de donner à cette solution une place dans leurs Actes, sous le titre « Nouveau Calcul des Mouvements Elliptiques ».[50] L'application de cette solution demande des tables spéciales et j'ai calculé pour chaque minute les sinus mesurés en degrés ; puis les valeurs des segments circulaires pour chacun des quarante mille minutes de la circonférence entière ; et aussi, pour faciliter les premières approximations, les anomalies moyennes pour chaque degré de l'arc de position en orbites de chaque degré d'ellipticité, avec leurs différences et leurs variations. Au moyen de ces tables on peut calculer très facilement, avec l'exactitude du centième d'une seconde, la position d'une planète à un temps donné, quelque soit l'excentricité de son orbite.

Pour rendre encore plus facile cette solution je voudrais calculer les rayons vecteurs et les anomalies vraies. Pour cela il m'a fallu des canons trigonométriques. La seule table décimale que je connais est celle donnée par Callet dans ses « Tables Portatives ». En m'en servant, j'ai observé une erreur evidente parmis les différences, et en examinant, j'ai trouvé vingt trois sur la même page, celle de 31 degrés. Me voila encore repoussé au calcul des sinus.

En interpolant pour les minutes, le calcul fut borné à quinze places qui suffisent pour toutes les affaires actuelles. Il n'y a guère une departement des sciences appliquées où l'exactitude dépasse celle de dix places, et le doute, quand nous avons à rejeter 50000 ne peut avoir aucun importance. Pour avoir une idée nette de cette exactitude, imaginons la distance solaire pour l'unité ; alors le chiffre 7 dans la quinzième place ne vaudra plus d'un millimètre.[51] Pour observer avec une telle precision, il nous faudra une lunette avec laquelle nous pouvons examiner des diatoms [= diatomées] à la surface de la lune.

Pour établir le titre de cette ouvrage à la confiance des mathématiciens, il nous faut en donner l'histoire dès son commencement.

La division du quadrant en dix milles parties exige quatre bisections et quatre quinquisections.[52] L'ordre dans lequel ces divisions doivent être efféctuées dé-

[50] See [77].
[51] 150 millions of km divided by 10^{15} and multiplied by 7 is about one millimeter.
[52] $10000 = 2^4 \times 5^4$.

pend de la commodité et l'exactitude des calculs. Or, pour la bisection nous avons les formules

$$\sin a = \sqrt{\left\{\frac{1}{2} - \frac{1}{2}\cos 2a\right\}}$$
$$\cos a = \sqrt{\left\{\frac{1}{2} + \frac{1}{2}\cos 2a\right\}}$$

tandis que pour les quinquisections il nous faut résoudre les equations du cinquième ordre

$$16\sin^5 a - 20\sin^3 a + 5\sin a - \sin 5a = 0$$
$$16\cos^5 a - 20\cos^3 a + 5\cos a - \cos 5a = 0.$$

On voit ici que le sinus du moitié provient du versinus de l'arc entier. Quand l'arc est petit l'expression pour son versinus manque des chiffres superieurs et sa racine ne peut pas être obtenue avec précision. Par exemple, dans un calcul à dix places, le versinus de $2'$ a la valeur[53] .00000 00506, et en extrayant la racine de sa moitié nous ne pouvons obtenir avec exactitude[54] que les six places .000157. Il nous faut donc porter les calculs antérieurs à plusieurs places supernumeraires. Mais pour la division en cinq l'exactitude du resultat depasse celle du donné ; consequemment les bisections doivent prendre precédence.

Néanmoins on connait que la division du cercle en cinq peut être faite au moyen des opérations geométriques : en effet[55]

$$\sin 20^{\mathrm{c}} = \frac{1}{4}(\sqrt{5} - 1)\,; \qquad \sin 40^{\mathrm{c}} = \frac{1}{4}\sqrt{(10 - 2\sqrt{5})}$$
$$\sin 60^{\mathrm{c}} = \frac{1}{4}(\sqrt{5} + 1)\,; \qquad \sin 80^{\mathrm{c}} = \frac{1}{4}\sqrt{(10 + 2\sqrt{5})}.$$

et apres ces calculs il nous restent les quatre bisections et trois quinquisections.

Pour les bisections, chaque racine quarrée fut extraite deux fois ; une fois par le procédé ordinaire, une fois à l'aide de Crelle's Rechentafeln ;[56] et, pour s'assurer des derniers chiffres on a verifié le calcul par la division

$$\sin a = \frac{\sin 2a}{2\cos a}.$$

Les opérations ont été portées à trente trois places. D'après cette procedé les sinus des quatre vingt parties du quadrant ont été obtenus.

[53] In fact, $\mathrm{ver}\,0^{\mathrm{c}}.02 = 0.000000049348\ldots$

[54] In fact, if we assume that the 10-place value is correctly rounded, then it is wrong by at most $5\cdot 10^{-11}$ and this leads to a maximal relative error of about 10^{-3}, that is about one or two units of the seventh place.

[55] In the original manuscript, the values of $\sin 40^{\mathrm{c}}$ and $\sin 60^{\mathrm{c}}$ were swapped.

[56] See [11].

Pour les quinquisections, la méthode que j'ai publiée en 1829 dans une brochure portant le titre « Solution of Algebraic Equations of all orders », a été employée.[57] C'est une forme ordonnée du Théorème de Taylor qui sert à résoudre les equations qui renferment des fonctions dont les dérivées se terminent ou se répétent.

En substituant pour $\sin a$ quelque valeur tentative x, et en ecrivant $\mathcal{E}$ pour l'erreur en resultant, nous avons

$$\mathcal{E} = 16x^5 - 20x^3 + 5x - \sin 5a$$

où $\mathcal{E}$ est une fonction de x.

En prenant les derivées successives nous trouvons,

$$\begin{aligned}
{}_1\mathcal{E} &= \ \ 80x^4 - \ \ 60x^2 + 5 \\
{}_2\mathcal{E} &= \ 320x^3 - 120x \\
{}_3\mathcal{E} &= \ 960x^2 - 120 \\
{}_4\mathcal{E} &= 1920x \\
{}_5\mathcal{E} &= 1920
\end{aligned}$$

Notre but est de donner à x quelque correction δx qui rendra $\mathcal{E}$ zero ; et pour attente [première] approximation nous posons $\delta x = \frac{-\mathcal{E}}{{}_1\mathcal{E}}$.

Le calcul des changements est répresenté par le schema ci-joint, où la nouvelle valeur de chaque derivée est obtenue en multipliant chaque terme de la derivée superieure par δx et en divisant le produit par l'indice de δx.

${}_5\mathcal{E}$	${}_4\mathcal{E}$	${}_3\mathcal{E}$	${}_2\mathcal{E}$	${}_1\mathcal{E}$	$\mathcal{E}$	x
	${}_5\mathcal{E}\ \frac{\delta x}{1}$	${}_4\mathcal{E}\ \frac{\delta x}{1}$ ${}_5\mathcal{E}\ \frac{\delta x^2}{1.2}$	${}_3\mathcal{E}\ \frac{\delta x}{1}$ ${}_4\mathcal{E}\ \frac{\delta x^2}{1.2}$ ${}_5\mathcal{E}\ \frac{\delta x^3}{1.2.3}$	${}_2\mathcal{E}\ \frac{\delta x}{1}$ ${}_3\mathcal{E}\ \frac{\delta x^2}{1.2}$ ${}_4\mathcal{E}\ \frac{\delta x^3}{1.2.3}$ ${}_5\mathcal{E}\ \frac{\delta x^3 4}{1.2.3.4}$	${}_1\mathcal{E}\ \frac{\delta x}{1}$ ${}_2\mathcal{E}\ \frac{\delta x^2}{1.2}$ ${}_3\mathcal{E}\ \frac{\delta x^3}{1.2.3}$ ${}_4\mathcal{E}\ \frac{\delta x^4}{1.2.3.4}$ ${}_5\mathcal{E}\ \frac{\delta x^5}{1.2.3.4.5}$	δx
${}_5\mathcal{E}$	${}_4\mathcal{E}'$	${}_3\mathcal{E}'$	${}_2\mathcal{E}'$	${}_1\mathcal{E}'$	$\mathcal{E}'$	x'

En appliquant ainsi des corrections successives à x nous parvenons à rendre $\mathcal{E}$ zero, et alors nous avons, avec la valeur de x celles des derivées, dont nous

[57] See [67].

trouvons aisément les puissances successives de x, et d'où l'on tire

$$\begin{aligned}
\cos^2 a &= \frac{7}{8} - \frac{1}{960}{}_3\mathcal{E}\\
\cos 2a &= \frac{3}{4} - \frac{1}{480}{}_3\mathcal{E}\\
(\sin 2a)^2 &= \frac{3}{8} + \frac{1}{960}{}_3\mathcal{E} - \frac{1}{20}{}_1\mathcal{E}\\
\sin 3a &= \frac{3}{2}x - \frac{1}{80}{}_2\mathcal{E}\\
\cos 4a &= \frac{1}{4} - \frac{1}{480}{}_3\mathcal{E} + \frac{1}{10}{}_1\mathcal{E}
\end{aligned}$$

qui peuvent servir pour les vérifications ultérieures.

La première application de cette méthode était au calcul du $\sin 25'$. Le sinus de $1^{\mathrm{c}}25'$ avait été trouvé

$$.01963\ 36924\ 60628\ 30208\ 54772\ 79238$$

et pour premier essai nous prenons $x_1 = .003926$ d'où

$$\begin{aligned}
{}_5\mathcal{E} &= \ 1920.\\
{}_4\mathcal{E} &= +\ \ 7.53792\\
{}_3\mathcal{E} &= -\ 119.98520\ 30630\ 4\\
{}_2\mathcal{E} &= -\ \ .47110\ 06357\ 41831\ 68\\
{}_1\mathcal{E} &= +\ \ 4.99907\ 52104\ 46019\ 39220\ 608\\
\mathcal{E} &= -\ \ .00000\ 49027\ 11840\ 29565\ 87170\ 65222
\end{aligned}$$

ce qui donne la correction approximative[58]

$$\delta x = x_2 - x_1 = .00000\ 09807\ 23.$$

[58]Below, the computation uses this truncated value of δx, and not the exact value of δx.

En appliquant cette correction, le calcul prend la forme suivante.

$_4\mathcal{E}$		$=$	$+$	7.53792
$_5\mathcal{E}$	$\frac{\delta x}{1}$	$=$	$+$	.00188 29881 6
$_3\mathcal{E}$			$-$	119.98520 30630 4
$_4\mathcal{E}$	$\frac{\delta x}{1}$		$+$	73926 11516 16
$_5\mathcal{E}$	$\frac{\delta x^2}{1.2}$		$+$	9 23344 89861 984
$_2\mathcal{E}$			$-$	.47110 06357 41831 68
$_3\mathcal{E}$	$\frac{\delta x}{1}$		$-$	11 76722 48303 59377 792
$_4\mathcal{E}$	$\frac{\delta x^2}{1.2}$		$+$	3625 05207 19814 9184
$_5\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$		$+$	30184 85263 36382
$_1\mathcal{E}$			$+$	4.99907 52104 46019 39220 608
$_2\mathcal{E}$	$\frac{\delta x}{1}$		$-$	4620 19228 78663 63907 0464
$_3\mathcal{E}$	$\frac{\delta x^2}{1.2}$		$-$	57701 94018 65227 00332
$_4\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$		$+$	118 50573 14397
$_5\mathcal{E}$	$\frac{\delta x^4}{1.2.3.4}$		$+$	740 07448
$\mathcal{E}$			$-$	.00000 49027 11840 29565 87170 65222
$_1\mathcal{E}$	$\frac{\delta x}{1}$		$+$	49027 08037 61425 14763 82523
$_2\mathcal{E}$	$\frac{\delta x^2}{1.2}$		$-$	226 55644 20566 58201
$_3\mathcal{E}$	$\frac{\delta x^3}{1.2.3}$		$-$	1886 32066 28516
$_4\mathcal{E}$	$\frac{\delta x^4}{1.2.3.4}$		$+$	2 90553
$_5\mathcal{E}$	$\frac{\delta x^5}{1.2.3.4.5}$		$+$	14

Les sommes, c'est à dire les nouvelles valeurs des fonctions $\mathcal{E}$ sont écrites ci dessous en laissant des spaces vides pour recevoir les corrections subsequentes ; ces sommes donnent pour seconde correction

$$x_3 - x_2 \quad = \qquad .00000\ 00000\ 00806\ 00049\ 315$$

qui est appliquée d'après le même procédé.

$_4\mathcal{E}$	$+$	7.53980 29881 6	
	$+$	15 47520 94684 8	
$_3\mathcal{E}$	$-$	119.98519 56695 05138 94138 016	
	$+$	6077 08492 67108 03611	
	$+$	6236 51323	
$_2\mathcal{E}$	$-$	.47121 83079 86509 91985 74121 71778	
	$-$	96708 12688 03203 86403	
	$+$	24 49067	
$_1\mathcal{E}$	$+$	4.99907 47483 69088 66656 82979 16873	
	$-$	379 80218 86184 35579	
	$-$	389 73399	
$\mathcal{E}$	$-$	.00000 00000 04029 25671 25036 78847	(should be 78849)
	$+$	4029 25671 24791 97623	
	$-$	1 53060	

Repondant à la valeur x_3 nous avons les resultats suivants, d'où nous tirons la correction[59]

$x_4 - x_3$	=	.00000 00000 00000 00000 00049 27769
et ensuite		
$_4\mathcal{E}$	+	7.53980 29897 07520 94684 8
	+	94613 15915
$_3\mathcal{E}$	−	119.98519 56694 99061 85645 28255 45066
	+	371 54405
$_2\mathcal{E}$	−	.47121 83080 83218 04673 77302 09114
	−	5912 59292
$_1\mathcal{E}$	+	4.99907 47483 68708 86437 96405 07895
	−	23 22055
$\mathcal{E}$	−	.00000 00000 00000 00000 00246 34284
	+	246 34284
		d'où dernièrement
$_5\mathcal{E}$	+1920.	
$_4\mathcal{E}$	+	7.53980 29897 07520 94685 74613 15915
$_3\mathcal{E}$	−	119.98519 56694 99061 85645 27883 90661
$_2\mathcal{E}$	−	.47121 83080 83218 04673 83213 68407
$_1\mathcal{E}$	+	4.99907 47483 68708 86437 96381 85840
$\mathcal{E}$		.00000 00000 00000 00000 00000 00000
x_1		.00392 6
$x_2 - x_1$		09807 23
$x_3 - x_2$		00806 00049 315
$x_4 - x_3$		00049 27769
$\sin 25'$	=	.00392 69807 23806 00049 31549 27769

Cet exemple demontre la rapidité de la méthode dès que la première approximation est trouvée. L'application aux racines ordinaires est expliquée dans mon traité de L'Arithmétique superieure (Higher Arithmetic, Edinburgh 1857).[60]

Le sinus et le cosinus de $25'$ étant trouvés on obtiens ceux de ses multiples au moyen des différences secondes. Or

$$\sin(P + a) - 2\sin P + \sin(P - a) = -2\,\mathrm{ver}\, a \cdot \sin P$$

et la différence second est le produit du sinus dernièrement trouvé par le constant $2\,\mathrm{ver}\, a$. On épargne le travail des multiplications en préparant une table des multiples de $2\,\mathrm{ver}\, a$. Pour $a = 25'$ cette table a été portée à cent fois ; mais pour $a = 5'$ et pour $a = 1'$ elle a été étendue à mille fois. Voici un échantillon du calcul :—

[59]What follows actually uses $x_4 - x_3 = .00000\ 00000\ 00000\ 00000\ 00049\ 27768\ 7057$ which is a (slightly) erroneous value of $\frac{0.000000000000000000000024634284}{4.999074748368708864379640507895}$. Consequently, the corrections to $_4\mathcal{E}$, $_3\mathcal{E}$, etc., are slightly wrong, but this does not prevent the algorithm from converging to $\sin 25'$.

[60]See [68].

2900′	.43993 91698 55915 14083 30457 65281
	+ 352 31447 76343 27439 92773 05436
	− 68387 37480 58022 90027 96021
2925	.44346 23146 32258 41523 23230 70717
	+ 351 63060 38862 69417 02745 09415
2950	.44697 86206 71121 10940 25975 80132

Ici une place est laissée vide pour recevoir le produit de sin 2950′ par 2 ver 25′ ; ce qui doit être soustraite de la différence première la dessus pour donner la différence première succédante.

Pour assurer l'exactitude du trentième chiffre les calculs ont été portés à trente trois places. À chaque cinquième pas les sinus ont été comparés avec ceux deja trouvés, et les petites divergences aux dernières places ont été corrigées empiriquement. Ce calcul fut donc une épreuve sevère du précedent, dans lequel aucun erreur n'a été trouvée.

Le même procédé fut suivi pour les sinus de cinq minutes et d'un minute ; mais avant d'en faire les applications on a verifié leurs valeurs par les series connues qui sont alors assez convergentes.

Le calcul des sinus de cinq en cinq minutes etait encore une revision complète des travails [travaux] précédents, jusqu'à trente places. Mais celui pour les minutes singuliers n'étant porté qu'à dix huit places (pour être exact à quinze) n'a verifié que cette partie de ses antecedents. La seule erreur exposée par ces verifications sevères fut erreur de transcription d'un chiffre dans le sinus de $80^c45'$.

Ainsi j'ose offrir aux mathématiciens cette canon des sinus avec toute confiance dans son exactitude. Elle contient en soi même les moyens pour sa verification, car on peut s'assurer d'un chiffre suspect par l'application des différences precédentes et succédentes.

Au moyen de ces canons à quinze places, des sinus et des logarithmes on peut construire des Tables trigonométriques justes à quatorze chiffres, qui suffisent pour toutes les affaires de la pratique ou des sciences.

Avec cette notice, je place devant les Académiciens la copie communiquée du Canon, afin qu'ils puissent juger de l'utilité de la publication. Des telles tables, quoique necéssaires au progrès des sciences exactes, sont d'une usage trop rare pour que les libraires voulussent en entreprendre les frais. L'opinion des mathématiciens peut renouveller l'intérêt que les Gouvernements de la France et d'Angleterre ont jadis montré à cet égard.

M. Lefort m'avait proposé de faire controller mes tables avec celles du Cadastre, et je serais très content d'accéder à cette proposition en cas que la publication des tables décimales soit entreprise serieusement ; car c'est dans l'intérêt de la science qu'aucun moyen pour obtenir de l'exactitude ne soit negligé.

Edimbourg
20 Fev. 1882

Edward Sang

3.8 On the need for decimal subdivisions (1884)

On the need for decimal subdivisions in astronomy and navigation, and on tables requisite therefor.[61]

The abstract question as to what number would have been most advantageously taken for the basis of an arithmetical system has been put aside by the universal preference shown for the number ten. All nations having any culture count in tens. In the English language, traces remain of the old numeration by dozens and scores;[62] the French still prefer to say "quatre-vingt seize," rather than "nonante-six." These vestiges serve to show that there has been change. But from the old Eastern languages all traces of any but the denary counting have disappeared.

It is in vain to argue that the number twelve is divisible by three and by four, or that the perfect number six has the preference; for, however strong the arguments may be, there is no likelihood that they shall overturn the universally adopted mode. Nay, when purely as arithmeticians, we come to look into the matter, and consider the needs and capabilities of mankind, we find arguments of no mean weight in favour of the denary mode.

But, whatever question there may be about the convenience of one or of another basis, there can be no question as to the principle of uniformity in the plan. To count our money in dozens and scores, our weights in sevens, and our distances in elevens, must necessarily entail trouble and confusion. Our unreasoning adherence to the medley of British monies, weights, and measures, is indeed a subject of wonder. If there be fourteen pounds to the stone, why not fourteen ounces to the pound? Five and a half yards go to a perche,[63] why not five perches and a half to the furlong?[64] We make our pound of seven thousand troy grains, and come down again with sixteen ounces to the pound!

The introduction of the Indian numerals and notation has brought the inconvenience of these haphazard schemes into strong relief. The whole power of this algorithm comes from its uniformity. The old scheme of counting by help of letters had proceeded decimally, its great convenience having led to its use among the Arabs, from whom it passed into Greece. In this scheme the value of the letters depends on their place in the Hebrew Elif Be, which place is fixed in the Arab's memory by the rhythm "ebjed hevves hota kelmen." &c., while the Greeks had to supply two new characters to fit it to their alphabet. The first group of nine letters are taken to signify units, the second group tens, and the third group hundreds. But the marks, in the Indian method, rise in value ten times for each step on the scale, and thus ten characters serve, and much more than serve, for the former twenty-eight.

[61] Published in 1884 [81].
[62] A score is 20.
[63] A rod (perche) is 16.5 feet, hence 5.5 yards.
[64] A furlong is actually equal to 40 rods.

We, who have never had to use the older method, can hardly appreciate the magnitude of the improvement. Adopted at once by men of science, it led to the decimal division of the radius, and to the construction of the canon of sines in its modern form. Passing to commercial men, it greatly facilitated their computations. In every branch of business its influence is felt. Thus Fahrenheit, when arranging his thermometers, divided the capacity of the bottle decimally, and estimated temperature by the expansion of mercury in ten-thousandth parts of its bulk; while Celsius, proceeding in another direction, placed one hundred degrees between the temperature of freezing and boiling water. The chemist makes his analyses in hundredths; the banker discounts per cent.; in every quarter the struggle is in favour of decimals. Gunter contrived his chain of one hundred links in order that there might be one hundred thousand square links in an acre; the engineer graduates his levelling staff not in feet, inches, and eighths, but in feet and hundredths.

There is no doing without decimals; when, in making a proportion, we have to compare two quantities of one kind, we, as the arithmeticians say, bring them both to one denomination:[65] 2 cwt. 3 qrs. 17 lbs. $11\frac{1}{4}$ oz. must be brought to quarter ounces, of which there are 20 845 in this quantity. That it to say, having found our old system to be unworkable, we have recourse to counting in tens; and, moreover, the trouble of converting our confused measures into decimals exceeds that of the real business at hand. Every such conversion is a protest in favour of uniformity.

Of all the affairs to which calculation is applied, trigonometry and astronomy have reaped most copiously the benefits of the Indian algorithm. We have only to compare the laborious process by which Archimedes determined the ratio of the circumference to the diameter of a circle, or the parallax and distance of the moon, to perceive how effectively the new numerals smoothed the rough road of alpha, beta; iota, kappa. Yet, great as these benefits were, they failed to satisfy the growing needs of science. Each step in exactitude added to the toil of the computer, till, discouraged by the swelling crowd of multiplications and divisions, of proportions among the sines and cosines, the mean distances, excentricities, anomalies, and periodic times, Kepler began to despair of the future of his science. Can we, then, afford to mar these benefits by a slavish adherence to a scheme of subdivision, beautiful in its uniformity, dignified by its age, but inept to the actual requirements?

The successive division by sixty, into parts of the first, second, and third degree of minuteness, dates back from before the reach of authentic history; it speaks of a great advance and subsequent decay of knowledge, for the ancient stadium and the Chinese li agree,[66] within an inch or two, with the third subdivision of the earth's circumference in this progression. The convenience of its numerous divisions has, no doubt, helped to retain it in use. Sixty combines, in this

[65] One hundredweight (cwt) is equal to 112 pounds. One quarter (qr) is a quarter of a hundredweight. A pound (lb) is sixteen ounces (oz).

[66] The "li" was about 500 m, but the Greek stadion was somewhere between 157 and 209 m, depending on which type was considered.

respect, the advantages of ten and twelve, but is far too large for numeration in the ordinary affairs of life. Its retention in the measurement of time and angle is a great hindrance to our progress.

In the exceedingly simple applications of trigonometry to land-measuring, we have very little to do even with the addition and subtraction of angles, nothing whatever with their ratios; and thus the character of the subdivision has, comparatively, little importance for the surveyor. Yet even he would be much helped by the centesimal division of the quadrant. It is a long time since the division of the azimuth circle into four quadrants of ninety degrees each was discarded; the bearings were then read, so many degrees to the east or west of north, so many degrees east or west of south, the same number of degrees indicating four different directions. The awkwardness of this is obvious to us; division into two parts of 180° each was substituted, and this again is now superseded by the graduation all round to 360°, so that a number applies to one direction only; this gives great clearness to the field operations. Having observed, from the station A, the bearings of various signals, among others that of the station B, and having carried the theodolite thither, we wish so to plant it as that it may again indicate the bearings of other signals. For this purpose we so place the azimuth circle as that, on looking back to A, the reading may be exactly the opposite of the previous reading from A to B. As the seamen phrase it, we must box the compass; we have to add or subtract 180° as the case may be.

In computing the co-ordinates of the stations, by help of the traverse table,[67] or by the logarithmic process, we have to note the change from addition to subtraction at 90°, 180°, 270°, 360°, and have to pass from the top to the bottom of the page at 45°, 135°, 225°, and 315°; changing sine into cosine, difference of latitude into departure. Whereas, with the centesimal division of the quadrant, the changes are at the hundreds and fifties, while the opposite directions differ by 200°. The improvement both in comfort and in freedom from mistakes needs not to be insisted on.

In astronomical work, the awkwardness of having two numerical systems is conspicuous. We observe a planet's opposition to the sun, and again another opposition; the interval of time is noted in days, hours, minutes, seconds; the change of longitude in signs, degrees, minutes, seconds; and thence, roughly, to compute the periodic time we have to make a proportion. If we had been habituated to count in sixties, and if the number 24 had not occurred, the calculation in sexagesimals would have been the natural one; our logarithms would, according to Nepair's own opinion, have had 60 for their basis, just as now they have 10. As things are, no one can make the calculation. We must turn the times and the angles into decimals, taking the day or the second as the unit of time, the degree perhaps or the second as that of angle; without decimals we are unable to move a single step.

Now these divisions are made for the purpose of calculation; intrinsically

[67] In 1864, Sang edited Shortrede's traverse table [85].

it is of no moment which way we count, the planetary phenomena are not thereby affected; the matter is one purely of arithmetical convenience. Had the subdivisions been according to the powers of ten, these conversions and their attendant labour would have been saved. But it is not now and then only that these irksome conversions occur; they pervade every calculation in geodesy, navigation, astronomy. The estimate is not too high, that they double the labour of computation.

In astronomical works there is abundant evidence of the need for a change. While the reckoning of longitude in signs, used sixty years ago, is discarded in favour of the counting in degrees all round, thirds are quite disused, the second is divided into tenths and hundredths. The arguments for the planetary disturbances are given, no in degrees, but in thousandths parts of the entire revolution.

There is no work having greater authority in these matters than that most admirable one, the *Nautical Almanac*, and every page thereof proclaims the need for decimals. The right ascensions, declinations, latitudes, and longitudes are given to decimals of the second. Now, if the division of the second into 100 parts be better than into 60, why should we not adopt, as John Newton did in the *Trigonometria Britannica*,[68] the centesimal division of the degree? There is, and there can be, no argument in favour of division by 60 down to seconds, that will not hold as well for thirds and for fourths; and the same instinct for convenience which leads to the decimal division of the second would, if it had its own way, lead to that of the degree, of the quadrant, and of the day.

But ease of calculation is not the only consideration. The sun's daily right-ascensions, to hundredths of a second, are accompanied by a column of variations in one hour; this, which is really needed for the sake of the inept computer, saves the division by 24. But this column occupies the place of the actual differences, needed by the strict computer for taking into account the variation of the variations. With decimal graduation one column would suffice for all, and the compiler of the almanac would be spared the labour. The same may be said of the sun's declinations and of the moon's hourly places, which are accompanied by variations in 10 minutes. In the last-mentioned there is a remarkably strong instance of the awkwardness of sixties. Thus the variation in declination is to be seen written $112'' \cdot 37$, rather than $1' \cdot 52'' \cdot 37$; it would be difficult to cite a more forcible example.

That triumph of skill, patience, and exactitude, the table of Lunar Distances, is a protest even stronger; therein the moon's distances from a star are given at intervals of three hours. In order to compute the Greenwich time of his observation, the mariner compares his observed distance (corrected for refraction and parallax) with those found in the almanac; he has therefore to make a proportion in sexagesimals. Seamen are understood to be so wedded to the present system, that they of all others would dislike a change; yet such are the torments of sexagesimals that, for the shunning of them, a column

[68] See [30].

of proportional logarithms is contrived, and a special logarithmic system is arranged. Instead of having to work out a simple proportion, the seaman is drilled to use the proportional logarithm, whose nature, in ninety-nine cases out of a hundred, he does not comprehend.

In the higher branches of astronomical calculations, and in the application of trigonometry to mechanical and physical problems, the arcs and their various functions have to be compared, the mode of comparison being suited to the particular cases. When the arcs are homologues of angles measured by help of graduated instruments, their natural unit is the entire circumference; but their sines and tangents, having reference to rectilineal measure, are most conveniently compared with the radius. Hence it is that, in ordinary trigonometry, two units are employed; and hence also the convenient though somewhat illogical expression, "the sine of an angle," instead of "the sine of the arc homologous to an angle." But in many cases, notably in analytical investigations, the radius of the circle is made the basis of comparison both for arcs and for sines. Also, in computing the anomalies of the planets, the areas passed over by the radius-vector have to be considered, and it is much preferable to measure the sines in parts of the circumference, the areas in parts of the surface of the circle.

Thus we have often to pass from one unit of measure to another; with no system of subdivision can the transitions be made more easily (if at all) than by that of uniform decimal subdivision.

From whichever point of view the matter may be studied, the desirability of the change is clear; but there are difficulties in the way; there are the prejudices of habit, the discomforts of transition, the existing mass of preparatory work suited to the old plan, and, above all, the mass of preparations needed for the new scheme, Here, indeed, the great obstacle lies.

Aside from the proposal of a change of system, a new computation of fundamental tables looms in the near future. The precision of modern measurements render it necessary, in astronomical speculations, to reckon to hundredths of a second of time, to tenths of a second of arc. Now when we determine an arc by help of (say) the seven-place logarithm of its tangent true in the last figure, the uncertainty arising from the omitted parts may amount to the fortieth part of a second; so that, since the logarithm itself is subject to several similar uncertainties, we may, notwithstanding all care, err by the tenth part of a second. But it is a sound principle that the accuracy of the arithmetical work should be clearly beyond that of observation, in order that no perceptible new error may be introduced, and thus the time is not very far distant when eight-place tables may be indispensable. Hence, in designing the canons for the decimal system, we must also look forward to increased precision.

Since by far the greater number of computations are done by help of logarithms, our first business is to see to the logarithmic canon. Beginning independently of all previous work, the logarithms of all primes up to 10,000 have been computed to 28 places, that they may be true to 25, each prime being put in relation to, at least, three others. The greatest discrepancy found amount to unit in the 27th place, so that this fundamental table may be regarded as altogether free from error. The volumes I., II., III., placed on the Society's table, contain all the articulate steps of the work, with indices to the primes and to the divisors used; so that, if in any subsequent computation one of these divisors should recur, we are spared the labour of a new division. Thus for the logarithm of 6563, the three equalities were used

$$\begin{aligned} 32\,130\,000\,001 &= 11 \cdot 599 \cdot 743 \cdot 6563 \\ 627\,600\,001 &= 7 \cdot 19 \cdot 719 \cdot 6563 \\ 36\,930\,001 &= 17 \cdot 331 \cdot 6563 \end{aligned}$$

and the agreement furnished presumptive evidence of the accuracy of the previous computations (themselves similarly checked) for the above eight primes 11, 599, 743; 7, 19, 719; 17, 331, and also for the prime divisors of 3213, 6276, 3693, namely, 17, 523, 1231. In this way the whole work is bound together by an intimate interlacing of tests. The search for the appropriate formulæ was greatly facilitated by Burckhardt's admirable "Table des Diviseurs," but the recent extensions of that table by Dase and by Glaisher would have been most welcome.[69]

By the combination of these primes and by interpolation to second differences, the logarithms, to 15 places, of all numbers from 100 000 to 370 000 have been computed. The actual calculations are contained in the twenty-seven volumes herewith presented, and the transfers in nine.

These logarithms are necessarily liable to residual errors, whose amount, however, cannot exceed three units in the fifteenth place. Among a large number of verifications, made for other purposes, no error exceeding two units has been found.

They are accompanied by the first and second differences — differences of the third order would only appear in the sixteenth place even at the beginning of the canon.

By help of these differences we can interpolate the logarithm of a number of more than six effective figures; the work consisting of three multiplications. For the converse operation, that of computing the number corresponding to a logarithm not found in the table, we need to resolve an equation of the second degree. Now the first differences have *ten*, the second difference have *five* effective figures, and therefore, when the utmost precision is required, either of these interpolations is necessarily laborious.

For the purposes of shortening the work, and of avoiding the solution of the quadratic equation, the expedient used by Nepair in the computation of his original *canon miri ficus*,[70] is had recourse in a form modified to suit the present circumstances.

[69] See [5, 2, 3, 4, 12, 13, 14, 17, 18, 19].
[70] See [25, 27, 37].

To get the logarithm of a number not in the table, it is enough to discover that of the ratio which it bears to the tabulated number immediately below it. Now this ratio itself is easily found by division, and, in our present case, is expressed by unit followed at an interval of at least five blanks, by other figures; its greatest possible value is 1,00001. In the volume marked *Auxiliary table* the logarithms of such ratios are given for each of the ten thousand numbers from 1 000 000 000 to 1 000 010 000. This list serves the purposes of both of Nepair's *Tabulæ prima et secunda*, and gives us, by help of this easy division, the fifteen-place logarithm of any number whatever. Not only so; it also enables us to solve the converse problem, by help of a multiplication as easy.

But we may approach to the required result, from the tabulated numbers immediately above. So, in order to supply the means of verification, the auxiliary table is carried, on the other side, to ten thousand numbers below the same 1 000 000 000.

This addition to the canon, besides greatly lessening the labour in interpolating, lends itself readily to systematic computation.

The fundamental canon for trigonometry is that of sines: these to 25 places for each two thousandth part of the quadrant, and to 15 places for each ten thousandth part, have been computed strictly by second differences, verified at short intervals. In the volumes placed before the Society the actual calculations are contained: they are recorded in such a form that each sine may be instantly examined. The manner of the calculation afforded a continuous and complete check, and the table is believed not to contain a single error. From these, the canon of logarithmic sines and the other usual trigonometrical tables may easily be compiled to an exactitude far beyond the requirements of practice.

In a paper entitled "Nouveau Calcul des Mouvements elliptiques," printed in the *Memoirs of the Turin Academy for 1879*,[71] the mean anomaly of a planet is deduced from its position by taking the sum or the difference of two circular segments. In order to reap the advantage of this exceedingly simple solution of Kepler's problem, we need first to compute the sines, measured, not in parts of the radius, but in parts of the circumference. The volume marked "Sines measured in Degrees" contains the whole calculation of this canon for each centesimal minute, and to ten decimal places of the quadrant.

From this table, that of circular segments, measured in degrees of the surface of the circle, for each of the 40 000 minutes of the entire circumference, has been composed. This table, though designed expressly for astronomical purposes, has its uses in other branches of science.

When the position of a planet in its orbit is given, the mean anomaly is obtained directly and almost by inspection; but when the mean anomaly is proposed, the position has to be got by the inverse use of the tables, that is by approximation. When the first estimate is reasonably near, the work

[71] See [77].

is scarcely more laborious than an ordinary interpolation; and when, as in preparing the *equations of the centre*, the computations are to be made at stated intervals, the labour is insignificant.

For the purpose of guiding the first estimate in sporadic cases, the mean anomalies corresponding to each degree of position, and in orbits of every degree of eccentricity, are given in the volume A, titled Mean Anomalies, the results being given to ten decimal places, and in volume B to eight places, with differences and variations.

In Kepler's time the details of only six elliptic orbits needed to be worked out; now we have forty times as many. The motions of the cloud of specks, so small as to be seen only by help of the telescope, afford an opportunity of verifying and correcting our estimates of the relative masses of the major planets, so much the more valuable that the disturbances exerted by these miniature worlds upon their giant neighbours escape our power of detection. This new mode of calculation vastly reduces the labour of comparing the purely elliptic with the observed motions.

For all analytical investigations the arc, as well as its sine, cosine, and tangent, is reckoned in parts of the radius—an arrangement also suited for several other applications of trigonometry. From this point of view, the sine and cosine take their place among functions with recurring derivatives; they are most easily and rapidly computed in this connection, without reference to the properties of the circle, being regarded as functions equal to their second derivatives with the signs changed. The volume titled "Recurring Functions"[72] contains their values to twelve decimal places, for each thousandth part of the radius, up to two radii.

For facilitating the change from the one unit to the other in the measurement of arcs, a table is here presented of the "Lengths of Circular Arcs" both for the ancient and for the modern graduation. The contrast in the arrangement of the two parts of this table affords an excellent example of the power and conciseness of the decimal system.

The lengths are given for each second of the ancient division up to one degree, in all 3600 values; thereafter for each degree up to 1800°, or five complete revolutions; the values for fractions of a second being got by the transposition of the numbers at the beginning of the table, this facility being due to the adoption of the decimal division for seconds.

For the modern division 1000 terms suffice, because by mere transposition the table may be extended indefinitely both ways.

In order to pass from the one system of subdivision of the quadrant to the other, a table of equivalent modern and ancien degrees is given, first from 10^c to 10^c or 9° to 9°; then for centesimal minutes up to 10° (computed for the sake of verification); next for each tenth decimal second up to the same limit;

[72] This is a reference an unpublished document, see National Library of Scotland, Edinburgh, Acc.10780/59 (not seen).

and, lastly, for each hundredth part of a second up to ten seconds. By this table the conversion of ancient into modern or of modern into ancient degrees is easily effected.

Similar tables for the conversion of ancient and decimal time are exhibited.

In the reduction of astronomical observations we have very often to exchange solar and sidereal time. In 1868 the writer published Time Conversion Tables for each tenth second of the whole day.[73] The counterpart to these is herewith presented; it is continued from day to day up to 1000 days; and this suffices for minutes, seconds, and fractions by simple transposition.

Lastly, there is appended a Traverse Table, for plain and mean latitude sailing, for each of the 400 degrees of azimuth and for distances up to 100.

These form at least a beginning in the collection of requisite decimal tables. That which is first wanted beyond them is the canon of logarithmic sines. The preparation of this canon would be greatly facilitated by the extension of the fifteen-place logarithms up to the whole million—that is, for all six-figure numbers. Those of them already prepared need the aid of the auxiliary multipliers 2 and 3; had they been carried to the half million, the auxiliary 2 would have sufficed.

In conclusion, it may be remarked that five and seven place tables are exact enough for almost all business purposes; but that, in order to have these true to the last figure, the original calculations must be carried several steps beyond; also, that while it is easy to abridge the lengthy results, it is impossible to extend those which have proved too short; then the work must be re-done from the beginning. Hence the great advantage already experienced in this: that Brigg's computations to fourteen places served for the preparation of Vlacq's ten-place table, and that again for those in common use, of seven and of five places.

[73] See [69].

3.9 Notice of fundamental tables (1889)

Notice of fundamental tables in trigonometry and astronomy, arranged according to the decimal division of the quadrant.[74]

Canon of Sines.

In January of 1878, there was laid on the Society's table the Canon of Sines to each fifth minute of the decimal division of the quadrant, computed to thirty-three for thirty places; along with a detailed record of every step in the process. During the years 1880-81, this work was continued for each single minute, but only to eighteen for fifteen places, and the record thereof to fifteen places is now submitted. When the rejected figures were from 497 to 503 a mark of interrogation is recorded, and it is believed that not a single error exists in the work. The arrangement of the sines with their first and second differences in position enables us instantly to detect an error.

That fifteen places suffice for all practical purposes, is made clear by this consideration, that the Earth's distance from the Sun, measured in inches, is represented by the number 6, twelve removes from the unit's place, that is 6 000 000 000 000; and that, if we take this as the radius of our circle, the figure 1 in the fifteenth place will represent ·006 or the 170th part of an inch. Thus the present canon gives, on this circle, the co-ordinates of the ten thousand points in the quadrant, each true to within the three-hundredth part of an inch.

The process followed in this work differs, in one very important respect, from that used by previous computers. The sine of the smallest tabular arc has hitherto been found indirectly by help of repeated bisections; in the present work the quinquisection of the arcs has been accomplished directly by the solution of the appropriate equations of the fifth degree, according to the method described in my treatise "On the Solution of Equations of all Orders."[75] The ease and rapidity of this method are well shown by the recorded details of the work for the various equations, to thirty decimal places.

A table of one thousand multiples of $2 \operatorname{ver} .1'$ having been prepared, the rest of the work was carried on in the usual manner with verifications at frequent intervals.

[74]Published in 1889 [82].
[75]See [67].

Logarithmic Sines.

The logarithmic sines were deduced from the sines themselves by help of my fifteen-place table of logarithms of numbers from 100 000 to 370 000, using the auxiliary table. Beginning at $100°00'$, the computations were made on scroll paper for each single minute down to $50°00'$, each step being verified by first, second, and third differences. The third differences were then copied into their places on the actual manuscript, and the others were thence reconstructed. In this way all errors of transcription were avoided, and any mistake in the previous work detected.

From this half of the canon, the other half, namely from $50°00'$ to $00°00'$, was deduced according to the formula $\sin a = \frac{1}{2}\sin 2a.\sec a$; the method of proceeding being to compute each tenth log sine directly, and to fill in the single terms by differences easily got from the differences already written in the first part. This operation supplied a check on all the previous work.

Logarithmic Tangents.

The logarithmic tangents were computed in the same way, that is to say, each tenth term was found directly, and the single terms by means of the preceding differences, thus furnishing another verification of the whole. But, seing that the log tangents of the one half are the arithmetical complements of those of the other half, it was enough to write out the log tangents of arcs from $50°00'$ to $00°00'$.

With the exception of the arrangement for computing by differences, and for assuring exactitude, this is the very process used by Nepair in the construction of his Canon Mirificus;[76] and, indeed, this volume of Logarithmic Sines and Tangents might, will all propriety, have been entitled:—

"John Nepair's wonder-working Canon, changed by his express desire, to suit the Denary System of Arithmetical Notation."

After a labour which must have occupied his leisure time for more than the quarter of a century, Nepair had published his Canon, had experienced its utility, had received the approval of the scientific world; and yet, foreseeing the advantage of accomodating his plan to the notation in common use, he at once recommended the putting of it aside for a far better plan. No stronger evidence can be adduced in favour of discarding the time-honoured division by 90 and by 60, and substituting the decimal division throughout.

Kepler's Problem.

While the Canon of Sines was still in progress, circumstances led to a repetition of the often fruitlessly made attack on Kepler's famous problem, and this

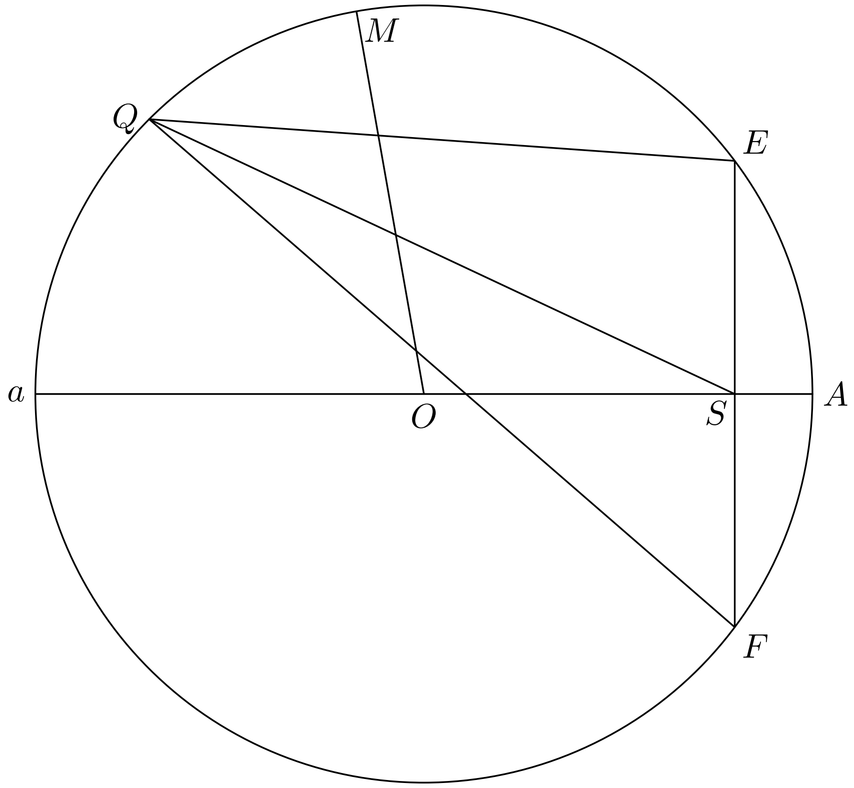
[76]See [25, 27, 37].

time an unexpectedly simple solution presented itself. The Royal Academy of Sciences of Turin did me the very great honour of giving that solution a place among their memoirs. The subject, however, may be treated more generally and even more simply, thus:—

Let us suppose ourselves to be studying the apparent relative motion of a binary system of stars; each one seems to describe round the other an ellipse, and the areas passed over the radius vector are proportional to the elapsed times. But, since the actual orbit may be inclined anyhow to the plane on which it appears to be projected, the one star does not appear to be in the focus of the orbit of the other; nor is the diameter drawn through its apparent place, necessarily be the major axis. If we divide the periodic time into equal portions, the corresponding vectors will similarly divide the area of the ellipse, and hence the problem virtually comes to be this, —"to subdivide the area of an ellipse by lines diverging from some point within it."

The line from the eye to the revolving star defines the surface of a cone, in our imaginary case sensibly of a cylinder and the planes passing though the eye, and along the vectors, subdivide this cylinder into wedges. If now this system be cut by any plane, the section so made will have its area also subdivided; now we can always cut a cylinder so that its section may be a circle, and this, ultimately, the problem becomes this, "to subdivide the area of a circle by lines diverging from a point within it."

If S represent the point given within the circle described round the centre O, the diameter $ASOa$ will represent the line of apsides, A being the perihelion, a the aphelion. Let now Q correspond to some position of the planet, then the surface comprised between AS, SQ, and the arc AQ, is proportional to the time elapsed from the planet, being in the position A, till its reaching the position corresponding to Q, so that this surface is the planet's mean anomaly.



Draw now ESF perpendicular to AO, then the arc AE, which has the excentricity OS for its cosine, may be called the arc of excentricity; we shall denote it by e; while AQ, the arc defining the planet's position,[77] may be denoted by p. Having joined EQ, FQ, it is seen, by mere inspection, that the mean anomaly[78] $AEQS$ is half the sum of the two circular segments cut off by the chords FQ, EQ, or that

$$\text{mean anomaly} = \frac{1}{2}\left\{\text{segm}(p+e) + \text{segm}(p-e)\right\},$$

and so a table of circular segments would enable us to determine the mean anomaly when the position is given, and conversely the position when the mean anomaly is given.[79]

In order to avoid the frequent multiplication and division by the number π, we measure the areas of the segments not in parts of the square of the radius, but in parts of the surface of the circle; a superficial degree being the sector standing on a degree of arc.

For the construction of such a table, it was necessary to compute the canon of sines measured in parts of the quadrant. The sines for the single degrees were therefore computed by simple multiplication of the ordinary sines, to serve as verifications of the subsequent work. Afterwards those for each quarter degree were obtained by using the previous multiples of $2\,\text{ver}\,.25'$ for second

[77] p is the eccentric anomaly.
[78] This is the surface of the sector bounded by SA and SQ.
[79] For details of this proof, see my reconstruction of Sang's table of circular segments [64].

differences; these two operations completely checked each other. Again the sines for each fifth minute were got by help of the multiples of $2\,\mathrm{ver}\,.5'$. But, as the computations were carried only to the tenth decimal of the quadrant, the products by $2\,\mathrm{ver}\,.1'$ were not needed.

Sines Measured in Degrees.

In this way the "Canon of Sines measured in degrees" now presented was completed,[80] the actual volume contains the whole details of the work, and it is hoped without any error exceeding two units in the tenth place.

Canon of Circular Segments.

Since the number which expresses the area of a segment in degrees of surface is the difference between those which express the arc and its sine, it follows that the second differences in the table of circular segments are identic with those of the sines; and therefore the canon of segments was constructed directly from those second differences. In the present volume[81] it is extended to the entire circumference, that is forty thousand minutes, and shows the value of each segment true to within two or three units in the ten-thousandth parts of the centesimal second. Its accuracy, thus, is very far beyond any requirement in actual astronomy. This work furnished another check on that for the canon of sines measured in degrees.

This table of circular segments enables us very easily to discover the mean anomaly when the angle of position is known. The converse problem, "to find the angle of position from the mean anomaly," has to be solved by approximation; which is sufficiently rapid if the first assumption be not very wide of the mark. When, for the orbit of some particular planet, we are computing the positions at equal intervals of time, attention to the differences reduces the labour to little more than that of writing out the results. It is only when a sporadic case is presented that the approximation is attended with any difficulty.

Mean Anomalies.

In order to obviate even this difficulty, a table has been constructed of the mean anomalies for orbits of each degree of excentricity, and for every degree of the angle of position, up to 200° in each of these orbits. This table enables us, in every possible case, to get at once a first assumption so near as to make the subsequent approximation quite easily.

[80]See my reconstruction of this table [63].
[81]See my reconstruction [64].

This table is presented in two forms. In the volume marked mean anomalies A,[82] the values are given to four decimal places of a second. In the corresponding volume marked B,[83] they are written only to the nearest second; but the differences and the variation from one orbit to another are inserted. Hence, by the ordinary method of interpolation for two variables, we can solve both the direct and the inverse problem with precision sufficient for all the purposes of practical astronomy.

My intention was to have computed also the radii vectors and the true anomalies. For this, however, the only available trigonometrical tables were those to seven places printed in a most inconvenient form, by Callet, in his Tables Portatives.[84] The work was scarcely begun when it became apparent that the precision attainable was not commensurate with the labour. Therefore, putting that work aside, I preferred to undertake the hopeless-looking task of computing the logarithmic sines and tangents to a greater number of places. This work is fortunately accomplished, although there still remain the transcription in the order usually adopted for convenient reference.

The application of these tables to the computation of the true anomalies, is a task far too great to be undertaken at the close of a long life, and, not without reluctance, it is left to the zeal of other computers. Enough, that I have been enabled to place within the reach of mathematicians some contributions to the progress of exact science.

The convenience of having the true anomaly, and the planet's distance alongside of the time-argument, would be so great as to dwarf altogether that of having merely the angle of position; this last mentioned forms, indeed, only a step toward the obtaining of the others. The remaining operation, implying only the solution of a right-angled triangle, is easy though laborious. It may, therefore, be not inopportune to indicate here the course most convenient to be followed in the subsequent work; particularly because that which may appear to be the most rapid in an isolated case, may not be the best for systematic work.

Distances.

If P be the planet's place in the ellipse $AsA's'$, having S and S' for its foci, and ss' for its minor axis, and if the ordinate HP be continued to meet the circle described on AA' as a diameter in Q, the arc AQ is p, the arc of position,[85] and we have[86]

$$SH = \cos p - \cos e \; ; \; PH = \sin p . \sin e .$$

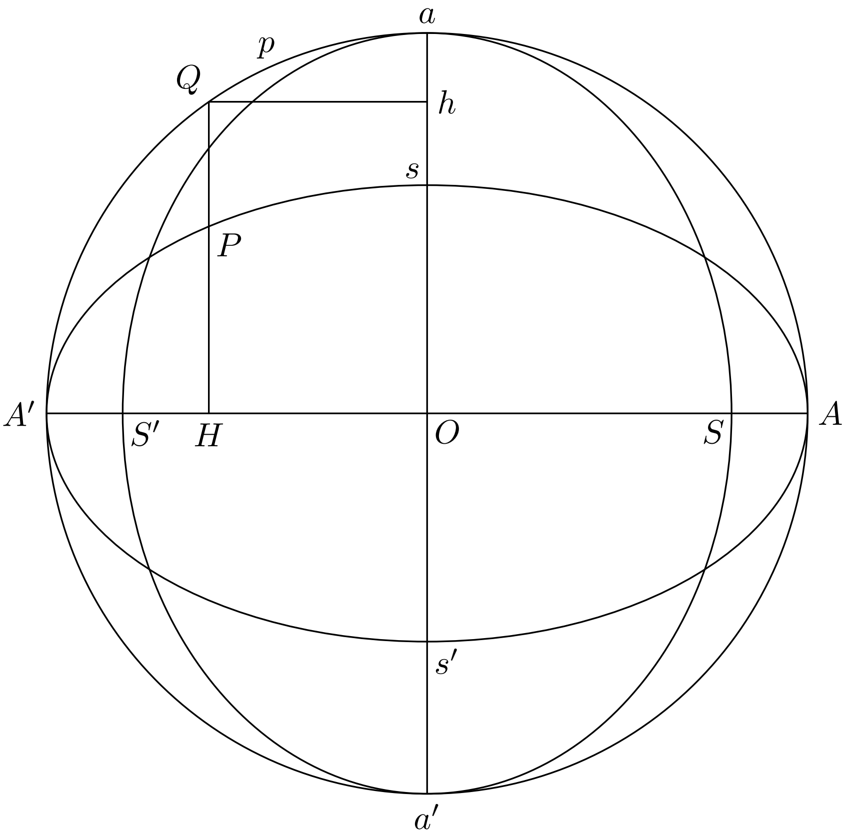
[82] See my reconstruction [65].

[83] See my reconstruction [66].

[84] See [7].

[85] This is also the eccentric anomaly.

[86] This is slightly incorrect, and we have $SH = \cos e - \cos p$, e being such that $OS = \cos e$. The radius of the circle is assumed to be 1.



From those we easily get the tangent of the true anomaly ASP, and thence the distance SP. Here the great part of the labour is in finding the logarithm of SH, the angle HSP from its log tangent, the log secant from the angle, and SP from its logarithm; that is to say, in using the tables of the logarithms of numbers, and of circular functions to a considerable number of decimal places. This labour, repeated for each of the twenty thousand cases to be tabulated, rises to a formidable total.

But if, on the perpendicular diameter AOA', we describe another ellipse having SS' for its minor axis, and consequently s and s' for its foci, and if from Q we draw the ordinate Qph, we have, according to the properties of the ellipse, $SP = OA \mp ph$, and conversely $SP = OA \doteq PH$. Thus the computation of the ordinates in the one of the two orbits gives us, with only the labour of writing the numbers in their places, the vectors of the other orbit, and we are now enabled to compute the true anomaly from its log sine. When following this course, it will be convenient to begin with the orbit $e = 50^\circ$, and to take the others in couples, $e = 49^\circ$, $e = 51^\circ$, and so on.

Our working formula then stand thus:—

$$\text{log ordinates} \quad \left\{ \begin{array}{l} \log\sin p + \log\sin e \\ \log\cos p + \log\cos e \end{array} \right\}, \text{ whence ordinates}$$

$$\text{distances} \quad \left\{ \begin{array}{l} 1 \mp \cos p.\cos e \\ 1 \mp \sin p.\sin e \end{array} \right\} \text{whence, log distances,}$$

If it were proposed to make these computations with all the precision obtainable from our fifteen-place tables, it might be economical, even for this single piece of work, to interpolate the logarithmic sines for each hundred-thousandth part of the quadrant.

4 References

The following list covers the most important references[87] related to Sang's table. Not all items of this list are mentioned in the text, and the sources which have not been seen are marked so. I have added notes about the contents of the articles in certain cases.

[1] Anonymous. Dr Edward Sang's logarithmic, trigonometrical, and astronomical tables. *Proceedings of the Royal Society of Edinburgh*, 28:183–196, 1908. [Possibly by Cargill Gilston Knott, reprinted in [21]. Reprints [83].].

[2] Johann Karl Burckhardt. *Table des diviseurs pour tous les nombres du deuxième million, etc.* Paris: Vve Courcier, 1814. [reconstructed in [41]].

[3] Johann Karl Burckhardt. *Table des diviseurs pour tous les nombres du troisième million, etc.* Paris: Vve Courcier, 1816. [reconstructed in [42]].

[4] Johann Karl Burckhardt. *Table des diviseurs pour tous les nombres des 1er, 2e et 3e million, etc.* Paris: Vve Courcier, 1817.

[5] Johann Karl Burckhardt. *Table des diviseurs pour tous les nombres du premier million, etc.* Paris: Vve Courcier, 1817. [reconstructed in [40]].

[6] Jean-François Callet. *Tables portatives de logarithmes, publiées à Londres par Gardiner, augmentées, et perfectionnées dans leur disposition par M. Callet, etc.* Paris: Firmin Ambroise Didot, 1783. [This edition does not yet contain tables for the decimal division.].

[7] Jean-François Callet. *Tables portatives de logarithmes, etc.* Paris: Firmin Didot, 1795. [There have been numerous later printings of these tables.].

[8] Alexander Duncan Davidson Craik. Edward Sang (1805–1890): calculator extraordinary. *Newsletter of the British Society for the History of Mathematics*, 45:32–43, Spring 2002.

[9] Alexander Duncan Davidson Craik. The logarithmic tables of Edward Sang and his daughters. *Historia Mathematica*, 30(1):47–84, February 2003.

[10] Alexander Duncan Davidson Craik. Sang, Knott and Spence on logarithmic and other tables, 2016. [article written for a joint meeting of the James Clerk Maxwell Society and the British Society for the History of Mathematics in celebration of the 400th anniversary of the publication of John Napier's Mirifici Logarithmorum Canonis Descriptio, 4th April 2014, Clerk Maxwell House, Edinburgh, `https://www.collectanea.eu/napier400memorial`].

[87] **Note on the titles of the works:** Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncapitalized. The names of the authors have also been homogenized and initials expanded, as much as possible.

The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but I have not done it here.

[11] August Leopold Crelle. *Rechentafeln, welche alles Multipliciren und Dividiren mit Zahlen unter Tausend ganz ersparen, bei größeren Zahlen aber die Rechnung erleichtern und sicherer machen.* Berlin: Maurer, 1820.

[12] Johann Martin Zacharias Dase. *Factoren-tafeln für alle Zahlen der siebenten Million etc.* Hamburg: Perthes-Besser & Mauke, 1862. [reconstructed in [45]].

[13] Johann Martin Zacharias Dase. *Factoren-tafeln für alle Zahlen der achten Million etc.* Hamburg: Perthes-Besser & Mauke, 1863. [reconstructed in [43]].

[14] Johann Martin Zacharias Dase and H. Rosenberg. *Factoren-tafeln für alle Zahlen der neunten Million etc.* Hamburg: Perthes-Besser & Mauke, 1865. [reconstructed in [44]].

[15] Jean-Charles de Borda and Jean-Baptiste Joseph Delambre. *Tables trigonométriques décimales : ou Table des logarithmes des sinus, sécantes et tangentes, suivant la division du quart de cercle en 100 degrés, du degré en 100 minutes, et de la minute en 100 secondes précédées de la table des logarithmes des nombres depuis dix mille jusqu'à cent mille, et de plusieurs tables subsidiaires.* Paris: Imprimerie de la République, 1801.

[16] Pierre-Simon de Laplace. *Traité de mécanique céleste*, volume 1. Paris: imprimerie de Crapelet, 1799.

[17] James Glaisher. *Factor table for the fourth million etc.* London: Taylor and Francis, 1879. [reconstructed in [47]].

[18] James Glaisher. *Factor table for the fifth million etc.* London: Taylor and Francis, 1880. [reconstructed in [46]].

[19] James Glaisher. *Factor table for the sixth million etc.* London: Taylor and Francis, 1883. [reconstructed in [48]].

[20] G. Govi. Rapport sur l'utilité des tables de logarithmes à plus de sept décimales ; à propos d'un projet publié par M. Sang. *Atti della Reale Accademia di Scienze di Torino*, 8:157–170, 1873. [Reprinted in [71].].

[21] Ellice Martin Horsburgh, editor. *Modern instruments and methods of calculation: a handbook of the Napier tercentenary exhibition.* London: G. Bell and sons, 1914.

[22] Cargill Gilston Knott. Edward Sang and his logarithmic calculations. [23], pages 261–268.

[23] Cargill Gilston Knott, editor. *Napier Tercentenary Memorial Volume.* London: Longmans, Green and company, 1915.

[24] Adrien Marie Legendre. *Traité des fonctions elliptiques et des intégrales eulériennes.* Paris: Huzard-Courcier, 1826. [volume 2].

[25] John Napier. *Mirifici logarithmorum canonis descriptio.* Edinburgh: Andrew Hart, 1614.

[26] John Napier. *A description of the admirable table of logarithmes*. London, 1616. [English translation of [25] by Edward Wright, reprinted in 1969 by Da Capo Press, New York. A second edition appeared in 1618.].

[27] John Napier. *Mirifici logarithmorum canonis constructio*. Edinburgh: Andrew Hart, 1619. [Reprinted in [28] and translated in [29]. A modern English translation by Ian Bruce is available on the web.].

[28] John Napier. *"Logarithmorum canonis descriptio" and "Mirifici logarithmorum canonis constructio"*. Lyon: Barthélemi Vincent, 1620. [Reprint of Napier's *descriptio* and *constructio*. At least the *constructio* was reprinted by A. Hermann in 1895.].

[29] John Napier. *The construction of the wonderful canon of logarithms*. Edinburgh: William Blackwood and sons, 1889. [Translation of [27] by William Rae Macdonald.].

[30] John Newton. *Trigonometria Britanica: or, doctrine of triangles*. London: R.& W. Leybourn, 1658.

[31] David Bruce Peebles. Edward Sang. *Proceedings of the Royal Society of Edinburgh*, 21:xvii–xxxii, 1897.

[32] Barbara J. Rhodes and William Wells Streeter. *Before photocopying: The art & history of mechanical copying, 1780-1938*. New Castle, Delaware: Oak Knoll Press, 1999.

[33] Denis Roegel. A construction of Edward Sang's projected table of nine-place logarithms to one million (1872). Technical report, LORIA, Nancy, 2010. [This construction is based on the specimen pages [71].].

[34] Denis Roegel. A reconstruction of Adriaan Vlacq's tables in the *Trigonometria artificialis* (1633). Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [87].].

[35] Denis Roegel. A reconstruction of De Decker-Vlacq's tables in the *Arithmetica logarithmica* (1628). Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [86].].

[36] Denis Roegel. A reconstruction of Edward Sang's table of logarithms (1871). Technical report, LORIA, Nancy, 2010. [This is a reconstruction of [70].].

[37] Denis Roegel. Napier's ideal construction of the logarithms. Technical report, LORIA, Nancy, 2010.

[38] Denis Roegel. Tables for the mutual conversion of solar and sidereal time, by Edward Sang. Technical report, LORIA, Nancy, 2010. [This is a reconstruction of [69].].

[39] Denis Roegel. The great logarithmic and trigonometric tables of the French Cadastre: a preliminary investigation. Technical report, LORIA, Nancy, 2010.

[40] Denis Roegel. A reconstruction of Burckhardt's table of factors (first million, 1817). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [5].].

[41] Denis Roegel. A reconstruction of Burckhardt's table of factors (second million, 1814). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [2].].

[42] Denis Roegel. A reconstruction of Burckhardt's table of factors (third million, 1816). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [3].].

[43] Denis Roegel. A reconstruction of Dase's table of factors (eighth million, 1863). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [13].].

[44] Denis Roegel. A reconstruction of Dase's table of factors (ninth million, 1865). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [14].].

[45] Denis Roegel. A reconstruction of Dase's table of factors (seventh million, 1862). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [12].].

[46] Denis Roegel. A reconstruction of Glaisher's table of factors (fifth million, 1880). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [18].].

[47] Denis Roegel. A reconstruction of Glaisher's table of factors (fourth million, 1879). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [17].].

[48] Denis Roegel. A reconstruction of Glaisher's table of factors (sixth million, 1883). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [19].].

[49] Denis Roegel. Mouton's table of logarithms and its extensions (ca. 1670). Technical report, LORIA, Nancy, 2011.

[50] Denis Roegel. A reconstruction of Shortrede's traverse tables (1864). Technical report, LORIA, Nancy, 2013. [This is a reconstruction of [85].].

[51] Denis Roegel. A guide to Edward Sang's tables and to their reconstructions. Technical report, LORIA, Nancy, 2020.

[52] Denis Roegel. Edward Sang's computation of the logarithms of integers. Technical report, LORIA, Nancy, 2020.

[53] Denis Roegel. Edward Sang's steps for the construction of the logarithms of the primes (K1-K3). Technical report, LORIA, Nancy, 2020.

[54] Denis Roegel. A reconstruction of Edward Sang's table of logarithms of the first 10000 primes (K4). Technical report, LORIA, Nancy, 2020.

[55] Denis Roegel. A reconstruction of Edward Sang's table of logarithms of the first myriad of integers (K5). Technical report, LORIA, Nancy, 2020.

[56] Denis Roegel. A reconstruction of Edward Sang's table of logarithms of the second myriad of integers (K6). Technical report, LORIA, Nancy, 2020.

[57] Denis Roegel. Introduction to Edward Sang's table of logarithms to 15 places. Technical report, LORIA, Nancy, 2020. [This document is supplemented by 90 volumes of tables, as well as a volume gathering the entire table.].

[58] Denis Roegel. A reconstruction of Edward Sang's auxiliary table for logarithms of almost unitary values (K39,1884). Technical report, LORIA, Nancy, 2020.

[59] Denis Roegel. A reconstruction of Edward Sang's canon of sines (K40/1,1876). Technical report, LORIA, Nancy, 2020.

[60] Denis Roegel. A reconstruction of Edward Sang's canon of sines (K40/2,1877). Technical report, LORIA, Nancy, 2020.

[61] Denis Roegel. A reconstruction of Edward Sang's canon of sines (K41-K42,1881). Technical report, LORIA, Nancy, 2020.

[62] Denis Roegel. A reconstruction of Edward Sang's table of logarithmic sines and tangents (K43,1888). Technical report, LORIA, Nancy, 2020.

[63] Denis Roegel. A reconstruction of Edward Sang's table of sines in degrees (K44,1879). Technical report, LORIA, Nancy, 2020.

[64] Denis Roegel. A reconstruction of Edward Sang's table of circular segments (K45,1879). Technical report, LORIA, Nancy, 2020.

[65] Denis Roegel. A reconstruction of Edward Sang's table of mean anomalies: volume A (K46,1880). Technical report, LORIA, Nancy, 2020.

[66] Denis Roegel. A reconstruction of Edward Sang's table of mean anomalies: volume B (K47,1880). Technical report, LORIA, Nancy, 2020.

[67] Edward Sang. *Solution of algebraic equations of all orders, whether involving one or more unknown quantities*. Edinburgh: William Tait, 1829.

[68] Edward Sang. *The higher arithmetic*. Edinburgh: William Blackwood and sons, 1857.

[69] Edward Sang. *Tables for the mutual conversion of solar and sidereal time*. Edinburgh: William Blackwood & sons, 1868. [Reconstruction by D. Roegel, 2010 [38].].

[70] Edward Sang. *A new table of seven-place logarithms of all numbers from 20 000 to 200 000.* London: Charles and Edwin Layton, 1871. [Reconstruction by D. Roegel, 2010 [36].].

[71] Edward Sang. Specimen pages of a table of the logarithms of all numbers up to one million...: shortened to nine figures from original calculations to fifteen places of decimals, 1872. [These specimen pages were reprinted in 1874 in a booklet which contained also a reprint of Govi's report [20], a reprint of Sang's article on Vlacq's errors [72], and several other letters by eminent scientists supporting the publication of Sang's table. The specimen pages were used to construct [33].].

[72] Edward Sang. On last-place errors in Vlacq's table of logarithms. *Proceedings of the Royal Society of Edinburgh*, 8:371–376, 1875. [First printed in the 1874 edition of [71].].

[73] Edward Sang. On a new investigation of the series for the sine and cosine of an arc. *Proceedings of the Royal Society of Edinburgh*, 9:400–402, 1878.

[74] Edward Sang. On the construction of the canon of sines, for the decimal division of the quadrant. *Proceedings of the Royal Society of Edinburgh*, 9:343–349, 1878.

[75] Edward Sang. On the precautions to be taken in recording and using the records of original computations. *Proceedings of the Royal Society of Edinburgh*, 9:349–352, 1878.

[76] Edward Sang. Description of new astronomical tables for the computation of anomalies. *Proceedings of the Royal Society of Edinburgh*, 10(107):726–727, 1880.

[77] Edward Sang. Nouveau calcul des mouvements elliptiques. *Memorie della Reale accademia delle scienze di Torino*, 32 (2nd series):187–199, 1880.

[78] Edward Sang. Addition au mémoire sur le calcul des mouvements elliptiques. *Memorie della Reale accademia delle scienze di Torino*, 32 (2nd series):305–307, 1880.

[79] Edward Sang. Notice du canon des sinus à quinze places, pour chaque minute. Contribution au système décimal., 1882. [National Library of Scotland, Acc10780/81, manuscript, 19 pages].

[80] Edward Sang. On the construction of the canon of logarithmic sines. *Proceedings of the Royal Society of Edinburgh*, 12:601–619, 1884.

[81] Edward Sang. On the need for decimal subdivisions in astronomy and navigation, and on tables requisite therefor. *Proceedings of the Royal Society of Edinburgh*, 12:533–544, 1884.

[82] Edward Sang. Notice of fundamental tables in trigonometry and astronomy, arranged according to the decimal division of the quadrant. *Proceedings of the Royal Society of Edinburgh*, 16:249–256, 1889.

[83] Edward Sang. List of trigonometrical and astronomical calculations, in manuscript, 1890. [Dated July 1890. National Library of Scotland: Acc10780/10. Reprinted in [1].].

[84] Henry Sherwin. *Mathematical tables.* London: Richard Mount and Thomas Page, 1706. [contains a table of logarithms by Abraham Sharp].

[85] Robert Shortrede. *Traverse tables to five places for every* $2'$ *of angle up to* 100 *of distance.* Edinburgh: William Blackwood and sons, 1864. [edited by Edward Sang, reconstruction by D. Roegel, 2013 [50]].

[86] Adriaan Vlacq. *Arithmetica logarithmica.* Gouda: Pieter Rammazeyn, 1628. [The introduction was reprinted in 1976 by Olms and the tables were reconstructed by D. Roegel in 2010. [35]].

[87] Adriaan Vlacq. *Trigonometria artificialis.* Gouda: Pieter Rammazeyn, 1633. [The tables were reconstructed by D. Roegel in 2010. [34]].