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# Lebesgue integration. Detailed proofs to be formalized in Coq

François Clément, Vincent Martin

**RESEARCH  
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January 2021

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# Lebesgue integration. Detailed proofs to be formalized in Coq

François Clément\*, Vincent Martin†

Project-Team Serena

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**Abstract:** To obtain the highest confidence on the correction of numerical simulation programs implementing the finite element method, one has to formalize the mathematical notions and results that allow to establish the soundness of the method. Sobolev spaces are the mathematical framework in which most weak formulations of partial derivative equations are stated, and where solutions are sought. These functional spaces are built on integration and measure theory. Hence, this chapter in functional analysis is a mandatory theoretical cornerstone for the definition of the finite element method. The purpose of this document is to provide the formal proof community with very detailed pen-and-paper proofs of the main results from integration and measure theory.

**Key-words:** measure theory, Lebesgue integration, detailed mathematical proof, formal proof in real analysis

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\* Inria & CERMICS, École des Ponts, 77455 Marne-la-Vallée Cedex 2, France. [Francois.Clement@inria.fr](mailto:Francois.Clement@inria.fr).

† Université de technologie de Compiègne, 60203 Compiègne, France. [Vincent.Martin@utc.fr](mailto:Vincent.Martin@utc.fr).

‡ Version 3 is a major revision of the document (see Foreword).

**RESEARCH CENTRE  
PARIS**

2 rue Simone Iff - CS 42112  
75589 Paris Cedex 12



# Intégrale de Lebesgue.

## Preuves détaillées en vue d'une formalisation en Coq

**Résumé :** Pour obtenir la plus grande confiance en la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il faut formaliser les notions et résultats mathématiques qui permettent d'établir la justesse de la méthode. Les espaces de Sobolev sont le cadre mathématique dans lequel la plupart des formulations faibles pour résoudre les équations aux dérivées partielles sont posées, et où les solutions sont recherchées. La construction de ces espaces fonctionnels repose sur le calcul intégral et la théorie de la mesure. Ce chapitre de l'analyse fonctionnelle est donc un fondement théorique nécessaire à la définition de la méthode des éléments finis. L'objectif de ce document est de fournir à la communauté des chercheurs en preuve formelle des preuves papiers très détaillées des principaux résultats du calcul intégral et de la théorie de la mesure.

**Mots-clés :** théorie de la mesure, intégrale de Lebesgue, preuve mathématique détaillée, preuve formelle en analyse réelle

# Foreword

This document is intended to evolve over time. Last version is release 2.0 (i.e. version 3). It is available at <https://hal.inria.fr/hal-03105815/>.

**Version 3** (release 2.0, 2022/12/16) is a major revision.

Main changes are:

- new Sections 3.1.4 on Bochner integral, and 4.5 on Bochner scheme;
- new Section 4.1 on subset systems;
- new Sections 5.1, 5.2 and 5.6 on the sketches of proofs of the Riesz–Fischer theorem, the Lebesgue’s extended dominated convergence theorem in  $\mathcal{L}^p$ , and the existence and uniqueness of the tensor product measure;
- new Figures in Sections 1.2 (finite element method), 4.1 (subset systems), 4.4 (Lebesgue scheme), 5.5 (Tonelli), and 5.6 (product measure);
- new Lemmas 212, 213, 217, 219, and 221, and generalized Lemma 215 in Section 7.1 on (naive) set theory;
- new Lemmas 231, and 238 in Section 7.2 on order theory;
- new Lemmas 245, 247, 249, 250, and 259, and generalized Lemmas 248, 252, 265, and 269 in Section 7.3.1 on topological basis;  
new Lemmas 264, 268, and 266 in Section 7.3.3 on metric spaces;
- new Sections 7.4.3.1, and 7.4.3.2 in Section 7.4 on algebraic structures;
- new Lemmas 300, 307, 308, 310, and 311 in Section 7.5.1 on real numbers;  
Section 7.5.2 on extended real numbers has been seriously expanded: new Definition 374, new Lemmas 314, 321, 323, 324, 328, 333, 335, 338, 343, 344, 345, 348, 350, 351, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 300, 373, 375, 377, 378, 379, 381, 382, 383, 385, 386, 390, 391, 392, 396, 399, 400, 335, 412, 413, 414, 420, 422, 429, 431, 432, 433, 434, 435, 436, 437, 438, 439, 441, 442, 443, 444, 445, 446, 448, 449, 455, and generalized Lemmas 380, 398, 417, 418, 419;  
new Lemma 457, and generalized Lemma 466 in Section 7.5.3 on the second countability of real numbers;  
new Section 7.5.4 on the second countability of extended real numbers (the proof of Lemma 479 is now completed);  
new Lemmas 483, 498, 499, 500, 501, 502, and 503, and generalized Lemmas 496, and 497 in Section 7.5.5 on infimum and supremum;  
new Lemmas 524, and 525 in Section 7.5.7 on masking numeric functions;
- Chapter 8 on generic subset systems is partly rewritten;  
new Remark 537 provides a mini-language to express properties of subset systems in a compact form;  
new Lemmas 544, 546, 550, 551, 552, 553, 554, 555, 556, 557, 561, 562, 563, 572, 574, 575, 576, and 577, and generalized Lemmas 569, 570, and 571, in Section 8.1 on basic properties of subset systems;  
new Section 8.2 implements the Remark 430 p. 75 of **Version 2**, and even goes a step further, this allows to factor many proofs in the remainder of the chapter (some statements are grouped together);  
new Lemmas 604, 607, and 609 in Section 8.3 on pi-systems;

- new Section 8.4 on rings of sets;
- new Lemmas 638, 640, 643, 645, 646, 647, 648, 649, 650, 651, new part on semi-algebras of sets (impacting the proof of Lemma 662), and generalized Lemmas 639, 652, and 662 in Section 8.5 on algebras of sets (previously called “set algebras”);
- new Lemmas 667, and 669 in Section 8.6 on monotone classes;
- new Lemmas 679, 683, 685, 686, 687, 688, and generalized Lemma 680 in Section 8.7 on lambda-systems;
- new Lemmas 703, 705, 706, 709, 722, 723, 724, and generalized Lemmas 696, 700, 714, 717, 720, 726, and 728, in Section 8.8 on sigma-algebras;
- new Definitions 739, and 748, and new Lemmas 740, and 742 in Section 9.1 on measurable spaces and Borel subsets;
  - new Lemma 760, and generalized Definition 758, and Lemmas 753, and 759 in Section 9.2 on measurable functions;
  - generalized Lemma 766 in Section 9.3 on measurable subspaces;
  - generalized Lemma 780 in Section 9.4 on products of measurable spaces;
  - new Lemmas 794, 800, and 806, and generalized Lemma 803 in Section 10.1 on Borel subsets of number spaces;
  - new Lemmas 808, 810, 837, 839, and 840, and generalized Definition 851, and Lemmas 809, 811, 813, 814, 841, and 853 in Section 10.2 on measurable numeric functions;
  - new Chapter 11 generalizes the old Section 11.1 on measures (Version 2, pp. 117–122), it is similar to Chapter 8 on subset systems, but for numeric functions over subset systems, such as measures; it includes an extension of the mini-language in Remark 854 providing compact forms of statements;
  - new Lemmas 915, 917, 918, 928, 929, and 932 in Section 12.1 on measure spaces;
  - new Lemmas 947, 955, 956, 957, 960, 963, 965, 966, 967, 970, 972, and 977, and generalized Lemma 962 in Section 12.3 on negligible subsets;
  - new Lemmas 993, 994, 995, 996, 1001, 1006, 1007, 1010, 1011, 1012, 1013, 1014, 1015, and 1016, and generalized Lemmas 998, and 1008 in Section 13.1 on negligibility in number spaces;
  - new Section 13.2 on essential suprema, including the specific case of nonnegative functions;
  - generalized Lemma 1081 in Section 13.3 on the construction of the Lebesgue measure;
  - new Chapter 14 on simple functions has been extracted from the old Chapter 13 on integration of nonnegative functions (Version 2, pp. 145–184);
  - generalized Section 14.1 on indicator functions from the only case of real numbers (Version 2, Section 13.1, pp. 145–146) to the case of measurable number spaces  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\overline{\mathbb{R}}$ , including new Lemma 1107;
  - generalized Section 14.2 on simple functions from the only case of real-values functions (Version 2, Section 13.2.1, pp. 149–153) to the case of vector-valued functions, including new Lemma 1121;
  - new Lemmas 1130, 1131, 1132, and 1134 in Section 14.3 on nonnegative simple functions;
  - new Lemmas 1137, 1138, 1139, and 1141 in Section 14.4 on adapted sequences;
  - new Lemmas 1147, and 1150, and generalized Lemmas 1146, and 1149 in Section 15.1 on the integration of indicator functions;
  - new Lemmas 1158, 1165, 1166, 1167, 1168, and 1169, generalized Lemma 1164, and modified Lemma 1163 in Section 15.2 on the integration of nonnegative simple functions;
  - new Lemmas 1192, 1193, 1194, 1201, 1203, 1204, 1205, 1206, and 1210, generalized Lemmas 1189, 1195, 1198, and modified Lemma 1175, and 1199 in Section 15.3 on the integration of nonnegative measurable functions;
  - new Definition 1216 in Section 15.4 on measure and integration over product spaces;
  - new Lemmas 1253, 1254, 1255, 1260, 1263, 1264, 1270, 1271, 1272, and 1273, and modified Lemmas 1267, and 1269 in Section 16.1 on the definition of the integral (for functions with possibly varying sign);
  - new Lemmas 1276, and 1280, and modified Lemmas 1274, and 1275 in Section 16.2 on notations for specific cases;

- generalized Section 16.3 on the seminormed vector space  $\mathcal{L}^p$  from the only case of  $p = 1$  (Version 2, Section 14.3, pp. 192–199) to the case of any  $p \in (0, \infty]$  (for the definition and basic properties of the seminorm  $N_p$  and  $\mathcal{L}^p$ ), and to the case of any  $p \in [1, \infty]$  for the Hölder and Minkowski inequalities and their variants, and for the resulting property of seminormed vector space of  $\mathcal{L}^p$ ;  
new Lemmas 1288, 1289, 1290, 1291, 1293, 1294, 1295, 1298, 1299 in Section 16.3.1 on the  $p$ -seminorm;  
new Section 16.3.2 on Hölder inequality;  
new Lemmas 1310, 1311, and 1312 in Section 16.3.3 on Minkowski inequality;  
new Lemmas 1317, 1318, 1320, 1321, 1324, 1326, 1334, 1335, 1340, and 1341, new Theorems 1343 (Beppo Levi monotone convergence in  $\mathcal{L}^1$ ), and 1345 (Fatou’s lemma in  $\mathcal{L}^1$ ), and generalized Lemmas 1322, 1329, and 1331 in Section 16.3.4 on the seminormed vector space  $\mathcal{L}^p$ ;  
new Theorems 1351, and 1352, and fixed Theorem 1349 in Section 16.3.5 on Lebesgue’s dominated convergence theorem;  
new Section 16.3.6 on the Riesz–Fischer theorem (in  $\mathcal{L}^p$ , thus without uniqueness);
- new Remarks 207, 208, 216, 218, 220, 294, 297, 237, 251, 253, 301, 309, 356, 384, 440, 447, 450, 467, 470, 472, 473, 481, 523, 537, 549, 560, 573, 636, 663, 673, 676, 767, 793, 795, 801, 805, 807, 812, 827, 846, 852, 950, 961, 964, 971, 973, 976, 1096, 1097, 1099, 1122, 1133, 1140, 1155, 1173, 1222, 1226, 1257, 1265, 1313, 1325, 1330, 1332, 1333, 1338, 1342, 1344, 1350, and modified Remarks 255, 263, 376, 536, 698, 781, 968, 1112, 1143, 1153, 1215, 1220, 1230, 1232, 1287;
- and possibly other slight modifications to improve the proof contents and their legibility.

Version 2 (release 1.1, 2021/04/01) is a minor revision.

Main changes are:

- addition of this foreword;
- reordering and sectioning:  
sections have become chapters and gathered into parts to ease the reading;  
all “complements” are gathered into a single chapter;  
Section 7.2 about algebraic structures is moved right after Section 7.1 about set theory;  
Chapter 8 on subset systems is separated from Chapter 9 on measurability;
- some color modifications to increase legibility after grayscale printing;
- new contents in the introduction (Chapter 1), and minor corrections in Chapters 2–5 (including a bunch of hyperlinks in Chapters 4 and 5);
- fix proofs of uniqueness in Lemma 668 (statement changed too) and Theorem 724;
- fix statement and proof of Theorem 846 and Lemma 847, with new Definition 844;
- fix statement of Theorem 817;
- alternate proof of additivity for the integral of nonnegative simple functions in Lemma 774 based on the new *disjoint* representation of simple functions of Lemma 754;  
this involves new Definition 207, new Lemmas 734, 735, 756, 757, 764, 766, 768, and 772 new Remarks 208, 750, 753, 755, 758, and 773, addition of uniqueness in Lemmas 752 and 765, thus impacting proofs of Lemmas 742, 744, 771, 779, 784, and 814;
- factor the proof on *layers* into new Lemmas 446, 480, and 771 thus impacting proofs of Lemmas 617, 715, and 719;
- new Lemma 644 and Remark 645 about reasoning with properties satisfied almost everywhere, now used in the proofs of Lemmas 651–654, 659, and 664 (statement changed too in the latter);
- new Lemmas 376, 377, 600, and 601 shortening proofs of Lemmas 827 and 828;
- additional Remarks 214, 430, 473, 507, 512, 689, 723, 730, 795, 816, 845, 850, 896, 898, and modification of Remarks 407, 436, 504, and 674;
- Lemmas 552 and 553 are moved from Section 13.4 to Section 9.4;
- and possibly other slight modifications to improve the proof contents and their legibility.

**Version 1** (release 1.0, 2021/01/14) is the first release.

It covers:

- the main basic results in measure theory, including the construction of Lebesgue measure in  $\mathbb{R}$  via Carathéodory's extension scheme;
- integration of nonnegative measurable functions, including the Beppo Levi (monotone convergence) theorem and Fatou's lemma;
- the integral over a product space, including the Tonelli theorem;
- integration of measurable functions with possibly changing sign, including the seminormed vector space  $\mathcal{L}^1$ , and Lebesgue's dominated convergence theorem.

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# Part I

# Overview



# Chapter 1

## Introduction

### 1.1 Formal proof

A formal proof is conducted in a logical framework that provides dedicated computer programs to mechanically check the validity of the proof, the so-called formal proof assistants. Such formal proofs may concern known mathematical theorems, but also properties of some piece of other computer programs, e.g. see [40], and [8, Glossary p. 343]. This field of computer science is extremely popular as it allows to certify with no doubt the behavior of critical programs.

Interactive theorem provers are now known to be able to tackle real analysis. For instance, in the field of Ordinary Differential Equations (ODEs) with Isabelle/HOL [43, 42, 44], and Coq [50], or in the field of Partial Differential Equations (PDEs), again with Isabelle/HOL [1], and Coq [7, 8]. In the latter example, the salient aspect is that the round-off error due to the use of IEEE-754 floating-point arithmetic can also be fully taken into account. But the price to pay is that all the details of the proofs has to be dealt with, and thus the availability of very detailed pen-and-paper proofs is a major asset.

### 1.2 Objective

Our current long-term purpose is to formally prove programs implementing the Finite Element Method (FEM), see Figure 1.1. The FEM is widely used to solve a broad class of PDEs, mainly because it has a sound mathematical foundation, e.g. see [61, 19, 53, 13, 27, 28, 29, 30]. The Lax–Milgram theorem, one of the key ingredients to establish the FEM, was already addressed in [20] for a detailed pen-and-paper proof, and in [9] for a formal proof in Coq. The present document is a further contribution to our ultimate goal.

The Lax–Milgram theorem claims existence and uniqueness of the solution to the weak formulation of a PDE problem, such as the Poisson problem, and its discrete approximation; it is stated on a Hilbert space, i.e. a complete inner product space. The simplest Hilbert functional spaces relevant for the resolution of PDEs are the spaces  $L^2$  and  $H^1$ . More generally, when stronger results such as the Banach–Nečas–Babuška theorem are involved, one may be interested in the Sobolev spaces  $W^{m,p}$  that are Banach spaces, i.e. complete normed vector spaces.

Let  $d \in \{1, 2, 3\}$  be the spatial dimension. Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . We consider real-valued functions defined almost everywhere over  $\Omega$ . For  $0 < p \leq \infty$ , the Lebesgue space  $L^p(\Omega)$  is the space of measurable functions for which the  $p$ -th power of the absolute value (or the essential supremum when  $p = \infty$ ) has a finite integral over  $\Omega$ . For any natural number  $m$ , the Sobolev space  $W^{m,p}(\Omega)$  is the vector subspace of  $L^p(\Omega)$  of functions for which all weak derivatives up to order  $m$  also belong to  $L^p(\Omega)$ . Thus, the Lebesgue spaces correspond to the case  $m = 0$ ,  $L^p(\Omega) = W^{0,p}(\Omega)$ . And the Hilbert spaces correspond to the case  $p = 2$ ,  $H^m(\Omega) \stackrel{\text{def.}}{=} W^{m,2}(\Omega)$ .

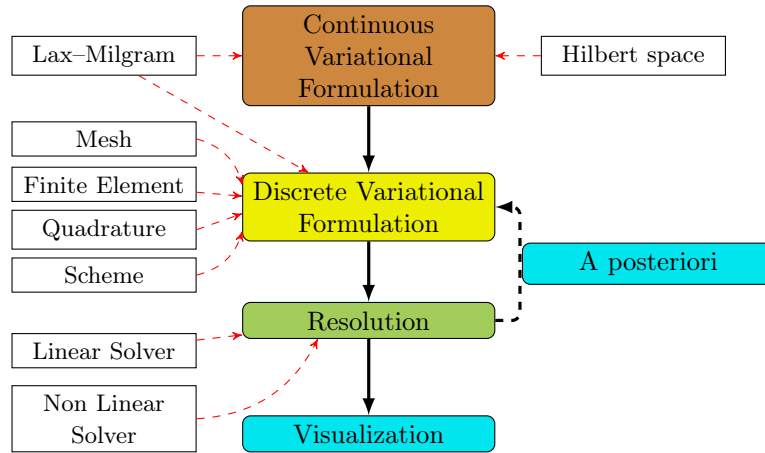


Figure 1.1: Brief recall of the various components of a typical (stationary) FEM formulation. The black thick arrows depict the usual construction steps when solving a problem via the FEM. The black thick dashed arrow shows the possible feedback loop that *a posteriori* analysis allows, to change the mesh, the finite element (i.e. the order of polynomial approximation, the quadrature formula or the numerical scheme). The red dashed arrows depict when a theorem or a numerical tool takes place.

### 1.3 Integration theories

There is a huge variety of concepts of integral and integrability in the literature, e.g. see [18, 15], and one may wonder which one to use. Some are overridden by others, some are equivalent, and some have been developed for specific situations, such as vector-valued functions or functions defined on an infinite-dimensional domain. But a few of them have gained popularity, be it for their appropriateness for teaching, or their general-purpose nature: namely Riemann, Lebesgue, and Henstock–Kurzweil integrations.

Lebesgue integral (with Lebesgue measure) is the traditional framework in which the Sobolev functional spaces are defined. Indeed, Riemann integral is disqualified because of its poor results on limit and integral exchange making completeness unreachable, and so is Henstock–Kurzweil integral, because of its not so obvious extension to the multidimensional case and construction of a complete normed vector space of HK-integrable functions [35, 52].

### 1.4 Contents

The present version of this document covers all material up to the first main properties of the seminormed vector space  $\mathcal{L}^p$  of integrable functions (before taking the quotient to obtain the normed vector space  $L^p$ ). Among the rich literature on Lebesgue integral theory, it was mostly derived from the textbooks [49, 34, 55, 59].

It includes results on the general concepts of measurability, measure and negligibility, and integration of nonnegative measurable functions culminating with the Beppo Levi (monotone convergence) theorem allowing to exchange limit and integral for nondecreasing sequences of measurable functions, and Fatou’s lemma that only gives an upper bound when the sequence is not monotone. The formalization in Coq of all these aspects is presented in [10].

It also includes product  $\sigma$ -algebras, product measures and their uniqueness, the construction of iterated integrals, up to the Tonelli theorem (for nonnegative measurable functions, the Fubini theorem which deals with integrable functions is not yet covered). The formalization in Coq of all these aspects is presented in [11].

In addition, this document also covers the building of the Lebesgue measure, the integral for measurable functions with possibly changing sign, the  $p$ -seminorms and the Lebesgue spaces  $\mathcal{L}^p$  for  $p \in (0, \infty]$ , up to Lebesgue’s dominated convergence theorem and its variants, and the Riesz–Fischer theorem.

It is planned to add more results in a forthcoming version.

## 1.5 Teaching

This document is not primarily meant for teaching usage. The objective was to be as comprehensive as possible in the proofs. This led to very detailed demonstrations, and to a compact style of writing that is not common, and may seem daunting to the uninformed reader.

However, the authors tried to give some insights on the integration theory and on the proofs and theorems in the next introductory sections. They also strove to give some indications in the proofs when they felt it necessary. They believe that this document could be useful for interested teachers, and dedicated students.

## 1.6 Disclaimer

Note that the manuscript itself is not formally proved (and will never be). Indeed,  $\text{\LaTeX}$  compilers are not formal proof tools!

Moreover, formalization is not just straightforward translation of mathematical texts and formulas. Some design choices have to be made and proof paths may differ, mainly to favor usability of `Coq` theorems and ease formal developments. Thus, there exist various differences between the mathematical setting presented here, and the formal setting developed in `Coq` [10, 11]. For instance, the `Coq` definition of measurability of subsets (i.e. of  $\sigma$ -algebra) takes the generators as parameter, and not only the basic axioms. The ubiquitous use of *total* functions (defined for *all* values of their arguments) may be surprising at first: e.g. the addition in  $\overline{\mathbb{R}}$  (“ $\infty - \infty$ ” is defined and takes the value 0), or the measure (one can take the measure of any subset, even nonmeasurable ones). Of course, just as in mathematical statements, unwanted behaviors are somehow prevented, for instance by adding an hypothesis stating the legality of the addition or the measurability of the subset (see [10] for details).

Hence, despite the care taken in its writing, this document might still be prone to errors or holes in the demonstrations. There could also exist simpler paths in the proofs. Please, feel free to inform the authors of any such issue, and to share any comments or suggestions...

## 1.7 Organization

Part I of this document is organized as follows. The notations are first collected in Chapter 2. Then, the literature on the subject is briefly reviewed in Chapter 3. Chapter 4 gathers some proof techniques. The chosen proof paths of the main results are then sketched in Chapter 5.

Part II (Chapters 6 to 16) is the core of this document. In this part, the definitions are presented, and the lemmas and theorems are stated with their detailed proofs. Its organization is described at the end of the introductory Chapter 6.

Chapter 17 concludes and gives some perspectives.

Finally, an appendix gathers the list of statements in Chapter A, and explicit dependencies (both ways) in Chapters B and C. The appendix is not intended for printing!



## Chapter 2

# Notations

In this chapter (as in most of the document), we use the following conventions:

- capital letters  $X$  and  $Y$  denote “surrounding” sets, typically the domain and codomain of functions,  $A$  and  $B$  are subsets of  $X$ , and  $I$  denotes a set of indices;
- calligraphic letter  $\mathcal{R}$  denotes a binary relation on  $X$  (e.g. equality or inequality);
- the capital letter  $P$  denotes a predicate, i.e. a function taking Boolean values;
- small letters  $x$ ,  $y$  and  $i$  denote elements of the set using the matching capital letter;
- small letters  $a$  and  $b$  are (extended) real numbers (such that  $a \leq b$ ), and  $n$  and  $p$  are (extended) natural numbers (such that  $n \leq p$ );
- small letter  $f$  denotes a function, e.g. from set  $X$  to set  $Y$ ;
- the capital Greek letter  $\Sigma$  denotes a  $\sigma$ -algebra on  $X$ , and the small Greek letter  $\mu$  denotes a measure on the measurable space  $(X, \Sigma)$ ;
- *positive/negative*, and *increasing/decreasing* are meant in their *strict* sense, i.e. without the possibility of equality. Otherwise, we use *nonnegative/nonpositive*, and *nondecreasing/nonincreasing*.

The following notations and conventions are used throughout this document.

- Logic:
  - Using a compound (tuple of elements of  $X$ , or subset of  $X$ ) in an expression at a location where only a single element makes sense is a shorthand for the same expression expanded for all elements of the compound; for instance, “ $\forall x, x' \in X$ ” means “ $\forall x \in X, \forall x' \in X$ ”, “ $\forall (x_i)_{i \in I} \in X$ ” means “ $\forall i \in I, x_i \in X$ ”, and “ $x, x' \mathcal{R} x''$ ” means  $x \mathcal{R} x''$  and  $x' \mathcal{R} x''$ ;
  - $x_0 \mathcal{R}_1 x_1 \dots \mathcal{R}_m x_m$  is a shorthand for  $x_0 \mathcal{R}_1 x_1 \wedge x_1 \mathcal{R}_2 x_2 \wedge \dots \wedge x_{m-1} \mathcal{R}_m x_m$ ;
  - “iff” is a shorthand for “if and only if”.
- Naive set theory:
  - $\mathcal{P}(X)$  denotes the power set of  $X$ , i.e. the set of its subsets;
  - subsets are denoted using  $\subset$ , and proper subsets by  $\subsetneq$ ;
  - big set operators ( $n$ -ary or more) based on union and intersection, have precedence over binary set operations (no need for big parentheses); for instance,  $\bigcup_{i \in I} A_i \cap B$  means  $(\bigcup_{i \in I} A_i) \cap B$ ;
  - the definition provides the nullary union law  $\bigcup_{\emptyset} \stackrel{\text{def.}}{=} \emptyset$  (no need for a convention), but unless specified otherwise, intersections are not considered with no operand; however, when a superset  $X$  is not ambiguous, we may use the nullary intersection convention  $\bigcap_{\emptyset} \stackrel{\text{def.}}{=} X$ ;
  - superscript  $^c$  denotes the absolute complement;
  - $\setminus$  denotes the set difference (or relative complement):  $A \setminus B \stackrel{\text{def.}}{=} A \cap B^c$ ;  
when  $B \subset A$ ,  $A \setminus B$  is also called the *local complement of  $B$  (in  $A$ )*;
  - $\Delta$  denotes the symmetric (set) difference:  $A \Delta B \stackrel{\text{def.}}{=} (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ ;
  - $\uplus$  denotes the disjoint union:  $A \uplus B$  means  $A \cup B$  with the assumption that  $A \cap B = \emptyset$ ;



- $\overline{\cap}$  is used to denote a set of traces of subsets, see Definition 226;
  - $\overline{\times}$  and  $\overline{\prod}$  are used to denote a set of Cartesian products of subsets, see Definition 227;
  - $\text{card}(X)$  is the cardinal of  $X$ , i.e. the number of its elements, with the convention  $\text{card}(X) \stackrel{\text{def.}}{=} \infty$  when  $X$  is infinite;
  - countable means either finite, or infinitely countable:  $X$  is countable iff there exists an injection from  $X$  to  $\mathbb{N}$  iff there exists  $I \subset \mathbb{N}$  and a bijection between  $I$  and  $X$ ;
  - $\mathbb{1}_A$ , or  $\mathbb{1}_A^X$ , denotes the indicator function of the subset  $A$  of  $X$ . It is the function from  $X$  to  $\mathbb{R}$ , or  $\overline{\mathbb{R}}$ , that takes the value 1 for all elements of  $A$ , and the value 0 elsewhere;
  - the set of functions from  $X$  to  $Y$  is either denoted  $Y^X$ , or through the type annotation “ $X \rightarrow Y$ ”. Both compact expressions “let  $f \in Y^X$ ” and “let  $f : X \rightarrow Y$ ” mean “let  $f$  be a function from  $X$  to  $Y$ ”;
  - to avoid double parentheses/braces, the expressions involving inverse images of singletons are simplified:  $f^{-1}(y)$  is a shorthand for  $f^{-1}(\{y\})$ ;
  - $\{P(f)\}$  is a shorthand for  $\{x \in X \mid P(f(x))\}$ , or  $f^{-1}(\{y \in Y \mid P(y)\})$ ; for instance,  $\{f < a\}$  means either  $f^{-1}[-\infty, a)$  or  $f^{-1}(-\infty, a)$ , see Lemma 816, and Lemma 820;
  - $f^+$  and  $f^-$  denote the nonnegative and nonpositive parts of a numeric function  $f$ , see Definition 528.
- Totally ordered set:
    - when the nonempty set  $X$  is totally ordered, intervals follow the notations used for real numbers; for instance,  $(x_1, x_2]$  represents the subset  $\{x \in X \mid x_1 < x \leq x_2\}$ ;
    - the square-and-round bracket notation is used for not specifying whether the bound is included or not:  $(x_1, x_2]$  means either  $(x_1, x_2)$  or  $(x_1, x_2]$ ;
    - the lower and greater bounds of  $X$  may be denoted  $-\infty$  and  $+\infty$  (as usual, the plus sign may be omitted); they may belong to  $X$  or not;  $\overline{X}$  denotes  $X \cup \{-\infty, \infty\}$ .
    - rays, or half-lines, are denoted using the square-and-round notation: whether  $\infty$  belongs to  $X$  or not,  $(x, \infty]$  represents  $\{x' \in X \mid x < x'\}$ , i.e. either  $(x, \infty]$  or  $(x, \infty)$ , see Definition 229;
    - $\mathcal{I}_X^{o,p}$  denotes the set of open proper intervals of  $X$ , see Definition 229;
    - $\mathcal{R}_X^o$  denotes the set of open rays of  $X$ , see Definition 229.
    - $\mathcal{I}_X^o$  denotes the set of open intervals of  $X$ , see Definition 229.
    - extrema of the empty set may be defined as  $\inf \emptyset$  equals the greater bound (usually denoted  $\infty$ ), and  $\sup \emptyset$  equals the lower bound (usually denoted  $-\infty$ , but it is 0 for  $\mathbb{R}_+$  and  $\overline{\mathbb{R}}_+$ ); this fulfills the monotonicity property, and the law  $\inf A \leq \sup A$  characterizes the nonempty subsets;
  - Numbers:
    - $\lfloor \cdot \rfloor$  denotes the floor function from  $\mathbb{R}$  to  $\mathbb{N}$ :  $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$ ;
    - $\llbracket n..p \rrbracket$  denotes the integer interval  $\llbracket n, p \rrbracket \cap \mathbb{N}$  (with the natural variants using parentheses and/or brackets when exclusion or inclusion of bounds is specified);
    - $\ell(\llbracket a, b \rrbracket)$  denotes the length of the interval  $\llbracket a, b \rrbracket$ , see Definition 1057;
    - $\mathbb{K}$  denotes a number space: either the fields  $\mathbb{R}$ , and  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$ .
  - General topology:
    - $\mathcal{T}_X(G)$  denotes the topology generated by  $G \subset \mathcal{P}(X)$ , see Definition 242;
  - Measure theory:
    - $\Pi_X(G)$  denotes the  $\pi$ -system generated by  $G \subset \mathcal{P}(X)$ , see Definition 606;
    - $\mathcal{R}_X(G)$  denotes the ring of sets generated by  $G \subset \mathcal{P}(X)$ , see Definition 617;
    - $\mathcal{A}_X(G)$  denotes the algebra of sets generated by  $G \subset \mathcal{P}(X)$ , see Definition 642;
    - $\mathcal{C}_X(G)$  denotes the monotone class generated by  $G \subset \mathcal{P}(X)$ , see Definition 666;
    - $\Lambda_X(G)$  denotes the  $\lambda$ -system generated by  $G \subset \mathcal{P}(X)$ , see Definition 682;
    - $\Sigma_X(G)$  denotes the  $\sigma$ -algebra generated by  $G \subset \mathcal{P}(X)$ , see Definition 708;
    - $\mathcal{B}(X)$  denotes the Borel  $\sigma$ -algebra generated by the open subsets, see Definition 743;
    - $\otimes_{i \in [1..m]} \Sigma_i$  denotes the tensor product of the  $\sigma$ -algebras  $(\Sigma_i)_{i \in [1..m]}$ , see Definition 774;
    - $\mathbb{N}$  denotes the set of negligible subsets, see Definition 942;

- the annotation “ $\mu$  a.e.” specifies that the proposition is only considered  $\mu$ -almost everywhere for some measure  $\mu$ , e.g.  $\mu \stackrel{\text{a.e.}}{=}$ ,  $\forall_{\mu \text{ a.e.}}$ ,  $(X \rightarrow Y)_{\mu \text{ a.e.}}$ , or  $\mathcal{R}_{\mu \text{ a.e.}}$ , see Definition 953;
- $\delta_Y$  denotes the counting measure associated with  $Y$ ,  $\delta_a$  denotes the Dirac measure at  $a$ , see Lemma 935 and Definition 939;
- $\mathcal{L}$  denotes the Lebesgue  $\sigma$ -algebra on  $\mathbb{R}$  and  $\lambda$  denotes the (Borel-)Lebesgue measure on  $\mathbb{R}$ , see Definitions 1063 and 1071, and Theorem 1090;
- $\mu_1 \otimes \mu_2$  denotes the tensor product of measures  $\mu_1$  and  $\mu_2$ , see Definition 1224, Definition 1225, and Lemma 1233;
- $\lambda^{\otimes 2}$  denotes the Lebesgue measure on  $\mathbb{R}^2$ , see Lemma 1235;
- Lebesgue integral:
  - $\mathcal{IF}_{\mathbb{K}}$  denotes the set of measurable indicator functions on the number space  $\mathbb{K}$ , see Definition 1098;
  - $\mathcal{SF}_E$  denotes the vector space of simple functions to the vector space  $E$ , see Lemma 1108;
  - $\mathcal{M}_Y$  denotes the set of measurable functions  $X \rightarrow Y$ , see Definition 758;
  - $\mathcal{SF}_+$  denotes the set of nonnegative simple functions, see Definition 1123;
  - $\mathcal{M}_+$  denotes the set of nonnegative measurable functions  $X \rightarrow \overline{\mathbb{R}}_+$ , see Definition 835;
  - $\mathcal{M}$  denotes the set of measurable functions  $X \rightarrow \overline{\mathbb{R}}$ , see Definition 817;
  - $\mathcal{M}_{\mathbb{R}}$  denotes the vector space of measurable functions  $X \rightarrow \mathbb{R}$ , see Lemma 815;
  - $\int f d\mu$  denotes the (Lebesgue) integral of  $f$  for the measure  $\mu$ , whether the function belongs to  $\mathcal{IF}$ ,  $\mathcal{SF}_+$ ,  $\mathcal{M}_+$ , or  $\mathcal{M}$ , see Section 4.4, Definition 1144, Lemmas 1154 and 1172, and Lemma 1256;
  - $\int_A f d\mu$  denotes the integral of the restriction of  $f$  to  $A$ , see Lemmas 1148, 1163, 1199, and 1274;
  - $\int_a^b f(x) d\mu(x)$  denotes the integral of  $f$  over the interval  $(a, b)$ , see Lemma 1278;
  - $\int f(x) dx$  denotes the integral of  $f$  for the Lebesgue measure, see Definition 1285;
  - $(\mathcal{L}^p, N_p)$  denotes the seminormed vector space of  $p$ -powered integrable functions for  $p$  in  $(0, \infty)$ , and of essentially bounded measurable functions for  $p = \infty$ , see Lemma 1286 and Definition 1314;
  - $\mathcal{I}$  denotes the integral operator for integrable functions, see Lemma 1319;
  - $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$  denotes the convergence of  $(f_n)_{n \in \mathbb{N}}$  towards  $f$  in  $p$ -seminorm  $N_p$  for  $p$  in  $[1, \infty]$ , see Lemma 1339.



## Chapter 3

# State of the art

After a brief survey of the wide variety of integrals, and of the means to build Lebesgue integral and Lebesgue measure, we review some works of a few authors, partly from the mathematical French school, that provide some details about results in measure theory, Lebesgue integration, and basics of functional analysis.

As usual, proofs provided in the literature are not comprehensive, and we have to cover a series of sources to collect all the details necessary for a formalization in a formal proof tool such as `Coq`. Usually, Lecture Notes in undergraduate mathematics are very helpful and we selected [37, 38, 39] among many other possible choices.

### 3.1 A zoology of integrals

The history of integral calculus dates back at least from the Greeks, with the method of exhaustion for the evaluation of areas, or volumes. Since then, mathematicians have constantly endeavored to develop new techniques. The objective is mainly to be able to integrate a wider class of functions. But also to establish more powerful results, such as convergence results, or the Fundamental Theorems of Calculus (expressing that derivation and integration are each other inverse operations). And possibly to fit specific contexts, e.g. driven by applications in physics, or for teaching purposes. For a wide panel, and pros and cons of different approaches, see for instance [21, 18, 15], in which more than a hundred named integrals are listed. But, as pointed out in [57], building a universal integral “will probably be unnecessarily complicated when restricted to a simple setting”.

Let us review some of the most popular ones, mainly for the case of numeric functions defined on some interval.

#### 3.1.1 Leibniz–Newton integral (late 17th century)

The Leibniz–Newton (LN-)integral is defined in a descriptive way as the difference of any primitive evaluated at the bounds of the integration interval. All continuous functions are LN-integrable.

This integral possesses interesting properties such as formulas for integration by parts or by substitution, and a uniform convergence theorem. But since the definition is not constructive, there is no regular process to build the primitive of a function, even if it is continuous. Moreover, other monotone or dominated convergence results must assume that the limit is LN-integrable. And it raises the question of integrability of functions with no primitives.

#### 3.1.2 Riemann integral (early 19th century)

The Riemann (R-)integral [54] consists in cutting the area “under” the graph of the function into vertical rectangular strips by choosing a subdivision of the integration interval, and in increasing the number of strips by making the step of the subdivisions go down to 0. Equivalently, it might

be more convenient to consider the Darboux integral [24], which checks that both upper and lower so-called Darboux sums have the same limit.

This simple approach has similar properties than the LN-integral, with for instance a uniform convergence result, and R-integrable functions are the piecewise continuous function which set of discontinuity points has null measure. But again, monotone convergence and dominated convergence theorems must assume R-integrability of the limit.

### 3.1.3 Lebesgue integral (early 20th century)

The key idea of the Lebesgue (L-) approach [46] is to consider subdivisions of the codomain of the function, i.e. cutting the area under the graph into horizontal pieces, which are no longer continuous strips when the function is not concave, and to define a measure for these pieces that generalizes the length of intervals.

This has the great advantage of providing powerful monotone and dominated convergence theorems, and to allow for an abstract setting in which one can handle functions defined on more general spaces than the Euclidean spaces  $\mathbb{R}^n$ . And this paves the road for probability theory. For instance, the Wiener measure used for the study of stochastic processes such as Brownian motion is defined on the infinite-dimensional Banach space of continuous functions from a compact interval to  $\mathbb{R}^n$ . Moreover, the concept of property satisfied *almost everywhere* opens the way to the  $L^p$  Lebesgue spaces as (complete) normed vector spaces (Banach spaces).

The main difficulty is the necessity to develop the measure theory to be able to associate a length to the “horizontal pieces”. In general, this is not possible for all subsets of the domain of the function, and leads to the concepts of *measurable subset* and *measurable function*.

Going back to the geometrical interpretation of the area under the graph of the function, it is interesting to note that the opposition Riemann (subdivision of the domain) versus Lebesgue (subdivision of the codomain) reflects in the field of ordinary differential equations where the classical numerical schemes based on a discretization of the time domain are now opposed to the recent quantized state system solvers that are based on the quantization of the state variables, see [17, 31, 32]. Moreover, we may also cite [36] in the field of signal processing where the Lebesgue geometrical scheme is seen as a nonlinear sampling qualified of “noise-free quantization”, that could have impact on electronic design.

### 3.1.4 Bochner integral (early mid-20th century)

The Bochner (B-) integral [6, 51, 59] is a generalization of the L-integral to the case of functions taking their values in a Banach space.

Similarly to the L-integral, its construction is based on the limit of integrals of simple functions (see Sections 4.4 and 4.5), that are measurable functions taking only a finite number of values. The main difference here is the absence of order in Banach spaces, which makes infinite values impracticable and prevents the use of the least-upper-bound property. Instead, the existence of the limit is ensured by completeness, and by the additional assumption of separability which provides the building of Cauchy sequences.

From the L-integral of nonnegative measurable functions, the B-integral has the great advantage to allow the handling of functions to  $\mathbb{R}$  (with possibly changing sign) or  $\mathbb{C}$ , to the Euclidean space  $\mathbb{R}^n$  or the Hermitian space  $\mathbb{C}^n$ , and to any separable Banach space. For instance, to  $L^p(\Omega)$  (with  $p \in [1, \infty)$ ), where  $\Omega$  is a regular enough domain in  $\mathbb{R}^n$ , or  $\mathbb{C}^n$ . Given a real number  $T > 0$ , this allows to define the space  $L^2([0, T], L^2(\Omega))$  of square-integrable functions from the time interval  $[0, T]$  to the square-integrable functions defined on the (space) domain  $\Omega$ . Such a cascading construction can be useful for the study of some evolutionary PDEs.

### 3.1.5 Henstock–Kurzweil integral (mid-20th century)

The Henstock–Kurzweil (HK-)integral, or gauge integral [45, 41, 4] is a generalization of the R-integral for which the subdivision of the integration interval is no longer uniform, but is driven by

the variations of the function to integrate through a so-called *gauge* function.

It is more powerful than the L-integral in the sense that a real-valued function is L-integrable if and only if the function and its absolute value are HK-integrable. Unlike the L-integral, this leads to a smoother formulation of the fundamental theorem of calculus, without the need for the concept of improper integral. The monotone and dominated convergence theorems are also valid. Moreover, the HK-integral is usually considered more suited for teaching, for it is hardly more complex than the R-integral.

Although less straightforward than with the L-integral, recent developments make some extensions feasible. We may cite the extension to functions defined on the Euclidean spaces  $\mathbb{R}^n$  [47], and the completion of HK-integrable functions into a Banach space [3, 56, 35, 52, 58].

### 3.1.6 Daniell integral (early 20th century)

The Daniell (D-) approach [23, 33] is a general scheme that extends an *elementary* integral defined for *elementary* functions to a much wider class of functions by checking that upper and lower elementary integrals of elementary functions share the same limit, in a way that is similar to the Darboux approach. When applied to the integral of simple functions, this is somehow equivalent to steps 3 and 4 of the Lebesgue scheme described later in Section 4.4. But it is enough to consider the R-integral for the compactly supported continuous functions (a.k.a. the Cauchy integral) to get back the L-integral for the Lebesgue measure on the Euclidean spaces  $\mathbb{R}^n$ .

This Daniell approach to the L-integral has the great advantage of not needing the concept of measure, which can be reconstructed afterwards by taking the integral of indicator functions of measurable subsets. Moreover, in the mid-20th century, this approach was shown in [5] to be compatible with constructive analysis.

In the present document, we choose the Lebesgue approach.

## 3.2 Lebesgue integral and Lebesgue measure

The Lebesgue integral was originally built using measure theory following the so-called Lebesgue scheme, described in Section 4.4. We have seen that it may also be built in a constructive way by applying the Daniell extension scheme to some elementary integral defined for elementary functions, such as the Riemann integral for compactly supported continuous functions. Another alternative using the same basic ingredients consists in the completion of the normed vector space of compactly supported continuous functions and the extension of the Riemann integral which is uniformly continuous [12, 25].

The Lebesgue measure on  $\mathbb{R}$  is a generalization of the length of bounded intervals to a much wider class of subsets. This extension is actually unique and complete. It is defined on the Lebesgue  $\sigma$ -algebra which is generated by the Borel subsets and the negligible subsets. The restriction to the sole Borel subsets (the  $\sigma$ -algebra generated by the open subsets, or the closed subsets) is sometimes called the Borel(-Lebesgue) measure. In the very same way, the Lebesgue measure on the Euclidean spaces  $\mathbb{R}^n$  is a generalization of the  $n$ -volume of rectangular boxes. It is the tensor product of the Lebesgue measure on  $\mathbb{R}$ .

There are three main techniques for the construction of the Lebesgue measure on  $\mathbb{R}$ . The most popular one for teaching is through Carathéodory's extension theorem [16, 26]. The process builds progressively the Lebesgue  $\sigma$ -algebra and the Lebesgue measure with their properties (see Section 4.3). A more abstract approach is based on the Riesz–Markov–Kakutani representation theorem, which associates any positive linear form on the space of compactly supported continuous functions with a unique measure, e.g. see [55]. The third way is not based on measure theory. It follows the Daniell approach to integration, which is also based on the Riemann integration of compactly supported continuous functions, and defines the Lebesgue measure of any measurable subset as the integral of its indicator function.

The construction of non-Lebesgue-measurable subsets requires the use of the axiom of choice, as for instance for the Vitali subset [60].

In the present document, we build the Lebesgue integral using the Lebesgue scheme (see Section 4.4), and the Lebesgue measure through Carathéodory's extension theorem (see Section 4.3).

### 3.3 Our main sources

The main ingredients of measure theory and Lebesgue integration are presented in an elementary manner in the textbook [49], which is very well suited for our purpose. However, some results such as the construction of the Lebesgue measure and the Tonelli theorem are admitted.

Details for the construction of the Lebesgue measure (through the Carathéodory approach), and for the proofs of the Fubini–Tonelli theorems can be found in [34]. This other textbook is quite comprehensive. Many results are presented as exercises with their solution.

We may also cite [55, 59, 2, 14]. The former for instance for a construction of the Lebesgue measure through the Riesz–Markov–Kakutani representation theorem. The second one for a fairly comprehensive presentation of the Bochner integral and the main results. And the latter two for the  $L^p$  Lebesgue spaces and the  $W^{m,p}$  Sobolev spaces.

Note that some original forms of statements have been developed for the present document, and even though most results are well known, their formulations are new to our knowledge. This includes:

- the individual treatment of constitutive properties (closedness under set operations) of subset systems in Section 8.1, and of numeric functions over subset systems in Section 11.1;
- the formalization of most concepts based on properties almost satisfied in Section 12.3 with abstract results such as Lemmas 959, 960, 962, 963, 970, 972, 984, 985, and 988, 1189 in Section 15.3, and 1264 in Section 16.1;
- the concept of almost sum in Section 13.1 with Lemmas 1002, 1003, and 1004, impacting Lemma 1187 in Section 15.3, Lemmas 1253, 1268, and 1269 in Section 16.1, Lemma 1274 in Section 16.2, and Lemmas 1309, and 1310 in Section 16.3;
- the concept of disjoint representation of simple functions in Section 14.2 with Lemmas 1114, 1116, and 1117.

# Chapter 4

## Proof techniques

In the framework of integration theory, some concepts and proof techniques are used repeatedly. This chapter describes the most important ones: “subset systems”, the “Dynkin  $\pi$ - $\lambda$  theorem / monotone class theorem scheme”, the “Carathéodory extension scheme”, the “Lebesgue scheme”, and the “Bochner scheme”.

### 4.1 Subset systems

A  $\pi$ -system, a ring of sets, an algebra of sets, a monotone class, a  $\lambda$ -system and a  $\sigma$ -algebra are all different kinds of subset systems (i.e. subsets of the power set), see Figure 4.1. They are defined in Chapter 8, where a mini-language to represent the properties in a compact form is introduced.

In the present document, the richest subset system is the  $\sigma$ -algebra (the others have fewer properties). The choice of some  $\sigma$ -algebra defines the measurable subsets (see Chapter 9), for which measures are then defined (in Chapter 12).

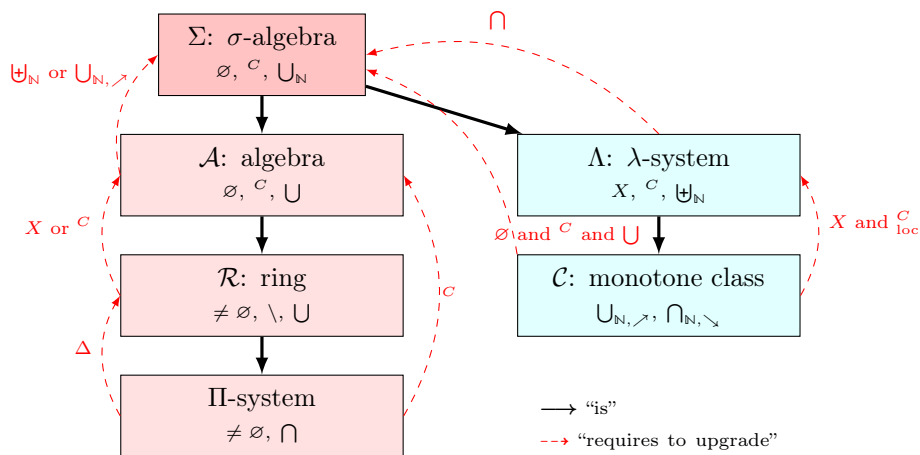


Figure 4.1: Description of the hierarchy of subset systems introduced in this document. In each box, a set of minimal properties to define the subset system is given: for instance, a ring is nonempty and closed under set difference and union. The red dashed arrows express some necessary properties to “upgrade” a system to a richer one: for instance, a monotone class is a  $\lambda$ -system if it contains the full set  $X$  and is compatible with local complement. We recall that  $\Delta$ ,  $C$ ,  $\mathfrak{U}_N$ ,  $\bigcup_{N, \rightrightarrows}$  and  $\bigcap_{N, \searrow}$  represent closedness under respectively symmetric difference, complement, disjoint countable union, monotone countable union and intersection.



## 4.2 Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem schemes

Roughly speaking, being a  $\lambda$ -system (resp. a monotone class) is what is missing to a  $\pi$ -system (resp. an algebra of subsets) to be a  $\sigma$ -algebra.

To prove that some property  $P$  holds on some  $\sigma$ -algebra  $\Sigma$  generated by some subset system  $G$ , the idea is to proceed in two steps. First, show that the property  $P$  holds on the simpler kind of subset system ( $\pi$ -system, or algebra of sets) generated by  $G$ , and then establish that the subset of  $\Sigma$  where  $P$  holds is of the less simple kind of subset system ( $\lambda$ -system, or monotone class). Indeed, the Dynkin  $\pi$ - $\lambda$  theorem states that the  $\lambda$ -system generated by a  $\pi$ -system is equal to the  $\sigma$ -algebra generated by the same  $\pi$ -system (in short  $\Lambda_X(\Pi) = \Sigma_X(\Pi)$ ), and similarly, the monotone class theorem states that the monotone class generated by an algebra of sets is equal to the  $\sigma$ -algebra generated by the same algebra of sets (in short  $\mathcal{C}_X(\mathcal{A}) = \Sigma_X(\mathcal{A})$ , see Section 5.10). Thus,  $P$  is established on the whole  $\sigma$ -algebra.

The five steps of the scheme of the Dynkin  $\pi$ - $\lambda$  theorem, and of the monotone class theorem are the following:

1. first define  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ ; obviously, we have  $\mathcal{S} \subset \Sigma$ ;
2. then, show that  $G \subset \mathcal{S}$ ;
3. then, show that the  $\pi$ -system (resp. algebra of sets)  $\mathcal{U}_X^2(G)$  is a subset of  $\mathcal{S}$ ;
4. then, show that  $\mathcal{S}$  is a  $\lambda$ -system (resp. monotone class);
5. and finally, obtain the other inclusion  $\Sigma \subset \mathcal{S}$ , because from monotonicity of subset system generation, and from the Dynkin  $\pi$ - $\lambda$  theorem (resp. monotone class theorem), i.e. the equality in the middle, we have

$$\Sigma = \Sigma_X(G) \subset \Sigma_X(\mathcal{U}_X^2(G)) = \mathcal{U}_X^1(\mathcal{U}_X^2(G)) \subset \mathcal{U}_X^1(\mathcal{S}) = \mathcal{S}$$

where  $\mathcal{U}_X^1 = \Lambda_X$  (resp.  $\mathcal{C}_X$ ), i.e. the generated  $\lambda$ -system (resp. monotone class), and  $\mathcal{U}_X^2 = \Pi_X$  (resp.  $\mathcal{A}_X$ ), i.e. the generated  $\pi$ -system (resp. algebra of sets).

In the present document, the fifth step is embodied by Lemma 734 for the Dynkin  $\pi$ - $\lambda$  theorem, and by Lemma 738 for the monotone class theorem. The Dynkin  $\pi$ - $\lambda$  theorem scheme is used once when extending the equality of two measures on a generator  $\pi$ -system to the whole  $\sigma$ -algebra in Lemma 992, which is then used to establish uniqueness of the Lebesgue measure on  $\mathbb{R}$  in Theorem 1090 (see also Section 5.9). The monotone class theorem scheme is used twice when building the tensor product measure in the context of the product of finite measure spaces: to establish first the measurability of the measure of sections in Lemma 1221, and then the uniqueness of the tensor product measure in Lemma 1231 (both in the finite case).

## 4.3 Carathéodory's extension scheme

The Carathéodory extension scheme consists in applying the eponymous theorem (see Section 5.9) to extend a premeasure (i.e. defined on a ring of subsets, and not on a  $\sigma$ -algebra) into a complete measure on a  $\sigma$ -algebra containing the initial ring of subsets.

Let  $X$  be the ambient set,  $\mu$  be the premeasure and  $\mathcal{R}$  be the ring of subsets (of  $X$ ). The two steps of the scheme are the following:

1. extend the premeasure to any subset of  $X$  by taking the infimum of the sum of premeasures of all coverings with elements of the ring  $\mathcal{R}$  (and the value  $\infty$  when there is no such covering):

$$\mu^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid (A_n)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_n \right\};$$

the theorem ensures that  $\mu^*$  is an outer measure on  $X$ , and that it is an extension of the premeasure  $\mu$ ;

2. define the set of Carathéodory-measurable subsets as

$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \mid \forall A \subset X, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)\};$$

the theorem ensures that  $\Sigma$  is a  $\sigma$ -algebra on  $X$  containing the ring  $\mathcal{R}$ , and that the restriction  $\mu^*|_{\Sigma}$  is a complete measure on  $(X, \Sigma)$ .

The Carathéodory extension scheme is a popular tool to build measures. The prominent example is the Lebesgue measure on  $\mathbb{R}$  that extends the length of open intervals (see Section 13.3). But for instance, the extension scheme may also be used to build the Lebesgue-Stieltjes measure associated with its cumulative distribution function, and Loeb measures in a nonstandard analysis framework, e.g. see [48, 22].

## 4.4 Lebesgue scheme

Consider a set  $X$  equipped with a  $\sigma$ -algebra  $\Sigma$  of measurable subsets, and a measure  $\mu$  that associates a “length” (in  $\overline{\mathbb{R}}_+$ ) to any measurable subset. The set of extended real numbers  $\overline{\mathbb{R}}$  is equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}(\overline{\mathbb{R}})$  that is generated by all open subsets. The purpose is to build, or prove properties of, the integral of “measurable” functions from  $X$  to  $\overline{\mathbb{R}}$ , i.e. for which the inverse image of measurable subsets (of  $\overline{\mathbb{R}}$ ) are measurable subsets (of  $X$ ).

The Lebesgue scheme consists in establishing facts about the integral by working successively inside three embedded functional subsets: the set  $\mathcal{IF}$  of indicator functions of measurable subsets, the vector space  $\mathcal{SF}$  of simple functions (the linear span of  $\mathcal{IF}$ ), and the set  $\mathcal{M}$  of measurable functions, see Figure 4.2.

The four steps of the scheme are the following:

1. establish the fact in the case of indicator functions (in  $\mathcal{IF}$ ) for which the integral is simply the measure of the support;
2. then, generalize the fact to the case of nonnegative simple functions (in  $\mathcal{SF}_+$ ) by nonnegative (finite) linear combination of indicator functions;
3. then again, generalize the fact to the case of nonnegative measurable functions (in  $\mathcal{M}_+$ ) by taking the supremum for all lower nonnegative simple functions;
4. finally, generalize the fact to the case of all measurable functions (in  $\mathcal{M}$ ) by taking the difference between the expressions involving nonnegative and nonpositive parts (integrable functions are those for which this difference is well-defined).

Actually, a fifth step can be added to generalize the fact to the case of numeric functions taking their values in  $\mathbb{R}^n$ ,  $\mathbb{C}$ , or  $\mathbb{C}^n$  by considering separately all components.

In the present document, the steps of Lebesgue scheme are mainly used to build the Lebesgue integral (steps 1 to 4, see Chapter 15 and Section 16.1), but also to establish some properties such as the Beppo Levi (monotone convergence) theorem (step 3, see Section 5.8 and Theorem 1180) and the Tonelli theorem (steps 1 to 3, see Section 5.5 and Theorem 1242).

## 4.5 Bochner scheme

Let us also consider now a separable Banach space  $E$  equipped with its Borel  $\sigma$ -algebra  $\mathcal{B}(E)$ . The Bochner integral extends the Lebesgue integral to the case of “integrable” functions from  $X$  to  $E$ .

The Bochner scheme is similar to the Lebesgue scheme. It also consists in establishing facts about the integral by working successively inside two embedded functional subsets: the subset of Bochner-integrable simple functions, and the subset of (Bochner-)integrable functions. The main difference here is the absence of order in the codomain  $E$  that prevents the consideration of

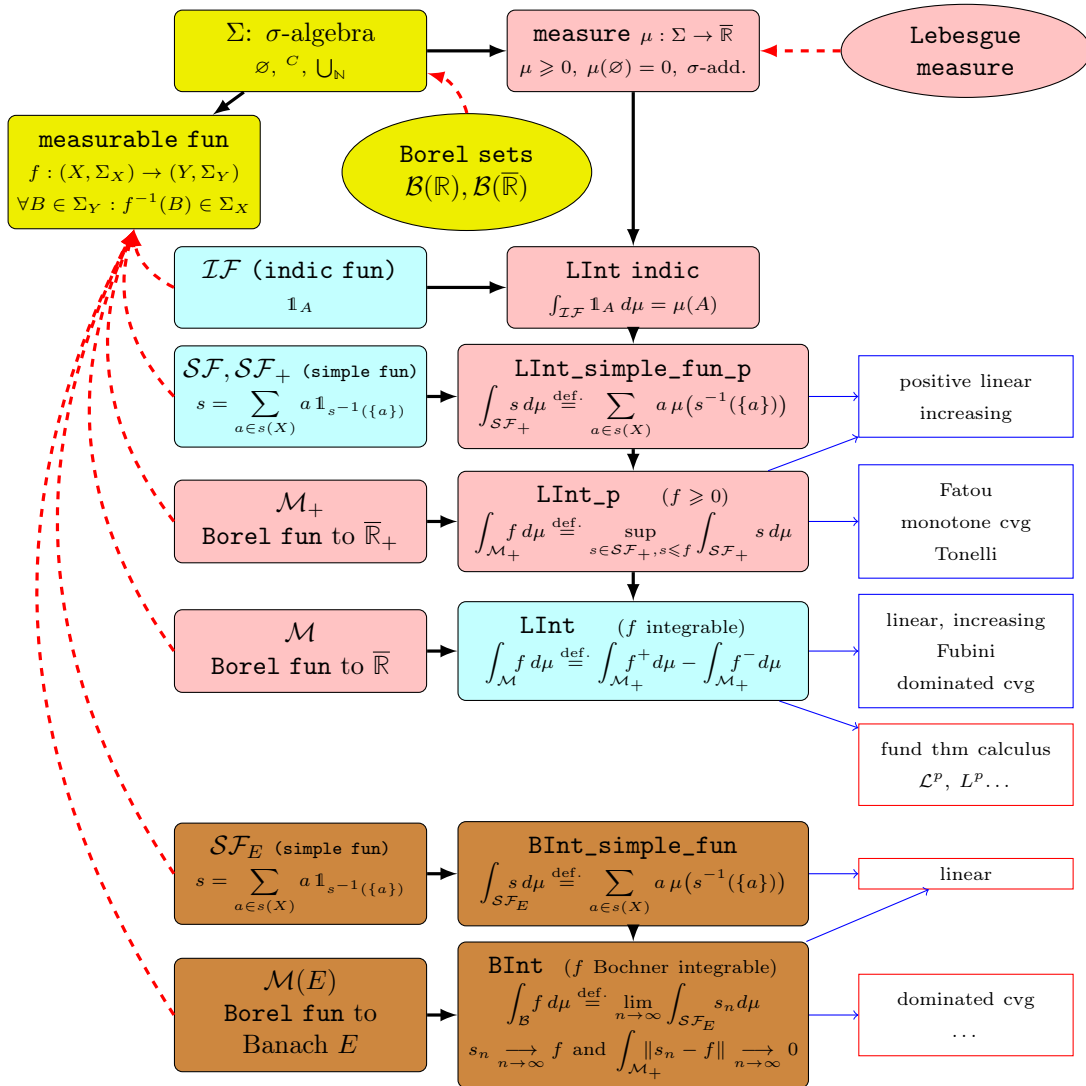


Figure 4.2: Description of the Lebesgue scheme to construct the Lebesgue integral, and insight into the construction of the Bochner integral.

In the boxes filled in light pink, the functions have values in  $\bar{\mathbb{R}}$ , whereas in the boxes filled in light blue, they have values in  $\mathbb{R}$ , and in the boxes filled in brown, they have values in a Banach space  $E$ . In the right column, the main lemmas over the integral are recalled (the blue boxes are stated and demonstrated in the present document, the red ones are not). The red dashed arrows show some instances of a general definition. Note that the construction of the Bochner integral is not included in the present version of the document.

nonnegative values, of supremum, and thus of (nonnegative) infinite values. However, the Bochner integral is based on the Lebesgue integral for nonnegative measurable functions.

The two steps of the scheme are the following:

1. establish the fact in the case of Bochner-integrable simple functions;
2. finally, generalize the fact to the case of all (Bochner-)integrable functions by taking the limit of integrable simple functions, using separability and completeness.

Note that the Bochner scheme is actually an alternative to step 4 of Section 4.4, and the informal fifth step for the extension of the integral to functions taking their values in  $\mathbb{R}$ ,  $\mathbb{R}^n$ ,  $\mathbb{C}$ , and  $\mathbb{C}^n$ .



## Chapter 5

# Statements and sketches of proofs

This chapter gathers the sketches of the proofs of the main results that are detailed in Part II. Namely: the Riesz–Fischer theorem, Lebesgue’s dominated convergence theorem and its extended versions for integrable functions and in  $\mathcal{L}^p$ , the Tonelli theorem, the Existence and uniqueness of the tensor product measure suite of lemmas, Fatou’s lemma, the Beppo Levi (monotone convergence) theorem, Carathéodory’s extension theorem, the Dynkin  $\pi$ – $\lambda$  theorem, and the monotone class theorem.

### 5.1 Sketch of the proof of the Riesz–Fischer theorem

#### Riesz–Fischer theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}^p$  be a Cauchy sequence.

Then, there exists  $f \in \mathcal{L}^p$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ .

Moreover, there exists  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  increasing and  $h \in \mathcal{L}^p$  such that for all  $n \in \mathbb{N}$ ,  $|f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h$ , and  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ . When  $p = \infty$ , one can take  $k_n \stackrel{\text{def.}}{=} n$ , for all  $n \in \mathbb{N}$ .

See Theorem 1354. The proof of the Riesz–Fischer theorem uses the arithmetic of  $\mathbb{R}$ , the Beppo Levi (monotone convergence) theorem set in  $\mathcal{L}^p$ , and Lebesgue’s extended dominated convergence theorem also set in  $\mathcal{L}^p$  (see Section 5.2). The proof of the Riesz–Fischer theorem goes as follows:

- for  $p \in [1, \infty)$ 
  - from the Cauchy assumption, build  $(k_n)_{n \in \mathbb{N}}$  increasing such that  $N_p(f_{k_{n+j}} - f_{k_n}) \leq \frac{1}{2^n}$ ;
  - apply the Beppo Levi (monotone convergence) theorem set in  $\mathcal{L}^p$  to the sequence

$$S_n \stackrel{\text{def.}}{=} \sum_{i \in [0..n)} |f_{k_{i+1}} - f_{k_i}|$$

to obtain the existence of  $S \in \mathcal{L}^p \cap \mathcal{M}_+$  such that  $S_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} S$  and  $|S_n| \stackrel{\mu \text{ a.e.}}{\leq} S$ ,

then establish  $|f_{k_{n+q}} - f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} S - S_n \leq S$  to show that  $(f_{k_n})$  is a Cauchy sequence (pointwise), and obtain the existence of  $f \in \mathcal{M}_{\mathbb{R}}$  such that  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ;

- then, let  $h_n \stackrel{\text{def.}}{=} |f_{k_n} - f|$  and  $g_q^n \stackrel{\text{def.}}{=} |f_{k_{n+q}} - f_{k_n}| \in \mathcal{L}^p$ , and show successively

$$h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} 0, \quad g_q^n \xrightarrow[q \rightarrow \infty]{\mu \text{ a.e.}} h_n, \quad h_n \stackrel{\mu \text{ a.e.}}{\leq} S, \quad f \in \mathcal{L}^p, \quad \text{and} \quad |f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h \stackrel{\text{def.}}{=} S + |f| \in \mathcal{L}^p,$$

and finally apply the Lebesgue’s extended dominated convergence theorem set in  $\mathcal{L}^p$  to  $(h_n)_{n \in \mathbb{N}}$  to obtain  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ , and then  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ ;

- for  $p = \infty$

- from the Cauchy assumption, establish  $|f_{n+q} - f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} 1$  to show that  $(f_n)$  is a Cauchy sequence (pointwise), and obtain the existence of  $f \in \mathcal{M}_{\mathbb{R}}$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ;
- then, let  $h_n \stackrel{\text{def.}}{=} |f_n - f|$  and  $g_q^n \stackrel{\text{def.}}{=} |f_{K_0+q} - f_{K_0+n}| \in \mathcal{L}^\infty$ , and show successively

$$h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} 0, \quad g_q^n \xrightarrow[q \rightarrow \infty]{\mu \text{ a.e.}} h_{K_0+n}, \quad h_{K_0+n} \stackrel{\mu \text{ a.e.}}{\leq} 1, \quad f \in \mathcal{L}^\infty,$$

$$\text{and } |f_n| \stackrel{\mu \text{ a.e.}}{\leq} h \stackrel{\text{def.}}{=} \max_{k < K_0} \{|f_k|, 1 + |f|\} \in \mathcal{L}^\infty,$$

and finally obtain  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^\infty} f$ .

For instance, the Riesz–Fischer theorem may be used to eventually apply the theory of Banach spaces to the Lebesgue spaces  $L^p$  (after taking the quotient with respect to the subspace of functions vanishing almost everywhere).

## 5.2 Sketch of the proof of Lebesgue’s extended dominated convergence theorem in $\mathcal{L}^p$

### Lebesgue’s extended dominated convergence theorem in $\mathcal{L}^p$ .

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty)$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{M}$ . Let  $g \in \mathcal{L}^p$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, there exists  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^p$  such that we have

$$(5.1) \quad \forall n \in \mathbb{N}, \quad \tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n \quad \wedge \quad N_p(\tilde{f}_n) = N_p(f_n) \leq N_p(g) < \infty,$$

$$(5.2) \quad \tilde{f} \stackrel{\mu \text{ a.e.}}{=} f \quad \wedge \quad N_p(\tilde{f}) = N_p(f) \leq N_p(g) < \infty,$$

$$(5.3) \quad \tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} \tilde{f},$$

$$(5.4) \quad \int |f|^p d\mu = \int |\tilde{f}|^p d\mu = \int \lim_{n \rightarrow \infty} |\tilde{f}_n|^p d\mu = \lim_{n \rightarrow \infty} \int |\tilde{f}_n|^p d\mu = \lim_{n \rightarrow \infty} \int |f_n|^p d\mu \quad (\text{in } \mathbb{R}).$$

See Theorem 1351. The proof of Lebesgue’s extended dominated convergence theorem in  $\mathcal{L}^p$  uses the arithmetic of  $\overline{\mathbb{R}}$  (but with functions that are almost everywhere finite), and Lebesgue’s extended dominated convergence theorem (see Section 5.3). The proof goes as follows:

- prove that  $(h_n \stackrel{\text{def.}}{=} |f_n|^p)_{n \in \mathbb{N}}, h \stackrel{\text{def.}}{=} |f|^p, |g|^p \in \mathcal{M}$ ,  $h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} h$ ,  $|g|^p$  is  $\mu$ -integrable, and for all  $n \in \mathbb{N}$ ,  $|h_n| \stackrel{\mu \text{ a.e.}}{\leq} |g|^p$ , then apply Lebesgue’s extended dominated convergence theorem to  $(h_n)_{n \in \mathbb{N}}, h$  and  $|g|^p$  to obtain (5.4), and finally pick finite representatives  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^p$  such that (5.1), (5.2), and  $\tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} \tilde{f}$  hold;
- prove that  $(k_n \stackrel{\text{def.}}{=} |\tilde{f}_n - \tilde{f}|^p)_{n \in \mathbb{N}}, k \stackrel{\text{def.}}{=} 0, 2^p |g|^p \in \mathcal{M}$ ,  $k_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} k$ ,  $2^p |g|^p$  is  $\mu$ -integrable, and for all  $n \in \mathbb{N}$ ,  $|k_n| \stackrel{\mu \text{ a.e.}}{\leq} 2^p |g|^p$ , and then apply Lebesgue’s extended dominated convergence theorem to  $(k_n)_{n \in \mathbb{N}}, k$  and  $2^p |g|^p$  to obtain the  $\mathcal{L}^p$  convergence (5.3).

Lebesgue’s extended dominated convergence theorem in  $\mathcal{L}^p$  is used in the present document to prove the variant of the Beppo Levi (monotone convergence) theorem set in  $\mathcal{L}^p$  (for  $p \in [1, \infty)$ , see Theorem 1352), and the Riesz–Fischer theorem (see Section 5.1 and Theorem 1354).

### 5.3 Sketch of the proof of Lebesgue's extended dominated convergence theorem

#### Lebesgue's extended dominated convergence theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ,  $g$  is  $\mu$ -integrable, and for all  $n \in \mathbb{N}$ ,  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then,  $(f_n)_{n \in \mathbb{N}}$  and  $f$  are  $\mu$ -integrable, and there exist  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^1$  such that, for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n$ ,  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ,  $\tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} \tilde{f}$ , and we have

$$(5.5) \quad \int f \, d\mu = \int \tilde{f} \, d\mu = \int \lim_{n \rightarrow \infty} \tilde{f}_n \, d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (\text{in } \mathbb{R}).$$

See Theorem 1349. The proof of Lebesgue's extended dominated convergence theorem uses the arithmetic of  $\overline{\mathbb{R}}$  (but with functions that are almost everywhere finite), and Lebesgue's dominated convergence theorem (see Section 5.4). The proof goes as follows:

- $A \stackrel{\text{def.}}{=} \{f = \liminf_{n \rightarrow \infty} f_n\} \cap \{f = \limsup_{n \rightarrow \infty} f_n\} \cap \bigcap_{n \in \mathbb{N}} \{|f_n| \leq g\} \cap g^{-1}(\mathbb{R}_+) \subset X$  is first shown to be measurable with  $\mu(A^c) = 0$ ; thus  $\tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} f_n$ ,  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} f$ , and  $\tilde{g} \stackrel{\text{def.}}{=} g \mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} g$ ; moreover,  $\tilde{g} \in \mathcal{L}^1$ ;
- then, since  $\lim_{n \rightarrow \infty} \tilde{f}_n = \tilde{f}$  and  $|\tilde{f}_n| \leq \tilde{g}$ , Lebesgue's dominated convergence theorem provides  $\tilde{f}_n, \tilde{f} \in \mathcal{L}^1$ , and the equality

$$\int \tilde{f} \, d\mu = \int \lim_{n \rightarrow \infty} \tilde{f}_n \, d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n \, d\mu;$$

- finally, the integrability of  $f_n$  and  $f$ , and identity (5.5) follow from the compatibility of the integral in  $\mathcal{M}$  with almost equality.

Lebesgue's extended dominated convergence theorem is used in the present document to prove the variant set in  $\mathcal{L}^p$  (for  $p \in [1, \infty)$ , see Section 5.2 and Theorem 1351).

For instance, Lebesgue's extended dominated convergence theorem may be used to prove the Leibniz integral rule (differentiation under the integral sign), e.g. for the study of integrals function of their upper bound.

### 5.4 Sketch of the proof of Lebesgue's dominated convergence theorem

#### Lebesgue's dominated convergence theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $g \in \mathcal{L}^1$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then,  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^1$ ,  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} f$ , and we have

$$(5.6) \quad \int f \, d\mu = \int \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (\text{in } \mathbb{R}).$$

See Theorem 1347. The proof of Lebesgue's dominated convergence theorem uses the arithmetic of  $\mathbb{R}$ , and Fatou's lemma (see Section 5.7). The proof goes as follows:

- monotonicity of integral in  $\mathcal{M}_+$  provides  $f_n, f \in \mathcal{L}^1$ ;
- the sequence  $g_n \stackrel{\text{def.}}{=} |f_n - f|$  is shown to be in  $\mathcal{L}^1 \cap \mathcal{M}_+$  with limit 0;



- the sequence  $2g - g_n$  is also shown to be in  $\mathcal{L}^1 \cap \mathcal{M}_+$ , with limit inferior  $2g$ ;
- then, Fatou's lemma and linearity of the integral in  $\mathcal{L}^1$  provide the inequality

$$2 \int g d\mu \leq 2 \int g d\mu - \limsup_{n \rightarrow \infty} \int g_n d\mu;$$

thus, a nonnegativeness argument provides the nullity of the limit of the integral of  $g_n$ 's, i.e. the convergence of the  $f_n$ 's towards  $f$  in  $\mathcal{L}^1$ ;

- finally, identity (5.6) follows from nondecreasingness of the integral in  $\mathcal{L}^1$ , the squeeze theorem, and linearity of the limit.

Lebesgue's dominated convergence theorem is used in the present document to prove an extended variant with lighter assumptions (see Section 5.3 and Theorem 1349).

Lebesgue's dominated convergence theorem admits another variant set in  $\mathcal{L}^p$  (for  $p \in [1, \infty)$ , see Theorem 1351).

For instance, it may be used to establish the Fourier inversion formula, or to study the Gibbs phenomenon.

## 5.5 Sketch of the proof of the Tonelli theorem

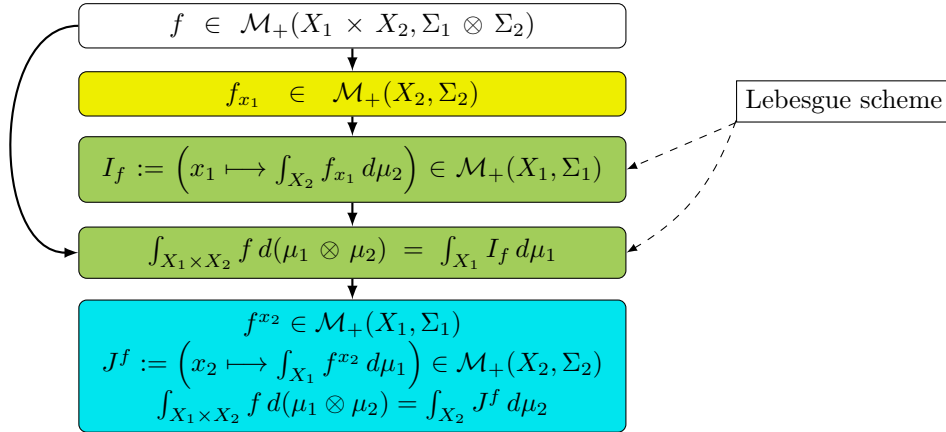


Figure 5.1: Path of the proof of Tonelli theorem for  $i = 1$ . The dashed arrows represent the main arguments for the proof.

### Tonelli theorem.

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $\psi$  be the permutation  $((x_i, x_j) \mapsto (x_1, x_2))$ . For all  $x_i \in X_i$ , let  $f_{x_i} \stackrel{\text{def.}}{=} (x_j \mapsto f \circ \psi(x_i, x_j))$ . Let  $I_{f,i} \stackrel{\text{def.}}{=} (x_i \mapsto \int f_{x_i} d\mu_j)$ . Then, for all  $x_i \in X_i$ ,  $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$ ,  $I_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have

$$(5.7) \quad \int f d(\mu_1 \otimes \mu_2) = \int I_{f,i} d\mu_i \quad (\text{in } \overline{\mathbb{R}}_+).$$

See Theorem 1242. The proof of the Tonelli theorem uses the arithmetic of  $\overline{\mathbb{R}}_+$ , and it follows steps 1 to 3 of the Lebesgue scheme (see Section 4.4). The proof goes as follows (see Figure 5.1):

- the result is first established for indicator functions from properties of the measure (see Section 5.6);

- then, the result is extended to nonnegative simple functions by taking (positive) linear combination of indicator functions and applying (positive) linearity of the integral in  $\mathcal{M}_+$ ;
- and finally, the result is extended to nonnegative measurable functions by taking the supremum of adapted sequences and applying the Beppo Levi (monotone convergence) theorem (see Section 5.8).

The Tonelli theorem is used in the present document to establish identities: for an integral over a subset in Lemma 1243, and for an integral of a tensor product function in Lemma 1244.

## 5.6 Sketch of the proof of existence and uniqueness of the tensor product measure

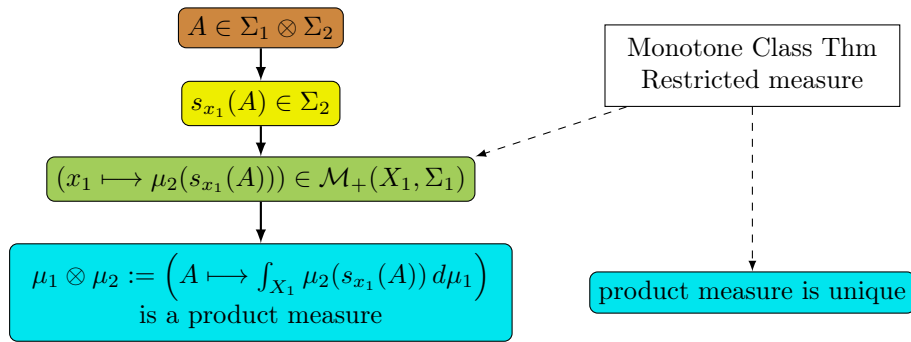


Figure 5.2: Description of the construction of the product measure induced by two  $\sigma$ -finite measures  $\mu_1$  and  $\mu_2$ . The dashed arrows represent the main arguments for the proof.

**Existence and uniqueness of the tensor product measure.**  
 Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Then, there exists a unique tensor product measure on  $(X, \Sigma)$ .  
 This measure is denoted  $\mu_1 \otimes \mu_2$ , and for all  $i \in \{1, 2\}$  with  $j \stackrel{\text{def.}}{=} 3 - i$ , we have

$$(5.8) \quad \forall A \in \Sigma, \quad (\mu_1 \otimes \mu_2)(A) = \int \mu_j(s_i(x_i, A)) d\mu_i.$$

It is called the (tensor) product measure of  $\mu_1$  and  $\mu_2$ , and  $(X, \Sigma, \mu_1 \otimes \mu_2)$  is called (tensor) product measure space.

See Lemmas 1221, 1223, 1227, 1231, and 1233. The proof of the existence and uniqueness of the tensor product measure suite of lemmas uses the arithmetic of  $\overline{\mathbb{R}}_+$ , the concepts of algebra of sets and monotone class (see Section 4.1), and the monotone class theorem (see Section 5.10). The proof goes as follows (see Figure 5.2):

- the tensor product algebra  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$  is the  $\sigma$ -algebra generated by  $\Sigma_1 \overline{\times} \Sigma_2 \stackrel{\text{def.}}{=} \{A_1 \times A_2 \mid A_1 \in \Sigma_1 \wedge A_2 \in \Sigma_2\}$ ;
- **existence:**
  - first, the sections of measurable sets of  $\Sigma$  are defined as  $s_{x_1}(A) \stackrel{\text{def.}}{=} \{x_2 \in X_2 \mid (x_1, x_2) \in A\}$ , see Figure 5.3;
  - then, the function  $(x_1 \mapsto \mu_2(s_{x_1}(A)))$  is proved to be measurable over  $(X_1, \Sigma_1)$ , in two stages:

1. if  $\mu_2$  is finite, the set  $\mathcal{S} := \{A \in \Sigma \mid (x_1 \mapsto \mu_2(s_{x_1}(A))) \in \mathcal{M}_+(X_1, \Sigma_1)\}$  is proved to be an algebra of set, and a monotone class (using continuity from below and from above of the measure); hence, thanks to the monotone class theorem,  $\mathcal{S} = \Sigma$ ;
  2. when  $\mu_2$  is  $\sigma$ -finite, the previous point is used with a restricted finite measure, and the result is extended by taking the supremum in  $\mathcal{M}_+(X_1, \Sigma_1)$ ;
- then, the tensor measure candidate can be defined as  $\mu(A) \stackrel{\text{def.}}{=} \int_{X_1} \mu_2(s_{x_1}(A)) d\mu_1$ ;
  - it is proved that  $\mu$  is indeed a product measure (i.e. such that for all  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ ,  $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ );

- **uniqueness:** the proof is done in two stages, that are similar to the ones for the existence.

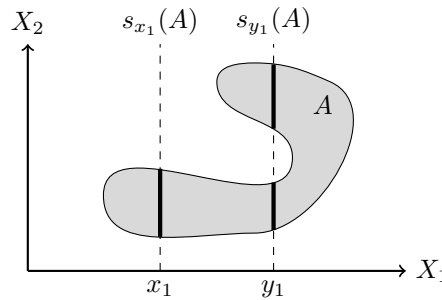


Figure 5.3: Examples of two sections of a set  $A \in \Sigma_1 \otimes \Sigma_2$ .

The Existence and uniqueness of the tensor product measure suite of lemmas is used in the present document to prove a property of measurable sections in Lemma 1234, the uniqueness of the Lebesgue measure on  $\mathbb{R}^2$  in Lemma 1235, and the Tonelli theorem (see Section 5.5 and Theorem 1242).

## 5.7 Sketch of the proof of Fatou's lemma

### Fatou's lemma.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then,  $\liminf_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have

$$(5.9) \quad \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \overline{\mathbb{R}}_+).$$

See Theorem 1208. The proof of Fatou's lemma uses the arithmetic of  $\overline{\mathbb{R}}_+$ , and the Beppo Levi (monotone convergence) theorem (see Section 5.8). The proof goes as follows:

- the sequence  $(\inf_{p \in \mathbb{N}} f_{n+p})_{n \in \mathbb{N}}$  is first shown to be pointwise nondecreasing in  $\mathcal{M}_+$ ;
- then, the Beppo Levi (monotone convergence) theorem provides the equality

$$\int \liminf_{n \in \mathbb{N}} f_n d\mu = \lim_{n \in \mathbb{N}} \int \inf_{p \in \mathbb{N}} f_{n+p} d\mu;$$

- and monotonicity of integral, and infimum and limit inferior definitions provide the inequality

$$\lim_{n \in \mathbb{N}} \int \inf_{p \in \mathbb{N}} f_{n+p} d\mu \leq \liminf_{n \in \mathbb{N}} \int f_n d\mu.$$

Fatou's lemma is used in the present document to prove properties of the integral in  $\mathcal{M}_+$ : identity for pointwise convergent sequences in Lemma 1209, Fatou's lemma variant set in  $\mathcal{L}^1$

(see Theorem 1345), and Lebesgue's dominated convergence theorem (see Section 5.4 and Theorem 1347).

The Beppo Levi (monotone convergence) theorem and Fatou's lemma can be established independently of one another, or each one can be obtained as a consequence of the other. Of course, those independent proofs do share a common ingredient to somehow obtain an upper bound for the integral of the limit by the limit of the integrals. We chose to establish the Beppo Levi (monotone convergence) theorem first.

## 5.8 Sketch of the proof of the Beppo Levi (monotone convergence) theorem

### Beppo Levi (monotone convergence) theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ .

Assume that the sequence is pointwise nondecreasing. Then,  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have

$$(5.10) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \overline{\mathbb{R}}_+).$$

See Theorem 1180. The proof of the Beppo Levi (monotone convergence) theorem uses the arithmetic of  $\overline{\mathbb{R}}_+$ , and it follows step 3 of the Lebesgue scheme by taking the ‘‘supremum’’ of quantities involving simple functions (see Section 4.4). The proof goes as follows:

- let  $f \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} f_n$ ; monotonicity and completeness arguments provide that  $f$  is also  $\sup_{n \rightarrow \infty} f_n$ , that it belongs to  $\mathcal{M}_+$ , and the ‘‘easy’’ inequality

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu;$$

- let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ ; using the nondecreasing sequence of measurable subsets  $(\{a\varphi \leq f_n\})_{n \in \mathbb{N}}$  for  $a \in (0, 1)$ , continuity from below of  $\mu$ , linearity, monotonicity and continuity arguments, allows to show the inequality

$$\int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu;$$

- then, taking the supremum over  $\varphi$ 's provides the other inequality

$$\int f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

The Beppo Levi (monotone convergence) theorem is used in the present document to prove properties of the integral in  $\mathcal{M}_+$  (possibly through Lemma 1182): homogeneity at  $\infty$  in Lemma 1181, additivity in Lemma 1183,  $\sigma$ -additivity in Lemma 1185, identity for the integral over a subset in Lemma 1199, Fatou's lemma (see Section 5.7 and Theorem 1208), identities for the integral with the trivial and counting measures in Lemmas 1210 and 1211, the Tonelli theorem (see Section 5.5 and Theorem 1242), and the Beppo Levi (monotone convergence) theorem variant set in  $\mathcal{L}^1$  (see Theorem 1343).

The Beppo Levi (monotone convergence) theorem admits another variant set in  $\mathcal{L}^p$  (for  $p$  in  $[1, \infty)$ , see Theorem 1352).

## 5.9 Sketch of the proof of Carathéodory's extension theorem

### Carathéodory's extension theorem.

Let  $X$  be a set. Let  $\mathcal{R}$  be a ring of subsets on  $X$  (i.e. closed under complement and union). Let  $\mu$  be a premeasure defined on  $\mathcal{R}$  (i.e. null on the empty set, and  $\sigma$ -additive). Then,  $\mu$  can be extended into a complete measure on a  $\sigma$ -algebra containing  $\mathcal{R}$  (i.e. for which all negligible subsets are measurable, and of measure 0). Moreover, this extension is unique when  $\mu$  is  $\sigma$ -finite.

See Theorem 1090 (in the specific case of  $X \stackrel{\text{def.}}{=} \mathbb{R}$  and  $\mathcal{R}$  is the ring generated by the intervals). The proof of Carathéodory's extension theorem uses the arithmetic of  $\overline{\mathbb{R}}_+$ , the concepts of  $\pi$ -systems and  $\lambda$ -systems (see Section 4.1), and the Dynkin  $\pi$ - $\lambda$  theorem (see Section 5.10); the existence part may also be known as the Carathéodory extension scheme (see Section 4.3). The proof goes as follows:

- the function  $\mu^*$  defined on  $\mathcal{P}(X)$  by (with the convention  $\inf \emptyset = \infty$ )

$$\forall A \subset X, \quad \mu^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid (A_n)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_n \right\}$$

is first shown to be an outer measure on  $X$  (i.e. null on the empty set, nonnegative, monotone, and  $\sigma$ -subadditive) whose restriction to  $\mathcal{R}$  is the premeasure  $\mu$ ;

- then, the set of subsets

$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \mid \forall A \subset X, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)\}$$

is shown to be a  $\sigma$ -algebra on  $X$  containing the ring  $\mathcal{R}$ , while the restriction  $\mu^*|_{\Sigma}$  is shown to be a complete measure on  $(X, \Sigma)$  (closedness of  $\Sigma$  under countable union comes from additivity of  $\mu^*$  on  $\Sigma$ );

- moreover, when  $\mu$  is  $\sigma$ -finite (with  $X = \biguplus_{n \in \mathbb{N}} X_n$  and  $\mu(X_n) < \infty$ ), two Carathéodory extensions  $\mu_1$  and  $\mu_2$  are first restricted to  $X_n$ , then the set  $\{\mu_1(A \cap X_n) = \mu_2(A \cap X_n)\}$  is shown to be equal to  $\Sigma$ , and finally uniqueness follows from  $\sigma$ -additivity of  $\mu_1$  and  $\mu_2$ .

In the present document, Carathéodory's extension theorem is stated (and proved) in the specific case of the building of the Lebesgue measure on  $\mathbb{R}$  (see Section 13.3).

## 5.10 Sketch of the proof of the Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem

Both theorems take advantage of the complementarity of being a  $\lambda$ -system and being a  $\pi$ -system (for the Dynkin  $\pi$ - $\lambda$  theorem), or being a monotone class and being an algebra of sets (for the monotone class theorem). Using the notation of Section 4.2 where  $\mathcal{U}_X^1$  represents the generated  $\lambda$ -system  $\Lambda_X$  (resp. the generated monotone class  $\mathcal{C}_X$ ), and  $\mathcal{U}_X^2$  represents the generated  $\pi$ -system  $\Pi_X$  (resp. the generated algebra of sets  $\mathcal{A}_X$ ), they both share the following abstract form.

### Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem.

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $\mathcal{U}_X^2(G) = G$ . Then,  $\mathcal{U}_X^1(G) = \Sigma_X(G)$ .

See Theorems 732 and 736. The proofs use the concepts of  $\pi$ -system and  $\lambda$ -system for the Dynkin  $\pi$ - $\lambda$  theorem, and of algebra of sets and monotone class for the monotone class theorem (see Section 4.1). Their common proof goes as follows:

- first, prove that  $\mathcal{U}_X^2(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$  (i.e. the property of  $G$  is transmitted to  $\mathcal{U}_X^1(G)$ );
- then, prove that  $\mathcal{U}_X^1(G) = \Sigma_X(G)$ .

In both cases, the most technical part is the second point. Note that we actually prove that for all  $G'$ ,  $\mathcal{U}_X^1(G') = G'$  and  $\mathcal{U}_X^2(G') = G'$  is equivalent to  $\Sigma_X(G') = G'$ .

The Dynkin  $\pi$ - $\lambda$  theorem is used in the present document (through Lemma 734) to extend the equality of two measures on a generator  $\pi$ -system to the whole  $\sigma$ -algebra in Lemma 992, which is used to establish uniqueness of the Lebesgue measure on  $\mathbb{R}$  in Theorem 1090 (see also Section 5.9). The monotone class theorem is used in the present document (through Lemma 738) to prove measurability of the measure of sections (in the case of finite measure spaces) in Lemma 1221, and uniqueness of the tensor product measure (also in the finite case) in Lemma 1231. All these three proofs follow the Dynkin  $\pi$ - $\lambda$  theorem / monotone class theorem scheme (see Section 4.2).

# Part II

## Detailed proofs

## Chapter 6

# Introduction

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

light gray is for remarks,   
 light green for definitions,  
light blue for lemmas,    and   
 light red for theorems.

Definitions and results have a number and a name. Inside the bodies of proofs, pertinent statements are referenced using both their number and name. When appropriate, some hints are given about the application, either to specify arguments, or to provide justification or consequences; they are underlined. Some useful definitions and results were already stated in [20], which was devoted to the detailed proof of the Lax–Milgram theorem. Those are numbered up to 206, and the statements in the present document are numbered starting from 207.

Furthermore, as in [20], the most basic results are supposed to be known and are not detailed further; they are displayed in **bold dark red**. These include:

- Logic: tautologies from propositional calculus.
- Set theory:
  - definition and properties of inclusion, intersection, (disjoint) union, complement, set difference, Cartesian product, power set, cardinality, and indicator function (that takes values 0 and 1), such as De Morgan’s laws, monotonicity of intersection and (partial) union, distributivity of intersection over (disjoint) union and set difference, distributivity of the Cartesian product over union, compatibility of intersection with Cartesian product,  $\sigma$ -additivity of the cardinality (with  $\infty$  absorbing element for addition in  $\overline{\mathbb{N}}$ );
  - definition and properties of equivalence and order binary relations;
  - definition and properties of function, composition of functions, inverse image (compatibility with set operations), injective and bijective functions, restriction and extension;
  - countability of finite Cartesian products of countable sets ( $\mathbb{N}^2$ ,  $\mathbb{Q}$ ,  $\mathbb{Q} \times \mathbb{Q}_+^*$ ).
- Algebraic structures: results from group theory.
- Topology: definition and properties of continuous functions.
- Real analysis:
  - properties of the ordered and valued field  $\mathbb{R}$  such as the Archimedean property, density of rational numbers, completeness, sum of the first terms of a geometric series;
  - definition and properties of basic numeric analytic functions such as square root, power function, exponential function, natural logarithm function, and exponentiation;
  - properties of limits in  $\mathbb{R}$  (cluster point), compactness of closed and bounded intervals, compatibility of limit with arithmetic operations, the squeeze theorem.



This part is organized as follows. Chapter 7 contains some results from various fields of mathematics (set theory, order theory, general topology, algebraic structures, real numbers and extended real numbers including second-countability), that are needed in the proofs of measure and integration theories. We recall that the material stated in [20] may also be used.

Then, Chapter 8 is devoted to subset systems, from  $\pi$ -system to  $\sigma$ -algebra. Measurability and measurable space are presented in Chapter 9, and the specific case of sets of real numbers and of extended real numbers is treated in Chapter 10. Chapter 11 is devoted to numeric functions over subset systems. Chapter 12 is dedicated to measure and measure space, and Chapter 13 to the specific cases involving sets of numbers, including the construction of the Lebesgue measure on  $\mathbb{R}$  through Carathéodory's extension theorem. Chapter 14 is devoted to simple functions, both in the scalar and vector cases. Then, the integral of nonnegative scalar functions is addressed in Chapter 15 with for instance the Beppo Levi (monotone convergence) theorem, Fatou's lemma, and the Tonelli theorem. Finally, Chapter 16 is dedicated to the integral of functions with arbitrary sign, with the seminormed vector spaces  $\mathcal{L}^p$ , Lebesgue's dominated convergence theorem and its variants, and the Riesz–Fischer theorem.

# Chapter 7

## Complements

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## 7.1 Complements on set theory

*Remark 207.* We recall that the definition of union provides the nullary union law  $\bigcup_{\emptyset} \stackrel{\text{def.}}{=} \emptyset$ . However, unless specified otherwise and provided a superset  $X$  is not ambiguous, we choose *not to use* the nullary intersection convention (“ $\bigcap_{\emptyset} \stackrel{\text{def.}}{=} X$ ” is generally not used).

*Remark 208.* We recall that  $\uplus$  denotes disjoint union, that  $A \setminus B \stackrel{\text{def.}}{=} A \cap B^c$  denotes the set difference, and that when  $B \subset A$ ,  $A \setminus B$  is the local complement of  $B$  (in  $A$ ), see Chapter 2.

### Definition 209 (pseudopartition).

Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Subsets  $(X_i)_{i \in I}$  of  $X$  are said to form a *pseudopartition* (of  $X$ ) iff

$$(7.1) \quad \forall i, j \in I, i \neq j \Rightarrow X_i \cap X_j = \emptyset, \quad \text{and} \quad X = \biguplus_{i \in I} X_i.$$

*Remark 210.* Note that, in contrast to proper partitions, parts of a pseudopartition may be empty.

**Lemma 211 (compatibility of pseudopartition with intersection).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $A, (X_i)_{i \in I} \subset X$ . Assume that  $(X_i)_{i \in I}$  form a pseudopartition of  $X$ . Then,  $(A \cap X_i)_{i \in I}$  form a pseudopartition of  $A$ , i.e.  $A = \biguplus_{i \in I} (A \cap X_i)$ .*

*Proof.* Direct consequence of Definition 209 (pseudopartition), and **distributivity of intersection over disjoint union**.  $\square$

**Lemma 212 (countable union is monotone).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I}, (B_i)_{i \in I} \subset X$ . Assume that for all  $i \in I$ ,  $\bigcup_{j \in [0..i]} A_j \subset \bigcup_{j \in [0..i]} B_j$ . Then, we have  $\bigcup_{i \in I} A_i \subset \bigcup_{i \in I} B_i$ .*

*Proof.* Let  $x \in \bigcup_{i \in I} A_i$ . Then, from **the definition of (countable) union**, there exists  $i_{A,x} \in I$  such that  $x \in A_{i_{A,x}} \subset \bigcup_{j \in [0..i_{A,x}]} A_j$ . Hence, there exists  $i_{B,x} \in [0..i_{A,x}]$  such that  $x \in B_{i_{B,x}}$ .

Therefore, we have  $\bigcup_{i \in I} A_i \subset \bigcup_{i \in I} B_i$ .  $\square$

**Lemma 213 (equality of countable union).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I}, (B_i)_{i \in I} \subset X$ . Assume that for all  $i \in I$ ,  $\bigcup_{j \in [0..i]} A_j = \bigcup_{j \in [0..i]} B_j$ . Then, we have  $\bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i$ .*

*Proof.* Direct consequence of Lemma 212 (countable union is monotone, applied twice, with  $A_i$ 's and  $B_i$ 's, then with  $B_i$ 's and  $A_i$ 's).  $\square$

*Remark 214.* The next lemma builds a pairwise disjoint sequence from any nonempty sequence of subsets. It is achieved by adding the set differences of the subsets layer by layer.

When the input sequence is nondecreasing, layers can be seen as onion peels.

**Lemma 215 (disjoint union of union).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \subset X$ . Let  $B_{\varphi(0)} \stackrel{\text{def.}}{=} A_{\varphi(0)}$ , and for all  $n \in J$  such that  $n+1 \in J$ , let  $B_{\varphi(n+1)} \stackrel{\text{def.}}{=} A_{\varphi(n+1)} \setminus \bigcup_{j \in [0..n]} B_{\varphi(j)}$ . Then,  $(B_i)_{i \in I}$  is pairwise disjoint, for all  $i \in I$ ,  $B_i \subset A_i$ , and we have*

$$(7.2) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)},$$

$$(7.3) \quad \bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i.$$

Moreover, if  $(A_i)_{i \in I}$  is nondecreasing, the  $B_{\varphi(n+1)}$  is the local complement of  $\biguplus_{j \in [0..n]} B_{\varphi(j)}$  in  $A_{\varphi(n+1)}$ .

*Proof. Pairwise disjointness.* Let  $p \in J$ . Assume that  $p + 1 \in J$ . Then, from **the definition of set difference**, and **De Morgan's laws**, we have

$$B_{\varphi(p+1)} = A_{\varphi(p+1)} \cap \left( \bigcup_{j \in [0..p]} B_{\varphi(j)} \right)^c = A_{\varphi(p+1)} \cap \bigcap_{j \in [0..p]} B_{\varphi(j)}^c.$$

Let  $j \in [0..p]$ . Then, from **properties of set operations**,  $B_{\varphi(p+1)}$  and  $B_{\varphi(j)}$  are disjoint.

Let  $m, n \in J$ . Assume that  $m \neq n$ , and let  $j \stackrel{\text{def.}}{=} \min(m, n)$  and  $p \stackrel{\text{def.}}{=} \max(m, n) - 1$ . Then, we have  $p + 1 \in J$  and  $j \in [0..p]$ . Hence, from **commutativity of intersection**, we have

$$B_{\varphi(m)} \cap B_{\varphi(n)} = B_{\varphi(p+1)} \cap B_{\varphi(j)} = \emptyset.$$

**Inclusion.** Direct consequence of the definition of the  $B_i$ 's, and **the definition of set difference**.

(7.2). For all  $n \in \mathbb{N}$ , let

$$P(n) \stackrel{\text{def.}}{=} \left( \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)} \right), \quad \text{and} \quad Q(n) \stackrel{\text{def.}}{=} (n \in J \Rightarrow P(n)).$$

**Induction:  $Q(0)$ .** Trivial.

**Induction:  $Q(n)$  implies  $Q(n + 1)$ .** Let  $n \in \mathbb{N}$ . Assume that  $Q(n)$  and  $n + 1 \in J$  hold.

Then,  $n \in J$  since  $J$  is an interval starting at 0. Thus,  $P(n)$  holds. Let  $B \stackrel{\text{def.}}{=} \biguplus_{j \in [0..n]} B_{\varphi(j)}$ . Then, from  $P(n)$ , and **properties of set operations (e.g.,  $A \cup B = A \uplus (B \setminus A)$ )**, we have

$$\bigcup_{j \in [0..n+1]} A_{\varphi(j)} = B \cup A_{\varphi(n+1)} = B \uplus (A_{\varphi(n+1)} \setminus B) = B \uplus B_{\varphi(n+1)} = \biguplus_{j \in [0..n+1]} B_{\varphi(j)}.$$

Thus, for all  $n \in \mathbb{N}$ ,  $Q(n)$  holds. Hence, for all  $n \in J$ ,  $P(n)$  holds.

(7.3). Direct consequence of **the definition of (countable) union**, the bijective property of  $\varphi$ , and Lemma 213 (**equality of countable union**, with  $I \stackrel{\text{def.}}{=} J$ ).

**With  $(A_i)_{i \in I}$  nondecreasing.** Let  $n \in J$ . Assume that  $n + 1 \in J$ .

Then, from monotonicity of  $\varphi$ , we have  $\biguplus_{j \in [0..n]} B_{\varphi(j)} = \bigcup_{j \in [0..n]} A_{\varphi(j)} = A_{\varphi(n)} \subset A_{\varphi(n+1)}$ .

Hence,  $B_{\varphi(n+1)}$  is the local complement of  $\biguplus_{j \in [0..n]} B_{\varphi(j)}$  in  $A_{\varphi(n+1)}$ .

Therefore, all five properties hold. □

*Remark 216.* The next lemma builds a nondecreasing sequence from any sequence of subsets. It is achieved by taking the partial unions.

**Lemma 217 (monotone union of union).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the (bijective) increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \subset X$ . For all  $n \in J$ , let  $B_{\varphi(n)} \stackrel{\text{def.}}{=} \bigcup_{j \in [0..n]} A_{\varphi(j)}$ .

Then,  $(B_i)_{i \in I}$  is nondecreasing, for all  $i \in I$ ,  $A_i \subset B_i$ , and we have

$$(7.4) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \bigcup_{j \in [0..n]} B_{\varphi(j)} (= B_{\varphi(n)}),$$

$$(7.5) \quad \bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i.$$

Moreover, if  $(A_i)_{i \in I}$  is pairwise disjoint, then the  $B_i$ 's are computed from the  $A_i$ 's using disjoint union.

*Proof. Nondecreasingness.* Let  $n \in J$ . Assume that  $n + 1 \in J$ . Then, from **associativity of union**, we have  $B_{\varphi(n+1)} = B_{\varphi(n)} \cup A_{\varphi(n+1)}$ . Hence, we have  $B_{\varphi(n)} \subset B_{\varphi(n+1)}$ .

**Inclusion.** Direct consequence of the definition of the  $B_i$ 's, and **the definition of union**.

(7.4). For all  $n \in J$ , let  $P(n) \stackrel{\text{def.}}{=} \left( \bigcup_{j \in [0..n]} B_{\varphi(j)} = B_{\varphi(n)} \right)$ .

**Induction:  $P(0)$ .** Trivial. **Induction:  $P(n)$  and  $n + 1 \in J$  implies  $P(n + 1)$ .** Let  $n \in J$ . Assume that  $P(n)$  holds and that  $n + 1 \in J$ . Then, from **associativity of union**, and (7.4), we have  $\bigcup_{j \in [0..n+1]} B_{\varphi(j)} = B_{\varphi(n)} \cup B_{\varphi(n+1)} = B_{\varphi(n+1)}$ . Hence, for all  $n \in J$ ,  $P(n)$  holds.

(7.5). Direct consequence of **the definition of (countable) union**, the bijective property of  $\varphi$ , and Lemma 213 (*equality of countable union*, with  $I \stackrel{\text{def.}}{=} J$ ).

**With  $(A_i)_{i \in I}$  pairwise disjoint.** Let  $n \in J$ . Then, we have  $B_{\varphi(n)} = \biguplus_{j \in [0..n]} A_{\varphi(j)}$ .

Therefore, all five properties hold.  $\square$

*Remark 218.* The next lemma builds an empty nonincreasing sequence from any nonempty sequence of subsets. It is achieved by taking local complements in the union.

Note that (7.7) cannot be compatible with the nullary intersection convention, hence the nonempty assumption on  $I$ . Moreover, note that in (7.8), the integer interval  $(n..n_I)$  is empty only when  $I$  is finite and  $n = n_I - 1$ , in which case we have  $B_{n_I-1} = \emptyset$ .

**Lemma 219 (empty monotone intersection of union).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \subset X$ . Let  $A \stackrel{\text{def.}}{=} \bigcup_{i \in I} A_i$ . Let  $B_{\varphi(0)} \stackrel{\text{def.}}{=} A \setminus A_{\varphi(0)}$ , and for all  $n \in J$  such that  $n + 1 \in J$ , let  $B_{\varphi(n+1)} \stackrel{\text{def.}}{=} B_{\varphi(n)} \setminus A_{\varphi(n+1)}$ .

Then,  $(B_i)_{i \in I}$  is nonincreasing, for all  $i \in I$ ,  $B_i \subset A$  and  $A_i \cap B_i = \emptyset$ , and we have

$$(7.6) \quad \forall n \in J, \quad A \setminus \bigcup_{j \in [0..n]} A_{\varphi(j)} = \bigcap_{j \in [0..n]} B_{\varphi(j)} (= B_{\varphi(n)}),$$

$$(7.7) \quad \emptyset = \bigcap_{i \in I} B_i.$$

Moreover, if  $(A_i)_{i \in I}$  is pairwise disjoint, then we have

$$(7.8) \quad \forall n \in J, \quad A = \biguplus_{j \in [0..n]} A_{\varphi(j)} \uplus B_{\varphi(n)} \quad \left( \text{i.e. } B_{\varphi(n)} = \biguplus_{j \in (n..n_I)} A_{\varphi(j)} \right).$$

*Proof. (1). Nonincreasingness, inclusion, and disjunction.*

Direct consequence of the definition of the  $B_i$ 's, and **the definition of local complement**.

(7.6). Direct consequence of the definition of the  $B_i$ 's, **the identity  $(A \setminus B) \setminus C = A \setminus (B \cup C)$** , (1), and **induction on  $n \in J$** .

(7.7). Direct consequence of **the lower bound property of intersection  $(\bigcap_{i \in I} B_i \subset B_i)$** , (1)  $(B_i \subset A, A_i^c)$ , **the greatest lower bound property of intersection  $(\bigcap_{i \in I} B_i \subset \bigcap_{i \in I} A_i^c)$** , and **De Morgan's laws  $(\bigcap_{i \in I} A_i^c = A^c)$** .

(7.8). Direct consequence of (7.6), and **the definition of local complement**.

Therefore, all six properties hold.  $\square$

*Remark 220.* The next lemma builds a nondecreasing sequence from any nonempty nonincreasing sequence of subsets. It is achieved by taking local complements.

**Lemma 221 (monotone union of monotone intersection).** Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \subset X$ . Let  $n_0 \in J$ . For all  $n \in J_{n_0} \stackrel{\text{def.}}{=} [n_0..n_I) \neq \emptyset$ , let  $B_n \stackrel{\text{def.}}{=} A_{\varphi(n_0)} \setminus A_{\varphi(n)}$ . Then, we have

$$(7.9) \quad \bigcup_{n \in J_{n_0}} B_n = A_{\varphi(n_0)} \setminus \bigcap_{n \in J_{n_0}} A_{\varphi(n)}.$$

Moreover, if  $(A_i)_{i \in I}$  is nonincreasing, then the  $B_n$ 's are computed from the  $A_i$ 's using local complement, and  $(B_n)_{n \in J_{n_0}}$  is nondecreasing.

*Proof.* Direct consequence of **left distributivity of set difference over intersection**, monotonicity of  $\varphi$ , and **monotonicity of set difference (right nonincreasingness)**.  $\square$

**Lemma 222 (technical inclusion for countable union).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \subset X$ . Let  $\varphi : I \rightarrow I$ . Then, we have  $\bigcup_{i \in I} A_{\varphi(i)} \subset \bigcup_{i \in I} A_i$ .

*Proof.* Let  $x \in \bigcup_{i \in I} A_{\varphi(i)}$ . then, from **the definition of union**, there exists  $i \in I$  such that  $x \in A_{\varphi(i)}$ . Thus, there exists  $j = \varphi(i) \in I$  such that  $x \in A_j$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{j \in I} A_j$ .

Therefore, we have  $\bigcup_{i \in I} A_{\varphi(i)} \subset \bigcup_{i \in I} A_i$ .  $\square$

**Lemma 223 (order is meaningless in countable union).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \subset X$ . Let  $\varphi : I \rightarrow I$ . Assume that  $\varphi$  is bijective. Then, we have  $\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_{\varphi(i)}$ .

*Proof.* Direct consequence of Lemma 222 (**technical inclusion for countable union**, with  $(A_i)_{i \in I}$  and  $\varphi$ , then  $(A_{\varphi(i)})_{i \in I}$  and  $\varphi^{-1}$  which satisfies  $\varphi \circ \varphi^{-1} = \text{Id}_I$ ).  $\square$

**Lemma 224 (definition of double countable union).**

Let  $X$  be a set. Let  $I, J, K \subset \mathbb{N}$ . For all  $(i, j) \in I \times J$ , let  $A_{i,j} \subset X$ . Let  $\varphi, \psi : K \rightarrow I \times J$ . Assume that  $\varphi$  and  $\psi$  are bijective. Then,  $\bigcup_{k \in K} A_{\varphi(k)} = \bigcup_{k \in K} A_{\psi(k)}$ . This union is denoted  $\bigcup_{(i,j) \in I \times J} A_{i,j}$ .

*Proof.* Direct consequence of **closedness of bijectivity by inverse and composition**, and Lemma 223 (**order is meaningless in countable union**, with  $I \stackrel{\text{def.}}{=} K$ ,  $(A_i)_{i \in I} \stackrel{\text{def.}}{=} (A_{\varphi(k)})_{k \in K}$  and  $\varphi \stackrel{\text{def.}}{=} \psi \circ \varphi^{-1}$ ).  $\square$

**Lemma 225 (double countable union).**

Let  $X$  be a set. Let  $I, J \subset \mathbb{N}$ . For all  $(i, j) \in I \times J$ , let  $A_{i,j} \subset X$ . Then, we have  $\bigcup_{i,j \in I \times J} A_{i,j} = \bigcup_{i \in I} \left( \bigcup_{j \in J} A_{i,j} \right)$ .

*Proof.* From **countability of  $\mathbb{N}^2$ , hence of  $I \times J$** , let  $K \subset \mathbb{N}$  and  $\varphi : K \rightarrow I \times J$  be a bijection. Then, from Lemma 224 (**definition of double countable union**), we have

$$\bigcup_{i,j \in I \times J} A_{i,j} = \bigcup_{k \in K} A_{\varphi(k)}$$

(and this does not depend on the choice for  $\varphi$ ).

Let  $x \in \bigcup_{k \in K} A_{\varphi(k)}$ . Then, from **the definition of union**, there exists  $k_0 \in K$  such that  $x \in A_{\varphi(k_0)}$ . Let  $(i_0, j_0) \stackrel{\text{def.}}{=} \varphi(k_0)$ . Thus, we have  $x \in A_{i_0, j_0}$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{j \in J} A_{i_0, j} \subset \bigcup_{i \in I} \left( \bigcup_{j \in J} A_{i, j} \right)$ .

Conversely, let  $x \in \bigcup_{i \in I} \left( \bigcup_{j \in J} A_{i, j} \right)$ . Then, from **the definition of union**, there exists  $i_0 \in I$  such that  $x \in \bigcup_{j \in J} A_{i_0, j}$ , and there exists  $j_0 \in J$  such that  $x \in A_{i_0, j_0}$ .

Let  $k_0 \stackrel{\text{def.}}{=} \varphi^{-1}(i_0, j_0)$ . Thus, we have  $x \in A_{\varphi(k_0)}$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{k \in K} A_{\varphi(k)}$ .

Therefore, we have  $\bigcup_{(i,j) \in I \times J} A_{i,j} = \bigcup_{i \in I} \left( \bigcup_{j \in J} A_{i,j} \right)$ .  $\square$

**Definition 226 (trace of subset system).** Let  $X$  be a set. Let  $Y \subset X$ . Let  $i$  be the canonical injection from  $Y$  to  $X$ . Let  $G \subset \mathcal{P}(X)$ . The notation  $G \bar{\cap} Y$  denotes the set

$$(7.10) \quad G \bar{\cap} Y \stackrel{\text{def.}}{=} i^{-1}(G) = \{A \cap Y \mid A \in G\}.$$

**Definition 227 (product of subset systems).**

Let  $m \in [2.. \infty)$ . For all  $i \in [1..m]$ , let  $X_i$  be a set, and  $G_i \subset \mathcal{P}(X_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . The notations  $G_1 \bar{\times} G_2$  and  $\bar{\prod}_{i \in [1..m]} G_i$  respectively denote the sets

$$(7.11) \quad G_1 \bar{\times} G_2 \stackrel{\text{def.}}{=} \{A_1 \times A_2 \in \mathcal{P}(X_1 \times X_2) \mid A_1 \in G_1 \wedge A_2 \in G_2\},$$

$$(7.12) \quad \bar{\prod}_{i \in [1..m]} G_i \stackrel{\text{def.}}{=} \left\{ \prod_{i \in [1..m]} A_i \in \mathcal{P}(X) \mid \forall i \in [1..m], A_i \in G_i \right\}.$$

When all the  $X_i$ 's are identical to some set  $Y$ , and all the  $G_i$ 's are identical to some  $G \subset \mathcal{P}(Y)$ , then the latter may be simply written  $G \bar{\times}^m$ ; it is a subset of  $\mathcal{P}(Y^m)$ .

## 7.2 Complements on order theory

*Remark 228.* In the next definition, the strict inequality “ $<$ ” naturally means “ $\leq$  and not equal” (equality is related to the set object).

We recall the notations  $\pm\infty$  to represent the infinite extrema of a totally ordered set; they may belong to the set, or not. See the notations in Chapter 2.

**Definition 229 (interval).** Let  $(X, \leq)$  be a totally ordered nonempty set. Let  $a, b \in X$ .

The subset  $(a, b) \stackrel{\text{def.}}{=} \{x \in X \mid a < x < b\}$  is called the *open proper interval from  $a$  to  $b$*  (with excluded bounds  $a$  and  $b$ ). The set of open proper intervals for all  $a, b \in X$  is denoted  $\mathcal{I}_X^{o,p}$ .

The subsets  $\{x \in X \mid x < b\}$  and  $\{x \in X \mid a < x\}$  are called *open left ray and open right ray* (with excluded bounds  $a$  and  $b$ ) (or *half-lines*); they are denoted  $(a, \infty]$  and  $[-\infty, b)$ . The set of open rays for all  $a, b \in X$  is denoted  $\mathcal{R}_X^o$ .

The notation  $[-\infty, \infty]$  may be used to represent the whole set  $X$ .

In all cases, square brackets “[, ]” are used to specify that bounds are included, and square-and-round brackets “[, )” are used to avoid specifying inclusion or exclusion of the bounds.

Let  $a, b \in X \cup \{\pm\infty\}$ . The subset  $[a, b]$  is either a proper interval, a ray, or the whole set; it is called *interval*. The set of open intervals is denoted  $\mathcal{I}_X^o \stackrel{\text{def.}}{=} \mathcal{I}_X^{o,p} \cup \mathcal{R}_X^o \cup \{X\}$ .

*Remark 230.* Note that if  $X$  contains at least two elements,  $\mathcal{I}_X^{o,p}$ ,  $\mathcal{R}_X^o$  and  $\{X\}$  are pairwise disjoint. In particular,  $(a, \infty)$  is an open ray when  $\infty \notin X$ , and an open proper interval when  $\infty \in X$ . More generally, open rays are never open proper intervals. For instance, when  $\infty \in X$ ,  $(a, \infty]$  is actually  $(a, \infty)$  (an open subset), which is distinct from the open proper interval  $(a, \infty)$ .

**Lemma 231 (equivalent definition of interval).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Let  $A \subset X$ .

Then,  $A$  is an interval iff for all  $a, b \in A$ ,  $(a, b) \subset A$ .

*Proof.* “Left” implies “right”.

Direct consequence of Definition 229 (interval), and **transitivity of order**.

“Right” implies “left”. From **the existence of the Dedekind–MacNeille completion of  $X$  (i.e. the smallest complete lattice that contains  $X$ , in which all subsets have both a supremum and an infimum)**, let  $\bar{X}$  be the Dedekind–MacNeille completion of  $X$ . Let  $\alpha \stackrel{\text{def.}}{=} \inf A$  and  $\beta \stackrel{\text{def.}}{=} \sup A$  in  $\bar{X}$ . Let us show that  $A = [\alpha, \beta]$ , or equivalently that  $(\alpha, \beta) \subset A$  (in  $X$ ), and  $A \subset [\alpha, \beta]$  (in  $\bar{X}$ ).

Let  $x \in X$ . Assume that  $\alpha < x < \beta$ . Then, from **the definition of infimum and supremum (greatest lower bound and least upper bound, contrapositive)** (e.g. see Definition 9 (infimum), and Definition 2 (supremum) in the case of  $X \stackrel{\text{def.}}{=} \mathbb{R}$ ), there exists  $a$  and  $b$  in  $A$  such that we have  $a < x < b$ . Thus,  $x \in A$ . Hence, we have  $(\alpha, \beta) \subset A$ .

Let  $x \in A$ . Then, from **the definition of infimum and supremum (lower and upper bounds)**, we have  $\alpha \leq x \leq \beta$ . Hence, we have  $A \subset [\alpha, \beta]$ .

Therefore, we have the equivalence. □

**Lemma 232 (empty open interval).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Assume that  $X$  is dense-in-itself:

$$(7.13) \quad \forall x, y \in X, \quad x < y \implies \exists z \in X, \quad x < z < y.$$

Let  $a, b \in X \cup \{\pm\infty\}$ . Then, the open interval  $(a, b)$  is empty iff  $b \leq a$ .



*Proof.* Then, from Definition 229 (*interval*), property (7.13), **transitivity of order (which provides the other implication)**, and **the definition of strict inequality**, we have

$$\begin{aligned} (a, b) = \emptyset &\iff \forall x \in X, \neg(a < x < b) \\ &\iff \neg(\exists x \in X, a < x < b) \iff \neg(a < b) \iff b \leq a. \end{aligned}$$

□

*Remark 233.* Of course, the previous statement is wrong in the presence of isolated points. For instance,  $(0, 1)$  is empty in the discrete sets  $\mathbb{N}$  and  $\mathbb{Z}$ .

*Remark 234.* In the next lemma, the left and right square-and-round brackets must remain the same on the sides of each interval. For instance “[<sub>1</sub>” denotes either “[” or “[”, but it remains identical in the three intervals of Equation (7.14).

**Lemma 235 (*intervals are closed under finite intersection*).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Then, the intersection of two proper intervals is a proper interval, the intersection of two rays is either a ray (if they both point towards the same direction), or a proper interval, and the intersection of a proper interval and a ray is a proper interval. In particular, for all  $a, b, c, d \in X \cup \{\pm\infty\}$ , we have (see remark above)

$$(7.14) \quad \llbracket_1 a, b \rrbracket_2 \cap \llbracket_1 c, d \rrbracket_2 = \llbracket_1 \max(a, c), \min(b, d) \rrbracket_2.$$

Hence, the closure of  $\mathcal{R}_X^o \cup \{X\}$  under finite intersection is  $\mathcal{I}_X^o$ . Moreover,  $\mathcal{I}_X^{o,p}$  and  $\mathcal{I}_X^o$  are closed under finite intersection.

*Proof.* Direct consequence of Definition 229 (*interval*), **totally ordered set properties of X**, and **induction on the number of operands of the finite intersection**. □

**Lemma 236 (*empty intersection of open intervals*).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Assume that  $X$  is dense-in-itself:

$$(7.15) \quad \forall x, y \in X, \quad x < y \implies \exists z \in X, \quad x < z < y.$$

Let  $a, b, c, d \in X \cup \{\pm\infty\}$ . Then,  $(a, b) \cap (c, d)$  is empty iff  $b \leq a$ ,  $d \leq c$ ,  $d \leq a$  or  $b \leq c$ .

*Proof.* Direct consequence of Lemma 235 (*intervals are closed under finite intersection*), Lemma 232 (*empty open interval*), and **totally ordered set properties of X**. □

*Remark 237.* We recall that  $\bar{\cap}$  denotes a set of traces of subsets, see Definition 226.

**Lemma 238 (*trace interval on a subset*).** Let  $(X, \leq)$  be a totally ordered nonempty set.

Let  $Y \subset X$ . Assume that  $Y$  is equipped with the induced total order.

Then, traces on  $Y$  of intervals of  $X$  coincide with intervals of  $Y$ , and in particular we have

$$(7.16) \quad \mathcal{I}_X^{o,p} \bar{\cap} Y = \mathcal{I}_Y^{o,p}, \quad \mathcal{R}_X^o \bar{\cap} Y = \mathcal{R}_Y^o, \quad \text{and} \quad \mathcal{I}_X^o \bar{\cap} Y = \mathcal{I}_Y^o.$$

*Proof.* Direct consequence of Definition 229 (*interval*), Definition 226 (*trace of subset system*), and **the definition of induced order**. □

## 7.3 Complements on general topology

### 7.3.1 Topological basis

*Remark 239.* In [20], we have only covered topology for metric spaces in which a subset is open iff it contains a nonempty open ball centered in each of its points. In the present document, we deal with the general case of topological spaces for which the collection of open subsets is given, e.g. via a subbase, see Definition 242, or a topological basis, see Definition 246.

**Definition 240 (topological space).** Let  $X$  be a set. A subset  $\mathcal{T}$  of  $\mathcal{P}(X)$  is called *topology* of  $X$  iff  $\emptyset, X \in \mathcal{T}$ , and  $\mathcal{T}$  is closed under any union and finite intersection.

In this case,  $(X, \mathcal{T})$  (or simply  $X$ ) is called *topological space*, elements of  $\mathcal{T}$  are called *open subsets of  $X$* , and the complement of elements of  $\mathcal{T}$  are called *closed subsets of  $X$* .

**Lemma 241 (intersection of topologies).**

Let  $X$  and  $I$  be sets. Let  $(\mathcal{T}_i)_{i \in I}$  be topologies on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{T}_i$  is a topology on  $X$ .

*Proof.* Direct consequence of Definition 240 (*topological space*), and **the definition and properties of union and intersection of subsets**.  $\square$

**Definition 242 (generated topology).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The *topology on  $X$  generated by  $G$*  is the intersection of all topologies on  $X$  containing  $G$ ; it is denoted  $\mathcal{T}_X(G)$ . The generator  $G$  is also called *subbase of the topology*.

**Lemma 243 (generated topology is minimum).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{T}_X(G)$  is the smallest topology on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 242 (*generated topology*), Lemma 241 (*intersection of topologies*), and **properties of the intersection**.  $\square$

**Lemma 244 (equivalent definition of generated topology).** Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Let  $O \subset X$ . Then,  $O \in \mathcal{T}_X(G)$  iff  $O$  is the union of finite intersections of elements of  $G \cup \{X\}$ .

*Proof.* Direct consequence of Definition 240 (*topological space*), **associativity and commutativity of union, distributivity of union and intersection (both ways, thus the set of unions of finite intersections of elements of  $G \cup \{X\}$  is a topology of  $X$  containing  $G$ , and contained in all topologies of  $X$  containing  $G$ )**, and Lemma 243 (*generated topology is minimum*).  $\square$

**Lemma 245 (topology generation is monotone).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1 \subset G_2$ . Then, we have  $\mathcal{T}_X(G_1) \subset \mathcal{T}_X(G_2)$ .

*Proof.* Direct consequence of **transitivity of inclusion**, Lemma 243 (*generated topology is minimum*), with  $G_2$ , thus  $G_2 \subset \mathcal{T}_X(G_2)$ , then with  $G_1$  and  $\mathcal{T}_X(G_2)$  being a topology containing  $G_1$ .  $\square$

**Definition 246 (topological basis).**

Let  $(X, \mathcal{T})$  be a topological space. A subset  $\mathcal{B}$  of  $\mathcal{T}$  is called *topological basis of  $(X, \mathcal{T})$*  iff

$$(7.17) \quad \forall O \in \mathcal{T}, \quad \exists J, \exists (B_j)_{j \in J} \in \mathcal{B}, \quad O = \bigcup_{j \in J} B_j.$$

**Lemma 247 (equivalent definition of topological basis).** *Let  $X$  and  $I$  be sets. Let  $\mathcal{B} \stackrel{\text{def.}}{=} \{B_i \mid i \in I\} \subset \mathcal{P}(X)$ . Then,  $\mathcal{B}$  is a topological basis of  $(X, \mathcal{T}_X(\mathcal{B}))$  iff  $\mathcal{B}$  covers  $X$ , and any intersection of two elements of  $\mathcal{B}$  is the union of elements of  $\mathcal{B}$ :*

$$(7.18) \quad X = \bigcup_{i \in I} B_i,$$

$$(7.19) \quad \forall B, B' \in \mathcal{B}, \quad \exists J \subset I, \quad B \cap B' = \bigcup_{j \in J} B_j.$$

If so,  $\mathcal{T}_X(\mathcal{B})$  reduces to the set of unions of elements of  $\mathcal{B}$ .

Moreover, if  $\mathcal{B} \cup \{\emptyset\}$  is closed under intersection, then (7.19) is satisfied.

*Proof.* **“Left” implies “right”.** Direct consequence of Lemma 243 (*generated topology is minimum*,  $\mathcal{T}_X(\mathcal{B})$  is a topology on  $X$  containing  $\mathcal{B}$ ), Definition 240 (*topological space*,  $X \in \mathcal{T}_X(\mathcal{B})$  and closedness under finite intersection, thus  $B \cap B' \in \mathcal{T}_X(\mathcal{B})$ ), and Definition 246 (*topological basis*, with  $O \stackrel{\text{def.}}{=} X$ , then  $O \stackrel{\text{def.}}{=} B \cap B'$ ).

**“Right” implies “left”.** Let  $\mathcal{T} \stackrel{\text{def.}}{=} \{\bigcup_{j \in J} B_j \mid J \subset I\}$ . Then, by construction, we have  $\mathcal{B} \subset \mathcal{T}$  (with  $\text{card}(J) = 1$ ),  $\emptyset \in \mathcal{T}$  (with  $\text{card}(J) = 0$ ),  $X \in \mathcal{T}$  (with  $J \stackrel{\text{def.}}{=} I$  and (7.18)).

Let  $K$  be a set. Let  $(A_k)_{k \in K} \in \mathcal{T}$ . Then, by construction, for each  $k \in K$ , there exists  $J_k \subset I$  such that  $A_k = \bigcup_{j_k \in J_k} B_{j_k}$ . Let  $L \stackrel{\text{def.}}{=} \bigcup_{k \in K} J_k$ . Then, from **the definition of union**, we have  $L \subset I$ , and  $\bigcup_{k \in K} A_k = \bigcup_{l \in L} B_l \in \mathcal{T}$ , i.e.  $\mathcal{T}$  is closed under any union.

Hence, from **distributivity of intersection over union**, (7.19), **associativity of union (thus  $\mathcal{T}$  is closed under intersection)**, **induction on the number of operands (thus  $\mathcal{T}$  is closed under finite intersection)**, Definition 240 (*topological space*,  $\mathcal{T}$  is a topology on  $X$ ), Lemma 243 (*generated topology is minimum*,  $\mathcal{T}_X(\mathcal{B}) \subset \mathcal{T}$ ), Lemma 244 (*equivalent definition of generated topology*,  $\mathcal{T} \subset \mathcal{T}_X(\mathcal{B})$ ), and Definition 246 (*topological basis*),  $\mathcal{B}$  is a topological basis of  $(X, \mathcal{T})$  with  $\mathcal{T} = \mathcal{T}_X(\mathcal{B})$ .

Therefore, we have the equivalence, and the characterization of  $\mathcal{T}_X(\mathcal{B})$ .

#### Additional property.

Trivial (with  $\text{card}(J) = 1$  when  $B \cap B' \in \mathcal{B}$ , and  $\text{card}(J) = 0$  when  $B \cap B' = \emptyset$ ).  $\square$

**Lemma 248 (augmented topological basis).** *Let  $(X, \mathcal{T})$  be a topological space. Let  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Let  $\mathcal{B}' \subset \mathcal{P}(X)$  such that  $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{T}$ . Then,  $\mathcal{B}'$  is a topological basis of  $(X, \mathcal{T})$ .*

*Proof.* Direct consequence of Definition 246 (*topological basis*, with  $\mathcal{B}' \stackrel{\text{def.}}{=} \{B_{i'} \mid i' \in I'\}$  such that  $I \subset I'$ ), and **transitivity of inclusion ( $J \subset I \Rightarrow J \subset I'$ )**.  $\square$

**Lemma 249 (topological basis is generator).** *Let  $(X, \mathcal{T})$  be a topological space. Let  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Then, we have  $\mathcal{T} = \mathcal{T}_X(\mathcal{B})$ .*

*Proof.* Direct consequence of Definition 246 (*topological basis*), and Lemma 244 (*equivalent definition of generated topology*).  $\square$

**Lemma 250 (topological subbasis).** *Let  $(X, \mathcal{T})$  be a topological space. Let  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Let  $\mathcal{B}' \subset \mathcal{P}(X)$ . Assume that elements of  $\mathcal{B}$  are unions of elements of  $\mathcal{B}'$ ,*

$$(7.20) \quad \forall B \in \mathcal{B}, \quad \exists J, \exists (B'_j)_{j \in J} \in \mathcal{B}', \quad B = \bigcup_{j \in J} B'_j.$$

*Then,  $\mathcal{B}'$  is a topological basis of  $(X, \mathcal{T})$ , and  $\mathcal{T} = \mathcal{T}_X(\mathcal{B}) = \mathcal{T}_X(\mathcal{B}')$ .*

*Proof.* Let  $O \in \mathcal{T}$ . Then, from Definition 246 (*topological basis*), let  $J$  be a set, and let  $(B_j)_{j \in J} \in \mathcal{B}$  such that  $O = \bigcup_{j \in J} B_j$ . Then, for all  $j \in J$ , let  $K_j$  be a set, and let  $(B'_{k_j})_{k_j \in K_j} \in \mathcal{B}'$  such that  $B_j = \bigcup_{k_j \in K_j} B'_{k_j}$ . Let  $K \stackrel{\text{def.}}{=} \bigcup_{j \in J} K_j$ . Hence,  $(B'_k)_{k \in K} \in \mathcal{B}'$ , and from **associativity of union**, we have  $O = \bigcup_{j \in J} \left( \bigcup_{k_j \in K_j} B'_{k_j} \right) = \bigcup_{k \in K} B'_k$ .

Therefore, from Definition 246 (*topological basis*), and Lemma 249 (*topological basis is generator*),  $\mathcal{B}'$  is a topological basis of  $(X, \mathcal{T})$ , and  $\mathcal{T} = \mathcal{T}_X(\mathcal{B}) = \mathcal{T}_X(\mathcal{B}')$ .  $\square$

*Remark 251.* We recall that  $\bar{\cap}$  denotes a set of traces of subsets, see Definition 226.

**Lemma 252 (trace topology on subset).**

Let  $(X, \mathcal{T})$  be a topological space, and  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Let  $Y \subset X$ .

Then,  $\mathcal{T}_Y \stackrel{\text{def.}}{=} \mathcal{T} \bar{\cap} Y$  is a topology of  $Y$ , and  $\mathcal{B} \bar{\cap} Y$  is a topological basis of  $(Y, \mathcal{T}_Y)$ .

Moreover, if  $Y \in \mathcal{T}$  is an open subset of  $X$ , then we have  $\mathcal{T}_Y \subset \mathcal{T}$ .

$\mathcal{T}_Y$  is called trace topology (induced by  $\mathcal{T}$ ), and  $(Y, \mathcal{T}_Y)$  is said topological subspace of  $(X, \mathcal{T})$ .

*Proof.* Direct consequence of Definition 240 (*topological space*), Definition 246 (*topological basis*), Definition 226 (*trace of subset system*), **distributivity of intersection over union**, and **commutativity of intersection**.  $\square$

*Remark 253.* We recall that  $\bar{\prod}$  denotes a set of products of subsets, see Definition 227.

The next lemma defines a topology on a finite Cartesian product of topological spaces based on the products of open subsets in each space. In the finite case, the box topology and the product topology are the same. However, when the set of indices is infinite, the former becomes too fine, and the latter is usually preferred as it has more “desirable” properties. The difference is that the product topology is generated by products where only a finite number of operands are distinct from the corresponding full space; it is the coarsest topology making projection maps continuous.

**Lemma 254 (box topology on Cartesian product).**

Let  $m \in [2, \infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \mathcal{T}_i)$  be a topological space. Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . Let  $\bar{\mathcal{T}} \stackrel{\text{def.}}{=} \bar{\prod}_{i \in [1..m]} \mathcal{T}_i$ . Let  $\mathcal{T} \stackrel{\text{def.}}{=} \mathcal{T}_X(\bar{\mathcal{T}})$ . Then,  $\bar{\mathcal{T}}$  is a topological basis of  $(X, \mathcal{T})$ .  $\mathcal{T}$  is called the box topology on  $X$  (induced by the  $\mathcal{T}_i$ 's), or the product topology on  $X$ .

Moreover, for all  $i \in [1..m]$ , let  $\mathcal{B}_i$  be a topological basis of  $(X_i, \mathcal{T}_i)$ . Let  $\bar{\mathcal{B}} \stackrel{\text{def.}}{=} \bar{\prod}_{i \in [1..m]} \mathcal{B}_i$ . Then,  $\bar{\mathcal{B}}$  is a topological basis of  $(X, \mathcal{T})$ .

*Proof.* From Definition 240 (*topological space*), for all  $i \in [1..m]$ , we have  $X_i \in \mathcal{T}_i$  and  $\mathcal{T}_i$  is closed under intersection. Then, from **compatibility of intersection with Cartesian product** ( $\prod_{i \in I} B_i \cap \prod_{i \in [1..m]} B'_i = \prod_{i \in [1..m]} B_i \cap B'_i$ ),  $\bar{\mathcal{T}}$  is closed under intersection. Hence, from Lemma 247 (*equivalent definition of topological basis, with  $X, \emptyset \in \bar{\mathcal{T}}$* ),  $\bar{\mathcal{T}}$  is a topological basis of  $(X, \mathcal{T})$ .

Let  $O \in \mathcal{T}$ . Then, from Definition 227 (*product of subset systems*), and Definition 246 (*topological basis, with  $\bar{\mathcal{T}}$* , then each  $\mathcal{B}_i$ ), we have

$$\begin{aligned} \exists K, \exists (O^k)_{k \in K} \in \bar{\mathcal{T}}, \quad O &= \bigcup_{k \in K} O^k, \\ \forall k \in K, \exists (O_i^k)_{i \in [1..m]} \in \bar{\mathcal{T}}, \quad O^k &= \prod_{i \in [1..m]} O_i^k, \\ \forall k \in K, \forall i \in [1..m], \exists J_i^k, \exists (B_i^{k, j_i})_{j_i \in J_i^k} \in \mathcal{B}_i, \quad O_i^k &= \bigcup_{j_i \in J_i^k} B_i^{k, j_i}. \end{aligned}$$

For all  $k \in K$ , let  $J^k \stackrel{\text{def.}}{=} J_1^k \times \dots \times J_m^k$ . Let  $L \stackrel{\text{def.}}{=} \{(k, j) \mid k \in K \wedge j \in J^k\}$ . Then, for all  $l \stackrel{\text{def.}}{=} (k, (j_1, \dots, j_m)) \in L$ , let  $C^l \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} B_i^{k, j_i}$ . Then, from **distributivity of Cartesian product over union**, Definition 227 (*product of subset systems*,  $C^l \in \overline{\mathcal{B}}$ ), **associativity of union**, and Definition 246 (*topological basis*), we have

$$O = \bigcup_{k \in K} \left( \prod_{i \in [1..m]} \left( \bigcup_{j_i \in J_i^k} B_i^{k, j_i} \right) \right) = \bigcup_{k \in K} \bigcup_{j \in J^k} \left( \prod_{i \in [1..m]} B_i^{k, j_i} \right) = \bigcup_{l \in L} C^l,$$

and  $\overline{\mathcal{B}}$  is a topological basis of  $(X, \mathcal{T})$ .  $\square$

*Remark 255.* In the next definition, see Definition 229 for the definition of  $\mathcal{R}_X^o$ .

Note that the order topology is the standard topology on the totally ordered sets of numbers  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ , see Lemmas 307 and 308.

**Definition 256 (order topology).**

Let  $(X, \leq)$  be a totally ordered set. The topology  $\mathcal{T}_X(\mathcal{R}_X^o)$  is called *order topology on X*.

*Remark 257.* In the next lemma, see Definition 229 (*interval*) for the definition of  $\mathcal{I}_X^o$ .

**Lemma 258 (topological basis of order topology).**

Let  $(X, \leq)$  be a totally ordered set. Then,  $\mathcal{I}_X^o$  is a topological basis for the order topology on  $X$ .

*Proof.* Direct consequence of Definition 256 (*order topology*), Lemma 244 (*equivalent definition of generated topology*), Lemma 235 (*intervals are closed under finite intersection*), and Definition 246 (*topological basis*).  $\square$

**Lemma 259 (trace of order topology).**

Let  $(X, \leq)$  be a totally ordered set. Let  $Y \subset X$ . Assume that  $X$  is equipped with the order topology  $\mathcal{T}$ , and that  $Y$  is equipped with the induced total order. Then, the trace topology  $\mathcal{T}_Y$  and the order topology  $\mathcal{T}_Y(\mathcal{R}_Y^o)$  on  $Y$  are the same,  $\mathcal{I}_Y^o$  is a topological basis of both, and when  $Y \in \mathcal{T}$ , we have  $\mathcal{T}_Y \subset \mathcal{T}$ .

*Proof.* Direct consequence of Lemma 258 (*topological basis of order topology*,  $\mathcal{I}_X^o$  is a topological basis of  $(X, \mathcal{T})$  and  $\mathcal{I}_Y^o$  is a topological basis of  $(Y, \mathcal{T}_Y(\mathcal{R}_Y^o))$ ), Lemma 252 (*trace topology on subset*,  $\mathcal{I}_X^o \cap Y$  is a topological basis of  $(Y, \mathcal{T}_Y)$  and  $Y \in \mathcal{T} \Rightarrow \mathcal{T}_Y \subset \mathcal{T}$ ), and Lemma 238 (*trace interval on a subset*,  $\mathcal{I}_X^o \cap Y = \mathcal{I}_Y^o$ , thus both topologies on  $Y$  share the same topological basis, they are the same).  $\square$

### 7.3.2 Second axiom of countability

**Definition 260 (second-countability).** A topological space  $(X, \mathcal{T})$  is said *second-countable*, or *completely separable*, iff it admits a countable topological basis,

**Lemma 261 (compatibility of second-countability with Cartesian product).**

Let  $m \in [2.. \infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \mathcal{T}_i)$  be a second-countable topological space, and let  $\mathcal{B}_i$  be a countable topological basis of  $(X_i, \mathcal{T}_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . Assume that  $X$  is equipped with the box topology  $\mathcal{T} \stackrel{\text{def.}}{=} \mathcal{T}_X(\prod_{i \in [1..m]} \mathcal{T}_i)$ .

Then,  $\prod_{i \in [1..m]} \mathcal{B}_i$  is a countable topological basis of  $(X, \mathcal{T})$ . Hence,  $(X, \mathcal{T})$  is second-countable.

*Proof.* Direct consequence of Lemma 254 (*box topology on Cartesian product*), Definition 227 (*product of subset systems*), **compatibility of finite Cartesian product with countability**, and Definition 260 (*second-countability*).  $\square$

### 7.3.3 Complements on metric space

**Definition 262 (pseudometric).**

Let  $X$  be a nonempty set. A function  $d : X \times X \rightarrow \mathbb{R}$  is called *pseudometric over  $X$*  iff it is nonnegative, symmetric, it is zero on the diagonal, and it satisfies the triangle inequality:

$$(7.21) \quad \forall x, y \in X, \quad d(x, y) \geq 0,$$

$$(7.22) \quad \forall x, y \in X, \quad d(y, x) = d(x, y),$$

$$(7.23) \quad \forall x \in X, \quad d(x, x) = 0,$$

$$(7.24) \quad \forall x, y, z \in X, \quad d(x, z) \leq d(x, y) + d(y, z).$$

If so,  $(X, d)$  (or simply  $X$ ) is called *pseudometric space*.

*Remark 263.* A pseudometric becomes a metric when it is also definite. Hence, most results from [20] on metric spaces are still valid on pseudometric spaces.

Note that a pseudometric does not separate points, hence there is no uniqueness of the limit (contrary to metric space, see Lemma 29).

**Lemma 264 (pseudo-point separation).** Let  $(X, d)$  be a pseudometric space. Let  $x, y \in X$ . Assume that for all  $\varepsilon > 0$ ,  $d(x, y) \leq \varepsilon$ . Then,  $d(x, y) = 0$ .

*Proof.* Assume that  $d(x, y) > 0$ . Let  $\varepsilon = \frac{d(x, y)}{2}$ . Then, from **ordered field properties of  $\mathbb{R}$  (with  $d(x, y) > 0$ )**, we have  $\varepsilon > 0$  and  $0 < d(x, y) \leq \varepsilon = \frac{d(x, y)}{2}$ , i.e.  $0 < 1 \leq \frac{1}{2}$ , which is wrong. Therefore, from Definition 262 (*pseudometric*,  $d$  is nonnegative), we have  $d(x, y) = 0$ .  $\square$

**Lemma 265 (equivalent definition of convergent sequence).**

Let  $(X, d)$  be a pseudometric space. Let  $(x_n)_{n \in \mathbb{N}}, l \in X$ . Then,  $x_n \xrightarrow[n \rightarrow \infty]{} l$  iff

$$(7.25) \quad \forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geq N \implies d(x_n, l) \leq \frac{1}{k+1}.$$

*Proof.* “Left” implies “right”.

Direct consequence of Definition 27 (*convergent sequence*, with  $\varepsilon \stackrel{\text{def.}}{=} \frac{1}{k+1}$ ).

“Right” implies “left”. Assume that property (7.25) holds. Let  $\varepsilon > 0$ . Then, from **the Archimedean property of  $\mathbb{R}$** , and **ordered field properties of  $\mathbb{R}$** , let  $k \in \mathbb{N}$  such that  $k \geq \frac{1}{\varepsilon} - 1$ , i.e.  $\frac{1}{k+1} \leq \varepsilon$ . Thus, from assumption, there exists  $N \in \mathbb{N}$  such that for all  $n \in [N, \infty)$ , we have  $d(x_n, l) \leq \frac{1}{k+1} \leq \varepsilon$ . Hence, from Definition 27 (*convergent sequence*), the sequence  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $l$ .

Therefore, we have the equivalence.  $\square$

**Lemma 266 (compatibility of subsequence with convergence).**

Let  $(X, d)$  be a pseudometric space. Let  $(x_n)_{n \in \mathbb{N}}, l \in X$ . Let  $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}$ .

Assume that  $x_n \xrightarrow[n \rightarrow \infty]{} l$ , and  $(n_k)_{k \in \mathbb{N}}$  is increasing. Then, we also have  $x_{n_k} \xrightarrow[k \rightarrow \infty]{} l$ .

*Proof.* Let  $\varepsilon > 0$ . Then, from Definition 27 (*convergent sequence*), let  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $d(x_n, x) \leq \varepsilon$ . Thus, since  $(n_k)_{k \in \mathbb{N}}$  is increasing, let  $K \in \mathbb{N}$  such that for all  $k \geq K$ , we have  $n_k > N$ . Hence, we have  $d(x_{n_k}, x) \leq \varepsilon$ . Therefore, from Definition 27 (*convergent sequence*),  $(x_{n_k})_{k \in \mathbb{N}}$  is convergent to  $x$ .  $\square$

**Definition 267 (cluster point).**

Let  $X$  be a topological space. Let  $(x_n)_{n \in \mathbb{N}} \in X$ . A *cluster point* of the sequence is the limit  $x$  of any convergent subsequence of  $(x_n)_{n \in \mathbb{N}}$ .

When  $(X, d)$  is a pseudometric space, this can take the following form:

$$(7.26) \quad \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, \quad d(x_n, x) \leq \varepsilon.$$

**Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)).**

Let  $(X, d)$  be a pseudometric space. Let  $(x_n)_{n \in \mathbb{N}} \in X$ . Then,  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence iff

$$(7.27) \quad \forall r \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n, k \in \mathbb{N}, \quad n \geq N \implies d(x_{n+k}, x_n) \leq \frac{1}{r+1}.$$

*Proof.* Same proof as Lemma 265 (equivalent definition of convergent sequence), but using Definition 35 (Cauchy sequence).  $\square$

**Lemma 269 (convergent subsequence of Cauchy sequence).**

Let  $(X, d)$  be a pseudometric space. Let  $(x_n)_{n \in \mathbb{N}}, l \in X$ . Let  $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}$ . Assume that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence,  $(n_k)_{k \in \mathbb{N}}$  is increasing, and  $x_{n_k} \xrightarrow[k \rightarrow \infty]{} l$ . Then, we also have  $x_n \xrightarrow[n \rightarrow \infty]{} x$ .

*Proof.* Let  $\varepsilon > 0$ . Then, from Definition 35 (Cauchy sequence), let  $N \in \mathbb{N}$  such that for all  $p, q \geq N$ , we have  $d(x_p, x_q) \leq \frac{\varepsilon}{2}$ . Thus, since  $(n_k)_{k \in \mathbb{N}}$  is increasing, let  $K' \in \mathbb{N}$  such that for all  $k \geq K'$ , we have  $n_k > N$ , and from Definition 27 (convergent sequence), let  $K'' \in \mathbb{N}$  such that for all  $k \geq K''$ , we have  $d(x_{n_k}, x) \leq \frac{\varepsilon}{2}$ . Let  $K \stackrel{\text{def.}}{=} \max(K', K'')$ . Then, from Definition 262 (pseudometric, triangle inequality), we have for all  $n \geq N$ ,

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from Definition 27 (convergent sequence),  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $x$ .  $\square$



## 7.4 Complements on algebraic structures

### 7.4.1 Vector space

**Definition 270 (relation compatible with vector operations).**

Let  $\mathbb{K}$  be a field. Let  $(E, +_E, \cdot_E)$  be a  $\mathbb{K}$ -vector space. An equivalence relation  $\mathcal{R}$  on  $E$  is said *compatible with the vector operations* iff

$$(7.28) \quad \forall u, u', v, v' \in E, \quad u \mathcal{R} u' \wedge v \mathcal{R} v' \implies (u +_E v) \mathcal{R} (u' +_E v'),$$

$$(7.29) \quad \forall \lambda \in \mathbb{K}, \forall u, u' \in E, \quad u \mathcal{R} u' \implies (\lambda \cdot_E u) \mathcal{R} (\lambda \cdot_E u').$$

**Lemma 271 (quotient vector operations).**

Let  $\mathbb{K}$  be a field. Let  $(E, +_E, \cdot_E)$  be a  $\mathbb{K}$ -vector space. Let  $\mathcal{R}$  be an equivalence relation on  $E$ . Assume that  $\mathcal{R}$  is compatible with the vector operations. Then, the mappings  $[+] : E/\mathcal{R} \times E/\mathcal{R} \rightarrow E/\mathcal{R}$  and  $[\cdot] : \mathbb{K} \times E/\mathcal{R} \rightarrow E/\mathcal{R}$  defined by

$$(7.30) \quad \forall u, v \in E, \quad [u] [+] [v] \stackrel{\text{def.}}{=} [u +_E v],$$

$$(7.31) \quad \forall \lambda \in \mathbb{K}, \forall u \in E, \quad \lambda [\cdot] [u] \stackrel{\text{def.}}{=} [\lambda \cdot_E u].$$

are well-defined. These mappings are called *quotient vector operations* induced on  $E/\mathcal{R}$ .

*Proof.* Let  $u, v \in E$ . Let  $\lambda \in \mathbb{K}$ . Let  $u', v' \in E$  such that  $u \mathcal{R} u'$  and  $v \mathcal{R} v'$ , i.e. such that  $[u'] = [u]$  and  $[v'] = [v]$ . Then, from Definition 270 (relation compatible with vector operations), we have  $(u +_E v) \mathcal{R} (u' +_E v')$  and  $(\lambda \cdot_E u) \mathcal{R} (\lambda \cdot_E u')$ , i.e.  $[u' +_E v'] = [u +_E v]$  and  $[\lambda \cdot_E u'] = [\lambda \cdot_E u]$ . Therefore, the quotient vector operations  $[+]$  and  $[\cdot]$  defined by (7.30) and (7.31) do not depend on the choice of the representative of classes, i.e. they are well-defined.  $\square$

**Lemma 272 (quotient vector space, equivalence relation).**

Let  $\mathbb{K}$  be a field. Let  $(E, +_E, \cdot_E)$  be a  $\mathbb{K}$ -vector space. Let  $\mathcal{R}$  be an equivalence relation on  $E$ . Assume that  $\mathcal{R}$  is compatible with the vector operations. Let  $[+]$  and  $[\cdot]$  be the quotient vector operations induced on the quotient set  $E/\mathcal{R}$ . Then,  $(E/\mathcal{R}, [+], [\cdot])$  is a  $\mathbb{K}$ -vector space.

*Proof.* From **group theory**,  $(E/\mathcal{R}, [+])$  is an abelian group with identity element  $[0_E]$ . Distributivity of the quotient scalar multiplication over quotient vector addition and field addition, compatibility of the quotient scalar multiplication with field multiplication, and 1 is the identity element for the quotient scalar multiplication are direct consequences of Definition 61 (vector space), and Lemma 271 (quotient vector operations). Therefore, from Definition 61 (vector space),  $(E/\mathcal{R}, [+], [\cdot])$  is a vector space.  $\square$

**Lemma 273 (quotient vector space).** Let  $\mathbb{K}$  be a field. Let  $E$  be a  $\mathbb{K}$ -vector space. Let  $F$  be a vector subspace of  $E$ . Let  $\mathcal{R}$  be the relation defined on  $E$  by for all  $u, v \in E$ ,  $u \mathcal{R} v \Leftrightarrow v - u \in F$ . Then,  $\mathcal{R}$  is an equivalence relation compatible with the vector operations of  $E$ .

The quotient set  $E/\mathcal{R}$  equipped with the quotient vector operations is thus a  $\mathbb{K}$ -vector space called *quotient vector space of  $E$  by  $F$* ; it is denoted  $E/F$ . For all  $u \in E$ , the class of  $u$  is denoted  $u + F$ . The quotient vector operations are still denoted  $+$  and  $\cdot$  (the latter may be omitted).

*Proof.* Let  $u, u', v, v' \in E$ . Let  $\lambda \in \mathbb{K}$ . Assume that  $u \mathcal{R} u'$  and  $v \mathcal{R} v'$ , i.e.  $u' - u, v' - v \in F$ . Then, from Definition 61 (vector space, vector addition is associative, and scalar multiplication is distributive over field addition), and Lemma 81 (closed under vector operations is subspace), we have  $(u' + v') - (u + v) = (u' - u) + (v' - v) \in F$  and  $(\lambda u') - (\lambda u) = \lambda(u' - u) \in F$ , i.e.  $(u + v) \mathcal{R} (u' + v')$  and  $(\lambda u) \mathcal{R} (\lambda u')$ . Hence, from Definition 270 (relation compatible with vector operations), the equivalence relation  $\mathcal{R}$  is compatible with the vector operations. Therefore, from Lemma 272 (quotient vector space, equivalence relation), the quotient set  $E/\mathcal{R}$  equipped with the induced quotient vector operations is a vector space.  $\square$



**Lemma 274 (linear map on quotient vector space).** *Let  $\mathbb{K}$  be a field. Let  $E, F$  be  $\mathbb{K}$ -vector spaces. Let  $G$  be a vector subspace of  $E$ . Let  $f$  be a linear map from  $E$  to  $F$ . Assume that  $G \subset \ker(f)$ . Then, the function  $[f] \stackrel{\text{def.}}{=} (u + G \mapsto f(u))$  is a linear map from  $E/G$  to  $F$ .*

*Proof.* Let  $u \in E$ . Let  $u' \in u + G$ . Then, from hypotheses, we have  $u' - u \in G \subset \ker(f)$ . Thus, from Definition 64 (linear map), and Definition 101 (kernel), we have  $f(u') = f(u)$ . Hence, the function  $[f]$  does not depend on the choice of the representative of equivalence classes, it is well-defined.

Let  $(u + G), (v + G) \in E/G$ . Let  $\lambda \in \mathbb{K}$ . Then, from Lemma 273 (quotient vector space), Lemma 271 (quotient vector operations), and Definition 64 (linear map), we have

$$\begin{aligned} [f](\lambda(u + G)) &= [f](\lambda u + G) = f(\lambda u) = \lambda f(u) = \lambda [f](u + G), \\ [f]((u + G) + (v + G)) &= [f]((u + v) + G) = f(u + v) = f(u) + f(v) = [f](u + G) + [f](v + G). \end{aligned}$$

Therefore, from Definition 64 (linear map),  $[f]$  is a linear map from  $E/G$  to  $F$ .  $\square$

## 7.4.2 Algebra over a field

*Remark 275.* The following algebraic structure “algebra over a field” is not to be confused with the concept of “algebra of sets” defined in Section 8.5.

*Remark 276.* Most results on algebras over a field are valid on algebras over a ring, for which the property of vector space over the field is replaced by that of module over the ring; especially when the ring is an integral domain, i.e. a nonzero commutative ring with no nonzero zero divisors, or a subring of a field.

**Definition 277 (algebra over a field).** Let  $\mathbb{K}$  be a field. A set  $E$  equipped with three algebra operations (a vector addition  $+$ , a scalar multiplication  $\cdot$ , and a vector multiplication  $*$ ), is called *algebra (over  $\mathbb{K}$ )*, or  *$\mathbb{K}$ -algebra*, iff  $(E, +, \cdot)$  is a  $\mathbb{K}$ -vector space, and vector multiplication is bilinear (or left and right distributive over vector addition, and compatible with scalars):

$$(7.32) \quad \forall u, v, w \in E, \quad (u + v) * w = (u * w) + (v * w),$$

$$(7.33) \quad \forall u, v, w \in E, \quad u * (v + w) = (u * v) + (u * w),$$

$$(7.34) \quad \forall \lambda, \mu \in \mathbb{K}, \forall u, v \in E, \quad (\lambda \cdot u) * (\mu \cdot v) = (\lambda \mu) \cdot (u * v).$$

*Remark 278.* The  $\cdot$  and  $*$  infix signs in the scalar and vector multiplications may be omitted.

**Lemma 279 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra).** *Let  $\mathbb{K}$  be a field. It is an algebra over itself.*

*Proof.* Direct consequence of Definition 277 (algebra over a field), and **field properties of  $\mathbb{K}$** .  $\square$

**Definition 280 (inherited algebra operations).**

Let  $X$  be a nonempty set. Let  $\mathbb{K}$  be a field. Let  $(E, +_E, \cdot_E, *_E)$  be an algebra over  $\mathbb{K}$ . The *algebra operations inherited on  $E^X$*  are the mappings  $+_{E^X}$  and  $\cdot_{E^X}$  of Definition 91 (inherited vector operations), and the mapping  $*_{E^X} : E^X \times E^X \rightarrow E^X$  defined by

$$(7.35) \quad \forall f, g \in E^X, \forall x \in X, \quad (f *_{E^X} g)(x) \stackrel{\text{def.}}{=} f(x) *_E g(x).$$

*Remark 281.* Usually, inherited algebra operations are denoted the same way as the algebra operations of the target algebra.

**Lemma 282 (algebra of functions to algebra).**

*Let  $\mathbb{K}$  be a field. Let  $X$  be a nonempty set. Let  $(E, +_E, \cdot_E, *_E)$  be a  $\mathbb{K}$ -algebra. Let  $+_{E^X}, \cdot_{E^X}$  and  $*_{E^X}$  be the algebra operations inherited on  $E^X$ . Then,  $(E^X, +_{E^X}, \cdot_{E^X}, *_{E^X})$  is a  $\mathbb{K}$ -algebra.*

*Proof.* From Lemma 93 (*space of functions to space*),  $(E^X, +_{E^X}, \cdot_{E^X})$  is a  $\mathbb{K}$ -vector space. Then, bilinearity of the inherited vector multiplication is a direct consequence of Definition 277 (*algebra over a field*), and Definition 280 (*inherited algebra operations*). Therefore, from Definition 277 (*algebra over a field*),  $E^X$  equipped with  $+_{E^X}$ ,  $\cdot_{E^X}$  and  $*_{E^X}$  is a  $\mathbb{K}$ -algebra.  $\square$

**Lemma 283 ( $\mathbb{K}^X$  is algebra).**

Let  $\mathbb{K}$  be a field. Let  $X$  be a nonempty set. Then,  $\mathbb{K}^X$  is a  $\mathbb{K}$ -algebra.

*Proof.* Direct consequence of Lemma 279 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra), and Lemma 282 (*algebra of functions to algebra*).  $\square$

**Definition 284 (subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $(E, +, \cdot, *)$  be a  $\mathbb{K}$ -algebra. A subset  $F$  of  $E$  equipped with the restrictions to  $F$   $+|_F$ ,  $\cdot|_F$  and  $*|_F$  of the algebra operations is called ( $\mathbb{K}$ -)subalgebra of  $E$  iff  $(F, +|_F, \cdot|_F, *|_F)$  is a  $\mathbb{K}$ -algebra.

*Remark 285.* Usually, restrictions  $+|_F$ ,  $\cdot|_F$ , and  $*|_F$  are still denoted  $+$ ,  $\cdot$  and  $*$ .

**Lemma 286 (vector subspace and closed under multiplication is subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $(E, +, \cdot, *)$  be a  $\mathbb{K}$ -algebra. Let  $F \subset E$ . Then,  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$  iff  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and  $F$  is closed under vector multiplication:

$$(7.36) \quad \forall u, v \in F, \quad u * v \in F.$$

*Proof.* “Left” implies “right”. Assume first that  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$ . Then, from Definition 284 (*subalgebra*),  $F$  is a  $\mathbb{K}$ -algebra), Definition 277 (*algebra over a field*),  $(F, +|_F, \cdot|_F)$  is a  $\mathbb{K}$ -vector space, and  $F$  is closed under the restriction to  $F$  of the three operations. Thus, from Definition 77 (*subspace*),  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and  $F$  is closed under vector multiplication.

“Right” implies “left”. Conversely, assume now that  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and that it is closed under vector multiplication. Then, from Definition 77 (*subspace*),  $(F, +|_F, \cdot|_F)$  is a  $\mathbb{K}$ -vector space. Moreover, from Lemma 81 (*closed under vector operations is subspace*),  $F$  is also closed under vector addition and scalar multiplication. Thus, since  $F$  is a subset of  $E$ , and  $E$  is a  $\mathbb{K}$ -algebra, properties (7.32) to (7.34) are trivially satisfied over  $F$  with the restrictions of the three operations. Hence, from Definition 277 (*algebra over a field*), and Definition 284 (*subalgebra*),  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$ .

Therefore, we have the equivalence.  $\square$

**Lemma 287 (closed under algebra operations is subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $E$  be a  $\mathbb{K}$ -algebra. Let  $F \subset E$ . Then,  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$  iff  $0_E \in F$ , and  $F$  is closed under vector addition, scalar and vector multiplications:

$$(7.37) \quad \forall u, v \in F, \quad u + v \in F,$$

$$(7.38) \quad \forall \lambda \in \mathbb{K}, \forall u \in F, \quad \lambda u \in F,$$

$$(7.39) \quad \forall u, v \in F, \quad u * v \in F.$$

*Proof.* Direct consequence of Lemma 286 (*vector subspace and closed under multiplication is subalgebra*), and Lemma 81 (*closed under vector operations is subspace*).  $\square$

### 7.4.3 Seminormed vector space

**Definition 288 (seminorm).**

Let  $\mathbb{K}$  be a valued field. Let  $E$  be a  $\mathbb{K}$ -vector space. A function  $\|\cdot\| : E \rightarrow \mathbb{R}$  is called *seminorm over  $E$*  iff it is absolutely homogeneous of degree 1, and it satisfies the triangle inequality:

$$(7.40) \quad \forall \lambda \in \mathbb{K}, \forall u \in E, \quad \|\lambda u\| = |\lambda| \|u\|,$$

$$(7.41) \quad \forall u, v \in E, \quad \|u + v\| \leq \|u\| + \|v\|.$$

If so,  $(E, \|\cdot\|)$  (or simply  $E$ ) is called *seminormed ( $\mathbb{K}$ -)vector space*.

*Remark 289.* Most results from [20] on normed vector spaces can be generalized to the case of seminormed vector space, sometimes with slight modifications of the statement. In particular, the associated distance becomes a pseudometric.

**Lemma 290 (seminorm is homogeneous).**

Let  $(E, \|\cdot\|)$  be a seminormed vector space. Then, we have  $\|(0_E) = 0_{\mathbb{R}}$ .

*Proof.* From Lemma 71 (*zero times yields zero*), Definition 288 (*seminorm, absolute homogeneity*), **definiteness of absolute values on a field**, and **field properties of  $\mathbb{R}$** , we have

$$\|0_E\| = \|0_{\mathbb{K}} \cdot 0_E\| = |0_{\mathbb{K}}| \|0_E\| = 0_{\mathbb{R}} \|0_E\| = 0_{\mathbb{R}}.$$

□

**Lemma 291 (seminorm is nonnegative).** Let  $\mathbb{K}$  be a field of characteristic distinct from 2. Let  $(E, \|\cdot\|)$  be a seminormed  $\mathbb{K}$ -vector space. Then,  $\|\cdot\|$  is nonnegative.

*Proof.* Let  $x \in E$ . Then, from Lemma 290 (*seminorm is homogeneous*), Definition 61 (*vector space*), Definition 288 (*seminorm, triangle inequality, then absolute homogeneity with  $\lambda \stackrel{\text{def.}}{=} -1$* ), we have  $0_{\mathbb{R}} = \|0_E\| = \|x + (-x)\| \leq \|x\| + \|-x\| = 2\|x\|$ . Hence, from **field properties of  $\mathbb{K}$  (2 is invertible)**, we have  $\|x\| \geq 0$ . □

**Lemma 292 (seminormed vector space is pseudometric space).**

Let  $(E, \|\cdot\|)$  be a seminormed vector space. Let  $d := ((x, y) \in E \mapsto \|x - y\|)$ . Then,  $(E, d)$  is a pseudometric space.

*Proof.* Direct consequence of Lemma 291 (*seminorm is nonnegative*, thus  $d$  is nonnegative), Definition 61 (*vector space*, thus  $x - x = 0_E$  and  $x - z = (x - y) + (y - z)$ ), Lemma 290 (*seminorm is homogeneous*, thus  $d$  is zero on the diagonal), Definition 288 (*seminorm, with  $\lambda \stackrel{\text{def.}}{=} -1$* , thus  $d$  is symmetric and satisfies the triangle inequality), and Definition 262 (*pseudometric*). □

**Lemma 293 (definite seminorm is norm).** Let  $(E, \|\cdot\|)$  be a seminormed vector space. Then,  $(E, \|\cdot\|)$  is a normed vector space iff  $\|\cdot\|$  is definite iff for all  $u \in E$ ,  $\|u\| = 0 \Leftrightarrow u = 0$ .

*Proof.* Direct consequence of Definition 288 (*seminorm*), Definition 106 (*normed vector space*), and Definition 105 (*norm*). □

#### 7.4.3.1 Banach space

*Remark 294.* The next definition depends on the definitions of complete metric space (see Definition 39), and of distance associated with a norm (see Definition 112).

**Definition 295 (Banach space).** A normed vector space  $(E, \|\cdot\|_E)$  is called *Banach space* iff  $(E, d_E)$  is a complete metric space, where  $d_E$  is the distance induced by the norm  $\|\cdot\|_E$ .

### 7.4.3.2 Semi-inner product vector space

**Definition 296 (semi-inner product).**

Let  $E$  be a real vector space. A function  $(\cdot, \cdot)_E : E \times E \rightarrow \mathbb{R}$  is called *semi-inner product on  $E$*  iff it is a bilinear form on  $E$  that is symmetric and nonnegative:

$$(7.42) \quad \forall u, v \in E, \quad (u, v)_E = (v, u)_E,$$

$$(7.43) \quad \forall u \in E, \quad (u, u)_E \geq 0.$$

If so,  $(E, (\cdot, \cdot)_E)$  (or simply  $E$ ) is called *semi-inner product vector space*.

*Remark 297.* Most results from [20] on inner product vector spaces are still valid on semi-inner product vector spaces.

**Lemma 298 (definite semi-inner product is inner product).**

Let  $(E, (\cdot, \cdot)_E)$  be a semi-inner product vector space. Then,  $(E, (\cdot, \cdot)_E)$  is an inner product vector space iff  $(\cdot, \cdot)_E$  is definite, i.e. for all  $u \in E$ ,  $(u, u)_E = 0$  implies  $u = 0$ .

*Proof.* Direct consequence of Definition 296 (semi-inner product), Definition 168 (inner product space), and Definition 166 (inner product).  $\square$

## 7.5 Complements on real numbers

### 7.5.1 Real numbers

#### Lemma 299 (finite cover of compact interval).

Let  $a, b, (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{R}$ . Assume that  $a \leq b$  and  $[a, b] \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ . Then, there exists  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b] \subset \bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a$ ,  $b < b_{i_q}$ , and for all  $p \in [0..q - 1]$ ,  $a_{i_{p+1}} < b_{i_p}$ .

*Proof.* From **the definition of compactness**, and **compactness of  $[a, b]$** , there exists  $n \in \mathbb{N}$  such that  $[a, b] \subset \bigcup_{j \in [0..n]} (a_j, b_j)$ .

Let  $J \subset [0..n]$  and  $x \in \mathbb{R}$ . Assume that  $[x, b] \subset \bigcup_{j \in J} (a_j, b_j)$  and  $x \leq b$ .

(1). **Next index:**  $\exists i \stackrel{\text{def.}}{=} i_J(x) \in J$ ,  $a_i < x < b_i$ .

Direct consequence of **the definition of union (with  $x \in [x, b]$ )**.

(2). **Next cover:**  $[b_i, b] \subset \bigcup_{j \in J \setminus \{i\}} (a_j, b_j)$ .

**Case  $b < b_i$ .** Trivial. **Case  $b_i \leq b$ .** Then, from (1), we have  $b_i \in (x, b]$ . Let  $y \in [b_i, b] \subset [x, b]$ . Then, from **the definition of union**, there exists  $j \in J$  such that  $a_j < y < b_j$ . Hence, from **transitivity of order in  $\mathbb{R}$** , we have  $b_i < b_j$ , i.e.  $j \neq i$ , and  $[b_i, b] \subset \bigcup_{j \in J \setminus \{i\}} (a_j, b_j)$ .

Let  $i_{-1} \stackrel{\text{def.}}{=} -1$ ,  $J_{-1} \stackrel{\text{def.}}{=} [0..n]$ , and  $b_{i_{-1}} = b_{-1} \stackrel{\text{def.}}{=} a$ . Let  $(i_p)_{p \in [0..q]}$  be the sequence of integers computed by the following algorithm:

```

p  $\stackrel{\text{def.}}{=} -1$ ;
repeat
  ip+1  $\stackrel{\text{def.}}{=} i_{J_p}(b_{i_p})$ ;
  Jp+1  $\stackrel{\text{def.}}{=} J_p \setminus \{i_{p+1}\}$ ;
  p  $\stackrel{\text{def.}}{=} p + 1$ ;
until b < bip;
return (i0, ..., ip);

```

For all  $p \in \{-1\} \cup \mathbb{N}$ , let  $I_p \stackrel{\text{def.}}{=} \{i_0, \dots, i_p\}$  (with  $I_{-1} \stackrel{\text{def.}}{=} \emptyset$ ), and  $P(p)$  be the property:

$$\text{card}(I_p) = p + 1 \quad \wedge \quad [0..n] = I_p \uplus J_p \quad \wedge \quad (\forall m \in [0..p], a_{i_m} < b_{i_{m-1}}) \quad \wedge \\ [a, b_{i_p}] \subset \bigcup_{j \in I_p} (a_j, b_j) \quad \wedge \quad [b_{i_p}, b] \subset \bigcup_{j \in J_p} (a_j, b_j).$$

Let us show that there exists  $q \in [0..n]$  such that  $P(q) \wedge b < b_{i_q}$  holds.

(3). **Initialization:**  $P(-1) \wedge b_{i_{-1}} \leq b$ . Trivial.

(4). **Iterations:**  $\forall p \in \{-1\} \cup \mathbb{N}$ ,  $P(p) \wedge b_{i_p} \leq b$  implies  $P(p + 1)$ .

Let  $p \in \{-1\} \cup \mathbb{N}$ . Assume that  $P(p) \wedge b_{i_p} \leq b$  holds.

Let  $J \stackrel{\text{def.}}{=} J_p$  and  $x \stackrel{\text{def.}}{=} b_{i_p}$ . Then, we have  $[x, b] \subset \bigcup_{j \in J} (a_j, b_j)$  and  $x \leq b$ . Thus, from (1), there exists  $i_{p+1} = i_J(x) \in J = J_p$  (i.e.  $i_{p+1} \notin I_p$ ) such that  $a_{i_{p+1}} < b_{i_p} < b_{i_{p+1}}$ . Moreover, from (2),

we have  $[b_{i_{p+1}}, b] \subset \bigcup_{j \in J_{p+1}} (a_j, b_j)$ . Then, we have

$$\begin{aligned} \text{card}(I_{p+1}) &= \text{card}(I_p) + 1 = p + 2, \\ [0..n] &= I_p \uplus \{i_{p+1}\} \uplus (J_p \setminus \{i_{p+1}\}) = I_{p+1} \uplus J_{p+1}. \\ a_{i_{p+1}} &< b_{i_p}, \\ [a, b_{i_{p+1}}] &= [a, b_{i_p}] \uplus [b_{i_p}, b_{i_{p+1}}] \subset \bigcup_{j \in I_p} (a_j, b_j) \cup (a_{i_{p+1}}, b_{i_{p+1}}) = \bigcup_{j \in I_{p+1}} (a_j, b_j), \\ [b_{i_{p+1}}, b] &\subset \bigcup_{j \in J_{p+1}} (a_j, b_j). \end{aligned}$$

Hence,  $P(p+1)$  holds.

**(5). Termination:**  $\exists q \in [0..n]$ ,  $P(q) \wedge b < b_{i_q}$ . Assume that  $P(p) \wedge b_{i_p} \leq b$  holds for all  $p$  in  $[0..n]$ . Then, from (4), we have  $P(n+1)$ . Thus, from **additivity of the cardinality**, we have

$$n+1 = \text{card}([0..n]) = \text{card}(I_{n+1}) + \text{card}(J_{p+1}) \geq \text{card}(I_{n+1}) = n+2.$$

Which is impossible. Hence, there exists  $q \in [0..n]$  such that  $P(q) \wedge b < b_{i_q}$  holds.

Moreover, we have  $[a, b] \subset [a, b_{i_q}] \subset \bigcup_{j \in I_q} (a_j, b_j)$ .

Therefore, there exists  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b]$  is included in  $\bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a$ ,  $b < b_{i_q}$ , and for all  $p \in [0..q-1]$ ,  $a_{i_{p+1}} < b_{i_p}$ .  $\square$

**Lemma 300 (power function satisfies relaxed triangle inequality).**

Let  $p \in [1, \infty)$ . Then, the power function of degree  $p$  satisfies the relaxed triangle inequality:

$$(7.44) \quad \forall a, b \in \mathbb{R}, \quad |a+b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Moreover, there is equality when  $a, b \geq 0$  for  $p = 1$ , and when  $a = b$  (for all  $p \geq 1$ ).

*Proof.* Let  $a, b \in \mathbb{R}$ . From **valued field properties of  $\mathbb{R}$** , and **convexity of the power function ( $x \mapsto |x|^p$ ) for  $p \geq 1$** , we have

$$|a+b|^p = \left| 2 \frac{a+b}{2} \right|^p = 2^p \left| \frac{a}{2} + \frac{b}{2} \right|^p \leq 2^p \left( \frac{|a|^p}{2} + \frac{|b|^p}{2} \right) = 2^{p-1}(|a|^p + |b|^p).$$

Moreover, assume first that  $p = 1$ , and  $a, b \geq 0$ . Then, from **valued field properties**, we have  $a+b \geq 0$ , and  $|a+b| = a+b = 2^0 \times (|a| + |b|)$ . Assume now that  $a = b$ . Then, from **valued field properties of  $\mathbb{R}$** , we have  $|a+b|^p = |2a|^p = 2^p |a|^p = 2^{p-1} (2|a|^p) = 2^{p-1}(|a|^p + |b|^p)$ .  $\square$

*Remark 301.*

The equality cases mentioned in Lemma 300 show that the constant  $2^{p-1}$  is actually optimal.

**Definition 302 (Hölder conjugates in  $\mathbb{R}$ ).**

Real numbers  $p, q \in (1, \infty)$  are said *Hölder conjugates in  $\mathbb{R}$*  iff  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Lemma 303 (2 is self-Hölder conjugate in  $\mathbb{R}$ ).**

The real number 2 is Hölder conjugate in  $\mathbb{R}$  with itself.

*Proof.* Direct consequence of Definition 302 (*Hölder conjugates in  $\mathbb{R}$* ), and since  $\frac{1}{2} + \frac{1}{2} = 1$ .  $\square$

**Lemma 304 (Young's inequality for products in  $\mathbb{R}$ ).** Let  $p, q \in (1, \infty)$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $a, b \in \mathbb{R}_+$ . Then, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .

*Proof.* **Case  $ab = 0$ .** Then, from **nonnegativeness of exponentiation in  $\mathbb{R}$ , closedness of multiplicative inverse in  $\mathbb{R}_+^*$ , and of the multiplication and addition in  $\mathbb{R}_+$** , The right-hand side is nonnegative. Hence, the inequality holds.

**Case  $ab > 0$ .** Then, from **the zero-product property in  $\mathbb{R}_+$  (contrapositive)**, we have  $a, b > 0$ . Thus, from **algebraic properties of the natural logarithm function**, Definition 302 (*Hölder conjugates in  $\mathbb{R}$* ), and **concavity of the natural logarithm function (with  $\frac{1}{p} + \frac{1}{q} = 1$ )**, we have

$$\ln(ab) = \ln a + \ln b = \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) \leq \ln \left( \frac{a^p}{p} + \frac{b^q}{q} \right).$$

Hence, from **monotonicity of the exponential function in  $\mathbb{R}$** , and since **the exponential and logarithm functions in  $\mathbb{R}$  are each other inverse**, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .

Therefore, the inequality always holds.  $\square$

**Lemma 305 (Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$ ).**

Let  $a, b \in \mathbb{R}_+$ . Let  $\varepsilon > 0$ . Then, we have  $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ .

*Proof.* Direct consequence of Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ), Lemma 304 (*Young's inequality for products in  $\mathbb{R}$ , with  $p = q \stackrel{\text{def.}}{=} 2$  and  $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}b \in \mathbb{R}_+$ ), and **properties of square root and multiplicative inverse in  $\mathbb{R}_+^*$ , and of multiplication in  $\mathbb{R}_+$** .  $\square$*

*Remark 306.* Note that a similar result also holds in the general case of Hölder conjugate numbers  $p, q \in (1, \infty)$ : for all  $a, b \in \mathbb{R}_+$ , for all  $\varepsilon > 0$ , we have  $ab \leq \frac{a^p}{p\varepsilon^{\frac{p-1}{p}}} + \frac{\varepsilon^{\frac{p-1}{p}} b^q}{q}$ .

**Lemma 307 (topology on  $\mathbb{R}$ ).** The order topology on  $\mathbb{R}$  is the same as the usual Euclidean topology in dimension 1. They admit the set  $\mathcal{I}_{\mathbb{R}}^o$  of open intervals as topological basis.

*Proof.* Direct consequence of Lemma 258 (*topological basis of order topology, the set  $\mathcal{I}_{\mathbb{R}}^o$  of open intervals is a topological basis of the order topology*), **the definition of the topology induced by a metric (the set  $\{(x-r, x+r) \mid x \in \mathbb{R} \wedge r \in \mathbb{R}_+^*\}$  of open balls is a topological basis of the usual Euclidean topology on  $\mathbb{R}$ )**, and Lemma 249 (*topological basis is generator*, both topological bases are identical, thus both topologies are the same).  $\square$

**Lemma 308 (trace topology on subset of  $\mathbb{R}$ ).**

Let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}$ , or the order topology. Let  $Y \subset \mathbb{R}$ .

Then,  $\mathcal{I}_Y^o$  is a topological basis of the trace topology  $\mathcal{T}_Y$ , and when  $Y$  is open, we have  $\mathcal{T}_Y \subset \mathcal{T}$ .

*Proof.* Direct consequence of Lemma 307 (*topology on  $\mathbb{R}$* ), and Lemma 252 (*trace topology on subset*).  $\square$

*Remark 309.* We recall that  $G^{\overline{\times} m}$  denotes a set of products of subsets, see Definition 227.

**Lemma 310 (topology on  $\mathbb{R}^m$ ).**

Let  $m \in [2, \infty)$ . Let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}$ , or the order topology.

Then, the box topology  $\mathcal{T}_{\mathbb{R}^m}(\mathcal{T}^{\overline{\times} m})$  on  $\mathbb{R}^m$  is the same as the usual Euclidean topology in dimension  $m$ . They admit the set  $(\mathcal{I}_{\mathbb{R}}^o)^{\overline{\times} m}$  of open  $m$ -boxes, or the set of open balls for the metric induced by any  $p$ -norm (for  $p \in [1, \infty]$ ), as topological bases.

*Proof.* Let  $\mathcal{T}^{\square}$  be the box topology on  $\mathbb{R}^m$ , and let  $\mathcal{B}^{\square} \stackrel{\text{def.}}{=} (\mathcal{I}_{\mathbb{R}}^o)^{\overline{\times} m}$  be the set of  $m$ -boxes. For all  $p \in [1, \infty]$ , let  $\mathcal{T}^p$  be the topology on  $\mathbb{R}^m$  induced by the  $p$ -norm, and let  $\mathcal{B}^p$  be the set of open balls for the same norm.

From Lemma 307 (*topology on  $\mathbb{R}$* ), and Lemma 254 (*box topology on Cartesian product*),  $\mathcal{B}^{\square}$  is a topological basis of  $\mathcal{T}^{\square}$ .

From **the equivalence of  $p$ -norms in finite-dimensional real vector space**, the usual Euclidean topology on  $\mathbb{R}^m$  is the same as the topology induced by any  $p$ -norm, thus they share the same bases, e.g. with the maximum norm (thus  $\mathcal{T}^2 = \mathcal{T}^\infty$ ). Hence, from **the definition of the topology induced by a norm**,  $\mathcal{B}^\infty$  is a topological basis of  $\mathcal{T}^2$ .

Let  $B \in \mathcal{B}^\square$  be an open  $m$ -box. **Case  $B = \emptyset$** . Then, from **the nullary union law**,  $B$  is a (nullary) union of elements of  $\mathcal{B}^\infty$ . **Case  $B \neq \emptyset$** . Then, from Lemma 254 (*box topology on Cartesian product*), and Definition 227 (*product of subset systems*), let  $(a_n)_{n \in [1..m]}$  and  $(b_n)_{n \in [1..m]}$  be in  $\mathbb{R}^m$  such that, for all  $n \in [1..m]$ ,  $a_n < b_n$ , and  $B = \prod_{n \in [1..m]} (a_n, b_n)$ . Without any loss of generality, assume that the smaller side of the  $m$ -box corresponds to the first dimension, and let  $r \stackrel{\text{def.}}{=} \frac{b_1 - a_1}{2}$ . Then, we have  $r > 0$ , and for all  $n \in [2..m]$ ,  $b_n - r - (a_n + r) = b_n - a_n - (b_1 - a_1) \geq 0$  (i.e.  $[a_n + r, b_n - r]$  is nonempty). For all  $x \in \mathbb{R}^m$ , let  $B_x \stackrel{\text{def.}}{=} \prod_{n \in [1..m]} (x_n - r, x_n + r)$ , and let

$$J \stackrel{\text{def.}}{=} \left\{ \left( \frac{a_1 + b_1}{2}, x_2, \dots, x_m \right) \in \mathbb{R}^m \mid \forall n \in [2..m], x_n \in [a_n + r, b_n - r] \right\}.$$

Then, from **the definition of the maximum norm**, for all  $x \in \mathbb{R}^m$ , we have  $B_x \in \mathcal{B}^\infty$ , and  $B = \bigcup_{j \in J} B_j$ .

Therefore, from Lemma 250 (*topological subbasis*), we have  $\mathcal{T}^\square = \mathcal{T}^\infty = \mathcal{T}^2$ , and all topologies share their respective bases.  $\square$

**Lemma 311 (trace topology on subset of  $\mathbb{R}^m$ ).** *Let  $m \in [2..\infty)$ . Let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}^m$ , or the box topology on  $\mathbb{R}^m$  induced by the usual Euclidean topology on  $\mathbb{R}$ , or by the order topology. Let  $Y \subset \mathbb{R}^m$ . Then, the set  $(\mathcal{I}_{\mathbb{R}}^o)^{\times m} \cap Y$  of traces of open  $m$ -boxes on  $Y$  is a topological basis of the trace topology  $\mathcal{T}_Y$ , and when  $Y \in \mathcal{T}$ , we have  $\mathcal{T}_Y \subset \mathcal{T}$ .*

*Proof.* Direct consequence of Lemma 310 (*topology on  $\mathbb{R}^m$* ), Lemma 252 (*trace topology on subset*, with  $X \stackrel{\text{def.}}{=} \mathbb{R}^m$ ),  $\square$



## 7.5.2 Extended real numbers

**Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ).** The set of *extended real numbers* is  $\overline{\mathbb{R}} \stackrel{\text{def.}}{=} \mathbb{R} \uplus \{-\infty, \infty\}$ , and the order in  $\mathbb{R}$  is extended to  $\overline{\mathbb{R}}$  with the following rule:

$$(7.45) \quad 1. \quad \forall a \in \mathbb{R}, \quad -\infty < a < \infty,$$

and, in this case,  $a$  is said to be *finite*.

**Lemma 313 (order in  $\overline{\mathbb{R}}$  is total).**  $(\overline{\mathbb{R}}, \leq)$  is a totally ordered set.

*Proof.* Direct consequence of **the definition of total order**, Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and **totality of order in  $\mathbb{R}$** .  $\square$

**Lemma 314 ( $\overline{\mathbb{R}}$  is complete).**  $(\overline{\mathbb{R}}, \leq)$  is the Dedekind–MacNeille completion of  $\mathbb{R}$ : every subset has both an infimum and a supremum.

*Proof.* Direct consequence of Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), **completeness of  $\mathbb{R}$  (bounded subsets have infimum and/or supremum)**, and **the definition of Dedekind–MacNeille completion (unbounded subsets have infimum  $-\infty$  and/or supremum  $\infty$ )**.  $\square$

*Remark 315.* The goal of this section is to clarify some properties of basic operations such as addition, multiplication, and exponentiation, when extended to  $\overline{\mathbb{R}}$ .

*Remark 316.* Note that, as with regular real numbers, all results on operations on extended real numbers can be lifted into similar results on functions taking their values in  $\overline{\mathbb{R}}$ .

**Definition 317 (addition in  $\overline{\mathbb{R}}$ ).**

Addition and subtraction in  $\mathbb{R}$  are extended to  $\overline{\mathbb{R}}$  with the following rules:

$$(7.46) \quad \left\{ \begin{array}{l} 1. \quad \forall a > -\infty, \quad a + \infty = \infty + a \stackrel{\text{def.}}{=} \infty, \\ 2. \quad \forall a < \infty, \quad a + (-\infty) = -\infty + a \stackrel{\text{def.}}{=} -\infty, \\ 3. \quad \infty + (-\infty) \text{ and } -\infty + \infty \text{ are undefined,} \\ 4. \quad -(\pm\infty) = \mp\infty, \\ 5. \quad \forall a, b \in \overline{\mathbb{R}}, \quad a + (-b) \text{ defined} \implies a - b \stackrel{\text{def.}}{=} a + (-b). \end{array} \right.$$

**Lemma 318 (zero is identity element for addition in  $\overline{\mathbb{R}}$ ).**

Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $a + 0 = 0 + a = a$ .

*Proof.* Direct consequence of **abelian additive group properties of  $\mathbb{R}$** , and Definition 317 (*addition in  $\overline{\mathbb{R}}$* , new rules 1, and 2 are compatible with the property).  $\square$

**Lemma 319 (addition in  $\overline{\mathbb{R}}$  is associative when defined).**

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a + (b + c)$  and  $(a + b) + c$  are either equal or both undefined.

*Proof.* Direct consequence of **associativity of addition in  $\mathbb{R}$** , and Definition 317 (*addition in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with associativity).  $\square$

**Lemma 320 (addition in  $\overline{\mathbb{R}}$  is commutative when defined).**

Let  $a, b \in \overline{\mathbb{R}}$ . Then,  $a + b$  and  $b + a$  are either equal or both undefined.

*Proof.* Direct consequence of **commutativity of addition in  $\mathbb{R}$** , and Definition 317 (*addition in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with commutativity).  $\square$

**Lemma 321 (addition in  $\overline{\mathbb{R}}$  is compatible with order when defined).** Let  $a, b, c \in \overline{\mathbb{R}}$ . Assume that  $a \leq b$ , and that  $a + c$  and  $b + c$  are well-defined. Then, we have  $a + c \leq b + c$ .

*Proof.* Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ), and **compatibility of addition with order in  $\mathbb{R}$** , and Definition 317 (addition in  $\overline{\mathbb{R}}$ , new rules 1, 2, and 3 are compatible with order).  $\square$

**Lemma 322 (infinity-sum property in  $\overline{\mathbb{R}}$ ).** Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.47) \quad a + b = \infty \iff (a = \infty \wedge b > -\infty) \vee (a > -\infty \wedge b = \infty)$$

$$(7.48) \quad a + b = -\infty \iff (a = -\infty \wedge b < \infty) \vee (a < \infty \wedge b = -\infty).$$

Moreover, if we assume that the sum  $a + b$  is well-defined, then we have

$$(7.49) \quad a + b = \pm\infty \iff a = \pm\infty \vee b = \pm\infty.$$

*Proof.* Direct consequence of Definition 317 (addition in  $\overline{\mathbb{R}}$ , rules 1, 2 and 3).  $\square$

**Lemma 323 (finite-sum property in  $\overline{\mathbb{R}}$ ).** Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $a + b \in \mathbb{R}$  iff  $a, b \in \mathbb{R}$ .

*Proof.* Direct consequence of Definition 317 (addition in  $\overline{\mathbb{R}}$ , new rules are not finite-sum rules).  $\square$

**Lemma 324 (additive inverse in  $\overline{\mathbb{R}}$  is involution).** Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $-(-a) = a$ .

*Proof.* Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ), **involution of additive inverse in  $\mathbb{R}$** , and Definition 317 (addition in  $\overline{\mathbb{R}}$ , rule 4).  $\square$

**Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone).**

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $a < b \iff -b < -a$ , and  $a \leq b \iff -b \leq -a$ .

*Proof.* Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ), **monotonicity of additive inverse in  $\mathbb{R}$** , and Definition 317 (addition in  $\overline{\mathbb{R}}$ , rule 4).  $\square$

**Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ).**

Multiplication and division by nonzero in  $\mathbb{R}$  are extended to  $\overline{\mathbb{R}}$  with the following rules:

$$(7.50) \quad \left\{ \begin{array}{l} 1. \quad \forall a > 0, \quad a \times (\pm\infty) = \pm\infty \times a \stackrel{\text{def.}}{=} \pm\infty, \\ 2. \quad \forall a < 0, \quad a \times (\pm\infty) = \pm\infty \times a \stackrel{\text{def.}}{=} \mp\infty, \\ 3. \quad 0 \times (\pm\infty) \text{ and } \pm\infty \times 0 \text{ are undefined,} \\ 4. \quad \frac{1}{0} \text{ is undefined, and } \frac{1}{\pm\infty} \stackrel{\text{def.}}{=} 0, \\ 5. \quad \forall a, b \in \overline{\mathbb{R}}, \quad \frac{1}{b} \text{ and } a \times \frac{1}{b} \text{ defined } \implies \frac{a}{b} \stackrel{\text{def.}}{=} a \times \frac{1}{b}. \end{array} \right.$$

*Remark 327.* The 5th rule in the previous definition implies that  $\frac{\infty}{\pm\infty}$ ,  $\frac{-\infty}{\pm\infty}$ , and  $\frac{a}{0}$  (for all  $a \in \overline{\mathbb{R}}$ ) are undefined. Note that rule 3 is modified in the context of measure theory in Definition 415.

**Lemma 328 (one is identity element for multiplication in  $\overline{\mathbb{R}}$ ).**

Let  $a \in \mathbb{R}$ . Then, we have  $a \times 1 = 1 \times a = a$ .

*Proof.* Direct consequence of **field properties of  $\mathbb{R}$** , and Definition 326 (multiplication in  $\overline{\mathbb{R}}$ , new rule 1 is compatible with the property).  $\square$

**Lemma 329** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a \times (b \times c)$  and  $(a \times b) \times c$  are either equal or both undefined.

*Proof.* Direct consequence of **associativity of multiplication in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with associativity).  $\square$

**Lemma 330** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*).

Let  $a, b \in \overline{\mathbb{R}}$ . Then,  $a \times b$  and  $b \times a$  are either equal or both undefined.

*Proof.* Direct consequence of **commutativity of multiplication in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with commutativity).  $\square$

**Lemma 331** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a \times (b + c)$  and  $(a \times b) + (a \times c)$  are either equal or both undefined.

*Proof.* Direct consequence of **left distributivity of multiplication over addition in  $\mathbb{R}$** , Definition 317 (*addition in  $\overline{\mathbb{R}}$* ), and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ) (new rules are compatible with left distributivity of multiplication over addition).  $\square$

**Lemma 332** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $(a + b) \times c$  and  $(a \times c) + (b \times c)$  are either equal or both undefined.

*Proof.* Direct consequence of Lemma 330 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*, used twice), and Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).  $\square$

**Lemma 333** (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*). Let  $a, b, c \in \overline{\mathbb{R}}$ . Assume that  $a \leq b$  and  $c \geq 0$ , and that  $a \times c$  and  $b \times c$  are well-defined. Then, we have  $a \times c \leq b \times c$ .

*Proof.* Direct consequence of Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), **compatibility of multiplication with order in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with order).  $\square$

**Lemma 334** (*zero-product property in  $\overline{\mathbb{R}}$* ).

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.51) \quad ab = 0 \iff ab \text{ is defined} \wedge (a = 0 \vee b = 0).$$

*Proof.* Direct consequence of **the zero-product property in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* , rule 3).  $\square$

**Lemma 335** (*one-product property in  $\overline{\mathbb{R}}$* ).

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.52) \quad ab = 1 \iff \left( a = \frac{1}{b} \vee b = \frac{1}{a} \right) \wedge a, b \notin \{0, \pm\infty\}.$$

*Proof.* Direct consequence of **the one-product property in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 336** (*infinity-product property in  $\overline{\mathbb{R}}$* ).

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.53) \quad ab = \infty \iff (a = \infty \wedge b > 0) \vee (a = -\infty \wedge b < 0) \vee (a > 0 \wedge b = \infty) \vee (a < 0 \wedge b = -\infty),$$

$$(7.54) \quad ab = -\infty \iff (a = -\infty \wedge b > 0) \vee (a = \infty \wedge b < 0) \vee (a > 0 \wedge b = -\infty) \vee (a < 0 \wedge b = \infty).$$

*Proof.* **“Left” implies “right”.** Assume that  $ab = \pm\infty$ . Then, from Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have  $a = \pm\infty$  or  $b = \pm\infty$ . Assume that the other operand is zero. Then, from Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rule 3*), the product  $ab$  is undefined. Which is impossible.

Assume that  $ab = \infty$ . Then, from Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have either  $a = \infty$  and  $b > 0$ ,  $a = -\infty$  and  $b < 0$ ,  $b = \infty$  and  $a > 0$ , or  $b = -\infty$  and  $a < 0$ .

Similarly, assume now that  $ab = -\infty$ . Then, again from Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have either  $a = -\infty$  and  $b > 0$ ,  $a = \infty$  and  $b < 0$ ,  $b = -\infty$  and  $a > 0$ , or  $b = \infty$  and  $a < 0$ .

**“Right” implies “left”.** Direct consequence of Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*).

Therefore, we have the equivalences.  $\square$

**Lemma 337 (finite-product property in  $\overline{\mathbb{R}}$ ).** Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $ab \in \mathbb{R}$  iff  $a, b \in \mathbb{R}$ .

*Proof.* Direct consequence of Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , new rules are not finite-product rules*).  $\square$

**Lemma 338 (multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined).**

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.55) \quad a < b < 0 \implies \frac{1}{b} < \frac{1}{a},$$

$$(7.56) \quad a < 0 < b \implies \frac{1}{a} < \frac{1}{b},$$

$$(7.57) \quad 0 < a < b \implies \frac{1}{b} < \frac{1}{a}.$$

*Proof.* Direct consequence of **monotonicity of the multiplicative inverse in  $\mathbb{R}$** , and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ) new rule 4 is compatible with monotonicity.  $\square$

**Definition 339 (absolute value in  $\overline{\mathbb{R}}$ ).**

The absolute value in  $\mathbb{R}$  is extended to  $\overline{\mathbb{R}}$  with the value  $\infty$  at  $\pm\infty$ .

**Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ).**

Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a| = \max(-a, a)$ .

*Proof.* Direct consequence of **the definition of the absolute value in  $\mathbb{R}$** , and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 341 (finite absolute value in  $\overline{\mathbb{R}}$ ).** Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a| \in \mathbb{R}_+$  iff  $a \in \mathbb{R}$ .

*Proof.* Direct consequence of **finiteness of the absolute value in  $\mathbb{R}$** , and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 342 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative).** Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a| \geq 0$ .

*Proof.* Direct consequence of **nonnegativeness of the absolute value in  $\mathbb{R}$** , and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative).**

Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a| = a$  iff  $a \geq 0$ .

*Proof.* Direct consequence of **valued field properties of  $\mathbb{R}$  (thus the property holds in  $\mathbb{R}$ )**, and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 344 (inverse of absolute value in  $\overline{\mathbb{R}}$ ).** *Let  $a \in \overline{\mathbb{R}}$ , and  $b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(7.58) \quad |a| = b \iff a = \pm b,$$

$$(7.59) \quad |a| < b \iff -b < a < b,$$

$$(7.60) \quad |a| \leq b \iff -b \leq a \leq b.$$

*Proof.* (7.58). Direct consequence of Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ), and Lemma 324 (additive inverse in  $\overline{\mathbb{R}}$  is involution).

(7.59). Direct consequence of Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ , thus  $|a| < b$  is equivalent to  $-a < b \wedge a < b$ ), Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone, thus  $-a < b$  is equivalent to  $-b < -(-a)$ ), and Lemma 324 (additive inverse in  $\overline{\mathbb{R}}$  is involution).

(7.60). Direct consequence of (7.58) and (7.59).  $\square$

**Lemma 345 (absolute value in  $\overline{\mathbb{R}}$  is idempotent).** *The absolute value in  $\overline{\mathbb{R}}$  is idempotent.*

*Proof.* Direct consequence of Lemma 342 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative), and Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative, with  $a \stackrel{\text{def.}}{=} |a|$ ).  $\square$

**Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even).** *The absolute value in  $\overline{\mathbb{R}}$  is even.*

*Proof.* Direct consequence of **evenness of the absolute value in  $\mathbb{R}$** , and Definition 339 (absolute value in  $\overline{\mathbb{R}}$ , new rule is compatible with evenness).  $\square$

**Lemma 347 (absolute value in  $\overline{\mathbb{R}}$  is definite).** *The absolute value in  $\overline{\mathbb{R}}$  is definite.*

*Proof.* Direct consequence of **definiteness of the absolute value in  $\mathbb{R}$** , and Definition 339 (absolute value in  $\overline{\mathbb{R}}$ , new rule is not a zero-absolute-value rule).  $\square$

**Lemma 348 (absolute value in  $\overline{\mathbb{R}}$  is multiplicative).**

*The absolute value in  $\overline{\mathbb{R}}$  is multiplicative (when defined):*

$$(7.61) \quad \forall a, b \in \overline{\mathbb{R}}, \quad ab \text{ is well-defined} \implies |a| |b| \text{ is well-defined} \wedge |ab| = |a| |b|.$$

*Proof.* Direct consequence of **multiplicativity of the absolute value in  $\mathbb{R}$** , Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ), Lemma 347 (absolute value in  $\overline{\mathbb{R}}$  is definite, contrapositive), and Definition 339 (absolute value in  $\overline{\mathbb{R}}$ , new rule is compatible with the property).  $\square$

**Lemma 349 (absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality).**

*Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a + b$  is well-defined. Then, we have  $|a + b| \leq |a| + |b|$ .*

*Proof.* Direct consequence of Definition 339 (absolute value in  $\overline{\mathbb{R}}$ ), **the triangle inequality for the absolute value in  $\mathbb{R}$** , Definition 317 (addition in  $\overline{\mathbb{R}}$ ,  $\infty$  is absorbing for addition in  $\overline{\mathbb{R}}_+$ ), and Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $\infty$  is the maximal element).  $\square$

**Lemma 350 (absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality).**

*Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a$  and  $b$  are not both infinite (i.e.  $(a, b) \notin \{(\pm\infty, \pm\infty)\}$ ).*

*Then, we have  $||a| - |b|| \leq |a - b|$ .*

*Proof.* **Case  $a, b \in \mathbb{R}$ .** Direct consequence of **the reverse triangle inequality for the absolute value in  $\mathbb{R}$** .

**Case  $a \in \mathbb{R}, b = \pm\infty$ .** Direct consequence of Definition 317 (addition in  $\overline{\mathbb{R}}$ ,  $a - \pm\infty = \mp\infty$ , and rule 3 does not occur (both sides of the inequality)), Definition 339 (absolute value in  $\overline{\mathbb{R}}$ , thus  $|a - \pm\infty| = \infty$ ), and Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $\infty$  is the maximal element).

**Case  $a = \pm\infty, b \in \mathbb{R}$ .** Direct consequence of the previous case (swapping  $a$  and  $b$ ), and Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even).  $\square$

**Lemma 351 (absolute value in  $\overline{\mathbb{R}}$  satisfies countable triangle inequality).** Let  $I \subset \mathbb{N}$ . Let  $(a_i)_{i \in I} \in \overline{\mathbb{R}}$ . Assume that  $\sum_{i \in I} a_n$  is well-defined. Then, we have  $|\sum_{i \in I} a_i| \leq \sum_{i \in I} |a_i|$ .

*Proof.* Direct consequence of Lemma 349 (absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality), and **induction on the number of operands.**  $\square$

**Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ).**

The exponential function in  $\mathbb{R}$  and the natural logarithm function in  $\mathbb{R}_+^*$  are respectively extended to  $\overline{\mathbb{R}}$  and  $\overline{\mathbb{R}}_+$  with the following rules:

$$(7.62) \quad \exp(-\infty) = 0, \quad \exp \infty = \infty, \quad \ln 0 = -\infty, \quad \text{and} \quad \ln \infty = \infty.$$

**Lemma 353 (exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse).**

The exponential function in  $\overline{\mathbb{R}}$  and the natural logarithm function in  $\overline{\mathbb{R}}_+$  are each other inverse.

*Proof.* Direct consequence of **properties of the exponential and natural logarithm functions in  $\mathbb{R}$** , and Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ , new rules are compatible with the property).  $\square$

**Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ).** Exponentiation in  $\mathbb{R}$  (with either positive base and any exponent, or zero base and positive exponent) is extended to  $\overline{\mathbb{R}}$  with the following rule:

$$(7.63) \quad \forall a \in \overline{\mathbb{R}}_+, \forall b \in \overline{\mathbb{R}}, \quad b \ln a \text{ defined} \implies a^b \stackrel{\text{def.}}{=} \exp(b \ln a).$$

**Lemma 355 (exponentiation in  $\overline{\mathbb{R}}$ ).**

Exponentiation in  $\overline{\mathbb{R}}$  is the function  $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_+$  defined by

$$(7.64) \quad \begin{cases} 1. \quad \forall a \in \mathbb{R}_+^*, \forall b \in \mathbb{R}, & a^b \stackrel{\text{def.}}{=} \exp(b \ln a) \in \mathbb{R}_+^*, \\ 2. \quad \forall a < 1, \forall b > 0, & a^\infty = 0^b \stackrel{\text{def.}}{=} 0 \quad \wedge \quad a^{-\infty} = \infty^b \stackrel{\text{def.}}{=} \infty, \\ 3. \quad \forall a > 1, \forall b < 0, & a^\infty = 0^b \stackrel{\text{def.}}{=} \infty \quad \wedge \quad a^{-\infty} = \infty^b \stackrel{\text{def.}}{=} 0, \\ 4. \quad 0^0, \infty^0, \text{ and } 1^{\pm\infty} & \text{are undefined.} \end{cases}$$

*Proof.* Direct consequence of **the definition of exponentiation in  $\mathbb{R}$** , Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ), and Definition 326 (multiplication in  $\mathbb{R}$ ).  $\square$

*Remark 356.* Note that in the previous lemma, rules 2 and 3 are overlapping for the definition of  $0^{-\infty}$  and  $\infty^\infty$ , but they are consistent as both are equal to  $\infty$  for both rules.

**Lemma 357 (exponentiation in  $\overline{\mathbb{R}}$  of power 1).**

Let  $a \in \overline{\mathbb{R}}_+$ . Then, we have  $a^1 = a$ .

*Proof.* Direct consequence of Lemma 355 (exponentiation in  $\overline{\mathbb{R}}$ , with  $b \stackrel{\text{def.}}{=} 1$ ).  $\square$

**Lemma 358 (zero-exponentiation property in  $\overline{\mathbb{R}}$ ).** Let  $a \in \overline{\mathbb{R}}_+$ . Let  $b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.65) \quad a^b = 0 \iff (a = 0 \quad \wedge \quad b > 0) \vee (a < 1 \quad \wedge \quad b = \infty) \vee \\ (a = \infty \quad \wedge \quad b < 0) \vee (a > 1 \quad \wedge \quad b = -\infty).$$

*Proof.* Direct consequence of Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ), **positiveness of the exponential function in  $\mathbb{R}$** , Lemma 336 (infinity-product property in  $\mathbb{R}$ , (7.54)), **increasingness of the natural logarithm function in  $\mathbb{R}$  (thus it is negative on  $(0, 1)$  and positive on  $(1, \infty)$ )**.  $\square$

**Lemma 359 (one-exponentiation property in  $\overline{\mathbb{R}}$ ).** Let  $a \in \overline{\mathbb{R}}_+$ . Let  $b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.66) \quad a^b = 1 \iff b \ln a \text{ is defined} \wedge (a = 1 \vee b = 0).$$

*Proof.* Direct consequence of Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ), **the one-exponential property in  $\mathbb{R}$** , Lemma 334 (zero-product property in  $\overline{\mathbb{R}}$ ), **the 0-natural logarithm property in  $\mathbb{R}$** .  $\square$

**Lemma 360 (infinity-exponentiation property in  $\overline{\mathbb{R}}$ ).** Let  $a \in \overline{\mathbb{R}}_+$ . Let  $b \in \overline{\mathbb{R}}$ . Then,

$$(7.67) \quad a^b = \infty \iff (a = 0 \wedge b < 0) \vee (a < 1 \wedge b = -\infty) \vee \\ (a = \infty \wedge b > 0) \vee (a > 1 \wedge b = \infty).$$

*Proof.* Direct consequence of Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ), **positiveness of the exponential function in  $\mathbb{R}$** , Lemma 336 (infinity-product property in  $\overline{\mathbb{R}}$ , (7.53)), **increasingness of the natural logarithm function in  $\mathbb{R}$  (thus it is negative on  $(0, 1)$  and positive on  $(1, \infty)$ )**.  $\square$

**Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).** Let  $p \in (0, \infty)$ . Then, the power function of degree  $p$ , ( $x \mapsto |x|^p$ ), is defined on  $\overline{\mathbb{R}}$ ; it is the same function in  $\mathbb{R}$  extended with the value  $\infty$  at  $\pm\infty$ .

*Proof.* Direct consequence of **the definition of the power function in  $\mathbb{R}$** , Definition 339 (absolute value in  $\overline{\mathbb{R}}$ ), and Lemma 355 (exponentiation in  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 362 (power function in  $\overline{\mathbb{R}}$  is absolute value).**  
The power function of degree 1 in  $\overline{\mathbb{R}}$  is the absolute value.

*Proof.* Direct consequence of Lemma 361 (power function in  $\overline{\mathbb{R}}$ ), and Lemma 357 (exponentiation in  $\overline{\mathbb{R}}$  of power 1).  $\square$

**Lemma 363 (finite power function in  $\overline{\mathbb{R}}$ ).**

Let  $p \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^p \in \mathbb{R}_+$  iff  $a \in \mathbb{R}$ .

*Proof.* Direct consequence of **finiteness of the power function in  $\mathbb{R}$** , and Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 364 (power function in  $\overline{\mathbb{R}}$  is nonnegative).**

Let  $p \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^p \geq 0$ .

*Proof.* Direct consequence of **nonnegativeness of the power function in  $\mathbb{R}$** , and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with nonnegativeness).  $\square$

**Lemma 365 (compatibility of power function in  $\overline{\mathbb{R}}$  with addition).**

Let  $p, q \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^{p+q} = |a|^p |a|^q$ .

*Proof.* Direct consequence of **compatibility of the power function in  $\mathbb{R}$  with addition**, and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with the property).  $\square$

**Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication).**

Let  $p, q \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^{pq} = (|a|^p)^q$ .

*Proof.* Direct consequence of **compatibility of the power function in  $\mathbb{R}$  with multiplication**, and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with the property).  $\square$



**Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone).** Let  $p \in (0, \infty)$ . Let  $a, b \in \overline{\mathbb{R}}$ .  
Then, we have  $|a| < |b| \Rightarrow |a|^p < |b|^p$ , and thus  $|a| \leq |b| \Rightarrow |a|^p \leq |b|^p$ .

*Proof.* Direct consequence of **monotonicity of the power function in  $\mathbb{R}_+$** , and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with monotonicity).  $\square$

**Lemma 368 (inverse of power function in  $\overline{\mathbb{R}}$ ).**  
Let  $p \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}$ , and  $b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.68) \quad |a|^p = b \iff a = \pm b^{\frac{1}{p}},$$

$$(7.69) \quad |a|^p < b \iff -b^{\frac{1}{p}} < a < b^{\frac{1}{p}},$$

$$(7.70) \quad |a|^p \leq b \iff -b^{\frac{1}{p}} \leq a \leq b^{\frac{1}{p}}.$$

*Proof.* Direct consequence of **field properties of  $\mathbb{R}$  (thus  $\frac{1}{p} \in (0, \infty)$  and  $p \times \frac{1}{p} = \frac{1}{p} \times p = 1$ )**, Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative, thus  $b = |b|$ ), Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication, with  $\frac{1}{p}$  and  $p$ , thus  $|a|^p = b \Leftrightarrow |a| = b^{\frac{1}{p}}$ ), Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone, with  $\frac{1}{p}$  and  $p$ , thus  $|a|^p < b \Leftrightarrow |a| < b^{\frac{1}{p}}$  and  $|a|^p \leq b \Leftrightarrow |a| \leq b^{\frac{1}{p}}$ ), Lemma 364 (power function in  $\overline{\mathbb{R}}$  is nonnegative, thus  $b^{\frac{1}{p}} \geq 0$ ), and Lemma 344 (inverse of absolute value in  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 369 (power function in  $\overline{\mathbb{R}}$  is definite).**

Let  $p \in (0, \infty)$ . Then, the power function of degree  $p$  in  $\overline{\mathbb{R}}$  is definite.

*Proof.* Direct consequence of **definiteness of the power function in  $\mathbb{R}$** , and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is not a zero-power-function rule).  $\square$

**Lemma 370 (power function in  $\overline{\mathbb{R}}$  is multiplicative).**

Let  $p \in (0, \infty)$ .

Then, the power function of degree  $p$  is multiplicative (when defined):

$$(7.71) \quad \forall a, b \in \overline{\mathbb{R}}, \quad ab \text{ is well-defined} \implies |a|^p |b|^p \text{ is well-defined} \quad \wedge \quad |ab|^p = |a|^p |b|^p.$$

*Proof.* Direct consequence of **multiplicativity of the power function in  $\mathbb{R}$** , Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ), Lemma 369 (power function in  $\overline{\mathbb{R}}$  is definite, contrapositive), and Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with the property).  $\square$

**Lemma 371 (power function satisfies extended relaxed triangle inequality).**

Let  $p \in [1, \infty)$ . Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a + b$  is well-defined. Then,  $|a + b|^p$  is well-defined, and the power function of degree  $p$  satisfies the extended relaxed triangle inequality:

$$(7.72) \quad |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Moreover, there is equality when  $a, b \geq 0$  for  $p = 1$ , and when  $a = b$  (for all  $p \geq 1$ ).

*Proof.* Direct consequence of Lemma 361 (power function in  $\overline{\mathbb{R}}$ , new rule is compatible with the property: when  $a$  or  $b$  are infinite and not opposite,  $|a + b|^p = |a|^p$  or  $|b|^p = \infty$ ), Definition 317 (addition in  $\overline{\mathbb{R}}$ , rule 1), Definition 326 (multiplication in  $\overline{\mathbb{R}}$ , rule 1), and Lemma 300 (power function satisfies relaxed triangle inequality).  $\square$

**Lemma 372 (topology on  $\overline{\mathbb{R}}$ ).**

The totally ordered set  $(\overline{\mathbb{R}}, \leq)$  is equipped with the order topology  $\mathcal{T}_{\overline{\mathbb{R}}}(\mathcal{R}_{\overline{\mathbb{R}}}^o)$  generated by the open rays, for which the open intervals  $\mathcal{I}_{\overline{\mathbb{R}}}^o$  constitute a topological basis.



*Proof.* Direct consequence of Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), Definition 256 (*order topology*), and Lemma 258 (*topological basis of order topology*).  $\square$

**Lemma 373 (trace topology on subset of  $\overline{\mathbb{R}}$ ).** *Let  $\overline{\mathcal{T}}$  be the order topology on  $\overline{\mathbb{R}}$ . Let  $Y \subset \overline{\mathbb{R}}$ . Then,  $\mathcal{I}_Y^\circ$  is a topological basis of the trace topology  $\overline{\mathcal{T}}_Y$ , and when  $Y \in \overline{\mathcal{T}}$ , we have  $\overline{\mathcal{T}}_Y \subset \overline{\mathcal{T}}$ .  
Moreover, let  $\mathcal{T}$  be the usual topology on  $\mathbb{R}$ , and assume that  $Y \subset \mathbb{R}$ .  
Then, the trace topology  $\overline{\mathcal{T}}_Y$  is equal to  $\mathcal{T}_Y$ . In particular, we have  $\overline{\mathcal{T}}_{\mathbb{R}} = \mathcal{T} \subset \overline{\mathcal{T}}$ .*

*Proof.* From Lemma 259 (*trace of order topology, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and  $\mathcal{T} \stackrel{\text{def.}}{=} \overline{\mathcal{T}}$* ),  $\mathcal{T}_Y$  is a topology on  $Y$  with topological basis  $\mathcal{I}_Y^\circ$ , and  $Y \in \overline{\mathcal{T}}$  implies  $\overline{\mathcal{T}}_Y \subset \overline{\mathcal{T}}$ .

Moreover, from Lemma 307 (*topology on  $\mathbb{R}$ , thus  $\overline{\mathcal{T}}_{\mathbb{R}} = \mathcal{T}$* ), Lemma 252 (*trace topology on subset, with  $Y \subset \overline{\mathbb{R}}$ ,  $\mathbb{R} \subset \overline{\mathbb{R}}$  and  $Y \subset \mathbb{R}$* ), and Definition 226 (*trace of subset system, with  $Y \subset \mathbb{R} \subset \overline{\mathbb{R}}$* ), we have  $\overline{\mathcal{T}}_Y = \overline{\mathcal{T}} \cap Y = (\overline{\mathcal{T}} \cap \mathbb{R}) \cap Y = \mathcal{T} \cap Y = \mathcal{T}_Y$ . Hence, since  $\mathbb{R} \in \overline{\mathcal{T}}$ , we have  $\overline{\mathcal{T}}_{\mathbb{R}} = \mathcal{T}_{\mathbb{R}} = \mathcal{T} \subset \overline{\mathcal{T}}$ .  $\square$

**Definition 374 (tangent and arctangent in  $\overline{\mathbb{R}}$ ).**

The tangent function from  $(-\frac{\pi}{2}, \frac{\pi}{2})$  to  $\mathbb{R}$ , and the arctangent function from  $\mathbb{R}$  to  $(-\frac{\pi}{2}, \frac{\pi}{2})$  are respectively extended with the following rules:

$$(7.73) \quad \tan\left(\pm\frac{\pi}{2}\right) = \pm\infty \quad \text{and} \quad \arctan(\pm\infty) = \pm\frac{\pi}{2}.$$

**Lemma 375 (tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms).** *The tangent and arctangent functions extended to  $\overline{\mathbb{R}}$  are each other inverse increasing homeomorphisms.*

*Proof.* Direct consequence of **the fact that the tangent function on  $\mathbb{R}$  restricted to an open interval of the form  $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$  for  $k \in \mathbb{Z}$  (here with  $k \stackrel{\text{def.}}{=} 0$ ) is an increasing homeomorphism which (increasing) inverse is the arctangent function**, Lemma 373 (*trace topology on subset of  $\overline{\mathbb{R}}$* ), the topology on  $\overline{\mathbb{R}}$  is an extension of the topology on  $\mathbb{R}$ , and Definition 374 (*tangent and arctangent in  $\overline{\mathbb{R}}$ , extension rules (7.73) are compatible with the property*).  $\square$

*Remark 376.* Note that  $\overline{\mathbb{R}}$  may also be metrized using the hyperbolic tangent, extended into an homeomorphism from  $\overline{\mathbb{R}}$  to  $[-1, 1]$ .

But the usual Euclidean metric on  $\mathbb{R}$  cannot be extended into a metric on  $\overline{\mathbb{R}}$ . Indeed, the extended metric would be a continuous function (e.g. using the box topology on  $\overline{\mathbb{R}}^2$ , see Lemma 385), and it would be bounded on  $\overline{\mathbb{R}}$  compact, but the usual Euclidean metric on  $\mathbb{R}$  is unbounded.

**Lemma 377 (metric on  $\overline{\mathbb{R}}$ ).** *Let  $\bar{d} : \overline{\mathbb{R}}^2 \rightarrow [0, \pi]$  be the function defined by*

$$(7.74) \quad \forall x, y \in \overline{\mathbb{R}}, \quad \bar{d}(x, y) \stackrel{\text{def.}}{=} |\arctan y - \arctan x|.$$

*Then,  $(\overline{\mathbb{R}}, \bar{d})$  is a metric space. The function  $\bar{d}$  is called the arctangent metric on  $\overline{\mathbb{R}}$ .*

*Proof.* From Definition 374 (*tangent and arctangent in  $\overline{\mathbb{R}}$* ), we have  $\arctan(\overline{\mathbb{R}}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Thus, the subtraction and absolute value in (7.74) are those of  $\mathbb{R}$ , i.e.  $\bar{d}$  is well-defined on  $\overline{\mathbb{R}}^2$ , and we have  $\bar{d}(\overline{\mathbb{R}} \times \{\infty\}) = [0, \pi]$ . Moreover, from **the triangle inequality for the absolute value in  $\mathbb{R}$** , we have  $\bar{d}(\overline{\mathbb{R}}^2) \subset [0, \pi]$ . Hence,  $\bar{d}(\overline{\mathbb{R}}^2) = [0, \pi]$ .

Then, from **nonnegativeness of the absolute value in  $\mathbb{R}$  (thus  $\bar{d}$  is nonnegative), evenness of the absolute value in  $\mathbb{R}$  (thus  $\bar{d}$  is symmetric), definiteness of the absolute value in  $\mathbb{R}$ , Lemma 375 (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms, arctan is bijective*), **the definition of bijectivity (arctan is injective, thus  $\bar{d}$  is definite), the triangle inequality for the absolute value in  $\mathbb{R}$  (thus  $\bar{d}$  satisfies the triangle inequality)**, and Definition 17 (*distance*),  $\bar{d}$  is a metric over  $\overline{\mathbb{R}}$ .**

Therefore, from Definition 18 (*metric space*),  $(\overline{\mathbb{R}}, \bar{d})$  is a metric space.  $\square$

**Lemma 378 (equivalent topologies on  $\overline{\mathbb{R}}$ ).**

The order topology on  $\overline{\mathbb{R}}$  is the same as the topology induced by the arctangent metric. They admit the set  $\mathcal{I}_{\overline{\mathbb{R}}}^o$  of open intervals as topological basis.

*Proof.* Direct consequence of Lemma 372 (topology on  $\overline{\mathbb{R}}$ , the set  $\mathcal{I}_{\overline{\mathbb{R}}}^o$  of open intervals is a topological basis of the order topology), Lemma 375 (tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms,  $\arctan^{-1} = \tan$  is increasing), the definition of the topology induced by a metric (the set  $\{(\tan(\arctan x - r), \tan(\arctan x + r)) \mid x \in \overline{\mathbb{R}} \wedge r \in \mathbb{R}_+^*\}$  of open balls is a topological basis of the topology induced by the arctangent metric on  $\overline{\mathbb{R}}$ ), and Lemma 249 (topological basis is generator, both topological bases are identical, thus both topologies are the same).  $\square$

**Lemma 379 (convergence towards  $\pm\infty$ ).**

Let  $(x_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ . Then, we have  $\lim_{n \rightarrow \infty} x_n = -\infty$  (resp.  $\lim_{n \rightarrow \infty} x_n = \infty$ ) iff

$$(7.75) \quad \forall l \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geq N, \quad x_n \leq l \quad (\text{resp. } l \leq x_n).$$

If so, the sequence is said convergent towards  $-\infty$  (resp.  $\infty$ ).

*Proof.* Direct consequence of Lemma 372 (topology on  $\overline{\mathbb{R}}$ ), and the definition of the limit using neighborhoods.  $\square$

**Lemma 380 (convergence towards  $\pm\infty$  (discrete)).**

Let  $(x_n)_{n \in \mathbb{N}} \in \mathbb{R}$ . Then, the sequence is convergent towards  $-\infty$  (resp.  $\infty$ ) iff

$$(7.76) \quad \forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geq N, \quad x_n \leq -k \quad (\text{resp. } k \leq x_n).$$

*Proof.* Direct consequence of Lemma 379 (convergence towards  $\pm\infty$ ), the Archimedean property of  $\mathbb{R}$ , and Lemma 313 (order in  $\overline{\mathbb{R}}$  is total, transitivity).  $\square$

**Lemma 381 (limit is monotone).** Let  $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ . Assume that both sequence converge in  $\overline{\mathbb{R}}$ , and satisfy  $\exists N \in \mathbb{N}, \forall n \geq N, u_n \leq v_n$ . Then, we have  $\lim_{n \rightarrow \infty} u_n \leq \lim_{n \rightarrow \infty} v_n$ .

*Proof.* Let  $u = \lim_{n \rightarrow \infty} u_n$ , and  $v = \lim_{n \rightarrow \infty} v_n$ .

**Case  $v = \infty$ , or  $u = -\infty$ .** Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ).

**Case  $u = \infty$ , or  $v = -\infty$ .** Direct consequence of Definition 380 (convergence towards  $\pm\infty$  (discrete)), thus, we respectively have  $v = \infty$ , or  $u = -\infty$ , and there is equality).

**Case  $u, v$  are finite.** Assume that  $u > v$ . Then, from Definition 27 (convergent sequence, with the distance induced by  $|\cdot|$  in  $\mathbb{R}$ , and  $\varepsilon = \frac{u-v}{3} > 0$ ), there exists  $N_u, N_v > 0$  such that,

$$\begin{aligned} \forall n \in \mathbb{N}, \quad n \geq N_u &\implies |u_n - u| \leq \frac{u-v}{3}, \\ \forall n \in \mathbb{N}, \quad n \geq N_v &\implies |v_n - v| \leq \frac{u-v}{3}. \end{aligned}$$

Hence, using the hypothesis  $u > v$ , and from boundedness of the absolute value in  $\mathbb{R}$ , we have for all  $n \geq \max(N, N_u, N_v)$ ,

$$v_n \leq v + \frac{u-v}{3} = \frac{u+2v}{3} < \frac{2u+v}{3} = u - \frac{u-v}{3} \leq u_n,$$

which contradicts the definition of  $N$ . Therefore, we have  $u \leq v$ .  $\square$

**Lemma 382 (cluster point at  $\pm\infty$  (discrete)).**

Let  $(x_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ . The sequence admits a cluster point at  $-\infty$  (resp.  $\infty$ ) iff

$$(7.77) \quad \forall k, N \in \mathbb{N}, \exists n \geq N, \quad x_n < -k \quad (\text{resp. } x_n > k).$$

*Proof.* Direct consequence of Definition 267 (*cluster point*), and Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*).  $\square$

**Lemma 383 (limit is invariant by translation).** *Let  $(x_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ . Let  $k \in \mathbb{N}$ . Then,  $(x_{k+n})_{n \in \mathbb{N}}$  is convergent in  $\overline{\mathbb{R}}$  iff  $(x_n)_{n \in \mathbb{N}}$  is convergent in  $\overline{\mathbb{R}}$ . If so, both limits are equal.*

*Proof.* “Left” implies “right”. Assume that  $(x_{k+n})_{n \in \mathbb{N}}$  converges towards some  $l \in \overline{\mathbb{R}}$ . Then, from **the definition of the limit using neighborhoods (denoted by  $\mathcal{V}$ )**, we have

$$\forall U \in \mathcal{V}(l), \exists N \in \mathbb{N}, \forall n \geq N, \quad x_{k+n} \in U.$$

Let  $N' \stackrel{\text{def.}}{=} k + N \in \mathbb{N}$ . Then, for all  $n \geq N'$ , we have  $x_n \in U$ . Hence, from **the definition of the limit using neighborhoods**, the sequence  $(x_n)_{n \in \mathbb{N}}$  is convergent with the same limit  $l$ .

“Right” implies “left”. Assume that  $(x_n)_{n \in \mathbb{N}}$  converges towards some  $l \in \overline{\mathbb{R}}$ . Then, from **the definition of the limit using neighborhoods**, we have

$$\forall U \in \mathcal{V}(l), \exists N \in \mathbb{N}, \forall n \geq N, \quad x_n \in U.$$

Let  $N' \stackrel{\text{def.}}{=} \max(0, -k + N) \in \mathbb{N}$ . Then, for all  $n \geq N'$ , we have  $x_{k+n} \in U$ . Hence, from **the definition of the limit using neighborhoods**, the sequence  $(x_{k+n})_{n \in \mathbb{N}}$  is convergent with the same limit  $l$ .

Therefore, we have the equivalence.  $\square$

*Remark 384.* We recall that  $G^{\times m}$  denotes a set of products of subsets, see Definition 227.

**Lemma 385 (topology on  $\overline{\mathbb{R}}^m$ ).** *Let  $m \in [2.. \infty)$ . Let  $\overline{\mathcal{T}}$  be the order topology on  $\overline{\mathbb{R}}$ , or the topology induced by the arctangent metric. Then, the box topology  $\overline{\mathcal{T}}_{\overline{\mathbb{R}}^m}(\overline{\mathcal{T}}^{\times m})$  on  $\overline{\mathbb{R}}^m$  admits the set  $(\mathcal{I}_{\overline{\mathbb{R}}}^o)^{\times m}$  of open  $m$ -boxes as a topological basis.*

*Proof.* Direct consequence of Lemma 378 (*equivalent topologies on  $\overline{\mathbb{R}}$* ), and Lemma 254 (*box topology on Cartesian product*).  $\square$

**Lemma 386 (trace topology on subset of  $\overline{\mathbb{R}}^m$ ).** *Let  $m \in [2.. \infty)$ . Let  $\overline{\mathcal{T}}$  be the box topology on  $\overline{\mathbb{R}}^m$ . Let  $Y \subset \overline{\mathbb{R}}^m$ . Then, the set  $(\mathcal{I}_{\overline{\mathbb{R}}}^o)^{\times m} \cap Y$  of traces of open  $m$ -boxes on  $Y$  is a topological basis of the trace topology  $\overline{\mathcal{T}}_Y$ , and when  $Y \in \overline{\mathcal{T}}$ , we have  $\overline{\mathcal{T}}_Y \subset \overline{\mathcal{T}}$ . Moreover, let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}^m$ , and assume that  $Y \subset \mathbb{R}^m$ . Then, the trace topology  $\overline{\mathcal{T}}_Y$  is equal to  $\mathcal{T}_Y$ . In particular, we have  $\overline{\mathcal{T}}_{\mathbb{R}^m} = \mathcal{T} \subset \overline{\mathcal{T}}$ .*

*Proof.* From Lemma 252 (*trace topology on subset, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}^m$  and  $\mathcal{T} \stackrel{\text{def.}}{=} \overline{\mathcal{T}}$* ), and Lemma 385 (*topology on  $\overline{\mathbb{R}}^m$* ),  $\overline{\mathcal{T}}_Y$  is a topology on  $Y$  with topological basis  $(\mathcal{I}_{\overline{\mathbb{R}}}^o)^{\times m} \cap Y$ , and  $Y \in \overline{\mathcal{T}}$  implies  $\overline{\mathcal{T}}_Y \subset \overline{\mathcal{T}}$ .

Moreover, from Lemma 310 (*topology on  $\mathbb{R}^m$ , thus  $\overline{\mathcal{T}}_{\mathbb{R}} = \mathcal{T}$* ), Lemma 252 (*trace topology on subset, with  $Y \subset \mathbb{R}^m$ ,  $\mathbb{R}^m \subset \overline{\mathbb{R}}^m$  and  $Y \subset \mathbb{R}^m$* ), and Definition 226 (*trace of subset system, with  $Y \subset \mathbb{R}^m \subset \overline{\mathbb{R}}^m$* ), we have  $\overline{\mathcal{T}}_Y = \overline{\mathcal{T}} \cap Y = (\overline{\mathcal{T}} \cap \mathbb{R}^m) \cap Y = \mathcal{T} \cap Y = \mathcal{T}_Y$ . Hence, since  $\mathbb{R}^m \in \overline{\mathcal{T}}$ , we have  $\overline{\mathcal{T}}_{\mathbb{R}^m} = \mathcal{T}_{\mathbb{R}^m} = \mathcal{T} \subset \overline{\mathcal{T}}$ .  $\square$

**Lemma 387 (addition in  $\overline{\mathbb{R}}$  is continuous).** *Addition in  $\overline{\mathbb{R}}$  is continuous when defined, i.e. on  $\overline{\mathbb{R}}^2 \setminus \{(-\infty, \infty), (\infty, -\infty)\}$ .*

*Proof.* Direct consequence of **continuity of addition in  $\mathbb{R}$ , unboundedness of addition when an operand tends towards  $\pm\infty$  and the other is not  $\mp\infty$  (using neighborhoods)**, and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rules 1 and 2*).  $\square$

**Lemma 388 (multiplication in  $\overline{\mathbb{R}}$  is continuous).**

Multiplication in  $\overline{\mathbb{R}}$  is continuous when defined, i.e. on  $\overline{\mathbb{R}}^2 \setminus \{(0, \pm\infty), (\pm\infty, 0)\}$ .

*Proof.* Direct consequence of **continuity of multiplication in  $\mathbb{R}$ , unboundedness of multiplication when an operand tends towards  $\pm\infty$  and the other is nonzero (using neighborhoods)**, and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*).  $\square$

**Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous).** The absolute value in  $\overline{\mathbb{R}}$  is continuous.

*Proof.* Direct consequence of **continuity of the absolute value in  $\mathbb{R}$** , and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$ , absolute value is closed in  $\{\pm\infty\}$* ).  $\square$

**Lemma 390 (power function in  $\overline{\mathbb{R}}$  is continuous).**

Let  $p \in (0, \infty)$ . Then, the power function of degree  $p$  is continuous on  $\overline{\mathbb{R}}$ .

*Proof.* Direct consequence of **continuity of the power function over  $\mathbb{R}$  for  $p \geq 0$ , unboundedness of the power function in  $\mathbb{R}$  for  $p > 0$** , and Lemma 361 (*power function in  $\overline{\mathbb{R}}$ , the value at  $\pm\infty$  is  $\infty$* ).  $\square$

**Lemma 391 (limit in  $\overline{\mathbb{R}}$  is linear).**

Limit in  $\overline{\mathbb{R}}$  is linear when defined.

*Proof.* Direct consequence of Lemma 387 (*addition in  $\overline{\mathbb{R}}$  is continuous*), Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*), and Lemma 45 (*compatibility of limit with continuous functions*).  $\square$

### 7.5.2.1 Nonnegative extended real numbers

**Lemma 392 ( $\overline{\mathbb{R}}_+$  is complete).**

$(\overline{\mathbb{R}}_+, \leq)$  is closed and complete.

*Proof.* Direct consequence of Lemma 372 (*topology on  $\overline{\mathbb{R}}$ ,  $\overline{\mathbb{R}}_+^c = [-\infty, 0)$  is an open ray*), and Lemma 314 ( *$\overline{\mathbb{R}}$  is complete, for all  $A \subset \overline{\mathbb{R}}_+$ ,  $\inf A$  and  $\sup A$  exist, and belong to  $\overline{\mathbb{R}}_+$* ).  $\square$

**Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed).**

Addition in  $\overline{\mathbb{R}}_+$  is closed.

*Proof.* Direct consequence of **closedness of addition in  $\mathbb{R}_+$** , and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 394 (addition in  $\overline{\mathbb{R}}_+$  is associative).**

Addition in  $\overline{\mathbb{R}}_+$  is associative.

*Proof.* Direct consequence of Lemma 319 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*), and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative).**

Addition in  $\overline{\mathbb{R}}_+$  is commutative.

*Proof.* Direct consequence of Lemma 320 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*), and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 396 (addition in  $\overline{\mathbb{R}}_+$  is compatible with order).**

Let  $a, b, c \in \overline{\mathbb{R}}_+$ . Assume that  $a \leq b$ . Then, we have  $a + c \leq b + c$ .

*Proof.* Direct consequence of Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*), and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 397 (infinity-sum property in  $\overline{\mathbb{R}}_+$ ).** Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.78) \quad a + b = \infty \iff a = \infty \vee b = \infty.$$

*Proof.* Direct consequence of Lemma 322 (infinity-sum property in  $\overline{\mathbb{R}}$ ). □

**Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ).** Let  $I \subset \mathbb{N}$ . Let  $(a_i)_{i \in I} \in \overline{\mathbb{R}}_+$ .  
 Then,  $\sup\{\sum_{j \in J} a_j \in \overline{\mathbb{R}}_+ \mid J \subset I \wedge J \text{ is finite}\}$  exists, and belongs to  $\overline{\mathbb{R}}_+$ .  
 It is called the sum of the series of terms  $(a_i)_{i \in I}$ ; it is denoted  $\sum_{i \in I} a_i$ .

*Proof.* Direct consequence of Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed), induction on the number of operands of the finite sum (thus  $\sum_{j \in J} a_j \in \overline{\mathbb{R}}_+$ ), and Lemma 392 ( $\overline{\mathbb{R}}_+$  is complete, thus the supremum exists, and belongs to  $\overline{\mathbb{R}}_+$ ). □

**Lemma 399 (series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition).**

Let  $(a_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $I, J \subset \mathbb{N}$ . Let  $(I_j)_{j \in J}$  be a pseudopartition of  $I$ .

Then, we have  $\sum_{i \in I} a_i = \sum_{j \in J} \left( \sum_{i \in I_j} a_i \right)$ .

*Proof.* From Definition 209 (pseudopartition, thus  $I = \bigsqcup_{j \in J} I_j$ ), and Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ , with  $I$ , the  $I_j$ 's and  $J$ ), we have

$$\begin{aligned} \sum_{i \in I} a_i &= \sup \left\{ \sum_{k_i \in K_I} a_{k_i} \mid K_I \subset I \wedge K_I \text{ is finite} \right\} \\ \sum_{j \in J} \left( \sum_{i \in I_j} a_i \right) &= \sup \left\{ \sum_{k_j \in K_J} \left( \sum_{k_i \in K_{I_{k_j}}} a_{k_i} \right) \mid K_J \subset J \wedge K_{I_{k_j}} \subset I_{k_j} \wedge K_J, K_{I_{k_j}} \text{ are finite} \right\}. \end{aligned}$$

Let  $K_J \subset J$ , and for all  $k_j \in K_J$ , let  $K_{I_{k_j}} \subset I_{k_j}$ , all finite. Let  $K_I \stackrel{\text{def.}}{=} \bigsqcup_{k_j \in K_J} K_{I_{k_j}}$ . Then, obviously,  $K_I \subset I$ ,  $K_I$  is finite, and from Lemma 394 (addition in  $\overline{\mathbb{R}}_+$  is associative), and Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative), we have

$$\sum_{k_j \in K_J} \left( \sum_{k_i \in K_{I_{k_j}}} a_{k_i} \right) = \sum_{k_i \in K_I} a_{k_i}.$$

Hence, from Definition 2 (supremum, upper bound for the sum on  $I$ , and the least upper bound for the sums on the  $I_j$ 's and  $J$ ), we have  $\sum_{j \in J} \left( \sum_{i \in I_j} a_i \right) \leq \sum_{i \in I} a_i$ .

Let  $K_I \subset I$  finite. For all  $j \in J$ , let  $K_{I_j} \stackrel{\text{def.}}{=} K_I \cap I_j$ . Let  $K_J \stackrel{\text{def.}}{=} \{j \in J \mid K_{I_j} \neq \emptyset\}$ . Then, we have  $K_J \subset J$ , for all  $k_j \in K_J$ ,  $K_{I_{k_j}} \subset I_{k_j}$ , from Definition 209 (pseudopartition), they are all finite, and from Lemma 394 (addition in  $\overline{\mathbb{R}}_+$  is associative), and Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative), we have

$$\sum_{k_i \in K_I} a_{k_i} = \sum_{k_j \in K_J} \left( \sum_{k_i \in K_{I_{k_j}}} a_{k_i} \right).$$

Hence, from Definition 2 (supremum, upper bound for the sums on the  $I_j$ 's and  $J$ , and the least upper bound for the sum on  $I$ ), we have  $\sum_{i \in I} a_i \leq \sum_{j \in J} \left( \sum_{i \in I_j} a_i \right)$ .

Therefore, we have the equality. □

**Lemma 400 (series in  $\overline{\mathbb{R}}_+$  is monotone).**

Let  $(a_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $I \subset J \subset \mathbb{N}$ . Then, we have  $\sum_{i \in I} a_i \leq \sum_{j \in J} a_j$ .

*Proof.* Direct consequence of Lemma 399 (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition, with  $J \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $I_0 \stackrel{\text{def.}}{=} I$  and  $I_1 \stackrel{\text{def.}}{=} J \setminus I$ ), and Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$ ,  $\sum_{j \in J \setminus I} a_j \geq 0$ ).  $\square$**

**Lemma 401** (*subseries in  $\overline{\mathbb{R}}_+$  are lower bounds*).

Let  $(a_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Assume that  $\varphi$  is injective. Then,  $\sum_{j \in \mathbb{N}} a_{\varphi(j)} \leq \sum_{n \in \mathbb{N}} a_n$ .

*Proof.* Direct consequence of Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ), and Lemma 400 (*series in  $\overline{\mathbb{R}}_+$  is monotone, with  $I \stackrel{\text{def.}}{=} \varphi(\mathbb{N})$  and  $J \stackrel{\text{def.}}{=} \mathbb{N}$ ).  $\square$*

**Lemma 402** (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* ).

Let  $(a_p)_{p \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Assume that  $\varphi$  is bijective. Then,  $\sum_{p \in \mathbb{N}} a_p = \sum_{j \in \mathbb{N}} a_{\varphi(j)}$ .

*Proof.* Direct consequence of Lemma 401 (*subseries in  $\overline{\mathbb{R}}_+$  are lower bounds*, first used with the sequence  $(a_p)_{p \in \mathbb{N}}$  and the function  $\varphi$ , and then with the sequence  $(b_j)_{j \in \mathbb{N}} \stackrel{\text{def.}}{=} (a_{\varphi(j)})_{j \in \mathbb{N}}$  and the function  $\varphi^{-1}$  which satisfies  $\varphi \circ \varphi^{-1} = \text{Id}_{\mathbb{N}}$ ).  $\square$

**Lemma 403** (*definition of double series in  $\overline{\mathbb{R}}_+$* ).

Let  $(a_{p,q})_{p,q \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}^2$ . Assume that  $\varphi$  and  $\psi$  are bijections. Then, we have  $\sum_{j \in \mathbb{N}} a_{\varphi(j)} = \sum_{j \in \mathbb{N}} a_{\psi(j)}$ . This sum is denoted  $\sum_{p,q \in \mathbb{N}} a_{p,q}$ .

*Proof.* Direct consequence of Lemma 402 (*order is meaningless in series in  $\overline{\mathbb{R}}_+$ , with  $(a_{\varphi(j)})_{j \in \mathbb{N}}$  and  $\varphi^{-1} \circ \psi$ ).  $\square$*

**Lemma 404** (*double series in  $\overline{\mathbb{R}}_+$* ).

Let  $(a_{p,q})_{p,q \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Then, we have  $\sum_{p,q \in \mathbb{N}} a_{p,q} = \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ .

*Proof.* From **countability of  $\mathbb{N}^2$** , let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}^2$  be a bijection. Then, from Lemma 403 (*definition of double series in  $\overline{\mathbb{R}}_+$* ), let  $A \stackrel{\text{def.}}{=} \sum_{p,q \in \mathbb{N}} a_{p,q} = \sum_{j \in \mathbb{N}} a_{\varphi(j)}$  (the sum does not depend on the choice for  $\varphi$ ). Let  $B \stackrel{\text{def.}}{=} \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ . Then, from Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ), we have  $A, B \in \overline{\mathbb{R}}_+$ .

Let  $i \in \mathbb{N}$ . Let  $n \stackrel{\text{def.}}{=} \max\{\pi_1(\varphi(j)) \mid j \in [0..i]\}$  and  $m \stackrel{\text{def.}}{=} \max\{\pi_2(\varphi(j)) \mid j \in [0..i]\}$  where  $\pi_1$  and  $\pi_2$  are the first and second canonical projections from  $\mathbb{N}^2$  onto  $\mathbb{N}$ . Then, from Definition 7 (*maximum*), we have  $\varphi([0..i]) \subset [0..n] \times [0..m] \subset \mathbb{N}^2$ . Thus, from Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*), we have

$$\sum_{j \in [0..i]} a_{\varphi(j)} = \sum_{j \in \varphi([0..i])} a_j \leq \sum_{p \in [0..n]} \left( \sum_{q \in [0..m]} a_{p,q} \right) \leq B.$$

Hence, from Lemma 381 (*limit is monotone, when  $i \rightarrow \infty$ , with  $v_i \stackrel{\text{def.}}{=} B$ ), we have  $A \leq B$ .*

Let  $n, m \in \mathbb{N}$ . Let  $i \stackrel{\text{def.}}{=} \max\{\varphi^{-1}(p, q) \mid (p, q) \in [0..n] \times [0..m]\}$ . Then, from Definition 7 (*maximum*), we have  $[0..n] \times [0..m] \subset \varphi([0..i]) = \mathbb{N}^2$ . Thus, from Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*), we have

$$\sum_{p \in [0..n]} \left( \sum_{q \in [0..m]} a_{p,q} \right) \leq \sum_{j \in [0..i]} a_{\varphi(j)} \leq A.$$

Hence, from Lemma 381 (*limit is monotone, when  $m \rightarrow \infty$ , then  $n \rightarrow \infty$* ), we have  $B \leq A$ .

Therefore, we have  $\sum_{p,q \in \mathbb{N}} a_{p,q} = \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ .  $\square$

**Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ ).**

Multiplication and division in  $\overline{\mathbb{R}}$  are restricted to  $\overline{\mathbb{R}}_+$  by replacing the fourth rule in (7.50) by

$$(7.79) \quad 4'. \quad \frac{1}{0} \stackrel{\text{def.}}{=} \infty, \text{ and } \frac{1}{\infty} \stackrel{\text{def.}}{=} 0.$$

*Remark 406.* When restricting to nonnegative extended numbers, making multiplicative inverse a bijection from  $\overline{\mathbb{R}}_+$  onto itself through Definition 405 implies that for all  $a > 0$ ,  $\frac{a}{0} = \infty$ . The expressions  $0 \times \infty$ ,  $\frac{\infty}{\infty}$  and  $\frac{0}{0}$  remain undefined.

**Lemma 407 (multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined).**

When they are defined, multiplication and division in  $\overline{\mathbb{R}}_+$  are closed.

*Proof.* Direct consequence of **closedness of multiplication and division by nonzero in  $\mathbb{R}_+$** , Definition 326 (multiplication in  $\mathbb{R}$ ), and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ ).  $\square$

**Lemma 408 (zero-product property in  $\overline{\mathbb{R}}_+$ ).**

Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.80) \quad ab = 0 \iff ab \text{ is defined} \wedge (a = 0 \vee b = 0).$$

*Proof.* Direct consequence of Lemma 334 (zero-product property in  $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not a zero-product rule).  $\square$

**Lemma 409 (one-product property in  $\overline{\mathbb{R}}_+$ ).**

Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.81) \quad ab = 1 \iff \left( a = \frac{1}{b} \vee b = \frac{1}{a} \right) \wedge a, b \notin \{0, \infty\}.$$

*Proof.* Direct consequence of Lemma 335 (one-product property in  $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not a one-product rule).  $\square$

**Lemma 410 (infinity-product property in  $\overline{\mathbb{R}}_+$ ).**

Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.82) \quad ab = \infty \iff (a = \infty \wedge b \neq 0) \vee (a \neq 0 \wedge b = \infty).$$

*Proof.* Direct consequence of Lemma 336 (infinity-product property in  $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not an infinity-product rule).  $\square$

**Lemma 411 (finite-product property in  $\overline{\mathbb{R}}_+$ ).**

Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have  $ab \in \mathbb{R}_+$  iff  $a, b \in \mathbb{R}_+$ .

*Proof.* Direct consequence of Lemma 337 (finite-product property in  $\overline{\mathbb{R}}$ ), **closedness of multiplication in  $\mathbb{R}_+$** , and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not a finite-product rule).  $\square$

**Lemma 412 (multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone).**

Multiplicative inverse in  $\overline{\mathbb{R}}_+$  is nonincreasing.

*Proof.* Direct consequence of Lemma 338 (multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined), and Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is compatible with monotonicity).  $\square$

**Lemma 413 (addition in  $\overline{\mathbb{R}}_+$  is continuous).**

Addition in  $\overline{\mathbb{R}}_+$  is continuous.

*Proof.* Direct consequence of Lemma 387 (addition in  $\overline{\mathbb{R}}$  is continuous), and Definition 317 (addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur).  $\square$

**Lemma 414 (limit in  $\overline{\mathbb{R}}_+$  is positive linear).**

Limit in  $\overline{\mathbb{R}}_+$  is positive linear when defined.

*Proof.* Direct consequence of Lemma 391 (limit in  $\overline{\mathbb{R}}$  is linear).  $\square$



### 7.5.2.2 In the context of measure theory

**Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory (and probability), the third rule in (7.50) is replaced by

$$(7.83) \quad 3'. \quad 0 \times (\pm\infty) = \pm\infty \times 0 \stackrel{\text{def.}}{=} 0.$$

*Remark 416.* In the context of measure theory, making 0 an absorbing element for multiplication in  $\overline{\mathbb{R}}$  through the previous definition implies that  $\frac{\infty}{\pm\infty} = \frac{-\infty}{\pm\infty} = 0$ . The expression  $\frac{a}{0}$  (for all  $a \in \overline{\mathbb{R}}$ ) remains undefined. Note that multiplication in  $\overline{\mathbb{R}}$  is always well-defined in this context.

**Lemma 417 (multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)).**

In the context of measure theory, multiplication in  $\overline{\mathbb{R}}$  is associative.

*Proof.* Direct consequence of Lemma 329 (multiplication in  $\overline{\mathbb{R}}$  is associative when defined), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 418 (multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)).**

In the context of measure theory, multiplication in  $\overline{\mathbb{R}}$  is commutative.

*Proof.* Direct consequence of Lemma 330 (multiplication in  $\overline{\mathbb{R}}$  is commutative when defined), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 419 (multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)).**

In the context of measure theory, multiplication is (left and right) distributive over addition in  $\overline{\mathbb{R}}$ .

*Proof.* Direct consequence of Lemma 331 (multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined), Lemma 332 (multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)).**

In the context of measure theory, let  $a, b, c \in \overline{\mathbb{R}}$ . Assume that  $a \leq b$  and  $c \geq 0$ .

Then, we have  $a \times c \leq b \times c$ .

*Proof.* Direct consequence of Lemma 333 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule 3' is compatible with the property).  $\square$

**Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.84) \quad ab = 0 \iff a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 334 (zero-product property in  $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule 3' is compatible with the property).  $\square$

**Lemma 422 (one-product property in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(7.85) \quad ab = 1 \iff \left( a = \frac{1}{b} \vee b = \frac{1}{a} \right) \wedge a, b \notin \{0, \pm\infty\}.$$

*Proof.* Direct consequence of Lemma 335 (one-product property in  $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule is not a one-product rule).  $\square$



**Lemma 423 (infinity-product property in  $\bar{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a, b \in \bar{\mathbb{R}}$ . Then, we have

$$(7.86) \quad ab = \infty \iff (a = \infty \wedge b > 0) \vee (a = -\infty \wedge b < 0) \vee (a > 0 \wedge b = \infty) \vee (a < 0 \wedge b = -\infty),$$

$$(7.87) \quad ab = -\infty \iff (a = -\infty \wedge b > 0) \vee (a = \infty \wedge b < 0) \vee (a > 0 \wedge b = -\infty) \vee (a < 0 \wedge b = \infty).$$

*Proof.* Direct consequence of Lemma 336 (infinity-product property in  $\bar{\mathbb{R}}$ ), and Definition 415 (multiplication in  $\bar{\mathbb{R}}$  (measure theory), new rule is not an infinity-product rule).  $\square$

**Lemma 424 (finite-product property in  $\bar{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a, b \in \bar{\mathbb{R}}$ . Then, we have

$$(7.88) \quad ab \in \mathbb{R} \iff a, b \in \mathbb{R} \vee a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 423 (infinity-product property in  $\bar{\mathbb{R}}$  (measure theory), contrapositive), **De Morgan's laws**, and **distributivity of logical conjunction over logical disjunction**.  $\square$

**Lemma 425 (multiplication in  $\bar{\mathbb{R}}_+$  is closed (measure theory)).**

In the context of measure theory, multiplication and division are total functions  $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$ . In particular, we have  $\frac{\infty}{\infty} = \frac{0}{0} = 0$ , and for all  $a > 0$ ,  $\frac{a}{0} = \infty$ .

*Proof.* Direct consequence of Definition 326 (multiplication in  $\bar{\mathbb{R}}$ ), Definition 405 (multiplication in  $\bar{\mathbb{R}}_+$ ), and Definition 415 (multiplication in  $\bar{\mathbb{R}}$  (measure theory)).  $\square$

*Remark 426.* Of course, in the context of measure theory, multiplication and division are no longer continuous on the whole boundary of  $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$ .

**Lemma 427 (infinity-product property in  $\bar{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \bar{\mathbb{R}}_+$ . Then, we have

$$(7.89) \quad ab = \infty \iff (a = \infty \wedge b > 0) \vee (b = \infty \wedge a > 0).$$

*Proof.* Direct consequence of Lemma 410 (infinity-product property in  $\bar{\mathbb{R}}_+$ ), and Definition 415 (multiplication in  $\bar{\mathbb{R}}$  (measure theory), new rule is not an infinity-product rule).  $\square$

**Lemma 428 (finite-product property in  $\bar{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \bar{\mathbb{R}}_+$ . Then, we have

$$(7.90) \quad ab \in \mathbb{R}_+ \iff a, b \in \mathbb{R}_+ \vee a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 424 (finite-product property in  $\bar{\mathbb{R}}$  (measure theory)), and **closedness of multiplication in  $\mathbb{R}_+$** .  $\square$

**Lemma 429 (absolute value in  $\bar{\mathbb{R}}$  is multiplicative (measure theory)).**

In the context of measure theory, the absolute value in  $\bar{\mathbb{R}}$  is multiplicative:

$$(7.91) \quad \forall a, b \in \bar{\mathbb{R}}, \quad |a| |b| \text{ is well-defined} \wedge |ab| = |a| |b|.$$

*Proof.* Direct consequence of Definition 415 (multiplication in  $\bar{\mathbb{R}}$  (measure theory), multiplication is always well-defined), and Lemma 348 (absolute value in  $\bar{\mathbb{R}}$  is multiplicative).  $\square$

**Lemma 430 (exponentiation in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, the fourth rule in (7.64) is replaced by

$$(7.92) \quad 4'. \quad 0^0 = \infty^0 = 1^{\pm\infty} \stackrel{\text{def.}}{=} 1.$$

Thus, for all  $a \in \overline{\mathbb{R}}_+$ , for all  $b \in \overline{\mathbb{R}}$ , we have

$$(7.93) \quad a^0 = 1^b = 1, \quad a < 1 \Rightarrow a^\infty = 0, \quad \text{and} \quad a > 1 \Rightarrow a^\infty = \infty.$$

*Proof.* Direct consequence of Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)), and Lemma 355 (exponentiation in  $\overline{\mathbb{R}}$ , new rule only affects rule 4).  $\square$

**Lemma 431 (one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a \in \overline{\mathbb{R}}_+$ , and  $b \in \overline{\mathbb{R}}$ . Then, we have  $a^b = 1$  iff  $a = 1$  or  $b = 0$ .

*Proof.* Direct consequence of Lemma 359 (one-exponentiation property in  $\overline{\mathbb{R}}$ ), and Lemma 430 (exponentiation in  $\overline{\mathbb{R}}$  (measure theory), new rule is a compatible one-exponentiation rule).  $\square$

**Lemma 432 (zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.94) \quad a^b = 0 \iff (a = 0 \wedge b > 0) \vee (a < 1 \wedge b = \infty).$$

*Proof.* Direct consequence of Lemma 358 (zero-exponentiation property in  $\overline{\mathbb{R}}$ , with  $b \geq 0$ ).  $\square$

**Lemma 433 (values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.95) \quad a^b = 0 \iff (a = 0 \wedge b > 0) \vee (a < 1 \wedge b = \infty),$$

$$(7.96) \quad 0 < a^b < 1 \iff 0 < a < 1 \wedge 0 < b < \infty,$$

$$(7.97) \quad a^b = 1 \iff a = 1 \vee b = 0,$$

$$(7.98) \quad 1 < a^b < \infty \iff 1 < a < \infty \wedge 0 < b < \infty,$$

$$(7.99) \quad a^b = \infty \iff (a > 1 \wedge b = \infty) \vee (a = \infty \wedge b > 0).$$

*Proof.*

Direct consequence of Lemma 432 (zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)), Lemma 431 (one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)), Lemma 360 (infinity-exponentiation property in  $\overline{\mathbb{R}}$ ), and **values of the exponentiation in  $\mathbb{R}_+$  (for values in  $(0, 1) \uplus (1, \infty)$ )**.  $\square$

**Lemma 434 (finite-positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.100) \quad 0 < a^b < \infty \iff (0 < a < \infty \wedge b < \infty) \vee (a = 1 \wedge b = \infty) \vee (a \in \{0, \infty\} \wedge b = 0).$$

*Proof.* Direct consequence of Lemma 433 (values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory), with (7.96) to (7.98)), and **ordered set properties of  $\mathbb{R}_+$** .  $\square$

**Lemma 435 (finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(7.101) \quad a^b < \infty \iff a \leq 1 \vee (1 \leq a < \infty \wedge b < \infty) \vee (a = \infty \wedge b = 0).$$

*Proof.* Direct consequence of Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*), with (7.95) to (7.98)), and **ordered set properties of  $\mathbb{R}_+$** .  $\square$

**Lemma 436 (positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(7.102) \quad 0 < a^b \iff a = b = 0 \vee (0 < a \leq 1 \wedge b < \infty) \vee a \geq 1.$$

*Proof.* Direct consequence of Direct consequence of Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*), with (7.96) to (7.99)), and **ordered set properties of  $\mathbb{R}_+$** .  $\square$

**Lemma 437 (less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(7.103) \quad a^b < 1 \iff a < 1 \wedge b > 0.$$

*Proof.* Direct consequence of Direct consequence of Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*), with (7.95) and (7.96)), and **ordered set properties of  $\mathbb{R}_+$** .  $\square$

**Lemma 438 (great-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(7.104) \quad a^b > 1 \iff a > 1 \wedge b > 0.$$

*Proof.* Direct consequence of Direct consequence of Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*), with (7.98) and (7.99)), and **ordered set properties of  $\mathbb{R}_+$** .  $\square$

**Lemma 439 (power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)).** *In the context of measure theory, let  $p \in \overline{\mathbb{R}}_+$ . Let  $a \in \overline{\mathbb{R}}$ . Then,  $|a|^p$  is well-defined and nonnegative.*

*Proof.* Direct consequence of Lemma 361 (*power function in  $\overline{\mathbb{R}}$ , for  $p \in (0, \infty)$* ), Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$ , for  $p = 0$  and  $|x| \notin \{0, \infty\}$ , and for  $p = \infty$  and  $|x| \neq 1$* ), and Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*, for  $p = 0$  and  $|x| \in \{0, \infty\}$ , and for  $p = \infty$  and  $|x| = 1$ ).  $\square$

*Remark 440.* Note that, in the context of measure theory, the previous lemma also states that the power function of any degree  $p \in \overline{\mathbb{R}}_+$  is a total function  $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_+$ . Of course, the power function is no longer continuous on the whole  $\overline{\mathbb{R}}$  when  $p \stackrel{\text{def.}}{=} \infty$ .

**Lemma 441 (compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)).**

*In the context of measure theory, let  $p, q \in \overline{\mathbb{R}}_+$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^{p+q} = |a|^p |a|^q$ .*

*Proof.* **Case  $p, q \in (0, \infty)$ .**

Direct consequence of Lemma 365 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition*).

**Case  $p = 0$  or  $q = 0$  or  $|a| = 1$ .** Direct consequence of Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ), Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*, thus  $|a|^0 = 1$  or  $1^{p+q} = 1 = 1^p = 1^q$ ), and Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ).

**Case  $p = \infty$  or  $q = \infty$ , and  $|a| < 1$ .** Direct consequence of Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$ , thus  $p+q = \infty$* ), Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*, thus  $|a|^\infty = 0$ ), and Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*, thus  $|a|^p |a|^q = 0$ ).

**Case  $p = \infty$  or  $q = \infty$ , and  $|a| > 1$ .** Direct consequence of Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$ , thus  $p+q = \infty$* ), Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*, thus  $|a|^\infty = \infty$ ), Lemma 436 (*positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*, thus  $|a|^p, |a|^q > 0$ ), and Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$ , thus  $|a|^p |a|^q = \infty$ ).  $\square$*

**Lemma 442 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)).** In the context of measure theory, let  $p, q \in \overline{\mathbb{R}}_+$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|^{pq} = (|a|^p)^q$ .

*Proof.* **Case  $p, q \in (0, \infty)$ .**

Direct consequence of Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication).

**Case  $p = 0$  or  $q = 0$  or  $|a| = 1$ .**

Direct consequence of Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)), and Lemma 430 (exponentiation in  $\overline{\mathbb{R}}$  (measure theory), thus  $|a|^{pq} = |a|^0 = 1$  or  $1^{pq} = 1 = 1^p = 1^q$ ).

**Case  $p = \infty$  and  $q > 0$ , or  $q = \infty$  and  $p > 0$ , and  $|a| < 1$ .**

Direct consequence of Lemma 427 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory), thus  $pq = \infty$ ), Lemma 437 (less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory), thus  $|a|^p < 1$ ), and Lemma 433 (values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory), value 0, thus  $|a|^\infty = 0 = (|a|^p)^q$ ).

**Case  $p = \infty$  and  $q > 0$ , or  $q = \infty$  and  $p > 0$ , and  $|a| > 1$ .** Direct consequence of Lemma 427 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory), thus  $pq = \infty$ ), Lemma 438 (greater-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory), thus  $|a|^p > 1$ ), and Lemma 433 (values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory), value  $\infty$ , thus  $|a|^\infty = \infty = (|a|^p)^q$ ).  $\square$

**Lemma 443 (compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)).**

In the context of measure theory, let  $p \in \overline{\mathbb{R}}_+$ . Let  $a \in \overline{\mathbb{R}}$ , and  $b \in \overline{\mathbb{R}}_+$ . Then, we have  $||a|^p| = |a|^p = ||a||^p$ , and  $|b|^p = b^p = |b|^p$ .

*Proof.* Direct consequence of Lemma 439 (power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)), Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative), Lemma 345 (absolute value in  $\overline{\mathbb{R}}$  is idempotent), and Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative).  $\square$

**Lemma 444 (power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)).**

In the context of measure theory, let  $p \in \overline{\mathbb{R}}_+$ . Let  $a, b \in \overline{\mathbb{R}}$ .

Then, we have  $|a| < |b| \Rightarrow |a|^p < |b|^p$ , and thus  $|a| \leq |b| \Rightarrow |a|^p \leq |b|^p$ .

*Proof.* Direct consequence of Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone, for  $p \in (0, \infty)$ ), and Lemma 430 (exponentiation in  $\overline{\mathbb{R}}$  (measure theory), new rule is compatible with monotonicity (for  $p \in \{0, \infty\}$ )).  $\square$

**Lemma 445 (power function in  $\overline{\mathbb{R}}$  is even (measure theory)).**

In the context of measure theory, let  $p \in \overline{\mathbb{R}}_+$ . Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|-a|^p = |a|^p$ .

*Proof.* Direct consequence of Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even).  $\square$

**Lemma 446 (power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)).**

In the context of measure theory, let  $p \in \overline{\mathbb{R}}_+$ . Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $|ab|^p = |a|^p |b|^p$ .

*Proof.* Direct consequence of Lemma 370 (power function in  $\overline{\mathbb{R}}$  is multiplicative, for  $p \in (0, \infty)$ ), Lemma 431 (one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory), for  $p \stackrel{\text{def.}}{=} 0$ ), and Lemma 328 (one is identity element for multiplication in  $\overline{\mathbb{R}}$ ).  $\square$

*Remark 447.* Note that the previous lemma does not hold when  $p \stackrel{\text{def.}}{=} \infty$  and logarithms of  $a$  and  $b$  are of opposite sign. Indeed, we have  $(\frac{1}{2} \times 2)^\infty = 1^\infty = 1 \neq 0 = 0 \times \infty = \frac{1}{2}^\infty \times 2^\infty$ .

**Lemma 448 (multiplication in  $\overline{\mathbb{R}}_+$  is continuous (measure theory)).**

In the context of measure theory, multiplication in  $\overline{\mathbb{R}}_+$  is continuous, except at  $(0, \infty)$  and  $(\infty, 0)$ .

*Proof.* Direct consequence of Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*), and Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*), new rule 3' is not compatible with continuity: for instance  $(\frac{1}{n} \times \infty)_{n \in \mathbb{N}}$  is a constant sequence of value  $\infty$ , which is distinct from 0, the value of the constant sequence  $(0 \times n)_{n \in \mathbb{N}}$ .  $\square$

**Lemma 449 (power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)).** *In the context of measure theory, let  $p \in \mathbb{R}_+$ . Then, the power function of degree  $p$  is continuous on  $\overline{\mathbb{R}}$ .*

*Proof.* Direct consequence of Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous, for  $p \in (0, \infty)$* ), Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*), the power function of degree 0 is a constant function, and **continuity of constant functions**.  $\square$

*Remark 450.* Note that the previous lemma does not hold when  $p \stackrel{\text{def.}}{=} \infty$  since the power function of degree  $\infty$  only takes the three values 0, 1 and  $\infty$ .

**Definition 451 (Hölder conjugates).**

Extended numbers  $p, q \in [1, \infty]$  are said *Hölder conjugates* iff  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Remark 452.* The previous definition implies that extended numbers 1 and  $\infty$  are Hölder conjugate, since from rule 4' of Definition 405, we have  $\frac{1}{\infty} = 0$ .

**Lemma 453 (Young's inequality for products (measure theory)).**

*In the context of measure theory, let  $p, q \in (1, \infty)$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .*

*Proof.* From Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*), Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$ , with  $p, q \neq 0, \pm\infty$* ), **closedness of multiplicative inverse in  $\mathbb{R}_+^*$** , Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*), Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rule 5 with  $b = p, q \in \mathbb{R}_+^*$* ), and Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  $ab$ ,  $a^p$ ,  $b^q$ ,  $\frac{a^p}{p}$ ,  $\frac{b^q}{q}$ , and  $\frac{a^p}{p} + \frac{b^q}{q}$  are well-defined, and belong to  $\overline{\mathbb{R}}_+$ .

**Case  $ab = \infty$ .** Then, from Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*,  $a = \infty$  or  $b = \infty$ ), Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$ , rule 2 with  $b = p, q > 0$* ), Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , rule 1 with  $a \stackrel{\text{def.}}{=} \frac{1}{p}, \frac{1}{q} > 0$* ), and Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rule 1, the other operand is not  $-\infty$* ), the right-hand side equals  $\infty$ . Thus, the (in)equality holds.

**Case  $ab = 0$ .** Then, the inequality holds since the right-hand side is nonnegative.

**Case  $ab \in \mathbb{R}_+^*$ .** Direct consequence of Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*, contrapositive), Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$ , contrapositive*), and Lemma 304 (*Young's inequality for products in  $\mathbb{R}$* ).

Therefore, the inequality always holds.  $\square$

**Lemma 454 (Young's inequality for products, case  $p = 2$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Let  $\varepsilon \in \mathbb{R}_+^*$ . Then, we have  $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ .*

*Proof.* Direct consequence of Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ), Lemma 453 (*Young's inequality for products (measure theory)*), with  $p = q \stackrel{\text{def.}}{=} 2$  and  $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}b \in \overline{\mathbb{R}}_+$ , **properties of square root and multiplicative inverse in  $\mathbb{R}_+^*$** , Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*), Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*), and Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*).  $\square$

**Lemma 455** (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* ). Let  $p, q \in (0, \infty]$ . Assume that  $p < q$ . Let  $r \in [p, q]$ . Let  $\alpha \stackrel{\text{def.}}{=} \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}$ . Then,  $\alpha \in [0, 1]$ , and  $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ .

*Proof.* Direct consequence of Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$ , all multiplicative inverses exist*), Lemma 412 (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone,  $\frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{p}$  belong to  $[0, 1]$ , thus are finite*), and **ordered field properties of  $\mathbb{R}$** .  $\square$

### 7.5.3 Second countability and real numbers

**Definition 456 (connected component in  $\mathbb{R}$ ).**

Let  $A \subset \mathbb{R}$ . Let  $x \in \mathbb{R}$ . The *connected component of  $A$  containing  $x$*  is the union of all open intervals  $I$  containing  $x$  and included in  $A$ ; it is denoted  $I_x^A$ .

**Lemma 457 (empty connected component).**

Let  $A \subset \mathbb{R}$ . Let  $x \in \mathbb{R}$ . Assume that  $x \notin A$ . Then, we have  $I_x^A = \emptyset$ .

*Proof.*

Direct consequence of Definition 456 (*connected component in  $\mathbb{R}$* ), and **the nullary union law**.  $\square$

**Lemma 458 (connected component of open subset of  $\mathbb{R}$  is open interval).**

Let  $O \subset \mathbb{R}$  be open. Let  $x \in \mathbb{R}$ . Assume that  $x \in O$ . Then,  $I_x^O$  is an open interval contained in  $O$ .

*Proof.* Note that from Definition 22 (*open subset*), there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subset O$ . Then, from Definition 456 (*connected component in  $\mathbb{R}$* ),  $(x - \varepsilon, x + \varepsilon) \subset I_x^O$ . Thus,  $I_x^O$  is nonempty, contains  $x$ . Moreover, from Definition 240 (*topological space, closedness under union*),  $I_x^O$  is open.

Let  $a, b, c \in \mathbb{R}$  such that  $a < c < b$  and  $a, b \in I_x^O$ . Then, from Definition 456 (*connected component in  $\mathbb{R}$* ), and **the definition of union**, there exist intervals  $I_a(x)$  and  $I_b(x)$  such that  $x, a \in I_a(x) \subset I_x^O \subset O$  and  $x, b \in I_b(x) \subset I_x^O \subset O$ . **Case  $c = x$ .** Then, we have  $c = x \in I_x^O$ . **Case  $c < x$ .** Then, we have  $a < c < x$ . Thus, since  $a, x \in I_a(x)$ , and  $I_a(x)$  is an interval, we have successively  $c \in (a, x) \subset I_a(x) \subset I_x^O$ . **Case  $x < c$ .** Then, we have  $x < c < b$ . Thus, since  $x, b \in I_b(x)$ , and  $I_b(x)$  is an interval, we have successively  $c \in (x, b) \subset I_b(x) \subset I_x^O$ . Hence, we always have  $c \in I_x^O$ . Thus, from Definition 229 (*interval, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$* ),  $I_x^O$  is an interval.

Therefore,  $I_x^O$  is an open interval contained in  $O$ .  $\square$

**Lemma 459 (connected component of open subset of  $\mathbb{R}$  is maximal).**

Let  $O \subset \mathbb{R}$  be open. Let  $x, y \in \mathbb{R}$ . Assume that  $x, y \in O$ , and  $y \in I_x^O$ . Then, we have  $I_y^O = I_x^O$ .

*Proof.* From Definition 456 (*connected component in  $\mathbb{R}$ , union*), there exists an open interval  $I$  such that  $y \in I$  and  $x \in I \subset O$ . Thus, we also have  $x \in I_y^O$ . But, from Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*),  $I_x^O$  is an open interval such that  $y \in I_x^O \subset O$ , and  $I_y^O$  is an open interval such that  $x \in I_y^O \subset O$ . Thus, from Definition 456 (*connected component in  $\mathbb{R}$* ), we have  $I_x^O \subset I_y^O$  and  $I_y^O \subset I_x^O$ . Therefore, we have  $I_x^O = I_y^O$ .  $\square$

**Lemma 460 (connected components of open subset of  $\mathbb{R}$  are equal or disjoint).**

Let  $O \subset \mathbb{R}$  be open. Let  $x, y \in \mathbb{R}$ . Assume that  $x, y \in O$ .

Then, we have either  $I_x^O = I_y^O$ , or  $I_x^O \cap I_y^O = \emptyset$ .

*Proof.* Assume that  $I_x^O \cap I_y^O \neq \emptyset$ . Then, there exists  $z \in I_x^O \cap I_y^O$ . Thus, from Lemma 459 (*connected component of open subset of  $\mathbb{R}$  is maximal*), we have  $I_z^O = I_x^O$  and  $I_z^O = I_y^O$ . Therefore,  $I_x^O$  and  $I_y^O$  are equal or disjoint.  $\square$

**Theorem 461 (countable connected components of open subsets of  $\mathbb{R}$ ).**

Let  $O \subset \mathbb{R}$  be open. Then,  $O$  is a countable union of disjoint open intervals.

*Proof.* **Case  $O$  empty.** Direct consequence of Definition 229 (*interval, thus  $O$  is an open interval*).

**Case  $O$  nonempty.** Let  $x \in O$ . Then, from Definition 22 (*open subset*), there exists  $\varepsilon > 0$  such that the open interval  $(x - \varepsilon, x + \varepsilon)$  is included in  $O$ . From **density of rational numbers in  $\mathbb{R}$** , there exists  $q \in \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon)$ . Then, by construction, we have  $x \in (x - \varepsilon, x + \varepsilon) \subset I_q^O$ . Thus, we have the inclusion  $O \subset \bigcup_{q \in \mathbb{Q} \cap O} I_q^O$ .



Conversely, let  $q \in \mathbb{Q} \cap O$ . Then, from Definition 456 (*connected component in  $\mathbb{R}$* ), we have  $I_q^O \subset O$ . Thus, we also have the other inclusion  $\bigcup_{q \in \mathbb{Q} \cap O} I_q^O \subset O$ , and the equality.

Hence, from **countability of  $\mathbb{Q}$** , Lemma 460 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*), and after eliminating doubles in the union,  $O$  is a countable union of disjoint open intervals.

Therefore, we always have the result.  $\square$

**Lemma 462 (rational approximation of lower bound of open interval).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ .

Then, there exists a sequence  $(a_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$  that is nonincreasing with limit  $a$ .

*Proof.* Note that  $-\infty \leq a < b \leq \infty$ . From **density of rational numbers in  $\mathbb{R}$** , let  $a_0 \in (a, b) \cap \mathbb{Q}$ .

**Case  $a = -\infty$ .** Then, from **density of rational numbers in  $\mathbb{R}$** , for all  $n \in \mathbb{N}$ , let  $a_{n+1}$  be in  $(-\infty, -2|a_n|) \cap \mathbb{Q}$ . Hence, from **ordered field properties of  $\mathbb{R}$** , and **the Archimedean property of  $\mathbb{R}$** , the sequence  $(a_n)_{n \in \mathbb{N}}$  belongs to  $(a, b)$ , and is nonincreasing with limit  $a = -\infty$ .

**Case  $a$  finite.** Then, from **density of rational numbers in  $\mathbb{R}$** , for all  $n \in \mathbb{N}$ , let  $a_{n+1}$  be in  $(a, \frac{a+a_n}{2}) \cap \mathbb{Q} \subset (a, b) \cap \mathbb{Q}$ . Let  $n \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have  $a_{n+1} < a_n$  and  $0 < a_{n+1} - a < \frac{a_n - a}{2}$ . Thus, a trivial finite induction on index  $i$  shows that

$$\forall i \in [0..n+1], \quad 0 < a_{n+1} - a < \frac{a_i - a}{2^{n+1-i}} < \frac{a_0 - a}{2^{n+1}}.$$

Hence, from **the squeeze theorem**, the sequence  $(a_n)_{n \in \mathbb{N}}$  is nonincreasing with limit  $a$ .  $\square$

**Lemma 463 (rational approximation of upper bound of open interval).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ .

Then, there exists a sequence  $(b_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$  that is nondecreasing with limit  $b$ .

*Proof.* Direct consequence of Lemma 462 (*rational approximation of lower bound of open interval*), with  $a \stackrel{\text{def.}}{=} -b$ ,  $b \stackrel{\text{def.}}{=} -a$ , and  $b_n \stackrel{\text{def.}}{=} -a_n$ .  $\square$

**Lemma 464 (open intervals with rational bounds cover open interval).** Let  $a, b \in \overline{\mathbb{R}}$ .

Assume that  $a < b$ . Then, there exist  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{Q}$  such that  $(a, b) = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ .

*Proof.* Direct consequence of Lemma 462 (*rational approximation of lower bound of open interval*), there exists  $(a_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$ , Lemma 463 (*rational approximation of upper bound of open interval*), there exists  $(b_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$ , Definition 27 (*convergent sequence*, with  $\varepsilon \stackrel{\text{def.}}{=} x - a$ , then  $\varepsilon \stackrel{\text{def.}}{=} b - x$  for all  $x \in (a, b)$  (hence,  $(a, b) \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ ), and Definition 240 (*topological space*, closedness under union ( $\bigcup_{n \in \mathbb{N}} (a_n, b_n) \subset (a, b)$ )).  $\square$

**Theorem 465 ( $\mathbb{R}$  is second-countable).**

Let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}$ , or the order topology.

Then,  $\{(a, b) \mid a, b \in \mathbb{Q} \wedge a < b\}$  is a topological basis of  $(\mathbb{R}, \mathcal{T})$ . Hence,  $(\mathbb{R}, \mathcal{T})$  is second-countable.

*Proof.* Direct consequence of Lemma 307 (*topology on  $\mathbb{R}$* , both topologies are equal), Lemma 464 (*open intervals with rational bounds cover open interval*), Lemma 250 (*topological subbasis*, with  $\mathcal{B} \stackrel{\text{def.}}{=} \mathcal{I}_{\mathbb{R}}^a$ ), **countability of  $\mathbb{Q} \times \mathbb{Q}_+^*$** , and Definition 260 (*second-countability*).  $\square$

**Lemma 466 ( $\mathbb{R}^m$  is second-countable).** Let  $m \in [2.. \infty)$ . Let  $\mathcal{T}$  be the usual Euclidean topology on  $\mathbb{R}^m$ , or the box topology on  $\mathbb{R}^m$  induced by the usual Euclidean topology on  $\mathbb{R}$ , or by the order topology. Then,  $\left\{ \prod_{n \in [1..m]} (a_n, b_n) \mid \forall n \in [1..m], a_n, b_n \in \mathbb{Q} \wedge a_n < b_n \right\}$  is a topological basis of  $(\mathbb{R}^m, \mathcal{T})$ . Hence,  $(\mathbb{R}^m, \mathcal{T})$  is second-countable.

*Proof.* Direct consequence of Lemma 310 (*topology on  $\mathbb{R}^m$* , all three topologies are equal), Theorem 465 ( $\mathbb{R}$  is second-countable), and Lemma 261 (*compatibility of second-countability with Cartesian product*).  $\square$



### 7.5.4 Second countability and extended real numbers

*Remark 467.*

Subsets of  $\overline{\mathbb{R}}$  are subsets of  $\mathbb{R}$ , or the union of subsets of  $\mathbb{R}$  with  $\{-\infty\}$ ,  $\{\infty\}$ , or  $\{-\infty, \infty\}$ .

**Definition 468 (projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ ).**

The *canonical projection from  $\mathcal{P}(\overline{\mathbb{R}})$  onto  $\mathcal{P}(\mathbb{R})$*  is denoted  $\downarrow$ , the *canonical injection of  $\mathcal{P}(\mathbb{R})$  into  $\mathcal{P}(\overline{\mathbb{R}})$*  is denoted  $\uparrow$ , and other injections are denoted  $\uparrow_-$ ,  $\uparrow_+$ , and  $\uparrow_{\pm}$ .

$$(7.105) \quad \forall B \subset \overline{\mathbb{R}}, \quad \downarrow B \stackrel{\text{def.}}{=} B \cap \mathbb{R} \quad \text{and} \quad \forall A \subset \mathbb{R}, \quad \uparrow A \stackrel{\text{def.}}{=} A,$$

$$(7.106) \quad \forall A \subset \mathbb{R}, \quad \uparrow_- A \stackrel{\text{def.}}{=} A \uplus \{-\infty\}, \quad \uparrow_+ A \stackrel{\text{def.}}{=} A \uplus \{\infty\}, \quad \uparrow_{\pm} A \stackrel{\text{def.}}{=} A \uplus \{-\infty, \infty\}.$$

The notation  $\uparrow_{\star}$  denote any of the four injections.

**Lemma 469 (compatibility between projection and injections in  $\overline{\mathbb{R}}$ ).**

We have the following identities:

$$(7.107) \quad \forall A \subset \mathbb{R}, \quad \downarrow \uparrow_{\star} A = A,$$

$$(7.108) \quad \forall B \subset \overline{\mathbb{R}}, \quad \begin{cases} -\infty \notin B \wedge \infty \notin B & \iff \uparrow \downarrow B = B, \\ -\infty \in B \wedge \infty \notin B & \iff \uparrow_- \downarrow B = B, \\ -\infty \notin B \wedge \infty \in B & \iff \uparrow_+ \downarrow B = B, \\ -\infty \in B \wedge \infty \in B & \iff \uparrow_{\pm} \downarrow B = B. \end{cases}$$

*Proof.* Direct consequence of Definition 468 (projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ ).  $\square$

*Remark 470.*

Note that in (7.110) in the next lemma, the complements are either in  $\mathbb{R}$  or in  $\overline{\mathbb{R}}$ , as appropriate.

For instance, it is in  $\overline{\mathbb{R}}$  in the left-hand side  $\downarrow B^c$ , and in  $\mathbb{R}$  in the right-hand side  $(\downarrow B)^c$ . And this is the converse for equalities on injections.

**Lemma 471 (compatibility of projection and injections in  $\overline{\mathbb{R}}$  with subset operations).**

We have the following identities:

$$(7.109) \quad \downarrow \emptyset = \emptyset, \quad \uparrow \emptyset = \emptyset,$$

$$(7.110) \quad \forall B \subset \overline{\mathbb{R}}, \downarrow B^c = (\downarrow B)^c, \quad \forall A \subset \mathbb{R}, \quad \begin{cases} \uparrow A^c = (\uparrow_{\pm} A)^c, & \uparrow_{\pm} A^c = (\uparrow A)^c, \\ \uparrow_- A^c = (\uparrow_+ A)^c, & \uparrow_+ A^c = (\uparrow_- A)^c, \end{cases}$$

$$(7.111) \quad \forall I \subset \mathbb{N}, \quad \begin{cases} \forall (B_i)_{i \in I} \subset \overline{\mathbb{R}}, \quad \downarrow \left( \bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} \downarrow B_i, & \downarrow \left( \bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} \downarrow B_i, \\ \forall (A_i)_{i \in I} \subset \mathbb{R}, \quad \uparrow_{\star} \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} \uparrow_{\star} A_i, & \uparrow_{\star} \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \uparrow_{\star} A_i. \end{cases}$$

*Proof.* Direct consequence of Definition 468 (projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ ).  $\square$

*Remark 472.* This remark deploys Definition 229 for the case of  $\overline{\mathbb{R}}$ .

Note that in  $\overline{\mathbb{R}}$ , inequalities with infinite values are useless since they either represent the full set, or the empty set. Thus, they are not taken into account.

The proper intervals of  $\overline{\mathbb{R}}$  are defined with two inequalities. They only contain finite values, thus they are the intervals of  $\mathbb{R}$  (either proper intervals, rays, or the full set), and they are open at their infinite ends (if ever).

The rays of  $\overline{\mathbb{R}}$  are defined with only one inequality. They contain exactly one infinite value, at which end they are closed, and the other end is open if infinite.

The full set  $\overline{\mathbb{R}}$  is the only interval with two infinite values.

*Remark 473.* In the sequel, let  $\mathcal{T}$  denote the usual topology of  $\mathbb{R}$  (i.e. either the order topology  $\mathcal{T}_{\mathbb{R}}(\mathcal{R}_{\mathbb{R}}^o)$ , or equivalently the Euclidean topology, see Lemma 307), and let  $\overline{\mathcal{T}}$  denote the usual topology of  $\overline{\mathbb{R}}$  (i.e. either the order topology  $\mathcal{T}_{\overline{\mathbb{R}}}(\mathcal{R}_{\overline{\mathbb{R}}}^o)$ , or equivalently the topology induced by the arctangent metric, see Lemma 378).

**Lemma 474 (open down).** *Let  $B \subset \overline{\mathbb{R}}$ . Then,  $B \in \overline{\mathcal{T}}$  implies  $\downarrow B \in \mathcal{T}$ .*

*Proof.* Direct consequence of Lemma 373 (trace topology on subset of  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 475 (open up).** *Let  $A \subset \mathbb{R}$ . Then,  $A \in \mathcal{T}$  iff  $\uparrow A \in \overline{\mathcal{T}}$ .*

*Proof.* “Left” implies “right”. Direct consequence of Lemma 373 (trace topology on subset of  $\overline{\mathbb{R}}$ ).

“Right” implies “left”. Direct consequence of Lemma 474 (open down), and Lemma 469 (compatibility between projection and injections in  $\overline{\mathbb{R}}$ ,  $\downarrow \uparrow A = A$ ).  $\square$

**Lemma 476 (open subsets of  $\overline{\mathbb{R}}$ ).**

*The open subsets of  $\overline{\mathbb{R}}$  with finite values only are the open subsets of  $\mathbb{R}$ .*

*The open subsets of  $\overline{\mathbb{R}}$  with exactly one infinite value are the union of open subsets of  $\mathbb{R}$  with an open ray (either left or right).*

*The open subsets of  $\overline{\mathbb{R}}$  with two infinite values are the union of open subsets of  $\mathbb{R}$  with a left open ray and a right open ray.*

*Proof.* “Left” implies “right”. Let  $\overline{O} \in \overline{\mathcal{T}}$ .

Then, from Lemma 474 (open down), we have  $\downarrow \overline{O} \in \mathcal{T}$ .

**Case  $\overline{O} \subset \mathbb{R}$ .**

Then, from Definition 468 (projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ ), we have  $\downarrow \overline{O} = \overline{O} \in \mathcal{T}$ .

**Case  $-\infty \in \overline{O}$  and  $\infty \notin \overline{O}$ .** Then, from Lemma 372 (topology on  $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exists  $b \in \mathbb{R}$  such that  $[-\infty, b) \subset \overline{O}$ , and we have  $\overline{O} = \downarrow \overline{O} \cup [-\infty, b)$ .

**Case  $-\infty \notin \overline{O}$  and  $\infty \in \overline{O}$ .** Then, from Lemma 372 (topology on  $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exists  $a \in \mathbb{R}$  such that  $(a, \infty] \subset \overline{O}$ , and we have  $\overline{O} = \downarrow \overline{O} \cup (a, \infty]$ .

**Case  $\{-\infty, \infty\} \subset \overline{O}$ .** Then, from Lemma 372 (topology on  $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exist  $a, b \in \mathbb{R}$  such that  $[-\infty, b), (a, \infty] \subset \overline{O}$ , and we have  $\overline{O} = [-\infty, b) \cup \downarrow \overline{O} \cup (a, \infty]$ .

“Right” implies “left”. Let  $O \in \mathcal{T}$ . Then, from Lemma 475 (open up), and **closedness of open subsets under union**,  $\uparrow O$ , and the union of  $\uparrow O$  with open rays are open subsets of  $\overline{\mathbb{R}}$ .  $\square$

**Lemma 477 (countable connected components of open subsets of  $\overline{\mathbb{R}}$ ).**

*All open subsets of  $\overline{\mathbb{R}}$  are countable disjoint unions of open intervals of  $\overline{\mathbb{R}}$ .*

*As a consequence, the unions in the previous lemma can be specified as disjoint unions.*

*Proof.* Let  $\overline{O} \in \overline{\mathcal{T}}$ . Then, from Lemma 476 (open subsets of  $\overline{\mathbb{R}}$ ), there exists  $O \in \mathcal{T}$ , and  $a, b \in \mathbb{R}$  such that  $\overline{O}$  equals either  $O$ ,  $[-\infty, b) \cup O$ ,  $O \cup (a, \infty]$ , or  $[-\infty, b) \cup O \cup (a, \infty]$ . Moreover, from Theorem 461 (countable connected components of open subsets of  $\mathbb{R}$ ), there exists  $I \subset \mathbb{N}$ , and disjoint open intervals  $(A_i)_{i \in I} \in \mathcal{I}^o(\mathbb{R})$  such that  $O = \bigsqcup_{i \in I} A_i$ .

Let  $J_a \stackrel{\text{def.}}{=} \{i \in I \mid A_i \cap (a, \infty] = \emptyset\}$ , and  $J_b \stackrel{\text{def.}}{=} \{i \in I \mid A_i \cap [-\infty, b) = \emptyset\}$ . If there exists  $i_0 \in I$  such that  $a \in A_{i_0} \stackrel{\text{def.}}{=} (a_{i_0}, b_{i_0})$ , then let  $a' \stackrel{\text{def.}}{=} a_{i_0}$ , otherwise let  $a' \stackrel{\text{def.}}{=} a$ . Symmetrically, if there exists  $i_1 \in I$  such that  $a \in A_{i_1} \stackrel{\text{def.}}{=} (a_{i_1}, b_{i_1})$ , then let  $b' \stackrel{\text{def.}}{=} b_{i_1}$ , otherwise let  $b' \stackrel{\text{def.}}{=} b$ .

Therefore, we have either

$$\overline{O} = \bigsqcup_{i \in I} A_i, \quad \overline{O} = [-\infty, b') \sqcup \bigsqcup_{j \in J_b} A_j, \quad \overline{O} = \bigsqcup_{j \in J_a} A_j \sqcup (a', \infty], \text{ or}$$

$$\overline{O} = [-\infty, b') \sqcup \bigsqcup_{j \in J_a \cap J_b} A_j \sqcup (a', \infty].$$

$\square$

**Lemma 478 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ ).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ . Then, there exist  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{Q}$  such that

$$(7.112) \quad (a, b) = \bigcup_{n \in \mathbb{N}} (a_n, b_n), \quad [-\infty, b) = \bigcup_{n \in \mathbb{N}} [-\infty, b_n), \quad \text{and} \quad (a, \infty] = \bigcup_{n \in \mathbb{N}} (a_n, \infty].$$

*Proof.* Direct consequence of Lemma 464 (open intervals with rational bounds cover open interval), Lemma 462 (rational approximation of lower bound of open interval), and Lemma 463 (rational approximation of upper bound of open interval).  $\square$

**Lemma 479 ( $\overline{\mathbb{R}}$  is second-countable).** The open intervals with rational bounds constitute a topological basis of  $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$ . Hence,  $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$  is second-countable.

*Proof.* Direct consequence of Lemma 372 (topology on  $\overline{\mathbb{R}}$ ), Lemma 477 (countable connected components of open subsets of  $\overline{\mathbb{R}}$ ), Lemma 478 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ ), Lemma 250 (topological subbasis, with  $\mathcal{B} \stackrel{\text{def.}}{=} \mathcal{I}_{\overline{\mathbb{R}}}^o$ ), **countability of  $\mathbb{Q}$  and  $\mathbb{Q} \times \mathbb{Q}_+^*$ , compatibility of union with countability**, and Definition 260 (second-countability).  $\square$

**Lemma 480 ( $\overline{\mathbb{R}}^m$  is second-countable).** Let  $m \in [2.. \infty)$ . Let  $\overline{\mathcal{T}}^{\times m}$  be the box topology on  $\overline{\mathbb{R}}^m$  induced by the usual topology on  $\overline{\mathbb{R}}$ . Then, the open  $m$ -boxes with rational bounds constitute a topological basis of  $(\overline{\mathbb{R}}^m, \overline{\mathcal{T}}^{\times m})$ . Hence,  $(\overline{\mathbb{R}}^m, \overline{\mathcal{T}}^{\times m})$  is second-countable.

*Proof.* Direct consequence of Lemma 385 (topology on  $\overline{\mathbb{R}}^m$ ), Lemma 479 ( $\overline{\mathbb{R}}$  is second-countable), and Lemma 261 (compatibility of second-countability with Cartesian product).  $\square$

### 7.5.5 Infimum, supremum

*Remark 481.*

Note that in this section, functions  $X \rightarrow \overline{\mathbb{R}}$  can actually be replaced by functions  $X \times Y \rightarrow \overline{\mathbb{R}}$  for any set  $Y$ . Then, extrema over  $X$  have to be understood as functions  $Y \rightarrow \overline{\mathbb{R}}$ , and (in)equalities are pointwise. This is used at the end of the section for extrema of sequence of functions.

**Lemma 482 (extrema of constant function).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $a \in \overline{\mathbb{R}}$ . Assume that  $f$  is constant of value  $a$ . Then,*

$$(7.113) \quad \sup(f(X)) = \max(f(X)) = \inf(f(X)) = \min(f(X)) = a.$$

*Proof.* Direct consequence of **the definition of constant function**, Lemma 8 (*equivalent definition of maximum*), Definition 7 (*maximum*), Lemma 15 (*equivalent definition of minimum*), and Definition 14 (*minimum*).  $\square$

**Lemma 483 (compatibility of extrema with affine transformation).**

*Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $a, b \in \mathbb{R}$ . Assume that  $a \geq 0$ . Then, we have*

$$(7.114) \quad \sup((af + b)(X)) = a \sup(f(X)) + b, \quad \inf((af + b)(X)) = a \inf(f(X)) + b,$$

$$(7.115) \quad \sup((-af + b)(X)) = -a \inf(f(X)) + b, \quad \inf((-af + b)(X)) = -a \sup(f(X)) + b.$$

*Proof.* (1). **Case  $a = 1$ :  $\sup((f + b)(X)) = \sup(f(X)) + b$ .** Direct consequence of Definition 2 (*supremum*), and Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined, with  $b \in \mathbb{R}$* ).

(2).  **$\sup((af + b)(X)) = a \sup(f(X)) + b$ .**

Direct consequence of (1), and Lemma 5 (*supremum is positive scalar multiplicative*).

(3).  **$\inf((af + b)(X)) = a \inf(f(X)) + b$ .** Direct consequence of Lemma 10 (*duality infimum-supremum*, first with  $af + b$ , then with  $f$ ), Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined, with  $a \stackrel{\text{def.}}{=} -1$ , everything is well-defined*), and (2).

(4).  **$\sup((-af + b)(X)) = -a \inf(f(X)) + b$ .** Direct consequence of Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined, with  $a \stackrel{\text{def.}}{=} -1$ , everything is well-defined*), Lemma 10 (*duality infimum-supremum, with  $af - b$* ), and (3).

(5).  **$\inf((-af + b)(X)) = -a \sup(f(X)) + b$ .** Direct consequence of Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined, with  $a \stackrel{\text{def.}}{=} -1$ , everything is well-defined*), Lemma 10 (*duality infimum-supremum, with  $-af + b$* ), and (2).  $\square$

**Lemma 484 (equivalent definition of finite infimum).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \mathbb{R}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$ .*

*Proof.* **“Left” implies “right”.** Assume first that  $l = \inf(f(X))$ . Then, from Definition 9 (*infimum*),  $l$  is a (finite) lower bound for  $f(X)$ . Thus, from Lemma 13 (*finite infimum discrete*), for all  $n \in \mathbb{N}$ , there exists  $x'_n \in X$  such that  $f(x'_n) < l + \frac{1}{n+1}$ . Let  $x_0 \stackrel{\text{def.}}{=} x'_0$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} \stackrel{\text{def.}}{=} \arg \min(f(x_n), f(x'_n))$ . Then, for all  $n \in \mathbb{N}$ , we have

$$f(x_{n+1}) \leq f(x_n) \leq f(x'_n) < l + \frac{1}{n+1}.$$

Thus, from **the definition of monotone sequence**, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing. Moreover, let  $k \in \mathbb{N}$ . Let  $N \stackrel{\text{def.}}{=} k$ . Let  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, from **ordered field**

**properties of  $\mathbb{R}$** , we have  $f(x_n) \leq f(x_N) = f(x_k) < l + \frac{1}{k+1}$ . Hence, from Lemma 265 (*equivalent definition of convergent sequence*), the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is convergent with limit  $l$ .

**“Right” implies “left”.** Conversely, assume now that  $l$  is a (finite) lower bound of  $f(X)$ , and that there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing with limit  $l$ . Let  $n \in \mathbb{N}$ , and let  $\varepsilon_n \stackrel{\text{def.}}{=} \frac{1}{2(n+1)} > 0$ . Then, from Definition 27 (*convergent sequence*), let  $N \in \mathbb{N}$  such that for all  $p \geq N$ ,  $|f(x_p) - l| < \varepsilon_n$ . Let  $x'_n \stackrel{\text{def.}}{=} x_N$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have  $f(x'_n) \leq l + \varepsilon_n < l + \frac{1}{n+1}$ . Hence, from Lemma 13 (*finite infimum discrete*), we have  $l = \inf(f(X))$ .

Therefore, we have the equivalence.  $\square$

**Lemma 485 (equivalent definition of finite infimum in  $\overline{\mathbb{R}}$ ).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \mathbb{R}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$ .*

*Proof.* **Case  $-\infty \in f(X)$ .** Then, from Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and Definition 9 (*infimum, lower bound*),  $-\infty$  is the only lower bound of  $f(X)$ . Thus, since  $l \in \mathbb{R}$ , both propositions “ $l = \inf(f(X))$ ” and “ $l$  is a lower bound of  $f(X)$ ” are wrong. Hence, we have the equivalence.

**Case  $f(X) \subset \mathbb{R}$ .** Direct consequence of Lemma 484 (*equivalent definition of finite infimum*).

**Case  $-\infty \notin f(X)$ , and  $\infty \in f(X)$ .** Then, from Lemma 482 (*extrema of constant function, contrapositive with  $\inf(f(X)) \neq \infty$* ), the function cannot be constant of value  $\infty$ . Thus, since  $-\infty \notin f(X)$ , let  $\tilde{x} \in X$  such that  $f(\tilde{x}) \in \mathbb{R}$ . Let  $\pi_f : X \rightarrow X$  be the projection defined by

$$\pi_f(x) \stackrel{\text{def.}}{=} \begin{cases} x & \text{when } f(x) \neq \infty, \\ \tilde{x} & \text{otherwise.} \end{cases}$$

Let  $\tilde{f} \stackrel{\text{def.}}{=} f \circ \pi_f$ . Then, by construction, we have  $\tilde{f}(X) \subset \mathbb{R}$  and  $\tilde{f} \leq f$ .

Let  $l$  be a lower bound of  $\tilde{f}(X)$ . Then,  $l$  is also a lower bound of  $f(X)$ . Conversely, let  $l$  be a lower bound of  $f(X)$ . Let  $x \in X$ . Then, from the definitions of  $\pi_f$  and  $\tilde{f}$ , we have

$$l \leq f(\pi_f(x)) = \tilde{f}(x).$$

Thus,  $l$  is also a lower bound of  $\tilde{f}(X)$ . Hence,  $f(X)$  and  $\tilde{f}(X)$  have the same lower bounds.

Moreover, from Definition 9 (*infimum, lower bound*),  $\inf(f(X))$  is also a lower bound of  $\tilde{f}(X)$ , and  $\inf(\tilde{f}(X))$  is also a lower bound of  $f(X)$ . Thus, from Definition 9 (*infimum, greatest lower bound*), we have  $\inf(f(X)) \leq \inf(\tilde{f}(X))$ , and  $\inf(\tilde{f}(X)) \leq \inf(f(X))$ . Hence, from Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total, antisymmetry*), we have  $\inf(\tilde{f}(X)) = \inf(f(X))$ .

Furthermore, let  $(\tilde{x}_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Then, from the definition of  $\pi_f$  and  $\tilde{f}$ , the sequence  $(x_n)_{n \in \mathbb{N}} \stackrel{\text{def.}}{=} \pi_f(\tilde{x}_n)_{n \in \mathbb{N}} \in X$  is such that both sequences  $(f(x_n))_{n \in \mathbb{N}}$  and  $(\tilde{f}(\tilde{x}_n))_{n \in \mathbb{N}}$  are identical. Conversely, assume that  $(x_n)_{n \in \mathbb{N}} \in X$  is such that  $(f(x_n))_{n \in \mathbb{N}}$  is in  $\mathbb{R}$ . Then, from the definition of  $\pi_f$  and  $\tilde{f}$ , both sequences  $(f(x_n))_{n \in \mathbb{N}}$  and  $(\tilde{f}(x_n))_{n \in \mathbb{N}}$  are identical.

Therefore, from what precedes, we have the equivalences  $l = \inf(f(X))$  iff  $l = \inf(\tilde{f}(X))$  iff  $l$  is a lower bound of  $\tilde{f}(X)$ , and there exists a sequence  $(\tilde{x}_n)_{n \in \mathbb{N}} \in X$  such that  $(\tilde{f}(\tilde{x}_n))_{n \in \mathbb{N}}$  is nonincreasing with limit  $l$  iff  $l$  is a lower bound of  $f(X)$ , and there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$   $\square$

**Lemma 486 (equivalent definition of infimum).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \overline{\mathbb{R}}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$  is nonincreasing with limit  $l$ .*

*Proof.* **Case  $\inf(f(X)) = -\infty$ .**

**“Left” implies “right”.** Assume first that  $l = \inf(f(X)) = -\infty$ . Then, from Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ),  $l$  is a lower bound of  $f(X)$ . Moreover, from Lemma 11 (*finite infimum, contrapositive*), for all  $m \in \mathbb{R}$ , there exists  $x \in X$  such that  $f(x) < m$ . For all  $n \in \mathbb{N}$ , let  $x'_n \in X$  such that  $f(x'_n) < -n$ . Let  $x_0 \stackrel{\text{def.}}{=} x'_0$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} \stackrel{\text{def.}}{=} \arg \min(f(x_n), f(x'_{n+1}))$ . Then, for all  $n \in \mathbb{N}$ , we have  $f(x_{n+1}) \leq f(x_n) \leq f(x'_n) < -n$ . Thus, from **the definition of monotone sequence**, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing. Moreover, let  $k \in \mathbb{N}$ . Let  $N \stackrel{\text{def.}}{=} k$ . Let  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, from Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total, transitivity*), we have

$$f(x_n) \leq f(x_N) = f(x_k) < -k.$$

Hence, from Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*), the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is convergent in  $\bar{\mathbb{R}}$  with limit  $-\infty = l$ .

**“Right” implies “left”.** Conversely, assume now that  $l$  is a lower bound of  $f(X)$ . Then, from Definition 9 (*infimum, greatest lower bound*), and Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ), we have  $-\infty \leq l \leq \inf(f(X)) = -\infty$ . Hence, we have  $l = -\infty = \inf(f(X))$ .

**Case  $\inf(f(X)) = \infty$ .**

Then, from Definition 9 (*infimum, lower bound*), and Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ), we have for all  $x \in X$ ,  $\infty = \inf(f(X)) \leq f(x) \leq \infty$ . Which means that  $f$  is a constant function of value  $\infty$ . Thus, from Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ), every extended number is a lower bound of  $f(X) = \{\infty\}$ . Moreover, for all sequence  $(x_n)_{n \in \mathbb{N}} \in X$ , the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is stationary of value  $\infty$ . Thus, from **the definition of monotone sequence**, and Lemma 34 (*stationary sequence is convergent*),  $(f(x_n))_{n \in \mathbb{N}} \in \bar{\mathbb{R}}$  is nonincreasing with limit  $\infty$ . For instance, the stationary sequence  $(\tilde{x})_{n \in \mathbb{N}}$  for some  $\tilde{x} \in X \neq \emptyset$  is such a sequence. Hence, we have the equivalence  $l = \infty (= \inf(f(X)))$  iff there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in X$  such that the sequence  $(f(x_n))_{n \in \mathbb{N}} \in \bar{\mathbb{R}}$  is nonincreasing with limit  $l$ .

**Case  $\inf(f(X)) \in \mathbb{R}$ .**

Direct consequence of Lemma 485 (*equivalent definition of finite infimum in  $\bar{\mathbb{R}}$* ).

Therefore, we always have the equivalence. □

**Lemma 487 (*infimum is smaller than supremum*).**

Let  $X$  be a nonempty set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $\inf(f(X)) \leq \sup(f(X))$ .

*Proof.* Direct consequence of Definition 9 (*infimum, lower bound*), Definition 2 (*supremum, upper bound*), and Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total, transitivity*). □

**Lemma 488 (*infimum is monotone (set)*).**

Let  $X$  and  $Y$  be nonempty sets.

Assume that  $Y \subset X$ . Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $\inf(f(X)) \leq \inf(f(Y))$ .

*Proof.* Direct consequence of Definition 9 (*infimum, lower bound with  $X$ , then greatest lower bound with  $Y$* ), and Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total, transitivity*). □

**Lemma 489 (*supremum is monotone (set)*).**

Let  $X$  and  $Y$  be nonempty sets.

Assume that  $Y \subset X$ . Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $\sup(f(Y)) \leq \sup(f(X))$ .

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 488 (*infimum is monotone (set)*), and Lemma 325 (*additive inverse in  $\bar{\mathbb{R}}$  is monotone*). □

**Lemma 490 (*infimum is monotone (function)*).**

Let  $X$  be a nonempty set.

Let  $f, g : X \rightarrow \bar{\mathbb{R}}$ . Assume that  $f \leq g$ . Then, we have  $\inf(f(X)) \leq \inf(g(X))$ .

*Proof.* Direct consequence of Definition 9 (*infimum, lower bound with  $f$ , and greatest lower bound with  $g$* ), and Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total, transitivity*). □



**Lemma 491 (supremum is monotone function).** *Let  $X$  be a nonempty set. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f \leq g$ . Then, we have  $\sup(f(X)) \leq \sup(g(X))$ .*

*Proof.* Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone), and Lemma 490 (infimum is monotone function), with  $-g \leq -f$ .  $\square$

**Lemma 492 (infimum of bounded is bounded).** *Let  $X$  be a nonempty set. Let  $a, b \in \overline{\mathbb{R}}$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . If  $f \leq b$ , then we have  $\inf(f(X)) \leq b$ . If  $a \leq f$ , then we have  $a \leq \inf(f(X))$ .*

*Proof.* Direct consequence of Lemma 482 (extrema of constant function), and Lemma 490 (infimum is monotone function).  $\square$

**Lemma 493 (supremum of bounded is bounded).** *Let  $X$  be a nonempty set. Let  $a, b \in \overline{\mathbb{R}}$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . If  $f \leq b$ , then we have  $\sup(f(X)) \leq b$ . If  $a \leq f$ , then we have  $a \leq \sup(f(X))$ .*

*Proof.* Direct consequence of Lemma 482 (extrema of constant function), and Lemma 491 (supremum is monotone function).  $\square$

**Lemma 494 (compatibility of infimum with absolute value).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $|\inf(f(X))| \leq \sup(|f(X)|)$ .*

*Proof.* From Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ,  $f, -f \leq |f|$ ), Lemma 490 (infimum is monotone function), Lemma 487 (infimum is smaller than supremum), and Lemma 491 (supremum is monotone function), we have

$$\inf(f(X)) \leq \inf |f(X)| \leq \sup |f(X)|, \quad \text{and} \quad \sup((-f)(X)) \leq \sup |f(X)|.$$

**Case  $\inf(f(X)) \geq 0$ .** Then, from Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ), we have  $|\inf(f(X))| = \inf(f(X)) \leq \sup |f(X)|$ . **Case  $\inf(f(X)) < 0$ .** Then, from Lemma 10 (duality infimum-supremum), we have  $|\inf(f(X))| = -\inf(f(X)) = \sup((-f)(X)) \leq \sup |f(X)|$ .  $\square$

**Lemma 495 (compatibility of supremum with absolute value).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $|\sup(f(X))| \leq \sup(|f(X)|)$ .*

*Proof.* Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 494 (compatibility of infimum with absolute value, with  $-f$ ), and Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even).  $\square$

**Lemma 496 (compatibility of translation with infimum of sequence).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(f_i)_{i \in I} : X \rightarrow \overline{\mathbb{R}}$ . Let  $p \in J$ . Let  $J_p \stackrel{\text{def.}}{=} [p..n_I] \neq \emptyset$ . Then, we have  $\inf_{i \in I} f_i \leq \inf_{j \in J_p} f_{\varphi(j)}$ .*

*Proof.* Direct consequence of Lemma 488 (infimum is monotone set), with  $\emptyset \neq \varphi(J_p) \subset I$ .  $\square$

**Lemma 497 (compatibility of translation with supremum of sequence).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(f_i)_{i \in I} : X \rightarrow \overline{\mathbb{R}}$ . Let  $p \in J$ . Let  $J_p \stackrel{\text{def.}}{=} [p..n_I] \neq \emptyset$ . Then, we have  $\sup_{j \in J_p} f_{\varphi(j)} \leq \sup_{i \in I} f_i$ .*

*Proof.* Direct consequence of Lemma 489 (supremum is monotone set), with  $\emptyset \neq \varphi(J_p) \subset I$ .  $\square$

**Lemma 498 (infimum of nonincreasing sequence).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(f_i)_{i \in I} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_i)_{i \in I}$  is nonincreasing. Let  $p \in J$ . Let  $J_p \stackrel{\text{def.}}{=} [p..n_I] \neq \emptyset$ . Then, we have  $\inf_{i \in I} f_i = \inf_{j \in J_p} f_{\varphi(j)}$ .*

*Proof.* Let  $i \in I$ . Let  $n \stackrel{\text{def.}}{=} \max(\varphi^{-1}(i), p)$  (i.e.  $n \in J_p$  and  $i \leq \varphi(n)$ ). Let  $x \in X$ . Then, from Definition 9 (*infimum, lower bound with  $X \stackrel{\text{def.}}{=} J_p$  and  $f \stackrel{\text{def.}}{=} (j \mapsto f_{\varphi(j)}(x))$ ), and monotonicity of the  $f_i$ 's, we have  $\inf_{j \in J_p} f_{\varphi(j)}(x) \leq f_{\varphi(n)}(x) \leq f_i(x)$ . Hence, from Definition 9 (*infimum, greatest lower bound with  $X \stackrel{\text{def.}}{=} I$  and  $f \stackrel{\text{def.}}{=} (i \mapsto f_i(x))$ ), we have  $\inf_{j \in J_p} f_{\varphi(j)}(x) \leq \inf_{i \in I} f_i(x)$ .**

Therefore, from Lemma 496 (*compatibility of translation with infimum of sequence*), we have the equality.  $\square$

**Lemma 499 (supremum of nondecreasing sequence).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(f_i)_{i \in I} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_i)_{i \in I}$  is nondecreasing. Let  $p \in J$ . Let  $J_p \stackrel{\text{def.}}{=} [p..n_I) \neq \emptyset$ . Then, we have  $\sup_{i \in I} f_i = \sup_{j \in J_p} f_{\varphi(j)}$ .*

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), and Lemma 498 (*infimum of nonincreasing sequence, with  $(-f_i)_{i \in I}$* ).  $\square$

**Lemma 500 (infimum is limit).** *Let  $X$  be a set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_n)_{n \in \mathbb{N}}$  is nonincreasing. Then,  $(f_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}$ , and we have  $\lim_{n \rightarrow \infty} f_n = \inf_{n \in \mathbb{N}} f_n$ .*

*Proof.* Let  $x \in X$ . Let  $m \stackrel{\text{def.}}{=} \inf_{n \in \mathbb{N}} f_n(x)$ .

**Case  $(f_n(x)) \equiv \infty$ .**

Direct consequence of Lemma 34 (*stationary sequence is convergent*, thus  $(f_n(x))_{n \in \mathbb{N}}$  is convergent with limit  $\infty$ ), and Lemma 482 (*extrema of constant function, thus  $m = \infty$* ).

**Case  $(f_n(x))_{n \in \mathbb{N}} \not\equiv \infty$  is bounded from below in  $\mathbb{R}$ .** Then,  $(f_n(x))_{n \in \mathbb{N}}$  is finite from some rank, say  $N_1 \in \mathbb{N}$  (and equals to  $\infty$  before). Let  $\varepsilon \in \mathbb{R}$  such that  $\varepsilon > 0$ . Then, from Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$ ,  $\infty$  is the greatest element*), and Lemma 11 (*finite infimum, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$* ), we have  $m = \inf_{k \in \mathbb{N}} f_{N_1+k}(x)$  is finite, for all  $n \in \mathbb{N}$ ,  $m \leq f_n(x)$ , and there exists  $N_2 \in \mathbb{N}$  such that  $f_{N_2}(x) < m + \varepsilon$ . Let  $N \stackrel{\text{def.}}{=} \max(N_1, N_2)$ . Let  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, since  $(f_n(x))$  is nonincreasing, and from Definition 27 (*convergent sequence*), we have

$$0 \leq f_n(x) - m \leq f_{N_2} - m < \varepsilon,$$

and thus  $(f_n(x))_{n \in \mathbb{N}}$  is convergent with limit  $m$ .

**Case  $(f_n(x))_{n \in \mathbb{N}}$  is unbounded from below in  $\mathbb{R}$ .** Then, we have

$$\forall l \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \implies f_n(x) \leq l.$$

Let  $l \in \mathbb{R}$ , and  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, from nonincreasingness of the sequence, and Lemma 379 (*convergence towards  $\pm\infty$* ), we have  $f_n(x) \leq f_N(x) \leq l$ , i.e. the sequence is convergent towards  $-\infty$ . Let  $l \in \overline{\mathbb{R}}$ . Then, from Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and Definition 9 (*infimum, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$* ), for all  $n \in \mathbb{N}$ , we have  $-\infty \leq f_n(x)$ , and  $l \leq f_n(x)$  for all  $n \in \mathbb{N}$  implies  $l \notin \mathbb{R}$  and  $l \neq \infty$ , thus  $l = \infty$ , i.e.  $m = -\infty$ .

Therefore, the sequence is always convergent, and the limit is the infimum.  $\square$

**Lemma 501 (supremum is limit).** *Let  $X$  be a set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_n)_{n \in \mathbb{N}}$  is nondecreasing. Then,  $(f_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}$ , and we have  $\lim_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} f_n$ .*

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 500 (*infimum is limit, with  $(-f_n)_{n \in \mathbb{N}}$* ), and Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous, with left operand equals the constant function of value  $-1$* ).  $\square$



**Lemma 502 (series in  $\overline{\mathbb{R}}_+$  is supremum).** Let  $I \subset \mathbb{N}$ . Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(a_i)_{i \in I} \in \overline{\mathbb{R}}_+$ . Then,  $\sum_{i \in I} a_i = \sup_{n \in J} \sum_{j \in [0..n]} a_{\varphi(j)}$ .

*Proof.* Let  $A \stackrel{\text{def.}}{=} \{\sum_{j \in [0..n]} a_{\varphi(j)} \mid n < \text{card}(I)\}$ , and  $B \stackrel{\text{def.}}{=} \{\sum_{k \in K} a_k \mid K \subset I \wedge \text{card}(K) < \infty\}$ .

From Lemma 488 (*infimum is monotone (set)*, with  $A \subset B$ ), we have  $\sup A \leq \sup B$ .

Let  $K$  be a finite subset of  $I$ . Let  $k_M \stackrel{\text{def.}}{=} \max(K) \in I$ . Then, from **the fact that an injective function between two sets of the same cardinality is bijective (thus  $\varphi$  is bijective)**, let  $n \stackrel{\text{def.}}{=} \varphi^{-1}(k_M) \in J$ . Thus, from Definition 2 (*supremum*, upper bound for  $\sup A$ ), we have  $K \subset [0, n]$ , i.e.  $\sum_{k \in K} a_k \leq \sum_{j \in [0..n]} a_{\varphi(j)} \leq \sup A$ . Hence, from Definition 2 (*supremum*, least upper bound for  $\sup B$ ), we have  $\sup B \leq \sup A$ .

Therefore, from Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ), we have  $\sum_{i \in I} a_i = \sup B = \sup A$ .  $\square$

**Lemma 503 (constant sum of monotone nonnegative functions).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $(f_i)_{i \in I}, (g_i)_{i \in I} : X \rightarrow \overline{\mathbb{R}}_+$ . Let  $\alpha \in \overline{\mathbb{R}}_+$ . Assume that  $(f_i)_{i \in I}$  is nondecreasing,  $(g_i)_{i \in I}$  is nonincreasing, and that for all  $i \in I$ , we have  $\alpha = f_i + g_i$ . Then, we have  $\alpha = \sup_{i \in I} f_i + \inf_{i \in I} g_i$ .

*Proof.* Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . We have  $n_I > 0$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ .

**Case  $\text{card}(I) = \infty$ .** Direct consequence of Lemma 501 (*supremum is limit*, with  $(f_{\varphi(n)})_{n \in \mathbb{N}}$ ), Lemma 500 (*infimum is limit*, with  $(g_{\varphi(n)})_{n \in \mathbb{N}}$ ), and Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*).

**Case  $\text{card}(I) < \infty$ .** Direct consequence of Lemma 8 (*equivalent definition of maximum*, with  $X \stackrel{\text{def.}}{=} I$ , for each  $x \in X$ ), Lemma 15 (*equivalent definition of minimum*, with  $X \stackrel{\text{def.}}{=} I$ , for each  $x \in X$ ), Definition 7 (*maximum*,  $\sup_{i \in I} f_i = \max_{i \in I} f_i = f_{\varphi(n_I-1)}$ ), and Definition 14 (*minimum*,  $\inf_{i \in I} g_i = \min_{i \in I} g_i = g_{\varphi(n_I-1)}$ ).  $\square$

### 7.5.6 Limit inferior, limit superior

**Lemma 504 (limit inferior).** *Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $x \in X$ ,  $(\inf_{p \in \mathbb{N}} f_{n+p}(x))_{n \in \mathbb{N}}$  is nondecreasing, and we have*

$$(7.116) \quad \forall x \in X, \quad \lim_{n \rightarrow \infty} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right) = \sup_{n \in \mathbb{N}} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right) \in \overline{\mathbb{R}}.$$

The limit inferior of the sequence is the function  $X \rightarrow \overline{\mathbb{R}}$  defined by

$$(7.117) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right).$$

*Proof.* Direct consequence of Lemma 488 (*infimum is monotone (set)*), and Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*, a nondecreasing sequence is convergent and its limit is its least upper bound).  $\square$

**Lemma 505 (limit inferior is  $\infty$ ).** *Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $x \in X$ . Assume that  $\liminf_{n \rightarrow \infty} f_n(x) = \infty$ . Then, we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$*

*Proof.* Let  $a \in \mathbb{R}$ . Then, from Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), we have

$$a < \liminf_{n \rightarrow \infty} f_n(x).$$

Thus, from Lemma 504 (*limit inferior*), and **the definition of the limit**, there exists  $N \in \mathbb{N}$  such that  $a < \inf_{p \in \mathbb{N}} f_{N+p}(x)$ . Hence, from Definition 9 (*infimum, lower bound*), we have for all  $p \in \mathbb{N}$ ,  $a < \inf_{p \in \mathbb{N}} f_{N+p}(x) \leq f_{N+p}(x)$ . Therefore, from **the definition of the limit**, we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$ .  $\square$

**Lemma 506 (equivalent definition of the limit inferior).**

*Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ .*

*Then, for all  $x \in X$ ,  $\liminf_{n \rightarrow \infty} f_n(x)$  is the smallest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .*

*Proof.* Let  $x \in X$ . For all  $n \in \mathbb{N}$ , let  $F_n^-(x) \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p}(x)$ .

Let  $\underline{f}(x) \stackrel{\text{def.}}{=} \liminf_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} F_n^-(x) \in \overline{\mathbb{R}}$ .

Let us first show that  $\underline{f}(x)$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x)$  is finite.** Let  $\varepsilon > 0$ . Let  $M \in \mathbb{N}$ . Then, from Lemma 504 (*limit inferior*,  $(F_n^-(x))_{n \in \mathbb{N}}$  is nondecreasing), and Definition 27 (*convergent sequence*, with  $(F_n^-(x))_{n \in \mathbb{N}}$ ), there exists  $N \in \mathbb{N}$  such that we have

$$\forall k \in [N.. \infty), \quad \underline{f}(x) - \varepsilon \leq F_k^-(x) \leq \underline{f}(x).$$

Let  $K \stackrel{\text{def.}}{=} \max(M, N)$ . Then, from Lemma 11 (*finite infimum*, for  $F_K^-$ ), there exists  $k \geq K \geq M$  such that

$$\underline{f}(x) - \varepsilon \leq F_k^-(x) \leq f_k(x) < F_k^-(x) + \varepsilon \leq \underline{f}(x) + \varepsilon.$$

Hence, from Definition 267 (*cluster point*),  $\underline{f}(x)$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x) = -\infty$ .** From Lemma 504 (*limit inferior*), and Definition 2 (*supremum, upper bound*), we have for all  $n \in \mathbb{N}$ ,  $F_n^-(x) = -\infty$ . Let  $n \in \mathbb{N}$ . Let  $a \in \mathbb{R}$ . Then,  $F_n^-(x) < a$ . Thus, from Definition 9 (*infimum, greatest lower bound, contrapositive*), there exists  $P \in \mathbb{N}$  such that  $f_{n+P}(x) < a$ . Hence, from Lemma 382 (*cluster point at  $\pm\infty$  (discrete)*),  $\underline{f}(x) = -\infty$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x) = \infty$ .** Then, from Lemma 505 (*limit inferior is  $\infty$* ), we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$ , and from **uniqueness of the cluster point of a convergent sequence**,  $\underline{f}(x) = \infty$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

Now, let  $f(x)$  be a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ . Let us show that  $\underline{f}(x) \leq f(x)$ .

**Case  $f(x) = \infty$ .** Then, from Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), we have  $\underline{f}(x) \leq f(x)$ .

**Case  $f(x) < \infty$ .** Let  $a \in \mathbb{R}$  such that  $f(x) < a$ . Let  $\varepsilon \stackrel{\text{def.}}{=} \frac{a-f(x)}{2} > 0$ . Let  $n \in \mathbb{N}$ . Then, from Definition 267 (*cluster point*), there exists  $k \geq n$  such that  $f_k(x) \leq f(x) + \varepsilon < a$ . Thus, from Definition 9 (*infimum, lower bound*),  $F_n^-(x) < a$ . Hence, from Lemma 381 (*limit is monotone*, with the stationary sequence  $v_n = a, \forall n \in \mathbb{N}$ ), Lemma 34 (*stationary sequence is convergent*), and Lemma 504 (*limit inferior,  $\lim_{n \rightarrow \infty} F_n^-(x)$  exists*), we have  $\underline{f}(x) = \lim_{n \rightarrow \infty} F_n^-(x) \leq a$ . Since this is true for all  $a > f(x)$ , we also have  $\underline{f}(x) \leq f(x)$ .

Therefore, for all  $x \in X$ ,  $\underline{f}(x)$  is the smallest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .  $\square$

**Lemma 507 (*limit inferior is invariant by translation*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(7.118) \quad \forall k \in \mathbb{N}, \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_{k+n}(x) = \liminf_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 506 (*equivalent definition of the limit inferior*), Definition 267 (*cluster point*), and Lemma 383 (*limit is invariant by translation*).  $\square$

**Lemma 508 (*limit inferior is monotone*).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_n)_{n \in \mathbb{N}} \leq (g_n)_{n \in \mathbb{N}}$  from some rank:

$$(7.119) \quad \exists N \in \mathbb{N}, \quad \forall n \in [N.. \infty), \quad \forall x \in X, \quad f_n(x) \leq g_n(x).$$

Then, we have

$$(7.120) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq \liminf_{n \rightarrow \infty} g_n(x).$$

*Proof.* Direct consequence of Lemma 507 (*limit inferior is invariant by translation*, with  $k \stackrel{\text{def.}}{=} N$ ), Lemma 504 (*limit inferior*), Lemma 490 (*infimum is monotone (function)*, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$ ), and Lemma 491 (*supremum is monotone (function)*, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$ ).  $\square$

**Lemma 509 (*limit superior*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ .

Then, for all  $x \in X$ ,  $(\sup_{p \in \mathbb{N}} f_{n+p}(x))_{n \in \mathbb{N}}$  is nonincreasing, and we have

$$(7.121) \quad \forall x \in X, \quad \lim_{n \rightarrow \infty} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right) = \inf_{n \in \mathbb{N}} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right) \quad \text{in } \overline{\mathbb{R}}.$$

The limit superior of the sequence is the function  $X \rightarrow \overline{\mathbb{R}}$  defined by

$$(7.122) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} \lim_{n \in \mathbb{N}} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right).$$

*Proof.* Let  $x \in X$ . For all  $n \in \mathbb{N}$ , let  $F_n(x) \stackrel{\text{def.}}{=} \{f_{n+p}(x) \mid p \in \mathbb{N}\}$ . Then, we have

$$\forall n \in \mathbb{N}, \quad \sup F_n(x) = \sup_{p \in \mathbb{N}} f_{n+p}(x) \quad \wedge \quad F_n(x) \subset F_{n+1}(x) \subset \overline{\mathbb{R}}.$$

Thus, the sequence  $(\sup F_n(x))_{n \in \mathbb{N}}$  is nonincreasing in  $\overline{\mathbb{R}}$ . Therefore, from Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*), the sequence is convergent and its limit is its greatest lower bound  $\inf_{n \in \mathbb{N}} (\sup F_n(x))$ .  $\square$

**Lemma 510 (duality limit inferior-limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(7.123) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) = - \liminf_{n \rightarrow \infty} (-f_n(x)).$$

*Proof.* Direct consequence of Lemma 509 (limit superior), Lemma 504 (limit inferior), and Lemma 10 (duality infimum-supremum, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$ ).  $\square$

**Lemma 511 (equivalent definition of limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ .

Then, for all  $x \in X$ ,  $\limsup_{n \rightarrow \infty} f_n(x)$  is the largest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

*Proof.* Let  $x \in X$ . Let  $\bar{f}(x) \stackrel{\text{def.}}{=} \limsup_{n \rightarrow \infty} f_n(x)$ . Then, from Lemma 510 (duality limit inferior-limit superior), and Lemma 506 (equivalent definition of the limit inferior),  $-\bar{f}(x)$  is the smallest cluster point of the sequence  $(-f_n(x))_{n \in \mathbb{N}}$ . Therefore, from Lemma 388 (multiplication in  $\overline{\mathbb{R}}$  is continuous, with  $-1$  as the first operand), and Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone),  $\bar{f}(x)$  is the largest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .  $\square$

**Lemma 512 (limit inferior is smaller than limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(7.124) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq \limsup_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 506 (equivalent definition of the limit inferior), Lemma 511 (equivalent definition of limit superior), and Lemma 313 (order in  $\overline{\mathbb{R}}$  is total, transitivity).  $\square$

**Lemma 513 (limit superior is monotone).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_n)_{n \in \mathbb{N}} \leq (g_n)_{n \in \mathbb{N}}$  from some rank:

$$(7.125) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq g_n(x).$$

Then, we have

$$(7.126) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq \limsup_{n \rightarrow \infty} g_n(x).$$

*Proof.* Direct consequence of Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone), Lemma 508 (limit inferior is monotone, with  $-g_n \leq -f_n$ ), and Lemma 510 (duality limit inferior-limit superior).  $\square$

**Lemma 514 (compatibility of limit inferior with absolute value).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(7.127) \quad \forall x \in X, \quad \left| \liminf_{n \rightarrow \infty} f_n(x) \right| \leq \limsup_{n \rightarrow \infty} |f_n(x)|.$$

*Proof.* From Lemma 494 (compatibility of infimum with absolute value, with  $X \stackrel{\text{def.}}{=} [n.. \infty) \times X$ ), for all  $x \in X$ , for all  $n \in \mathbb{N}$ , we have

$$\left| \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| \leq \sup_{p \in \mathbb{N}} |f_{n+p}(x)|.$$

Therefore, from Lemma 504 (limit inferior), Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous), Lemma 381 (limit is monotone, with  $u_n \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} |f_{n+p}(x)|$  and  $v_n \stackrel{\text{def.}}{=} \sup_{p \in \mathbb{N}} |f_{n+p}(x)|$ , and  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  are convergent in  $\overline{\mathbb{R}}$ , and Lemma 509 (limit superior), we have

$$\left| \liminf_{n \rightarrow \infty} f_n(x) \right| = \left| \lim_{n \rightarrow \infty} \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| = \lim_{n \rightarrow \infty} \left| \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| \leq \lim_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} |f_{n+p}(x)| = \limsup_{n \rightarrow \infty} |f_n(x)|.$$

□

**Lemma 515 (compatibility of limit superior with absolute value).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \mathbb{R}_+$ . Then, we have

$$(7.128) \quad \forall x \in X, \quad \left| \limsup_{n \rightarrow \infty} f_n(x) \right| \leq \limsup_{n \rightarrow \infty} |f_n(x)|.$$

*Proof.* Direct consequence of Lemma 510 (*duality limit inferior-limit superior*), Lemma 514 (*compatibility of limit inferior with absolute value, with  $-f_n$* ), and Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*). □

**Definition 516 (pointwise convergence).**

Let  $X$  be a nonempty set. A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions  $X \rightarrow \overline{\mathbb{R}}$  is said *pointwise convergent* iff for all  $x \in X$ ,  $(f_n(x))_{n \in \mathbb{N}}$  is convergent in  $\overline{\mathbb{R}}$ ; it is denoted  $f_n \xrightarrow[n \rightarrow \infty]{} f$ .

**Lemma 517 (limit inferior and limit superior of pointwise convergent).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ .

Assume that the sequence is pointwise convergent. Then, we have

$$(7.129) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Definition 516 (*pointwise convergence*), Lemma 506 (*equivalent definition of the limit inferior*), Lemma 511 (*equivalent definition of limit superior*), and **uniqueness of the cluster point of a convergent sequence**. □

**Lemma 518 (limit inferior bounded from below).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $m \in \overline{\mathbb{R}}$ .

Assume that the sequence is bounded by  $m$  from below from some rank:

$$(7.130) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad m \leq f_n(x).$$

Then, we have

$$(7.131) \quad \forall x \in X, \quad m \leq \liminf_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 508 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence, constant sequence is stationary ( $N = 0$ )*), and Lemma 517 (*limit inferior and limit superior of pointwise convergent, with the constant sequence of value  $m$* ). □

**Lemma 519 (limit inferior bounded from above).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $M \in \overline{\mathbb{R}}$ .

Assume that the sequence is bounded by  $M$  from above from some rank:

$$(7.132) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq M.$$

Then, we have

$$(7.133) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq M.$$

*Proof.* Direct consequence of Lemma 508 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence, constant sequence is stationary ( $N = 0$ )*), and Lemma 517 (*limit inferior and limit superior of pointwise convergent, with the constant sequence of value  $M$* ). □

**Lemma 520 (limit superior bounded from below).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $m \in \overline{\mathbb{R}}$ .

Assume that the sequence is bounded by  $m$  from below from some rank:

$$(7.134) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad m \leq f_n(x).$$

Then, we have

$$(7.135) \quad \forall x \in X, \quad m \leq \limsup_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 519 (limit inferior bounded from above, with  $f_n \stackrel{\text{def.}}{=} -f_n$  and  $M \stackrel{\text{def.}}{=} -m$ ), and Lemma 510 (duality limit inferior-limit superior).  $\square$

**Lemma 521 (limit superior bounded from above).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $M \in \overline{\mathbb{R}}$ .

Assume that the sequence is bounded by  $M$  from above from some rank:

$$(7.136) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq M.$$

Then, we have

$$(7.137) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq M.$$

*Proof.* Direct consequence of Lemma 518 (limit inferior bounded from below, with  $f_n \stackrel{\text{def.}}{=} -f_n$  and  $m \stackrel{\text{def.}}{=} -M$ ), and Lemma 510 (duality limit inferior-limit superior).  $\square$

**Lemma 522 (limit inferior, limit superior and pointwise convergence).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that

$$(7.138) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq \liminf_{n \rightarrow \infty} f_n(x).$$

Then, the sequence is pointwise convergent and

$$(7.139) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(x).$$

*Proof.* Let  $x \in X$ . Then, from Lemma 512 (limit inferior is smaller than limit superior), we have  $\liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} f(x) \in \overline{\mathbb{R}}$ .

**Case  $f(x) = \infty$ .** Then, from Lemma 505 (limit inferior is  $\infty$ ), we have

$$\lim_{n \rightarrow \infty} f_n(x) = \infty = f(x).$$

**Case  $f(x) = -\infty$ .** Then, from Lemma 510 (duality limit inferior-limit superior), and Lemma 505 (limit inferior is  $\infty$ ), we have  $\lim_{n \rightarrow \infty} f_n(x) = -\infty = f(x)$ .

**Case  $f(x) \in \mathbb{R}$ .** Let  $\varepsilon > 0$ . Then, we have

$$f(x) - \varepsilon < \liminf_{n \rightarrow \infty} f_n(x) = f(x) = \limsup_{n \rightarrow \infty} f_n(x) < f(x) + \varepsilon.$$

Thus, from Lemma 506 (equivalent definition of the limit inferior), there exists  $N^- \in \mathbb{N}$  such that for all  $n \geq N^-$ , we have  $f(x) - \varepsilon < f_n(x)$ , and from Lemma 511 (equivalent definition of limit superior), there exists  $N^+ \in \mathbb{N}$  such that for all  $n \geq N^+$ , we have  $f_n(x) < f(x) + \varepsilon$ .

Let  $N \stackrel{\text{def.}}{=} \max(N^-, N^+)$ . Then, for all  $n \geq N$ , we have  $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$ . Therefore, from Definition 27 (convergent sequence), the sequence  $(f_n(x))_{n \in \mathbb{N}}$  is convergent with limit  $f(x)$ .  $\square$

### 7.5.7 Truncating a numeric function

*Remark 523.* Note that in the next lemma, masking must be denoted acting on the left ( $\mathbb{1}_A f$ ) when  $E$  is of dimension greater than 1, but it is often on the right ( $f \mathbb{1}_A$ ) in the scalar case.

**Lemma 524 (restriction is masking).**

Let  $X$  be a set. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ , and  $E$  be a vector space over  $\mathbb{K}$ , or let  $E \stackrel{\text{def.}}{=} \bar{\mathbb{R}}$ . Let  $A \subset Y \subset X$ . Let  $f : Y \rightarrow E$  and  $\hat{f} : X \rightarrow E$ . Assume that  $\hat{f}|_Y = f$ .

Then, we have  $(\mathbb{1}_A \hat{f})|_A = f|_A$  and  $(\mathbb{1}_A f)|_{A^c} \equiv 0_E$ .

*Proof.* Direct consequence of **the definition of the indicator function**, and **the definition of restriction of function**.  $\square$

**Lemma 525 (masking is  $\sigma$ -additive).**

Let  $X$  be a set. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ , and  $E$  be a vector space over  $\mathbb{K}$ , or let  $E \stackrel{\text{def.}}{=} \bar{\mathbb{R}}$ . Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \subset X$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ . Let  $f : X \rightarrow E$ .

Then, we have  $\mathbb{1}_A = \sum_{i \in I} \mathbb{1}_{A_i}$ , and  $\mathbb{1}_A f = \sum_{i \in I} \mathbb{1}_{A_i} f$ .

*Proof.* Direct consequence of **the formula for the indicator of disjoint union (sum)**, Definition 209 (**pseudopartition**, either  $x \notin A$ , or  $x$  belongs to a single  $A_i$ ), **the definition of indicator function (either the identity is  $0 = 0$ , or the sums reduce to the single term  $\mathbb{1}_{A_i}(x) f(x) = f(x)$ )**.  $\square$

**Definition 526 (finite part).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . The *finite part* of  $f$  is the function  $f \mathbb{1}_{f^{-1}(\mathbb{R})}$ .

**Lemma 527 (finite part is finite).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, the finite part of  $f$  is finite.

*Proof.* Direct consequence of Definition 526 (**finite part**), Definition 312 (**extended real numbers**,  $\bar{\mathbb{R}}, \bar{\mathbb{R}} = \{\pm\infty\} \sqcup \mathbb{R}$ ), **properties of inverse image ( $X = f^{-1}(\bar{\mathbb{R}}) = f^{-1}(\pm\infty) \sqcup f^{-1}(\mathbb{R})$ )**, and **the definition of the indicator function**.  $\square$

**Definition 528 (nonnegative and nonpositive parts).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . The functions  $f^+ \stackrel{\text{def.}}{=} \max(f, 0)$  and  $f^- \stackrel{\text{def.}}{=} \max(-f, 0)$  are respectively called the *nonnegative part* of  $f$  and the *nonpositive part* of  $f$ .

**Lemma 529 (equivalent definition of nonnegative and nonpositive parts).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $f^+ = f \mathbb{1}_{f^{-1}(\bar{\mathbb{R}}_+)}$  and  $f^- = -f \mathbb{1}_{f^{-1}(\bar{\mathbb{R}}_-)}$ .

*Proof.* Direct consequence of Definition 528 (**nonnegative and nonpositive parts**), Definition 7 (**maximum**), and **the definition of the indicator function**.  $\square$

**Lemma 530 (nonnegative and nonpositive parts are nonnegative).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $f^+, f^- \geq 0$ .

*Proof.* Direct consequence of Definition 528 (**nonnegative and nonpositive parts**), and Definition 7 (**maximum**).  $\square$

**Lemma 531 (nonnegative and nonpositive parts are orthogonal).**

Let  $X$  be a set. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Let  $x \in X$ . Then, we have

$$(7.140) \quad f^+(x) = 0 \text{ (and } f^-(x) = -f(x)) \quad \vee \quad f^-(x) = 0 \text{ (and } f^+(x) = f(x)).$$



*Proof.* Direct consequence of Definition 7 (*maximum*), Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and **the partition of  $\mathbb{R}$  into negative, zero, and positive numbers**.  $\square$

**Lemma 532 (decomposition into nonnegative and nonpositive parts).**

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ .

*Proof.* Direct consequence of Lemma 531 (*nonnegative and nonpositive parts are orthogonal,  $f^+$  and  $f^-$  cannot take value  $\infty$  at the same point*), Definition 317 (*addition in  $\overline{\mathbb{R}}$* ), and Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 533 (compatibility of nonpositive and nonnegative parts with addition).**

Let  $X$  be a set. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  such that their sum is well-defined. Then, we have

$$(7.141) \quad (f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+.$$

*Proof.* Let  $x \in X$ .

**Case  $f(x)$  and  $g(x)$  finite.** From Lemma 532 (*decomposition into nonnegative and nonpositive parts*), with  $f + g$ , then  $f$  and  $g$ , we have

$$(f + g)^+(x) - (f + g)^-(x) = f(x) + g(x) = (f^+(x) - f^-(x)) + (g^+(x) - g^-(x)).$$

Then, from Definition 528 (*nonnegative and nonpositive parts*),  $(f + g)^+(x)$ ,  $(f + g)^-(x)$ ,  $f^+(x)$ ,  $f^-(x)$ ,  $g^+(x)$  and  $g^-(x)$ , are finite. Hence, from **abelian additive group properties of  $\mathbb{R}$** , we have

$$(f + g)^+(x) + f^-(x) + g^-(x) = (f + g)^-(x) + f^+(x) + g^+(x).$$

**Case  $f(x)$  or  $g(x)$  is  $\infty$ .** Then, from Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rule 3 cannot occur*), and Lemma 531 (*nonnegative and nonpositive parts are orthogonal*), we have

$$(f + g)^+(x) = f(x) + g(x) = \infty, \quad \text{and} \quad (f^+(x) = \infty \vee g^+(x) = \infty).$$

Thus, from Lemma 530 (*nonnegative and nonpositive parts are nonnegative*), Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ), Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*), and Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*), we have

$$(f + g)^+(x) + f^-(x) + g^-(x) = \infty, \quad \text{and} \quad (f + g)^-(x) + f^+(x) + g^+(x) = \infty.$$

Hence, property (7.141) is satisfied.

**Case  $f(x)$  or  $g(x)$  is  $-\infty$ .** Same reasoning with the nonpositive parts.

Therefore, we always have the identity.  $\square$

**Lemma 534 (compatibility of nonpositive and nonnegative parts with mask).**

Let  $X$  be a set. Let  $A \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $(f \mathbb{1}_A)^\pm = f^\pm \mathbb{1}_A$ .

*Proof.* Direct consequence of Definition 528 (*nonnegative and nonpositive parts*), and **nonnegativity of the indicator function**.  $\square$

**Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction).**

Let  $X$  be a set. Let  $A \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $(f|_A)^\pm = f^\pm|_A$ .

*Proof.* Direct consequence of Definition 528 (*nonnegative and nonpositive parts*), and **compatibility of restriction of function with min/max**.  $\square$





## Chapter 8

# Subset system

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*Remark 536.* Subset systems of a set  $X$  are subsets of its power set  $\mathcal{P}(X)$ .

Interesting subset systems are those closed under a family of set operations such as complement, union or intersection. The simplest one is  $\pi$ -system (see Section 8.3) that is nonempty and only closed under intersection. The most elaborate one is  $\sigma$ -algebra (see Section 8.8) that is somehow closed under all set operations, including countable union and intersection.

Measure theory and Lebesgue integration are usually based on  $\sigma$ -algebra, that is the most general subset system concept with which one can build desirable properties such as  $\sigma$ -additivity for measures, and linearity and powerful convergence theorems for the integral. Thus, the last two sections of this chapter are dedicated to ways to extend weaker subset systems, such as algebra of sets or  $\lambda$ -system, into a  $\sigma$ -algebra.

Note that when the full set is not required to be measurable,  $\sigma$ -rings (i.e. rings that are also closed under countable union) can be used instead of  $\sigma$ -algebras, and  $\delta$ -rings (i.e. rings that are also closed under countable intersection) when subsets of infinite measure are not allowed.

## 8.1 Basic properties

*Remark 537.* Mini-language to represent properties of subset systems under a compact form. All the following keywords contain an uppercase letter.

$Ne, wE, wF$  are for nonemptiness, content of the empty set, and of the full set;

$C, D', D, sD$  are for closedness under complement, local complement, set difference (or relative complement), and symmetric difference (see definitions in Chapter 2);

$Ud, U, I$  are for closedness under (binary) disjoint union, union, and intersection;

$PtfC, PtfD', PtfD$  are for the capabilities of expressing any complement, local complement, or set difference, as a finite disjoint union (i.e. a pseudopartition) of elements of the same subset system;

$Udf, Uf, If$  are for closedness under finite disjoint union, and finite union and intersection;

$Udc, Umc, Imc, Uc, Ic$  are for closedness under countable disjoint union, countable monotone union and intersection, and countable union and intersection;

$e_1; e_2 \dots$  denotes a sequence of expressions;

$[e_1 \mid e_2 \dots]$  denotes syntactic alternative: the complete expression is true for each case; for instance, the expression  $e_1([e_2 \mid e_3])$  is equivalent to  $e_1(e_2); e_1(e_3)$ .

Compact formulas express properties of subset systems, they should be understood as universally quantified. For instance, the first item of the compact formula of Lemma 538 should be read as for all  $S \subset \mathcal{P}(X)$ ,  $wE(S)$  implies  $Ne(S)$ , and  $wF(S)$  implies  $Ne(S)$ .

**Lemma 538 (nonempty and with empty or full).** *Let  $X$  be a set. Let  $S \subset \mathcal{P}(X)$ . If  $S$  contains either the empty set or the full set  $X$ , then it is nonempty. Assume that  $S$  is nonempty and closed under complement. Then, it contains the empty set if it is closed under intersection, and it contains the full set if it is closed under union. Assume that  $S$  is nonempty and closed under set difference. Then, it contains the empty set. Compact form:  $[wE \mid wF] \Rightarrow Ne; [C \wedge I \mid D] \Rightarrow Ne \Rightarrow wE; C \Rightarrow U \Rightarrow Ne \Rightarrow wF$ .*

*Proof.* Direct consequence of **the identities  $A \cap A^c = A \setminus A = \emptyset$ , and  $A \cup A^c = X$** . □

**Lemma 539 (with empty and full).** *Let  $X$  be a set. Let  $S \subset \mathcal{P}(X)$ . Assume that  $S$  is closed under complement. Then, it contains the empty set iff it contains the full set. Compact form:  $C \Rightarrow wF \Leftrightarrow wE$ .*

*Proof.* Direct consequence of **the identities  $\emptyset^c = X$ , and  $X^c = \emptyset$** . □

*Remark 540.* The local complement is the set difference when the first operand contains the second one.

**Lemma 541 (closedness under local complement and complement).** *Let  $X$  be a set. Let  $S \subset \mathcal{P}(X)$ . Assume that  $S$  contains the full set and is closed under local complement. Then, it is closed under complement. Compact form:  $wF \Rightarrow D' \Rightarrow C$ .*

*Proof.* Direct consequence of **the identity  $A^c = X \setminus A$** . □

**Lemma 542 (closedness under disjoint union and local complement).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then, it is closed under disjoint union iff it is closed under local complement.

Compact form:  $C \Rightarrow Ud \Leftrightarrow D'$ .

*Proof.* Direct consequence of **the identity  $A \setminus B = (A^c \cup B)^c$ , monotonicity of complement (then  $A \cap B = \emptyset \Leftrightarrow B \subset A^c$ ), and the identity  $A \cup B = (A^c \setminus B)^c$ .**  $\square$

**Lemma 543 (closedness under set difference and local complement).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

If  $\mathcal{S}$  is closed under set difference, then it is closed under local complement.

If  $\mathcal{S}$  is closed under intersection and local complement, then it is closed under set difference.

Compact form:  $D \Rightarrow D'$ ;  $I \Rightarrow D' \Rightarrow D$ .

*Proof.* Direct consequence of **the identity  $A \setminus B = A \setminus (A \cap B)$  with  $A \cap B \subset A$ .**  $\square$

**Lemma 544 (closedness under symmetric difference, local complement, and disjoint union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . If  $\mathcal{S}$  is closed under symmetric difference, then it is closed under local complement and disjoint union. If  $\mathcal{S}$  is closed under intersection, local complement, and disjoint union, then it is closed under symmetric difference.

Compact form:  $sD \Rightarrow [D' \mid Ud]$ ;  $I \Rightarrow D' \Rightarrow Ud \Rightarrow sD$ .

*Proof.* Direct consequence of **the identities**

$$B \subset A \implies A \setminus B = A \Delta B, \quad A \cap B = \emptyset \implies A \uplus B = A \Delta B, \\ \text{and } A \Delta B = (A \setminus (A \cap B)) \uplus (B \setminus (A \cap B)) \text{ with } A \cap B \subset A, B.$$

$\square$

**Lemma 545 (closedness under intersection and set difference).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

If  $\mathcal{S}$  is closed under complement and intersection, then it is closed under set difference.

If  $\mathcal{S}$  is closed under set difference, then it is closed under intersection.

Compact form:  $C \Rightarrow I \Rightarrow D$ ;  $D \Rightarrow I$ .

*Proof.* Direct consequence of **the identity  $A \cap B = A \setminus (A \setminus B)$ .**  $\square$

**Lemma 546 (closedness under union and disjoint union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

If  $\mathcal{S}$  is closed under set difference and disjoint union, then it is closed under union.

If  $\mathcal{S}$  is closed under union, then it is closed under disjoint union.

Compact form:  $D \Rightarrow Ud \Rightarrow U$ ;  $U \Rightarrow Ud$ .

*Proof.* Direct consequence of **the identity  $A \cup B = (A \setminus B) \uplus B$ .**  $\square$

**Lemma 547 (closedness under union and intersection).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement.

Then, it is closed under union iff it is closed under intersection.

Compact form:  $C \Rightarrow U \Leftrightarrow I$ .

*Proof.* Direct consequence of **De Morgan's laws.**  $\square$

**Lemma 548 (closedness under union and set difference).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then, it is closed under union iff it is closed under set difference.

Compact form:  $C \Rightarrow U \Leftrightarrow D$ .

*Proof.* Direct consequence of Lemma 547 (closedness under union and intersection,  $C \Rightarrow U \Leftrightarrow I$ ), and Lemma 545 (closedness under intersection and set difference,  $C \Rightarrow I \Leftrightarrow D$ ).  $\square$

*Remark 549.*

We recall that  $\Delta$  denotes the symmetric difference of subsets, see its definition in Chapter 2.

**Lemma 550 (closedness under symmetric difference and set difference).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under union. Then, it is closed under symmetric difference iff it is closed under set difference.

Compact form:  $U \Rightarrow sD \Leftrightarrow D$ .

*Proof.* Direct consequence of **the identity  $A \setminus B = (A \cup B) \Delta B$** , and **the definition of symmetric difference**.  $\square$

**Lemma 551 (closedness under symmetric difference and intersection).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under symmetric difference and intersection.

Then, it is closed under union.

Compact form:  $sD \Rightarrow I \Rightarrow U$ .

*Proof.* Direct consequence of **the identity  $A \cup B = (A \Delta B) \Delta (A \cap B)$** .  $\square$

### 8.1.1 Finite properties

**Lemma 552 (nonempty is with empty).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then,  $\mathcal{S}$  contains the empty set if it is nonempty.

Compact form:  $Pt f D \Rightarrow Ne \Rightarrow wE$ .

*Proof.* Direct consequence of **the identity  $A \setminus A = \emptyset$** , and **the upper bound property of union ( $B_i \subset \biguplus_{i \in [0..n]} B_i = \emptyset$  with  $B_i \in \mathcal{S}$ )**.  $\square$

**Lemma 553 (with full is with empty).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that the complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then,  $\mathcal{S}$  contains the empty set if it contains the full set.

Compact form:  $Pt f C \Rightarrow wF \Rightarrow wE$ .

*Proof.* Direct consequence of **the identity  $X^c = \emptyset$** , and **the upper bound property of union ( $B_i \subset \biguplus_{i \in [0..n]} B_i = \emptyset$  with  $B_i \in \mathcal{S}$ )**.  $\square$

**Lemma 554 (disjoint decomposition).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . If  $\mathcal{S}$  is closed under complement (resp. local complement, or set difference), then the complement (resp. local complement, or set difference) of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Compact form:  $C \Rightarrow Pt f C$ ;  $D' \Rightarrow Pt f D'$ ;  $D \Rightarrow Pt f D$ .

*Proof.* Direct consequence of **the identity  $A \setminus B = A \cap B^c$** , and **distributivity of intersection over disjoint union**.  $\square$

**Lemma 555 (disjoint decomposition of complement and set difference).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under intersection, and that the complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then, the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Compact form:  $I \Rightarrow \text{Pt}fC \Rightarrow \text{Pt}fD$ .

*Proof.* Direct consequence of **the identity  $A \setminus B = A \cap B^c$** , and **distributivity of intersection over disjoint union**.  $\square$

**Lemma 556 (disjoint decomposition of local complement and complement).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set, and that the local complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then, the complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Compact form:  $wF \Rightarrow \text{Pt}fD' \Rightarrow \text{Pt}fC$ .

*Proof.* Direct consequence of **the identity  $A^c = X \setminus A$** .  $\square$

**Lemma 557 (disjoint decomposition of set difference and local complement).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then, the local complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Compact form:  $\text{Pt}fD \Rightarrow \text{Pt}fD'$ .

*Proof.* Direct consequence of **the definitions of set difference and local complement**.  $\square$

**Lemma 558 (closedness under finite operations).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

$\mathcal{S}$  is closed under finite intersection iff it is closed under intersection.

$\mathcal{S}$  is closed under finite union iff it is closed under union.

$\mathcal{S}$  is closed under finite disjoint union iff it is closed under disjoint union.

Compact form:  $\text{If} \Leftrightarrow I$ ;  $\text{Uf} \Leftrightarrow U$ ;  $\text{Udf} \Leftrightarrow \text{Ud}$ .

*Proof.* Direct consequence of **induction on the number of operands**.  $\square$

**Lemma 559 (closedness under finite union and intersection).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement.

Then, it is closed under finite union iff it is closed under finite intersection.

Compact form:  $C \Rightarrow \text{Uf} \Leftrightarrow \text{If}$ .

*Proof.* Direct consequence of Lemma 547 (**closedness under union and intersection**,  $C \Rightarrow U \Leftrightarrow I$ ), and Lemma 558 (**closedness under finite operations**,  $\text{Uf} \Leftrightarrow U$  and  $\text{If} \Leftrightarrow I$ ).  $\square$

*Remark 560.* Note that the compact formula in the next lemma should actually be read as for all subset system  $\mathcal{S}$ ,  $I(\mathcal{S})$  implies  $I(\mathcal{U}^+(\mathcal{S}))$ .

**Lemma 561 (closedness under intersection and finite disjoint union closure).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under intersection.

Then, the set  $\mathcal{U}^+$  of finite disjoint unions of elements of  $\mathcal{S}$  is closed under intersection.

Compact form:  $I \Rightarrow I(\mathcal{U}^+)$ .

*Proof.* Let  $A^0, A^1 \in \mathcal{U}^+$ . Let  $\alpha \in \{0, 1\}$ . Let  $n^\alpha \in \mathbb{N}$  and let  $(A_{i^\alpha}^\alpha)_{i^\alpha \in [0..n^\alpha]}$  be pairwise disjoint elements of  $\mathcal{S}$  such that  $A^\alpha = \bigsqcup_{i^\alpha \in [0..n^\alpha]} A_{i^\alpha}^\alpha$ .

For all  $(i^0, i^1) \in [0..n^0] \times [0..n^1]$ , let  $B_{i^0, i^1} \stackrel{\text{def.}}{=} A_{i^0}^0 \cap A_{i^1}^1$ . Then, from hypothesis,  $B_{i^0, i^1}$  belongs to  $\mathcal{S}$ . Moreover, from **distributivity of intersection over disjoint union**, we have

$$A^0 \cap A^1 = \bigsqcup_{i^0 \in [0..n^0]} A_{i^0}^0 \cap \bigsqcup_{i^1 \in [0..n^1]} A_{i^1}^1 = \bigsqcup_{\substack{i^0 \in [0..n^0] \\ i^1 \in [0..n^1]}} B_{i^0, i^1}.$$

Hence, from the definition of  $\mathcal{U}^+$ , we have  $A^0 \cap A^1 \in \mathcal{U}^+$ .

Therefore,  $\mathcal{U}^+$  is closed under intersection.  $\square$

**Lemma 562 (finite disjoint union closure is closed under finite disjoint union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

Then, the set  $\mathcal{U}^+$  of finite disjoint unions of elements of  $\mathcal{S}$  is closed under disjoint union.

Compact form:  $Ud(\mathcal{U}^+)$ .

*Proof.* Direct consequence of **properties of disjoint union (disjoint union of disjoint union is disjoint union)**.  $\square$

**Lemma 563 (closedness under set difference and finite disjoint union closure).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under intersection, and that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Then, the set  $\mathcal{U}^+$  of finite disjoint unions of elements of  $\mathcal{S}$  is closed under set difference.

Compact form:  $I \Rightarrow \text{Ptfd} \Rightarrow D(\mathcal{U}^+)$ .

*Proof.* Let  $A^0, A^1 \in \mathcal{U}^+$ . Let  $\alpha \in \{0, 1\}$ . Let  $n^\alpha \in \mathbb{N}$  and let  $(A_{i^\alpha}^\alpha)_{i^\alpha \in [0..n^\alpha]}$  be pairwise disjoint elements of  $\mathcal{S}$  such that  $A^\alpha = \bigsqcup_{i^\alpha \in [0..n^\alpha]} A_{i^\alpha}^\alpha$ .

For all  $(i^0, i^1) \in [0..n^0] \times [0..n^1]$ , let  $B_{i^0, i^1} \stackrel{\text{def.}}{=} A_{i^0}^0 \setminus A_{i^1}^1$ . Then, from hypothesis,  $B_{i^0, i^1}$  is a finite disjoint union of elements of  $\mathcal{S}$ , and thus belongs to  $\mathcal{U}^+$ . Moreover, from **left and right distributivity of set difference over disjoint union** ( $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$  and  $(A \sqcup B) \setminus C = (A \setminus C) \sqcup (B \setminus C)$ ), we have

$$A^0 \setminus A^1 = \bigsqcup_{i^0 \in [0..n^0]} A_{i^0}^0 \setminus \bigsqcup_{i^1 \in [0..n^1]} A_{i^1}^1 = \bigcap_{i^1 \in [0..n^1]} \left( \bigsqcup_{i^0 \in [0..n^0]} A_{i^0}^0 \setminus A_{i^1}^1 \right) = \bigcap_{i^1 \in [0..n^1]} \left( \bigsqcup_{i^0 \in [0..n^0]} B_{i^0, i^1} \right).$$

Hence, from Lemma 562 (finite disjoint union closure is closed under finite disjoint union), and Lemma 561 (closedness under intersection and finite disjoint union closure), we have  $A^0 \setminus A^1 \in \mathcal{U}^+$ .

Therefore,  $\mathcal{U}^+$  is closed under set difference.  $\square$

## 8.1.2 Countable properties

*Remark 564.*

Note that obviously, closedness under finite or countable union implies closedness under finite or countable disjoint union. In the same way, closedness under countable union (resp. intersection) implies closedness under countable monotone union (resp. intersection).

Compact form:  $Uf \Rightarrow Udf$ ;  $Uc \Rightarrow [Udc \mid Umc]$ ;  $Ic \Rightarrow Imc$ .

*Remark 565.* Note that closedness under a countable subset operation actually means that the subset system is closed under this operation with at most a countable number of operands, i.e. it is also valid for a finite number (larger than or equal to two).

Compact form:  $Udc \Rightarrow [Udf \mid Ud]$ ;  $Uc \Rightarrow [Uf \mid U]$ ;  $Ic \Rightarrow [If \mid I]$ .

**Lemma 566 (closedness under countable disjoint union and local complement).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement and countable disjoint union. Then, it is closed under local complement.

Compact form:  $C \Rightarrow Udc \Rightarrow D'$ .

*Proof.* Direct consequence of Lemma 542 (closedness under disjoint union and local complement,  $C \Rightarrow (Udc \Rightarrow) Ud \Rightarrow D'$ ).  $\square$

**Lemma 567 (closedness under countable union and intersection).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then, it is closed under countable (monotone) union iff it is closed under countable (monotone) intersection.

Compact form:  $C \Rightarrow Umc \Leftrightarrow Imc$ ;  $C \Rightarrow Uc \Leftrightarrow Ic$ .

*Proof.* Direct consequence of **De Morgan's laws**, and **monotonicity of complement**.  $\square$

**Lemma 568 (closedness under countable disjoint and monotone union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under local complement and countable disjoint union. Then, it is closed under countable monotone union.

Compact form:  $D' \Rightarrow Udc \Rightarrow Umc$ .

*Proof.* Let  $I \subset \mathbb{N}$  be nonempty. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Assume that  $(A_i)_{i \in I}$  is nondecreasing. Then, from Lemma 215 (disjoint union of union), there exists  $(B_i)_{i \in I} \subset X$  pairwise disjoint, that is computed from the  $A_i$ 's using local complement and disjoint union (thus belonging to  $\mathcal{S}$ ), and such that  $\bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable monotone union.  $\square$

**Lemma 569 (closedness under countable monotone and disjoint union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under disjoint union and countable monotone union. Then, it is closed under countable disjoint union.

Compact form:  $Ud \Rightarrow Umc \Rightarrow Udc$ .

*Proof.* Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Assume that  $(A_i)_{i \in I}$  is pairwise disjoint. Then, from Lemma 217 (monotone union of union), there exists  $(B_i)_{i \in I} \subset X$  nondecreasing, that is computed from the  $A_i$ 's using disjoint union (thus belonging to  $\mathcal{S}$ ), and such that  $\biguplus_{i \in I} A_i = \bigcup_{i \in I} B_i \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable disjoint union.  $\square$

**Lemma 570 (closedness under countable disjoint union and countable union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under set difference and countable disjoint union. Then, it is closed under countable union.

Compact form:  $D \Rightarrow Udc \Rightarrow Uc$ .

*Proof.* Let  $I \subset \mathbb{N}$  be nonempty. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Then, from Lemma 215 (disjoint union of union), there exists  $(B_i)_{i \in I} \subset X$  pairwise disjoint, that is computed from the  $A_i$ 's using set difference and disjoint union (thus belonging to  $\mathcal{S}$ ), and such that  $\bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable union.  $\square$

**Lemma 571 (closedness under countable monotone union and countable union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under set difference, disjoint union and countable monotone union. Then, it is closed under countable union.

Compact form:  $D \Rightarrow Ud \Rightarrow Umc \Rightarrow Uc$ .

*Proof.* Direct consequence of Lemma 569 (closedness under countable monotone and disjoint union,  $Ud \wedge Umc \Rightarrow Udc$ ), and Lemma 570 (closedness under countable disjoint union and countable union,  $D \wedge Udc \Rightarrow Uc$ ).  $\square$



**Lemma 572 (disjoint union of union in subset system).** *Let  $X$  be a set.*

*Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0.. \text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under set difference, and disjoint union. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Then, there exists  $(B_i)_{i \in I} \in \mathcal{S}$  pairwise disjoint such that we have*

$$(8.1) \quad \forall i \in I, \quad B_i \subset A_i,$$

$$(8.2) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)} \in \mathcal{S},$$

$$(8.3) \quad \bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i.$$

*Moreover, when the sequence  $(A_i)_{i \in I}$  is nondecreasing, it is sufficient to assume that  $\mathcal{S}$  is closed under local complement (instead of set difference); and if  $\mathcal{S}$  is additionally closed under countable disjoint union, then equality (8.3) also occurs in  $\mathcal{S}$ .*

*Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} D \text{ (or } D') \wedge \text{Ud}$ .*

*Proof.* Direct consequence of Lemma 215 (disjoint union of union), and Lemma 558 (closedness under finite operations,  $\text{Ud} \Rightarrow \text{Udf}$ ).  $\square$

*Remark 573.* The previous lemma is intended to be applied in rings of sets (or in  $(\sigma)$ -algebras, see next sections). The next one is dedicated to semirings of sets (or to semi-algebras of sets).

**Lemma 574 (disjoint union of union in subset system (variant)).** *Let  $X$  be a set.*

*Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is nonempty, closed under intersection, and that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .*

*Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Then, there exist  $n_K \in \overline{\mathbb{N}}$ ,  $K \stackrel{\text{def.}}{=} [0..n_K)$ ,  $\psi : [0..n_I] \rightarrow [0..n_K]$  nondecreasing (with  $\psi(0) = 0$  and  $\psi(n_I) = n_K$ ), and  $(C_k)_{k \in K} \in \mathcal{S}$  pairwise disjoint such that we have*

$$(8.4) \quad \forall n \in J, \quad \biguplus_{k \in [\psi(n).. \psi(n+1))} C_k \subset A_{\varphi(n)},$$

$$(8.5) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{\substack{j \in [0..n] \\ k \in [\psi(j).. \psi(j+1))}} C_k = \biguplus_{k \in [0.. \psi(n+1))} C_k,$$

$$(8.6) \quad \bigcup_{i \in I} A_i = \biguplus_{\substack{n \in J \\ k \in [\psi(n).. \psi(n+1))}} C_k = \biguplus_{k \in K} C_k.$$

*Moreover, when the sequence  $(A_i)_{i \in I}$  is nondecreasing, it is sufficient to assume that the local complement (instead of the set difference) of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .*

*Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} \text{Ne} \wedge I \wedge \text{Ptfd}$  (or  $\text{Ptfd}'$ ).*

*Proof.* From Lemma 552 (nonempty is with empty), we have  $\emptyset \in \mathcal{S}$ ,

From Lemma 215 (disjoint union of union), let  $B_{\varphi(0)} \stackrel{\text{def.}}{=} A_{\varphi(0)}$ , and for all  $n \in J$  such that  $n+1 \in J$ , let  $B_{\varphi(n+1)} \stackrel{\text{def.}}{=} A_{\varphi(n+1)} \setminus \bigcup_{j \in [0..n]} B_{\varphi(j)}$ . Then,  $(B_i)_{i \in I}$  is pairwise disjoint, for all  $i \in I$ ,  $B_i \subset A_i$ , and we have

$$\forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)}, \quad \text{and} \quad \bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i.$$

For all  $n \in \mathbb{N}$ , let  $P(n)$  be the property:

$$\forall j \in [0..n], \exists k_j \in \mathbb{N}, \exists (C_k^j)_{k \in [0..k_j]} \in \mathcal{S}, \quad (\forall k \in [0..k_j], C_k^j \subset A_{\varphi(j)}) \\ \wedge \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{\substack{j \in [0..n] \\ k \in [0..k_j]}} C_k^j.$$

Let  $Q(n) \stackrel{\text{def.}}{=} (n \in J \Rightarrow P(n))$ .

**Induction:  $Q(0)$ .** Trivial (with  $k_0 \stackrel{\text{def.}}{=} 1$  and  $C_0^0 \stackrel{\text{def.}}{=} A_{\varphi(0)}$ ).

**Induction:  $Q(n)$  implies  $Q(n+1)$ .** Let  $n \in \mathbb{N}$ . Assume that  $Q(n)$  and  $n+1 \in J$  hold.

Then,  $n \in J$  since  $J$  is an interval starting at 0. Thus,  $P(n)$  holds. Hence, from **left distributivity of set difference over union** ( $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ ), and assumption (*PtfD*), let  $l_{n+1}^{j,k} \in \mathbb{N}$ , and  $(D_l^{j,k})_{l \in [0..l_{n+1}^{j,k}]} \in \mathcal{S}$  such that we have

$$B_{\varphi(n+1)} = \bigcap_{\substack{j \in [0..n] \\ k \in [0..k_j]}} (A_{\varphi(n+1)} \setminus C_k^j) = \bigcap_{\substack{j \in [0..n] \\ k \in [0..k_j]}} \biguplus_{l \in [0..l_{n+1}^{j,k}]} D_l^{j,k}.$$

Let  $k_{n+1} \stackrel{\text{def.}}{=} \max_{j \in [0..n], k \in [0..k_j]} l_{n+1}^{j,k}$ , and

$$\forall j \in [0..n], \forall k \in [0..k_j], \forall l \in [l_{n+1}^{j,k}..k_{n+1}], \quad \text{let } D_l^{j,k} \stackrel{\text{def.}}{=} \emptyset, \\ \forall l \in [0..k_{n+1}], \quad \text{let } C_l^{n+1} \stackrel{\text{def.}}{=} \bigcap_{\substack{j \in [0..n] \\ k \in [0..k_j]}} D_l^{j,k}.$$

Then, since **the empty set is the identity element for union**, and from **distributivity of intersection over disjoint union**, we have

$$B_{\varphi(n+1)} = \bigcap_{\substack{j \in [0..n] \\ k \in [0..k_j]}} \biguplus_{l \in [0..k_{n+1}]} D_l^{j,k} = \biguplus_{k \in [0..k_{n+1}]} C_k^{n+1}.$$

Let  $k \in [0..k_{n+1}]$ . Then, since  $\emptyset \in \mathcal{S}$ , and from **transitivity of inclusion, associativity of union**, and  $P(n)$ , we have  $C_k^{n+1} \in \mathcal{S}$ ,  $C_k^{n+1} \subset A_{\varphi(n+1)}$ , and

$$\bigcup_{j \in [0..n+1]} A_{\varphi(j)} = \biguplus_{\substack{j \in [0..n] \\ k \in [0..k_j]}} C_k^j \uplus B_{\varphi(n+1)} = \biguplus_{\substack{j \in [0..n+1] \\ k \in [0..k_j]}} C_k^j.$$

Thus, for all  $n \in \mathbb{N}$ ,  $Q(n)$  holds. Hence, for all  $n \in J$ ,  $P(n)$  holds.

Let  $\psi(0) \stackrel{\text{def.}}{=} 0$ . For all  $n \in (0..n_I]$ , let  $\psi(n) \stackrel{\text{def.}}{=} \sum_{j \in [0..n]} k_j$ . Let  $n_K \stackrel{\text{def.}}{=} \psi(n_I)$ . For all  $n \in J$ , for all  $k \in [\psi(n).. \psi(n+1))$  (i.e. we have  $k - \psi(n) \in [0..k_n)$ ), let  $C_k \stackrel{\text{def.}}{=} C_{k - \psi(n)}^n$ . Then, the  $C_k$ 's belong to  $\mathcal{S}$ , are pairwise disjoint, and properties (8.1), (8.2), and (8.3) are satisfied.  $\square$

**Lemma 575 (monotone union of union in subset system).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $J \stackrel{\text{def.}}{=} [0.. \text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under union. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Then, there exists  $(B_i)_{i \in I} \in \mathcal{S}$  nondecreasing such that for all  $i \in I$ ,  $A_i \subset B_i$ , and we have

$$(8.7) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \bigcup_{j \in [0..n]} B_{\varphi(j)} (= B_{\varphi(n)}) \in \mathcal{S},$$

$$(8.8) \quad \bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i.$$

Moreover, when the sequence  $(A_i)_{i \in I}$  is pairwise disjoint, it is sufficient to assume that  $\mathcal{S}$  is closed under disjoint union (instead of union); and if  $\mathcal{S}$  is additionally closed under countable monotone union, then equality (8.8) also occurs in  $\mathcal{S}$ .

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} U$  (or  $Ud$ ).

*Proof.* Direct consequence of Lemma 217 (monotone union of union), and Lemma 558 (closedness under finite operations,  $U \Rightarrow Uf$  and  $Ud \Rightarrow Udf$ ).  $\square$

**Lemma 576 (empty monotone intersection of union in subset system).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under local complement. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Assume that  $A \stackrel{\text{def.}}{=} \bigcup_{i \in I} A_i \in \mathcal{S}$ . Then, there exists  $(B_i)_{i \in I} \in \mathcal{S}$  nonincreasing such that for all  $i \in I$ ,  $B_i \subset A$  and  $A_i \cap B_i = \emptyset$ , and we have

$$(8.9) \quad \forall n \in J, \quad A \setminus \bigcup_{j \in [0..n]} A_{\varphi(j)} = \bigcap_{j \in [0..n]} B_{\varphi(j)} (= B_{\varphi(n)}) \in \mathcal{S},$$

$$(8.10) \quad \emptyset = \bigcap_{i \in I} B_i.$$

Moreover, if  $(A_i)_{i \in I}$  is pairwise disjoint, then we have

$$(8.11) \quad \forall n \in J, \quad A = \biguplus_{j \in [0..n]} A_{\varphi(j)} \uplus B_{\varphi(n)} \quad \left( \text{i.e. } B_{\varphi(n)} = \biguplus_{j \in (n..n_I)} A_{\varphi(j)} \in \mathcal{S} \right).$$

And if  $\mathcal{S}$  contains the empty set, then equality (8.10) also occurs in  $\mathcal{S}$ .

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} D'$ .

*Proof.* Direct consequence of Lemma 219 (empty monotone intersection of union).  $\square$

**Lemma 577 (monotone union of monotone intersection in subset system).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I)$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under set difference. Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Let  $n_0 \in J$ . Let  $J_{n_0} \stackrel{\text{def.}}{=} [n_0..n_I) \neq \emptyset$ . For all  $n \in J_{n_0}$ , let  $B_n \stackrel{\text{def.}}{=} A_{\varphi(n_0)} \setminus A_{\varphi(n)}$ . Then,  $(B_n)_{n \in J_{n_0}} \in \mathcal{S}$ , and we have

$$(8.12) \quad \bigcup_{n \in J_{n_0}} B_n = A_{\varphi(n_0)} \setminus \bigcap_{n \in J_{n_0}} A_{\varphi(n)}.$$

Moreover, when the sequence  $(A_i)_{i \in I}$  is nonincreasing, it is sufficient to assume that  $\mathcal{S}$  is closed under local complement (instead of set difference), and then  $(B_n)_{n \in J_{n_0}}$  is nondecreasing; and if  $\mathcal{S}$  is additionally closed under countable (monotone) union or intersection, then equality (8.12) occurs in  $\mathcal{S}$ .

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} D$  (or  $D'$ ).

*Proof.* Direct consequence of Lemma 221 (monotone union of monotone intersection).  $\square$

## 8.2 Generated subset system

**Definition 578 (subset system).**

Let  $X$  be a set. Subsets of the power set  $\mathcal{P}(X)$  are called *subset systems (on  $X$ )*.

*Remark 579.* Note that subset systems are elements of  $\mathcal{P}(\mathcal{P}(X))$ .

**Definition 580 (subset system class).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ , i.e. a function from  $\mathcal{P}(\mathcal{P}(X))$  to a Boolean domain. The *subset system class (on  $X$ ) corresponding to predicate  $P$*  (in short the *subset systems of class  $P$* , or  *$P$ -class subset systems*) is denoted  $\mathfrak{C}_{P,X}$  (or simply  $\mathfrak{C}_P$ ); it is the collection of subset systems on  $X$  satisfying  $P$ :

$$(8.13) \quad \mathfrak{C}_{P,X} \stackrel{\text{def.}}{=} \{\mathcal{S} \subset \mathcal{P}(X) \mid P(\mathcal{S})\}.$$

*Remark 581.* Note that subset system classes are elements of  $\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ .

**Lemma 582 (intersection of subset system classes).**

Let  $X$  be a set.

Let  $P$  and  $Q$  be predicates on subset systems on  $X$ . Then, we have  $\mathfrak{C}_{P \wedge Q} = \mathfrak{C}_P \cap \mathfrak{C}_Q$ .

*Proof.* Direct consequence of Definition 580 (*subset system class*), and **compatibility of logical conjunction with intersection**.  $\square$

**Lemma 583 (subset system classes are monotone).**

Let  $X$  be a set. Let  $P$  and  $Q$  be

predicates on subset systems on  $X$ . Assume that  $P$  implies  $Q$ . Then, we have  $\mathfrak{C}_P \subset \mathfrak{C}_Q$ .

*Proof.* Direct consequence of Definition 580 (*subset system class*), and **compatibility of implication with inclusion**.  $\square$

**Definition 584 (generated subset system).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ . Let  $G \subset \mathcal{P}(X)$ . The  *$P$ -class subset system (on  $X$ ) generated by  $G$*  is denoted  $\mathfrak{G}_{P,X}(G)$  (or simply  $\mathfrak{G}_P(G)$ ); it is the intersection of all  $P$ -class subset systems on  $X$  containing  $G$ :

$$(8.14) \quad \mathfrak{G}_{P,X}(G) \stackrel{\text{def.}}{=} \bigcap_{G \subset \mathcal{S} \in \mathfrak{C}_P} \mathcal{S}.$$

**Lemma 585 (subset system generation preserves class).**

Let  $X$  be a set.

Let  $P$  be a predicate on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Then, generated  $P$ -class subset systems are  $P$ -class subset systems containing the generator:

$$(8.15) \quad \forall G \subset \mathcal{P}(X), \quad G \subset \mathfrak{G}_P(G) \in \mathfrak{C}_P.$$

*Proof.* Direct consequence of Definition 584 (*generated subset system*), Definition 580 (*subset system class*), Definition 578 (*subset system*), and **the lower bound property of intersection**.  $\square$

**Lemma 586 (properties of subset system generation).**

Let  $X$  be a set.

Let  $P$  be a predicate on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Then,  $P$ -class subset system generation is minimum, monotone, and idempotent:

$$(8.16) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_P, \quad G \subset \mathcal{S} \implies \mathfrak{G}_P(G) \subset \mathcal{S},$$

$$(8.17) \quad \forall G_1, G_2 \subset \mathcal{P}(X), \quad G_1 \subset G_2 \implies \mathfrak{G}_P(G_1) \subset \mathfrak{G}_P(G_2),$$

$$(8.18) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_P \iff \mathfrak{G}_P(\mathcal{S}) = \mathcal{S}.$$

*Proof.* (8.16). Direct consequence of Definition 584 (*generated subset system*), Definition 580 (*subset system class*), Definition 578 (*subset system*), and **properties of the intersection**.

(8.17). Direct consequence of Lemma 585 (*subset system generation preserves class*, with  $G_2$ , thus  $G_2 \subset \mathfrak{G}_P(G_2) \in \mathfrak{C}_P$ ), and (8.16) (with  $G_1$  and  $\mathcal{S} \stackrel{\text{def.}}{=} \mathfrak{G}_P(G_2)$ ).

(8.18). Direct consequence of Lemma 585 (*subset system generation preserves class*, we always have  $\mathcal{S} \subset \mathfrak{G}_P(\mathcal{S}) \in \mathfrak{C}_P$ , thus  $\mathfrak{G}_P(\mathcal{S}) \subset \mathcal{S} \Rightarrow \mathcal{S} \in \mathfrak{C}_P$ ), **reflexivity of inclusion**, and (8.16) (with  $G \stackrel{\text{def.}}{=} \mathcal{S}$ , thus  $\mathcal{S} \in \mathfrak{C}_P \Rightarrow \mathfrak{G}_P(\mathcal{S}) \subset \mathcal{S}$ ).  $\square$

**Lemma 587 (other properties of subset system generation).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.19) \quad G_1 \subset \mathfrak{G}_P(G_2) \quad \wedge \quad G_2 \subset \mathfrak{G}_P(G_1) \quad \Longrightarrow \quad \mathfrak{G}_P(G_1) = \mathfrak{G}_P(G_2),$$

$$(8.20) \quad G_2 \subset \mathfrak{G}_P(G_1) \quad \Longrightarrow \quad \mathfrak{G}_P(G_1 \cup G_2) = \mathfrak{G}_P(G_1).$$

*Proof.* (8.19). Direct consequence of Lemma 586 (*properties of subset system generation*, monotonicity and idempotence:  $\mathfrak{G}_P(G_i) \subset \mathfrak{G}_P(\mathfrak{G}_P(G_j)) = \mathfrak{G}_P(G_j)$  with  $i \in \{1, 2\}$  and  $j \stackrel{\text{def.}}{=} 3 - i$ ), and Lemma 585 (*subset system generation preserves class*,  $\mathfrak{G}_P(G_j) \in \mathfrak{C}_P$ ).

(8.20). Direct consequence of Lemma 585 (*subset system generation preserves class*,  $G_1$  is included in  $\mathfrak{G}_P(G_1)$ ), **monotonicity of union** ( $G_1 \cup G_2 \subset \mathfrak{G}_P(G_1) \cup G_2 = \mathfrak{G}_P(G_1)$ ), Lemma 586 (*properties of subset system generation*, monotonicity:  $G_1 \subset \mathfrak{G}_P(G_1) \subset \mathfrak{G}_P(G_1 \cup G_2)$ ), and (8.19) (with  $G_1 \stackrel{\text{def.}}{=} G_1 \cup G_2$  and  $G_2 \stackrel{\text{def.}}{=} G_1$ ).  $\square$

**Lemma 588 (generated subset systems are monotone).** Let  $X$  be a set. Let  $P$  and  $Q$  be predicates on subset systems on  $X$ . Assume that  $P$  implies  $Q$ . Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathfrak{G}_Q(G) \subset \mathfrak{G}_P(G)$ .

*Proof.* Direct consequence of Definition 584 (*generated subset system*), and Lemma 583 (*subset system classes are monotone*).  $\square$

*Remark 589.* Note that the previous lemma provides  $\mathfrak{G}_P(G), \mathfrak{G}_Q(G) \subset \mathfrak{G}_{P \wedge Q}(G)$ , but in general, we do not have the inclusion the other way around.

Hence,  $\mathfrak{G}_{P \wedge Q}(G)$  is generally distinct from  $\mathfrak{G}_P(G) \cap \mathfrak{G}_Q(G)$ .

**Lemma 590 (upgrade generated subset system).**

Let  $X$  be a set.

Let  $P, Q$  and  $R$  be predicates on subset systems on  $X$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $\mathfrak{C}_P$  and  $\mathfrak{C}_Q$  are closed under any intersection, that  $Q \wedge R$  implies  $P$ , and that  $R$  holds on  $\mathfrak{G}_Q(G)$ . Then, we have  $\mathfrak{G}_P(G) \subset \mathfrak{G}_Q(G)$ .

*Proof.* Direct consequence of Lemma 585 (*subset system generation preserves class*,  $\mathfrak{G}_Q(G)$  is a  $Q$ -class subset system containing  $G$ ), Definition 580 (*subset system class*,  $\mathfrak{G}_Q(G)$  is a  $P$ -class subset system), Lemma 586 (*properties of subset system generation*, minimality with  $P$ ).  $\square$

**Lemma 591 (cascading generated subset systems).**

Let  $X$  be a set. Let  $P$  and  $Q$  be predicates on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  and  $\mathfrak{C}_Q$  are closed under any intersection, and that  $P$  implies  $Q$ . Let  $G \subset \mathcal{P}(X)$ .

Then, we have  $\mathfrak{G}_P(G) = \mathfrak{G}_P(\mathfrak{G}_Q(G))$ .

*Proof.*  $\mathfrak{G}_P(G) \subset \mathfrak{G}_P(\mathfrak{G}_Q(G))$ . Direct consequence of Lemma 585 (*subset system generation preserves class*, with  $Q$ , thus  $G \subset \mathfrak{G}_Q(G)$ ), and Lemma 586 (*properties of subset system generation*, monotonicity with  $P$ ,  $G_1 \stackrel{\text{def.}}{=} G$  and  $G_2 \stackrel{\text{def.}}{=} \mathfrak{G}_Q(G)$ ).

$\mathfrak{G}_P(\mathfrak{G}_Q(G)) \subset \mathfrak{G}_P(G)$ . Direct consequence of Lemma 585 (*subset system generation preserves class*, with  $P$ , thus  $G \subset \mathfrak{G}_P(G) \in \mathfrak{C}_P$ ), Lemma 583 (*subset system classes are monotone*,  $\mathfrak{C}_P \subset \mathfrak{C}_Q$ ), Lemma 586 (*properties of subset system generation*, minimality, first with  $Q$ ,  $G$  and  $\mathcal{S} \stackrel{\text{def.}}{=} \mathfrak{G}_P(G) \in \mathfrak{C}_Q$ , thus  $\mathfrak{G}_Q(G) \subset \mathfrak{G}_P(G)$ , and then with  $P$ ,  $G \stackrel{\text{def.}}{=} \mathfrak{G}_Q(G)$  and  $\mathcal{S} \stackrel{\text{def.}}{=} \mathfrak{G}_P(G) \in \mathfrak{C}_P$ ).

Therefore, we have the equality.  $\square$

**Lemma 592 (double cascading generated subset systems).**

Let  $X$  be a set. Let  $P$ ,  $Q$  and  $R$  be predicates on subset systems on  $X$ . Assume that

$$(8.21) \quad \forall G \subset \mathcal{P}(X), \quad \begin{cases} \mathfrak{G}_P(G) = \mathfrak{G}_P(\mathfrak{G}_Q(G)), \\ \mathfrak{G}_Q(G) = \mathfrak{G}_Q(\mathfrak{G}_R(G)). \end{cases}$$

Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathfrak{G}_P(G) = \mathfrak{G}_P(\mathfrak{G}_R(G))$ .

*Proof.* Direct consequence of the identity

$$\mathfrak{G}_P(G) = \mathfrak{G}_P(\mathfrak{G}_Q(G)) = \mathfrak{G}_P(\mathfrak{G}_Q(\mathfrak{G}_R(G))) = \mathfrak{G}_P(\mathfrak{G}_R(G)).$$

$\square$

*Remark 593.*

Note that when the predicate  $P$  involves nonemptiness (denoted as  $Ne$ , see Section 8.1), the premise on closedness under intersection cannot be met, and one has to add nonemptiness of the generator as expressed in the next lemmas. This is useful for  $\pi$ -systems (see Section 8.3).

**Lemma 594 (class of nonempty subset systems).**

Let  $X$  be a set.

Let  $P$  be a predicate on subset systems on  $X$ . Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is nonempty.

Then, we have  $\mathcal{S} \in \mathfrak{C}_{P \wedge Ne} \Leftrightarrow \mathcal{S} \in \mathfrak{C}_P$ .

*Proof.* Direct consequence of Lemma 582 (*intersection of subset system classes*), and Definition 580 (*subset system class*, thus  $\mathcal{S} \in \mathfrak{C}_{Ne}$ ).  $\square$

**Lemma 595 (generated nonempty subset systems).**

Let  $X$  be a set.

Let  $P$  be a predicate on subset systems on  $X$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is nonempty.

Then, we have  $\mathfrak{G}_{P \wedge Ne}(G) = \mathfrak{G}_P(G)$ .

*Proof.* Direct consequence from Definition 584 (*generated subset system*), **transitivity of (strict) inclusion** ( $\emptyset \subsetneq G \subset \mathcal{S}$  implies  $\emptyset \subsetneq \mathcal{S}$ ), and Lemma 594 (*class of nonempty subset systems*).  $\square$

**Lemma 596 (subset system generation preserves class (with nonemptiness)).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Let  $P' \stackrel{\text{def.}}{=} P \wedge Ne$ . Then, generated  $P'$ -class subset systems are  $P'$ -class subset systems containing the nonempty generator:

$$(8.22) \quad \forall G \subset \mathcal{P}(X), \quad G \neq \emptyset \implies G \subset \mathfrak{G}_{P'}(G) \in \mathfrak{C}_{P'}.$$

*Proof.* Direct consequence of Lemma 585 (*subset system generation preserves class*, with  $P$ , thus  $G \subset \mathfrak{G}_P(G) \in \mathfrak{C}_P$ ), **transitivity of (strict) inclusion** ( $\mathfrak{G}_P(G) \neq \emptyset$ ), Lemma 594 (*class of nonempty subset systems*, thus  $\mathfrak{G}_P(G) \in \mathfrak{C}_{P'}$ ), and Lemma 595 (*generated nonempty subset systems*,  $\mathfrak{G}_{P'}(G) = \mathfrak{G}_P(G)$ ).  $\square$

**Lemma 597 (properties of generated subset system (with nonemptiness)).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Let  $P' \stackrel{\text{def.}}{=} P \wedge Ne$ . Then,  $P'$ -class subset system generation (or  $P$ -class subset system generation with nonemptiness) is minimum, monotone, and idempotent:

$$(8.23) \quad \forall G \subset \mathcal{P}(X), \quad \forall \mathcal{S} \in \mathfrak{C}_{P'}, \quad \emptyset \neq G \subset \mathcal{S} \implies \mathfrak{G}_{P'}(G) \subset \mathcal{S},$$

$$(8.24) \quad \forall G_1, G_2 \subset \mathcal{P}(X), \quad \emptyset \neq G_1 \subset G_2 \implies \mathfrak{G}_{P'}(G_1) \subset \mathfrak{G}_{P'}(G_2),$$

$$(8.25) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \neq \emptyset \implies (\mathcal{S} \in \mathfrak{C}_{P'} \iff \mathfrak{G}_{P'}(\mathcal{S}) = \mathcal{S}).$$

*Proof.* Direct consequence of **transitivity of (strict) inclusion** ( $\emptyset \neq G \subset G' \Rightarrow G' \neq \emptyset$ ), Lemma 594 (*class of nonempty subset systems*), Lemma 595 (*generated nonempty subset systems*, with  $G, G_1, G_2$ , and  $\mathcal{S}$ ), and Lemma 586 (*properties of subset system generation*).  $\square$

**Lemma 598 (other properties of subset system generation (with nonemptiness)).**

Let  $X$  be a set. Let  $P$  be a predicate on subset systems on  $X$ . Let  $P' \stackrel{\text{def.}}{=} P \wedge Ne$ . Assume that  $\mathfrak{C}_P$  is closed under any intersection. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1$  and  $G_2$  are nonempty. Then, we have

$$(8.26) \quad G_1 \subset \mathfrak{G}_{P'}(G_2) \quad \wedge \quad G_2 \subset \mathfrak{G}_{P'}(G_1) \quad \Longrightarrow \quad \mathfrak{G}_{P'}(G_1) = \mathfrak{G}_{P'}(G_2),$$

$$(8.27) \quad G_2 \subset \mathfrak{G}_{P'}(G_1) \quad \Longrightarrow \quad \mathfrak{G}_{P'}(G_1 \cup G_2) = \mathfrak{G}_{P'}(G_1).$$

*Proof.* Direct consequence of **the upper bound property of union** ( $\emptyset \neq G_1, G_2 \subset G_1 \cup G_2$ , thus  $G_1 \cup G_2 \neq \emptyset$ ), Lemma 595 (*generated nonempty subset systems*, with  $G_1, G_2$ , and  $G_1 \cup G_2$ ), and Lemma 587 (*other properties of subset system generation*).  $\square$

**Lemma 599 (generated subset systems are monotone (with nonemptiness)).**

Let  $X$  be a set. Let  $P$  and  $Q$  be predicates on subset systems on  $X$ . Assume that  $P$  implies  $Q$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is nonempty. Then, we have  $\mathfrak{G}_{Q \wedge Ne}(G) \subset \mathfrak{G}_P(G)$ .

*Proof.* Direct consequence of Lemma 595 (*generated nonempty subset systems*, with  $Q$ ), and Lemma 588 (*generated subset systems are monotone*).  $\square$

**Lemma 600 (upgrade generated subset system (with nonemptiness)).**

Let  $X$  be a set. Let  $P, Q$  and  $R$  be predicates on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  and  $\mathfrak{C}_Q$  are closed under any intersection, and that  $Q \wedge R$  implies  $P$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $R$  holds on  $\mathfrak{G}_Q(G)$ , and that  $G$  is nonempty. Then, we have  $\mathfrak{G}_P(G) \subset \mathfrak{G}_{Q \wedge Ne}(G)$ .

*Proof.* Direct consequence of Lemma 596 (*subset system generation preserves class (with non-emptiness)*, with  $Q$ , thus  $\mathfrak{G}_{Q \wedge Ne}(G) \neq \emptyset$ ), Lemma 595 (*generated nonempty subset systems*, with  $G \neq \emptyset$  and  $Q$ , thus  $\mathfrak{G}_{Q \wedge Ne}(G) = \mathfrak{G}_Q(G)$ ), and Lemma 590 (*upgrade generated subset system*, with  $Q \wedge (Ne \wedge R) \Rightarrow P$ ).  $\square$

**Lemma 601 (cascading generated subset systems (with nonemptiness)).**

Let  $X$  be a set. Let  $P$  and  $Q$  be predicates on subset systems on  $X$ . Assume that  $\mathfrak{C}_P$  and  $\mathfrak{C}_Q$  are closed under any intersection, and that  $P$  implies  $Q$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is nonempty. Then, we have  $\mathfrak{G}_P(G) = \mathfrak{G}_P(\mathfrak{G}_{Q \wedge Ne}(G))$ .

*Proof.* Direct consequence of Lemma 595 (*generated nonempty subset systems*, with  $G \neq \emptyset$  and  $Q$ , thus  $\mathfrak{G}_{Q \wedge Ne}(G) = \mathfrak{G}_Q(G)$ ), and Lemma 591 (*cascading generated subset systems*, with  $P \Rightarrow Q$ ).  $\square$



### 8.3 Pi-system

**Definition 602 ( $\pi$ -system).** Let  $X$  be a set. Let  $P_\Pi$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of nonemptiness and closedness under finite intersection. The class of  $\pi$ -systems on  $X$  is  $\mathfrak{C}_{P_\Pi}$ : a subset  $\Pi$  of  $\mathcal{P}(X)$  is a  $\pi$ -system on  $X$  iff

$$(8.28) \quad \Pi \neq \emptyset,$$

$$(8.29) \quad \forall n \in \mathbb{N}, \forall (A_i)_{i \in [0..n]} \in \Pi, \quad \bigcap_{i \in [0..n]} A_i \in \Pi.$$

Compact form:  $P_\Pi \stackrel{\text{def.}}{=} Ne \wedge If$ .

*Remark 603.* Note that adding the empty set to a  $\pi$ -system keeps it a  $\pi$ -system. We may replace in the previous definition the nonempty condition (8.28) by the “contain-the-empty-set” condition, and still maintain the Dynkin  $\pi$ - $\lambda$  theorem (see Theorem 732).

**Lemma 604 (equivalent definition of  $\pi$ -system).** Let  $X$  be a set. Let  $\Pi \subset \mathcal{P}(X)$ . Then,  $\Pi$  is a  $\pi$ -system iff it is nonempty and closed under (binary) intersection.

Compact form:  $P_\Pi \Leftrightarrow Ne \wedge I$ .

*Proof.* Direct consequence of Definition 602 ( $\pi$ -system,  $P_\Pi = Ne \wedge If$ ), and Lemma 558 (closedness under finite operations,  $If \Leftrightarrow I$ ).  $\square$

**Lemma 605 (intersection of  $\pi$ -systems).** Let  $X$  and  $I$  be sets. Let  $(\Pi_i)_{i \in I}$  be  $\pi$ -systems on  $X$ . Then,  $\bigcap_{i \in I} \Pi_i$  is closed under intersection, i.e. it is a  $\pi$ -system iff it is nonempty.

*Proof.* Direct consequence of Lemma 604 (equivalent definition of  $\pi$ -system), and the definition of intersection.  $\square$

**Definition 606 (generated  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . The  $\pi$ -system generated by  $G$  is denoted  $\Pi_X(G) \stackrel{\text{def.}}{=} \mathfrak{G}_{P_\Pi, X}(G)$ ; it is the intersection of all  $\pi$ -systems on  $X$  containing  $G$ .

**Lemma 607 (generated  $\pi$ -system is  $\pi$ -system).** Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then,  $\Pi_X(G)$  is a  $\pi$ -system on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 606 (generated  $\pi$ -system), Lemma 605 (intersection of  $\pi$ -systems), and Lemma 596 (subset system generation preserves class (with nonemptiness)).  $\square$

**Lemma 608 (properties of  $\pi$ -system generation).**

Let  $X$  be a set. Then,  $\pi$ -system generation on  $X$  is minimum, monotone, and idempotent:

$$(8.30) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_\Pi, X}, \quad \emptyset \neq G \subset \mathcal{S} \implies \Pi_X(G) \subset \mathcal{S},$$

$$(8.31) \quad \forall G_1, G_2 \subset \mathcal{P}(X), \quad \emptyset \neq G_1 \subset G_2 \implies \Pi_X(G_1) \subset \Pi_X(G_2),$$

$$(8.32) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \neq \emptyset \implies (\mathcal{S} \in \mathfrak{C}_{P_\Pi, X} \iff \Pi_X(\mathcal{S}) = \mathcal{S}).$$

*Proof.* Direct consequence of Definition 606 (generated  $\pi$ -system), Lemma 605 (intersection of  $\pi$ -systems), and Lemma 597 (properties of generated subset system (with nonemptiness)).  $\square$

**Lemma 609 (other properties of  $\pi$ -system generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1, G_2 \neq \emptyset$ . Then, we have

$$(8.33) \quad G_1 \subset \Pi_X(G_2) \quad \wedge \quad G_2 \subset \Pi_X(G_1) \implies \Pi_X(G_1) = \Pi_X(G_2),$$

$$(8.34) \quad G_2 \subset \Pi_X(G_1) \implies \Pi_X(G_1 \cup G_2) = \Pi_X(G_1).$$

*Proof.* Direct consequence of Definition 606 (*generated  $\pi$ -system*), Lemma 605 (*intersection of  $\pi$ -systems*), and Lemma 598 (*other properties of subset system generation (with nonemptiness)*).  $\square$

## 8.4 Ring of sets

*Remark 610.*

The next concept of “ring of sets” is not to be confused with the algebraic structure “ring”.

**Definition 611 (ring of sets).**

Let  $X$  be a set. Let  $P_{\mathcal{R}}$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of nonemptiness, and closedness under set difference and finite union. The class of *rings (of (sub)sets) on  $X$*  is  $\mathfrak{C}_{P_{\mathcal{R}}}$ : a subset  $\mathcal{R}$  of  $\mathcal{P}(X)$  is a ring of sets on  $X$  iff

$$(8.35) \quad \mathcal{R} \neq \emptyset,$$

$$(8.36) \quad \forall A, B \in \mathcal{R}, \quad A \setminus B \in \mathcal{R},$$

$$(8.37) \quad \forall n \in \mathbb{N}, \forall (A_i)_{i \in [0..n]} \in \mathcal{R}, \quad \bigcup_{i \in [0..n]} A_i \in \mathcal{R}.$$

Compact form:  $P_{\mathcal{R}} \stackrel{\text{def.}}{=} Ne \wedge D \wedge Uf$ .

**Lemma 612 (equivalent definition of ring of sets).**

Let  $X$  be a set. Let  $\mathcal{R} \subset \mathcal{P}(X)$ . Then,  $\mathcal{R}$  is a ring of sets on  $X$  iff

$$(8.38) \quad \mathcal{R} \neq \emptyset, \quad \text{or} \quad \emptyset \in \mathcal{R},$$

$$(8.39) \quad \left\{ \begin{array}{l} \forall A, B \in \mathcal{R}, \quad A \setminus B \in \mathcal{R}, \\ \forall A, B \in \mathcal{R}, \quad A \cup B \in \mathcal{R}, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \forall A, B \in \mathcal{R}, \quad A \setminus B \in \mathcal{R}, \\ \forall A, B \in \mathcal{R}, \quad A \cap B = \emptyset \implies A \uplus B \in \mathcal{R}, \end{array} \right.$$

$$\text{or} \quad \left\{ \begin{array}{l} \forall A, B \in \mathcal{R}, \quad A \Delta B \in \mathcal{R}, \\ \forall A, B \in \mathcal{R}, \quad A \cap B \in \mathcal{R}, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \forall A, B \in \mathcal{R}, \quad B \subset A \implies A \setminus B \in \mathcal{R}, \\ \forall A, B \in \mathcal{R}, \quad A \cap B = \emptyset \implies A \uplus B \in \mathcal{R}, \\ \forall A, B \in \mathcal{R}, \quad A \cap B \in \mathcal{R}. \end{array} \right.$$

Compact form:  $P_{\mathcal{R}} \Leftrightarrow [Ne \mid wE] \wedge [D \wedge U \mid D \wedge Ud \mid sD \wedge I \mid D' \wedge Ud \wedge I]$ .

*Proof.* Direct consequence of Definition 611 (ring of sets,  $P_{\mathcal{R}} = Ne \wedge D \wedge Uf$ ), Lemma 538 (non-empty and with empty or full,  $Ne \wedge D \Rightarrow wE$  and  $wE \Rightarrow Ne$ ), Lemma 558 (closedness under finite operations,  $Uf \Leftrightarrow U$ ), Lemma 546 (closedness under union and disjoint union,  $U \Rightarrow Ud$  and  $D \wedge Ud \Rightarrow U$ ), Lemma 545 (closedness under intersection and set difference,  $D \Rightarrow I$ ), Lemma 550 (closedness under symmetric difference and set difference,  $U \Rightarrow D \Leftrightarrow sD$ ), Lemma 551 (closedness under symmetric difference and intersection,  $sD \wedge I \Rightarrow U$ ), and Lemma 544 (closedness under symmetric difference, local complement, and disjoint union,  $sD \Rightarrow D'$ ,  $sD \Rightarrow Ud$  and  $D' \wedge Ud \wedge I$  implies  $sD$ ).  $\square$

*Remark 613.* Note that the previous lemma means that the predicate  $P_{\mathcal{R}}$  characterizing rings of sets is equivalent to any of the eight alternatives summarized by the conjunction (8.38) and (8.39).

**Lemma 614 (ring of sets is  $\pi$ -system).**

Let  $X$  be a set. Then, rings of sets on  $X$  are  $\pi$ -systems on  $X$ .

Compact form:  $P_{\mathcal{R}} \Rightarrow P_{\Pi}$ .

*Proof.* Direct consequence of Lemma 612 (equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow [Ne \mid I]$ ), and Lemma 604 (equivalent definition of  $\pi$ -system,  $P_{\Pi} = Ne \wedge I$ ).  $\square$

**Lemma 615 (other properties of ring of sets).**

Let  $X$  be a set. Let  $\mathcal{R}$  be a ring of sets on  $X$ .

Then,  $\mathcal{R}$  is closed under finite disjoint union and finite intersection.

Compact form:  $P_{\mathcal{R}} \Rightarrow [Udf \mid If]$ .

*Proof.* Direct consequence of Lemma 612 (*equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow Ud$* ), Lemma 558 (*closedness under finite operations,  $Ud \Rightarrow Uf$* ), Lemma 614 (*ring of sets is  $\pi$ -system,  $P_{\mathcal{R}} \Rightarrow P_{\Pi}$* ), and Definition 602 ( *$\pi$ -system,  $P_{\Pi} \Rightarrow If$* ).  $\square$

**Lemma 616 (intersection of rings of sets).**

Let  $X$  and  $I$  be sets. Let  $(\mathcal{R}_i)_{i \in I}$  be rings of sets on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{R}_i$  is a ring of sets on  $X$ .

*Proof.* Direct consequence of Lemma 612 (*equivalent definition of ring of sets, with  $wE$* ), and **the definition of intersection**.  $\square$

**Definition 617 (generated ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The ring of sets generated by  $G$  is denoted  $\mathcal{R}_X(G) \stackrel{\text{def.}}{=} \mathfrak{G}_{P_{\mathcal{R}}, X}(G)$ ; it is the intersection of all rings of sets on  $X$  containing  $G$ .

**Lemma 618 (generated ring of sets is ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{R}_X(G)$  is a ring of sets on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 617 (*generated ring of sets*), Lemma 616 (*intersection of rings of sets*), and Lemma 585 (*subset system generation preserves class*).  $\square$

**Lemma 619 (properties of ring of sets generation).**

Let  $X$  be a set. Then, ring of sets generation on  $X$  is minimum, monotone, and idempotent:

$$(8.40) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{R}}, X}, G \subset \mathcal{S} \implies \mathcal{R}_X(G) \subset \mathcal{S},$$

$$(8.41) \quad \forall G_1, G_2 \subset \mathcal{P}(X), G_1 \subset G_2 \implies \mathcal{R}_X(G_1) \subset \mathcal{R}_X(G_2),$$

$$(8.42) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{R}}, X} \iff \mathcal{R}_X(\mathcal{S}) = \mathcal{S}.$$

*Proof.* Direct consequence of Definition 617 (*generated ring of sets*), Lemma 612 (*equivalent definition of ring of sets, with  $wE$* ), Lemma 616 (*intersection of rings of sets*), and Lemma 586 (*properties of subset system generation*).  $\square$

**Lemma 620 (other properties of ring of sets generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.43) \quad G_1 \subset \mathcal{R}_X(G_2) \wedge G_2 \subset \mathcal{R}_X(G_1) \implies \mathcal{R}_X(G_1) = \mathcal{R}_X(G_2),$$

$$(8.44) \quad G_2 \subset \mathcal{R}_X(G_1) \implies \mathcal{R}_X(G_1 \cup G_2) = \mathcal{R}_X(G_1).$$

*Proof.* Direct consequence of Definition 617 (*generated ring of sets*), Lemma 616 (*intersection of rings of sets*), and Lemma 587 (*other properties of subset system generation*).  $\square$

**Lemma 621 (ring of sets contains  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Pi_X(G) \subset \mathcal{R}_X(G)$ .

*Proof.* Direct consequence of Definition 617 (*generated ring of sets*), Definition 606 (*generated  $\pi$ -system*), Lemma 614 (*ring of sets is  $\pi$ -system*), and Lemma 599 (*generated subset systems are monotone (with nonemptiness), with  $P_{\mathcal{R}} \Rightarrow P_{\Pi}$* ).  $\square$

**Lemma 622 (upgrade  $\pi$ -system to ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G \neq \emptyset$ , and that  $\Pi_X(G)$  is closed under symmetric difference.

Then, we have  $\Pi_X(G) = \mathcal{R}_X(G)$ .

Similar compact form:  $P_{\Pi} \wedge sD \Leftrightarrow P_{\mathcal{R}}$ .

*Proof.* Direct consequence of Lemma 621 (*ring of sets contains  $\pi$ -system*,  $\Pi_X(G) \subset \mathcal{R}_X(G)$ ), Definition 617 (*generated ring of sets*), Definition 606 (*generated  $\pi$ -system*), Lemma 616 (*intersection of rings of sets*), Lemma 605 (*intersection of  $\pi$ -systems*), Lemma 612 (*equivalent definition of ring of sets*,  $\underline{P_{\mathcal{R}}} = Ne \wedge sD \wedge I$ ), Lemma 604 (*equivalent definition of  $\pi$ -system*,  $\underline{P_{\Pi}} = Ne \wedge I$ ), and Lemma 600 (*upgrade generated subset system (with nonemptiness)*, with  $P_{\Pi} \wedge sD = \underline{P_{\mathcal{R}}}$ , thus  $\mathcal{R}_X(G) \subset \Pi_X(G)$ ).  $\square$

**Lemma 623 (ring of sets generated by  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\mathcal{R}_X(\Pi_X(G)) = \mathcal{R}_X(G)$ .

*Proof.* Direct consequence of Definition 617 (*generated ring of sets*), Definition 606 (*generated  $\pi$ -system*), Lemma 616 (*intersection of rings of sets*), Lemma 605 (*intersection of  $\pi$ -systems*), Lemma 614 (*ring of sets is  $\pi$ -system*), and Lemma 601 (*cascading generated subset systems (with nonemptiness)*, with  $P_{\mathcal{R}} \Rightarrow P_{\Pi}$ ).  $\square$

**Lemma 624 (disjoint union of union in ring of sets).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0.. \text{card}(I)]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{R}$  be a ring of sets on  $X$ . Let  $(A_i)_{i \in I} \subset \mathcal{R}$ . Then, there exists  $(B_i)_{i \in I} \in \mathcal{R}$  pairwise disjoint such that for all  $i \in I$ ,  $B_i \subset A_i$ , and we have

$$(8.45) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)} \in \mathcal{R}.$$

*Proof.* Direct consequence of Lemma 612 (*equivalent definition of ring of sets*,  $\underline{P_{\mathcal{R}}} \Rightarrow [D \mid Ud]$ ), and Lemma 572 (*disjoint union of union in subset system*,  $\underline{Hyp} = D \wedge Ud$ ).  $\square$

*Remark 625.* The next concept of “semiring of sets” is not to be confused with the algebraic structure “semiring”.

**Definition 626 (semiring of sets).**

Let  $X$  be a set. Let  $P_{\mathcal{R}/2}$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  expressing nonemptiness, closedness under intersection, and that any set difference is a finite disjoint union. The class of *semirings (of (sub)sets) on  $X$*  is  $\mathfrak{C}_{P_{\mathcal{R}/2}}$ : a subset  $\mathcal{R}$  of  $\mathcal{P}(X)$  is a semiring of sets on  $X$  iff

$$(8.46) \quad \mathcal{R} \neq \emptyset,$$

$$(8.47) \quad \forall A, B \in \mathcal{R}, \quad A \cap B \in \mathcal{R},$$

$$(8.48) \quad \forall A, B \in \mathcal{R}, \quad \exists n \in \mathbb{N}, \exists (C_i)_{i \in [0..n]} \in \mathcal{R}, \quad \left\{ \begin{array}{l} (C_i)_{i \in [0..n]} \text{ is pairwise disjoint} \\ A \setminus B = \biguplus_{i \in [0..n]} C_i. \end{array} \right.$$

Compact form:  $P_{\mathcal{R}/2} \stackrel{\text{def.}}{=} Ne \wedge I \wedge PtfD$ .

**Lemma 627 (equivalent definition of semiring of sets).** Let  $X$  be a set. Let  $\mathcal{R} \subset \mathcal{P}(X)$ . Then,  $\mathcal{R}$  is a semiring of sets on  $X$  iff (8.47), and (8.48) hold and

$$(8.49) \quad \mathcal{R} \neq \emptyset, \quad \text{or} \quad \emptyset \in \mathcal{R}.$$

Compact form:  $P_{\mathcal{R}/2} \Leftrightarrow [Ne \mid wE] \wedge I \wedge PtfD$ .

*Proof.* Direct consequence of Definition 626 (*semiring of sets*,  $\underline{P_{\mathcal{R}/2}} = Ne \wedge I \wedge PtfD$ ), Lemma 552 (*nonempty is with empty*,  $\underline{Ne \wedge PtfD} \Rightarrow wE$ ), and Lemma 538 (*nonempty and with empty or full*,  $wE \Rightarrow Ne$ ).  $\square$

*Remark 628.* Note that (8.48) is not a property compatible with intersection: the intersection of two semirings of sets on  $X$  is not necessarily a semiring of sets on  $X$ . Thus, the concept of generated semiring of sets is not practical. The main use of semirings of sets is to ease the building of rings of sets, e.g. through Lemma 632.

**Lemma 629 (semiring of sets is  $\pi$ -system).**

Let  $X$  be a set. Then, semirings of sets on  $X$  are  $\pi$ -systems on  $X$ .

Compact form:  $P_{\mathcal{R}/2} \Rightarrow P_{\Pi}$ .

*Proof.* Direct consequence of Definition 626 (semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow [Ne \mid I]$ ), and Lemma 604 (equivalent definition of  $\pi$ -system,  $P_{\Pi} = Ne \wedge I$ ).  $\square$

**Lemma 630 (other properties of semiring of sets).**

Let  $X$  be a set. Let  $\mathcal{R}$  be a semiring of sets on  $X$ . Then, any local complement of elements of  $\mathcal{R}$  is a finite disjoint union of elements of  $\mathcal{R}$ , and  $\mathcal{R}$  is closed under finite intersection.

Compact form:  $P_{\mathcal{R}/2} \Rightarrow [Ptfd' \mid If]$ .

*Proof.* Direct consequence of Definition 626 (semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow PtfD$ ), Lemma 557 (disjoint decomposition of set difference and local complement,  $PtfD \Rightarrow PtfD'$ ), Lemma 629 (semiring of sets is  $\pi$ -system,  $P_{\mathcal{R}/2} \Rightarrow P_{\Pi}$ ), and Definition 602 ( $\pi$ -system,  $P_{\Pi} \Rightarrow If$ ).  $\square$

**Lemma 631 (ring of sets is semiring of sets).**

Let  $X$  be a set. Then, rings of sets on  $X$  are semirings of sets on  $X$ .

Compact form:  $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R}/2}$ .

*Proof.* Direct consequence of Lemma 612 (equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow [Ne \mid I \mid D]$ ), Lemma 554 (disjoint decomposition,  $D \Rightarrow PtfD$ ), and Definition 626 (semiring of sets,  $P_{\mathcal{R}/2}$  equals  $Ne \wedge I \wedge PtfD$ ).  $\square$

**Lemma 632 (explicit ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a semiring of sets on  $X$ .

Then,  $\mathcal{R}_X(G)$  is the set of finite (disjoint) unions of elements of  $G$ :

$$(8.50) \quad \mathcal{R}_X(G) = \left\{ \bigcup_{i \in [0..n]} A_i \mid n \in \mathbb{N} \quad \wedge \quad (A_i)_{i \in [0..n]} \in G \right\} \\ = \left\{ \biguplus_{i \in [0..n]} A_i \mid n \in \mathbb{N} \quad \wedge \quad (A_i)_{i \in [0..n]} \in G \text{ is pairwise disjoint} \right\}.$$

*Proof.* Let  $\mathcal{R} \stackrel{\text{def.}}{=} \mathcal{R}_X(G)$ . Let  $\mathcal{U}$  be the set of finite unions of elements of  $G$ , and  $\mathcal{U}^+$  be the set of finite disjoint unions of elements of  $G$ .

(0a).  $\mathcal{U}^+ \subset \mathcal{U}$ . Direct consequence of the definitions of  $\mathcal{U}^+$  and  $\mathcal{U}$ .

(0b).  $\mathcal{U} \subset \mathcal{R}$ . Direct consequence of Lemma 618 (generated ring of sets is ring of sets,  $G \subset \mathcal{R}$ ), and Definition 611 (ring of sets,  $P_{\mathcal{R}} \Rightarrow Uf$  (for  $\mathcal{R}$ )).

(1a).  $G \subset \mathcal{U}^+$ . Direct consequence of the definition of  $\mathcal{U}^+$  (with  $n \stackrel{\text{def.}}{=} 0$ ).

(1b).  $\emptyset \in \mathcal{U}^+$ . Direct consequence of Lemma 627 (equivalent definition of semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow wE$ , thus  $\emptyset \in G$ ), and (1a).

(1c).  $\mathcal{U}^+$  is closed under set difference. Direct consequence of Definition 626 (*semiring of sets*,  $\underline{P_{\mathcal{R}/2}} \Rightarrow [I \mid PtfD]$  (for  $G$ )), and Lemma 563 (*closedness under set difference and finite disjoint union closure*,  $\underline{I \wedge PtfD} \Rightarrow D(\mathcal{U}^+)$ ).

(1d).  $\mathcal{U}^+$  is closed under disjoint union. Direct consequence of Lemma 562 (*finite disjoint union closure is closed under finite disjoint union*,  $\underline{Ud}(\mathcal{U}^+)$ ).

(2).  $\mathcal{R} \subset \mathcal{U}^+$ . Direct consequence of (1b), (1c), (1d), Lemma 612 (*equivalent definition of ring of sets*,  $\underline{P_{\mathcal{R}} = wE \wedge D \wedge Ud}$ , thus  $\mathcal{U}^+$  is ring of sets), (1a), and Lemma 619 (*properties of ring of sets generation*, *minimality*).

Therefore, from (0a), (0b) and (2), we have  $\mathcal{R} = \mathcal{U} = \mathcal{U}^+$ . □

## 8.5 Algebra of sets

*Remark 633.* The next concept of “algebra of sets” is not to be confused with the algebraic structure “algebra over a field” defined in Section 7.4.2.

Note that the same concept is sometimes called “field of sets”, which is of course not to be confused with the algebraic structure either.

**Definition 634 (algebra of sets).** Let  $X$  be a set. Let  $P_{\mathcal{A}}$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of containing the empty set, and of closedness under complement and finite union. The class of *algebras (of (sub)sets) on  $X$*  is  $\mathfrak{C}_{P_{\mathcal{A}}}$ : a subset  $\mathcal{A}$  of  $\mathcal{P}(X)$  is an *algebra of sets on  $X$*  iff

$$(8.51) \quad \emptyset \in \mathcal{A},$$

$$(8.52) \quad \forall A \in \mathcal{A}, \quad A^c \in \mathcal{A},$$

$$(8.53) \quad \forall n \in \mathbb{N}, \forall (A_i)_{i \in [0..n]} \in \mathcal{A}, \quad \bigcup_{i \in [0..n]} A_i \in \mathcal{A}.$$

Compact form:  $P_{\mathcal{A}} \stackrel{\text{def.}}{=} wE \wedge C \wedge Uf$ .

**Lemma 635 (equivalent definition of algebra of sets).**

Let  $X$  be a set. Let  $\mathcal{A} \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}$  is an algebra of sets on  $X$  iff (8.52) holds and

$$(8.54) \quad \mathcal{A} \neq \emptyset, \quad \text{or} \quad \emptyset \in \mathcal{A}, \quad \text{or} \quad X \in \mathcal{A},$$

$$(8.55) \quad \forall A, B \in \mathcal{A}, \quad A \cup B \in \mathcal{A} \quad \text{or} \quad \forall A, B \in \mathcal{A}, \quad A \cap B \in \mathcal{A}.$$

Compact form:  $P_{\mathcal{A}} \Leftrightarrow [Ne \mid wE \mid wF] \wedge C \wedge [U \mid I]$ .

*Proof.* Direct consequence of Definition 634 (*algebra of sets*,  $P_{\mathcal{A}} = wE \wedge C \wedge Uf$ ), Lemma 558 (*closedness under finite operations*,  $Uf \Leftrightarrow U$ ), Lemma 547 (*closedness under union and intersection*,  $C \Rightarrow U \Leftrightarrow I$ ), Lemma 538 (*nonempty and with empty or full*,  $wE \Rightarrow Ne$  and  $C \wedge I \wedge Ne$  implies  $wE$ ), and Lemma 539 (*with empty and full*,  $C \Rightarrow wE \Leftrightarrow wF$ ).  $\square$

*Remark 636.*

Note that the previous lemma means that the predicate  $P_{\mathcal{A}}$  characterizing algebras of sets is equivalent to any of the six alternatives summarized by the conjunction (8.54), (8.52), and (8.55).

**Lemma 637 (other equivalent definition of algebra of sets).**

Let  $X$  be a set. Let  $\mathcal{A} \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}$  is an algebra of sets on  $X$  iff it contains the full set, and it is closed under set difference:

$$(8.56) \quad X \in \mathcal{A}.$$

$$(8.57) \quad \forall A, B \in \mathcal{A}, \quad A \setminus B \in \mathcal{A}.$$

Compact form:  $P_{\mathcal{A}} \Leftrightarrow wF \wedge D$ .

*Proof.* Direct consequence of Lemma 635 (*equivalent definition of algebra of sets*,  $P_{\mathcal{A}}$  equals  $wF \wedge C \wedge I$ ), Lemma 545 (*closedness under intersection and set difference*,  $C \wedge I \Rightarrow D$  and  $D \Rightarrow I$ ), Lemma 543 (*closedness under set difference and local complement*,  $D \Rightarrow D'$ ), and Lemma 541 (*closedness under local complement and complement*,  $wF \wedge D' \Rightarrow C$ ).  $\square$

**Lemma 638 (algebra of sets is ring of sets).**

Let  $X$  be a set. Then, algebras of sets on  $X$  are rings of sets on  $X$ .

Compact form:  $P_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$ .



*Proof.* Direct consequence of Lemma 635 (equivalent definition of algebra of sets,  $\underline{P_A} \Rightarrow [wE \mid U]$ ), Lemma 637 (other equivalent definition of algebra of sets,  $\underline{P_A} \Rightarrow D$ ), and Lemma 612 (equivalent definition of ring of sets,  $\underline{P_R} = wE \wedge D \wedge U$ ).  $\square$

**Lemma 639 (other properties of algebra of sets).**

Let  $X$  be a set. Let  $\mathcal{A}$  be an algebra of sets on  $X$ . Then,  $\mathcal{A}$  is closed under local complement, symmetric difference, disjoint union, finite disjoint union, and finite intersection.

Compact form:  $\underline{P_A} \Rightarrow [D' \mid sD \mid Ud \mid Udf \mid If]$ .

*Proof.* Direct consequence of Lemma 638 (algebra of sets is ring of sets,  $\underline{P_A} \Rightarrow \underline{P_R}$ ), Lemma 612 (equivalent definition of ring of sets,  $\underline{P_R} \Rightarrow [D' \mid sD]$ ), Lemma 615 (other properties of ring of sets,  $\underline{P_R} \Rightarrow If$ ), Lemma 635 (equivalent definition of algebra of sets,  $\underline{P_A} \Rightarrow U$ ), Lemma 546 (closedness under union and disjoint union,  $\underline{U} \Rightarrow Ud$ ), and Lemma 558 (closedness under finite operations,  $Ud \Rightarrow Udf$ ).  $\square$

**Lemma 640 (algebra of sets is  $\pi$ -system).**

Let  $X$  be a set. Then, algebras of sets on  $X$  are  $\pi$ -systems on  $X$ .

Compact form:  $\underline{P_A} \Rightarrow \underline{P_\Pi}$ .

*Proof.* Direct consequence of Lemma 638 (algebra of sets is ring of sets,  $\underline{P_A} \Rightarrow \underline{P_R}$ ), and Lemma 614 (ring of sets is  $\pi$ -system,  $\underline{P_R} \Rightarrow \underline{P_\Pi}$ ).  $\square$

**Lemma 641 (intersection of algebras of sets).**

Let  $X$  and  $I$  be sets.

Let  $(\mathcal{A}_i)_{i \in I}$  be algebras of sets on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{A}_i$  is an algebra of sets on  $X$ .

*Proof.* Direct consequence of Definition 634 (algebra of sets), and **the definition of intersection**.  $\square$

**Definition 642 (generated algebra of sets).**

Let  $X$  be a set.

Let  $G \subset \mathcal{P}(X)$ . The algebra of sets generated by  $G$  is denoted  $\mathcal{A}_X(G) \stackrel{\text{def.}}{=} \mathfrak{G}_{P_A, X}(G)$ ; it is the intersection of all algebras of sets on  $X$  containing  $G$ .

**Lemma 643 (generated algebra of sets is algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}_X(G)$  is an algebra of sets on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 642 (generated algebra of sets), Lemma 641 (intersection of algebras of sets), and Lemma 585 (subset system generation preserves class).  $\square$

**Lemma 644 (properties of algebra of sets generation).**

Let  $X$  be a set. Then, algebra of sets generation on  $X$  is minimum, monotone, and idempotent:

$$(8.58) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_A, X}, G \subset \mathcal{S} \implies \mathcal{A}_X(G) \subset \mathcal{S},$$

$$(8.59) \quad \forall G_1, G_2 \subset \mathcal{P}(X), G_1 \subset G_2 \implies \mathcal{A}_X(G_1) \subset \mathcal{A}_X(G_2),$$

$$(8.60) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \mathcal{S} \in \mathfrak{C}_{P_A, X} \iff \mathcal{A}_X(\mathcal{S}) = \mathcal{S}.$$

*Proof.* Direct consequence of Definition 642 (generated algebra of sets), Lemma 641 (intersection of algebras of sets), and Lemma 586 (properties of subset system generation).  $\square$

**Lemma 645 (other properties of algebra of sets generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.61) \quad G_1 \subset \mathcal{A}_X(G_2) \wedge G_2 \subset \mathcal{A}_X(G_1) \implies \mathcal{A}_X(G_1) = \mathcal{A}_X(G_2),$$

$$(8.62) \quad G_2 \subset \mathcal{A}_X(G_1) \implies \mathcal{A}_X(G_1 \cup G_2) = \mathcal{A}_X(G_1).$$

*Proof.* Direct consequence of Definition 642 (*generated algebra of sets*), Lemma 641 (*intersection of algebras of sets*), and Lemma 587 (*other properties of subset system generation*).  $\square$

**Lemma 646 (algebra of sets contains ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{R}_X(G) \subset \mathcal{A}_X(G)$ .

*Proof.* Direct consequence of Definition 642 (*generated algebra of sets*), Definition 617 (*generated ring of sets*), Lemma 638 (*algebra of sets is ring of sets*), and Lemma 588 (*generated subset systems are monotone*, with  $P_A \Rightarrow P_R$ ).  $\square$

**Lemma 647 (upgrade ring of sets to algebra of sets).** Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{R}_X(G)$  either contains the full set, or is closed under complement.

Then, we have  $\mathcal{R}_X(G) = \mathcal{A}_X(G)$ .

Similar compact form:  $P_R \wedge [wF \mid C] \Leftrightarrow P_A$ .

*Proof.* Direct consequence of Lemma 646 (*algebra of sets contains ring of sets*,  $\mathcal{R}_X(G) \subset \mathcal{A}_X(G)$ ), Definition 642 (*generated algebra of sets*), Definition 617 (*generated ring of sets*), Lemma 641 (*intersection of algebras of sets*), Lemma 616 (*intersection of rings of sets*), Lemma 637 (*other equivalent definition of algebra of sets*,  $P_A = wF \wedge D$ ), Lemma 635 (*equivalent definition of algebra of sets*,  $P_A = Ne \wedge C \wedge U$ ), Lemma 612 (*equivalent definition of ring of sets*,  $P_R = Ne \wedge D \wedge U$ ), Lemma 590 (*upgrade generated subset system*, with  $P_R \wedge [wF \mid C] \Rightarrow P_A$ , thus  $\mathcal{A}_X(G) \subset \mathcal{R}_X(G)$ ), and Lemma 638 (*algebra of sets is ring of sets*,  $P_A \Rightarrow P_R$ , thus  $P_A \Rightarrow [P_R \mid wF \mid C]$ ).  $\square$

**Lemma 648 (algebra of sets generated by ring of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{A}_X(\mathcal{R}_X(G)) = \mathcal{A}_X(G)$ .

*Proof.* Direct consequence of Definition 642 (*generated algebra of sets*), Definition 617 (*generated ring of sets*), Lemma 641 (*intersection of algebras of sets*), Lemma 616 (*intersection of rings of sets*), Lemma 638 (*algebra of sets is ring of sets*), and Lemma 591 (*cascading generated subset systems*, with  $P_A \Rightarrow P_R$ ).  $\square$

**Lemma 649 (algebra of sets contains  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Pi_X(G) \subset \mathcal{A}_X(G)$ .

*Proof.* Direct consequence of Lemma 621 (*ring of sets contains  $\pi$ -system*,  $\Pi_X(G) \subset \mathcal{R}_X(G)$ ), Lemma 646 (*algebra of sets contains ring of sets*,  $\mathcal{R}_X(G) \subset \mathcal{A}_X(G)$ ), and **transitivity of inclusion**.  $\square$

**Lemma 650 (upgrade  $\pi$ -system to algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ , and that  $\Pi_X(G)$  is closed under complement. Then, we have  $\Pi_X(G) = \mathcal{A}_X(G)$ .

Similar compact form:  $P_\Pi \wedge C \Leftrightarrow P_A$ .

*Proof.* Direct consequence of Lemma 604 (*equivalent definition of  $\pi$ -system*,  $P_\Pi \Rightarrow I$ ), Lemma 547 (*closedness under union and intersection*,  $I \wedge C \Rightarrow U$ ), Lemma 545 (*closedness under intersection and set difference*,  $I \wedge C \Rightarrow D$ ), Lemma 550 (*closedness under symmetric difference and set difference*,  $U \wedge D \Rightarrow sD$ ), Lemma 622 (*upgrade  $\pi$ -system to ring of sets*,  $P_\Pi \wedge C \Rightarrow P_\Pi \wedge sD = P_R$ , thus  $\Pi_X(G) = \mathcal{R}_X(G)$ ), Lemma 614 (*ring of sets is  $\pi$ -system*,  $P_R \Rightarrow P_\Pi$ ), and Lemma 647 (*upgrade ring of sets to algebra of sets*,  $P_\Pi \wedge C \Leftrightarrow P_R \wedge C \Leftrightarrow P_A$ , thus  $\mathcal{R}_X(G) = \mathcal{A}_X(G)$ ).  $\square$

**Lemma 651 (algebra of sets generated by  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\mathcal{A}_X(\Pi_X(G)) = \mathcal{A}_X(G)$ .

*Proof.* Direct consequence of Definition 642 (*generated algebra of sets*), Definition 617 (*generated ring of sets*), Definition 606 (*generated  $\pi$ -system*), Lemma 648 (*algebra of sets generated by ring of sets*), Lemma 623 (*ring of sets generated by  $\pi$ -system*), and Lemma 592 (*double cascading generated subset systems*).  $\square$

**Lemma 652 (*disjoint union of union in algebra of sets*).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0.. \text{card}(I)]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\mathcal{A}$  be an algebra of sets on  $X$ . Let  $(A_i)_{i \in I} \subset \mathcal{A}$ .

Then, there exists  $(B_i)_{i \in I} \in \mathcal{A}$  pairwise disjoint such that for all  $i \in I$ ,  $B_i \subset A_i$ , and we have

$$(8.63) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)} \in \mathcal{A}.$$

*Proof.* Direct consequence of Lemma 638 (*algebra of sets is ring of sets*), and Lemma 624 (*disjoint union of union in ring of sets*).  $\square$

*Remark 653.*

Note that the next concept of “semi-algebra<sup>a</sup> of sets” is sometimes called “semifield of sets”.

<sup>a</sup>We use a hyphen here because “semialgebra” looks awful (thus applying the algorithm from [https://www.grammar-monster.com/lessons/hyphens\\_in\\_prefixes.htm](https://www.grammar-monster.com/lessons/hyphens_in_prefixes.htm)).

**Definition 654 (*semi-algebra of sets*).**

Let  $X$  be a set.

Let  $P_{A/2}$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of being a semiring of sets and containing the full set.

The class of *semi-algebras (of (sub)sets)* on  $X$  is  $\mathfrak{C}_{P_{A/2}}$ .

Compact form:  $P_{A/2} \stackrel{\text{def.}}{=} P_{\mathcal{R}/2} \wedge wF$ .

**Lemma 655 (*equivalent definition of semi-algebra of sets*).**

Let  $X$  be a set. Let  $\mathcal{A} \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}$  is a semi-algebra of sets on  $X$  iff

$$(8.64) \quad X \in \mathcal{A},$$

$$(8.65) \quad \forall A, B \in \mathcal{A}, \quad A \cap B \in \mathcal{A},$$

$$(8.66) \quad \left\{ \begin{array}{l} \forall A \in \mathcal{A}, \quad \exists n \in \mathbb{N}, \exists (B_i)_{i \in [0..n]} \in \mathcal{A}, \quad \left\{ \begin{array}{l} (B_i)_{i \in [0..n]} \text{ is pairwise disjoint} \\ A^c = \biguplus_{i \in [0..n]} B_i, \end{array} \right. \\ \text{or} \\ \forall A, B \in \mathcal{A}, \quad \exists n \in \mathbb{N}, \exists (C_i)_{i \in [0..n]} \in \mathcal{A}, \quad \left\{ \begin{array}{l} (C_i)_{i \in [0..n]} \text{ is pairwise disjoint} \\ A \setminus B = \biguplus_{i \in [0..n]} C_i. \end{array} \right. \end{array} \right.$$

Compact form:  $P_{A/2} \Leftrightarrow wF \wedge I \wedge [Ptfc \mid PtfD]$ .

*Proof.* Direct consequence of Definition 654 (*semi-algebra of sets*,  $P_{A/2} = P_{\mathcal{R}/2} \wedge wF$ ), Definition 626 (*semiring of sets*,  $P_{\mathcal{R}/2} = Ne \wedge I \wedge PtfD$ ), Lemma 557 (*disjoint decomposition of set difference and local complement*,  $PtfD \Rightarrow PtfD'$ ), Lemma 556 (*disjoint decomposition of local complement and complement*,  $wF \wedge PtfD' \Rightarrow PtfC$ ). Lemma 538 (*nonempty and with empty or full*,  $wF \Rightarrow Ne$ ), and Lemma 555 (*disjoint decomposition of complement and set difference*,  $I \wedge PtfC$  implies  $PtfD$ ).  $\square$

*Remark 656.* Note that (8.66) is not a property compatible with intersection: the intersection of two semi-algebras of sets on  $X$  is not necessarily a semi-algebra of sets on  $X$ . Thus, the concept of generated semi-algebra of sets is not practical. The main use of semi-algebras of sets is to ease the building of algebras of sets, e.g. through Lemma 662.

**Lemma 657 (semi-algebra of sets is semiring of sets).**

Let  $X$  be a set. Then, semi-algebras of sets on  $X$  are semirings of sets on  $X$ .

Compact form:  $P_{A/2} \Rightarrow P_{\mathcal{R}/2}$ .

*Proof.* Direct consequence of Definition 654 (semi-algebra of sets,  $P_{A/2} \Rightarrow P_{\mathcal{R}/2}$ ).  $\square$

**Lemma 658 (other properties of semi-algebra of sets).**

Let  $X$  be a set. Let  $\mathcal{A}$  be a semi-algebra of sets on  $X$ . Then, any local complement of elements of  $\mathcal{A}$  is a finite disjoint union of elements of  $\mathcal{A}$ , and  $\mathcal{A}$  is closed under finite intersection.

Compact form:  $P_{A/2} \Rightarrow [PtFD' \mid If]$ .

*Proof.* Direct consequence of Lemma 657 (semi-algebra of sets is semiring of sets,  $P_{A/2} \Rightarrow P_{\mathcal{R}/2}$ ), and Lemma 630 (other properties of semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow [PtFD' \mid If]$ ).  $\square$

**Lemma 659 (semi-algebra of sets is  $\pi$ -system).**

Let  $X$  be a set. Then, semi-algebras of sets on  $X$  are  $\pi$ -systems on  $X$ .

Compact form:  $P_{A/2} \Rightarrow P_{\Pi}$ .

*Proof.* Direct consequence of Lemma 657 (semi-algebra of sets is semiring of sets,  $P_{A/2} \Rightarrow P_{\mathcal{R}/2}$ ), and Lemma 629 (semiring of sets is  $\pi$ -system,  $P_{\mathcal{R}/2} \Rightarrow P_{\Pi}$ ).  $\square$

**Lemma 660 (algebra of sets is semi-algebra of sets).**

Let  $X$  be a set. Then, algebras of sets on  $X$  are semi-algebras of sets on  $X$ .

Compact form:  $P_{\mathcal{A}} \Rightarrow P_{A/2}$ .

*Proof.* Direct consequence of Lemma 635 (equivalent definition of algebra of sets,  $P_{\mathcal{A}}$  implies  $[wF \mid I \mid C]$ ), Lemma 554 (disjoint decomposition,  $C \Rightarrow PtFC$ ), and Lemma 655 (equivalent definition of semi-algebra of sets,  $P_{A/2} = wF \wedge I \wedge PtFC$ ).  $\square$

**Lemma 661 (algebra of sets is semiring of sets).**

Let  $X$  be a set. Then, algebras of sets on  $X$  are semirings of sets on  $X$ .

Compact form:  $P_{\mathcal{A}} \Rightarrow P_{\mathcal{R}/2}$ .

*Proof.* Direct consequence of Lemma 660 (algebra of sets is semi-algebra of sets,  $P_{\mathcal{A}} \Rightarrow P_{A/2}$ ), and Lemma 657 (semi-algebra of sets is semiring of sets,  $P_{A/2} \Rightarrow P_{\mathcal{R}/2}$ ).  $\square$

**Lemma 662 (explicit algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a semi-algebra of sets on  $X$ .

Then,  $\mathcal{A}_X(G)$  is the set of finite (disjoint) unions of elements of  $G$ :

$$(8.67) \quad \mathcal{A}_X(G) = \left\{ \bigcup_{i \in [0..n]} A_i \mid n \in \mathbb{N} \wedge \forall i \in [0..n], A_i \in G \right\} \\ = \left\{ \biguplus_{p \in [0..n]} A_p \mid n \in \mathbb{N} \wedge (A_i)_{i \in [0..n]} \in G \text{ is pairwise disjoint} \right\}.$$

*Proof.* Direct consequence of Definition 654 (semi-algebra of sets,  $P_{A/2} = P_{\mathcal{R}/2} \wedge wF$ ), Lemma 632 (explicit ring of sets,  $\mathcal{R}_X(G) = \mathcal{U} = \mathcal{U}^+$ ), Lemma 618 (generated ring of sets is ring of sets, thus  $X \in G \subset \mathcal{R}_X(G)$ ), and Lemma 647 (upgrade ring of sets to algebra of sets,  $\mathcal{A}_X(G) = \mathcal{R}_X(G)$ ).  $\square$

## 8.6 Monotone class

*Remark 663.* Note that in the next definition, nonemptiness of the set of indices  $I$  is important since a monotone class is not required to contain the empty set, or the full set.

**Definition 664 (monotone class).** Let  $X$  be a set. Let  $P_{\mathcal{C}}$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of closedness under countable monotone union and intersection. The class of *monotone classes on  $X$*  is  $\mathfrak{C}_{P_{\mathcal{C}}}$ : a subset  $\mathcal{C}$  of  $\mathcal{P}(X)$  is a monotone class on  $X$  iff

$$(8.68) \quad \forall I \subset \mathbb{N} \text{ nonempty}, \forall (A_i)_{i \in I} \in \mathcal{C}, (A_i)_{i \in I} \text{ nondecreasing} \implies \bigcup_{i \in I} A_i \in \mathcal{C},$$

$$(8.69) \quad \forall I \subset \mathbb{N} \text{ nonempty}, \forall (A_i)_{i \in I} \in \mathcal{C}, (A_i)_{i \in I} \text{ nonincreasing} \implies \bigcap_{i \in I} A_i \in \mathcal{C}.$$

Compact form:  $P_{\mathcal{C}} \stackrel{\text{def.}}{=} Umc \wedge Imc.$

**Lemma 665 (intersection of monotone classes).** Let  $X$  and  $I$  be sets. Let  $(\mathcal{C}_i)_{i \in I}$  be a family of monotone classes on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{C}_i$  is a monotone class on  $X$ .

*Proof.*

Direct consequence of Definition 664 (*monotone class*), and **the definition of intersection**.  $\square$

**Definition 666 (generated monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The *monotone class generated by  $G$*  is denoted  $\mathcal{C}_X(G) \stackrel{\text{def.}}{=} \mathfrak{C}_{P_{\mathcal{C}}, X}(G)$ ; it is the intersection of all monotone classes on  $X$  containing  $G$ .

**Lemma 667 (generated monotone class is monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{C}_X(G)$  is a monotone class on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 666 (*generated monotone class*), Lemma 665 (*intersection of monotone classes*), and Lemma 585 (*subset system generation preserves class*).  $\square$

**Lemma 668 (properties of monotone class generation).**

Let  $X$  be a set. Then, monotone class generation on  $X$  is minimum, monotone, and idempotent:

$$(8.70) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{C}}, X}, G \subset \mathcal{S} \implies \mathcal{C}_X(G) \subset \mathcal{S},$$

$$(8.71) \quad \forall G_1, G_2 \subset \mathcal{P}(X), G_1 \subset G_2 \implies \mathcal{C}_X(G_1) \subset \mathcal{C}_X(G_2),$$

$$(8.72) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{C}}, X} \iff \mathcal{C}_X(\mathcal{S}) = \mathcal{S}.$$

*Proof.* Direct consequence of Definition 666 (*generated monotone class*), Lemma 665 (*intersection of monotone classes*), and Lemma 586 (*properties of subset system generation*).  $\square$

**Lemma 669 (other properties of monotone class generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.73) \quad G_1 \subset \mathcal{C}_X(G_2) \wedge G_2 \subset \mathcal{C}_X(G_1) \implies \mathcal{C}_X(G_1) = \mathcal{C}_X(G_2),$$

$$(8.74) \quad G_2 \subset \mathcal{C}_X(G_1) \implies \mathcal{C}_X(G_1 \cup G_2) = \mathcal{C}_X(G_1).$$

*Proof.* Direct consequence of Definition 666 (*generated monotone class*), Lemma 665 (*intersection of monotone classes*), and Lemma 587 (*other properties of subset system generation*).  $\square$

**Definition 670 (monotone class and symmetric set difference).**

Let  $X$  be a set. Let  $\mathcal{C}$  be a monotone class on  $X$ . For all  $A \subset X$ , let  $\mathcal{C}_A^\setminus$  be the subset system defined by

$$(8.75) \quad \mathcal{C}_A^\setminus \stackrel{\text{def.}}{=} \{B \subset X \mid A \setminus B \in \mathcal{C} \wedge B \setminus A \in \mathcal{C}\}.$$

**Lemma 671** ( $\mathcal{C}^{\setminus}$  is symmetric). Let  $X$  be a set.  
Let  $\mathcal{C}$  be a monotone class on  $X$ . Then, for all  $A, B \subset X$ , we have  $B \in \mathcal{C}_A^{\setminus}$  iff  $A \in \mathcal{C}_B^{\setminus}$ .

*Proof.* Direct consequence of Definition 670 (monotone class and symmetric set difference), and **commutativity of logical conjunction**.  $\square$

**Lemma 672** ( $\mathcal{C}^{\setminus}$  is monotone class). Let  $X$  be a set.  
Let  $\mathcal{C}$  be a monotone class on  $X$ . Then, for all  $A \subset X$ ,  $\mathcal{C}_A^{\setminus}$  is a monotone class on  $X$ .

Compact form:  $P_{\mathcal{C}} \Rightarrow P_{\mathcal{C}_A^{\setminus}}$ .

*Proof.* Let  $A \subset X$ . Let  $I \subset \mathbb{N}$  be nonempty. Let  $(B_i)_{i \in I} \in \mathcal{C}_A^{\setminus}$ . Let  $i \in I$ . Then, from Definition 670 (monotone class and symmetric set difference), we have  $A \setminus B_i, B_i \setminus A \in \mathcal{C}$ .

Assume first that  $(B_i)_{i \in I}$  is nondecreasing. Then, from **monotonicity of set difference**,  $(A \setminus B_i)_{i \in I}$  is nonincreasing, and  $(B_i \setminus A)_{i \in I}$  is nondecreasing. Let  $B \stackrel{\text{def.}}{=} \bigcup_{i \in I} B_i$ . Then, from **the definition of set difference, De Morgan's laws**, Definition 670 (monotone class and symmetric set difference), and Definition 664 (monotone class), we have  $A \setminus B = \bigcap_{i \in I} (A \setminus B_i) \in \mathcal{C}$  and  $B \setminus A = \bigcup_{i \in I} (B_i \setminus A) \in \mathcal{C}$ . Thus, from Definition 670 (monotone class and symmetric set difference), we have  $B \in \mathcal{C}_A^{\setminus}$ . Hence,  $\mathcal{C}_A^{\setminus}$  is closed under countable monotone union.

Assume now that  $(B_i)_{i \in I}$  is nonincreasing. Then, from **monotonicity of set difference**,  $(A \setminus B_i)_{i \in I}$  is nondecreasing, and  $(B_i \setminus A)_{i \in I}$  is nonincreasing. Let  $B \stackrel{\text{def.}}{=} \bigcap_{i \in I} B_i$ . Then, from **the definition of set difference, De Morgan's laws**, Definition 670 (monotone class and symmetric set difference), and Definition 664 (monotone class), we have  $A \setminus B = \bigcup_{n \in \mathbb{N}} (A \setminus B_n) \in \mathcal{C}$  and  $B \setminus A = \bigcap_{n \in \mathbb{N}} (B_n \setminus A) \in \mathcal{C}$ . Thus, from Definition 670 (monotone class and symmetric set difference), we have  $B \in \mathcal{C}_A^{\setminus}$ . Hence,  $\mathcal{C}_A^{\setminus}$  is closed under countable monotone intersection.

Therefore, from Definition 664 (monotone class),  $\mathcal{C}_A^{\setminus}$  is a monotone class on  $X$ .  $\square$

*Remark 673.* Note that the compact formula in the next lemma should actually be read as for all subset system  $G$ ,  $D(G)$  implies  $I(\mathcal{C}_X(G))$ .

**Lemma 674** (monotone class is closed under set difference).

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is closed under set difference. Then,  $\mathcal{C}_X(G)$  is closed under set difference.

Compact form:  $D \Rightarrow D(\mathcal{C}_X)$ .

*Proof.* For all  $A \subset X$ , we use the simplified notation  $\mathcal{C}_A^{\setminus} \stackrel{\text{def.}}{=} (\mathcal{C}_X(G))_A^{\setminus}$ .

(1).  $\forall A \in G, \mathcal{C}_X(G) \subset \mathcal{C}_A^{\setminus}$ . Let  $A \in G$ .

Let  $B \in G$ . Then, we have  $A \setminus B, B \setminus A \in G$ . Thus, from Lemma 667 (generated monotone class is monotone class,  $G \subset \mathcal{C}_X(G)$ ), and Definition 670 (monotone class and symmetric set difference), we have  $B \in \mathcal{C}_A^{\setminus}$ , i.e.  $G \subset \mathcal{C}_A^{\setminus}$ . Hence, from Lemma 672 ( $\mathcal{C}^{\setminus}$  is monotone class), and Lemma 668 (properties of monotone class generation, minimality), we have  $\mathcal{C}_X(G) \subset \mathcal{C}_A^{\setminus}$ .

(2).  $\forall B \in \mathcal{C}_X(G), \mathcal{C}_X(G) \subset \mathcal{C}_B^{\setminus}$ . Let  $B \in \mathcal{C}_X(G)$ .

Let  $A \in G$ . Then, from (1), we have  $B \in \mathcal{C}_A^{\setminus}$ , and from Lemma 671 ( $\mathcal{C}^{\setminus}$  is symmetric), we have  $A \in \mathcal{C}_B^{\setminus}$ , i.e.  $G \subset \mathcal{C}_B^{\setminus}$ . Hence, from Lemma 672 ( $\mathcal{C}^{\setminus}$  is monotone class), and Lemma 668 (properties of monotone class generation, minimality), we have  $\mathcal{C}_X(G) \subset \mathcal{C}_B^{\setminus}$ .

Let  $A, B \in \mathcal{C}_X(G)$ . Then, from (2), we have  $B \in \mathcal{C}_A^{\setminus}$ , and from Definition 670 (monotone class and symmetric set difference), we have  $A \setminus B \in \mathcal{C}_X(G)$ .

Therefore,  $\mathcal{C}_X(G)$  is closed under set difference.  $\square$

**Lemma 675 (monotone class generated by algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is an algebra of sets on  $X$ .

Then,  $\mathcal{C}_X(G)$  is an algebra of sets on  $X$ .

Compact form:  $P_A \Rightarrow P_A(\mathcal{C}_X)$ .

*Proof.* Direct consequence of Lemma 637 (*other equivalent definition of algebra of sets,  $P_A$  equals  $wF \wedge D$  (for  $G$ , then for  $\mathcal{C}_X(G)$ )*), Lemma 667 (*generated monotone class is monotone class,  $G \subset \mathcal{C}_X(G)$ , thus  $wF(\mathcal{C}_X(G))$* ), and Lemma 674 (*monotone class is closed under set difference, thus  $D(\mathcal{C}_X(G))$* ).  $\square$



## 8.7 Lambda-system

*Remark 676.* Note that the next concept of “ $\lambda$ -system” is sometimes called “Dynkin-system”.

**Definition 677 ( $\lambda$ -system).** Let  $X$  be a set. Let  $P_\Lambda$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of containing the full set, and of closedness under complement and countable disjoint union. The class of  $\lambda$ -systems on  $X$  is  $\mathfrak{C}_{P_\Lambda}$ : a subset  $\Lambda$  of  $\mathcal{P}(X)$  is a  $\lambda$ -system on  $X$  iff

$$(8.76) \quad X \in \Lambda,$$

$$(8.77) \quad \forall A \in \Lambda, \quad A^c \in \Lambda,$$

$$(8.78) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Lambda, \quad (A_i)_{i \in I} \text{ pairwise disjoint} \implies \biguplus_{i \in I} A_i \in \Lambda.$$

Compact form:  $P_\Lambda \stackrel{\text{def.}}{=} wF \wedge C \wedge Udc$ .

**Lemma 678 (equivalent definition of  $\lambda$ -system).**

Let  $X$  be a set. Let  $\Lambda \subset \mathcal{P}(X)$ . Then,  $\Lambda$  is a  $\lambda$ -system on  $X$  iff (8.76) holds, and it is closed under local complement and countable monotone union:

$$(8.79) \quad \forall A, B \in \Lambda, \quad B \subset A \implies A \setminus B \in \Lambda,$$

$$(8.80) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Lambda, \quad (A_i)_{i \in I} \text{ nondecreasing} \implies \bigcup_{i \in I} A_i \in \Lambda.$$

Compact form:  $P_\Lambda \Leftrightarrow wF \wedge D' \wedge Umc$ .

*Proof.* “Left” implies “right”.

Direct consequence of Definition 677 ( $\lambda$ -system,  $P_\Lambda \Rightarrow [C \mid Udc]$ ), Lemma 566 (closedness under countable disjoint union and local complement,  $C \wedge Udc \Rightarrow D'$ ), and Lemma 568 (closedness under countable disjoint and monotone union,  $Udc \wedge D' \Rightarrow Umc$ ).

“Right” implies “left”.

Direct consequence of Lemma 541 (closedness under local complement and complement,  $wF \wedge D'$  implies  $C$ ), Lemma 542 (closedness under disjoint union and local complement,  $D' \wedge C \Rightarrow Ud$ ), Lemma 569 (closedness under countable monotone and disjoint union,  $Ud \wedge Umc \Rightarrow Udc$ ), and Definition 677 ( $\lambda$ -system,  $P_\Lambda = wF \wedge C \wedge Udc$ ).  $\square$

**Lemma 679 ( $\lambda$ -system is monotone class).**

Let  $X$  be a set. Then,  $\lambda$ -systems on  $X$  are monotone classes on  $X$ .

Compact form:  $P_\Lambda \Rightarrow P_C$ .

*Proof.* Direct consequence of Definition 677 ( $\lambda$ -system,  $P_\Lambda \Rightarrow C$ ), Lemma 678 (equivalent definition of  $\lambda$ -system,  $P_\Lambda \Rightarrow Umc$ ), Lemma 567 (closedness under countable union and intersection,  $C \wedge Umc \Rightarrow Imc$ ), and Definition 664 (monotone class,  $P_C = Umc \wedge Imc$ ).  $\square$

**Lemma 680 (other properties of  $\lambda$ -system).**

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then,  $\Lambda$  is nonempty, contains the empty set, and is closed under disjoint union, finite disjoint union, and countable monotone intersection.

Compact form:  $P_\Lambda \Rightarrow [Ne \mid wE \mid Ud \mid Udf \mid Imc]$ .

*Proof.* Direct consequence of Definition 677 ( $\lambda$ -system,  $P_\Lambda \Rightarrow [wF \mid C \mid Udc]$ ), Lemma 538 (non-empty and with empty or full,  $wF \Rightarrow Ne$ ), Lemma 539 (with empty and full,  $wF \wedge C \Rightarrow wE$ ), Lemma 558 (closedness under finite operations,  $(Udc \Rightarrow)Udf \Rightarrow Ud$ ), Lemma 679 ( $\lambda$ -system is monotone class,  $P_\Lambda \Rightarrow P_C$ ), and Definition 664 (monotone class,  $P_C \Rightarrow Imc$ ).  $\square$



**Lemma 681 (intersection of  $\lambda$ -systems).**

Let  $X$  and  $I$  be sets. Let  $(\Lambda_i)_{i \in I}$  be  $\lambda$ -systems on  $X$ . Then, if  $\bigcap_{i \in I} \Lambda_i$  is a  $\lambda$ -system on  $X$ .

*Proof.* Direct consequence of Definition 677 ( $\lambda$ -system), and **the definition of intersection**.  $\square$

**Definition 682 (generated  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The  $\lambda$ -system generated by  $G$  is denoted  $\Lambda_X(G) \stackrel{\text{def.}}{=} \mathfrak{G}_{P_\Lambda, X}(G)$ ; it is the intersection of all  $\lambda$ -systems on  $X$  containing  $G$ .

**Lemma 683 (generated  $\lambda$ -system is  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\Lambda_X(G)$  is a  $\lambda$ -system on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 682 (generated  $\lambda$ -system), Lemma 681 (intersection of  $\lambda$ -systems), and Lemma 585 (subset system generation preserves class).  $\square$

**Lemma 684 (properties of  $\lambda$ -system generation).**

Let  $X$  be a set. Then,  $\lambda$ -system generation on  $X$  is minimum, monotone, and idempotent:

$$(8.81) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_\Lambda, X}, G \subset \mathcal{S} \implies \Lambda_X(G) \subset \mathcal{S},$$

$$(8.82) \quad \forall G_1, G_2 \subset \mathcal{P}(X), G_1 \subset G_2 \implies \Lambda_X(G_1) \subset \Lambda_X(G_2),$$

$$(8.83) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \mathcal{S} \in \mathfrak{C}_{P_\Lambda, X} \iff \Lambda_X(\mathcal{S}) = \mathcal{S}.$$

*Proof.* Direct consequence of Definition 682 (generated  $\lambda$ -system), Lemma 681 (intersection of  $\lambda$ -systems), and Lemma 586 (properties of subset system generation).  $\square$

**Lemma 685 (other properties of  $\lambda$ -system generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.84) \quad G_1 \subset \Lambda_X(G_2) \quad \wedge \quad G_2 \subset \Lambda_X(G_1) \implies \Lambda_X(G_1) = \Lambda_X(G_2),$$

$$(8.85) \quad G_2 \subset \Lambda_X(G_1) \implies \Lambda_X(G_1 \cup G_2) = \Lambda_X(G_1).$$

*Proof.* Direct consequence of Definition 682 (generated  $\lambda$ -system), Lemma 681 (intersection of  $\lambda$ -systems), and Lemma 587 (other properties of subset system generation).  $\square$

**Lemma 686 ( $\lambda$ -system contains monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{C}_X(G) \subset \Lambda_X(G)$ .

*Proof.* Direct consequence of Definition 682 (generated  $\lambda$ -system), Definition 666 (generated monotone class), Lemma 679 ( $\lambda$ -system is monotone class), and Lemma 599 (generated subset systems are monotone (with nonemptiness), with  $P_\Lambda \Rightarrow P_C$ ).  $\square$

**Lemma 687 (upgrade monotone class to  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{C}_X(G)$  contains the full set, and is closed under local complement.

Then, we have  $\mathcal{C}_X(G) = \Lambda_X(G)$ .

Similar compact form:  $P_C \wedge wF \wedge D' \Leftrightarrow P_\Lambda$ .

*Proof.* Direct consequence of Lemma 686 ( $\lambda$ -system contains monotone class,  $\mathcal{C}_X(G) \subset \Lambda_X(G)$ ), Definition 682 (generated  $\lambda$ -system), Definition 666 (generated monotone class), Lemma 681 (intersection of  $\lambda$ -systems), Lemma 665 (intersection of monotone classes), Lemma 678 (equivalent definition of  $\lambda$ -system,  $P_\Lambda = wF \wedge D' \wedge Umc$ ), Definition 664 (monotone class,  $P_C = Umc \wedge Imc$ ), Lemma 590 (upgrade generated subset system, with  $P_C \wedge wF \wedge D' \Rightarrow P_\Lambda$ , thus  $\Lambda_X(G) \subset \mathcal{C}_X(G)$ ), Lemma 541 (closedness under local complement and complement,  $wF \wedge D' \Rightarrow C$ ), and Lemma 567 (closedness under countable union and intersection,  $C \wedge Umc \Rightarrow Imc$ , thus  $P_\Lambda \Rightarrow [P_C \mid wF \mid D']$ ).  $\square$

**Lemma 688** ( *$\lambda$ -system generated by monotone class*).

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Lambda_X(\mathcal{C}_X(G)) = \Lambda_X(G)$ .

*Proof.* Direct consequence of Definition 682 (*generated  $\lambda$ -system*), Definition 666 (*generated monotone class*), Lemma 681 (*intersection of  $\lambda$ -systems*), Lemma 665 (*intersection of monotone classes*), Lemma 679 ( *$\lambda$ -system is monotone class*), and Lemma 591 (*cascading generated subset systems, with  $P_\Lambda \Rightarrow P_{\mathcal{C}}$* ).  $\square$

**Definition 689** ( *$\lambda$ -system and intersection*).Let  $X$  be a set.

Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . For all  $A \subset X$ , let  $\Lambda_A^\cap$  be the subset system defined by

$$(8.86) \quad \Lambda_A^\cap \stackrel{\text{def.}}{=} \{B \in \Lambda \mid A \cap B \in \Lambda\}.$$

**Lemma 690** ( *$\Lambda^\cap$  is symmetric*).

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then, for all  $A, B \in \Lambda$ , we have  $B \in \Lambda_A^\cap$  iff  $A \in \Lambda_B^\cap$ .

*Proof.* Direct consequence of Definition 689 ( *$\lambda$ -system and intersection*), and **commutativity of intersection**.  $\square$

**Lemma 691** ( *$\Lambda^\cap$  is  $\lambda$ -system*).

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then, for all  $A \in \Lambda$ ,  $\Lambda_A^\cap$  is a  $\lambda$ -system on  $X$ .

Compact form:  $P_\Lambda \Rightarrow P_\Lambda(\Lambda_A^\cap)$ .

*Proof.* Let  $A \in \Lambda$ .

From Definition 677 ( *$\lambda$ -system,  $X \in \Lambda$* ), and **the identity  $A \cap X = A \in \Lambda$** , we have  $X \in \Lambda_A^\cap$ .

Let  $B, C \in \Lambda_A^\cap$ , i.e.  $B, C, A \cap B, A \cap C \in \Lambda$ . Assume that  $C \subset B$ .

Then, from **monotonicity of intersection, distributivity of intersection over set difference**, and Lemma 678 (*equivalent definition of  $\lambda$ -system, closedness under local complement*), we have  $(A \cap C) \subset (A \cap B)$ ,  $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) \in \Lambda$ , and  $B \setminus C \in \Lambda$ . Hence,  $\Lambda_A^\cap$  is closed under local complement.

Let  $I \subset \mathbb{N}$ . Let  $(B_i)_{i \in I} \in \Lambda_A^\cap$ , i.e. for all  $i \in I$ ,  $B_i, A \cap B_i \in \Lambda$ . Assume that  $(B_i)_{i \in I}$  is non-decreasing. Then, from **monotonicity of intersection, distributivity of intersection over union**, and Lemma 678 (*equivalent definition of  $\lambda$ -system, closedness under countable monotone union*),  $(A \cap B_i)_{i \in I}$  is nondecreasing, and we have  $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i) \in \Lambda$ , and  $\bigcup_{i \in I} B_i \in \Lambda$ . Hence,  $\Lambda_A^\cap$  is closed under countable monotone union.

Therefore, from Lemma 678 (*equivalent definition of  $\lambda$ -system*),  $\Lambda_A^\cap$  is a  $\lambda$ -system on  $X$ .  $\square$

**Lemma 692** ( *$\lambda$ -system with intersection*).Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G$  is closed under intersection. Then,  $\forall A \in \Lambda_X(G)$ ,  $(\Lambda_X(G))_A^\cap = \Lambda_X(G)$ .

*Proof.* For all  $A \subset X$ , we use the simplified notation  $\Lambda_A^\cap \stackrel{\text{def.}}{=} (\Lambda_X(G))_A^\cap$ .

(0). From Definition 689 ( *$\lambda$ -system and intersection*), we have  $\forall A \in \Lambda_X(G)$ ,  $\Lambda_A^\cap \subset \Lambda_X(G)$ .

(1).  $\forall A \in G$ ,  $\Lambda_X(G) \subset \Lambda_A^\cap$ . Let  $A \in G$ .

Let  $B \in G$ . Then, from Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system,  $G \subset \Lambda_X(G)$* ), and Definition 689 ( *$\lambda$ -system and intersection*), we have  $B \in \Lambda_A^\cap$ , i.e.  $G \subset \Lambda_A^\cap$ . Hence, from Lemma 691 ( *$\Lambda^\cap$  is  $\lambda$ -system, with  $A \in G \subset \Lambda_X(G)$* ), and Lemma 684 (*properties of  $\lambda$ -system generation, minimality*), we have  $\Lambda_X(G) \subset \Lambda_A^\cap$ .

(2).  $\forall B \in \Lambda_X(G)$ ,  $\Lambda_X(G) \subset \Lambda_B^\cap$ . Let  $B \in \Lambda_X(G)$ .

Let  $A \in G$ . Then, from (1), we have  $B \in \Lambda_X(G) \subset \Lambda_A^\cap$ . Thus, from Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system,  $G \subset \Lambda_X(G)$* ), and Lemma 690 ( *$\Lambda^\cap$  is symmetric, with  $A, B \in \Lambda_X(G)$* ), we have  $A \in \Lambda_B^\cap$ , i.e.  $G \subset \Lambda_B^\cap$ . Hence, from Lemma 691 ( *$\Lambda^\cap$  is  $\lambda$ -system*), and Lemma 684 (*properties of  $\lambda$ -system generation, minimality*), we have  $\Lambda_X(G) \subset \Lambda_B^\cap$ .

Therefore, from (0) and (2), we have the equality.  $\square$

**Lemma 693** ( *$\lambda$ -system is closed under intersection*). *Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is closed under intersection. Then,  $\Lambda_X(G)$  is closed under intersection.*

*Compact form:  $I \Rightarrow I(\Lambda_X)$ .*

*Proof.* Let  $A, B \in \Lambda_X(G)$ . Then, from Lemma 692 ( *$\lambda$ -system with intersection*,  $B$  belongs to  $\Lambda_X(G) = (\Lambda_X(G))_A^{\cap}$ ), and Definition 689 ( *$\lambda$ -system and intersection*), we have  $A \cap B \in \Lambda_X(G)$ .

Therefore,  $\Lambda_X(G)$  is closed under intersection.  $\square$

**Lemma 694** ( *$\lambda$ -system generated by  $\pi$ -system*). *Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a  $\pi$ -system on  $X$ . Then,  $\Lambda_X(G)$  is a  $\pi$ -system on  $X$ .*

*Compact form:  $P_{\Pi} \Rightarrow P_{\Pi}(\Lambda_X)$ .*

*Proof.* Direct consequence of Lemma 604 (*equivalent definition of  $\pi$ -system*,  $P_{\Pi} = Ne \wedge I$  (for  $G$ , then for  $\Lambda_X(G)$ )), Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*,  $G \subset \Lambda_X(G)$ , thus  $Ne(\Lambda_X(G))$ ), and Lemma 693 ( *$\lambda$ -system is closed under intersection*, thus  $I(\Lambda_X(G))$ ).  $\square$

## 8.8 Sigma-algebra

*Remark 695.* Note that the next concept of “ $\sigma$ -algebra” is sometimes called “ $\sigma$ -field”.

**Lemma 696 ( $\sigma$ -algebra).** *Let  $X$  be a set. Let  $P_\Sigma$  be the predicate on  $\mathcal{P}(\mathcal{P}(X))$  of being an algebra of sets, and of closedness under countable union.*

*The class of  $\sigma$ -algebras on  $X$  is  $\mathfrak{C}_{P_\Sigma}$ : a subset  $\Sigma$  of  $\mathcal{P}(X)$  is a  $\sigma$ -algebra on  $X$  iff*

$$(8.87) \quad \emptyset \in \Sigma,$$

$$(8.88) \quad \forall A \in \Sigma, \quad A^c \in \Sigma,$$

$$(8.89) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \quad \bigcup_{i \in I} A_i \in \Sigma.$$

*Compact form:  $P_\Sigma \stackrel{\text{def.}}{=} P_A \wedge U_c$ ;  $P_\Sigma \Leftrightarrow wE \wedge C \wedge U_c$ .*

*Proof.* Direct consequence of Definition 634 (*algebra of sets*). □

**Lemma 697 (equivalent definition of  $\sigma$ -algebra).**

*Let  $X$  be a set. Let  $\Sigma \subset \mathcal{P}(X)$ . Then,  $\Sigma$  is a  $\sigma$ -algebra on  $X$  iff (8.88) holds and*

$$(8.90) \quad \Sigma \neq \emptyset, \quad \text{or} \quad \emptyset \in \Sigma, \quad \text{or} \quad X \in \Sigma,$$

$$(8.91) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \quad \bigcup_{i \in I} A_i \in \Sigma, \quad \text{or} \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \quad \bigcap_{i \in I} A_i \in \Sigma.$$

*Compact form:  $P_\Sigma \Leftrightarrow [Ne \mid wE \mid wF] \wedge C \wedge [U_c \mid Ic]$ .*

*Proof.* Direct consequence of Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma = wE \wedge C \wedge U_c$ ), Lemma 538 (*nonempty and with empty or full,  $wE \Rightarrow Ne$  and  $C \wedge Ic \wedge Ne \Rightarrow C \wedge I \wedge Ne \Rightarrow wE$* ), Lemma 539 (*with empty and full,  $C \Rightarrow wE \Leftrightarrow wF$* ), and Lemma 567 (*closedness under countable union and intersection,  $C \Rightarrow U_c \Leftrightarrow Ic$* ). □

*Remark 698.*

Note that the previous lemma means that the predicate  $P_\Sigma$  characterizing  $\sigma$ -algebras is equivalent to any of the six alternatives summarized by the conjunction (8.90), (8.88), and (8.91).

**Lemma 699 ( $\sigma$ -algebra is algebra of sets).**

*Let  $X$  be a set. Then,  $\sigma$ -algebras of sets on  $X$  are algebras of sets on  $X$ .*

*Compact form:  $P_\Sigma \Rightarrow P_A$ .*

*Proof.* Direct consequence of Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma \Rightarrow P_A$ ). □

**Lemma 700 (other properties of  $\sigma$ -algebra).**

*Let  $X$  be a set. Let  $\Sigma$  be a  $\sigma$ -algebra on  $X$ . Then,  $\Sigma$  is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, and countable monotone union and intersection.*

*Compact form:  $P_\Sigma \Rightarrow [D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid Udc \mid Umc \mid Imc]$ .*

*Proof.* Direct consequence of Lemma 699 ( $\sigma$ -algebra is algebra of sets,  $P_\Sigma \Rightarrow P_A$ ), Lemma 639 (*other properties of algebra of sets,  $P_A \Rightarrow [D' \mid sD \mid Ud \mid Udf \mid If]$* ). Lemma 637 (*other equivalent definition of algebra of sets,  $P_A \Rightarrow D$* ), Lemma 635 (*equivalent definition of algebra of sets,  $P_A \Rightarrow [U \mid I]$* ), Definition 634 (*algebra of sets,  $P_A \Rightarrow Uf$* ), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow [Uc \mid Ic] \Rightarrow [Udc \mid Umc \mid Imc]$* ). □

**Lemma 701 ( $\sigma$ -algebra is  $\lambda$ -system).**

Let  $X$  be a set. Then,  $\sigma$ -algebras on  $X$  are  $\lambda$ -systems on  $X$ :  $P_\Sigma \Rightarrow P_\Lambda$ .

*Proof.*

Direct consequence of Lemma 697 (*equivalent definition of  $\sigma$ -algebra*,  $P_\Sigma \Rightarrow [wF \mid C]$ ), Lemma 700 (*other properties of  $\sigma$ -algebra*,  $P_\Sigma \Rightarrow Udc$ ), and Definition 677 ( *$\lambda$ -system*,  $P_\Lambda = wF \wedge C \wedge Udc$ ).  $\square$

**Lemma 702 ( $\sigma$ -algebra is monotone class).**

Let  $X$  be a set. Then,  $\sigma$ -algebras on  $X$  are monotone classes on  $X$ :  $P_\Sigma \Rightarrow P_C$ .

*Proof.* Direct consequence of Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*,  $P_\Sigma \Rightarrow P_\Lambda$ ), and Lemma 679 ( *$\lambda$ -system is monotone class*,  $P_\Lambda \Rightarrow P_C$ ).  $\square$

**Lemma 703 ( $\sigma$ -algebra is ring of sets).**

Let  $X$  be a set. Then,  $\sigma$ -algebras on  $X$  are rings of sets on  $X$ :  $P_\Sigma \Rightarrow P_R$ .

*Proof.* Direct consequence of Lemma 699 ( *$\sigma$ -algebra is algebra of sets*,  $P_\Sigma \Rightarrow P_A$ ), and Lemma 638 (*algebra of sets is ring of sets*,  $P_A \Rightarrow P_R$ ).  $\square$

**Lemma 704 ( $\sigma$ -algebra is  $\pi$ -system).**

Let  $X$  be a set. Then,  $\sigma$ -algebras on  $X$  are  $\pi$ -systems on  $X$ :  $P_\Sigma \Rightarrow P_\Pi$ .

*Proof.* Direct consequence of Lemma 699 ( *$\sigma$ -algebra is algebra of sets*,  $P_\Sigma \Rightarrow P_A$ ), and Lemma 640 (*algebra of sets is  $\pi$ -system*,  $P_A \Rightarrow P_\Pi$ ).  $\square$

**Lemma 705 ( $\sigma$ -algebra is semi-algebra of sets).**

Let  $X$  be a set. Then,  $\sigma$ -algebras of sets on  $X$  are semi-algebras of sets on  $X$ .

Compact form:  $P_\Sigma \Rightarrow P_{A/2}$ .

*Proof.* Direct consequence of Lemma 699 ( *$\sigma$ -algebra is algebra of sets*,  $P_\Sigma \Rightarrow P_A$ ), and Lemma 660 (*algebra of sets is semi-algebra of sets*,  $P_A \Rightarrow P_{A/2}$ ).  $\square$

**Lemma 706 ( $\sigma$ -algebra is semiring of sets).**

Let  $X$  be a set. Then,  $\sigma$ -algebras of sets on  $X$  are semirings of sets on  $X$ .

Compact form:  $P_\Sigma \Rightarrow P_{R/2}$ .

*Proof.* Direct consequence of Lemma 699 ( *$\sigma$ -algebra is algebra of sets*,  $P_\Sigma \Rightarrow P_A$ ), and Lemma 661 (*algebra of sets is semiring of sets*,  $P_A \Rightarrow P_{R/2}$ ).  $\square$

**Lemma 707 (intersection of  $\sigma$ -algebras).**

Let  $X$  and  $I$  be sets. Let  $(\Sigma_i)_{i \in I}$  be  $\sigma$ -algebras on  $X$ . Then,  $\bigcap_{i \in I} \Sigma_i$  is a  $\sigma$ -algebra on  $X$ .

*Proof.* Direct consequence of Lemma 696 ( *$\sigma$ -algebra*), and **the definition of intersection**.  $\square$

**Definition 708 (generated  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The  $\sigma$ -algebra generated by  $G$  is denoted  $\Sigma_X(G) \stackrel{\text{def}}{=} \mathfrak{G}_{P_\Sigma, X}(G)$ ; it is the intersection of all  $\sigma$ -algebras on  $X$  containing  $G$ .

**Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\Sigma_X(G)$  is a  $\sigma$ -algebra on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Lemma 707 (*intersection of  $\sigma$ -algebras*), and Lemma 585 (*subset system generation preserves class*).  $\square$

**Lemma 710 (properties of  $\sigma$ -algebra generation).**

Let  $X$  be a set. Then,  $\sigma$ -algebra generation on  $X$  is minimum, monotone, and idempotent:

$$(8.92) \quad \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\Sigma}, X}, \quad G \subset \mathcal{S} \implies \Sigma_X(G) \subset \mathcal{S},$$

$$(8.93) \quad \forall G_1, G_2 \subset \mathcal{P}(X), \quad G_1 \subset G_2 \implies \Sigma_X(G_1) \subset \Sigma_X(G_2),$$

$$(8.94) \quad \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P_{\Sigma}, X} \iff \Sigma_X(\mathcal{S}) = \mathcal{S}.$$

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Lemma 707 (*intersection of  $\sigma$ -algebras*), and Lemma 586 (*properties of subset system generation*).  $\square$

**Lemma 711 (other properties of  $\sigma$ -algebra generation).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Then, we have

$$(8.95) \quad G_1 \subset \Sigma_X(G_2) \quad \wedge \quad G_2 \subset \Sigma_X(G_1) \implies \Sigma_X(G_1) = \Sigma_X(G_2),$$

$$(8.96) \quad G_2 \subset \Sigma_X(G_1) \implies \Sigma_X(G_1 \cup G_2) = \Sigma_X(G_1).$$

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Lemma 707 (*intersection of  $\sigma$ -algebras*), and Lemma 587 (*other properties of subset system generation*).  $\square$

**Lemma 712 (countable union  $\sigma$ -algebra generator).**

Let  $X$  be a set.

Let  $G_1 \subset G_2 \subset \mathcal{P}(X)$ . Assume that all elements of  $G_2$  are countable unions of elements of  $G_1$ . Then, we have  $\Sigma_X(G_1) = \Sigma_X(G_2)$ .

*Proof.* Direct consequence of Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra,  $G_1 \subset G_2 \subset \Sigma_X(G_2)$ ), Lemma 696 ( *$\sigma$ -algebra, closedness under countable union, thus  $G_2 \subset \Sigma_X(G_1)$* ), and Lemma 711 (*other properties of  $\sigma$ -algebra generation, with  $G_1 \cup G_2 = G_2$* ).  $\square$*

**Lemma 713 ( $\sigma$ -algebra contains  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Lambda_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Definition 682 (*generated  $\lambda$ -system*), Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*), and Lemma 588 (*generated subset systems are monotone, with  $P_{\Sigma} \Rightarrow P_{\Lambda}$* ).  $\square$

**Lemma 714 (upgrade  $\lambda$ -system to  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\Lambda_X(G)$  is closed under intersection. Then, we have  $\Lambda_X(G) = \Sigma_X(G)$ .

Similar compact form:  $P_{\Lambda} \wedge I \Leftrightarrow P_{\Sigma}$ .

*Proof.* Direct consequence of Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system,  $\Lambda_X(G) \subset \Sigma_X(G)$ ), Definition 708 (*generated  $\sigma$ -algebra*), Definition 682 (*generated  $\lambda$ -system*), Lemma 707 (*intersection of  $\sigma$ -algebras*), Lemma 681 (*intersection of  $\lambda$ -systems*), Lemma 697 (*equivalent definition of  $\sigma$ -algebra,  $P_{\Sigma} = wF \wedge C \wedge Uc$ ), Definition 677 ( *$\lambda$ -system,  $P_{\Lambda} = wF \wedge C \wedge Udc$ ), Lemma 545 (*closedness under intersection and set difference,  $C \wedge I \Rightarrow D$* ), Lemma 570 (*closedness under countable disjoint union and countable union,  $D \wedge Udc \Rightarrow Uc$* ), Lemma 590 (*upgrade generated subset system, with  $P_{\Lambda} \wedge I \Rightarrow P_{\Sigma}$ , thus  $\Sigma_X(G) \subset \Lambda_X(G)$* ), Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system,  $P_{\Sigma} \Rightarrow P_{\Lambda}$* ), and Lemma 700 (*other properties of  $\sigma$ -algebra,  $P_{\Sigma} \Rightarrow I$ , thus  $P_{\Sigma} \Rightarrow [P_{\Lambda} | I]$* ).  $\square$***

**Lemma 715 ( $\sigma$ -algebra generated by  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\Lambda_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Definition 682 (*generated  $\lambda$ -system*), Lemma 707 (*intersection of  $\sigma$ -algebras*), Lemma 681 (*intersection of  $\lambda$ -systems*), Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*), and Lemma 591 (*cascading generated subset systems, with  $P_{\Sigma}$  implies  $P_{\Lambda}$* ).  $\square$

**Lemma 716 ( $\sigma$ -algebra contains monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{C}_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 713 ( $\sigma$ -algebra contains  $\lambda$ -system,  $\Lambda_X(G) \subset \Sigma_X(G)$ ), Lemma 686 ( $\lambda$ -system contains monotone class,  $\mathcal{C}_X(G) \subset \Lambda_X(G)$ ), and **transitivity of inclusion**.  $\square$

**Lemma 717 (upgrade monotone class to  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{C}_X(G)$  contains the empty set, and is closed under complement and union.

Then, we have  $\mathcal{C}_X(G) = \Sigma_X(G)$ .

Similar compact form:  $P_C \wedge wE \wedge C \wedge U \Leftrightarrow P_\Sigma$ .

*Proof.* Direct consequence of Lemma 539 (with empty and full,  $wE \wedge C \Rightarrow wF$  and  $wF \wedge C \Rightarrow wE$ ), Lemma 546 (closedness under union and disjoint union,  $U \Rightarrow Ud$ ), Lemma 542 (closedness under disjoint union and local complement,  $C \wedge Ud \Rightarrow D'$ ), Lemma 687 (upgrade monotone class to  $\lambda$ -system,  $P_C \wedge wE \wedge C \wedge U \Rightarrow P_C \wedge wF \wedge D' (\Leftrightarrow P_\Lambda)$ , thus  $\mathcal{C}_X(G) = \Lambda_X(G)$ ), Lemma 547 (closedness under union and intersection,  $C \wedge U \Rightarrow I$  and  $C \wedge I \Rightarrow U$ ), Definition 677 ( $\lambda$ -system,  $P_\Lambda$  equals  $wF \wedge C \wedge Ud$ ), and Lemma 714 (upgrade  $\lambda$ -system to  $\sigma$ -algebra,  $P_C \wedge wE \wedge C \wedge U \Leftrightarrow P_\Lambda \wedge I \Leftrightarrow P_\Sigma$ , thus  $\Lambda_X(G) = \Sigma_X(G)$ ).  $\square$

**Lemma 718 ( $\sigma$ -algebra generated by monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\mathcal{C}_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (generated  $\sigma$ -algebra), Definition 682 (generated  $\lambda$ -system), Definition 666 (generated monotone class), Lemma 715 ( $\sigma$ -algebra generated by  $\lambda$ -system), Lemma 688 ( $\lambda$ -system generated by monotone class), and Lemma 592 (double cascading generated subset systems).  $\square$

**Lemma 719 ( $\sigma$ -algebra contains algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{A}_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (generated  $\sigma$ -algebra), Definition 642 (generated algebra of sets), Lemma 699 ( $\sigma$ -algebra is algebra of sets), and Lemma 588 (generated subset systems are monotone, with  $P_\Sigma \Rightarrow P_A$ ).  $\square$

**Lemma 720 (upgrade algebra of sets to  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{A}_X(G)$  is closed under countable disjoint or monotone union.

Then, we have  $\mathcal{A}_X(G) = \Sigma_X(G)$ .

Similar compact form:  $P_A \wedge [Udc \mid Umc] \Leftrightarrow P_\Sigma$ .

*Proof.* Direct consequence of Lemma 719 ( $\sigma$ -algebra contains algebra of sets,  $\mathcal{A}_X(G) \subset \Sigma_X(G)$ ), Definition 708 (generated  $\sigma$ -algebra), Definition 642 (generated algebra of sets), Lemma 707 (intersection of  $\sigma$ -algebras), Lemma 641 (intersection of algebras of sets), Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma = P_A \wedge Uc$ ), Lemma 637 (other equivalent definition of algebra of sets,  $P_A \Rightarrow D$ ), Lemma 570 (closedness under countable disjoint union and countable union,  $D \wedge Udc \Rightarrow Uc$ ), Lemma 639 (other properties of algebra of sets,  $P_A \Rightarrow Ud$ ), Lemma 571 (closedness under countable monotone union and countable union,  $D \wedge Ud \wedge Umc \Rightarrow Uc$ ), Lemma 590 (upgrade generated subset system, with  $P_A \wedge [Udc \mid Umc] \Rightarrow P_\Sigma$ , thus  $\Sigma_X(G) \subset \mathcal{A}_X(G)$ ), Lemma 699 ( $\sigma$ -algebra is algebra of sets,  $P_\Sigma \Rightarrow P_A$ ), and Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow [Udc \mid Umc]$ , thus  $P_\Sigma \Rightarrow [P_A \mid Udc \mid Umc]$ ).  $\square$

**Lemma 721 ( $\sigma$ -algebra generated by algebra of sets).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\mathcal{A}_X(G)) = \Sigma_X(G)$ .



*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Definition 642 (*generated algebra of sets*), Lemma 707 (*intersection of  $\sigma$ -algebras*), Lemma 641 (*intersection of algebras of sets*), Lemma 699 ( *$\sigma$ -algebra is algebra of sets*), and Lemma 591 (*cascading generated subset systems*, with  $P_\Sigma \Rightarrow P_A$ ).  $\square$

**Lemma 722 ( *$\sigma$ -algebra contains ring of sets*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{R}_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*), Lemma 646 (*algebra of sets contains ring of sets*), and **transitivity of inclusion**.  $\square$

**Lemma 723 (*upgrade ring of sets to  $\sigma$ -algebra*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{R}_X(G)$  either contains the full set or is closed under complement, and is closed under countable disjoint or monotone union. Then, we have  $\mathcal{R}_X(G) = \Sigma_X(G)$ .

Similar compact form:  $P_{\mathcal{R}} \wedge [wF \mid C] \wedge [Udc \mid Umc] \Leftrightarrow P_\Sigma$ .

*Proof.* Direct consequence of Lemma 647 (*upgrade ring of sets to algebra of sets*,  $\mathcal{R}_X(G) = \mathcal{A}_X(G)$  and  $P_{\mathcal{R}} \wedge [wF \mid C] \Leftrightarrow P_A$ ), and Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*,  $\mathcal{A}_X(G) = \Sigma_X(G)$  and  $P_A \wedge [Udc \mid Umc] \Leftrightarrow P_\Sigma$ ).  $\square$

**Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\mathcal{R}_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Definition 642 (*generated algebra of sets*), Definition 617 (*generated ring of sets*), Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*), Lemma 648 (*algebra of sets generated by ring of sets*), and Lemma 592 (*double cascading generated subset systems*).  $\square$

**Lemma 725 ( *$\sigma$ -algebra contains  $\pi$ -system*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Pi_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 722 ( *$\sigma$ -algebra contains ring of sets*), Lemma 621 (*ring of sets contains  $\pi$ -system*), and **transitivity of inclusion**.  $\square$

**Lemma 726 (*upgrade  $\pi$ -system to  $\sigma$ -algebra*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G \neq \emptyset$ , and that  $\Pi_X(G)$  is closed under complement and countable disjoint or monotone union. Then, we have  $\Pi_X(G) = \Sigma_X(G)$ .

Similar compact form:  $P_\Pi \wedge C \wedge [Udc \mid Umc] \Leftrightarrow P_\Sigma$ .

*Proof.* Direct consequence of Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*,  $\Pi_X(G) = \mathcal{A}_X(G)$  and  $P_\Pi \wedge C \Leftrightarrow P_A$ ), and Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*,  $\mathcal{A}_X(G) = \Sigma_X(G)$  and  $P_A \wedge [Udc \mid Umc] \Leftrightarrow P_\Sigma$ ).  $\square$

**Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Sigma_X(\Pi_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 708 (*generated  $\sigma$ -algebra*), Definition 617 (*generated ring of sets*), Definition 606 (*generated  $\pi$ -system*), Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*), Lemma 623 (*ring of sets generated by  $\pi$ -system*), and Lemma 592 (*double cascading generated subset systems*).  $\square$



**Lemma 728 (disjoint union of union in  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I)]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $\Sigma$  be a  $\sigma$ -algebra on  $X$ . Let  $(A_i)_{i \in I} \subset \Sigma$ .

Then, there exists  $(B_i)_{i \in I} \in \Sigma$  pairwise disjoint such that for all  $i \in I$ ,  $B_i \subset A_i$ , and we have

$$(8.97) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} A_{\varphi(j)} = \biguplus_{j \in [0..n]} B_{\varphi(j)} \in \Sigma,$$

$$(8.98) \quad \bigcup_{i \in I} A_i = \biguplus_{i \in I} B_i \in \Sigma.$$

*Proof.* Direct consequence of Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow [D \mid Ud \mid Udc]$ ), and Lemma 572 (disjoint union of union in subset system,  $Hyp = D \wedge Ud \wedge Udc$ ).  $\square$

*Remark 729.* Note that in the next proof, the point (1) does not depend on the hypotheses. It could be an independent lemma that characterizes  $\Sigma_1 \bar{\times} \Sigma_2$ .

Note also that  $\Sigma_1 \bar{\times} \Sigma_2$  may not be a  $\sigma$ -algebra. See Section 9.4 for a definition of a  $\sigma$ -algebra on the product of measurable spaces.

**Lemma 730 (algebra of sets generated by product of  $\sigma$ -algebras).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ , and  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$ .

Then,  $\mathcal{A}_X(\bar{\Sigma})$  is the set of finite (disjoint) unions of elements of  $\bar{\Sigma}$ .

*Proof.* (1).  $X \in \bar{\Sigma}$ . Direct consequence of Definition 227 (product of subset systems), and Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow wF$ , for  $\Sigma_1$  and  $\Sigma_2$ ).

**(2).  $\bar{\Sigma}$  is closed under intersection.**

Direct consequence of Definition 227 (product of subset systems), **compatibility of intersection with Cartesian product** ( $(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2)$ ), and Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow I$ , for  $\Sigma_1$  and  $\Sigma_2$ ).

**(3). The complement of elements of  $\bar{\Sigma}$  is the disjoint union of elements of  $\bar{\Sigma}$ .**

Direct consequence of Definition 227 (product of subset systems), **the identity  $(A_1 \times A_2)^c$  equals  $(X_1 \times A_2^c) \uplus (A_1^c \times A_2)$** , and Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow wF$ , for  $\Sigma_1$ , and  $P_\Sigma \Rightarrow C$ , for  $\Sigma_1$  and  $\Sigma_2$ ),

Therefore, from (1), (2), (3), Lemma 655 (equivalent definition of semi-algebra of sets,  $P_{\mathcal{A}/2}$  equals  $wF \wedge I \wedge PtfC$  (with  $n \stackrel{\text{def.}}{=} 1$ )), and Lemma 662 (explicit algebra of sets, with  $G \stackrel{\text{def.}}{=} \bar{\Sigma}$ ),  $\bar{\Sigma}$  is a semi-algebra on  $X$ , and  $\mathcal{A}_X(\bar{\Sigma})$  is the set of finite (disjoint) unions of elements of  $\bar{\Sigma}$ .  $\square$

## 8.9 Dynkin $\pi$ - $\lambda$ theorem

*Remark 731.* See the sketch of next proof in Section 5.10.

**Theorem 732 (Dynkin  $\pi$ - $\lambda$  theorem).**

Let  $X$  be a set. Let  $\Pi$  be a  $\pi$ -system on  $X$ . Then, we have  $\Lambda_X(\Pi) = \Sigma_X(\Pi)$ .

Similar compact form:  $P_\Pi \wedge P_\Lambda \Leftrightarrow P_\Sigma$ .

*Proof.* Direct consequence of Lemma 694 ( $\lambda$ -system generated by  $\pi$ -system,  $P_\Pi \Rightarrow P_\Pi(\Lambda_X)$ ), Lemma 604 (equivalent definition of  $\pi$ -system,  $P_\Pi \Rightarrow I$ , for  $\Lambda_X(\Pi)$ ), Lemma 714 (upgrade  $\lambda$ -system to  $\sigma$ -algebra,  $\Lambda_X(\Pi) = \Sigma_X(\Pi)$  and  $P_\Pi \wedge P_\Lambda \Rightarrow P_\Sigma$ , for  $\Lambda_X(\Pi)$ ), Lemma 704 ( $\sigma$ -algebra is  $\pi$ -system,  $P_\Sigma \Rightarrow P_\Pi$ , for  $\Lambda_X(\Pi)$ ), and Lemma 701 ( $\sigma$ -algebra is  $\lambda$ -system,  $P_\Sigma \Rightarrow P_\Lambda$ , for  $\Lambda_X(\Pi)$ ).  $\square$

*Remark 733.* Note that the Dynkin  $\pi$ - $\lambda$  theorem may take the following form: if a  $\lambda$ -system contains a  $\pi$ -system, then it also contains the  $\sigma$ -algebra generated by the  $\pi$ -system.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemma 992 in Section 12.4, itself later used to prove uniqueness of the Lebesgue measure.

**Lemma 734 (usage of Dynkin  $\pi$ - $\lambda$  theorem).**

Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate over  $\mathcal{P}(X)$ , and let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ . Let  $G \subset \mathcal{P}(X)$  be a nonempty generator of  $\Sigma$ . Assume that  $\Pi_X(G) \subset \mathcal{S}$ , and that  $\mathcal{S}$  is a  $\lambda$ -system. Then, we have  $\mathcal{S} = \Sigma$ , i.e.  $P$  holds for all subsets in  $\Sigma$ .

*Proof.* From the definition of  $\mathcal{S}$ , we have  $\mathcal{S} \subset \Sigma$ .

Let  $\Pi \stackrel{\text{def.}}{=} \Pi_X(G)$ . Then, from Definition 708 (generated  $\sigma$ -algebra,  $\Sigma = \Sigma_X(G)$ ), Lemma 607 (generated  $\pi$ -system is  $\pi$ -system,  $\Pi$  is a  $\pi$ -system containing  $G$ ), Lemma 710 (properties of  $\sigma$ -algebra generation, monotonicity:  $\Sigma_X(G) \subset \Sigma_X(\Pi)$ ), Theorem 732 (Dynkin  $\pi$ - $\lambda$  theorem,  $\Sigma_X(\Pi)$  equals  $\Lambda_X(\Pi)$ ), and Lemma 684 (properties of  $\lambda$ -system generation, monotonicity:  $\Lambda_X(\Pi)$  is included in  $\Lambda_X(\mathcal{S})$ , and idempotence:  $\Lambda_X(\mathcal{S}) = \mathcal{S}$ ), we have  $\Sigma \subset \mathcal{S}$ .

Therefore, we have the equality.  $\square$

## 8.10 Monotone class theorem

*Remark 735.* See the sketch of next proof in Section 5.10.

**Theorem 736 (monotone class).**

Let  $X$  be a set. Let  $\mathcal{A}$  be an algebra of sets on  $X$ . Then,  $\mathcal{C}_X(\mathcal{A}) = \Sigma_X(\mathcal{A})$ .

Similar compact form:  $P_{\mathcal{A}} \wedge P_{\mathcal{C}} \Leftrightarrow P_{\Sigma}$ .

*Proof.* Direct consequence of Lemma 675 (*monotone class generated by algebra of sets*,  $P_{\mathcal{A}}$  implies  $P_{\mathcal{A}}(\mathcal{C}_X)$ ), Lemma 635 (*equivalent definition of algebra of sets*,  $P_{\mathcal{A}} \Rightarrow [wE \mid C \mid U]$ , for  $\mathcal{C}_X(\mathcal{A})$ ), Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*,  $\mathcal{C}_X(\mathcal{A}) = \Sigma_X(\mathcal{A})$  and  $P_{\mathcal{A}} \wedge P_{\mathcal{C}} \Rightarrow P_{\Sigma}$ , for  $\mathcal{C}_X(\mathcal{A})$ ), Lemma 699 ( *$\sigma$ -algebra is algebra of sets*,  $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ , for  $\mathcal{C}_X(\mathcal{A})$ ), and Lemma 702 ( *$\sigma$ -algebra is monotone class*,  $P_{\Sigma} \Rightarrow P_{\mathcal{C}}$ , for  $\mathcal{C}_X(\mathcal{A})$ ).  $\square$

*Remark 737.* Note that the monotone class theorem may take the following form: if a monotone class contains an algebra of sets, then it also contains the  $\sigma$ -algebra generated by the algebra.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemmas 1221 and 1231 in Section 15.4 in the context of product spaces.

**Lemma 738 (usage of monotone class theorem).**

Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate over  $\mathcal{P}(X)$ , and let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ . Let  $G \subset \mathcal{P}(X)$  be a generator of  $\Sigma$ . Assume that  $\mathcal{A}_X(G) \subset \mathcal{S}$ , and that  $\mathcal{S}$  is a monotone class. Then, we have  $\mathcal{S} = \Sigma$ , i.e.  $P$  holds for all subsets in  $\Sigma$ .

*Proof.* From the definition of  $\mathcal{S}$ , we have  $\mathcal{S} \subset \Sigma$ .

Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(G)$ . Then, from Definition 708 (*generated  $\sigma$ -algebra*,  $\Sigma = \Sigma_X(G)$ ), Lemma 643 (*generated algebra of sets is algebra of sets*,  $\mathcal{A}$  is algebra of sets containing  $G$ ), Lemma 710 (*properties of  $\sigma$ -algebra generation*, monotonicity:  $\Sigma_X(G) \subset \Sigma_X(\mathcal{A})$ ), Theorem 736 (*monotone class*,  $\Sigma_X(\mathcal{A}) = \mathcal{C}_X(\mathcal{A})$ ), and Lemma 668 (*properties of monotone class generation*, monotonicity:  $\mathcal{C}_X(\mathcal{A}) \subset \mathcal{C}_X(\mathcal{S})$ , and idempotence:  $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$ ), we have  $\Sigma \subset \mathcal{S}$ .

Therefore, we have the equality.  $\square$

## Chapter 9

# Measurability

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## 9.1 Measurable space and Borel subset

**Definition 739 (premeasurable space).** Let  $X$  be a set. Let  $\mathcal{R}$  be a ring of sets on  $X$ . Then,  $(X, \mathcal{R})$  is called *premeasurable space*, and elements of  $\mathcal{R}$  are said *( $\mathcal{R}$ -)premeasurable*.

**Lemma 740 (properties of premeasurable space).**

Let  $(X, \mathcal{R})$  be a premeasurable space. Then,  $\mathcal{R}$  is a (semi)ring of sets, and a  $\pi$ -system.

Moreover,  $\mathcal{R}$  is nonempty, contains the empty set, is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, and finite union and intersection, and local complement and set difference of elements of  $\mathcal{R}$  can be expressed as a finite disjoint union of elements of  $\mathcal{R}$ .

Compact form:  $\text{Concl} \stackrel{\text{def.}}{=} [P_{\mathcal{R}} \mid P_{\mathcal{R}/2} \mid P_{\Pi} \mid$   
 $Ne \mid wE \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid PtfD' \mid PtfD].$

*Proof.* Direct consequence of Definition 739 (premeasurable space, thus  $P_{\mathcal{R}}$ ), Lemma 631 (ring of sets is semiring of sets,  $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R}/2}$ ), Lemma 614 (ring of sets is  $\pi$ -system,  $P_{\mathcal{R}} \Rightarrow P_{\Pi}$ ), Lemma 612 (equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow [Ne \mid wE \mid D' \mid D \mid sD \mid Ud \mid U \mid I]$ ), Lemma 615 (other properties of ring of sets,  $P_{\mathcal{R}} \Rightarrow [Udf \mid If]$ ), Definition 611 (ring of sets,  $P_{\mathcal{R}} \Rightarrow Uf$ ), and Lemma 554 (disjoint decomposition,  $D' \Rightarrow PtfD'$  and  $D \Rightarrow PtfD$ ).  $\square$

**Definition 741 (measurable space).** Let  $X$  be a set. Let  $\Sigma$  be a  $\sigma$ -algebra on  $X$ . Then,  $(X, \Sigma)$  is called *measurable space*, and elements of  $\Sigma$  are said *( $\Sigma$ -)measurable*.

**Lemma 742 (properties of measurable space).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $(X, \Sigma)$  is a premeasurable space, and  $\Sigma$  is a  $\lambda$ -system, a monotone class, a (semi-)algebra of sets, a (semi)ring of sets, and a  $\pi$ -system.

Moreover,  $\Sigma$  is nonempty, contains the empty set and the full set, is closed under complement, local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, countable monotone union and intersection, and countable union and intersection, and complement, local complement and set difference of elements of  $\Sigma$  can be expressed as a finite disjoint union of elements of  $\Sigma$ .

Compact form:  $\text{Concl} \stackrel{\text{def.}}{=} [P_{\Sigma} \mid P_{\Lambda} \mid P_C \mid P_A \mid P_{A/2} \mid P_{\mathcal{R}} \mid P_{\mathcal{R}/2} \mid P_{\Pi} \mid$   
 $Ne \mid wE \mid wF \mid C \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid$   
 $Udc \mid Umc \mid Imc \mid Uc \mid Ic \mid PtfC \mid PtfD' \mid PtfD].$

*Proof.* Direct consequence of Definition 741 (measurable space, thus  $P_{\Sigma}$ ), Lemma 701 ( $\sigma$ -algebra is  $\lambda$ -system,  $P_{\Sigma} \Rightarrow P_{\Lambda}$ ), Lemma 702 ( $\sigma$ -algebra is monotone class,  $P_{\Sigma} \Rightarrow P_C$ ), Lemma 699 ( $\sigma$ -algebra is algebra of sets,  $P_{\Sigma} \Rightarrow P_A$ ), Lemma 660 (algebra of sets is semi-algebra of sets,  $P_A \Rightarrow P_{A/2}$ ), Lemma 703 ( $\sigma$ -algebra is ring of sets,  $P_{\Sigma} \Rightarrow P_{\mathcal{R}}$ ), Lemma 631 (ring of sets is semiring of sets,  $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R}/2}$ ), Lemma 704 ( $\sigma$ -algebra is  $\pi$ -system,  $P_{\Sigma} \Rightarrow P_{\Pi}$ ), Definition 739 (premeasurable space), Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $P_{\Sigma} \Rightarrow [Ne \mid wE \mid wF \mid C \mid Uc \mid Ic]$ ), Lemma 700 (other properties of  $\sigma$ -algebra,  $P_{\Sigma} \Rightarrow [D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid Udc \mid Umc \mid Imc]$ ) and Lemma 554 (disjoint decomposition,  $C \Rightarrow PtfC$ ,  $D' \Rightarrow PtfD'$  and  $D \Rightarrow PtfD$ ).  $\square$

**Definition 743 (Borel  $\sigma$ -algebra).**

Let  $(X, \mathcal{T})$  be a topological space. The *Borel  $\sigma$ -algebra of  $X$*  is the  $\sigma$ -algebra generated by the open subsets; it is denoted  $\mathcal{B}(X) \stackrel{\text{def.}}{=} \Sigma_X(\mathcal{T})$ .

$\mathcal{B}(X)$ -measurable subsets are called *Borel subsets of  $X$* .

**Lemma 744 (some Borel subsets).**

Let  $(X, \mathcal{T})$  be a topological space. Then, open and closed subsets are Borel subsets of  $X$ .

Moreover, if the space  $X$  is separable, then countable subsets of  $X$  are Borel subsets of  $X$ .

*Proof.* From Definition 743 (*Borel  $\sigma$ -algebra*), and Lemma 696 ( *$\sigma$ -algebra*), the Borel subsets are the elements of the Borel  $\sigma$ -algebra.

Let  $Y$  be an open subset of  $X$ . Then, from Definition 743 (*Borel  $\sigma$ -algebra*), and Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have  $Y \in \mathcal{T} \subset \mathcal{B}(X)$ .

Let  $Y$  be a closed subset of  $X$ . Then, from Definition 240 (*topological space, closed subset*), Definition 743 (*Borel  $\sigma$ -algebra*), and Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have  $Y^c \in \mathcal{T} \subset \mathcal{B}(X)$ . Hence, from Lemma 696 ( *$\sigma$ -algebra, closedness under complement*), and from **involutiveness of complement**, we have  $Y = (Y^c)^c \in \mathcal{B}(X)$ .

Assume now that  $X$  is separable, i.e.  $X$  contains a countable dense subset. Let  $Y$  be a countable subset of  $X$ . Then, from **the definition of countability**, there exist  $I \subset \mathbb{N}$ , and  $(x_i)_{i \in I} \in X$  such that  $Y = \bigcup_{i \in I} \{x_i\}$ . Let  $i \in I$ . Then, from **closedness of singletons in a separable space**,  $\{x_i\}$  is closed, thus it belongs to  $\mathcal{B}(X)$ . Hence, from Lemma 696 ( *$\sigma$ -algebra, closedness under countable union*), we have  $Y \in \mathcal{B}(X)$ .

Therefore, open, closed, and countable subsets (when  $X$  is separable) are Borel subsets of  $X$ .  $\square$

**Lemma 745 (countable union Borel  $\sigma$ -algebra generator).**

Let  $(X, \mathcal{T})$  be a topological space. Let  $G \subset \mathcal{T}$ . Assume that all open subsets of  $X$  are countable unions of elements of  $G$ . Then, we have  $\mathcal{B}(X) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Definition 743 (*Borel  $\sigma$ -algebra,  $\mathcal{B}(X) = \Sigma_X(\mathcal{T})$* ), and Lemma 712 (*countable union  $\sigma$ -algebra generator, with  $G_1 \stackrel{\text{def.}}{=} G$  and  $G_2 \stackrel{\text{def.}}{=} \mathcal{T}$* ).  $\square$

*Remark 746.* The previous lemma means that if  $G$  contains a countable topological basis of  $(X, \mathcal{T})$  (and possibly other open subsets) (see Definitions 246 and 260), then the Borel  $\sigma$ -algebra of  $X$  is generated by  $G$ .

*Remark 747.* Note that metric spaces are separable topological spaces, thus both previous lemmas apply in metric spaces.

**Definition 748 (measurability and structure).**

Let  $(X, \mathcal{T})$  be a topological space. The measurable space  $(X, \mathcal{B}(X))$  (or  $X$ ), is called *measurable topological space*.

Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $(E, \|\cdot\|_E)$  be a normed vector space over  $\mathbb{K}$  equipped with the induced topology. The measurable space  $(E, \mathcal{B}(E))$  (or  $E$ ) is called *measurable normed vector space*.

Let  $\mathbb{K}$  be  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$  equipped with the usual topology. The measurable space  $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$  (or  $\mathbb{K}$ ) is called *measurable number space*. For  $\mathbb{R}$  and  $\mathbb{C}$ , it is also called *measurable valued field*.

## 9.2 Measurable function

**Definition 749 (measurable function).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. A function  $f : X \rightarrow X'$  is said *measurable* (for  $\Sigma$  and  $\Sigma'$ ) iff

$$(9.1) \quad f^{-1}(\Sigma') \subset \Sigma \quad \text{i.e. } \forall A' \in \Sigma', f^{-1}(A') \in \Sigma.$$

If  $\Sigma$  and  $\Sigma'$  are the Borel  $\sigma$ -algebras of  $X$  and  $X'$ , then  $f$  is called *Borel function*.

**Lemma 750 (inverse  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $(X', \Sigma')$  be a measurable space. Let  $f : X \rightarrow X'$ . Then,  $\Sigma = f^{-1}(\Sigma')$  is the smallest  $\sigma$ -algebra of  $f^{-1}(X') \subset X$  such that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

It is called the inverse  $\sigma$ -algebra of  $\Sigma'$  by  $f$ .

*Proof.* Direct consequence of Definition 741 (*measurable space*), Lemma 696 ( *$\sigma$ -algebra*), **homogeneity of inverse image, compatibility of inverse image with complement, union and intersection**, and Definition 749 (*measurable function*).  $\square$

**Lemma 751 (image  $\sigma$ -algebra).**

Let  $(X, \Sigma)$  be a measurable space. Let  $X'$  be a set. Let  $f : X \rightarrow X'$ . Then,  $\Sigma' = \Sigma_f \stackrel{\text{def.}}{=} \{A' \subset X' \mid f^{-1}(A') \in \Sigma\}$  is the largest  $\sigma$ -algebra of  $f(X)$  such that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . It is called the image  $\sigma$ -algebra of  $\Sigma$  by  $f$ .

*Proof.* Direct consequence of Definition 741 (*measurable space*), Lemma 696 ( *$\sigma$ -algebra*), **compatibility of inverse image with complement, union and intersection, homogeneity of inverse image** ( $f^{-1}(\emptyset) = \emptyset$ ), and Definition 749 (*measurable function*).  $\square$

**Lemma 752 (identity function is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Then, the identity function is measurable for  $\Sigma$  (and  $\Sigma$ ).

*Proof.* Direct consequence of **the definition of the identity function**, and Definition 749 (*measurable function*).  $\square$

**Lemma 753 (countable range is measurable).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Assume that  $f(X)$  is countable, and that for all  $x' \in f(X)$ ,  $f^{-1}(x') \in \Sigma$ . Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

*Proof.* Let  $I \subset \mathbb{N}$ , and  $(x'_i)_{i \in I} \in X'$  be pairwise distinct such that  $f(X) = \{x'_i \mid i \in I\}$ . For all  $i \in I$ , let  $A_i \stackrel{\text{def.}}{=} f^{-1}(x'_i) \in \Sigma$ . Then, from **properties of inverse image**, we have  $X = \bigsqcup_{i \in I} A_i$ . Let  $A' \in \Sigma'$ . Let  $I_{A'} \stackrel{\text{def.}}{=} \{i \in I \mid x'_i \in A'\} \subset I \subset \mathbb{N}$ . Then, from **properties of inverse image**, we have  $f^{-1}(A') = \bigsqcup_{i \in I_{A'}} A_i$  (which is empty when  $I_{A'} = \emptyset$ ). Therefore, from Definition 741 (*measurable space*), Lemma 700 (*other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow Udc$* ), Lemma 696 ( *$\sigma$ -algebra,  $P_\Sigma \Rightarrow wE$* ), and Definition 749 (*measurable function*),  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .  $\square$

**Lemma 754 (constant function is measurable).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Assume that  $f$  is constant. Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

*Proof.* Direct consequence of Definition 741 (*measurable space*), Lemma 697 (*equivalent definition of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow wF$* ), and Lemma 753 (*countable range is measurable, with  $f(X) \stackrel{\text{def.}}{=} \{x'\}$  and  $f^{-1}(x') = X \in \Sigma$* ).  $\square$

**Lemma 755 (inverse image of generating family).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Then, we have

$$(9.2) \quad \forall G' \subset \mathcal{P}(X'), \quad \Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G')).$$

*Proof. “Left” included in “right”.* Let  $G' \subset \mathcal{P}(X')$ . Then, from Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have  $G' \subset \Sigma_{X'}(G')$ . Thus, from **monotonicity of inverse image**,  $f^{-1}(G')$  is a subset of  $f^{-1}(\Sigma_{X'}(G'))$ . Hence, from Lemma 710 (*properties of  $\sigma$ -algebra generation, monotonicity and idempotence*), and Lemma 750 (*inverse  $\sigma$ -algebra,  $f^{-1}(\Sigma_{X'}(G'))$  is  $\sigma$ -algebra*), we have  $\Sigma_X(f^{-1}(G')) \subset \Sigma_X(f^{-1}(\Sigma_{X'}(G'))) = f^{-1}(\Sigma_{X'}(G'))$ .

**“Right” included in “left”.** Conversely, let  $A' \in G' \subset \mathcal{P}(X')$ . Then, from **monotonicity of inverse image**, and Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have

$$f^{-1}(A') \in f^{-1}(G') \subset \Sigma_X(f^{-1}(G')).$$

Thus, from Lemma 751 (*image  $\sigma$ -algebra*), we have  $G' \subset (\Sigma_X(f^{-1}(G')))_f$ , and from Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have  $\Sigma_{X'}(G') \subset (\Sigma_X(f^{-1}(G')))_f$ . Hence, from Lemma 751 (*image  $\sigma$ -algebra*), and **the definition of inverse image**, we have

$$f^{-1}(\Sigma_{X'}(G')) \subset \Sigma_X(f^{-1}(G')).$$

Therefore, we have  $\Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G'))$ . □

**Lemma 756 (equivalent definition of measurable function).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$  iff

$$(9.3) \quad \forall G' \subset \mathcal{P}(X'), \quad \Sigma_{X'}(G') = \Sigma' \implies f^{-1}(G') \subset \Sigma.$$

*Proof. “Left” implies “right”.* Assume first that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . Let  $G' \subset \mathcal{P}(X')$  such that  $\Sigma_{X'}(G') = \Sigma'$ . Then, from Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*), we have  $G' \subset \Sigma'$ . Hence, from **monotonicity of inverse image**, and Definition 749 (*measurable function*), we have  $f^{-1}(G') \subset f^{-1}(\Sigma') \subset \Sigma$ .

**“Right” implies “left”.** Conversely, assume now that there exists  $G' \subset \mathcal{P}(X')$  such that  $\Sigma_{X'}(G') = \Sigma'$  and  $f^{-1}(G') \subset \Sigma$ . Then, from Lemma 755 (*inverse image of generating family*), and Lemma 710 (*properties of  $\sigma$ -algebra generation, monotonicity and idempotence*), we have  $f^{-1}(\Sigma') = f^{-1}(\Sigma_{X'}(G')) = \Sigma_X(f^{-1}(G')) \subset \Sigma_X(\Sigma) = \Sigma$ .

Therefore, we have the equivalence. □

**Lemma 757 (compatibility of measurability with composition).**

Let  $(X, \Sigma)$ ,  $(X', \Sigma')$ , and  $(X'', \Sigma'')$  be measurable spaces. Let  $f : X \rightarrow X'$  and  $g : X' \rightarrow X''$ . Assume that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ , and that  $g$  is measurable for  $\Sigma'$  and  $\Sigma''$ . Then,  $g \circ f$  is measurable for  $\Sigma$  and  $\Sigma''$ .

*Proof.* Direct consequence of **the definition of composition**, Definition 749 (*measurable function*), with  $f$  and  $g$ , and **monotonicity of inverse image**. □

**Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ ).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(Y, \mathcal{B}(Y))$  be a measurable topological space. The set of functions  $X \rightarrow Y$  that are measurable for  $\Sigma$  and  $\mathcal{B}(Y)$  is called the *set of measurable functions (from  $X$ ) to  $Y$* ; it is denoted  $\mathcal{M}_Y(X, \Sigma)$  (or simply  $\mathcal{M}_Y$ ).

**Lemma 759 ( $\mathcal{M}_Y$  inverse image of countable is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(Y, \mathcal{B}(Y))$  be a measurable separable topological space. Let  $f \in \mathcal{M}_Y$ . Then, we have

$$(9.4) \quad \forall I \subset \mathbb{N}, \forall (y_i)_{i \in I} \in Y, \quad \{f = (y_i)_{i \in I}\} \stackrel{\text{def.}}{=} f^{-1}\{y_i \mid i \in I\} \in \Sigma.$$

*Proof.* Direct consequence of Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ ), Definition 749 (*measurable function*), Lemma 744 (*some Borel subsets, countable subsets are measurable*). □



**Lemma 760 ( $\mathcal{M}_Y$  countable range).** *Let  $(X, \Sigma)$  be a measurable space. Let  $(Y, \mathcal{B}(Y))$  be a measurable separable topological space. Let  $f : X \rightarrow Y$ . Assume that  $f(X)$  is countable. Then, we have  $f \in \mathcal{M}_Y$  iff for all  $y \in f(X)$ ,  $f^{-1}(y) \in \Sigma$ .*

*Proof.* Direct consequence of Lemma 759 ( $\mathcal{M}_Y$  inverse image of countable is measurable, with  $\text{card}(I) = 1$ ), and Lemma 753 (countable range is measurable).  $\square$

**Lemma 761 (continuous is measurable).** *Let  $(X, \mathcal{B}(X))$  and  $(X', \mathcal{B}(X'))$  be measurable topological spaces. Let  $f : X \rightarrow X'$ . Assume that  $f$  is continuous. Then,  $f$  is a Borel function.*

*Proof.* Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies of  $X$  and  $X'$ . Let  $O' \in \mathcal{T}'$ . Then, from **the definition of continuity**,  $f^{-1}(O') \in \mathcal{T}$ . Thus, from Definition 743 (Borel  $\sigma$ -algebra), and Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra), we have  $\mathcal{B}(X') = \Sigma_{X'}(\mathcal{T}')$  and  $f^{-1}(\mathcal{T}') \subset \mathcal{T} \subset \Sigma_X(\mathcal{T}) = \mathcal{B}(X)$ . Therefore, from Lemma 756 (equivalent definition of measurable function), and Definition 749 (measurable function),  $f$  is a Borel function from  $X$  to  $X'$ .  $\square$

### 9.3 Measurable subspace

*Remark 762.* We recall that  $\bar{\cap}$  denotes a set of traces of subsets, see Definition 226.

**Lemma 763 (trace  $\sigma$ -algebra).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then,  $\Sigma \bar{\cap} Y$  is a  $\sigma$ -algebra of  $Y$ , and  $i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma$ .

$\Sigma \bar{\cap} Y$  is called trace  $\sigma$ -algebra of  $\Sigma$  on  $Y$ . The measurable space  $(Y, \Sigma \bar{\cap} Y)$  is said measurable subspace of  $(X, \Sigma)$ .

*Proof.* From Definition 741 (measurable space,  $\Sigma$  is a  $\sigma$ -algebra), Lemma 696 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), and since  $\emptyset$  is absorbing for intersection, we have  $\emptyset = \emptyset \cap Y \in \Sigma \bar{\cap} Y$ .

Let  $A \in \Sigma \bar{\cap} Y$ . Then, from Definition 226 (trace of subset system), let  $B \in \Sigma$  such that  $A = B \cap Y$ . Then, from the definition of set difference, De Morgan's laws, distributivity of intersection over union, the definition of complement, and commutativity of intersection, we have

$$Y \setminus A = Y \setminus (B \cap Y) = Y \cap (B \cap Y)^c = Y \cap (B^c \cup Y^c) = (Y \cap B^c) \cup (Y \cap Y^c) = B^c \cap Y.$$

Hence, from Lemma 696 ( $\sigma$ -algebra, closedness under complement), we have  $Y \setminus A \in \Sigma \bar{\cap} Y$ .

Let  $I$  be a set. Let  $(A_i)_{i \in I} \in \Sigma \bar{\cap} Y$ . Then, from Definition 226 (trace of subset system), for all  $i \in I$ , let  $B_i \in \Sigma$  such that  $A_i = B_i \cap Y$ . Then, from distributivity of intersection over union, we have

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} (B_i \cap Y) = \bigcup_{i \in I} B_i \cap Y.$$

Hence, from Lemma 696 ( $\sigma$ -algebra, closedness under countable union), we have  $\bigcup_{i \in I} A_i \in \Sigma \bar{\cap} Y$ .

Therefore, from Lemma 696 ( $\sigma$ -algebra),  $\Sigma \bar{\cap} Y$  is a  $\sigma$ -algebra of  $Y$ . Moreover, from the definition of the canonical injection, and Definition 749 (measurable function),  $i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma$ .  $\square$

**Lemma 764 (measurability of measurable subspace).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Then, we have  $Y \in \Sigma$  iff  $\Sigma \bar{\cap} Y = \Sigma \cap \mathcal{P}(Y)$ .

*Proof.* From Definition 226 (trace of subset system), we always have  $\Sigma \cap \mathcal{P}(Y) \subset \Sigma \bar{\cap} Y$ .

**“Left” implies “right”.** Assume first that  $Y \in \Sigma$ . Let  $A \in \Sigma \bar{\cap} Y$ . Then, from Definition 226 (trace of subset system), there exists  $B \in \Sigma$  such that  $A = B \cap Y \subset Y$ . Thus, from Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection), we have  $A \in \Sigma$ , and  $\Sigma \bar{\cap} Y \subset \Sigma \cap \mathcal{P}(Y)$ . Hence, we have  $\Sigma \bar{\cap} Y = \Sigma \cap \mathcal{P}(Y)$ .

**“Right” implies “left”.** Conversely, assume now that  $\Sigma \bar{\cap} Y = \Sigma \cap \mathcal{P}(Y)$ . Then, from Lemma 763 (trace  $\sigma$ -algebra), and Lemma 697 (equivalent definition of  $\sigma$ -algebra, contains full set), we have  $Y \in \Sigma \bar{\cap} Y \subset \Sigma$ .

Therefore, we have the equivalence.  $\square$

**Lemma 765 (generating measurable subspace).** Let  $X$  be a set,  $Y \subset X$ , and  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_Y(G \bar{\cap} Y) = \Sigma_X(G) \bar{\cap} Y$ , and  $Y \in \Sigma_X(G)$  iff  $\Sigma_Y(G \bar{\cap} Y) = \Sigma_X(G) \cap \mathcal{P}(Y)$ .

*Proof.* Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then, from Definition 226 (trace of subset system), we have  $i^{-1}(G) = G \bar{\cap} Y$  and  $i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \bar{\cap} Y$ . Therefore, from Lemma 755 (inverse image of generating family), we have

$$\Sigma_Y(G \bar{\cap} Y) = \Sigma_Y(i^{-1}(G)) = i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \bar{\cap} Y.$$

And the equivalence directly follows from Lemma 764 (measurability of measurable subspace).  $\square$

**Lemma 766 (generating on a partition).** *Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Let  $I \subset \mathbb{N}$ . Let  $(Y_i)_{i \in I} \in \Sigma_X(G)$ . Assume that  $X = \bigsqcup_{i \in I} Y_i$ . Let  $A \subset X$ . Then, we have*

$$(9.5) \quad A \in \Sigma_X(G) \iff \forall i \in I, \quad A \cap Y_i \in \Sigma_{Y_i}(G \cap Y_i).$$

*Proof. “Left” implies “right”.*

Assume first that  $A \in \Sigma_X(G)$ . Let  $i \in I$ . Then, from Definition 226 (*trace of subset system*), and Lemma 765 (*generating measurable subspace*), we have  $A \cap Y_i \in \Sigma_X(G) \cap Y_i = \Sigma_{Y_i}(G \cap Y_i)$ .

**“Right” implies “left”.** Conversely, assume now that for all  $i \in I$ ,  $A \cap Y_i \in \Sigma_{Y_i}(G \cap Y_i)$ . Let  $i \in I$ . Then, from Lemma 765 (*generating measurable subspace*, with  $Y_i \in \Sigma_X(G)$ ), we have  $A \cap Y_i \in \Sigma_X(G)$ . Hence, from Lemma 211 (*compatibility of pseudopartition with intersection*), and Lemma 696 ( *$\sigma$ -algebra, closedness under countable union*), we have  $A = \bigsqcup_{i \in I} (A \cap Y_i) \in \Sigma_X(G)$ .

Therefore, we have the equivalence.  $\square$

*Remark 767.* We recall that  $\mathcal{T}_Y \stackrel{\text{def.}}{=} \mathcal{T} \cap Y$  denotes the trace topology, see Lemma 252.

**Lemma 768 (Borel sub- $\sigma$ -algebra).** *Let  $(X, \mathcal{T})$  be a topological space. Let  $Y \subset X$ . Then, we have  $\mathcal{B}(Y) \stackrel{\text{def.}}{=} \Sigma_Y(\mathcal{T}_Y) = \mathcal{B}(X) \cap Y$ , and  $Y \in \mathcal{B}(X)$  iff  $\mathcal{B}(Y) = \mathcal{B}(X) \cap \mathcal{P}(Y)$ .*

*Proof.* Direct consequence of Definition 743 (*Borel  $\sigma$ -algebra*), Lemma 252 (*trace topology on subset, definition*), and Lemma 765 (*generating measurable subspace, with  $G \stackrel{\text{def.}}{=} \mathcal{T}$* ).  $\square$

**Lemma 769 (Borel subsets of partition).** *Let  $(X, \mathcal{T})$  be a topology space. Let  $I \subset \mathbb{N}$ . Let  $(Y_i)_{i \in I} \in \mathcal{B}(X)$ . Assume that  $X = \bigsqcup_{i \in I} Y_i$ . Let  $A \subset X$ . Then, we have*

$$(9.6) \quad A \in \mathcal{B}(X) \iff \forall i \in I, \quad A \cap Y_i \in \mathcal{B}(Y_i).$$

*Proof.* Direct consequence of Lemma 766 (*generating on a partition, with  $G \stackrel{\text{def.}}{=} \mathcal{T}$* ).  $\square$

**Lemma 770 (source restriction of measurable function).** *Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Let  $Y \subset X$ . Let  $f|_Y$  be the restriction of  $f$  to the source  $Y$ . Assume that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . Then,  $f|_Y$  is measurable for  $\Sigma \cap Y$  and  $\Sigma'$ .*

*Proof.* Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then, from Definition 226 (*trace of subset system*), and Lemma 763 (*trace  $\sigma$ -algebra*),  $i^{-1}(\Sigma) = \Sigma \cap Y$  is a  $\sigma$ -algebra of  $Y$ . Thus, from Definition 749 (*measurable function*),  $i$  is measurable for  $\Sigma \cap Y$  and  $\Sigma$ . Therefore, from Lemma 757 (*compatibility of measurability with composition*),  $f|_Y = f \circ i$  is measurable for  $\Sigma \cap Y$  and  $\Sigma'$ .  $\square$

**Lemma 771 (destination restriction of measurable function).** *Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Let  $Y' \subset X'$ . Assume that  $f(X) \subset Y'$ . Let  $f|^{Y'}$  be the restriction of  $f$  to the destination  $Y'$ . Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$  iff  $f|^{Y'}$  is measurable for  $\Sigma$  and  $\Sigma' \cap Y'$ .*

*Proof.* Let  $A' \in \Sigma'$ . Then, from **the definition of set difference**, and **monotonicity of inverse image and complement**, we have

$$f^{-1}(A' \setminus Y') = f^{-1}(A' \cap Y'^c) \subset f^{-1}(Y'^c) \subset f^{-1}(f(X)^c) = \emptyset.$$

Then, from **compatibility of inverse image with disjoint union**, and **the definition of destination restriction**, we have

$$f^{-1}(A') = f^{-1}((A' \cap Y') \uplus (A' \setminus Y')) = f^{-1}(A' \cap Y') = \left(f|^{Y'}\right)^{-1}(A' \cap Y').$$

Hence,  $f^{-1}(\Sigma') = \left(f|^{Y'}\right)^{-1}(\Sigma' \cap Y')$ . Therefore, from Definition 749 (*measurable function*),  $f|^{Y'}$  is measurable for  $\Sigma$  and  $\Sigma' \cap Y'$ .  $\square$

**Lemma 772 (measurability of function defined on a pseudopartition).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $X_i \in \Sigma$  and  $f_i : X \rightarrow X'$ . Assume that for all  $i \in I$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'$ , and that  $X = \bigsqcup_{i \in I} X_i$ . Then, the function defined by  $f_i$  on  $X_i$  for all  $i \in I$  is measurable for  $\Sigma$  and  $\Sigma'$ .

*Proof.* Let  $f$  be the function defined by for all  $i \in I$ , for all  $x \in X_i$ ,  $f(x) \stackrel{\text{def.}}{=} f_i(x)$ . Let  $A' \in \Sigma'$ . Let  $i \in I$ . Then, from Definition 749 (measurable function, with  $f_i$ ), we have  $f_i^{-1}(A') \in \Sigma$ . Thus, from Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection), we have  $f^{-1}(A') = \bigcup_{i \in I} (X_i \cap f_i^{-1}(A')) \in \Sigma$ . Hence,  $f^{-1}(\Sigma') \subset \Sigma$ .

Therefore, from Definition 749 (measurable function),  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .  $\square$

## 9.4 Product of measurable spaces

*Remark 773.* We recall that  $[n..p]$  denotes an interval of integers, and that  $\overline{\prod}$  and  $\overline{\times}$  denote a set of Cartesian products of subsets, see Definition 227.

The concepts presented in this section are used mainly in Section 15.4.

### Definition 774 (tensor product of $\sigma$ -algebras).

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space. Let  $X \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} X_i$  and  $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} \Sigma_i$ . The (tensor) product of the  $\sigma$ -algebras  $(\Sigma_i)_{i \in [1..m]}$  on  $X$  is the  $\sigma$ -algebra generated by  $\overline{\Sigma}$ ; it is denoted  $\bigotimes_{i \in [1..m]} \Sigma_i \stackrel{\text{def.}}{=} \Sigma_X(\overline{\Sigma})$ .

### Lemma 775 (product of measurable subsets is measurable).

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and  $A_i \in \Sigma_i$ .

Let  $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} \Sigma_i$  and  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ . Then,  $\prod_{i \in [1..m]} A_i \in \overline{\Sigma} \subset \Sigma$ .

*Proof.* Direct consequence of Definition 227 (product of subset systems), Definition 774 (tensor product of  $\sigma$ -algebras), and Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra,  $\overline{\Sigma} \subset \Sigma_X(\overline{\Sigma})$ ).  $\square$

### Lemma 776 (measurability of function to product space).

Let  $m \in [2..\infty)$ . Let  $(X, \Sigma)$  be a measurable space. For all  $i \in [1..m]$ , let  $(X'_i, \Sigma'_i)$  be a measurable space, and  $f_i : X \rightarrow X'_i$ . Then,  $(f_i)_{i \in [1..m]}$  is measurable for  $\Sigma$  and  $\bigotimes_{i \in [1..m]} \Sigma'_i$  iff for all  $i \in [1..m]$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

*Proof.* Let  $\Sigma' \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma'_i$ .

“Left” implies “right”. Assume first that  $(f_i)_{i \in [1..m]}$  is measurable for  $\Sigma$  and  $\Sigma'$ .

Let  $i \in [1..m]$ . Let  $A'_i \in \Sigma'_i$ . Let  $A' \stackrel{\text{def.}}{=} X'_1 \times \dots \times A'_i \times \dots \times X'_m$ . Then, from Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $\underline{X'_i} \in \underline{\Sigma'_i}$ ), Definition 774 (tensor product of  $\sigma$ -algebras), and Lemma 775 (product of measurable subsets is measurable), we have  $A' \in \Sigma'$ . Thus, from Definition 749 (measurable function, with  $f$ ), we have  $f_i^{-1}(A'_i) = f^{-1}(A') \in \Sigma$ . Hence, from Definition 749 (measurable function),  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

“Right” implies “left”. Assume now that for all  $i \in [1..m]$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

Let  $\overline{\Sigma'} \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} \Sigma'_i$  and  $f \stackrel{\text{def.}}{=} (f_i)_{i \in [1..m]}$ . Let  $A' \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} A'_i \in \overline{\Sigma'}$ . From Definition 227 (product of subset systems), for all  $i \in [1..m]$ , we have  $A'_i \in \Sigma'_i$ . Let  $x \in X$ . Then,  $f(x) \in A'$  iff for all  $i \in [1..m]$ ,  $f_i(x) \in A'_i$ . Thus, we have  $f^{-1}(A') = \bigcap_{i \in [1..m]} f_i^{-1}(A'_i)$ . Hence, from Definition 749 (measurable function, with  $f_i$ ), and Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection), we have  $f^{-1}(\overline{\Sigma'}) = \bigcap_{i \in [1..m]} f_i^{-1}(\Sigma'_i) \subset \Sigma$ . Thus, from Lemma 756 (equivalent definition of measurable function, with  $G' \stackrel{\text{def.}}{=} \overline{\Sigma'}$ ),  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

Therefore, we have the equivalence.  $\square$

### Lemma 777 (canonical projection is measurable).

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and let  $\pi_i$  be the canonical projection from  $X \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} X_i$  onto  $X_i$ . Let  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ .

Then,  $\Sigma$  is the smallest  $\sigma$ -algebra on  $X$  such that for all  $i \in [1..m]$ ,  $\pi_i$  is measurable for  $\Sigma$  and  $\Sigma_i$ .

*Proof.* Let  $\Sigma'$  be a  $\sigma$ -algebra on  $X$ . Then, from Lemma 776 (measurability of function to product space, with  $\Sigma \stackrel{\text{def.}}{=} \Sigma'$ ,  $X'_i \stackrel{\text{def.}}{=} X_i$ ,  $\Sigma'_i \stackrel{\text{def.}}{=} \Sigma_i$ , and  $f \stackrel{\text{def.}}{=} \text{Id}_X = (\pi_i)_{i \in [1..m]}$ ), Definition 749 (measurable function), and reflexivity of inclusion we have

$$\begin{aligned} \forall i \in [1..m], \quad \pi_i \text{ is measurable for } \Sigma' \text{ and } \Sigma_i &\iff \text{Id}_X \text{ is measurable for } \Sigma' \text{ and } \Sigma \\ &\iff \text{Id}_X^{-1}(\Sigma) = \Sigma \subset \Sigma'. \end{aligned}$$

□

**Lemma 778 (permutation is measurable).** *Let  $m \in [2..∞)$ . Let  $\psi$  be a permutation of  $[1..m]$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and let  $\pi_i$  be the canonical projection from  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$  onto  $X_i$ . Let  $\Sigma \stackrel{\text{def.}}{=} \otimes_{i \in [1..m]} \Sigma_i$ , and  $\Sigma^\psi \stackrel{\text{def.}}{=} \otimes_{i \in [1..m]} \Sigma_{\psi(i)}$ . Then, the permutation of coordinates  $(\pi_{\psi(i)})_{i \in [1..m]}$  is measurable for  $\Sigma$  and  $\Sigma^\psi$ .*

*Proof.* Direct consequence of Lemma 776 (measurability of function to product space), and Lemma 777 (canonical projection is measurable, with  $X'_i \stackrel{\text{def.}}{=} X_{\psi(i)}$  and  $f_i \stackrel{\text{def.}}{=} \pi_{\psi(i)}$ ). □

**Lemma 779 (generating product measurable space).** *Let  $m \in [2..∞)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space,  $G_i \subset \mathcal{P}(X_i)$ , and assume that  $X_i \in G_i$  and  $\Sigma_i = \Sigma_{X_i}(G_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ ,  $\Sigma \stackrel{\text{def.}}{=} \otimes_{i \in [1..m]} \Sigma_i$ , and  $\overline{G} \stackrel{\text{def.}}{=} \overline{\prod_{i \in [1..m]} G_i}$ . Then, we have  $\Sigma = \Sigma_X(\overline{G})$ .*

*Proof.*  $\Sigma \subset \Sigma_X(\overline{G})$ . Let  $i \in [1..m]$ . Let  $\pi_i$  be the canonical projection from  $X$  onto  $X_i$ . Then, from Lemma 755 (inverse image of generating family), we have

$$\pi_i^{-1}(\Sigma_i) = \pi_i^{-1}(\Sigma_{X_i}(G_i)) = \Sigma_X(\pi_i^{-1}(G_i)).$$

Moreover, since  $X_i \in G_i$ , and from Definition 227 (product of subset systems), we have

$$\pi_i^{-1}(G_i) = \{X_1 \times \dots \times A_i \times \dots \times X_m \mid A_i \in G_i\} \subset \overline{G}.$$

Hence, from Lemma 710 (properties of  $\sigma$ -algebra generation, monotonicity, thus  $\pi_i^{-1}(\Sigma_i)$  is a subset of  $\Sigma_X(\overline{G})$ ), Definition 749 (measurable function,  $\pi_i$  is measurable for  $\Sigma_X(\overline{G})$  and  $\Sigma_i$ ), and Lemma 777 (canonical projection is measurable, smallest  $\sigma$ -algebra), we have  $\Sigma \subset \Sigma_X(\overline{G})$ .

$\Sigma_X(\overline{G}) \subset \Sigma$ . Let  $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod_{i \in [1..m]} \Sigma_i}$ .

Direct consequence of Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra,  $G_i \subset \Sigma_i$ ), Definition 227 (product of subset systems,  $\overline{G} \subset \overline{\Sigma}$ ), Lemma 710 (properties of  $\sigma$ -algebra generation, monotonicity, thus  $\Sigma_X(\overline{G}) \subset \Sigma_X(\overline{\Sigma})$ ), and Definition 774 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\overline{\Sigma})$ ),

Therefore, we have the equality. □

**Lemma 780 (Borel subsets in product measurable subspace).** *Let  $m \in [2..∞)$ . For all  $i \in [1..m]$ , let  $(X_i, \mathcal{T}_i)$  be a topological space, and  $Y_i \subset X_i$ . Assume that the  $Y_i$ 's are equipped with the trace topology, and  $Y \stackrel{\text{def.}}{=} \prod_{i \in [0..m]} Y_i$  with the box topology. Then, we have*

$$(9.7) \quad \mathcal{B}(Y) = \bigotimes_{i \in [1..m]} \mathcal{B}(X_i) \overline{\cap} Y_i.$$

*When the subspaces are all the same, we define the notation  $\mathcal{B}(Z)^{m \otimes} \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} (\mathcal{B}(X) \overline{\cap} Z)$ .*

*Proof.* From Lemma 252 (trace topology on subset), and Lemma 254 (box topology on Cartesian product), we have  $\mathcal{T}_Y \stackrel{\text{def.}}{=} \overline{\prod_{i \in [0..m]} \mathcal{T}_i \overline{\cap} Y_i}$ . Therefore, from Definition 743 (Borel  $\sigma$ -algebra,  $\mathcal{B}(Y) = \Sigma_Y(\mathcal{T}_Y)$  and  $\mathcal{B}(X_i) = \Sigma_{X_i}(\mathcal{T}_i)$ ), Lemma 779 (generating product measurable space), and Lemma 765 (generating measurable subspace), we have

$$\mathcal{B}(Y) = \bigotimes_{i \in [0..m]} \Sigma_{Y_i}(\mathcal{T}_i \overline{\cap} Y_i) = \bigotimes_{i \in [0..m]} \mathcal{B}(X_i) \overline{\cap} Y_i.$$

□

*Remark 781.*

For instance, the previous lemma states  $\mathcal{B}(\mathbb{R}_+^n) = (\mathcal{B}(\mathbb{R}_+))^{n \otimes}$  (with  $Y_i = \mathbb{R}_+ \subset \mathbb{R} = X_i$ ).

*Remark 782.* For the sake of simplicity, we only present the remainder of this section in the case of the product of two measure spaces. When  $i \in \{1, 2\}$ , the complement  $\{1, 2\} \setminus \{i\}$  is  $\{3 - i\}$ .

**Definition 783 (section in Cartesian product).** Let  $X_1$  and  $X_2$  be sets. Let  $A \subset X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . The  $i$ -th section of  $A$  in  $x_i$  is the subset

$$(9.8) \quad s_i(x_i, A) \stackrel{\text{def.}}{=} \{x_j \in X_j \mid (x_1, x_2) \in A\}.$$

**Lemma 784 (section of product).** Let  $X_1$  and  $X_2$  be sets. Let  $A_1 \in X_1$  and  $A_2 \in X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(9.9) \quad s_i(x_i, A_1 \times A_2) = \begin{cases} A_j & \text{when } x_i \in A_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

*Proof.* Direct consequence of Definition 783 (section in Cartesian product).  $\square$

**Lemma 785 (compatibility of section with set operations).** Let  $X_1$  and  $X_2$  be sets. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $x_i \in X_i$ . Then, the sections are compatible with the empty set, the complement, countable union and intersection, and are monotone:

$$(9.10) \quad s_i(x_i, \emptyset) = \emptyset,$$

$$(9.11) \quad \forall A \subset X, \quad s_i(x_i, A^c) = s_i(x_i, A)^c,$$

$$(9.12) \quad \forall I \subset \mathbb{N}, \forall (A_n)_{n \in I} \subset X, \quad s_i\left(x_i, \bigcup_{n \in I} A_n\right) = \bigcup_{n \in I} s_i(x_i, A_n),$$

$$(9.13) \quad \forall I \subset \mathbb{N}, \forall (A_n)_{n \in I} \subset X, \quad s_i\left(x_i, \bigcap_{n \in I} A_n\right) = \bigcap_{n \in I} s_i(x_i, A_n),$$

$$(9.14) \quad \forall A, B \subset X, \quad A \subset B \implies s_i(x_i, A) \subset s_i(x_i, B).$$

*Proof.* Direct consequences of Definition 783 (section in Cartesian product).  $\square$

**Lemma 786 (measurability of section).** Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have  $s_i(x_i, A) \in \Sigma_j$ .

*Proof.* Let  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \subset X \mid s_i(x_i, A) \in \Sigma_j\}$ .

From Definition 741 (measurable space),  $\Sigma_j$  is a  $\sigma$ -algebra.

Let  $A_1 \in \Sigma_1$ , and  $A_2 \in \Sigma_2$ . Then, from Lemma 784 (section of product,  $s_i(x_i, A_1 \times A_2)$  belongs to  $\{\emptyset, A_j\}$ ), Lemma 696 ( $\sigma$ -algebra,  $\emptyset, A_j \in \Sigma_j$ ), and the definition of  $\mathcal{S}_i$ , we have  $A_1 \times A_2 \in \mathcal{S}_i$ . Hence, from Definition 227 (product of subset systems), we have  $\bar{\Sigma} \subset \mathcal{S}_i$ .

From Lemma 785 (compatibility of section with set operations), and Lemma 696 ( $\sigma$ -algebra),  $\mathcal{S}_i$  contains  $\emptyset$ , and is closed under complement and countable union. Thus, from Lemma 696 ( $\sigma$ -algebra),  $\mathcal{S}_i$  is a  $\sigma$ -algebra on  $X$ . Hence, from Definition 774 (tensor product of  $\sigma$ -algebras,  $\Sigma$  is a generated by  $\bar{\Sigma}$ ), Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra), we have  $\Sigma \subset \mathcal{S}_i$ .

Therefore, for all  $A \in \Sigma$  we have  $s_i(x_i, A) \in \Sigma_j$ .  $\square$

**Lemma 787 (countable union of sections is measurable).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $I \subset \mathbb{N}$ . Let  $(A_n)_{n \in I} \in \Sigma_1 \otimes \Sigma_2$ . Let  $i$  be in  $\{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(9.15) \quad s_i\left(x_i, \bigcup_{n \in I} A_n\right) = \bigcup_{n \in I} s_i(x_i, A_n) \in \Sigma_j.$$

*Proof.* Direct consequence of Lemma 785 (*compatibility of section with set operations, with countable union*), Definition 741 (*measurable space,  $\Sigma_j$  is a  $\sigma$ -algebra*), Lemma 696 ( *$\sigma$ -algebra, closedness under countable union*), and Lemma 786 (*measurability of section*).  $\square$

**Lemma 788 (countable intersection of sections is measurable).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma_1 \otimes \Sigma_2$ . Let  $i$  be in  $\{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(9.16) \quad s_i \left( x_i, \bigcap_{n \in \mathbb{N}} A_n \right) = \bigcap_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j.$$

*Proof.* Direct consequence of Lemma 785 (*compatibility of section with set operations, with countable intersection*), Lemma 697 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), and Lemma 786 (*measurability of section*).  $\square$

**Lemma 789 (indicator of section).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $A \subset X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, we have

$$(9.17) \quad \forall x_1 \in X_1, \quad \forall x_2 \in X_2, \quad \mathbb{1}_A(x_1, x_2) = \mathbb{1}_{s_i(x_i, A)}(x_j).$$

*Proof.* Direct consequence of Definition 783 (*section in Cartesian product*), and **the definition of the indicator function**.  $\square$

**Lemma 790 (measurability of function from product space).**

Let  $(X_1, \Sigma_1)$ ,  $(X_2, \Sigma_2)$  and  $(X', \Sigma')$  be measurable spaces. Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $f : X_1 \times X_2 \rightarrow X'$ . Assume that  $f$  is measurable for  $\Sigma_1 \otimes \Sigma_2$  and  $\Sigma'$ . Then, for all  $x_i \in X_i$ , the function  $(x_j \mapsto f(x_1, x_2))$  is measurable for  $\Sigma_j$  and  $\Sigma'$ .

*Proof.* Direct consequence of **properties of inverse image** ( $(f_{x_i}^j)^{-1}(A') = s_i(x_i, f^{-1}(A'))$ ) **where  $f_{x_i}^j \stackrel{\text{def.}}{=} (x_j \mapsto f(x_1, x_2))$  and  $A' \in \Sigma'$** , Lemma 786 (*measurability of section,  $(f_{x_i}^j)^{-1}(A') \in \Sigma_j$* ), and Definition 749 (*measurable function*).  $\square$

*Remark 791.* Note that the reciprocal of the previous lemma is false.

Indeed, let us build a counter-example. Let  $X_1 = X_2 = X' \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $\Sigma_1 = \Sigma_2 \stackrel{\text{def.}}{=} \Sigma_{\mathbb{R}}(\{\{x\}\}_{x \in \mathbb{R}})$ , and  $\Sigma' \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$ . Let  $D \stackrel{\text{def.}}{=} \{(x, x) \mid x \in \mathbb{R}\} \subset X_1 \times X_2$ . Then, the tensor product  $\sigma$ -algebra  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$  is also generated by all singletons of  $\mathbb{R}^2$ , and  $D$  is not measurable. Let  $f = \mathbb{1}_D$ . Then,  $f$  is not measurable for  $\Sigma$  and  $\Sigma'$ , but for all  $x \in X$ ,  $(y \mapsto f(x, y)) = (y \mapsto f(y, x)) = \mathbb{1}_{\{x\}}$  is obviously measurable for  $\Sigma$  and  $\Sigma'$ .





## Chapter 10

# Measurability and numbers

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*Remark 792.* In this chapter, we address the specific case of the measurable number space  $\mathbb{K}$  (i.e.  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$ , see Definition 748), and of measurable functions to it.

## 10.1 Borel subset of numbers

### 10.1.1 Borel subset of numbers

*Remark 793.* In the next lemma,  $\mathbb{K}$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$  (see Definition 748).

**Lemma 794 (some Borel subsets of numbers).** *Let  $\mathbb{K}$  be a measurable number space. Then, open subsets, closed subsets, and countable subsets are Borel subsets of  $\mathbb{K}$ .*

*Proof.* Direct consequence of Lemma 744 (some Borel subsets, with  $Y \stackrel{\text{def.}}{=} \mathbb{R} = \mathbb{Q}$ ,  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}} = \mathbb{Q}$  and  $Y \stackrel{\text{def.}}{=} \mathbb{C} = \overline{\{a + ib \mid a, b \in \mathbb{Q}\}}$ , all separable).  $\square$

### 10.1.2 Borel subset of real numbers

*Remark 795.* In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

**Lemma 796 (Borel subsets of  $\mathbb{R}$ ).**  *$\mathcal{B}(\mathbb{R})$  is generated by any of the following sets of intervals:  $\{(a, b)\}_{a < b}$ ,  $\{(-\infty, b)\}_{b \in \mathbb{R}}$ ,  $\{(a, \infty)\}_{a \in \mathbb{R}}$ .*

*Proof.* From Theorem 461 (countable connected components of open subsets of  $\mathbb{R}$ ), and Lemma 745 (countable union Borel  $\sigma$ -algebra generator), we have  $\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b})$ .

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following countable unions:

$$\begin{aligned} [a, b) &= \{a\} \cup (a, b), & [a, b] &= \{a\} \cup (a, b) \cup \{b\}, & (a, b] &= (a, b) \cup \{b\}, \\ (a, b) &= \bigcup_{n \in \mathbb{N}} \left[ a + \frac{1}{n+1}, b \right) = \bigcup_{n \in \mathbb{N}} \left[ a + \frac{1}{n+1}, b - \frac{1}{n+1} \right] = \bigcup_{n \in \mathbb{N}} \left( a, b - \frac{1}{n+1} \right]. \end{aligned}$$

Thus, from Definition 708 (generated  $\sigma$ -algebra), Lemma 696 ( $\sigma$ -algebra, closedness under countable union), and Lemma 794 (some Borel subsets of numbers, singletons of  $\mathbb{R}$  are Borel subsets),

$$\begin{aligned} \{(a, b)\}_{a < b}, \{[a, b)\}_{a < b}, \{(a, b]\}_{a < b} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(a, b)\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{[a, b)\}_{a < b}), \Sigma_{\mathbb{R}}(\{(a, b]\}_{a < b}), \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}). \end{aligned}$$

Hence, from Lemma 711 (other properties of  $\sigma$ -algebra generation), we have

$$\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{[a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(a, b]\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}).$$

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following set differences:

$$\begin{aligned} [a, b) &= (-\infty, b) \setminus (-\infty, a) = [a, \infty) \setminus [b, \infty), \\ (a, b] &= (-\infty, b] \setminus (-\infty, a] = (a, \infty) \setminus (b, \infty). \end{aligned}$$

Thus, since  $(-\infty, b)$  and  $(a, \infty)$  are open,  $(-\infty, b]$  and  $[a, \infty)$  are closed, and from Definition 708 (generated  $\sigma$ -algebra), Lemma 700 (other properties of  $\sigma$ -algebra, closedness under set difference), and Lemma 794 (some Borel subsets of numbers, closed subsets of  $\mathbb{R}$  are Borel subsets), we have

$$\begin{aligned} \{(-\infty, b)\}_{b \in \mathbb{R}}, \{(a, \infty)\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(-\infty, b]\}_{b \in \mathbb{R}}, \{(a, \infty)\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(a, b)\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{(-\infty, b)\}_{b \in \mathbb{R}}), \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}), \\ \{(a, b)\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{(-\infty, b]\}_{b \in \mathbb{R}}), \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}). \end{aligned}$$

Hence, from Lemma 711 (other properties of  $\sigma$ -algebra generation), we have

$$\begin{aligned} \mathcal{B}(\mathbb{R}) &= \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(-\infty, b)\}_{b \in \mathbb{R}}) = \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}), \\ \mathcal{B}(\mathbb{R}) &= \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(-\infty, b]\}_{b \in \mathbb{R}}) = \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}). \end{aligned}$$

Therefore, all eight sets of intervals generate the Borel subsets of  $\mathbb{R}$ .  $\square$

**Lemma 797 (countable generator of Borel subsets of  $\mathbb{R}$ ).**

In Lemma 796 (Borel subsets of  $\mathbb{R}$ ), finite bounds of intervals,  $a$  and  $b$ , may be taken rational.

*Proof.* Same proof as for Lemma 796 (Borel subsets of  $\mathbb{R}$ ), but with using the countability of the topological basis of  $\mathbb{R}$  equipped with the Euclidean distance through Theorem 465 ( $\mathbb{R}$  is second-countable) instead of Theorem 461 (countable connected components of open subsets of  $\mathbb{R}$ ).  $\square$

### 10.1.3 Borel subset of extended real numbers

**Lemma 798 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ).** Let  $A \subset \overline{\mathbb{R}}$ . Then,  $A \in \mathcal{B}(\overline{\mathbb{R}})$  iff  $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ .

*Proof.* Direct consequence of Definition 743 (Borel  $\sigma$ -algebra,  $\mathbb{R}$  (open subset) belongs to  $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 696 ( $\sigma$ -algebra,  $\{-\infty, \infty\} = \mathbb{R}^c \in \mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 769 (Borel subsets of partition, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 \stackrel{\text{def.}}{=} \mathbb{R}$ , and  $Y_1 = \{-\infty, \infty\}$ ), Lemma 768 (Borel sub- $\sigma$ -algebra, with  $X$  set to  $\overline{\mathbb{R}}$  and  $Y \stackrel{\text{def.}}{=} \{-\infty, \infty\}$ , i.e.  $\mathcal{B}(\{-\infty, \infty\}) = \mathcal{P}(\{-\infty, \infty\})$ , and thus  $A \cap \{-\infty, \infty\}$  always belongs to  $\mathcal{B}(\{-\infty, \infty\})$ ).  $\square$

*Remark 799.* In other words, Borel subsets of  $\overline{\mathbb{R}}$  are Borel subsets of  $\mathbb{R}$ , or the union of Borel subsets of  $\mathbb{R}$  with  $\{-\infty\}$ ,  $\{\infty\}$ , or  $\{-\infty, \infty\}$ .

**Lemma 800 (generators of Borel subsets of  $\overline{\mathbb{R}}$ ).** Let  $\overline{G} \subset \mathcal{P}(\overline{\mathbb{R}})$ . Assume that  $\overline{G} \cap \{-\infty, \infty\}$  is one of the two singletons  $\{\pm\infty\}$ , and that  $\overline{G} \cap \mathbb{R}$  is a generator of  $\mathcal{B}(\mathbb{R})$ . Then,  $\overline{G}$  is a generator of  $\mathcal{B}(\overline{\mathbb{R}})$ .

*Proof.* Let  $A \subset \overline{\mathbb{R}}$ . Then, from Definition 708 (generated  $\sigma$ -algebra), we have

$$A \cap \{-\infty, \infty\} \in \mathcal{P}(\{-\infty, \infty\}) = \Sigma_{\{-\infty, \infty\}}(\overline{G} \cap \{-\infty, \infty\}).$$

Hence, from Lemma 766 (generating on a partition, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 \stackrel{\text{def.}}{=} \mathbb{R}$ , and  $Y_1 \stackrel{\text{def.}}{=} \{-\infty, \infty\}$ ), and Lemma 798 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ), we have

$$A \in \Sigma_{\overline{\mathbb{R}}}(\overline{G}) \iff A \cap \mathbb{R} \in \Sigma_{\mathbb{R}}(\overline{G} \cap \mathbb{R}) = \mathcal{B}(\mathbb{R}) \iff A \in \mathcal{B}(\overline{\mathbb{R}}).$$

$\square$

*Remark 801.* In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

**Lemma 802 (Borel subsets of  $\overline{\mathbb{R}}$ ).**

$\mathcal{B}(\overline{\mathbb{R}})$  is generated by any of the following sets of intervals:  $\{[-\infty, b]\}_{b \in \mathbb{R}}$ ,  $\{[a, \infty)\}_{a \in \mathbb{R}}$ .

*Proof.* Direct consequence of Lemma 800 (generators of Borel subsets of  $\overline{\mathbb{R}}$ , with  $\overline{G}$  being any of the four families of rays above), Definition 226 (trace of subset system,  $\overline{G} \cap \{-\infty, \infty\}$  is one of the two singletons  $\{\pm\infty\}$ ), and Lemma 796 (Borel subsets of  $\mathbb{R}$ ,  $\overline{G} \cap \mathbb{R}$  is either  $\{(-\infty, b)\}_{b \in \mathbb{R}}$ , or  $\{[a, \infty)\}_{a \in \mathbb{R}}$ , i.e. a generator of  $\mathcal{B}(\mathbb{R})$ ).  $\square$

### 10.1.4 Borel subset of nonnegative numbers

**Lemma 803 (Borel subsets of  $\mathbb{R}_+$ ).**  $\mathcal{B}(\mathbb{R}_+) = \mathcal{B}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R}_+)$  is generated by any of the following sets of intervals:  $\{[0, b]\}_{b > 0}$ ,  $\{(a, \infty)\}_{a > 0}$ .

*Proof.* Direct consequence of Lemma 796 (Borel subsets of  $\mathbb{R}$ ,  $\mathbb{R}_+ = [0, \infty) \in \mathcal{B}(\mathbb{R})$ ), Lemma 768 (Borel sub- $\sigma$ -algebra, with  $Y \stackrel{\text{def.}}{=} \mathbb{R}_+$ ), and Lemma 765 (generating measurable subspace, with  $Y \stackrel{\text{def.}}{=} \mathbb{R}_+$  and  $G$  a generator of  $\mathcal{B}(\mathbb{R})$ ).  $\square$

**Lemma 804 (Borel subsets of  $\overline{\mathbb{R}}_+$ ).**  $\mathcal{B}(\overline{\mathbb{R}}_+) = \mathcal{B}(\overline{\mathbb{R}}) \cap \mathcal{P}(\overline{\mathbb{R}}_+)$  is generated by any of the following sets of intervals:  $\{[0, b]\}_{b>0}$ ,  $\{[a, \infty]\}_{a>0}$ .

*Proof.* Direct consequence of Lemma 802 (Borel subsets of  $\overline{\mathbb{R}}$ ,  $\overline{\mathbb{R}}_+ = [0, \infty] \in \mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 768 (Borel sub- $\sigma$ -algebra, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}_+$ ), and Lemma 765 (generating measurable subspace, with  $Y$  set to  $\overline{\mathbb{R}}_+$  and  $G$  a generator of  $\mathcal{B}(\overline{\mathbb{R}})$ ).  $\square$

### 10.1.5 Borel subset of complex numbers

*Remark 805.* In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

Of course, there are several other possibly interesting generators...

**Lemma 806 (Borel subsets of  $\mathbb{C}$ ).**

$\mathcal{B}(\mathbb{C})$  can be identified to  $\mathcal{B}(\mathbb{R})^{2\otimes}$ , which is generated by any of the following sets of boxes:

$$(10.1) \quad \{[a, b] \times [c, d]\}_{a<b, c<d}, \quad \{(-\infty, b] \times (-\infty, d]\}_{b, d \in \mathbb{R}}, \quad \{[a, \infty) \times [c, \infty)\}_{a, c \in \mathbb{R}}.$$

*Proof.* Direct consequence of **the canonical isometry**  $((x, y) \mapsto x + iy) : \mathbb{R}^2 \rightarrow \mathbb{C}$ , Lemma 780 (Borel subsets in product measurable subspace, with  $Y_i = X_i \stackrel{\text{def.}}{=} \mathbb{R}$  and  $m \stackrel{\text{def.}}{=} 2$ ), and Lemma 796 (Borel subsets of  $\mathbb{R}$ ).  $\square$

## 10.2 Measurable numeric function

### 10.2.1 Measurable numeric function

*Remark 807.* In the next four lemmas,  $\mathbb{K}$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$  (see Definition 748).

**Lemma 808 (inverse image of countable is measurable).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $f \in \mathcal{M}_{\mathbb{K}}$ . Then, we have

$$(10.2) \quad \forall I \subset \mathbb{N}, \forall (y_i)_{i \in I} \in \mathbb{K}, \quad \{f = (y_i)_{i \in I}\} \stackrel{\text{def.}}{=} f^{-1}\{y_i \mid i \in I\} \in \Sigma.$$

*Proof.* Direct consequence of Lemma 759 ( $\mathcal{M}_Y$  inverse image of countable is measurable, with  $Y \stackrel{\text{def.}}{=} \mathbb{R} = \mathbb{Q}$ ,  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}} = \mathbb{Q}$  and  $Y \stackrel{\text{def.}}{=} \mathbb{C} = \overline{\{a + ib \mid a, b \in \mathbb{Q}\}}$ , all separable).  $\square$

**Lemma 809 (inverse image is measurable).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $f \in \mathcal{M}_{\mathbb{K}}$ . Then, we have

$$(10.3) \quad \forall y \in \mathbb{K}, \quad \{f = y\} \stackrel{\text{def.}}{=} f^{-1}(y) \in \Sigma.$$

*Proof.*

Direct consequence of Lemma 808 (inverse image of countable is measurable, with  $\text{card}(I) = 1$ ).  $\square$

**Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$  countable range).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $f : X \rightarrow \mathbb{K}$ . Assume that  $f(X)$  is countable. Then, we have  $f \in \mathcal{M}_{\mathbb{K}}$  iff for all  $y \in f(X)$ ,  $f^{-1}(y) \in \Sigma$ .

*Proof.* Direct consequence of Lemma 760 ( $\mathcal{M}_Y$  countable range, with  $Y \stackrel{\text{def.}}{=} \mathbb{R} = \mathbb{Q}$ ,  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}} = \mathbb{Q}$  and  $Y \stackrel{\text{def.}}{=} \mathbb{C} = \overline{\{a + ib \mid a, b \in \mathbb{Q}\}}$ , all separable).  $\square$

**Lemma 811 (measurability of indicator function).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $A \subset X$ . Then, we have  $\mathbb{1}_A \in \mathcal{M}_{\mathbb{K}}$  iff  $A \in \Sigma$ .

*Proof.* Direct consequence of Definition 741 (measurable space), Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $\mathcal{P}_{\Sigma} \Rightarrow \mathcal{C}$ ), and Definition 748 (measurability and structure), Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$  countable range, with  $f(X) = \{0_{\mathbb{K}}, 1_{\mathbb{K}}\}$ ,  $f^{-1}(0_{\mathbb{K}}) = A^c \in \Sigma$ ,  $f^{-1}(1_{\mathbb{K}}) = A \in \Sigma$ ).  $\square$

*Remark 812.* In the next two lemmas, the measurable valued field  $\mathbb{K}$  is  $\mathbb{R}$ , or  $\mathbb{C}$  (see Definition 748, thus  $\overline{\mathbb{R}}$  is not involved here).

Note also that “(sub)algebra” is the algebraic structure.

**Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable valued field. Then,  $\mathcal{M}_{\mathbb{K}}$  is a subalgebra of  $\mathbb{K}^X$ , i.e. an algebra over  $\mathbb{K}$  with the  $0_{\mathbb{K}}$ -function as zero.

*Proof.* Let  $f, g \in \mathcal{M}_{\mathbb{K}}$ . Then, from Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ , with  $Y \stackrel{\text{def.}}{=} \mathbb{K}$ ), Lemma 776 (measurability of function to product space), and Lemma 780 (Borel subsets in product measurable subspace, with  $Y_i = X_i \stackrel{\text{def.}}{=} \mathbb{K}$  and  $m \stackrel{\text{def.}}{=} 2$ ), the function  $(f, g) : X \rightarrow \mathbb{K}^2$  is measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{K}^2)$ . Moreover, from **continuity of addition and multiplication from  $\mathbb{K}^2$  to  $\mathbb{K}$** , Lemma 761 (continuous is measurable), and Lemma 757 (compatibility of measurability with composition),  $f + g$  and  $fg$  are measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{K})$ .

Let  $f : X \rightarrow \mathbb{K}$ . Let  $a \in \mathbb{K}$ . Then, from Lemma 754 (constant function is measurable), and Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ , with  $Y \stackrel{\text{def.}}{=} \mathbb{K}$ ), we have  $a \in \mathcal{M}_{\mathbb{K}}$ . Thus, from the previous result, we have  $af \in \mathcal{M}_{\mathbb{K}}$ .

Therefore, from Lemma 279 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$  or  $\mathbb{C}$ ), Lemma 282 (algebra of functions to algebra), and Lemma 287 (closed under algebra operations is subalgebra),  $\mathcal{M}_{\mathbb{K}}$  is a subalgebra of  $(\mathbb{K}^X, +_{\mathbb{K}}, \cdot_{\mathbb{K}}, *_{\mathbb{K}})$ .  $\square$

**Lemma 814** ( $\mathcal{M}_{\mathbb{K}}$  is vector space).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable valued field.

Then,  $\mathcal{M}_{\mathbb{K}}$  is a vector subspace of  $\mathbb{K}^X$ , i.e. a vector space over  $\mathbb{K}$  with the  $0_{\mathbb{K}}$ -function as zero.

*Proof.* Direct consequence of Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra), Definition 284 (subalgebra), Definition 277 (algebra over a field,  $\mathcal{M}_{\mathbb{K}}$  is a vector space), and Definition 77 (subspace).  $\square$

**10.2.2 Measurable numeric function to  $\mathbb{R}$** 

**Lemma 815** ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ). Let  $(X, \Sigma)$  be a measurable space. The set  $\mathcal{M}_{\mathbb{R}}(X, \Sigma)$  (or simply  $\mathcal{M}_{\mathbb{R}}$ ) of measurable functions from  $X$  to  $\mathbb{R}$  is a vector space; it is called the vector space of finite-valued measurable functions (over  $X$ ).

*Proof.* Direct consequence of Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and **valued field properties of  $\mathbb{R}$** .  $\square$

**Lemma 816** (measurability of numeric function to  $\mathbb{R}$ ). Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then,  $f \in \mathcal{M}_{\mathbb{R}}$  iff one of the following conditions is satisfied:

$$(10.4) \quad \forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a) \in \Sigma,$$

$$(10.5) \quad \forall a \in \mathbb{R}, \quad \{f \leq a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a] \in \Sigma,$$

$$(10.6) \quad \forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty) \in \Sigma,$$

$$(10.7) \quad \forall a \in \mathbb{R}, \quad \{f \geq a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty) \in \Sigma.$$

*Proof.* **“Left” implies “right”.** Direct consequence of Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 749 (measurable function), Definition 743 (Borel  $\sigma$ -algebra), Lemma 794 (some Borel subsets of numbers), **open intervals of  $\mathbb{R}$  are open subsets**, and **closed intervals of  $\mathbb{R}$ , are closed subsets**.

**“Right” implies “left”.** Direct consequence of Lemma 796 (Borel subsets of  $\mathbb{R}$ ), Lemma 756 (equivalent definition of measurable function), and Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ).

Therefore, we have the equivalence.  $\square$

**10.2.3 Measurable numeric function to  $\overline{\mathbb{R}}$** **Definition 817** ( $\mathcal{M}$ , set of measurable numeric functions).

Let  $(X, \Sigma)$  be a measurable space. The set  $\mathcal{M}_{\overline{\mathbb{R}}}(X, \Sigma)$  of measurable functions from  $X$  to  $\overline{\mathbb{R}}$  is called the set of measurable functions (over  $X$ ); it is simply denoted  $\mathcal{M}(X, \Sigma)$  (or  $\mathcal{M}$ ).

*Remark 818.* We use the convention of identifying real-valued functions and their extension as functions  $X \rightarrow \overline{\mathbb{R}}$ . This allows us to consider  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$  and  $\mathbb{R}^X \subset \overline{\mathbb{R}}^X$  in the next lemma.

**Lemma 819** ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ).

Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathcal{M}_{\mathbb{R}} = \mathcal{M} \cap \mathbb{R}^X$ .

*Proof.*  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M} \cap \mathbb{R}^X$ . Let  $f \in \mathcal{M}_{\mathbb{R}}$ . Let  $A \in \mathcal{B}(\overline{\mathbb{R}})$ . Then, from Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ , with  $Y \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $\overline{\mathbb{R}} = \mathbb{R} \uplus \{\pm\infty\}$ ), **compatibility of inverse image with union**, Lemma 798 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ,  $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ ), and Definition 749 (measurable function), we have  $f \in \mathbb{R}^X$ , and

$$f^{-1}(A) = f^{-1}(A \cap \mathbb{R}) \uplus f^{-1}(A \cap \{\pm\infty\}) = f^{-1}(A \cap \mathbb{R}) \in \Sigma.$$

Hence, from Definition 749 (*measurable function*), Definition 758 ( $\mathcal{M}_Y$ , *set of measurable functions to  $Y$* , with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ), and Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*),  $f \in \mathcal{M} \cap \mathbb{R}^X$ .

$\mathcal{M} \cap \mathbb{R}^X \subset \mathcal{M}_{\mathbb{R}}$ . Let  $f \in \mathcal{M} \cap \mathbb{R}^X$ . Let  $A \in \mathcal{B}(\mathbb{R})$ . Then, from Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*), Definition 758 ( $\mathcal{M}_Y$ , *set of measurable functions to  $Y$* , with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ), Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ,  $A \cap \mathbb{R} = A \in \mathcal{B}(\mathbb{R})$ ), and Definition 749 (*measurable function*), we have  $A \in \mathcal{B}(\mathbb{R})$ , and  $f^{-1}(A) \in \Sigma$ . Hence, from Definition 749 (*measurable function*), Definition 758 ( $\mathcal{M}_Y$ , *set of measurable functions to  $Y$* , with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ), and Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ), we have  $f \in \mathcal{M}_{\mathbb{R}}$ .*

Therefore, we have the equality.  $\square$

**Lemma 820 (measurability of numeric function).** *Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f \in \mathcal{M}$  iff one of the following conditions is satisfied:*

$$(10.8) \quad \forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a) \in \Sigma,$$

$$(10.9) \quad \forall a \in \mathbb{R}, \quad \{f \leq a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a] \in \Sigma,$$

$$(10.10) \quad \forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty] \in \Sigma,$$

$$(10.11) \quad \forall a \in \mathbb{R}, \quad \{f \geq a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty] \in \Sigma.$$

*Proof. “Left” implies “right”.* Direct consequence of Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*), Definition 749 (*measurable function*), Definition 743 (*Borel  $\sigma$ -algebra*), Lemma 794 (*some Borel subsets of numbers, with  $\mathbb{K} \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$* ), Lemma 372 (*topology on  $\overline{\mathbb{R}}$ , open intervals are open subsets and closed intervals are closed subsets*).

*“Right” implies “left”.* Direct consequence of Lemma 802 (*Borel subsets of  $\overline{\mathbb{R}}$* ), Lemma 756 (*equivalent definition of measurable function*), and Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*).

Therefore, we have the equivalence.  $\square$

**Lemma 821 ( $\mathcal{M}$  is closed under finite part).** *Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Let  $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then,  $A \in \Sigma$  and (the finite part of  $f$ )  $f \mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}(\subset \mathcal{M})$ .*

*Proof.* From Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), we have  $\mathbb{R} = \{\pm\infty\}^c$ . Let

$$A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}), \quad \text{and} \quad B \stackrel{\text{def.}}{=} f^{-1}(\{\pm\infty\}).$$

Then, from Lemma 808 (*inverse image of countable is measurable, with  $\text{card}(I) = 2$* ), **compatibility of inverse image with complement**, **involutiveness of complement**, Lemma 696 ( *$\sigma$ -algebra, closedness under complement*), and Definition 209 (*pseudopartition*), we have  $A = B^c$ ,  $A, B \in \Sigma$ , and  $(A, B)$  forms a pseudopartition of  $X$ .

From Definition 526 (*finite part*), let  $\hat{f} \stackrel{\text{def.}}{=} f \mathbb{1}_A$  be the finite part of  $f$ . Therefore, from Lemma 527 (*finite part is finite*), **the definition of the indicator function**, Lemma 754 (*constant function is measurable,  $0 \in \mathcal{M}$* ), Lemma 772 (*measurability of function defined on a pseudopartition, with  $(X', \Sigma') \stackrel{\text{def.}}{=} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$* ), and the pseudopartition  $X = A \uplus B$ , and Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ), we have  $\hat{f} \in \mathcal{M} \cap \mathbb{R}^X = \mathcal{M}_{\mathbb{R}}$ .  $\square$

**Lemma 822 ( $\mathcal{M}$  is closed under addition when defined).**

*Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$ . Assume that  $f$  and  $g$  never take opposite infinite values at the same point. Then,  $f + g$  is well-defined, and belongs to  $\mathcal{M}$ .*



*Proof.* By hypothesis, the subsets  $\{f = \infty\} \cap \{g = -\infty\}$  and  $\{f = -\infty\} \cap \{g = \infty\}$  are empty. Thus, from Definition 209 (*pseudopartition*), the whole space reduces to the pseudopartition

$$X = X_{\pm\infty}^c \uplus X_{\infty} \uplus X_{-\infty}$$

(we recall that the notation  $\uplus$  is for disjoint union) where

$$\begin{aligned} X_{\pm\infty}^c &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}), \\ X_{\infty} &\stackrel{\text{def.}}{=} \{f = \infty\} \cup \{g = \infty\}, \\ X_{-\infty} &\stackrel{\text{def.}}{=} \{f = -\infty\} \cup \{g = -\infty\}. \end{aligned}$$

Then, from Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*), Definition 749 (*measurable function*, for  $f$  and  $g$ ), Definition 743 (*Borel  $\sigma$ -algebra*,  $\mathbb{R}$  open in  $\overline{\mathbb{R}}$  belongs to  $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable union and intersection), and Lemma 809 (*inverse image is measurable*, with  $y \stackrel{\text{def.}}{=} \pm\infty$ ), we have  $X_{\pm\infty}^c, X_{\infty}, X_{-\infty} \in \Sigma$ .

Let  $\tilde{f}$  and  $\tilde{g}$  be the finite parts of  $f$  and  $g$ . Then, from Definition 526 (*finite part*), and Definition 317 (*addition in  $\overline{\mathbb{R}}$* ), the sum  $f + g$  can be defined on the pseudopartition by

$$f + g \stackrel{\text{def.}}{=} \begin{cases} \tilde{f} + \tilde{g} & \text{on } X_{\pm\infty}^c, \\ \infty & \text{on } X_{\infty}, \\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 821 ( $\mathcal{M}$  *is closed under finite part*, with  $f$  and  $g$ ), Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  *is algebra*, sum with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Lemma 754 (*constant function is measurable*), and Lemma 772 (*measurability of function defined on a pseudopartition*), we have  $f + g \in \mathcal{M}$ .  $\square$

**Lemma 823** ( $\mathcal{M}$  *is closed under finite sum when defined*).

Let  $(X, \Sigma)$  be a measurable space. Let  $I$  be a finite set. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Assume that the  $f_i$ 's never take opposite infinite values at the same point, i.e. for all  $i, j \in I$ ,  $i \neq j$  implies  $\{f_i = \infty\} \cap \{f_j = -\infty\} = \emptyset$ . Then,  $\sum_{i \in I} f_i$  is well-defined, and belongs to  $\mathcal{M}$ .

*Proof.* Direct consequence of Lemma 822 ( $\mathcal{M}$  *is closed under addition when defined*), and induction on the cardinality of  $I$ .  $\square$

**Lemma 824** ( $\mathcal{M}$  *is closed under multiplication*).

Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$ . Then,  $fg$  is well-defined, and belongs to  $\mathcal{M}$ .

*Proof.* From Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*), and Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ), the whole space can be pseudopartitioned into  $X = X_{\pm\infty}^c \uplus X_0 \uplus X_{\infty} \uplus X_{-\infty}$  (the notation  $\uplus$  is for disjoint union) where

$$\begin{aligned} X_{\pm\infty}^c &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}), \\ X_0 &\stackrel{\text{def.}}{=} (\{|f| = \infty\} \cap \{g = 0\}) \cup (\{f = 0\} \cap \{|g| = \infty\}), \\ X_{\infty} &\stackrel{\text{def.}}{=} (\{f = \infty\} \cap \{g > 0\}) \cup (\{f = -\infty\} \cap \{g < 0\}) \\ &\quad \cup (\{f > 0\} \cap \{g = \infty\}) \cup (\{f < 0\} \cap \{g = -\infty\}), \\ X_{-\infty} &\stackrel{\text{def.}}{=} (\{f = -\infty\} \cap \{g > 0\}) \cup (\{f = \infty\} \cap \{g < 0\}) \\ &\quad \cup (\{f > 0\} \cap \{g = -\infty\}) \cup (\{f < 0\} \cap \{g = \infty\}). \end{aligned}$$

Then, from Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*), Definition 749 (*measurable function*, for  $f$  and  $g$ ), Definition 743 (*Borel  $\sigma$ -algebra*,  $\mathbb{R}$  open in  $\overline{\mathbb{R}}$  belongs to  $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 809 (*inverse image is measurable*, with  $y \stackrel{\text{def.}}{=} \pm\infty, 0$ ) Lemma 820 (*measurability of numeric function*,

with  $a \stackrel{\text{def.}}{=} 0$ ), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection*), we have  $X_{\pm\infty}^c, X_0, X_\infty, X_{-\infty} \in \Sigma$ .

Let  $\tilde{f}$  and  $\tilde{g}$  be the finite parts of  $f$  and  $g$ . Then, from Definition 526 (*finite part*), Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ), and Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*), the product  $fg$  can be defined on the pseudopartition by

$$fg \stackrel{\text{def.}}{=} \begin{cases} \tilde{f}\tilde{g} & \text{on } X_{\pm\infty}^c, \\ 0 & \text{on } X_0, \\ \infty & \text{on } X_\infty, \\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 821 ( *$\mathcal{M}$  is closed under finite part, with  $f$  and  $g$* ), Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra, product with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$* ), Lemma 754 (*constant function is measurable*), and Lemma 772 (*measurability of function defined on a pseudopartition*), we have  $fg \in \mathcal{M}$ .  $\square$

**Lemma 825 ( $\mathcal{M}$  is closed under finite product).**

Let  $(X, \Sigma)$  be a measurable space. Let  $I$  be a finite set. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then,  $\prod_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*), and induction on the cardinality of  $I$ .  $\square$

**Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication).**

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in \overline{\mathbb{R}}$ . Let  $f \in \mathcal{M}$ . Then, we have  $af \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*), and Lemma 754 (*constant function is measurable*).  $\square$

*Remark 827.* Note that in the next two lemmas, the index set  $I$  could also be the empty set, provided the use of the convention  $\inf \emptyset \stackrel{\text{def.}}{=} \infty$  and  $\sup \emptyset \stackrel{\text{def.}}{=} -\infty$ , and Lemma 754 (*constant function is measurable*).

**Lemma 828 ( $\mathcal{M}$  is closed under infimum).**

Let  $(X, \Sigma)$  be a measurable space.

Let  $I \subset \mathbb{N}$  be nonempty. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then, we have  $\inf_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, from Definition 9 (*infimum*), Lemma 820 (*measurability of numeric function, with  $f_i$* ), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), we have

$$\left\{ \inf_{i \in I} f_i \geq a \right\} = \bigcap_{i \in I} \{f_i \geq a\} \in \Sigma.$$

Therefore, from Lemma 820 (*measurability of numeric function, with  $\inf_{i \in I} f_i$* ),  $\inf_{i \in I} f_i$  belongs to  $\mathcal{M}$ .  $\square$

**Lemma 829 ( $\mathcal{M}$  is closed under supremum).**

Let  $(X, \Sigma)$  be a measurable space.

Let  $I \subset \mathbb{N}$  be nonempty. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then, we have  $\sup_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, from Definition 2 (*supremum*), Lemma 820 (*measurability of numeric function, with all the  $f_i$ 's*), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*),

$$\left\{ \sup_{i \in I} f_i \leq a \right\} = \bigcap_{i \in I} \{f_i \leq a\} \in \Sigma.$$

Therefore, from Lemma 820 (*measurability of numeric function, with  $\sup_{i \in I} f_i$* ),  $\sup_{i \in I} f_i$  belongs to  $\mathcal{M}$ .  $\square$

**Lemma 830 ( $\mathcal{M}$  is closed under limit inferior).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Then, we have  $\liminf_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 828 ( $\mathcal{M}$  is closed under infimum,  $F_n^- \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p} \in \mathcal{M}$ ), Lemma 829 ( $\mathcal{M}$  is closed under supremum,  $f \stackrel{\text{def.}}{=} \sup_{n \in \mathbb{N}} F_n^- \in \mathcal{M}$ ), and Lemma 504 (*limit inferior*,  $\liminf_{n \rightarrow \infty} f_n = f$ ).  $\square$

**Lemma 831 ( $\mathcal{M}$  is closed under limit superior).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Then, we have  $\limsup_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 829 ( $\mathcal{M}$  is closed under supremum,  $F_n^+ \stackrel{\text{def.}}{=} \sup_{p \in \mathbb{N}} f_{n+p}$  belongs to  $\mathcal{M}$ ), Lemma 828 ( $\mathcal{M}$  is closed under infimum,  $\bar{f} \stackrel{\text{def.}}{=} \inf_{n \in \mathbb{N}} F_n^+ \in \mathcal{M}$ ), and Lemma 509 (*limit superior*,  $\limsup_{n \rightarrow \infty} f_n = \bar{f}$ ).  $\square$

**Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Assume that  $(f_n(x))_{n \in \mathbb{N}}$  is pointwise convergent in  $\bar{\mathbb{R}}$ . Then, we have  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 522 (*limit inferior, limit superior and pointwise convergence*), Lemma 830 ( $\mathcal{M}$  is closed under limit inferior), or Lemma 831 ( $\mathcal{M}$  is closed under limit superior).  $\square$

**Lemma 833 (measurability and masking).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then,  $f \in \mathcal{M}$  iff for all  $A \in \Sigma$ ,  $f \mathbb{1}_A \in \mathcal{M}$ .

*Proof.* “Left” implies “right”. Direct consequence of Lemma 811 (*measurability of indicator function*), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), and Lemma 824 ( $\mathcal{M}$  is closed under multiplication).

“Right” implies “left”. Direct consequence of Definition 741 (*measurable space*,  $\Sigma$  is a  $\sigma$ -algebra), Lemma 696 ( $\sigma$ -algebra,  $X \in \Sigma$ ), **the definition of the indicator function** ( $\mathbb{1}_X \equiv \mathbb{1}$ ), and Definition 326 (*multiplication in  $\bar{\mathbb{R}}$* ,  $\mathbb{1}$  is unity).

Therefore, we have the equivalence.  $\square$

**Lemma 834 (measurability of restriction).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \bar{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \bar{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f} \mathbb{1}_A \in \mathcal{M}(X, \Sigma)$ .

*Proof.* Let  $B' \in \mathcal{B}(\bar{\mathbb{R}})$ . **Case  $0 \notin B'$ .** Then, from **the definition of the indicator function**, we have  $(\hat{f} \mathbb{1}_A)^{-1}(B') = f^{-1}(B') \cap A$ . **Case  $0 \in B'$ .** Then, from **compatibility of inverse image with disjoint union**, **the definition of the indicator function**, and **associativity of disjoint union**, we have

$$\begin{aligned} (\hat{f} \mathbb{1}_A)^{-1}(B') &= (\hat{f} \mathbb{1}_A)^{-1}(B' \setminus \{0\}) \uplus (\hat{f} \mathbb{1}_A)^{-1}(0) \\ &= (f^{-1}(B' \setminus \{0\}) \cap A) \uplus ((f^{-1}(0) \cap A) \uplus A^c) \\ &= (f^{-1}(B') \cap A) \uplus A^c. \end{aligned}$$

Moreover, from **the definition of restriction of function**, we have  $f|_A^{-1}(B') = f^{-1}(B') \cap A$ . Hence, we have  $(\hat{f} \mathbb{1}_A)^{-1}(B') = f|_A^{-1}(B') \uplus C$  with  $C \in \{\emptyset, A^c\}$ .

Let  $B \subset A$ . Assume first that  $B \in \Sigma$ . Then, from Definition 741 (*measurable space*,  $\Sigma$  is a  $\sigma$ -algebra), and Lemma 696 ( $\sigma$ -algebra, closedness under complement and countable union), we have  $B \uplus A^c \in \Sigma$ . Conversely, assume now that  $B \uplus A^c \in \Sigma$ . Then, from Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection), we have  $(B \uplus A^c) \cap A = B \in \Sigma$ . Hence, we have the equivalence  $B \in \Sigma$  iff  $B \uplus A^c \in \Sigma$ .

Therefore, from Definition 749 (*measurable function*), and Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*), we have the equivalence  $f|_A \in \mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f} \mathbb{1}_A \in \mathcal{M}(X, \Sigma)$ .  $\square$

### 10.2.4 Nonnegative measurable numeric function

**Definition 835** ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).

Let  $(X, \Sigma)$  be a measurable space. The subset of nonnegative measurable functions (over  $X$ ) is denoted  $\mathcal{M}_+(X, \Sigma)$  (or simply  $\mathcal{M}_+$ ); it is defined by  $\mathcal{M}_+(X, \Sigma) \stackrel{\text{def.}}{=} \{f \in \mathcal{M} \mid f(X) \subset \overline{\mathbb{R}}_+\}$ .

**Lemma 836** (measurability of nonnegative and nonpositive parts).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f \in \mathcal{M}$  iff  $f^+, f^- \in \mathcal{M}_+$ .

*Proof.* “Left” implies “right”. Direct consequence of Definition 528 (nonnegative and nonpositive parts), Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication,  $-f \in \mathcal{M}$ ), Lemma 754 (constant function is measurable,  $0 \in \mathcal{M}$ ), Lemma 829 ( $\mathcal{M}$  is closed under supremum, maximum is supremum), Lemma 530 (nonnegative and nonpositive parts are nonnegative), and Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).

“Right” implies “left”. Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $\mathcal{M}_+ \subset \mathcal{M}$ ), Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication,  $-f^- \in \mathcal{M}$ ), Lemma 532 (decomposition into nonnegative and nonpositive parts,  $f^+ - f^-$  is well-defined), and Lemma 822 ( $\mathcal{M}$  is closed under addition when defined).

Therefore, we have the equivalence.  $\square$

**Lemma 837** ( $\mathcal{M}_+$  is closed under masking).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ . Then,  $f \in \mathcal{M}_+$  iff for all  $A \in \Sigma$ ,  $f \mathbf{1}_A \in \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions, thus  $\mathcal{M}_+ = \mathcal{M} \cap \overline{\mathbb{R}}_+^X$ ), Lemma 833 (measurability and masking), and the definition of the indicator function (thus  $f \geq 0$  iff for all  $A \in \Sigma$ ,  $f \mathbf{1}_A \geq 0$ ).  $\square$

**Lemma 838** ( $\mathcal{M}_+$  is closed under finite part).

Let  $(X, \Sigma)$  be a measurable space.

Let  $f \in \mathcal{M}_+$ . Let  $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then,  $A \in \Sigma$  and (the finite part of  $f$ )  $f \mathbf{1}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 821 ( $\mathcal{M}$  is closed under finite part), nonnegativeness of the indicator function, and closedness of multiplication in  $\mathbb{R}_+$ .  $\square$

**Lemma 839** ( $\mathcal{M}$  is closed under nonnegative finite part).

Let  $(X, \Sigma)$  be a measurable

space. Let  $f \in \mathcal{M}$ . Let  $A \in \Sigma$ . Let  $A_f \stackrel{\text{def.}}{=} A \cap f^{-1}(\mathbb{R}_+)$ . Let  $\tilde{f}_A \stackrel{\text{def.}}{=} f \mathbf{1}_{A_f}$ .

Then,  $A_f \in \Sigma$  and  $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Let  $B \stackrel{\text{def.}}{=} f^{-1}(\overline{\mathbb{R}}_+)$  and  $C \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then, from Lemma 820 (measurability of numeric function, with  $\overline{\mathbb{R}}_+ = [0, \infty]$ ), and Lemma 838 ( $\mathcal{M}_+$  is closed under finite part, thus  $C \in \Sigma$ ), Definition 741 (measurable space,  $\Sigma$  is a  $\sigma$ -algebra), Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under intersection), compatibility of inverse image with intersection, the definition of  $A_f$ , and properties of the indicator function, we have

$$B, C \in \Sigma, \quad f^{-1}(\mathbb{R}_+) = B \cap C \in \Sigma, \quad A_f = A \cap B \cap C \in \Sigma, \quad \text{and} \quad \mathbf{1}_{A_f} = \mathbf{1}_A \mathbf{1}_B \mathbf{1}_C.$$

Hence, from Lemma 529 (equivalent definition of nonnegative and nonpositive parts), and Definition 526 (finite part),  $f_A$  is the finite part of  $(f \mathbf{1}_A)^+$ . Therefore, from Lemma 833 (measurability and masking,  $f \mathbf{1}_A \in \mathcal{M}$ ), Lemma 836 (measurability of nonnegative and nonpositive parts, with  $f \mathbf{1}_A$ ), and Lemma 838 ( $\mathcal{M}_+$  is closed under finite part, with  $(f \mathbf{1}_A)^+$ ), we have  $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .  $\square$

**Lemma 840 (power function in  $\overline{\mathbb{R}}$  is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Let  $p \in \overline{\mathbb{R}}_+$ . Then, we have  $(x \mapsto |x|^p) \in \mathcal{M}_+(\overline{\mathbb{R}}) \subset \mathcal{M}(\overline{\mathbb{R}})$ .

*Proof.* Let  $\varphi_p \stackrel{\text{def.}}{=} (x \in \overline{\mathbb{R}} \mapsto |x|^p)$ .

**Case  $p \in [0, \infty)$ .** Direct consequence of Lemma 449 (power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)), and Lemma 761 (continuous is measurable).

**Case  $p = \infty$ .** Direct consequence of Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ), Lemma 433 (values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory), thus  $\varphi_\infty(\overline{\mathbb{R}}) = \{0, 1, \infty\}$ , and  $\varphi_\infty^{-1}(\{0\}) = (-1, 1)$ ,  $\varphi_\infty^{-1}(\{1\}) = \{\pm 1\}$ , and  $\varphi_\infty^{-1}(\{\infty\}) = \overline{\mathbb{R}} \setminus [-1, 1]$ ), Lemma 796 (Borel subsets of  $\mathbb{R}$ ), Lemma 798 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ , thus all previous preimages belong to  $\mathcal{B}(\overline{\mathbb{R}})$ ), and Lemma 753 (countable range is measurable).

Therefore, from Lemma 439 (power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory), thus  $\varphi_p \geq 0$ ), and Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), we always have  $\varphi_p \in \mathcal{M}_+(\overline{\mathbb{R}}) \subset \mathcal{M}(\overline{\mathbb{R}})$ .  $\square$

**Lemma 841 ( $\mathcal{M}$  is closed under power function).** Let  $(X, \Sigma)$  be a measurable space.

Let  $p \in \overline{\mathbb{R}}_+$ . Let  $f \in \mathcal{M}$ , and  $g \in \mathcal{M}_+$ . Then,  $|f|^p, g^p \in \mathcal{M}_+ \subset \mathcal{M}$ .

Moreover, if  $p$  is finite and  $f \in \mathcal{M}_{\mathbb{R}}$ , then we have  $|f|^p \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions, thus  $g \geq 0$ ), Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative, thus  $|g| = g$ ), Lemma 840 (power function in  $\overline{\mathbb{R}}$  is measurable), Lemma 757 (compatibility of measurability with composition), Definition 817 ( $\mathcal{M}$ , set of measurable numeric functions,  $|f|^p, g^p \in \mathcal{M}$ ), Lemma 439 (power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)), Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $|f|^p, g^p \in \mathcal{M}_+$ ), **closedness of the power function of nonnegative degree in  $\mathbb{R}$  (for  $p \in (0, \infty)$ )**, Lemma 431 (one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory), the power function of degree 0 is finite), and Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ,  $|f|^p \in \mathcal{M}_{\mathbb{R}}$ ).  $\square$

**Lemma 842 ( $\mathcal{M}$  is closed under absolute value).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then, we have  $|f| \in \mathcal{M}_+ \subset \mathcal{M}$ .

Moreover, if  $f \in \mathcal{M}_{\mathbb{R}}$ , then we have  $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $p \stackrel{\text{def.}}{=} 1$ ), and Lemma 362 (power function in  $\overline{\mathbb{R}}$  is absolute value).  $\square$

**Lemma 843 ( $\mathcal{M}_+$  is closed under addition).** Let  $(X, \Sigma)$  be a measurable space.

Let  $f, g \in \mathcal{M}_+$ . Then,  $f + g$  is well-defined, and belongs to  $\mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 822 ( $\mathcal{M}$  is closed under addition when defined,  $f$  and  $g$  cannot take opposite infinite values), and Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed).  $\square$

**Lemma 844 ( $\mathcal{M}_+$  is closed under multiplication).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}_+$ . Then, we have  $fg \in \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 824 ( $\mathcal{M}$  is closed under multiplication), and Lemma 425 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)).  $\square$

**Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication).**

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in \overline{\mathbb{R}}_+$ . Let  $f \in \mathcal{M}_+$ . Then, we have  $af \in \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication), and Lemma 425 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)).  $\square$

*Remark 846.* Note that in the next two lemmas, the index set  $I$  could also be the empty set, provided the use of the convention  $\inf \emptyset \stackrel{\text{def.}}{=} \infty$  and  $\sup \emptyset \stackrel{\text{def.}}{=} 0$ , and Lemma 754 (constant function is measurable).

**Lemma 847 ( $\mathcal{M}_+$  is closed under infimum).** *Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$  be nonempty. Let  $(f_i)_{i \in I} \in \mathcal{M}_+$ . Then, we have  $\inf_{i \in I} f_i \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Lemma 828 ( $\mathcal{M}$  is closed under infimum), and Lemma 492 (infimum of bounded is bounded, with  $X \stackrel{\text{def.}}{=} I$  and  $a \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Lemma 848 ( $\mathcal{M}_+$  is closed under supremum).** *Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$  be nonempty. Let  $(f_i)_{i \in I} \in \mathcal{M}_+$ . Then, we have  $\sup_{i \in I} f_i \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Lemma 829 ( $\mathcal{M}$  is closed under supremum), and Lemma 493 (supremum of bounded is bounded, with  $X \stackrel{\text{def.}}{=} I$  and  $a \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent).** *Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Assume that  $(f_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}_+$ . Then, we have  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent), and Lemma 392 ( $\overline{\mathbb{R}}_+$  is complete,  $\cdot$ ).  $\square$

**Lemma 850 ( $\mathcal{M}_+$  is closed under countable sum).** *Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$ . Let  $(f_i)_{i \in I} \in \mathcal{M}_+$ . Then,  $\sum_{i \in I} f_i$  is well-defined, and belongs to  $\mathcal{M}_+$ .*

*Proof.* Direct consequence of Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ), Lemma 843 ( $\mathcal{M}_+$  is closed under addition), induction on  $n \in \mathbb{N}$  (with  $g_n \stackrel{\text{def.}}{=} \sum_{i \in I \cap [0..n]} f_i \in \mathcal{M}_+$ , with the convention that a sum indexed by  $\emptyset$  is the identity element 0), and Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent, with  $f_n \stackrel{\text{def.}}{=} g_n$ ).  $\square$

## 10.2.5 Tensor product of measurable numeric functions

**Definition 851 (tensor product of numeric functions).** Let  $\mathbb{K}$  be a set equipped with a multiplication. Let  $m \in [2.. \infty)$ . For all  $i \in [1..m]$ , let  $X_i$  be a set, and  $f_i : X_i \rightarrow \mathbb{K}$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . The *tensor product of  $(f_i)_{i \in [1..m]}$*  is the function  $\otimes_{i \in [1..m]} f_i : X \rightarrow \mathbb{K}$  defined by

$$(10.12) \quad \forall x \stackrel{\text{def.}}{=} (x_i)_{i \in [1..m]} \in X, \quad \left( \otimes_{i \in [1..m]} f_i \right) (x) \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} f_i(x_i).$$

*Remark 852.* In the next lemma, the measurable number space  $\mathbb{K}$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$  (see Definition 748).

**Lemma 853 (measurability of tensor product of numeric functions).** *Let  $\mathbb{K}$  be a measurable number space. Let  $m \in [2.. \infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and  $f_i \in \mathcal{M}_{\mathbb{K}}(X_i, \Sigma_i)$ . Then, we have  $\otimes_{i \in [1..m]} f_i \in \mathcal{M}_{\mathbb{K}} \left( \prod_{i \in [1..m]} X_i, \otimes_{i \in [1..m]} \Sigma_i \right)$ .*

*Proof.* Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$  and  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ . Let  $i \in [1..m]$ . Let  $\pi_i$  be the canonical projection from  $X$  onto  $X_i$ . Then, from Definition 748 (*measurability and structure*), Definition 758 ( $\mathcal{M}_Y$ , *set of measurable functions to  $Y$ , with  $Y \stackrel{\text{def.}}{=} \mathbb{K}$* ), Lemma 777 (*canonical projection is measurable*), and Lemma 757 (*compatibility of measurability with composition*),  $f_i \circ \pi_i$  belongs to  $\mathcal{M}_{\mathbb{K}}(X, \Sigma)$ . Therefore, from Definition 851 (*tensor product of numeric functions*), Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  *is algebra*, *multiplicativity*, when  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$  or  $\mathbb{C}$ ), and Lemma 825 ( $\mathcal{M}$  *is closed under finite product*, with  $I \stackrel{\text{def.}}{=} [1..m]$ , when  $\mathbb{K} \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ),

$$\bigotimes_{i \in [1..m]} f_i = \prod_{i \in [1..m]} (f_i \circ \pi_i) \in \mathcal{M}_{\mathbb{K}}(X, \Sigma).$$

□



## Chapter 11

# Numeric function over subset system

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## 11.1 Basic properties

*Remark 854.* Extension of the mini-language of Section 8.1 to represent properties of numeric functions over subset systems under a compact form. None of the following new keywords contain an uppercase letter ( $A$ , below, is an argument to continuity).

$pos, hom, diff, fin$  are for nonnegativeness, homogeneity, diffuseness, and finiteness;

$add, \overline{add}, add$  are for subadditivity, superadditivity, and additivity;

$sub, monot$  are for subtractivity, and monotonicity;

$\sigma add, \sigma \overline{add}, \sigma add, \sigma fin$  are for  $\sigma$ -subadditivity,  $\sigma$ -superadditivity,  $\sigma$ -additivity, and  $\sigma$ -finiteness;

$cont\uparrow(A), cont\downarrow(A)$  are for continuity from below (at  $A$ ), and from above (at  $A$ ).

When there is only one subset system and/or only one numeric function over a subset system in the context, it is not necessary to specify to which one the properties apply. For instance, the compact formula of Lemma 867 should be read as  $\mathcal{S}$  is  $Ptfd'$  and  $\mu$  is both  $pos$  and  $add$  implies that  $\mu$  is nondecreasing.

**Definition 855 (homogeneity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *homogeneous* iff  $\emptyset \in \mathcal{S}$  implies  $\mu(\emptyset) = 0$ .

**Definition 856 (diffuseness).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains all singletons of  $X$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *diffuse* iff for all  $x \in X$ ,  $\mu(\{x\}) = 0$ .

**Definition 857 (finiteness).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *finite* iff  $X \in \mathcal{S}$  and  $\mu(X) \in \mathbb{R}$ .

**Lemma 858 (finite is bounded).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, nondecreasing, and finite. Then,  $\mu$  is bounded.

*Compact form:*  $Hyp \stackrel{\text{def.}}{=} wF \wedge pos \wedge monot \wedge fin$ .

*Proof.* Direct consequence of Definition 857 (*finiteness*), and **the definition of nondecreasingness** ( $\emptyset \subset A \subset X \Rightarrow 0 \leq \mu(\emptyset) \leq \mu(A) \leq \mu(X) < \infty$ ).  $\square$

### 11.1.1 Finite properties

**Definition 859 (subadditivity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *subadditive* iff for all  $n \in \mathbb{N}$ , for all  $(A_i)_{i \in [0..n]} \in \mathcal{S}$ ,

$$(11.1) \quad \bigcup_{i \in [0..n]} A_i \in \mathcal{S} \implies \mu \left( \bigcup_{i \in [0..n]} A_i \right) \leq \sum_{i \in [0..n]} \mu(A_i).$$

*Remark 860.* We recall that  $\uplus$  denotes disjoint union.

**Definition 861 (superadditivity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *superadditive* iff for all  $n \in \mathbb{N}$ , for all  $(A_i)_{i \in [0..n]} \in \mathcal{S}$  pairwise disjoint,

$$(11.2) \quad \biguplus_{i \in [0..n]} A_i \in \mathcal{S} \implies \mu \left( \biguplus_{i \in [0..n]} A_i \right) \geq \sum_{i \in [0..n]} \mu(A_i).$$

**Definition 862 (additivity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *additive* iff for all  $n \in \mathbb{N}$ , for all  $(A_i)_{i \in [0..n]} \in \mathcal{S}$  pairwise disjoint,

$$(11.3) \quad \biguplus_{i \in [0..n]} A_i \in \mathcal{S} \implies \mu \left( \biguplus_{i \in [0..n]} A_i \right) = \sum_{i \in [0..n]} \mu(A_i).$$

**Lemma 863 (decomposition over finite pseudopartition).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is additive. Let  $n \in \mathbb{N}$ . Let  $A, (B_i)_{i \in [0..n]} \in \mathcal{S}$ . Assume that for all  $i \in [0..n]$ ,  $A \cap B_i \in \mathcal{S}$ , and that  $X = \biguplus_{i \in [0..n]} B_i$ . Then, we have

$$(11.4) \quad \mu(A) = \sum_{i \in [0..n]} \mu(A \cap B_i) = \sum_{i \in [0..n]} \mu(B_i \cap A).$$

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} \text{add.}$

*Proof.* Direct consequence of Lemma 211 (*compatibility of pseudopartition with intersection*), Definition 862 (*additivity*), and **commutativity of intersection**.  $\square$

**Lemma 864 (binary inclusion-exclusion principle).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is additive. Let  $A, B \in \mathcal{S}$ . Assume that  $B \setminus A$  is a finite disjoint union of elements of  $\mathcal{S}$ . Then,  $\mu$  satisfies the binary inclusion-exclusion principle with  $A$  and  $B$ :

$$(11.5) \quad A \cup B, A \cap B \in \mathcal{S} \implies \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

When this holds for all  $A, B \in \mathcal{S}$ ,  $\mu$  is said to satisfy the binary inclusion-exclusion principle.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} \text{PtfD} \wedge \text{add.}$

*Proof.* From **the definition of set difference**, we have

$$A \cup B = A \uplus (B \setminus A), \quad \text{and} \quad B = (B \setminus A) \uplus (A \cap B).$$

Let  $n \in \mathbb{N}$ , and  $(C_i)_{i \in [0..n]} \in \mathcal{S}$  such that  $B \setminus A = \biguplus_{i \in [0..n]} C_i$ . Assume that  $A \cup B, A \cap B \in \mathcal{S}$ . Then, from Definition 862 (*additivity, with  $n+2$* ), and Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \sum_{i \in [0..n]} \mu(C_i) + \mu(A \cap B) = \mu(A) + \mu(B).$$

$\square$

**Definition 865 (subtractivity).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *subtractive* iff for all  $A, B \in \mathcal{S}$ ,

$$(11.6) \quad A \subset B \wedge \mu(A) \in \mathbb{R} \wedge B \setminus A \in \mathcal{S} \implies \mu(B \setminus A) = \mu(B) - \mu(A).$$

**Lemma 866 (subtractivity).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is additive. Then,  $\mu$  is subtractive.

Compact form:  $\text{add} \Rightarrow \text{sub}$ .

*Proof.* Direct consequence of **the definition of set difference ( $B = A \uplus (B \setminus A)$  when  $A \subset B$ )**, Definition 862 (*additivity, with  $n \stackrel{\text{def.}}{=} 1$* ), Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rule 5 applies*), and Definition 865 (*subtractivity*).  $\square$

**Lemma 867 (monotonicity).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and additive. Let  $A, B \in \mathcal{S}$ . Assume that  $A \subset B$ , and that  $B \setminus A$  is a finite disjoint union of elements of  $\mathcal{S}$ . Then, we have  $\mu(A) \leq \mu(B)$ .*

*Compact form:  $PtfD' \Rightarrow pos \Rightarrow add \Rightarrow monot.$*

*Proof.* Direct consequence of **the definition of set difference ( $B = (A \cap B) \uplus (B \setminus A)$  equals  $A \uplus \biguplus_{i \in [0..n]} C_i$ )**, Definition 862 (**additivity, with  $n + 1$** ), and nonnegativeness ( $\mu(C_i) \geq 0$ ).  $\square$

*Remark 868.* Note that in the next lemma,  $\mathcal{S}$  is actually a semiring of sets. This lemma is the core of the finite Boole inequality in Section 11.2.

**Lemma 869 (subadditivity).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is nonempty, closed under intersection, and that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and additive. Then,  $\mu$  is subadditive (see (11.1)).*

*Compact form:  $Ne \Rightarrow I \Rightarrow PtfD \Rightarrow pos \Rightarrow add \Rightarrow \underline{add}$ .*

*Proof.* Let  $n \in \mathbb{N}$ . Let  $(A_i)_{i \in [0..n]} \in \mathcal{S}$ . Assume that  $\bigcup_{i \in [0..n]} A_i \in \mathcal{S}$ .

Then, from Lemma 574 (**disjoint union of union in subset system (variant)**), *Hyp* = *Ne, I, and PtfD*), let  $n_K \in \mathbb{N}$ , let  $\psi : [0..n + 1] \rightarrow [0..n_K + 1]$  be nondecreasing with  $\psi(0) = 0$  and  $\psi(n + 1) = n_K + 1$ , and  $(C_k)_{k \in [0..n_K]} \in \mathcal{S}$  be pairwise disjoint such that we have

$$\forall i \in [0..n], \quad \biguplus_{k \in [\psi(i)..\psi(i+1)]} C_k \subset A_i, \quad \text{and} \quad \bigcup_{i \in [0..n]} A_i = \biguplus_{\substack{i \in [0..n] \\ k \in [\psi(i)..\psi(i+1)]}} C_k \in \mathcal{S}.$$

Hence, from Definition 862 (**additivity**), Lemma 557 (**disjoint decomposition of set difference and local complement,  $PtfD \Rightarrow PtfD'$** ), Lemma 867 (**monotonicity**), and Lemma 396 (**addition in  $\overline{\mathbb{R}}_+$  is compatible with order**), we have

$$\mu \left( \bigcup_{i \in [0..n]} A_i \right) = \sum_{i \in [0..n]} \mu \left( \biguplus_{k \in [\psi(i)..\psi(i+1)]} C_k \right) \leq \sum_{i \in [0..n]} \mu(A_i).$$

Therefore, from Definition 859 (**subadditivity**),  $\mu$  is subadditive.  $\square$

### 11.1.2 Countable properties

**Definition 870 ( $\sigma$ -subadditivity).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said  $\sigma$ -subadditive iff for all nonempty  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \mathcal{S}$ ,*

$$(11.7) \quad \bigcup_{i \in I} A_i \in \mathcal{S} \implies \mu \left( \bigcup_{i \in I} A_i \right) \leq \sum_{i \in I} \mu(A_i).$$

**Lemma 871 ( $\sigma$ -subadditive is subadditive).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is  $\sigma$ -subadditive. Then,  $\mu$  is subadditive (see(11.1)).*

*Compact form:  $\sigma \underline{add} \Rightarrow \underline{add}$ .*

*Proof.* Direct consequence of Definition 870 ( **$\sigma$ -subadditivity, with  $I \stackrel{\text{def.}}{=} [0..n]$** ), and Definition 859 (**subadditivity**).  $\square$

**Definition 872 ( $\sigma$ -superadditivity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said  $\sigma$ -superadditive iff for all nonempty  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \mathcal{S}$  pairwise disjoint,

$$(11.8) \quad \bigsqcup_{i \in I} A_i \in \mathcal{S} \implies \mu \left( \bigsqcup_{i \in I} A_i \right) \geq \sum_{i \in I} \mu(A_i).$$

**Lemma 873 ( $\sigma$ -superadditive is superadditive).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is  $\sigma$ -superadditive. Then,  $\mu$  is superadditive (see(11.2)).*

*Compact form:  $\sigma \overline{add} \Rightarrow \overline{add}$ .*

*Proof.* Direct consequence of Definition 872 ( $\sigma$ -superadditivity, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), and Definition 861 (superadditivity).  $\square$

**Definition 874 ( $\sigma$ -additivity).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said  $\sigma$ -additive iff for all nonempty  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \mathcal{S}$  pairwise disjoint,

$$(11.9) \quad \bigsqcup_{i \in I} A_i \in \mathcal{S} \implies \mu \left( \bigsqcup_{i \in I} A_i \right) = \sum_{i \in I} \mu(A_i).$$

**Lemma 875 ( $\sigma$ -additive is additive).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is  $\sigma$ -additive. Then,  $\mu$  is additive (see(11.3)).*

*Compact form:  $\sigma add \Rightarrow add$ .*

*Proof.* Direct consequence of Definition 874 ( $\sigma$ -additivity, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), and Definition 862 (additivity).  $\square$

**Lemma 876 (decomposition over countable pseudopartition).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is  $\sigma$ -additive. Let  $I \subset \mathbb{N}$  be nonempty. Let  $A, (B_i)_{i \in I} \in \mathcal{S}$ . Assume that for all  $i \in I$ ,  $A \cap B_i \in \mathcal{S}$ , and that  $X = \bigsqcup_{i \in I} B_i$ . Then, we have*

$$(11.10) \quad \mu(A) = \sum_{i \in I} \mu(A \cap B_i) = \sum_{i \in I} \mu(B_i \cap A).$$

*Compact form:  $Hyp \stackrel{\text{def.}}{=} \sigma add$ .*

*Proof.* Direct consequence of Lemma 211 (compatibility of pseudopartition with intersection), Definition 874 ( $\sigma$ -additivity), and **commutativity of intersection**.  $\square$

**Remark 877.** Note that in the next lemma,  $\mathcal{S}$  is actually a semiring of sets.

The next lemma is the core of the Boole inequality in Sections 11.3 and 12.1.

**Lemma 878 ( $\sigma$ -subadditivity).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is nonempty, closed under intersection, and that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and  $\sigma$ -additive. Then,  $\mu$  is  $\sigma$ -subadditive (see(11.7)).*

*Compact form:  $Ne \Rightarrow I \Rightarrow PtfD \Rightarrow pos \Rightarrow \sigma add \Rightarrow \sigma \underline{add}$ .*

*Proof.* Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I = \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$ . Assume that  $\bigcup_{i \in I} A_i \in \mathcal{S}$ .

Then, from Lemma 574 (*disjoint union of union in subset system (variant)*),  $\underline{Hyp} = Ne, I$ , and  $\underline{Ptfd}$ , let  $n_K \in \overline{\mathbb{N}}$ ,  $\psi : [0..n_I] \rightarrow [0..n_K]$  be nondecreasing with  $\psi(0) = 0$  and  $\psi(n_I) = n_K$ , and  $(C_k)_{k \in K} \in \mathcal{S}$  be pairwise disjoint such that we have

$$\forall n \in J, \quad \bigsqcup_{k \in [\psi(n).. \psi(n+1)]} C_k \subset A_{\varphi(n)}, \quad \text{and} \quad \bigcup_{i \in I} A_i = \bigsqcup_{k \in [\psi(n).. \psi(n+1)]} C_k \in \mathcal{S}.$$

Hence, from Definition 874 ( $\sigma$ -additivity), Lemma 557 (*disjoint decomposition of set difference and local complement*,  $\underline{Ptfd} \Rightarrow \underline{Ptfd}'$ ), Lemma 875 ( $\sigma$ -additive is additive), Lemma 867 (*monotonicity*), and Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*), we have

$$\mu \left( \bigcup_{i \in I} A_i \right) = \sum_{n \in J} \mu \left( \bigsqcup_{k \in [\psi(n).. \psi(n+1)]} C_k \right) \leq \sum_{i \in I} \mu(A_i).$$

Therefore, from Definition 870 ( $\sigma$ -subadditivity),  $\mu$  is  $\sigma$ -subadditive.  $\square$

**Definition 879 (continuity from below).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $A \in \mathcal{S}$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *continuous from below* at  $A$  iff for all nonempty  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \mathcal{S}$  nondecreasing,

$$(11.11) \quad A = \bigcup_{i \in I} A_i \in \mathcal{S} \quad \Longrightarrow \quad \mu(A) = \sup_{i \in I} \mu(A_i).$$

It is said *continuous from below* iff it is continuous from below at all  $A \in \mathcal{S}$ .

**Lemma 880 (continuity from below).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{S}$  is closed under local complement, and disjoint union. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and  $\sigma$ -additive. Then,  $\mu$  is continuous from below.

Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$  nondecreasing, we have

$$(11.12) \quad A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S} \quad \Longrightarrow \quad \mu(A) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Compact form:  $D' \Rightarrow Ud \Rightarrow pos \Rightarrow \sigma add \Rightarrow cont \uparrow$ .

*Proof.* From Lemma 875 ( $\sigma$ -additive is additive),  $\mu$  is additive.

Let  $A \in \mathcal{S}$ . Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I)]$ . Let  $\varphi : J \rightarrow I$  be the (bijective) increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$  be nondecreasing. Assume that  $A = \bigcup_{i \in I} A_i \in \mathcal{S}$ . Then, from Lemma 572 (*disjoint union of union in subset system*,  $\underline{Hyp} = D' \wedge Ud$ ), let  $(B_i)_{i \in I} \in \mathcal{S}$  be pairwise disjoint such that

$$\forall n \in J, \quad \bigsqcup_{j \in [0..n]} B_{\varphi(j)} = \bigcup_{j \in [0..n]} A_{\varphi(j)} = A_{\varphi(n)} \in \mathcal{S}, \quad \text{and} \quad \bigsqcup_{i \in I} B_i = \bigcup_{i \in I} A_i = A \in \mathcal{S}.$$

Hence, from Definition 874 ( $\sigma$ -additivity), Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*, with  $a_i \stackrel{\text{def.}}{=} \mu(B_i)$ ), Definition 862 (*additivity*), and the bijective property of  $\varphi$ ,

$$\mu(A) = \sum_{i \in I} \mu(B_i) = \sup_{n \in J} \sum_{j \in [0..n]} \mu(B_{\varphi(j)}) = \sup_{n \in J} \mu(A_{\varphi(n)}) = \sup_{i \in I} \mu(A_i).$$

Therefore, from Definition 879 (*continuity from below*),  $\mu$  is continuous from below.

Moreover, when  $I \stackrel{\text{def.}}{=} \mathbb{N} = J$  (i.e.  $\varphi$  is the identity map), property (11.12) is a direct consequence of Lemma 554 (*disjoint decomposition*,  $\underline{D}' \Rightarrow \underline{Ptfd}'$ ), Lemma 867 (*monotonicity*), and Lemma 501 (*supremum is limit*).  $\square$

**Lemma 881 ( $\sigma$ -additivity (from below)).** Let  $X$  be a set.  
 Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under local complement and disjoint union.  
 Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative.  
 Then,  $\mu$  is  $\sigma$ -additive iff it is additive and continuous from below.  
 Compact form:  $D' \Rightarrow Ud \Rightarrow pos \Rightarrow \sigma add \Leftrightarrow (add \wedge cont\uparrow)$ .

*Proof.* “Left” implies “right”.

Direct consequence of Lemma 875 ( $\sigma$ -additive is additive), and Lemma 880 (continuity from below).

“Right” implies “left”. Assume that  $\mu$  is additive and continuous from below.

Let  $I \subset \mathbb{N}$  be nonempty. Let  $J \stackrel{\text{def.}}{=} [0..card(I)]$ . Let  $\varphi : J \rightarrow I$  be the (bijective) increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$  be pairwise disjoint. Assume that  $\bigsqcup_{i \in I} A_i \in \mathcal{S}$ . Then, from Lemma 575 (monotone union of union in subset system, Hyp = Ud), let  $(B_i)_{i \in I} \in \mathcal{S}$  be nondecreasing such that we have

$$\forall n \in J, \quad B_{\varphi(n)} = \bigcup_{j \in [0..n]} B_{\varphi(j)} = \bigsqcup_{j \in [0..n]} A_{\varphi(j)} \in \mathcal{S}, \quad \text{and} \quad \bigcup_{i \in I} B_i = \bigsqcup_{i \in I} A_i \in \mathcal{S}.$$

Thus, from Definition 879 (continuity from below), the bijective property of  $\varphi$ , Definition 862 (additivity), and Lemma 502 (series in  $\overline{\mathbb{R}}_+$  is supremum, with  $a_i \stackrel{\text{def.}}{=} \mu(A_i)$ ), we have

$$\mu \left( \bigsqcup_{i \in I} A_i \right) = \sup_{i \in I} \mu(B_i) = \sup_{n \in J} \mu(B_{\varphi(n)}) = \sup_{n \in J} \sum_{j \in [0..n]} \mu(A_{\varphi(j)}) = \sum_{i \in I} \mu(A_i).$$

Hence, from Definition 874 ( $\sigma$ -additivity),  $\mu$  is  $\sigma$ -additive.

Therefore, we have the equivalence. □

**Definition 882 (continuity from above).**  
 Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $A \in \mathcal{S}$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said *continuous from above at  $A$*  iff for all nonempty  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \mathcal{S}$  nonincreasing,

$$(11.13) \quad (\exists i_0 \in I, \mu(A_{i_0}) \in \mathbb{R}) \quad \wedge \quad A = \bigcap_{i \in I} A_i \in \mathcal{S} \quad \implies \quad \mu(A) = \inf_{i \in I} \mu(A_i) < \infty.$$

It is said *continuous from above* iff it is continuous from above at all  $A \in \mathcal{S}$ .

**Lemma 883 (continuity from above).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .  
 Assume that  $\mathcal{S}$  is closed under local complement, and disjoint union. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and  $\sigma$ -additive. Then,  $\mu$  is continuous from above.  
 Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$  nonincreasing, we have

$$(11.14) \quad (\exists n_0 \in \mathbb{N}, \mu(A_{n_0}) < \infty) \quad \wedge \quad A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{S} \quad \implies \quad \mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) < \infty.$$

Compact form:  $D' \Rightarrow Ud \Rightarrow pos \Rightarrow \sigma add \Rightarrow cont\downarrow$ .

*Proof.* From Lemma 875 ( $\sigma$ -additive is additive),  $\mu$  is additive. Then, from Lemma 554 (disjoint decomposition,  $D' \Rightarrow Pt\downarrow D'$ ), and Lemma 867 (monotonicity),  $\mu$  is nondecreasing.

Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} card(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$  be nonincreasing. Assume that there exists  $i_0 \in I$  such that  $\mu(A_{i_0}) < \infty$ , and that  $A = \bigcap_{i \in I} A_i \in \mathcal{S}$ . Let  $n_0 \stackrel{\text{def.}}{=} \varphi^{-1}(i_0) \in J$  (i.e.  $i_0 = \varphi(n_0)$ ). Let  $J_{n_0} \stackrel{\text{def.}}{=} [n_0..n_I] \neq \emptyset$ .

Let  $n \in J_{n_0}$ . Then, from **the lower bound property of intersection**, and monotonicity of  $\varphi$ , we have  $\bigcap_{j \in J_{n_0}} A_{\varphi(j)} \subset A_{\varphi(n)} \subset A_{\varphi(n_0)}$ . Let  $p \in [0..n_0]$ . Then, from **the right intersection law** ( $A \supset B \Leftrightarrow A \cap B = B$ ), monotonicity of  $\varphi$ , and **finite induction on  $p$** , we have  $A = \bigcap_{j \in [p..n_I]} A_{\varphi(j)} = \bigcap_{j \in J_{n_0}} A_{\varphi(j)} \in \mathcal{S}$ . Hence, from monotonicity of  $\mu$ , we have

$$0 \leq \mu(A) = \mu \left( \bigcap_{j \in J_{n_0}} A_{\varphi(j)} \right) \leq \mu(A_{\varphi(n)}) \leq \mu(A_{\varphi(n_0)}) < \infty.$$

For all  $n \in J_{n_0}$ , let  $B_n \stackrel{\text{def.}}{=} A_{\varphi(n_0)} \setminus A_{\varphi(n)}$ . Then, from Lemma 577 (*monotone union of monotone intersection in subset system, Hyp = D'*),  $(B_n)_{n \in J_{n_0}} \in \mathcal{S}$  is nondecreasing, and such that we have  $\bigcup_{n \in J_{n_0}} B_n = A_{\varphi(n_0)} \setminus \bigcap_{n \in J_{n_0}} A_{\varphi(n)} \in \mathcal{S}$ . Thus, from Lemma 866 (*subtractivity*), Definition 865 (*subtractivity, twice*), Lemma 880 (*continuity from below*), Definition 879 (*continuity from below*), and Lemma 483 (*compatibility of extrema with affine transformation, with  $a \stackrel{\text{def.}}{=} 1$  and  $b \stackrel{\text{def.}}{=} \mu(A_{\varphi(n_0)})$* ), we have

$$\begin{aligned} \mu(A_{\varphi(n_0)}) - \mu \left( \bigcap_{n \in J_{n_0}} A_{\varphi(n)} \right) &= \mu \left( \bigcup_{n \in J_{n_0}} B_n \right) = \sup_{n \in J_{n_0}} \mu(B_n) = \sup_{n \in J_{n_0}} (\mu(A_{\varphi(n_0)}) - \mu(A_{\varphi(n)})) \\ &= \mu(A_{\varphi(n_0)}) - \inf_{n \in J_{n_0}} \mu(A_{\varphi(n)}). \end{aligned}$$

Hence, from **additive group properties of  $\mathbb{R}$** , Lemma 498 (*infimum of nonincreasing sequence*), and Definition 9 (*infimum, lower bound*),

$$\mu(A) = \inf_{n \in J_{n_0}} \mu(A_{\varphi(n)}) = \inf_{i \in I} \mu(A_i) \leq \mu(A_{i_0}) < \infty.$$

Therefore, from Definition 882 (*continuity from above*),  $\mu$  is continuous from above.

Moreover, when  $I \stackrel{\text{def.}}{=} \mathbb{N} = J$  (i.e.  $\varphi$  is the identity map), property (11.14) is a direct consequence of monotonicity of  $\mu$ , and Lemma 500 (*infimum is limit*).  $\square$

**Lemma 884 ( $\sigma$ -additivity (from above)).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the empty set, and is closed under local complement, and disjoint union. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, homogeneous, and finite. Then,  $\mu$  is  $\sigma$ -additive iff it is additive and continuous from above at  $\emptyset$ .*  
*Compact form:  $wE \Rightarrow D' \Rightarrow Ud \Rightarrow pos \Rightarrow hom \Rightarrow fin \Rightarrow \sigma add \Leftrightarrow (add \wedge cont \downarrow \emptyset)$ .*

*Proof. “Left” implies “right”.*

Direct consequence of Lemma 875 ( *$\sigma$ -additive is additive*), and Lemma 883 (*continuity from above*).

**“Right” implies “left”.** Assume that  $\mu$  is additive and continuous from above at  $\emptyset$ .

Let  $I \subset \mathbb{N}$  be nonempty. Let  $n_I \stackrel{\text{def.}}{=} \text{card}(I)$ . Let  $J \stackrel{\text{def.}}{=} [0..n_I]$ . Let  $\varphi : J \rightarrow I$  be the (bijective) increasing numbering of  $I$ . Let  $(A_i)_{i \in I} \in \mathcal{S}$  be pairwise disjoint. Assume that  $A \stackrel{\text{def.}}{=} \bigsqcup_{i \in I} A_i \in \mathcal{S}$ . Then, from Lemma 576 (*empty monotone intersection of union in subset system, Hyp = D'*), let  $(B_i)_{i \in I} \in \mathcal{S}$  be nonincreasing such that we have

$$\forall n \in J, \quad A = \bigsqcup_{j \in [0..n]} A_{\varphi(j)} \sqcup B_{\varphi(n)}, \quad \text{and} \quad \bigcap_{i \in I} B_i = \emptyset \in \mathcal{S}.$$

Thus, from Definition 862 (*additivity, with  $n + 2$* ), Lemma 503 (*constant sum of monotone non-negative functions, with  $I \stackrel{\text{def.}}{=} J, f_n \stackrel{\text{def.}}{=} \sum_{j \in [0..n]} \mu(A_{\varphi(j)}), g_n \stackrel{\text{def.}}{=} \mu(B_{\varphi(n)})$* ), the bijective property of  $\varphi$ , Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum, with  $a_i \stackrel{\text{def.}}{=} \mu(A_i)$* ), finiteness of  $\mu$ , Definition 882



(*continuity from above*, with  $(B_i)_{i \in I}$  and  $i_0 \stackrel{\text{def.}}{=} \varphi(0)$ ), homogeneity of  $\mu$ , and **additive group properties of  $\mathbb{R}$** , we have for all  $n \in J$ ,

$$\mu(A) = \sum_{j \in [0..n]} \mu(A_{\varphi(j)}) + \mu(B_{\varphi(n)}) = \sup_{n \in J} \sum_{j \in [0..n]} \mu(A_{\varphi(j)}) + \inf_{n \in J} \mu(B_{\varphi(n)}) = \sum_{i \in I} \mu(A_i).$$

Hence, from Definition 874 ( *$\sigma$ -additivity*),  $\mu$  is  $\sigma$ -additive.

Therefore, we have the equivalence.  $\square$

**Definition 885 ( $\sigma$ -finiteness).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is said  *$\sigma$ -finite* iff

$$(11.15) \quad \exists I \subset \mathbb{N}, \exists (A_i)_{i \in I} \in \mathcal{S}, \quad (\forall i \in I, \mu(A_i) \in \mathbb{R}) \quad \wedge \quad X = \bigcup_{i \in I} A_i.$$

**Lemma 886 (equivalent definition of  $\sigma$ -finiteness).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{S}$  is closed under union. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is subadditive.

Then,  $\mu$  is  $\sigma$ -finite iff there exist  $I \subset \mathbb{N}$  and  $(B_i)_{i \in I} \in \mathcal{S}$  nondecreasing such that we have

$$(11.16) \quad (\forall i \in I, \mu(B_i) \in \mathbb{R}) \quad \wedge \quad X = \bigcup_{i \in I} B_i.$$

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} U \wedge \text{add}$ .

*Proof. “Left” implies “right”.* From Definition 885 ( *$\sigma$ -finiteness*), let  $I \subset \mathbb{N}$  and  $(A_i)_{i \in I} \in \mathcal{S}$  such that for all  $i \in I$ ,  $\mu(A_i) \in \mathbb{R}$ , and  $X = \bigcup_{i \in I} A_i$ . Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I)]$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Then, from Lemma 575 (*monotone union of union in subset system*,  $\text{Hyp} = U$ ), let  $(B_i)_{i \in I} \in \mathcal{S}$  be nondecreasing such that we have

$$\forall n \in J, \quad \bigcup_{i \in [0..n]} A_{\varphi(i)} = \bigcup_{i \in [0..n]} B_{\varphi(i)} = B_{\varphi(n)} \in \mathcal{S}, \quad \text{and} \quad \bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i.$$

Let  $i \in I$ . Let  $n \stackrel{\text{def.}}{=} \varphi^{-1}(i) \in J$ . Then, from Definition 859 (*subadditivity*), and **closedness of addition in  $\mathbb{R}$** , we have,  $\mu(B_i) \leq \sum_{j \in [0..n]} \mu(A_{\varphi(j)}) \in \mathbb{R}$ , and  $X = \bigcup_{i \in I} B_i$ .

*“Right” implies “left”.* Trivial.

Therefore, we have the equivalence.  $\square$

**Lemma 887 (finite is  $\sigma$ -finite).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set. Let  $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is finite. Then,  $\mu$  is  $\sigma$ -finite.

Compact form:  $wF \Rightarrow \text{fin} \Rightarrow \sigma \text{fin}$ .

*Proof.* Direct consequence of Definition 857 (*finiteness*), and Definition 885 ( *$\sigma$ -finiteness*, with  $I \stackrel{\text{def.}}{=} \{0\}$  and  $A_0 \stackrel{\text{def.}}{=} X$ ).  $\square$



## 11.2 Content

**Definition 888 (content).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\gamma : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is called *content on*  $(X, \mathcal{S})$  iff it is nonnegative, homogeneous, and additive.  
Compact form:  $P_\gamma \stackrel{\text{def.}}{=} \text{pos} \wedge \text{hom} \wedge \text{add}$ .

*Remark 889.* Note that the concept of content is mostly used when the subset system  $\mathcal{S}$  is a (semi)ring of sets, or a (semi-)algebra of sets.

**Lemma 890 (content over finite pseudopartition).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under intersection. Let  $n \in \mathbb{N}$ . Let  $A, (B_i)_{i \in [0..n]} \in \mathcal{S}$ . Assume that  $X = \biguplus_{i \in [0..n]} B_i$ . Then, we have

$$(11.17) \quad \forall i \in [0..n], \quad A \cap B_i = B_i \cap A \in \mathcal{S} \quad \wedge \quad \gamma(A) = \sum_{i \in [0..n]} \gamma(A \cap B_i) = \sum_{i \in [0..n]} \gamma(B_i \cap A).$$

For instance, this holds when  $\mathcal{S}$  is a  $\pi$ -system.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} I \wedge P_\gamma$ .

*Proof.* Direct consequence of Definition 888 (content,  $P_\gamma \Rightarrow \text{add}$ ), Lemma 863 (decomposition over finite pseudopartition,  $\text{Hyp} = \text{add}$ ), and Definition 602 ( $\pi$ -system,  $P_\Pi \Rightarrow I$ ).  $\square$

**Lemma 891 (content satisfies the binary inclusion-exclusion principle).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Assume that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then,  $\gamma$  satisfies the binary inclusion-exclusion principle (11.5).  
For instance, this holds when  $\mathcal{S}$  is a semiring of sets (and  $A \cap B \in \mathcal{S}$  always holds).

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} \text{Ptfd} \wedge P_\gamma$ .

*Proof.* Direct consequence of Definition 888 (content,  $P_\gamma \Rightarrow \text{add}$ ), Lemma 864 (binary inclusion-exclusion principle,  $\text{Hyp} = \text{Ptfd} \wedge \text{add}$ ), and Definition 626 (semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow \text{Ptfd}$ ).  $\square$

**Lemma 892 (content is subtractive).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under local complement. Then,  $\gamma$  is subtractive (and  $B \setminus A \in \mathcal{S}$  always holds when  $A \subset B$ , see (11.6)).  
For instance, this holds when  $\mathcal{S}$  is a ring of sets.

Compact form:  $D' \Rightarrow P_\gamma \Rightarrow \text{sub}$ .

*Proof.* Direct consequence of Definition 888 (content,  $P_\gamma \Rightarrow \text{add}$ ), Lemma 866 (subtractivity,  $\text{Hyp}$  equals  $\text{add}$ ), Definition 865 (subtractivity), and Lemma 612 (equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow D'$ ).  $\square$

**Lemma 893 (content is monotone).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Assume that the local complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Then,  $\gamma$  is nondecreasing.  
For instance, this holds when  $\mathcal{S}$  is a semiring of sets.

Compact form:  $\text{Ptfd}' \Rightarrow P_\gamma \Rightarrow \text{monot}$ .

*Proof.*

Direct consequence of Definition 888 (content,  $P_\gamma \Rightarrow [\text{pos} \mid \text{add}]$ ), Lemma 867 (monotonicity,  $\text{Hyp}$  equals  $\text{Ptfd}' \wedge \text{pos} \wedge \text{add}$ ), and Lemma 630 (other properties of semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow \text{Ptfd}'$ ).  $\square$

**Lemma 894 (content satisfies the finite Boole inequality).**

Let  $X$  be a set. Let  $\mathcal{S}$  be a semiring of sets on  $X$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ .

Then,  $\gamma$  satisfies the finite Boole inequality, i.e. it is subadditive (see (11.1)).

Compact form:  $P_{\mathcal{R}/2} \Rightarrow P_\gamma \Rightarrow \underline{add}$ .

*Proof.* Direct consequence of Definition 626 (semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow [Ne \mid I \mid PtfD]$ ), Definition 888 (content,  $P_\gamma \Rightarrow [pos \mid add]$ ), and Lemma 869 (subadditivity,  $\underline{Hyp} = Ne \wedge I \wedge PtfD$  and  $pos \wedge add$ ).  $\square$

**Lemma 895 (finite content is bounded).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set, and that the local complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Assume that  $\gamma$  is finite. Then,  $\gamma$  is bounded.

For instance, this holds when  $\mathcal{S}$  is a semi-algebra of sets.

Compact form:  $\underline{Hyp} \stackrel{\text{def.}}{=} wF \wedge PtfD' \wedge P_\gamma \wedge \underline{fin}$ .

*Proof.* Direct consequence of Definition 888 (content,  $P_\gamma \Rightarrow pos$ ), Lemma 893 (content is monotone,  $PtfD' \wedge P_\gamma \Rightarrow \underline{monot}$ ), Lemma 858 (finite is bounded,  $\underline{Hyp} \stackrel{\text{def.}}{=} wF \wedge pos \wedge \underline{monot} \wedge \underline{fin}$ ), Definition 654 (semi-algebra of sets,  $P_{A/2} \Rightarrow wF$ ), and Lemma 658 (other properties of semi-algebra of sets,  $P_{A/2} \Rightarrow PtfD'$ ).  $\square$

**Lemma 896 ( $\sigma$ -finite content).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is a semiring of sets, and is closed under union. Let  $\gamma$  be a content on  $(X, \mathcal{S})$ . Then,  $\gamma$  is  $\sigma$ -finite iff there exist  $I \subset \mathbb{N}$  and  $(B_i)_{i \in I} \in \mathcal{S}$  nondecreasing such that we have

$$(11.18) \quad (\forall i \in I, \gamma(B_i) \in \mathbb{R}) \quad \wedge \quad X = \bigcup_{i \in I} B_i.$$

For instance, this holds when  $\mathcal{S}$  is a ring of sets.

Compact form:  $\underline{Hyp} \stackrel{\text{def.}}{=} P_{\mathcal{R}/2} \wedge U \wedge P_\gamma$ .

*Proof.* Direct consequence of Lemma 894 (content satisfies the finite Boole inequality,  $P_{\mathcal{R}/2} \wedge P_\gamma$  implies  $\underline{add}$ ), Lemma 886 (equivalent definition of  $\sigma$ -finiteness,  $\underline{Hyp} = U \wedge \underline{add}$ ), Lemma 631 (ring of sets is semiring of sets,  $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R}/2}$ ), and Lemma 612 (equivalent definition of ring of sets,  $P_{\mathcal{R}} \Rightarrow U$ ).  $\square$

### 11.3 Premeasure

**Lemma 897 (premeasure).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . A function  $\mu_0 : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  is called premeasure on  $(X, \mathcal{S})$  iff it is a content on  $(X, \mathcal{S})$  that is  $\sigma$ -additive, i.e. iff it is nonnegative, homogeneous, and  $\sigma$ -additive.

When  $\mu_0$  is a premeasure on the premeasurable space  $(X, \mathcal{R})$ , i.e. when  $\mathcal{R}$  is a ring of sets on  $X$ , then  $(X, \mathcal{R}, \mu_0)$  is called premeasure space. Moreover, if the premeasure  $\mu_0$  is diffuse (resp. finite, or  $\sigma$ -finite), then the premeasure space  $(X, \mathcal{R}, \mu_0)$  is also said diffuse (resp. finite, or  $\sigma$ -finite).

Compact form:  $P_{\mu_0} \stackrel{\text{def.}}{=} P_\gamma \wedge \sigma \text{ add}$ ;  $P_{\mu_0} \Leftrightarrow \text{pos} \wedge \text{hom} \wedge \sigma \text{ add}$ .

*Proof.* Direct consequence of Lemma 875 ( $\sigma$ -additive is additive,  $\sigma \text{ add} \Rightarrow \text{add}$ ), Definition 888 (content,  $P_\gamma = \text{pos} \wedge \text{hom} \wedge \text{add}$ ), and Definition 739 (premeasurable space).  $\square$

*Remark 898.* Note that the concept of premeasure is mostly used when the subset system  $\mathcal{S}$  is a ring of sets, an algebra of sets, or a  $\sigma$ -algebra (in which case it is called measure, see Section 12.1).

**Lemma 899 (premeasure is additive).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Then,  $\mu_0$  is additive.

Compact form:  $P_{\mu_0} \Rightarrow \text{add}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), and Definition 888 (content,  $P_\gamma \Rightarrow \text{add}$ ).  $\square$

**Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that the set difference of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ .

Then,  $\mu_0$  satisfies the binary inclusion-exclusion principle (11.5).

For instance, this holds when  $\mathcal{S}$  is a ring of sets (i.e. on a premeasure space).

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} \text{Ptfd} \wedge P_{\mu_0}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), Lemma 891 (content satisfies the binary inclusion-exclusion principle,  $\text{Hyp} = \text{Ptfd} \wedge P_\gamma$ , and  $P_{\mathcal{R}/2}$ ), and Lemma 740 (properties of premeasurable space, thus  $P_{\mathcal{R}/2}$ ).  $\square$

**Lemma 901 (premeasure is subtractive).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .

Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under local complement.

Then,  $\mu_0$  is subtractive (and  $B \setminus A \in \mathcal{S}$  always holds when  $A \subset B$ , see (11.6)).

For instance, this holds on a premeasure space.

Compact form:  $D' \Rightarrow P_{\mu_0} \Rightarrow \text{sub}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), Lemma 892 (content is subtractive,  $\text{Hyp} = D' \wedge P_\gamma$ , and  $P_{\mathcal{R}}$ ), and Lemma 740 (properties of premeasurable space, thus  $P_{\mathcal{R}}$ ).  $\square$

**Lemma 902 (premeasure is monotone).** Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under local complement. Then,  $\mu_0$  is nondecreasing.

For instance, this holds on a premeasure space.

Compact form:  $D' \Rightarrow P_{\mu_0} \Rightarrow \text{monot}$ .

*Proof.* Direct consequence of Lemma 554 (disjoint decomposition,  $D' \Rightarrow \text{Ptfd}'$ ), Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), Lemma 893 (content is monotone,  $\text{Hyp} = \text{Ptfd}' \wedge P_\gamma$ , and  $P_{\mathcal{R}/2}$ ), and Lemma 740 (properties of premeasurable space, thus  $P_{\mathcal{R}/2}$ ).  $\square$

**Lemma 903 (finite premeasure is bounded).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set, and that the local complement of elements of  $\mathcal{S}$  can be expressed as a finite disjoint union of elements of  $\mathcal{S}$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mu_0$  is finite. Then,  $\mu_0$  is bounded.

For instance, this holds when  $\mathcal{S}$  is an algebra of sets, and on a measurable space.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} wF \wedge \text{Ptfd}' \wedge P_{\mu_0} \wedge \text{fin}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), Lemma 895 (finite content is bounded,  $\text{Hyp} \stackrel{\text{def.}}{=} wF \wedge \text{Ptfd}' \wedge P_\gamma \wedge \text{fin}$ , and  $P_{A/2}$ ), Lemma 660 (algebra of sets is semi-algebra of sets,  $P_A \Rightarrow P_{A/2}$ ), and Lemma 742 (properties of measurable space, thus  $P_A$ ).  $\square$

**Lemma 904 ( $\sigma$ -finite premeasure).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is a semiring of sets, and is closed under union. Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Then,  $\mu_0$  is  $\sigma$ -finite iff there exist  $I \subset \mathbb{N}$  and  $(B_i)_{i \in I} \in \mathcal{S}$  nondecreasing such that we have

$$(11.19) \quad (\forall i \in I, \mu_0(B_i) \in \mathbb{R}) \quad \wedge \quad X = \bigcup_{i \in I} B_i.$$

For instance, this holds on a premeasure space.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} P_{\mathcal{R}/2} \wedge U \wedge P_{\mu_0}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow P_\gamma$ ), Lemma 896 ( $\sigma$ -finite content,  $\text{Hyp} = P_{\mathcal{R}/2} \wedge U \wedge P_\gamma$ , and  $P_{\mathcal{R}}$ ), and Lemma 740 (properties of premeasurable space, thus  $P_{\mathcal{R}}$ ).  $\square$

**Lemma 905 (premeasure over countable pseudopartition).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under intersection. Let  $I \subset \mathbb{N}$ . Let  $A, (B_i)_{i \in I} \in \mathcal{S}$ . Assume that  $X = \bigsqcup_{i \in I} B_i$ . Then, we have

$$(11.20) \quad \forall i \in I, \quad A \cap B_i = B_i \cap A \in \mathcal{S} \quad \wedge \quad \mu_0(A) = \sum_{i \in I} \mu_0(A \cap B_i) = \sum_{i \in I} \mu_0(B_i \cap A).$$

For instance, this holds when  $\mathcal{S}$  is a  $\pi$ -system, and on a premeasure space.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} I \wedge P_{\mu_0}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow \sigma \text{ add}$ ), Lemma 876 (decomposition over countable pseudopartition,  $\text{Hyp} = \sigma \text{ add}$ ), Definition 602 ( $\pi$ -system,  $P_\Pi \Rightarrow I$ ), and Lemma 740 (properties of premeasurable space, thus  $P_\Pi$ ).  $\square$

**Lemma 906 (premeasure satisfies the Boole inequality).**

Let  $X$  be a set. Let  $\mathcal{S}$  be a semiring of sets on  $X$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Then,  $\mu_0$  satisfies the Boole inequality, i.e. it is  $\sigma$ -subadditive (see (11.7), and subadditive, see (11.1)).

For instance, this holds on a premeasure space.

Compact form:  $P_{\mathcal{R}/2} \Rightarrow P_{\mu_0} \Rightarrow [\sigma \text{ add} \mid \text{add}]$ .

*Proof.* Direct consequence of Definition 626 (semiring of sets,  $P_{\mathcal{R}/2} \Rightarrow [Ne \mid I \mid \text{Ptfd}]$ ), Lemma 897 (premeasure,  $P_{\mu_0} \Rightarrow [\text{pos} \mid \sigma \text{ add}]$ ), Lemma 878 ( $\sigma$ -subadditivity,  $\text{Hyp} = Ne \wedge I \wedge \text{Ptfd} \wedge \text{pos}$  and  $\sigma \text{ add}$ ), Lemma 871 ( $\sigma$ -subadditive is subadditive,  $\sigma \text{ add} \Rightarrow \text{add}$ ), and Lemma 740 (properties of premeasurable space, thus  $P_{\mathcal{R}/2}$ ).  $\square$

**Lemma 907 (compatibility of null premeasure with countable union).**

Let  $X$  be a set. Let  $\mathcal{S}$  be a semiring of sets on  $X$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Let  $I \subset \mathbb{N}$ . For all  $i \in I$ , let  $A_i \in \mathcal{S}$  such that  $\mu_0(A_i) = 0$ , and  $\bigcup_{i \in I} A_i \in \mathcal{S}$ . Then, we have  $\mu_0(\bigcup_{i \in I} A_i) = 0$ .

For instance, this holds on a premeasure space.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} P_{\mathcal{R}/2} \wedge P_{\mu_0}$ .

*Proof.* Direct consequence of Lemma 897 (*premeasure, nonnegativeness*), Lemma 906 (*premeasure satisfies the Boole inequality*), **additive group properties of  $\mathbb{R}$** , Lemma 34 (*stationary sequence is convergent, countable sum of zero terms is zero*), and Lemma 740 (*properties of premeasurable space, thus  $P_{\mathbb{R}/2}$* ).  $\square$

**Lemma 908 (premeasure is continuous from below).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under local complement, and disjoint union. Then,  $\mu_0$  is continuous from below (see (11.11)).

Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$  nondecreasing, we have

$$(11.21) \quad A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S} \implies \mu_0(A) = \lim_{n \rightarrow \infty} \mu_0(A_n).$$

For instance, this holds on a premeasure space.

Compact form:  $D' \Rightarrow Ud \Rightarrow P_{\mu_0} \Rightarrow \text{cont}\uparrow$ .

*Proof.* Direct consequence of Lemma 897 (*premeasure,  $P_{\mu_0} \Rightarrow [\text{pos} \mid \sigma \text{add}]$* ), Lemma 880 (*continuity from below,  $\text{Hyp} = D' \wedge Ud \wedge \text{pos} \wedge \sigma \text{add}$* ), and Lemma 740 (*properties of premeasurable space, thus  $[D' \mid Ud]$* ).  $\square$

**Lemma 909 (equivalent definition of premeasure (from below)).**

Let  $X$  be a set.

Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under local complement, and disjoint union.

Let  $\mu_0 : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu_0$  is nonnegative and homogeneous.

Then,  $\mu_0$  is a premeasure on  $(X, \mathcal{S})$  iff it is additive and continuous from below.

For instance, this holds on a premeasure space.

Compact form:  $D' \Rightarrow Ud \Rightarrow \text{pos} \Rightarrow \text{hom} \Rightarrow P_{\mu_0} \Leftrightarrow (\text{add} \wedge \text{cont}\uparrow)$ .

*Proof.* Direct consequence of Lemma 897 (*premeasure,  $P_{\mu_0} = \text{pos} \wedge \text{hom} \wedge \sigma \text{add}$* ), Lemma 881 ( *$\sigma$ -additivity (from below),  $D' \wedge Ud \wedge \text{pos} \Rightarrow \sigma \text{add} \Leftrightarrow (\text{add} \wedge \text{cont}\uparrow)$* ), and Lemma 740 (*properties of premeasurable space, thus  $[D' \mid Ud]$* ).  $\square$

**Lemma 910 (premeasure is continuous from above).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Let  $\mu_0$  be a premeasure on  $(X, \mathcal{S})$ . Assume that  $\mathcal{S}$  is closed under local complement, and disjoint union. Then,  $\mu_0$  is continuous from above (see (11.13)).

Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$  nonincreasing, we have

$$(11.22) \quad (\exists n_0 \in \mathbb{N}, \mu(A_{n_0}) < \infty) \wedge A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{S} \implies \mu_0(A) = \lim_{n \rightarrow \infty} \mu_0(A_n) < \infty.$$

For instance, this holds on a premeasure space.

Compact form:  $D' \Rightarrow Ud \Rightarrow P_{\mu_0} \Rightarrow \text{cont}\downarrow$ .

*Proof.* Direct consequence of Lemma 897 (*premeasure,  $P_{\mu_0} \Rightarrow [\text{pos} \mid \sigma \text{add}]$* ), Lemma 883 (*continuity from above,  $\text{Hyp} = D' \wedge Ud \wedge \text{pos} \wedge \sigma \text{add}$* ), and Lemma 740 (*properties of premeasurable space, thus  $[D' \mid Ud]$* ).  $\square$

**Lemma 911 (equivalent definition of premeasure (from above)).**

Let  $X$  be a set.

Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the empty set, and is closed under local complement, and disjoint union. Let  $\mu_0 : \mathcal{S} \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu_0$  is nonnegative, homogeneous, and finite.

Then,  $\mu_0$  is a finite premeasure on  $(X, \mathcal{S})$  iff it is additive and continuous from above at  $\emptyset$ .

For instance, this holds on a premeasure space.

Compact form:  $wE \Rightarrow D' \Rightarrow Ud \Rightarrow \text{pos} \Rightarrow \text{hom} \Rightarrow \text{fin} \Rightarrow P_{\mu_0} \Leftrightarrow (\text{add} \wedge \text{cont}\downarrow\emptyset)$ .

*Proof.* Direct consequence of Lemma 897 (*premeasure*,  $\underline{P_{\mu_0} = pos \wedge hom \wedge \sigma add}$ ), Lemma 884 ( $\sigma$ -*additivity (from above)*),  $\underline{wE \wedge D' \wedge Ud \wedge pos \wedge hom \wedge fin \Rightarrow \sigma add \Leftrightarrow (add \wedge cont \downarrow \emptyset)}$ , and Lemma 740 (*properties of premeasurable space*, thus  $\underline{wE \mid D' \mid Ud}$ ).  $\square$

**Lemma 912 (properties of premeasure space).** *Let  $(X, \mathcal{R}, \mu_0)$  be a premeasure space. Then,  $\mu_0$  is a premeasure, and a content, and  $\mathcal{R}$  is a (semi)ring a sets, and a  $\pi$ -system.*

*Moreover,  $\mu_0$  is additive, subtractive, monotone, continuous from below (see (11.11)) and above (see (11.13)), and satisfies the binary inclusion-exclusion principle (see (11.5)), the equivalent  $\sigma$ -finiteness definition (see (11.19)), the formula of decomposition against a countable pseudopartition (see (11.20)), and the Boole inequality (see (11.7)); and  $\mathcal{R}$  is nonempty, contains the empty set, is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, and finite union and intersection, and local complement and set difference of elements of  $\mathcal{R}$  can be expressed as a finite disjoint union of elements of  $\mathcal{R}$ .*

*Compact form:  $Concl \stackrel{\text{def.}}{=} [P_{\mu_0} \mid P_{\gamma} \mid P_{\mathcal{R}} \mid P_{\mathcal{R}/2} \mid P_{\Pi} \mid$   
 $pos \mid hom \mid add \mid \sigma add \mid sub \mid monot \mid cont \uparrow \mid cont \downarrow \mid \underline{add} \mid \underline{\sigma add} \mid$   
 $Ne \mid wE \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid PtfD' \mid PtfD].$*

*Proof.* Direct consequence of Lemma 897 (*premeasure*, thus  $\underline{P_{\mu_0} \mid P_{\gamma} \mid pos \mid hom \mid \sigma add}$ ), Lemma 740 (*properties of premeasurable space*, thus  $\underline{P_{\mathcal{R}} \mid P_{\mathcal{R}/2} \mid P_{\Pi} \mid Ne \mid wE \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Udf \mid Uf \mid If \mid PtfD' \mid PtfD}$ ), Lemma 899 (*premeasure is additive*, thus  $\underline{add}$ ), Lemma 901 (*premeasure is subtractive*, thus  $\underline{sub}$ ), Lemma 902 (*premeasure is monotone*, thus  $\underline{monot}$ ), Lemma 908 (*premeasure is continuous from below*, thus  $\underline{cont \uparrow}$ ), Lemma 910 (*premeasure is continuous from above*, thus  $\underline{cont \downarrow}$ ), Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*), Lemma 904 ( $\sigma$ -*finite premeasure*), Lemma 905 (*premeasure over countable pseudopartition*), and Lemma 906 (*premeasure satisfies the Boole inequality*).  $\square$



# Chapter 12

# Measure

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## 12.1 Measure space

### Lemma 913 (measure).

Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \bar{\mathbb{R}}$  is called *measure* on  $(X, \Sigma)$  iff it is a premeasure on  $(X, \Sigma)$ , i.e. iff it is nonnegative, homogeneous, and  $\sigma$ -additive  $\biguplus_{n \in I} A_i \in \mathcal{S} \stackrel{\text{def.}}{=} \Sigma$  always holds). If so,  $(X, \Sigma, \mu)$  is called *measure space*.

Moreover, if the measure  $\mu$  is diffuse (resp. finite, or  $\sigma$ -finite), then the measure space  $(X, \Sigma, \mu)$  is also said *diffuse* (resp. *finite*, or  *$\sigma$ -finite*).

Compact form:  $P_\mu \stackrel{\text{def.}}{=} P_\Sigma \wedge P_{\mu_0}$ ;  $P_\mu \Leftrightarrow P_\Sigma \wedge \text{pos} \wedge \text{hom} \wedge \sigma\text{add}$ .

*Proof.* Direct consequence of Lemma 897 (premeasure), Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow \text{Udc}$ ), and Definition 874 ( $\sigma$ -additivity).  $\square$

*Remark 914.* The previous definition is actually that of *nonnegative* measure, but as we do not consider “signed” measures in this document, we omit the qualifier “nonnegative”.

### Lemma 915 (measure space is premeasure space).

Let  $(X, \Sigma, \mu)$  be a measure space. Then, it is a premeasure space.

Compact form:  $P_\mu \Rightarrow P_{\mathcal{R}} \wedge P_{\mu_0}$ .

*Proof.* Direct consequence of Lemma 913 (measure,  $\mu$  is a premeasure), Definition 741 (measurable space,  $\Sigma$  is  $\sigma$ -algebra), Lemma 703 ( $\sigma$ -algebra is ring of sets), and Lemma 897 (premeasure).  $\square$

### Lemma 916 (measure is additive).

Let  $(X, \Sigma, \mu)$  be a measure space.

Then,  $\mu$  is additive (see (11.3) where  $\biguplus_{n \in [0..n]} A_i \in \Sigma$  always holds).

Compact form:  $P_\mu \Rightarrow \text{add}$ .

*Proof.* Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 899 (premeasure is additive), Definition 862 (additivity), and Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow \text{Udf}$ ).  $\square$

### Lemma 917 (measure satisfies the binary inclusion-exclusion principle).

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  satisfies the binary inclusion-exclusion principle (see (11.5) where  $A \cup B, A \cap B \in \Sigma$  always holds).

*Proof.* Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle), Lemma 864 (binary inclusion-exclusion principle), and Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow [I \mid U]$ ).  $\square$

### Lemma 918 (measure is subtractive).

Let  $(X, \Sigma, \mu)$  be a measure space.

Then,  $\mu$  is subtractive (see (11.6) where  $B \setminus A \in \Sigma$  always holds).

Compact form:  $P_\mu \Rightarrow \text{sub}$ .

*Proof.* Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 901 (premeasure is subtractive), Definition 865 (subtractivity), and Lemma 700 (other properties of  $\sigma$ -algebra,  $P_\Sigma \Rightarrow \text{D}'$ ).  $\square$

### Lemma 919 (measure is monotone).

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is nondecreasing.

Compact form:  $P_\mu \Rightarrow \text{monot}$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 902 (*premeasure is monotone*).  $\square$

**Lemma 920 (finite measure is bounded).**

Let  $(X, \Sigma, \mu)$  be a measure space. Assume that  $\mu$  is finite. Then,  $\mu$  is bounded.

Compact form:  $\text{Hyp} \stackrel{\text{def.}}{=} P_\mu \wedge \text{fin}$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 903 (*finite premeasure is bounded*).  $\square$

**Lemma 921 ( $\sigma$ -finite measure).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Then,  $\mu$  is  $\sigma$ -finite iff there exist  $I \subset \mathbb{N}$  and  $(B_i)_{i \in I} \in \mathcal{S}$  nondecreasing such that we have

$$(12.1) \quad (\forall i \in I, \mu(B_i) \in \mathbb{R}) \quad \wedge \quad X = \bigcup_{i \in I} B_i.$$

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 904 ( *$\sigma$ -finite premeasure*).  $\square$

**Lemma 922 (measure over countable pseudopartition).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (B_i)_{i \in I} \in \Sigma$ . Assume that  $X = \bigsqcup_{i \in I} B_i$ . Then, for all  $i \in I$ ,  $A \cap B_i = B_i \cap A \in \Sigma$ , and we have

$$(12.2) \quad \mu(A) = \sum_{i \in I} \mu(A \cap B_i) = \sum_{i \in I} \mu(B_i \cap A).$$

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 905 (*premeasure over countable pseudopartition*).  $\square$

**Lemma 923 (measure satisfies the Boole inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  satisfies the Boole inequality, i.e. it is  $\sigma$ -subadditive (see (11.7) where  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$  always holds).

Compact form:  $P_\mu \Rightarrow [\sigma\text{add} \mid \text{add}]$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), Lemma 906 (*premeasure satisfies the Boole inequality*), Definition 870 ( *$\sigma$ -subadditivity*), and Lemma 696 ( *$\sigma$ -algebra,  $P_\Sigma \Rightarrow Uc$* ).  $\square$

**Lemma 924 (compatibility of null measure with countable union).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . For all  $i \in I$ , let  $A_i \in \Sigma$  such that  $\mu(A_i) = 0$ . Then, we have  $\mu(\bigcup_{i \in I} A_i) = 0$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), Lemma 907 (*compatibility of null premeasure with countable union*), and Lemma 696 ( *$\sigma$ -algebra,  $P_\Sigma \Rightarrow Uc$* ).  $\square$

**Lemma 925 (measure is continuous from below).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Then,  $\mu$  is continuous from below (see (11.11) where  $\bigcup_{n \in \mathbb{N}} A_n \in \Sigma$  always holds).

Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \Sigma$  nondecreasing, we have

$$(12.3) \quad \bigcup_{n \in \mathbb{N}} A_n \in \Sigma \quad \wedge \quad \mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Compact form:  $P_\mu \Rightarrow \text{cont}\uparrow$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), Lemma 908 (*premeasure is continuous from below*), and Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma \Rightarrow Uc$ ).  $\square$

**Lemma 926 (equivalent definition of measure (from below)).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mu : \Sigma \rightarrow \bar{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative and homogeneous. Then,  $\mu$  is a measure on  $(X, \Sigma)$  iff it is additive and continuous from below.

Compact form:  $P_\Sigma \Rightarrow pos \Rightarrow hom \Rightarrow P_\mu \Leftrightarrow (add \wedge cont\uparrow)$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 909 (*equivalent definition of premeasure (from below)*).  $\square$

**Lemma 927 (measure is continuous from above).** Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is continuous from above (see (11.13) where  $\bigcap_{n \in I} A_i \in \Sigma$  always holds).

Moreover, for all  $(A_n)_{n \in \mathbb{N}} \in \Sigma$  nonincreasing, we have

$$(12.4) \quad \bigcap_{n \in \mathbb{N}} A_n \in \Sigma \quad \wedge \quad \left[ (\exists n_0 \in \mathbb{N}, \mu(A_{n_0}) < \infty) \implies \mu \left( \bigcap_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n) < \infty \right].$$

Compact form:  $P_\mu \Rightarrow cont\downarrow$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), Lemma 910 (*premeasure is continuous from above*), and Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma \Rightarrow Ic$ ).  $\square$

**Lemma 928 (equivalent definition of measure (from above)).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mu : \Sigma \rightarrow \bar{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, homogeneous and finite. Then,  $\mu$  is a finite measure on  $(X, \Sigma)$  iff it is additive and continuous from above at  $\emptyset$ .

Compact form:  $P_\Sigma \Rightarrow pos \Rightarrow hom \Rightarrow fin \Rightarrow P_\mu \Leftrightarrow (add \wedge cont\downarrow\emptyset)$ .

*Proof.* Direct consequence of Lemma 915 (*measure space is premeasure space*), and Lemma 911 (*equivalent definition of premeasure (from above)*).  $\square$

**Lemma 929 (properties of measure space).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is a measure, a premeasure and a content, and  $\Sigma$  is a  $\lambda$ -system, a monotone class, a (semi-)algebra of sets, a (semi)ring of sets, and a  $\pi$ -system.

Moreover,  $\mu$  is additive, subtractive, monotone, continuous from below (see (11.11)) and above (see (11.13)), and satisfies the binary inclusion-exclusion principle (see (11.5)), the equivalent  $\sigma$ -finiteness definition (see (11.19)), the formula of decomposition against a countable pseudopartition (see (11.20)), and the Boole inequality (see (11.7)); and  $\Sigma$  is nonempty, contains the empty set and the full set, is closed under complement, local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, countable monotone union and intersection, and countable union and intersection, and complement, local complement and set difference of elements of  $\Sigma$  can be expressed as a finite disjoint union of elements of  $\Sigma$ .

Compact form:  $Concl \stackrel{\text{def.}}{=} [P_\mu \mid P_{\mu_0} \mid P_\gamma \mid P_\Sigma \mid P_\Lambda \mid P_C \mid P_A \mid P_{A/2} \mid P_R \mid P_{R/2} \mid P_\Pi \mid pos \mid hom \mid add \mid \sigma add \mid sub \mid monot \mid cont\uparrow \mid cont\downarrow \mid add \mid \sigma add \mid Ne \mid wE \mid wF \mid C \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Ud\uparrow \mid Uf \mid If \mid Udc \mid Umc \mid Imc \mid Uc \mid Ic \mid PtfC \mid PtfD' \mid PtfD]$ .

*Proof.*

Direct consequence of Lemma 913 (*measure, thus  $P_\mu$* ), Lemma 912 (*properties of premeasure space, thus  $[P_{\mu_0} \mid P_\gamma \mid pos \mid hom \mid add \mid \sigma add \mid sub \mid monot \mid cont\uparrow \mid cont\downarrow \mid add \mid \sigma add]$* ), and Lemma 742 (*properties of measurable space, thus  $[P_\Sigma \mid P_\Lambda \mid P_C \mid P_A \mid P_{A/2} \mid P_R \mid P_{R/2} \mid P_\Pi \mid Ne \mid wE \mid wF \mid C \mid D' \mid D \mid sD \mid Ud \mid U \mid I \mid Ud\uparrow \mid Uf \mid If \mid Udc \mid Umc \mid Imc \mid Uc \mid Ic \mid PtfC \mid PtfD' \mid PtfD]$* ).  $\square$

**Lemma 930 (trace measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y \in \Sigma$ . Then,  $\mu|_{\Sigma \cap Y}$  is a measure on  $(Y, \Sigma \cap Y)$ .

The measure  $\mu_Y \stackrel{\text{def.}}{=} \mu|_{\Sigma \cap Y}$  is called trace measure on  $Y$ . The measure space  $(Y, \Sigma \cap Y, \mu_Y)$  is called trace measure space on  $Y$ .

*Proof.* Direct consequence of Lemma 763 (trace  $\sigma$ -algebra), Lemma 764 (measurability of measurable subspace,  $\Sigma \cap Y \subset \Sigma$ ), and Lemma 913 (measure).  $\square$

**Lemma 931 (restricted measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y \in \Sigma$ .

Then, the function  $\mu'_Y \stackrel{\text{def.}}{=} (A \in \Sigma \mapsto \mu(A \cap Y))$  is a measure on  $(X, \Sigma)$ .

*Proof.* Direct consequence of Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $\text{card}(I) = 2$ )), **properties of intersection**, and Lemma 913 (measure).  $\square$

**Lemma 932 (trace measure is restricted measure).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $Y \in \Sigma$ . Let  $B \in \Sigma \cap Y$ . Then, we have  $\mu_Y(B) = \mu(B) = \mu'_Y(B)$ .

*Proof.* Direct consequence of Lemma 930 (trace measure), Lemma 931 (restricted measure), and Lemma 764 (measurability of measurable subspace, thus  $B \in \Sigma$  and  $B \subset Y$ ).  $\square$

## 12.2 Some measures

### Lemma 933 (trivial measure).

Let  $(X, \Sigma)$  be a measurable space. Then, the zero function on  $\Sigma$  is a measure on  $(X, \Sigma)$ .  
It is called the trivial measure on  $(X, \Sigma)$ , and  $(X, \Sigma, 0)$  is called trivial measure space.

*Proof.*

Direct consequence of Lemma 913 (measure), and **ordered additive group properties of  $\mathbb{R}$** .  $\square$

### Lemma 934 (equivalent definition of trivial measure).

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is the trivial measure iff  $\mu(X) = 0$ .

*Proof.* Direct consequence of Lemma 933 (trivial measure), Lemma 913 (measure, nonnegativity), and Lemma 919 (measure is monotone,  $\mu$  is nonpositive).  $\square$

### Lemma 935 (counting measure).

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $\delta_Y$  be the function defined by

$$(12.5) \quad \forall A \in \Sigma, \quad \delta_Y(A) \stackrel{\text{def.}}{=} \text{card}(A \cap Y)$$

with the convention that the cardinality of an infinite set is  $\infty$ . Then,  $\delta_Y$  is a measure on  $(X, \Sigma)$ .  
It is called the counting measure (associated with  $Y$ ).

*Proof.* From **the definition of the cardinality**, and Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ), the function  $\delta_Y$  is nonnegative. Moreover, from Lemma 913 (measure), Definition 741 (measurable space,  $\Sigma$  is a  $\sigma$ -algebra), Lemma 696 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), and **the definition of  $\emptyset$** , we have

$$\delta_Y(\emptyset) = \text{card}(\emptyset) = 0.$$

Let  $I \subset \mathbb{N}$  be nonempty. Let  $(A_i)_{i \in I} \in \Sigma$ . Assume that the  $A_i$ 's are pairwise disjoint. Then, from **distributivity of intersection over disjoint union**, and  **$\sigma$ -additivity of the cardinality (with the convention that  $\infty$  is absorbing element for addition in  $\overline{\mathbb{N}}$ )**,

$$\delta_Y \left( \bigsqcup_{i \in I} A_i \right) = \text{card} \left( \bigsqcup_{i \in I} A_i \cap Y \right) = \text{card} \left( \bigsqcup_{i \in I} (A_i \cap Y) \right) = \sum_{i \in I} \text{card}(A_i \cap Y) = \sum_{i \in I} \delta_Y(A_i).$$

Therefore, from Definition 874 ( $\sigma$ -additivity), and Lemma 913 (measure),  $\delta_Y$  is a measure on  $(X, \Sigma)$ .  $\square$

### Lemma 936 (finiteness of counting measure).

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Then,  $\delta_Y$  is finite iff the set  $Y$  is finite.

*Proof.* Direct consequence of Lemma 697 (equivalent definition of  $\sigma$ -algebra, contains full set), Lemma 935 (counting measure,  $\delta_Y(X) = \text{card}(Y)$ ), and Definition 857 (finiteness).  $\square$

### Lemma 937 ( $\sigma$ -finite counting measure).

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Assume that  $\Sigma$  contains all singletons of  $X$ . Then,  $\delta_Y$  is  $\sigma$ -finite implies  $Y$  is countable, and  $X$  is countable implies  $\delta_Y$  is  $\sigma$ -finite.

*Proof.* From Lemma 913 (measure), Definition 741 (measurable space), and Lemma 696 ( $\sigma$ -algebra, closedness under countable union), the  $\sigma$ -algebra  $\Sigma$  contains all countable subsets of  $X$ . Hence, if  $X$  is countable, we have  $\Sigma = \mathcal{P}(X)$ .

**First implication.** Assume that  $\delta_Y$  is  $\sigma$ -finite.

Then, from Lemma 886 (equivalent definition of  $\sigma$ -finiteness), let  $(B_n)_{n \in \mathbb{N}} \in \Sigma$  such that for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ ,  $\delta_Y(B_n) = \text{card}(B_n \cap Y) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} B_n$ . For all  $n \in \mathbb{N}$ ,

let  $A_n \stackrel{\text{def.}}{=} B_n \cap Y$ . Let  $n \in \mathbb{N}$ . Then, from **De Morgan's laws**, and Lemma 935 (*counting measure*), we have  $Y = \bigcup_{n \in \mathbb{N}} A_n$ , and  $\text{card}(A_n) = \delta_Y(B_n) < \infty$ . Hence, from **countability of countable union of finite subsets**,  $Y$  is countable.

**Second implication.** Assume that  $X$  is countable. Then, we have  $\Sigma = \mathcal{P}(X)$ .

From **the definition of countability**, let  $\varphi$  be a bijection from  $I \subset \mathbb{N}$  onto  $X$ . For all  $n \in \mathbb{N}$ , let  $A_n \stackrel{\text{def.}}{=} \varphi([0..n] \cap I) \in \Sigma$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 935 (*counting measure*), **monotonicity of the cardinality**, **preservation of the cardinality by bijection**, **compatibility of direct image with union**, and **De Morgan's laws**, we have

$$\delta_Y(A_n) = \text{card}(A_n \cap Y) \leq \text{card}(A_n) = \text{card}([0..n] \cap I) \leq n + 1 < \infty,$$

$$\bigcup_{n \in \mathbb{N}} A_n = \varphi \left( \bigcup_{n \in \mathbb{N}} ([0..n] \cap I) \right) = \varphi \left( \bigcup_{n \in \mathbb{N}} [0..n] \cap I \right) = \varphi(I) = X.$$

Hence, from Definition 885 ( *$\sigma$ -finiteness*), the counting measure  $\delta_Y$  is  $\sigma$ -finite.

Therefore, we have both implications.  $\square$

*Remark 938.*

Note that in the previous lemmas, the set  $Y$  is not required to be measurable. Note also that the previous lemmas are valid for any  $\sigma$ -algebra  $\Sigma$ , including the discrete  $\sigma$ -algebra  $\mathcal{P}(X)$ .

Counting measures are usually considered in the case where  $Y = X$  is countable, and  $\Sigma$  is the discrete  $\sigma$ -algebra  $\mathcal{P}(X)$ . A typical example is the  $\sigma$ -finite measure space  $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \delta_{\mathbb{N}})$ .

**Definition 939 (Dirac measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in X$ . The counting measure associated with  $\{a\}$  is called *Dirac measure (at  $a$ )*; it is also denoted  $\delta_a \stackrel{\text{def.}}{=} \delta_{\{a\}}$ .

**Lemma 940 (equivalent definition of Dirac measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in X$ , and let  $A \in \Sigma$ . Then, we have  $\delta_a(A) = \mathbb{1}_A(a)$ .

*Proof.* Direct consequence of Definition 939 (*Dirac measure*), and Lemma 935 (*counting measure*,  $\text{card}(A \cap \{a\}) = 1$  when  $a \in A$  and 0 otherwise).  $\square$

**Lemma 941 (Dirac measure is finite).**

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in X$ . Then,  $\delta_a$  is finite and  $\delta_a(X) = 1$ .

*Proof.* Direct consequence of Definition 939 (*Dirac measure*), Lemma 936 (*finiteness of counting measure*,  $\delta_a$  is finite), and Lemma 935 (*counting measure*,  $\text{card}(X \cap \{a\}) = 1$ ).  $\square$

## 12.3 Negligible subset

**Definition 942 (negligible subset).** Let  $(X, \Sigma, \mu)$  be a measure space. A subset  $A$  of  $X$  is said  $(\mu)$ -negligible iff there exists  $B \in \Sigma$  such that  $A \subset B$  and  $\mu(B) = 0$ .  
The set of  $\mu$ -negligible subsets is denoted  $\mathbf{N}(X, \Sigma, \mu)$  (or simply  $\mathbf{N}$ ).

**Definition 943 (complete measure).** Let  $(X, \Sigma, \mu)$  be a measure space. The measure  $\mu$  is said *complete* iff  $\mathbf{N}(X, \Sigma, \mu) \subset \Sigma$ . If so, the measure space  $(X, \Sigma, \mu)$  is also said *complete*.

*Remark 944.* In the next definition, “considerable” naturally means “non-negligible”.

**Definition 945 (considerable subset).** Let  $(X, \Sigma, \mu)$  be a measure space. A subset  $A$  of  $X$  is said  $(\mu)$ -considerable iff  $A \notin \mathbf{N}(X, \Sigma, \mu)$ .

**Lemma 946 (equivalent definition of considerable subset).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset X$ . Then,  $A$  is  $\mu$ -considerable iff for all  $B \in \Sigma$ ,  $A \subset B$  implies  $\mu(B) > 0$ .

*Proof.* Direct consequence of Definition 945 (*considerable subset*), Definition 942 (*negligible subset*), Lemma 913 (*measure, nonnegativeness*), and **the tautology**  $\neg(P \wedge Q) \Leftrightarrow (P \Rightarrow \neg Q)$ .  $\square$

**Lemma 947 (negligibility with trivial measure).** Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathbf{N}(X, \Sigma, 0) = \mathcal{P}(X)$ .

*Proof.* Direct consequence of Lemma 696 ( *$\sigma$ -algebra,  $X \in \Sigma$* ), Lemma 933 (*trivial measure,  $X$  is of measure zero*), and Definition 942 (*negligible subset, with  $B \stackrel{\text{def.}}{=} X$* ).  $\square$

**Lemma 948 (negligibility of measurable subset).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $A \in \mathbf{N}$  iff  $\mu(A) = 0$ .

*Proof.* Direct consequence of Definition 942 (*negligible subset, with  $B = A$* ), Lemma 919 (*measure is monotone, with  $A \subset B$ , in  $\Sigma$* ), and Lemma 913 (*measure, nonnegativeness*).  $\square$

**Lemma 949 (empty set is negligible).** Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\emptyset \in \mathbf{N}$ .

*Proof.* Direct consequence of Lemma 948 (*negligibility of measurable subset*), Lemma 913 (*measure*), Definition 741 (*measurable space,  $\Sigma$  is a  $\sigma$ -algebra*), Lemma 696 ( *$\sigma$ -algebra,  $\emptyset \in \Sigma$* ), and Lemma 913 (*measure, homogeneity*).  $\square$

*Remark 950.* Note that from Lemma 913 (*measure, homogeneity*), and Lemma 949 (*empty set is negligible*), the next two lemmas are compliant with the nullary union law ( $\bigcup_{\emptyset} = \emptyset$ ).

**Lemma 951 (subset of negligible is negligible).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \mathbf{N}$ . Then, we have  $\mathcal{P}(A) \subset \mathbf{N}$ .

*Proof.* Direct consequence of Definition 942 (*negligible subset*), **the definition of the power set**, and **transitivity of the inclusion**.  $\square$

**Lemma 952 ( $\mathbf{N}$  is closed under countable union).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbf{N}$ . Let  $(A_i)_{i \in I} \in \mathbf{N}$ . Then, we have  $\bigcup_{i \in I} A_i \in \mathbf{N}$ .

*Proof.* Direct consequence of Definition 942 (*negligible subset*), **monotonicity of union**, and Lemma 924 (*compatibility of null measure with countable union*).  $\square$



**Definition 953 (property almost satisfied).**

Let  $(X, \Sigma, \mu)$  be a measure space. A predicate  $P$  defined on  $X$  is said *satisfied ( $\mu$ -)almost everywhere* iff  $\{\neg P\} \stackrel{\text{def.}}{=} \{x \in X \mid \neg P(x)\} \in \mathbf{N}$ ; this is denoted either “ $P \mu$  a.e.”, “ $\forall x \in X, P(x) \mu$  a.e.”, “ $\forall_{\mu \text{ a.e.}} x \in X, P(x)$ ”, or “for  $\mu$ -almost all  $x \in X, P(x)$ ” (or simply without the mention of the measure  $\mu$ ).

*Remark 954.* When a single binary relation (using infix notation) is involved in  $P$ , the annotation “ $\mu$  a.e.” may be put above the infix operator, as in “ $\stackrel{\mu \text{ a.e.}}{=}$ ” (or simply as in “ $\stackrel{\text{a.e.}}{=}$ ”).

**Lemma 955 (equivalent definition of property almost satisfied).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  be a predicate on  $X$ . Then, we have

$$(12.6) \quad P \mu \text{ a.e.} \iff \exists A \in \Sigma, \quad A \subset \{P\} \quad \wedge \quad \mu(A^c) = 0.$$

*Proof.* Direct consequence of Definition 953 (*property almost satisfied*,  $\{P\}^c \in \mathbf{N}$ ), Definition 942 (*negligible subset*, there exists  $B \in \Sigma$  such that  $\{P\}^c \subset B$  and  $\mu(B) = 0$ ), **involutiveness of complement (with  $A \stackrel{\text{def.}}{=} B^c$ )**, Lemma 913 (*measure*), Definition 741 (*measurable space*), Lemma 696 ( *$\sigma$ -algebra*,  $A, A^c \in \Sigma$ ), and **monotonicity of complement ( $A \subset \{P\}$ )**.  $\square$

**Lemma 956 (property almost satisfied with trivial measure).**

Let  $(X, \Sigma)$  be a

measurable space. Let  $P$  be a predicate on  $X$ . Then,  $P$  is satisfied 0-almost everywhere.

*Proof.* Direct consequence of Lemma 947 (*negligibility with trivial measure*,  $\{\neg P\} \in \mathbf{N} = \mathcal{P}(X)$ ), and Definition 953 (*property almost satisfied*).  $\square$

**Lemma 957 (measurable property almost satisfied).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  be a predicate on  $X$ . Assume that  $\{P\} \in \Sigma$ . Then, we have  $P \mu$  a.e. iff  $\mu(\{\neg P\}) = 0$ .

*Proof.* Direct consequence of Definition 953 (*property almost satisfied*), and Lemma 948 (*negligibility of measurable subset*).  $\square$

**Lemma 958 (everywhere is almost everywhere).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  be a predicate on  $X$ . Then, we have

$$(12.7) \quad (\forall x \in X, P(x)) \implies P \mu \text{ a.e.}$$

*Proof.* Direct consequence of Definition 953 (*property almost satisfied*), and Lemma 949 (*empty set is negligible*).  $\square$

**Lemma 959 (almost modus ponens).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  and  $Q$  be predicates on  $X$ . Then, we have

$$(12.8) \quad (\forall x \in X, P(x) \implies Q(x)) \quad \wedge \quad P \mu \text{ a.e.} \implies Q \mu \text{ a.e.}$$

*Proof.* Direct consequence of **monotonicity of complement ( $\{Q\}^c \subset \{P\}^c$ )**, Definition 953 (*property almost satisfied*), and Lemma 951 (*subset of negligible is negligible*, with  $A \stackrel{\text{def.}}{=} \{P\}^c$ ).  $\square$

**Lemma 960 (countable almost properties is almost property).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbf{N}$ , and let  $(P_i)_{i \in I}$  be predicates on  $X$ . Then, we have

$$(12.9) \quad (\forall i \in I, P_i \mu \text{ a.e.}) \iff \bigwedge_{i \in I} P_i \mu \text{ a.e.}$$

$$(12.10) \quad \iff \exists A \in \Sigma, \quad A \subset \left\{ \bigwedge_{i \in I} P_i \right\} = \bigcap_{i \in I} \{P_i\} \quad \wedge \quad \mu(A^c) = 0.$$



*Proof.* Direct consequence of Definition 953 (*property almost satisfied*), Lemma 952 ( *$\mathbf{N}$  is closed under countable union*), Lemma 951 (*subset of negligible is negligible*), and Lemma 955 (*equivalent definition of property almost satisfied*).  $\square$

*Remark 961.*

The previous lemma allows to split a property satisfied almost everywhere into all its components (each satisfied almost everywhere) when it is a countable conjunction of subproperties.

**Lemma 962 (*almost everywhere for almost the same*).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $P$  be a predicate on  $Y$ , lifted into a predicate on  $Y^X$ . Then, for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I}, g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.11) \quad P(f) \mu \text{ a.e.} \quad \wedge \quad f \stackrel{\mu \text{ a.e.}}{=} g \quad \implies \quad P(g) \mu \text{ a.e.}$$

In particular, we have  $(\forall x \in X, P(f(x))) \wedge f \stackrel{\mu \text{ a.e.}}{=} g \implies P(g) \mu \text{ a.e.}$

*Proof.* Direct consequence of **the tautology**  $P(f) \wedge f = g \implies P(g)$ , Lemma 959 (*almost modus ponens*, with  $P(x) \stackrel{\text{def.}}{=} P(f(x)) \wedge f(x) = g(x)$ ), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} P$  and  $P_1 \stackrel{\text{def.}}{=} (f = g)$ ), and Lemma 958 (*everywhere is almost everywhere*).  $\square$

**Lemma 963 (*extended almost modus ponens*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $y \stackrel{\text{def.}}{=} (y_i)_{i \in I} \in Y$ . Let  $P$  and  $Q$  be predicates on  $Y$ , lifted into predicates on  $Y^X$ . Let  $R$  and  $S$  be predicates on  $Y^X$ . Assume that  $P(y)$  holds. Assume that  $R$  and  $S$  are compatible with almost equality ( $f \stackrel{\mu \text{ a.e.}}{=} g \wedge R(f) \implies R(g)$ , and the same for  $S$ ). Assume that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.12) \quad (\forall x \in X, P(f(x))) \quad \wedge \quad R(f) \quad \implies \quad Q(f) \mu \text{ a.e.} \quad \wedge \quad S(f).$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.13) \quad P(g) \mu \text{ a.e.} \quad \wedge \quad R(g) \quad \implies \quad Q(g) \mu \text{ a.e.} \quad \wedge \quad S(g).$$

*Proof.* Let  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ . Assume that  $P(g) \mu \text{ a.e.}$  and  $R(g)$  hold.

Let  $B \stackrel{\text{def.}}{=} \{P(g)\}$ . Then, from Lemma 955 (*equivalent definition of property almost satisfied*), there exists  $A \subset B$  such that  $A, A^c \in \Sigma$ , and  $\mu(A^c) = 0$ .

For all  $i \in I$ , let  $f_i(x) \stackrel{\text{def.}}{=} g_i(x)$  when  $x \in A$ , and let  $f_i(x) \stackrel{\text{def.}}{=} y_i$  otherwise. Let  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I}$ . Then, by construction,  $P(f)$  holds everywhere, and from Lemma 955 (*equivalent definition of property almost satisfied*, with  $P \stackrel{\text{def.}}{=} (f = g)$  and  $A \subset \{f = g\}$ , thus  $f \stackrel{\mu \text{ a.e.}}{=} g$ ), we have  $R(f)$ .

Therefore, from (12.12), and Lemma 962 (*almost everywhere for almost the same*), we have  $Q(f) \mu \text{ a.e.}$  and  $S(f)$ , thus  $Q(g) \mu \text{ a.e.}$ , and  $S(g)$ .  $\square$

*Remark 964.* In the previous lemma,  $P$  and  $Q$  are pointwise predicates acting on  $Y$ , while  $R$  and  $S$  are “functionwise” predicates acting on functions  $X \rightarrow Y$  (such as properties involving integrals when  $Y_i \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ). Of course, any of them may be the always-true predicate, and thus disappears from the conjunctions.

**Lemma 965 (extended almost modus ponens (simplified 0)).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $y \stackrel{\text{def.}}{=} (y_i)_{i \in I} \in Y$ . Let  $P$  be a predicate on  $Y$ , lifted into predicates on  $Y^X$ . Let  $S$  be a predicates on  $Y^X$ . Assume that  $P(y)$  holds. Assume that  $R$  is compatible with almost equality. Assume that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.14) \quad (\forall x \in X, P(f(x))) \implies S(f).$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ , we have  $P(g) \mu \text{ a.e.} \implies S(g)$ .

*Proof.* Direct consequence of Lemma 963 (extended almost modus ponens, with  $Q = R \stackrel{\text{def.}}{=} \text{True}$ ).  $\square$

**Lemma 966 (extended almost modus ponens (simplified 1)).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $y \stackrel{\text{def.}}{=} (y_i)_{i \in I} \in Y$ . Let  $P$  and  $Q$  be predicates on  $Y$ , lifted into predicates on  $Y^X$ . Assume that  $P(y)$  holds. Assume that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.15) \quad (\forall x \in X, P(f(x))) \implies Q(f) \mu \text{ a.e.}$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ , we have  $P(g) \mu \text{ a.e.} \implies Q(g) \mu \text{ a.e.}$

*Proof.* Direct consequence of Lemma 963 (extended almost modus ponens, with  $R = S \stackrel{\text{def.}}{=} \text{True}$ ).  $\square$

**Lemma 967 (extended almost modus ponens (simplified 2)).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $y \stackrel{\text{def.}}{=} (y_i)_{i \in I} \in Y$ . Let  $P$  and  $Q$  be predicates on  $Y$ , lifted into predicates on  $Y^X$ . Assume that  $P(y)$  holds. Assume that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} : X \rightarrow Y$ , we have

$$(12.16) \quad (\forall x \in X, P(f(x))) \implies \forall x \in X, Q(f(x)).$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : X \rightarrow Y$ , we have  $P(g) \mu \text{ a.e.} \implies Q(g) \mu \text{ a.e.}$

*Proof.* Direct consequence of Lemma 966 (extended almost modus ponens (simplified 1)), and Lemma 958 (everywhere is almost everywhere, with  $P \stackrel{\text{def.}}{=} Q$ ).  $\square$

*Remark 968.* The four previous lemmas allow to still use modus ponens in reasoning when predicates are only valid almost everywhere.

**Definition 969 (almost definition).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. A function  $f : X \rightarrow Y$  is said *defined ( $\mu$ -)almost everywhere in  $X$*  iff the property “ $f(x)$  is defined” is satisfied  $\mu$ -almost everywhere.

The set of functions  $X \rightarrow Y$  defined ( $\mu$ -)almost everywhere is denoted  $(Y^X)_{\mu \text{ a.e.}}$ , or through the type annotation  $(X \rightarrow Y)_{\mu \text{ a.e.}}$  (or simply without the mention of the measure  $\mu$ ).

**Lemma 970 (extension to almost defined).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y$  be a nonempty set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $P$  be a predicate on  $Y$ , lifted into a predicate on  $Y^X$ . Then, we have

$$(12.17) \quad (\forall (f_i)_{i \in I} : X \rightarrow Y, P(f_i)_{i \in I}) \implies \forall (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}, P(g_i)_{i \in I} \mu \text{ a.e.}$$

*Proof.* Let  $y \stackrel{\text{def.}}{=} (y_i)_{i \in I} \in Y$ . Assume that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} : X \rightarrow Y$ ,  $P(f)$  holds.

Let  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ .

Let  $i \in I$ . Let  $B_i \stackrel{\text{def.}}{=} \{g_i \text{ is defined}\}$ . Let  $f_i$  be the function defined by  $f_i(x) \stackrel{\text{def.}}{=} g_i(x)$  when  $x \in B_i$ , and  $f_i(x) \stackrel{\text{def.}}{=} y_i$  otherwise. Then, by construction, and from Definition 953 (*property almost satisfied*, with  $\{f_i = g_i\} = B_i$ ),  $f_i$  is a total function on  $X$ , and we have  $f_i \stackrel{\mu \text{ a.e.}}{=} g_i$ .

Therefore, from Lemma 960 (*countable almost properties is almost property*, with  $P_i$  set to  $\{f_i = g_i\}$ , thus  $f \stackrel{\mu \text{ a.e.}}{=} g$ ), and Lemma 959 (*almost modus ponens*, with  $P \stackrel{\text{def.}}{=} (f = g)$ ,  $Q \stackrel{\text{def.}}{=} P(g)$  and  $(f = g) \Rightarrow P(g)$ ), we have  $P(g) \mu \text{ a.e.}$   $\square$

*Remark 971.* The previous lemma allows to replace a property on total functions by the same property satisfied almost everywhere for functions defined almost everywhere.

**Lemma 972 (generalized almost modus ponens).** *Let  $(X, \Sigma, \mu)$  be a measure space.*

*Let  $I \subset \mathbb{N}$  be nonempty. For all  $i \in I$ , let  $Y_i$  be a nonempty set. Let  $Y \stackrel{\text{def.}}{=} \prod_{i \in I} Y_i$ . Let  $P$  and  $Q$  be predicates on  $Y$ , lifted into predicates on  $Y^X$ . Assume that for all  $(f_i)_{i \in I} : X \rightarrow Y$ , we have*

$$(12.18) \quad \forall x \in X, \quad P(f_i(x))_{i \in I} \implies Q(f_i(x))_{i \in I}.$$

*Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ , we have*

$$(12.19) \quad P(g) \mu \text{ a.e.} \implies Q(g) \mu \text{ a.e.}$$

*Proof.* Direct consequence of Lemma 970 (*extension to almost defined*), **modus ponens ( $P$  implies  $Q$  and  $P$ , implies  $Q$ )**, Lemma 959 (*almost modus ponens*, with  $P \stackrel{\text{def.}}{=} (P \Rightarrow Q) \wedge P$ ), and Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} (P \Rightarrow Q)$  and  $P_1 \stackrel{\text{def.}}{=} P$ ).  $\square$

*Remark 973.* The previous lemma also allows to still use modus ponens in reasoning when predicates are only valid almost everywhere.

**Definition 974 (almost binary relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . The functions  $f, g : (X \rightarrow Y)_{\mu \text{ a.e.}}$  are said *in relation ( $\mu$ -)almost everywhere through  $\mathcal{R}$*  iff the property “ $f(x) \mathcal{R} g(x)$ ” is satisfied  $\mu$ -almost everywhere, i.e.

$$\{x \in X \mid f(x) \mathcal{R} g(x)\}^c \in \mathbf{N}.$$

This is denoted  $f \mathcal{R}_{\mu \text{ a.e.}} g$  (or simply  $f \mathcal{R}_{\text{a.e.}} g$ ).

**Lemma 975 (compatibility of almost binary relation with reflexivity).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is reflexive. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also reflexive,*

$$\forall f : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad f \mathcal{R}_{\mu \text{ a.e.}} f.$$

*Proof.* Direct consequence of Definition 974 (*almost binary relation*), **the definition of reflexivity**, and Lemma 970 (*extension to almost defined*, with  $\text{card}(I) = 1$  and  $P(y) \stackrel{\text{def.}}{=} y \mathcal{R} y$ ).  $\square$

*Remark 976.* Note that reflexivity can also take a form symmetrical to that of antisymmetry:

$$\forall x, y \in X, \quad x = y \implies x \mathcal{R} y \wedge y \mathcal{R} x.$$

This leads to the next lemma, which is actually stronger than the previous one.

**Lemma 977 (compatibility of almost binary relation with extended reflexivity).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is reflexive. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is “almost” reflexive (where equality is replaced by almost equality), i.e.

$$\forall f, g : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad f \stackrel{\mu \text{ a.e.}}{=} g \implies f \mathcal{R}_{\mu \text{ a.e.}} g \wedge g \mathcal{R}_{\mu \text{ a.e.}} f.$$

*Proof.* Direct consequence of **the definition of reflexivity**, Definition 974 (*almost binary relation*), Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} Y$ ,  $P(f, g)$  is the predicate  $(f = g)$  and  $Q(f, g) \stackrel{\text{def.}}{=} f \mathcal{R} g \wedge g \mathcal{R} f$ ), and Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} f \mathcal{R} g$  and  $P_1 \stackrel{\text{def.}}{=} g \mathcal{R} f$ ).  $\square$

**Lemma 978 (compatibility of almost binary relation with symmetry).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is symmetric. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also symmetric,

$$\forall f, g : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad f \mathcal{R}_{\mu \text{ a.e.}} g \implies g \mathcal{R}_{\mu \text{ a.e.}} f.$$

*Proof.* Direct consequence of **the definition of symmetry**, Definition 974 (*almost binary relation*), and Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} Y$ ,  $P(f, g) \stackrel{\text{def.}}{=} f \mathcal{R} g$  and  $Q(f, g) \stackrel{\text{def.}}{=} g \mathcal{R} f$ ).  $\square$

**Lemma 979 (compatibility of almost binary relation with extended antisymmetry).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is antisymmetric. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is “almost” antisymmetric (where equality is replaced by almost equality), i.e.

$$\forall f, g : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad f \mathcal{R}_{\mu \text{ a.e.}} g \wedge g \mathcal{R}_{\mu \text{ a.e.}} f \implies f \stackrel{\mu \text{ a.e.}}{=} g.$$

*Proof.* Direct consequence of **the definition of antisymmetry**, Definition 974 (*almost binary relation*), Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} Y$ ,  $P(f, g)$  is the predicate  $f \mathcal{R} g \wedge g \mathcal{R} f$  and  $Q(f, g) \stackrel{\text{def.}}{=} (f = g)$ ), and Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} f \mathcal{R} g$  and  $P_1 \stackrel{\text{def.}}{=} g \mathcal{R} f$ ).  $\square$

**Lemma 980 (compatibility of almost binary relation with transitivity).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is transitive. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also transitive,

$$\forall f, g, h : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad f \mathcal{R}_{\mu \text{ a.e.}} g \wedge g \mathcal{R}_{\mu \text{ a.e.}} h \implies f \mathcal{R}_{\mu \text{ a.e.}} h.$$

*Proof.* Direct consequence of **the definition of transitivity**, Definition 974 (*almost binary relation*), Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \{0, 1, 2\}$ ,  $Y_0 = Y_1 = Y_2 \stackrel{\text{def.}}{=} Y$ ,  $P(f, g, h) \stackrel{\text{def.}}{=} f \mathcal{R} g \wedge g \mathcal{R} h$  and  $Q(f, g, h) \stackrel{\text{def.}}{=} f \mathcal{R} h$ ), and Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} f \mathcal{R} g$  and  $P_1 \stackrel{\text{def.}}{=} g \mathcal{R} h$ ).  $\square$

**Lemma 981 (almost equivalence is equivalence relation).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is an equivalence relation. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also an equivalence relation.

*Proof.* Direct consequence of **the definition of equivalence relation**, Lemma 975 (*compatibility of almost binary relation with reflexivity*), Lemma 978 (*compatibility of almost binary relation with symmetry*), and Lemma 980 (*compatibility of almost binary relation with transitivity*).  $\square$

**Lemma 982 (almost equality is equivalence relation).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Then,  $\stackrel{\mu \text{ a.e.}}{=}$  is an equivalence relation over  $Y^X$ .

*Proof.* Direct consequence of Lemma 981 (almost equivalence is equivalence relation, with equality over  $Y$ ).  $\square$

**Lemma 983 (almost order is order relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is an order relation. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is an “almost” order relation (where equality is replaced by almost equality in antisymmetry).

*Proof.* Direct consequence of **the definition of order relation**, Lemma 975 (compatibility of almost binary relation with reflexivity), Lemma 979 (compatibility of almost binary relation with extended antisymmetry), and Lemma 980 (compatibility of almost binary relation with transitivity).  $\square$

**Lemma 984 (compatibility of almost binary relation with operator).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $\mathcal{R}$  and  $\mathcal{R}'$  be binary relations over  $Y$ , lifted into binary relations over  $Y^X$ . Let  $I \subset \mathbb{N}$  be nonempty. Let  $\diamond : Y^I \rightarrow Y$ , lifted into an operator  $(Y^X)^I \rightarrow Y^X$ . Assume that for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : X \rightarrow Y$ , for all  $x \in X$ , we have

$$(12.20) \quad (\forall i \in I, f_i(x) \mathcal{R} g_i(x)) \implies \diamond(f_i(x))_{i \in I} \mathcal{R}' \diamond(g_i(x))_{i \in I}.$$

Then, for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ , we have

$$(12.21) \quad (\forall i \in I, f_i \mathcal{R}_{\mu \text{ a.e.}} g_i) \implies \diamond(f_i)_{i \in I} \mathcal{R}'_{\mu \text{ a.e.}} \diamond(g_i)_{i \in I}.$$

*Proof.* Direct consequence of Definition 974 (almost binary relation), Lemma 972 (generalized almost modus ponens, with  $f_i \stackrel{\text{def.}}{=} (f_i, g_i)$ ,  $P \stackrel{\text{def.}}{=} (\forall i, f_i \mathcal{R} g_i)$  and  $Q \stackrel{\text{def.}}{=} (\diamond(f_i)_{i \in I} \mathcal{R}' \diamond(g_i)_{i \in I})$ , and Lemma 960 (countable almost properties is almost property, with  $P_i \stackrel{\text{def.}}{=} f_i \mathcal{R} g_i$ ).  $\square$

**Lemma 985 (compatibility of almost equality with operator).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $I \subset \mathbb{N}$  be nonempty. Let  $\diamond : Y^I \rightarrow Y$ , lifted into an operator  $(Y^X)^I \rightarrow Y^X$ . Let  $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ . Assume that for all  $i \in I$ ,  $f_i \stackrel{\mu \text{ a.e.}}{=} g_i$ . Then, we have  $\diamond(f_i)_{i \in I} \stackrel{\mu \text{ a.e.}}{=} \diamond(g_i)_{i \in I}$ .

*Proof.* Direct consequence of Lemma 984 (compatibility of almost binary relation with operator, with  $\mathcal{R}, \mathcal{R}' \stackrel{\text{def.}}{=} \text{equality}$ ), and **the definition of function** ((for all  $i \in I$ ,  $f_i = g_i$ ) implies  $\diamond(f_i)_{i \in I} = \diamond(g_i)_{i \in I}$ ).  $\square$

*Remark 986.* The previous generic results apply to unary operators  $\diamond$  such as the scalar multiplication, or the absolute value (for which  $\text{card}(I) = 1$ ), as well as to binary (or  $n$ -ary) operators  $\diamond$  such as the addition (when defined), the multiplication, the maximum, or the minimum (for which  $\text{card}(I) \geq 2$  is finite). But it also applies to operators  $\diamond$  taking a countable number of arguments such as the infimum, the supremum, or the limit (assuming that the sequence is pointwise convergent).

*Remark 987.* Note that Lemma 984 may be useful with  $\mathcal{R}'$  distinct from  $\mathcal{R}$ . For instance, when  $\mathcal{R} \stackrel{\text{def.}}{=} \leq$  and  $\diamond$  is the scalar multiplication by a negative number, we need  $\mathcal{R}' \stackrel{\text{def.}}{=} \geq$ .

**Lemma 988 (definiteness is almost definiteness).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Assume that  $0 \in Y$ . Let  $\diamond : Y \rightarrow Y$ , lifted into an operator  $Y^X \rightarrow Y^X$ . Assume that  $\diamond$  is definite. Then,  $\diamond$  is also “almost definite” (where equality is replaced by almost equality), i.e.

$$(12.22) \quad \forall f : (X \rightarrow Y)_{\mu \text{ a.e.}}, \quad \diamond(f) \stackrel{\mu \text{ a.e.}}{=} 0 \implies f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Direct consequence of Lemma 972 (*generalized almost modus ponens*, with  $\text{card}(I) = 1$ ,  $P(f) \stackrel{\text{def.}}{=} (\diamond(f) = 0)$  and  $Q(f) \stackrel{\text{def.}}{=} (f = 0)$ ).  $\square$

*Remark 989.* Note that Lemma 985 provides the implication in the other direction.

## 12.4 Uniqueness condition

*Remark 990.* The next lemma is useful to establish in Section 13.3 uniqueness of the Lebesgue measure that generalizes the length of bounded intervals (see Theorem 1090).

*Remark 991.* The next proof follows the Dynkin  $\pi$ - $\lambda$  theorem scheme (see Section 4.2).

**Lemma 992 (uniqueness of measures extended from a  $\pi$ -system).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mu_1$  and  $\mu_2$  be measures on  $(X, \Sigma)$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a  $\pi$ -system, a generator of  $\Sigma$ , contains a countable pseudopartition  $(X_n)_{n \in \mathbb{N}}$  of  $X$  with finite measure  $\mu_1$ , and that both measures coincide on  $G$ . Then, we have  $\mu_1 = \mu_2$ .

*Proof.* Let  $A \in G$ . Assume that  $\mu_1(A) < \infty$  (e.g.  $A \stackrel{\text{def.}}{=} X_0$ ).

Let  $B \in \Sigma$ . Then, from Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*,  $A \in G \subset \Sigma$ ), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection with card( $I$ ) equals 2), we have  $B \cap A \in \Sigma$ . Let  $\mathcal{S}_A \stackrel{\text{def.}}{=} \{B \in \Sigma \mid \mu_1(B \cap A) = \mu_2(B \cap A)\}$ .

(1).  $G \neq \emptyset \wedge \Pi_X(G) \subset \mathcal{S}_A$ .

Let  $B \in G$ . Then, from Definition 602 ( *$\pi$ -system*, closedness under intersection),  $B \cap A \in G$ , i.e.  $B \in \mathcal{S}_A$ . Thus, we have  $G \subset \mathcal{S}_A$ . Hence, from Definition 602 ( *$\pi$ -system*, nonemptiness), and Lemma 608 (*properties of  $\pi$ -system generation*, idempotence), we have  $G \neq \emptyset$  and  $\Pi_X(G) = G \subset \mathcal{S}_A$ .

(2).  $\mathcal{S}_A$  is  $\lambda$ -system.

From **the identity**  $X \cap A = A$ , and Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, contains full set), we have  $X \in \mathcal{S}_A$ .

Let  $B_1, B_2 \in \mathcal{S}_A$ . Assume that  $B_1 \subset B_2$ . Then, from **distributivity of intersection over local complement**, and Lemma 919 (*measure is monotone*, with  $\mu_1$  and  $\mu_2$ ), we have

$$\begin{aligned} \mu_2(B_1 \cap A) &= \mu_1(B_1 \cap A) \leq \mu_1(A) < \infty, & \text{and} \\ \mu_1((B_2 \setminus B_1) \cap A) &= \mu_1(B_2 \cap A \setminus B_1 \cap A) = \mu_1(B_2 \cap A) - \mu_1(B_1 \cap A) \\ &= \mu_2(B_2 \cap A) - \mu_2(B_1 \cap A) = \mu_2(B_2 \cap A \setminus B_1 \cap A) = \mu_2((B_2 \setminus B_1) \cap A). \end{aligned}$$

Then,  $B_2 \setminus B_1 \in \mathcal{S}_A$ . Thus,  $\mathcal{S}_A$  is closed under local complement.

Let  $(B_n)_{n \in \mathbb{N}} \in \mathcal{S}_A$ . Assume that for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ . Then, from **distributivity of intersection over union**, Lemma 925 (*measure is continuous from below*, with  $\mu_1$  and  $\mu_2$ , and nondecreasing  $(B_n \cap A)_{n \in \mathbb{N}}$ ), and Definition 879 (*continuity from below*), we have

$$\begin{aligned} \mu_1\left(\bigcup_{n \in \mathbb{N}} B_n \cap A\right) &= \mu_1\left(\bigcup_{n \in \mathbb{N}} (B_n \cap A)\right) = \lim_{n \rightarrow \infty} \mu_1(B_n \cap A) \\ &= \lim_{n \rightarrow \infty} \mu_2(B_n \cap A) = \mu_2\left(\bigcup_{n \in \mathbb{N}} (B_n \cap A)\right) = \mu_2\left(\bigcup_{n \in \mathbb{N}} B_n \cap A\right). \end{aligned}$$

Then,  $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{S}_A$ . Thus,  $\mathcal{S}_A$  is closed under countable monotone union.

Hence, from Lemma 678 (*equivalent definition of  $\lambda$ -system*),  $\mathcal{S}_A$  is a  $\lambda$ -system on  $X$ .

(3).  $\mathcal{S}_A = \Sigma$ . Direct consequence of (1), (2), and Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*, with  $P$  being the function  $B \mapsto \mu_1(B \cap A) = \mu_2(B \cap A)$ ).

Hence, for all  $B \in \Sigma$ , for all  $A \in G$  such that  $\mu_1(A) < \infty$ , we have  $\mu_1(B \cap A) = \mu_2(B \cap A)$ .

(4).  $\mu_1 = \mu_2$ . Let  $B \in \Sigma$ .

For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} B \cap X_n$ . Then, from Definition 209 (*pseudopartition*), Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*,  $X_n \in G \subset \Sigma$ ), Lemma 697 (*equivalent definition of  $\sigma$ -algebra*,

closedness under countable intersection with  $\text{card}(I)$  equals 2), **compatibility of intersection with pairwise disjunction**, **left distributivity of intersection over union**, and **the identity law for intersection**,  $(B_n)_{n \in \mathbb{N}} \in \Sigma$  is pairwise disjoint, and we have  $\biguplus_{n \in \mathbb{N}} B_n = B$ .

Let  $n \in \mathbb{N}$ . Then, from (3) (with  $B \in \Sigma$ , and  $A \stackrel{\text{def.}}{=} X_n \in G$  such that  $\mu_1(X_n) < \infty$ ), we have  $\mu_1(B_n) = \mu_2(B_n)$ . Hence, from Lemma 913 (*measure*,  $\sigma$ -additivity with  $\mu_1$  and  $\mu_2$ ), and Definition 874 ( *$\sigma$ -additivity*), we have

$$\mu_1(B) = \sum_{n \in \mathbb{N}} \mu_1(B_n) = \sum_{n \in \mathbb{N}} \mu_2(B_n) = \mu_2(B).$$

Therefore, we have the equality  $\mu_1 = \mu_2$ . □





## Chapter 13

# Measure and numbers

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### 13.1 Negligibility and numbers

**Lemma 993 (transitivity of almost nonnegativeness).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}$ . Then, we have

$$(13.1) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \wedge f \stackrel{\mu \text{ a.e.}}{\geq} 0 \implies g \stackrel{\mu \text{ a.e.}}{\geq} 0.$$

*Proof.* Direct consequence of Lemma 980 (compatibility of almost binary relation with transitivity, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $f \stackrel{\text{def.}}{=} (x \mapsto 0)$ ,  $g \stackrel{\text{def.}}{=} f$  and  $h \stackrel{\text{def.}}{=} g$ ).  $\square$

**Lemma 994 (transitivity of almost inequality and equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g, h : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}$ . Then, we have

$$(13.2) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \stackrel{\mu \text{ a.e.}}{=} h \vee f \stackrel{\mu \text{ a.e.}}{=} g \stackrel{\mu \text{ a.e.}}{\leq} h \implies f \stackrel{\mu \text{ a.e.}}{\leq} h,$$

$$(13.3) \quad f \leq g \stackrel{\mu \text{ a.e.}}{=} h \vee f \stackrel{\mu \text{ a.e.}}{=} g \leq h \implies f \stackrel{\mu \text{ a.e.}}{\leq} h.$$

*Proof.* **Property (13.2).** Direct consequence of Lemma 977 (compatibility of almost binary relation with extended reflexivity, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and  $\mathcal{R} \stackrel{\text{def.}}{=} \leq$ , thus  $f \stackrel{\mu \text{ a.e.}}{\leq} g \stackrel{\mu \text{ a.e.}}{\leq} h$ ), and Lemma 980 (compatibility of almost binary relation with transitivity, with  $\mathcal{R} \stackrel{\text{def.}}{=} \leq$ ).

**Property (13.3).** Direct consequence of (13.2), and Lemma 958 (everywhere is almost everywhere).  $\square$

**Lemma 995 (compatibility of almost nonnegativeness with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}$ . Then, we have

$$(13.4) \quad f \stackrel{\mu \text{ a.e.}}{=} g \implies (f \stackrel{\mu \text{ a.e.}}{\geq} 0 \iff g \stackrel{\mu \text{ a.e.}}{\geq} 0).$$

*Proof.* Direct consequence of Lemma 977 (compatibility of almost binary relation with extended reflexivity, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} g$  and  $g \stackrel{\mu \text{ a.e.}}{\leq} f$ ), and Lemma 993 (transitivity of almost nonnegativeness, with  $f, g$ , then with  $f \stackrel{\text{def.}}{=} g$  and  $g \stackrel{\text{def.}}{=} f$ ).  $\square$

**Lemma 996 (almost finite).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then, we have  $|f| \stackrel{\mu \text{ a.e.}}{<} \infty$  iff  $\mu(\{f = \pm\infty\}) = 0$ .

*Proof.* Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ), Lemma 341 (finite absolute value in  $\overline{\mathbb{R}}$ , thus  $\{|f| < \infty\} = \{f \neq \pm\infty\}$ ), Lemma 808 (inverse image of countable is measurable, thus  $\{f = \pm\infty\} \in \Sigma$ ), and Lemma 957 (measurable property almost satisfied).  $\square$

**Definition 997 (summability domain).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$ . The summability domain of  $f$  and  $g$  is denoted  $\mathcal{D}^+(f, g)$ ; it is defined by

$$(13.5) \quad \mathcal{D}^+(f, g) \stackrel{\text{def.}}{=} [(\{f = \infty\} \cap \{g = -\infty\}) \cup (\{f = -\infty\} \cap \{g = \infty\})]^c.$$

**Lemma 998 (summability on summability domain).**

Let  $f, g : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f(x) + g(x)$  is well-defined iff  $x \in \mathcal{D}^+(f, g)$ .

*Proof.* Direct consequence of Definition 317 (addition in  $\overline{\mathbb{R}}$ , rule 3 cannot occur), and Definition 997 (summability domain).  $\square$

**Lemma 999 (measurability of summability domain).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Then,  $\mathcal{D}^+(f, g) \in \Sigma$ .

*Proof.* Direct consequence of Lemma 809 (*inverse image is measurable*, with  $f, g$ , and  $y \stackrel{\text{def.}}{=} \pm\infty$ ), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection, union and complement).  $\square$

**Lemma 1000 (negligibility and summability domain).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f + g$  is well-defined almost everywhere iff  $[\mathcal{D}^+(f, g)]^c \in \mathbf{N}$ .*

*Proof.* Direct consequence of Lemma 998 (*summability on summability domain*), and Definition 953 (*property almost satisfied*).  $\square$

**Lemma 1001 (absolute value and almost inequality).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g, h : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}$ . Then, we have*

$$(13.6) \quad |f| \stackrel{\mu \text{ a.e.}}{\leq} g \iff -g \stackrel{\mu \text{ a.e.}}{\leq} f \stackrel{\mu \text{ a.e.}}{\leq} g.$$

*If  $\mathcal{D}^+(g, h)^c \in \mathbf{N}$ , then we have*

$$(13.7) \quad |f| \stackrel{\mu \text{ a.e.}}{\leq} |g + h| \implies |f| \stackrel{\mu \text{ a.e.}}{\leq} |g| + |h|,$$

*and if  $\mathcal{D}^+(f, \pm g)^c, \mathcal{D}^+(h, \pm g)^c \in \mathbf{N}$ , then we have*

$$(13.8) \quad |f - g| \stackrel{\mu \text{ a.e.}}{\leq} h \implies ||f| - |g|| \stackrel{\mu \text{ a.e.}}{\leq} h \iff -h + |g| \stackrel{\mu \text{ a.e.}}{\leq} |f| \stackrel{\mu \text{ a.e.}}{\leq} h + |g|.$$

*Proof. Property (13.6).* Direct consequence of Lemma 972 (*generalized almost modus ponens*, with  $\text{card}(I) = 2$ ), and Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* , thus  $|f| \leq g \Leftrightarrow -g \leq f \leq g$ ).

*Property (13.7).* Direct consequence of Lemma 1000 (*negligibility and summability domain*), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} (g + h \text{ well-defined})$  and  $P_1 \stackrel{\text{def.}}{=} |f| \leq |g + h|$ ), Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*), Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*, transitivity, thus  $P_0 \wedge P_1 \Rightarrow |f| \leq |g + h|$ ), and Lemma 972 (*generalized almost modus ponens*, with  $\text{card}(I) = 3$  and  $P \stackrel{\text{def.}}{=} P_0 \wedge P_1$ ).

*Property (13.8).* Direct consequence of Lemma 1000 (*negligibility and summability domain*), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} [0..4]$ ,  $P_0 \stackrel{\text{def.}}{=} (f + g \text{ well-defined})$ ,  $P_1 \stackrel{\text{def.}}{=} (f - g \text{ well-defined})$ ,  $P_2 \stackrel{\text{def.}}{=} (h + g \text{ well-defined})$ ,  $P_3 \stackrel{\text{def.}}{=} (h - g \text{ well-defined})$  and  $P_4 \stackrel{\text{def.}}{=} (|f - g| \leq h)$ ), Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*), Lemma 983 (*almost order is order relation*, transitivity), Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* ), and Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*, thus  $P_0 \wedge P_1 \wedge P_2 \wedge P_3 \wedge P_4$  implies  $||f| - |g|| \leq h \Leftrightarrow -h + |g| \leq |f| \leq h + |g|$ ), and Lemma 972 (*generalized almost modus ponens*, with  $\text{card}(I) = 3$ ),

Therefore, all properties hold.  $\square$

**Lemma 1002 (almost sum).**

*Let  $(X, \Sigma, \mu)$  be a measure space.*

*Let  $f, g \in \mathcal{M}$ . Let  $A \stackrel{\text{def.}}{=} \mathcal{D}^+(f, g)$ . Let  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_A$  and  $\tilde{g} \stackrel{\text{def.}}{=} g \mathbf{1}_A$ . Assume that  $f + g$  is well-defined almost everywhere. Then, we have  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ,  $\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g$ ,  $\mathcal{D}^+(\tilde{f}, \tilde{g}) = X$ , and  $\tilde{f}, \tilde{g}, \tilde{f} + \tilde{g} \in \mathcal{M}$ .*

*The expression  $f \stackrel{\mu \text{ a.e.}}{+} g \stackrel{\text{def.}}{=} \tilde{f} + \tilde{g}$  is called the almost sum of  $f$  and  $g$ .*

*Proof.* From Lemma 1000 (*negligibility and summability domain*), and Definition 953 (*property almost satisfied*, with  $\{\tilde{f} \neq f\} = \{\tilde{g} \neq g\} = A^c$ ), we have  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$  and  $\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g$ . From Lemma 999 (*measurability of summability domain*), Lemma 811 (*measurability of indicator function*), and Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*), we have  $\tilde{f}, \tilde{g} \in \mathcal{M}$ . Hence, from Definition 317 (*addition in  $\overline{\mathbb{R}}$* ), and Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  $\tilde{f} + \tilde{g}$  is well-defined, and we have  $\tilde{f} + \tilde{g} \in \mathcal{M}$ .  $\square$

**Lemma 1003 (compatibility of almost sum with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g, f', g' \in \mathcal{M}$ . Assume that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f' + g' \in \mathcal{M}$ ,  $f + g$  is well-defined almost everywhere,  $f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and we have  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ .

*Proof.* From Lemma 998 (summability on summability domain), and Lemma 822 ( $\mathcal{M}$  is closed under addition when defined),  $f' + g'$  is well-defined, and we have  $f' + g' \in \mathcal{M}$ .

From Lemma 998 (summability on summability domain), and Lemma 962 (almost everywhere for almost the same, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $f \stackrel{\text{def.}}{=} (f', g')$ ,  $P(f', g') \stackrel{\text{def.}}{=} (f' + g')$  is well-defined) and  $g \stackrel{\text{def.}}{=} (f, g)$ ,  $f + g$  is well-defined almost everywhere.

Let  $C \stackrel{\text{def.}}{=} \mathcal{D}^+(f, g)$ ,  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_C$  and  $\tilde{g} \stackrel{\text{def.}}{=} g \mathbf{1}_C$ . Then, from Lemma 1002 (almost sum), Lemma 982 (almost equality is equivalence relation, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ , transitivity and reflexivity), we have  $f' \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} \tilde{g}$ , and  $f \stackrel{\mu \text{ a.e.}}{+} g = \tilde{f} + \tilde{g}$  is well-defined, and belongs to  $\mathcal{M}$ .

Therefore, from Lemma 985 (compatibility of almost equality with operator, with the binary operator addition, well-defined for  $(f', g')$  and  $(\tilde{f}, \tilde{g})$ ), we have  $f' + g' \stackrel{\mu \text{ a.e.}}{=} \tilde{f} + \tilde{g} = f \stackrel{\mu \text{ a.e.}}{+} g$ .  $\square$

**Lemma 1004 (almost sum is sum).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $\mathcal{D}^+(f, g) = X$ .

Then,  $f + g \in \mathcal{M}$ , and we have  $f \stackrel{\mu \text{ a.e.}}{+} g = f + g$ . In particular, this is the case when  $f, g \in \mathcal{M}_{\mathbb{R}}$ .

*Proof.* Direct consequence of Definition 997 (summability domain,  $f + g$  is well-defined everywhere), Lemma 1002 (almost sum, with  $\tilde{f} = f$  and  $\tilde{g} = g$ ), Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ , thus  $f + g$  is well-defined everywhere), and Lemma 998 (summability on summability domain, thus  $\mathcal{D}^+(f, g) = X$ ).  $\square$

**Lemma 1005 (absolute value is almost definite).** Let  $(X, \Sigma, \mu)$  be a measure space. Then, the absolute value, lifted into an operator  $\overline{\mathbb{R}}^X \rightarrow \overline{\mathbb{R}}^X$ , is almost definite, i.e.

$$(13.9) \quad \forall f : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}, \quad |f| \stackrel{\mu \text{ a.e.}}{=} 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Direct consequence of Lemma 988 (definiteness is almost definiteness, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and the unary operator absolute value), Lemma 347 (absolute value in  $\overline{\mathbb{R}}$  is definite), and Lemma 985 (compatibility of almost equality with operator, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $I \stackrel{\text{def.}}{=} \{0\}$ , the unary operator absolute value and  $g_0 \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Lemma 1006 (absolute value is almost equal to almost nonnegative function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then, we have

$$(13.10) \quad \forall f : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}, \quad |f| \stackrel{\mu \text{ a.e.}}{=} f \iff f \stackrel{\mu \text{ a.e.}}{\geq} 0.$$

*Proof.* Direct consequence of Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative), and Lemma 972 (generalized almost modus ponens, with  $\text{card}(I) = 1$  and  $P, Q$  being  $(a \mapsto |a| = a)$  or  $(a \mapsto a \geq 0)$ ).  $\square$

**Lemma 1007 (power function is almost definite).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty)$ .

Then, the power function of degree  $p$ , lifted into an operator  $\overline{\mathbb{R}}^X \rightarrow \overline{\mathbb{R}}^X$ , is almost definite, i.e.

$$(13.11) \quad \forall f : (X \rightarrow \overline{\mathbb{R}})_{\mu \text{ a.e.}}, \quad |f|^p \stackrel{\mu \text{ a.e.}}{=} 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Direct consequence of Lemma 988 (*definiteness is almost definiteness*, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and the unary operator power function of degree  $p$ ), Lemma 369 (*power function in  $\mathbb{R}$  is definite*), and Lemma 985 (*compatibility of almost equality with operator*, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $I \stackrel{\text{def.}}{=} \{0\}$ , the unary operator power function of degree  $p$  and  $g_0 \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Lemma 1008 (masking almost nowhere).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $A \in \Sigma$ . Assume that  $\mu(A^c) = 0$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{=} f \mathbb{1}_A$ .

*Proof.* Direct consequence of Lemma 913 (*measure*), Definition 741 (*measurable space*,  $\Sigma$  is a  $\sigma$ -algebra), Lemma 696 ( *$\sigma$ -algebra*, closedness under complement), **the definition of the indicator function ( $A \subset \{f = f \mathbb{1}_A\}$ )**, **monotonicity of complement ( $\{f = f \mathbb{1}_A\}^c$  is included in  $A^c$ )**, Definition 942 (*negligible subset*). and Definition 953 (*property almost satisfied*).  $\square$

**Lemma 1009 (finite nonnegative part).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $A \in \Sigma$ . Assume that  $0 \stackrel{\mu \text{ a.e.}}{\leq} f \stackrel{\mu \text{ a.e.}}{<} \infty$ , and  $\mu(A^c) = 0$ . Let  $A_f \stackrel{\text{def.}}{=} A \cap f^{-1}(\mathbb{R}_+)$ . Let  $\tilde{f}_A \stackrel{\text{def.}}{=} f \mathbb{1}_{A_f}$ . Then,  $A_f \in \Sigma$ ,  $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ , and we have  $\mu(A_f^c) = 0$ , i.e.  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_A$ .

*Proof.* From Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*), we have  $A_f \in \Sigma$  and  $\tilde{f}_A$  belongs to  $\mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

Assume that  $0 \leq f < \infty$ , i.e.  $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ . Then, since  $f^{-1}(\mathbb{R}_+) = X$ , we have  $A_f = A$ , and from Lemma 955 (*equivalent definition of property almost satisfied*, with  $A_f \subset \{f = \tilde{f}_A\}$  and  $\mu(A_f^c) = \mu(A^c) = 0$ ), we have  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_A$ .

Therefore, from Lemma 966 (*extended almost modus ponens (simplified 1)*, with  $\text{card}(I) = 1$ ,  $P(f) \stackrel{\text{def.}}{=} (0 \leq f < \infty)$  and  $Q(f) \stackrel{\text{def.}}{=} (f = \tilde{f}_A)$ ), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0(f) \stackrel{\text{def.}}{=} (0 \leq f)$  and  $P_1(f) \stackrel{\text{def.}}{=} (f < \infty)$ ), and Lemma 955 (*equivalent definition of property almost satisfied*), we have  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_A$ , i.e.  $\mu(A_f^c) = 0$ .  $\square$

**Lemma 1010 (compatibility of almost limit with subsequence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f : X \rightarrow \overline{\mathbb{R}}$ . Let  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  be increasing. Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ . Then, we have  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

*Proof.* Direct consequence of Lemma 266 (*compatibility of subsequence with convergence*, thus  $f_n \xrightarrow[n \rightarrow \infty]{} f$  implies  $f_{k_n} \xrightarrow[n \rightarrow \infty]{} f$ ), and Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$ ,  $P((u_n)_n, l) \stackrel{\text{def.}}{=} (\lim_{n \rightarrow \infty} u_n = l)$  and  $Q((u_n)_n, l) \stackrel{\text{def.}}{=} (\lim_{n \rightarrow \infty} u_{k_n} = l)$ ).  $\square$

**Lemma 1011 (compatibility of almost limit with continuous function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ . For  $\mu$ -almost all  $x \in X$ , let  $g_x : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$  be a continuous function. Then,  $(x \mapsto g_x(f_n(x))) \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} (x \mapsto g_x(f(x)))$ .

*Proof.* Let  $x \in X$ . Assume that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ , and  $g_x$  is continuous. Then, from Lemma 45 (*compatibility of limit with continuous functions*), we have  $\lim g_x(f_n(x)) = g_x(f(x))$ .

Therefore, from Lemma 517 (*limit inferior and limit superior of pointwise convergent*), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1, 2\}$ ,  $P_0(x)$  is the predicate  $(\liminf_{n \rightarrow \infty} f_n(x) = f(x))$ ,  $P_1(x) \stackrel{\text{def.}}{=} (\limsup_{n \rightarrow \infty} f_n(x) = f(x))$  and  $P_2(x) \stackrel{\text{def.}}{=} (g_x \text{ continuous})$ ), Lemma 972 (*generalized almost modus ponens*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 \stackrel{\text{def.}}{=} \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ ,  $Y_1 \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $f_0 \stackrel{\text{def.}}{=} f$ ,  $f_1 = g \stackrel{\text{def.}}{=} (x \mapsto g_x)$ ,  $P(f, g) \stackrel{\text{def.}}{=} P_0 \wedge P_1 \wedge P_2$ ,  $Q(f, g) \stackrel{\text{def.}}{=} (\lim_{n \rightarrow \infty} g_x(f_n(x)) = g_x(f(x)))$ ), and Lemma 958 (*everywhere is almost everywhere*, thus we can have  $f$  defined everywhere), we have  $\lim_{n \rightarrow \infty} g_x(f_n(x)) \stackrel{\mu \text{ a.e.}}{=} g_x(f(x))$ .  $\square$

**Lemma 1012 (compatibility of almost limit with power function).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f : X \rightarrow \overline{\mathbb{R}}$ . Let  $g : X \rightarrow \mathbb{R}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

Then, for all  $p \in \mathbb{R}_+$  and  $n \in \mathbb{N}$ ,  $f_n - g$ , and  $f - g$  are well defined, and  $|f_n - g|^p \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} |f - g|^p$ .

*Proof.* Direct consequence of Definition 317 (addition in  $\overline{\mathbb{R}}$ , rule 3 cannot occur, thus for all  $n \in \mathbb{N}$   $f_n - g$  and  $f - g$  are well defined), continuity of translation (with  $\tau_x(z) \stackrel{\text{def.}}{=} z - g(x)$ ), Lemma 449 (power function in  $\overline{\mathbb{R}}$  is continuous (measure theory),  $\varphi(y) \stackrel{\text{def.}}{=} |y|^p$  is continuous), compatibility of composition with continuity, and Lemma 1011 (compatibility of almost limit with continuous function, with  $g(x, z) \stackrel{\text{def.}}{=} \varphi \circ \tau_x(z)$ ).  $\square$

**Lemma 1013 (almost limits are almost equal).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f, g$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{=} g$ .

*Proof.* Direct consequence of Lemma 517 (limit inferior and limit superior of pointwise convergent, thus limit is unique), Lemma 960 (countable almost properties is almost property, with  $I \stackrel{\text{def.}}{=} \{0, 1, 2, 3\}$ ,  $P_0 \stackrel{\text{def.}}{=} (\liminf_{n \rightarrow \infty} f_n = f)$ ,  $P_1 \stackrel{\text{def.}}{=} (\limsup_{n \rightarrow \infty} f_n = f)$ ,  $P_2$  is the predicate  $(\liminf_{n \rightarrow \infty} f_n = g)$  and  $P_3 \stackrel{\text{def.}}{=} (\limsup_{n \rightarrow \infty} f_n = g)$ ), and Lemma 967 (extended almost modus ponens (simplified 2), with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $f_0 \stackrel{\text{def.}}{=} f$ ,  $f_1 \stackrel{\text{def.}}{=} g$  and  $P(f, g)$  is the predicate  $(P_0 \wedge P_1 \wedge P_2 \wedge P_3)$ ).  $\square$

**Lemma 1014 (compatibility of almost limit with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}, f, g : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ,  $f \stackrel{\mu \text{ a.e.}}{=} g$ , and there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $f_n \stackrel{\mu \text{ a.e.}}{=} g_n$ . Then, we have  $g_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} g$ .

In particular, the last assumption holds when  $(g_n)_{n \in \mathbb{N}} = (f_n)_{n \in \mathbb{N}}$ . Then, we have  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} g$ .

*Proof.* Direct consequence of Lemma 960 (countable almost properties is almost property, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$ ,  $P_0 \stackrel{\text{def.}}{=} (f = g)$  and  $P_{n+1} \stackrel{\text{def.}}{=} (f_n = g_n)$ ), Lemma 962 (almost everywhere for almost the same, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$  and  $P(f, (f_n)_{n \in \mathbb{N}}) \stackrel{\text{def.}}{=} (\lim_{n \rightarrow \infty} f_n = f)$ ), and Lemma 958 (everywhere is almost everywhere).  $\square$

**Lemma 1015 (compatibility of almost limit with almost inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, we have  $|f| \stackrel{\mu \text{ a.e.}}{\leq} g$ .

*Proof.* Direct consequence of Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous), Lemma 45 (compatibility of limit with continuous functions, with the absolute value), Lemma 34 (stationary sequence is convergent, with stationary value  $g$ ), Lemma 381 (limit is monotone, thus  $P_0 \stackrel{\text{def.}}{=} (\lim_{n \rightarrow \infty} f_n = f)$  and  $P_1 \stackrel{\text{def.}}{=} (|f_n| \leq g \text{ from some rank})$  implies  $|f| \leq g$ ), Lemma 967 (extended almost modus ponens (simplified 2), with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $Y_0 = Y_1 \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $f_0 \stackrel{\text{def.}}{=} f$ ,  $f_1 \stackrel{\text{def.}}{=} g$  and  $P(f, g) \stackrel{\text{def.}}{=} (P_0 \wedge P_1)$ ), and Lemma 960 (countable almost properties is almost property, with  $\text{card}(I) = 2$ ).  $\square$

**Lemma 1016 (compatibility of almost limit with measurability).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

Then, there exists  $\tilde{f} \in \mathcal{M}$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} \tilde{f}$ , and  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ .

Moreover, if  $f$  is finite, then  $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ , and if  $f$  is nonnegative, then  $\tilde{f} \in \mathcal{M}_+$ .

*Proof.* Let  $B \stackrel{\text{def.}}{=} \{f = \liminf_{n \rightarrow \infty} f_n\} \cap \{f = \limsup_{n \rightarrow \infty} f_n\}$ . Then, from Lemma 517 (*limit inferior and limit superior of pointwise convergent*), Definition 953 (*property almost satisfied*, thus  $B^c \in \mathbf{N}$ ), Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ),  $P_0 \stackrel{\text{def.}}{=} (f = \liminf_{n \rightarrow \infty} f_n)$  and  $P_1 \stackrel{\text{def.}}{=} (f = \limsup_{n \rightarrow \infty} f_n)$ , there exists  $A \in \Sigma$  such that  $\mu(A^c)$  equals 0, and  $A \subset B$ .

For all  $n \in \mathbb{N}$ , let  $\tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbb{1}_A$ , and let  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbb{1}_A$ . Let  $x \in X$ . Then, from **the definition of indicator function**, and Lemma 34 (*stationary sequence is convergent*, with stationary value 0), we have  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x) = \tilde{f}(x)$  when  $x \in A$ , and  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = 0 = \tilde{f}(x)$  otherwise. Hence,  $(\tilde{f}_n)_{n \in \mathbb{N}}$  is pointwise convergent towards  $\tilde{f}$ .

Let  $n \in \mathbb{N}$ . Then, from Lemma 833 (*measurability and masking*, with  $\tilde{f}_n$ ), Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent), Lemma 1008 (*masking almost nowhere*), and Lemma 1014 (*compatibility of almost limit with almost equality*, with  $g \stackrel{\text{def.}}{=} \tilde{f}$ ), we have  $\tilde{f}_n, \tilde{f} \in \mathcal{M}$ ,  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ , and  $(f_n)_{n \in \mathbb{N}}$  is almost everywhere pointwise convergent towards  $\tilde{f}$ .

Moreover, from **the definition of indicator function**, Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), and Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), if  $f$  is finite, then  $\tilde{f}$  is also finite and  $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ , and if  $f$  is nonnegative, then  $\tilde{f}$  is also nonnegative and  $\tilde{f} \in \mathcal{M}_+$ .  $\square$



## 13.2 Essential supremum

### 13.2.1 Essential supremum

**Lemma 1017 (essential supremum).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then,  $B_f \stackrel{\text{def.}}{=} \{a \in \overline{\mathbb{R}} \mid \mu(\{f > a\}) = 0\}$ , and  $\text{ess sup } f \stackrel{\text{def.}}{=} \inf B_f$  are well-defined in  $\overline{\mathbb{R}}$ . Moreover, we have  $\infty \in B_f$ , and  $a \in B_f \Rightarrow [a, \infty] \subset B_f$ . The extended number  $\text{ess sup } f$  is called the essential supremum of  $f$ .*

*Proof.* From Lemma 820 (*measurability of numeric function*), Lemma 913 (*measure*), and Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*), the functions  $\psi_f \stackrel{\text{def.}}{=} (a \mapsto \{f > a\}) : \overline{\mathbb{R}} \rightarrow \Sigma$ , and  $\mu \circ \psi_f : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_+$  are well-defined,  $B_f \stackrel{\text{def.}}{=} (\mu \circ \psi_f)^{-1}(\{0\})$  is a well-defined subset of  $\overline{\mathbb{R}}$ , and  $\inf B_f$  is well-defined in  $\overline{\mathbb{R}}$ .

From Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and Lemma 913 (*measure, homogeneity*), we have  $\mu(\{f > \infty\}) = \mu(\emptyset) = 0$ . Hence, we have  $\infty \in B_f$ .

Let  $a \in B_f$ , i.e.  $\mu(\{f > a\}) = 0$ . Let  $b \in \overline{\mathbb{R}}$ . Assume that  $b > a$ . Then, from **monotonicity of inverse image (with  $(b, \infty] \subset (a, \infty]$ )**, we have  $\{f > b\} \subset \{f > a\}$ . Thus, from Lemma 913 (*measure, nonnegativeness*), and Lemma 919 (*measure is monotone*), we have

$$0 \leq \mu(\{f > b\}) \leq \mu(\{f > a\}) = 0,$$

i.e.  $b \in B_f$ . Hence, we have  $[a, \infty] \subset B_f$ . □

*Remark 1018.* Note that in the previous lemma, the essential supremum could be defined for any nonmeasurable function  $f$  by specifying  $B_f \stackrel{\text{def.}}{=} \{a \in \overline{\mathbb{R}} \mid \{f > a\} \in \mathbf{N}\}$ .

**Lemma 1019 (essential supremum with trivial measure).** *Let  $(X, \Sigma, \mu)$  be a trivial measure space. Let  $f \in \mathcal{M}$ . Then, we have  $\text{ess sup } f = -\infty$ , and  $B_f = \overline{\mathbb{R}}$ .*

*Proof.* Direct consequence of Lemma 933 (*trivial measure,  $\mu(\{f > -\infty\}) = 0$* ), and Lemma 1017 (*essential supremum,  $-\infty \in B_f, B_f = \overline{\mathbb{R}}$  and  $\text{ess sup } f = -\infty$* ). □

**Lemma 1020 (essential supremum is bounded by supremum).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then, we have  $\text{ess sup } f \leq \sup f$ .*

*Proof.* Let  $M \stackrel{\text{def.}}{=} \sup f \in \overline{\mathbb{R}}$ . Then, from Definition 2 (*supremum, upper bound*), Lemma 913 (*measure, homogeneity*), Lemma 1017 (*essential supremum,  $\text{ess sup}$  is infimum*), and Definition 9 (*infimum, lower bound*), we have  $f \leq M$  and  $\mu(\{f > M\}) = \mu(\emptyset) = 0$ , thus  $M \in B_f$  and  $\text{ess sup } f \leq M$ . □

**Lemma 1021 (essential supremum is minimum).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then,  $\mu(\{f > \text{ess sup } f\}) = 0$ , i.e.  $\text{ess sup } f \in B_f$ . Moreover, we have  $B_f = [\text{ess sup } f, \infty]$ .*

*Proof.* Let  $M \stackrel{\text{def.}}{=} \text{ess sup } f \in \overline{\mathbb{R}}$ .

**Case  $M = \infty$ .** Then, from Lemma 1017 (*essential supremum*), we have  $\infty \in B_f$ .

**Case  $M < \infty$ .** Then, from Lemma 1017 (*essential supremum,  $M$  is an infimum*), Lemma 486 (*equivalent definition of infimum*), let  $(a_n)_{n \in \mathbb{N}} \in B_f$  be a nonincreasing sequence of extended numbers such that  $\lim_{n \rightarrow \infty} a_n = \inf_{n \in \mathbb{N}} a_n = M$ . Then, from **compatibility of inverse image with union (with  $(M, \infty] = \bigcup_{n \in \mathbb{N}} (a_n, \infty]$ )**, we have  $\{f > M\} = \bigcup_{n \in \mathbb{N}} \{f > a_n\}$ . Therefore, from Lemma 924 (*compatibility of null measure with countable union*), we have  $\mu(\{f > M\}) = 0$ , i.e.  $M \in B_f$ .

Moreover, from Lemma 1017 (*essential supremum*), we have  $[M, \infty] \subset B_f$ . Conversely, let  $a$  be in  $B_f$ . Then, from Lemma 1017 (*essential supremum,  $M$  is infimum*), and Definition 9 (*infimum, lower bound*), we have  $M \leq a$ . Thus,  $B_f \subset [M, \infty]$ . Hence, we have  $B_f = [M, \infty]$ . □

**Lemma 1022 (essential supremum is almost upper bound).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $b \in B_f$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{\leq} b$ , and in particular  $f \stackrel{\mu \text{ a.e.}}{\leq} \text{ess sup } f$ .

*Proof.* Direct consequence of Lemma 1017 (essential supremum,  $\mu(\{f > b\}) = 0$ ), Lemma 948 (negligibility of measurable subset,  $\{f \leq b\}^c \in \mathbf{N}$ ), Definition 974 (almost binary relation), and Lemma 1021 (essential supremum is minimum,  $\text{ess sup } f \in B_f$ ).  $\square$

**Lemma 1023 (essential supremum of constant function).** Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let  $y \in \overline{\mathbb{R}}$ . Let  $f$  be the constant function from  $X$  to  $\overline{\mathbb{R}}$  of value  $y$ . Then we have  $f \in \mathcal{M}$ ,  $\text{ess sup } f = y$ , and  $B_f = [y, \infty]$ .

*Proof.* From Lemma 754 (constant function is measurable), and Lemma 1017 (essential supremum),  $\text{ess sup } f$  is well-defined. Then, From the definition of complement, Lemma 913 (measure), and Lemma 1017 (essential supremum), we have  $\mu(\{f > y\}) = \mu(\{f \leq y\}^c) = \mu(\emptyset) = 0$ ,  $y \in B_f$ , and  $[y, \infty] \subset B_f$ .

Let  $a < y$ . Then, from Lemma 934 (equivalent definition of trivial measure, contrapositive), and Lemma 913 (measure, nonnegativeness), we have  $\mu(\{f > a\}) = \mu(X) > 0$ . Therefore, from Lemma 1017 (essential supremum), we have  $a \notin B_f$ ,  $B_f = [y, \infty]$ , and  $\text{ess sup } f = \inf B_f = y$ .  $\square$

**Lemma 1024 (essential supremum of extended simple functions).**

Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let a triple  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbf{N} \times \Sigma \times \overline{\mathbb{R}}$  such that  $f : X \rightarrow \overline{\mathbb{R}}$  is defined by

$$(13.12) \quad \forall i \in [0..n], \quad A_i \subset \{f = a_i\},$$

$$(13.13) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(13.14) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}.$$

Then, we have  $f \in \mathcal{M}$ , and  $\text{ess sup } f = \max_{i \in [0..n] \mid \mu(A_i) > 0} a_i$ .

*Proof.* From Lemma 811 (measurability of indicator function), Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication,  $a_i \mathbb{1}_{A_i} \in \mathcal{M}$ ), Definition 317 (addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur), and Lemma 823 ( $\mathcal{M}$  is closed under finite sum when defined), we have  $f \in \mathcal{M}$ .

From Lemma 934 (equivalent definition of trivial measure, contrapositive), Definition 209 (pseudopartition), Lemma 916 (measure is additive), Definition 862 (additivity), and Lemma 913 (measure, nonnegativeness), we have  $0 < \mu(X) = \sum_{i \in [0..n]} \mu(A_i)$ , and thus there exists  $i_0 \in [0..n]$  such that  $\mu(A_{i_0}) > 0$ .

Let  $J \stackrel{\text{def.}}{=} \{i \in [0..n] \mid \mu(A_i) > 0\} \neq \emptyset$ , and  $j \stackrel{\text{def.}}{=} \arg \max_{i \in J} a_i$  (note that  $j$  may not be unique). Then, we have  $\forall x \in \bigsqcup_{i \in J} A_i$ ,  $f(x) \leq a_j$ , the pseudopartition  $X = \bigsqcup_{i \in J} A_i \sqcup \bigsqcup_{i \in [0..n] \setminus J} A_i$ , and thus  $\{f > a_j\} \subset \bigsqcup_{i \in [0..n] \setminus J} A_i$ . Thus, from Lemma 913 (measure, nonnegativeness), Lemma 919 (measure is monotone), Lemma 916 (measure is additive), and Definition 862 (additivity), we have  $0 \leq \mu(\{f > a_j\}) \leq \mu(\bigsqcup_{i \in I \setminus J} A_i) = \sum_{i \in I \setminus J} \mu(A_i) = 0$ . Hence, from Lemma 1017 (essential supremum), we have  $a_j \in B_f$ .

**Case  $a_j = -\infty$ .** Then, from Lemma 1017 (essential supremum,  $B_f = \overline{\mathbb{R}}$ ),  $\text{ess sup } f = -\infty = a_j$ .

**Case  $-\infty < a_j$ .** Let  $y < a_j$ . Then,  $A_j \subset \{f = a_j\} \subset \{f > y\}$ , and from Lemma 919 (measure is monotone), we have  $0 < \mu(A_j) \leq \mu(\{f > y\})$ . Hence, from Lemma 1017 (essential supremum), we have  $y \notin B_f$ , and  $\text{ess sup } f = a_j$ .

Therefore, we always have  $\text{ess sup } f = a_j$ .  $\square$

*Remark 1025.* Note that both previous lemmas do not hold when the measure is trivial, as all essential suprema are then equal to  $-\infty$ .

**Lemma 1026 (essential supremum of infinite times function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then,  $\infty f \in \mathcal{M}$ , and we have

$$(13.15) \quad \text{ess sup}(\infty f) = \begin{cases} -\infty & \text{if } \mu(\{f \geq 0\}) = 0, \\ 0 & \text{if } \mu(\{f > 0\}) = 0 \wedge \mu(\{f = 0\}) > 0, \\ \infty & \text{if } \mu(\{f > 0\}) > 0. \end{cases}$$

*Proof.* From Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication), and Lemma 1017 (essential supremum), we have  $\infty f \in \mathcal{M}$ , and  $\text{ess sup}(\infty f)$  is well-defined.

**Case  $\mu$  trivial.** Direct consequence of Lemma 933 (trivial measure,  $\mu(\{f > 0\}) = 0$ ), and Lemma 1019 (essential supremum with trivial measure).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 209 (pseudopartition,  $X$  is the disjoint union of  $\{f < 0\}$ ,  $\{f = 0\}$  and  $\{f > 0\}$ ), Lemma 820 (measurability of numeric function,  $\{f > 0\}$  and  $\{f \geq 0\}$  belong to  $\Sigma$ ), Lemma 809 (inverse image is measurable,  $\{f = 0\} \in \Sigma$ ), and Lemma 1024 (essential supremum of extended simple functions, with  $\infty f = -\infty \mathbf{1}_{\{f < 0\}} + 0 \mathbf{1}_{\{f = 0\}} + \infty \mathbf{1}_{\{f > 0\}}$ ).  $\square$

**Lemma 1027 (essential supremum is positive homogeneous).** Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let  $\lambda \in \mathbb{R}_+$ . Let  $f \in \mathcal{M}$ . Then,  $\lambda f \in \mathcal{M}$ , and we have  $\text{ess sup}(\lambda f) = \lambda \text{ess sup } f$ .

*Proof.* From Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication), we have  $\lambda f \in \mathcal{M}$ . Hence, from Lemma 1017 (essential supremum),  $\text{ess sup } f$  and  $\text{ess sup}(\lambda f)$  are well-defined.

Let  $M \stackrel{\text{def.}}{=} \text{ess sup } f \in \overline{\mathbb{R}}$ , and let  $M_\lambda \stackrel{\text{def.}}{=} \text{ess sup}(\lambda f) \in \overline{\mathbb{R}}$ . **Case  $\lambda = 0$ .** Then, from Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)), and Lemma 1023 (essential supremum of constant function,  $\mu$  is nontrivial),  $0f$  is the zero function, and we have  $M_0 = 0 = 0M$ . **Case  $\lambda \neq 0$ .** Let  $a \in \overline{\mathbb{R}}$ . Then, from Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)), and Lemma 1017 (essential supremum), we have  $\{\lambda f > a\} = \{f > \frac{a}{\lambda}\}$ , and  $a \in B_{\lambda f} \Leftrightarrow \frac{a}{\lambda} \in B_f$ . Thus, from Lemma 1021 (essential supremum is minimum), we have

$$\frac{M_\lambda}{\lambda} \in B_f = [M, \infty], \quad \text{and} \quad \lambda M \in B_{\lambda f} = [M_\lambda, \infty].$$

Hence, from Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)), and **the definition of order (antisymmetry)**, we have  $M_\lambda = \lambda M$ .

Therefore, we always have  $M_\lambda = \lambda M$ .  $\square$

*Remark 1028.* Note that the result of Lemma 1027 does not hold when  $\mu \stackrel{\text{def.}}{=} 0$ . Indeed, from Lemma 1019, and Definition 415, we have then  $\text{ess sup}(0f) = -\infty \neq 0 = 0 \text{ess sup } f$ .

Note also that the result does not hold either when  $\lambda \stackrel{\text{def.}}{=} \infty$ . Indeed, let  $f \stackrel{\text{def.}}{=} \arctan x - \frac{\pi}{2}$ . Then, with the Lebesgue measure, we have  $\text{ess sup } f = \sup f = 0$ , and  $\mu(\{f \geq 0\}) = 0$ . Hence, from Lemma 1026, we have  $\text{ess sup}(\infty f) = -\infty \neq 0 = \infty \text{ess sup } f$ .

**Lemma 1029 (essential supremum is almost monotone).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, we have  $\text{ess sup } f \leq \text{ess sup } g$ .

*Proof.* From Definition 974 (almost binary relation, with inequality), we have  $A \stackrel{\text{def.}}{=} \{f \leq g\}^c \in \mathbf{N}$ .

Let  $a \in \overline{\mathbb{R}}$ . Then, from **distributivity of intersection over union**, and **the definition of order (transitivity)**, we have

$$\{f > a\} = (\{f > a\} \cap A^c) \uplus (\{f > a\} \cap A) \subset \{g > a\} \cup A.$$

Assume that  $\mu(\{g > a\}) = 0$ . Then, from Lemma 820 (*measurability of numeric function,  $\{g > a\}, \{f > a\} \in \Sigma$* ), Lemma 948 (*negligibility of measurable subset,  $\{g > a\} \in \mathbf{N}$* ), Lemma 952 ( *$\mathbf{N}$  is closed under countable union*), Lemma 951 (*subset of negligible is negligible,  $\{f > a\} \in \mathbf{N}$* ), we have  $\mu(\{f > a\}) = 0$ . Hence, from Lemma 1017 (*essential supremum*), we have  $B_g \subset B_f$ .

Therefore, from Lemma 1021 (*essential supremum is minimum*), we have  $\text{ess sup } f \leq \text{ess sup } g$ .  $\square$

**Lemma 1030 (essential supremum bounded by constant).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ , and  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(13.16) \quad \mu \text{ nontrivial} \wedge a \stackrel{\mu \text{ a.e.}}{\leq} f \implies a \leq \text{ess sup } f,$$

$$(13.17) \quad f \stackrel{\mu \text{ a.e.}}{\leq} b \iff \text{ess sup } f \leq b.$$

*Proof.* “Left” implies “right”. **Case  $\mu$  trivial.** Direct consequence of Lemma 1019 (*essential supremum with trivial measure, thus  $\text{ess sup } f = -\infty$* ), and Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$ , thus  $-\infty \leq b$* ). **Case  $\mu$  nontrivial.** Direct consequence of Lemma 1029 (*essential supremum is almost monotone, with  $f \stackrel{\text{def.}}{=} a$  and  $g \stackrel{\text{def.}}{=} f$ , then with  $f$  and  $g \stackrel{\text{def.}}{=} b$* ), and Lemma 1023 (*essential supremum of constant function,  $\text{ess sup}(a) = a$  and  $\text{ess sup}(b) = b$* ).

“Right” implies “left”. Direct consequence of Lemma 1022 (*essential supremum is almost upper bound*), and Lemma 983 (*almost order is order relation, transitivity*).  $\square$

**Lemma 1031 (essential supremum of absolute value is nonnegative).** Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let  $f \in \mathcal{M}$ . Then,  $|f| \in \mathcal{M}$ , and we have  $\text{ess sup } |f| \geq 0$ .

*Proof.* Direct consequence of Lemma 842 ( *$\mathcal{M}$  is closed under absolute value,  $|f| \in \mathcal{M}$  and  $0 \leq |f|$* ), Lemma 958 (*everywhere is almost everywhere,  $0 \stackrel{\mu \text{ a.e.}}{\leq} |f|$* ), and Lemma 1030 (*essential supremum bounded by constant, with  $a \stackrel{\text{def.}}{=} 0$* ).  $\square$

**Lemma 1032 (compatibility of essential supremum with almost equality).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then, we have  $\text{ess sup } f = \text{ess sup } g$ .

*Proof.* Direct consequence of Lemma 977 (*compatibility of almost binary relation with extended reflexivity, with  $\leq$* ), Lemma 1029 (*essential supremum is almost monotone*), and **the definition of order (antisymmetry)**.  $\square$

### 13.2.2 Nonnegative essential supremum for nonnegative functions

*Remark 1033.* In this section, the definition of essential supremum is restricted to nonnegative functions. This provides a more regular treatment of extreme cases, such as when the measure is trivial, or when dealing with infinite values (e.g. for positive homogeneity).

This will be useful when defining the infinite seminorm  $N_\infty$  in Section 16.3.1.

**Definition 1034 (nonnegative essential supremum).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+ \subset \mathcal{M}$ . The (*nonnegative*) essential supremum of  $f$  is its usual essential supremum  $\text{ess sup } f$  when the measure  $\mu$  is nontrivial, and 0 otherwise; it is denoted  $\text{ess sup}^+ f$ .

*Remark 1035.* Note that from the previous definition, the essential supremum and the nonnegative essential supremum coincide on nonnegative measurable functions when the measure is nontrivial. Thus, when no confusion is possible, we may drop the “+” superscript and simply write  $\text{ess sup}$ .

**Lemma 1036 (equivalent definition of nonnegative essential supremum).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $f \in \mathcal{M}$ ,  $C_f \stackrel{\text{def.}}{=} \{a \in \overline{\mathbb{R}}_+ \mid \mu(\{f > a\}) = 0\}$  and  $\inf C_f$  are well-defined in  $\overline{\mathbb{R}}_+$ , and we have  $\text{ess sup}^+ f = \inf C_f$ .

Moreover, we have  $\infty \in C_f$ , and  $a \in C_f \implies [a, \infty] \subset C_f$ .

*Proof.* From Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), and Lemma 1017 (essential supremum), we have  $f \in \mathcal{M}$ ,  $B_f$  is well-defined in  $\overline{\mathbb{R}}$ , and  $C_f = B_f \cap \overline{\mathbb{R}}_+$  and  $\inf C_f$  are well-defined in  $\overline{\mathbb{R}}_+$ .

**Case  $\mu$  trivial.** Direct consequence of Lemma 933 (trivial measure, thus  $C_f = \overline{\mathbb{R}}_+$ ), and Definition 9 (infimum, thus  $\text{ess sup}^+ f \stackrel{\text{def.}}{=} 0 = \inf C_f$ ,  $\infty \in C_f$  and  $a \in C_f \Rightarrow [a, \infty] \subset C_f$ ).

**Case  $\mu$  nontrivial.** Let  $a \in \overline{\mathbb{R}}_-^*$ . Then, from Lemma 913 (measure, nonnegativeness), and Lemma 934 (equivalent definition of trivial measure), we have  $\mu(\{f > a\}) = \mu(X) > 0$ . Hence, from Lemma 1017 (essential supremum), we have  $a \notin B_f$ , i.e.  $B_f \subset \overline{\mathbb{R}}_+$ , thus  $C_f = B_f$ ,

$$\begin{aligned} \text{ess sup}^+ f &\stackrel{\text{def.}}{=} \text{ess sup } f = \inf B_f = \inf C_f, \\ \infty \in B_f = C_f, \quad \text{and} \quad a \in C_f = B_f &\Rightarrow [a, \infty] \subset B_f = C_f. \end{aligned}$$

Therefore, we always have the result.  $\square$

**Lemma 1037 (nonnegative essential supremum with trivial measure).**

Let  $(X, \Sigma, \mu)$  be a trivial measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\text{ess sup}^+ f = 0$ , and  $C_f = \overline{\mathbb{R}}_+$ .

*Proof.* Direct consequence of Definition 1034 (nonnegative essential supremum,  $\text{ess sup}^+ f = 0$ ), Lemma 933 (trivial measure, thus  $\mu(\{f > a\}) = 0$  for all  $a$ ), Lemma 1036 (equivalent definition of nonnegative essential supremum, thus  $C_f = \overline{\mathbb{R}}_+$ ).  $\square$

**Lemma 1038 (nonnegative essential supremum is minimum).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then, we have  $\mu(\{f > \text{ess sup}^+ f\}) = 0$ , i.e.  $\text{ess sup}^+ f \in C_f$ . Moreover, we have  $C_f = [\text{ess sup}^+ f, \infty]$ .

*Proof.* **Case  $\mu$  trivial.**

Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1021 (essential supremum is minimum).

Therefore, we always have the result.  $\square$

**Lemma 1039 (nonnegative essential supremum is almost upper bound).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $b \in C_f = [\text{ess sup}^+ f, \infty]$ .

Then, we have  $f \stackrel{\mu \text{ a.e.}}{\leq} b$  (e.g.  $f \stackrel{\mu \text{ a.e.}}{\leq} \text{ess sup}^+ f$ ).

In particular, for all  $g \in \mathcal{M}$ , we have  $|g| \in \mathcal{M}_+$ , and for all  $b \in C_{|g|} = [\text{ess sup}^+ |g|, \infty]$ , we have  $|g| \stackrel{\mu \text{ a.e.}}{\leq} b$  (e.g.  $|g| \stackrel{\mu \text{ a.e.}}{\leq} \text{ess sup}^+ |g|$ ).

*Proof.* **Case  $\mu$  trivial.** Direct consequence of Lemma 956 (property almost satisfied with trivial measure, with  $P \stackrel{\text{def.}}{=} (f \leq b)$ ).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1022 (essential supremum is almost upper bound).

Therefore, we always have the result.

Let  $g \in \mathcal{M}$ . Then, from Lemma 842 ( $\mathcal{M}$  is closed under absolute value,  $|g| \in \mathcal{M}_+$ ), the previous result applies with  $f \stackrel{\text{def.}}{=} |g|$ .  $\square$

**Lemma 1040 (nonnegative essential supremum of constant function).** Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let  $y \in \overline{\mathbb{R}}_+$ . Let  $f$  be the constant function from  $X$  to  $\overline{\mathbb{R}}_+$  of value  $y$ . Then,  $f \in \mathcal{M}_+$ , and we have  $\text{ess sup}^+ f = y$  and  $C_f = [y, \infty]$ .

*Proof.*

Direct consequence of Lemma 754 (*constant function is measurable*,  $f \in \mathcal{M}_+$ ), Definition 1034 (*nonnegative essential supremum*), and Lemma 1023 (*essential supremum of constant function*).  $\square$

**Lemma 1041 (*nonnegative essential supremum for extended simple functions*).**

Let  $(X, \Sigma, \mu)$  be a nontrivial measure space. Let a triple  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \overline{\mathbb{R}}_+$  such that  $f : X \rightarrow \overline{\mathbb{R}}_+$  is defined by

$$(13.18) \quad \forall i \in [0..n], \quad A_i \subset \{f = a_i\},$$

$$(13.19) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(13.20) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}.$$

Then,  $f \in \mathcal{M}_+$ , and we have  $\text{ess sup}^+ f = \max_{i \in [0..n] \mid \mu(A_i) > 0} a_i$ .

*Proof.* Direct consequence of Definition 1034 (*nonnegative essential supremum*), and Lemma 1024 (*essential supremum of extended simple functions*).  $\square$

*Remark 1042.* Note that both previous lemmas do not hold when the measure is trivial, as all nonnegative essential suprema are then equal to 0.

**Lemma 1043 (*nonnegative essential supremum of infinite times function*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\infty f \in \mathcal{M}_+$ , and we have

$$(13.21) \quad \text{ess sup}^+(\infty f) = \begin{cases} 0 & \text{if } \mu(\{f > 0\}) = 0, \\ \infty & \text{if } \mu(\{f > 0\}) > 0. \end{cases}$$

*Proof.* From Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication), and Definition 1034 (*nonnegative essential supremum*), we have  $\infty f \in \mathcal{M}_+$ , and  $\text{ess sup}^+(\infty f)$  is well-defined.

**Case  $\mu$  trivial.** Direct consequence of Lemma 933 (*trivial measure*, thus  $\mu(\{f > 0\}) = 0$ ), and Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 1034 (*nonnegative essential supremum*), Lemma 934 (*equivalent definition of trivial measure*, thus  $\mu(X) \neq 0$ ), and Lemma 1026 (*essential supremum of infinite times function*,  $\mu(\{f \geq 0\}) = 0$  cannot occur).

Therefore, we always have the result.  $\square$

**Lemma 1044 (*nonnegative essential supremum is positive homogeneous*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $\lambda \in \overline{\mathbb{R}}_+$ . Let  $f \in \mathcal{M}_+$ .

Then,  $\lambda f \in \mathcal{M}_+$ , and we have  $\text{ess sup}^+(\lambda f) = \lambda \text{ess sup}^+ f$ .

*Proof.* From Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication,  $\lambda f \in \mathcal{M}_+$ ), and Definition 1034 (*nonnegative essential supremum*),  $\text{ess sup}^+ f$  and  $\text{ess sup}^+(\lambda f)$  are well-defined.

Let  $M \stackrel{\text{def.}}{=} \text{ess sup}^+ f \in \overline{\mathbb{R}}_+$ , and  $M_\lambda \stackrel{\text{def.}}{=} \text{ess sup}^+(\lambda f) \in \overline{\mathbb{R}}_+$ .

**Case  $\mu$  trivial.** Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*,  $M = M_\lambda = 0$ ), and Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Case  $\mu$  nontrivial, and  $\lambda \in [0, \infty)$ .** Direct consequence of Definition 1034 (*nonnegative essential supremum*), and Lemma 1027 (*essential supremum is positive homogeneous*).

**Case  $\mu$  nontrivial,  $\lambda = \infty$ , and  $\mu(\{f > 0\}) = 0$ .**

Then, from Lemma 1036 (*equivalent definition of nonnegative essential supremum*), Lemma 1043



(*nonnegative essential supremum of infinite times function*), and Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*), we have  $0 \in C_f$ ,  $C_f = \overline{\mathbb{R}}_+$ ,  $M = 0$ , and  $M_\infty = 0 = \infty M$ .

**Case  $\mu$  nontrivial,  $\lambda = \infty$ , and  $\mu(\{f > 0\}) > 0$ .** Then, from Lemma 1036 (*equivalent definition of nonnegative essential supremum*), Lemma 1038 (*nonnegative essential supremum is minimum*), Lemma 1043 (*nonnegative essential supremum of infinite times function*), and Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*), we have  $0 \notin C_f = [M, \infty]$ , thus  $M > 0$  and  $M_\infty = \infty = \infty M$ .

Therefore, we always have  $M_\lambda = \lambda M$ . □

**Lemma 1045 (*nonnegative essential supremum is almost monotone*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, we have  $\text{ess sup}^+ f \leq \text{ess sup}^+ g$ .

*Proof.* **Case  $\mu$  trivial.** Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, thus  $\text{ess sup}^+ f = \text{ess sup}^+ g = 0$ ), and Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total, reflexivity*).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 1034 (*nonnegative essential supremum*), and Lemma 1029 (*essential supremum is almost monotone*).

Therefore, we always have the result. □

**Lemma 1046 (*nonnegative essential supremum bounded by constant*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(13.22) \quad \mu \text{ nontrivial} \quad \wedge \quad a \stackrel{\mu \text{ a.e.}}{\leq} f \quad \implies \quad a \leq \text{ess sup}^+ f,$$

$$(13.23) \quad f \stackrel{\mu \text{ a.e.}}{\leq} b \quad \iff \quad \text{ess sup}^+ f \leq b.$$

In particular, for all  $g \in \mathcal{M}$ , we have  $|g| \in \mathcal{M}_+$ , and  $|g| \stackrel{\mu \text{ a.e.}}{\leq} b$  iff  $\text{ess sup}^+ |g| \leq b$ .

*Proof.* **Case  $\mu$  trivial.** Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, thus  $\text{ess sup}^+ f = 0$ ), Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$ , thus  $0 \leq b$* ), and Lemma 956 (*property almost satisfied with trivial measure*, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} b$  always holds).

**Case  $\mu$  nontrivial.** Direct consequence of Definition 1034 (*nonnegative essential supremum*), and Lemma 1030 (*essential supremum bounded by constant*).

Let  $g \in \mathcal{M}$ . Then, from Lemma 842 ( *$\mathcal{M}$  is closed under absolute value,  $|g| \in \mathcal{M}_+$* ), (13.23) applies. □

**Definition 1047 (*essentially bounded*).**

Let  $(X, \Sigma, \mu)$  be a measure space. A function  $f \in \mathcal{M}$  is said *essentially bounded* iff there exists  $M \in \mathbb{R}_+$  such that  $\text{ess sup}^+ |f| \leq M$ .

**Lemma 1048 (*essentially bounded is almost finite*).** Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f \in \mathcal{M}$ . Then,  $f$  is essentially bounded iff  $|f| \stackrel{\mu \text{ a.e.}}{<} \infty$  iff  $\mu(\{f = \pm\infty\}) = 0$ .

*Proof.* Direct consequence of Definition 1047 (*essentially bounded*), Lemma 1046 (*nonnegative essential supremum bounded by constant*, with  $g \stackrel{\text{def.}}{=} f$  and  $b \stackrel{\text{def.}}{=} M \in \mathbb{R}_+$ ), Lemma 958 (*everywhere is almost everywhere*), and Lemma 983 (*almost order is order relation, transitivity*). □

**Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then, we have  $\text{ess sup}^+ f = \text{ess sup}^+ g$ .

*Proof. Case  $\mu$  trivial.* Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, thus  $\text{ess sup}^+ f = 0 = \text{ess sup}^+ g$ ).

*Case  $\mu$  nontrivial.* Direct consequence of Definition 1034 (*nonnegative essential supremum*), and Lemma 1032 (*compatibility of essential supremum with almost equality*).  $\square$

**Lemma 1050 (*nonnegative essential supremum is almost definite*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then, we have

$$(13.24) \quad \text{ess sup}^+ f = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0 \iff \mu(\{f > 0\}) = 0.$$

In particular, for all  $g \in \mathcal{M}$ , we have  $|g| \in \mathcal{M}_+$ , and

$$(13.25) \quad \text{ess sup}^+ |g| = 0 \iff |g| \stackrel{\mu \text{ a.e.}}{=} 0 \iff \mu(\{|g| > 0\}) = 0.$$

*Proof.* From Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $f \geq 0$ ), Definition 209 (*pseudopartition*,  $X = \{f = 0\} \uplus \{f > 0\}$ ), Lemma 809 (*inverse image is measurable*,  $\{f = 0\} \in \Sigma$ ), Lemma 820 (*measurability of numeric function*,  $\{f > 0\} \in \Sigma$ ), Definition 953 (*property almost satisfied*), and Lemma 948 (*negligibility of measurable subset*), we have the equivalence  $f \stackrel{\mu \text{ a.e.}}{=} 0 \Leftrightarrow \mu(\{f > 0\}) = 0$ .

**Case  $\mu$  trivial.**

Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, thus  $\text{ess sup}^+ f = 0$ ), and Lemma 956 (*property almost satisfied with trivial measure*, thus  $f \stackrel{\mu \text{ a.e.}}{=} 0$ ).

**Case  $\mu$  nontrivial.** Direct consequence of Lemma 1038 (*nonnegative essential supremum is minimum*, thus  $\text{ess sup}^+ f = 0$  implies  $\mu(\{f > 0\}) = 0$ ), Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*, with  $g \stackrel{\text{def.}}{=} 0$ ), and Lemma 1040 (*nonnegative essential supremum of constant function*, thus  $\text{ess sup}^+(x \mapsto 0) = 0$ ).

Therefore, we always have the result.

Let  $g \in \mathcal{M}$ . Then, from Lemma 842 ( $\mathcal{M}$  is closed under absolute value,  $|g| \in \mathcal{M}_+$ ), (13.24) applies.  $\square$

**Lemma 1051 (*nonnegative essential supremum is subadditive*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ .

Then,  $f + g \in \mathcal{M}_+$ , and we have  $\text{ess sup}^+(f + g) \leq \text{ess sup}^+ f + \text{ess sup}^+ g$ .

*Proof. Case  $\mu$  trivial.* Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, all the  $\text{ess sup}^+$  are 0), Definition 317 (*addition in  $\overline{\mathbb{R}}$* , thus  $0 + 0 = 0$ ), and **the definition of order (reflexivity)**.

**Case  $\mu$  nontrivial.** Let  $M_f = \text{ess sup}^+ f$ , and  $M_g = \text{ess sup}^+ g$ .

Direct consequence of Lemma 1039 (*nonnegative essential supremum is almost upper bound*, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} M_f$  and  $g \stackrel{\mu \text{ a.e.}}{\leq} M_g$ ), Lemma 984 (*compatibility of almost binary relation with operator*, with  $Y \stackrel{\text{def.}}{=} \overline{\mathbb{R}}_+$ ,  $\mathcal{R}, \mathcal{R}' \stackrel{\text{def.}}{\leq}$  and the binary operator addition, thus  $f + g \stackrel{\mu \text{ a.e.}}{\leq} M_f + M_g$ ), Lemma 843 ( $\mathcal{M}_+$  is closed under addition,  $f + g \in \mathcal{M}_+$ ), and Lemma 1046 (*nonnegative essential supremum bounded by constant*, with  $b \stackrel{\text{def.}}{=} M_f + M_g$ ).

Therefore, we always have the result.  $\square$

*Remark 1052.* Note that a counterpart of Lemma 1051 probably holds over  $\mathcal{M}$  with the essential supremum (instead of the nonnegative essential supremum), at the cost of handling *almost sum* of the functions. It is not stated.



**Lemma 1053** (*nonnegative essential supremum is submultiplicative*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ .

Then,  $fg \in \mathcal{M}_+$ , and we have  $\text{ess sup}^+(fg) \leq \text{ess sup}^+ f \times \text{ess sup}^+ g$ .

*Proof.* **Case  $\mu$  trivial.** Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, all the  $\text{ess sup}^+$  are 0), Definition 326 (*multiplication in  $\bar{\mathbb{R}}$* , thus  $0 \times 0 = 0$ ), and **the definition of order (reflexivity)**.

**Case  $\mu$  nontrivial.** Let  $M_f = \text{ess sup}^+ f$ , and  $M_g = \text{ess sup}^+ g$ .

Direct consequence of Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*, thus  $f, g \geq 0$ ), Lemma 1039 (*nonnegative essential supremum is almost upper bound*, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} M_f$  and  $g \stackrel{\mu \text{ a.e.}}{\leq} M_g$ ), Lemma 984 (*compatibility of almost binary relation with operator*, with  $Y \stackrel{\text{def.}}{=} \bar{\mathbb{R}}_+$ ,  $\mathcal{R}, \mathcal{R}' \stackrel{\text{def.}}{\leq}$  and the binary operator multiplication, thus  $fg \stackrel{\mu \text{ a.e.}}{\leq} M_f M_g$ ), Lemma 844 ( $\mathcal{M}_+$  is *closed under multiplication*,  $fg \in \mathcal{M}_+$ ), and Lemma 1046 (*nonnegative essential supremum bounded by constant*, with  $b \stackrel{\text{def.}}{=} M_f M_g$ ).

Therefore, we always have the result.  $\square$

**Lemma 1054** (*nonnegative essential supremum is compatible with power function*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in \bar{\mathbb{R}}_+$ . Let  $f \in \mathcal{M}_+$ .

Then,  $f^p \in \mathcal{M}_+$ , and we have

$$(13.26) \quad \mu \text{ trivial} \quad \wedge \quad 0 < p \quad \implies \quad \text{ess sup}^+ f^p = (\text{ess sup}^+ f)^p,$$

$$(13.27) \quad \mu \text{ nontrivial} \quad \wedge \quad p < \infty \quad \implies \quad \text{ess sup}^+ f^p = (\text{ess sup}^+ f)^p.$$

*Proof.* **Case  $\mu$  trivial, and  $0 < p$ .** Direct consequence of Lemma 1037 (*nonnegative essential supremum with trivial measure*, all the  $\text{ess sup}^+$  are 0), and Lemma 432 (*zero-exponentiation property in  $\bar{\mathbb{R}}_+$*  (measure theory), thus  $(\text{ess sup}^+ f)^p = 0$ ).

**Case  $\mu$  nontrivial, and  $p = 0$ .** Direct consequence of Lemma 431 (*one-exponentiation property in  $\bar{\mathbb{R}}$*  (measure theory), thus  $f^0 = (\text{ess sup}^+ f)^0 = 1$ ), and Lemma 1040 (*nonnegative essential supremum of constant function*, with  $y \stackrel{\text{def.}}{=} 1$ , thus  $\text{ess sup}^+ f^0 = 1$ ).

**Case  $\mu$  nontrivial, and  $p \in (0, \infty)$ .** Let  $M = \text{ess sup}^+ f$ , and  $N = \text{ess sup}^+(f^p)$ .

Then, from Lemma 841 ( $\mathcal{M}$  is *closed under power function*, thus  $f^p \in \mathcal{M}_+$ ), Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*, thus  $f, f^p \geq 0$ ), and Lemma 1039 (*nonnegative essential supremum is almost upper bound*), we have  $f \stackrel{\mu \text{ a.e.}}{\leq} M$  and  $f^p \stackrel{\mu \text{ a.e.}}{\leq} N$ . Thus, from Lemma 442 (*compatibility of power function in  $\bar{\mathbb{R}}$  with multiplication* (measure theory), with  $q \stackrel{\text{def.}}{=} \frac{1}{p}$ ), Lemma 422 (*one-product property in  $\bar{\mathbb{R}}$*  (measure theory), thus  $p \times \frac{1}{p} = 1$ ), Lemma 444 (*power function in  $\bar{\mathbb{R}}$  is monotone* (measure theory), with  $p$  and  $\frac{1}{p}$ ), Lemma 984 (*compatibility of almost binary relation with operator*, with  $Y \stackrel{\text{def.}}{=} \bar{\mathbb{R}}_+$ ,  $\mathcal{R}, \mathcal{R}' \stackrel{\text{def.}}{\leq}$  and the unary operators power function of degrees  $p$  and  $\frac{1}{p}$ , thus  $f^p \stackrel{\mu \text{ a.e.}}{\leq} M^p$  and  $f \stackrel{\mu \text{ a.e.}}{\leq} N^{\frac{1}{p}}$ ), and Lemma 1046 (*nonnegative essential supremum bounded by constant*), we have  $N \leq M^p$  and  $M \leq N^{\frac{1}{p}}$ . Hence, from Lemma 444 (*power function in  $\bar{\mathbb{R}}$  is monotone* (measure theory)), and Lemma 442 (*compatibility of power function in  $\bar{\mathbb{R}}$  with multiplication* (measure theory), with  $(p, q) \stackrel{\text{def.}}{=} (\frac{1}{p}, p)$ ), we have the other inequality  $M^p \leq N$ , and the equality holds.

Therefore, we have the result in both cases.  $\square$

### 13.3 The Lebesgue measure

*Remark 1055.* This section follows Carathéodory's extension scheme (see Section 4.3).

*Remark 1056.* We recall the notation  $[\cdot, \cdot]$  for not specifying open or closed bounds for intervals.

**Definition 1057 (length of interval).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . The *length of interval* from  $a$  to  $b$  is  $\ell([a, b]) \stackrel{\text{def.}}{=} b - a$ .

**Lemma 1058 (length is nonnegative).** Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then,  $\ell((a, b)) \geq 0$ .

*Proof.* Direct consequence of Definition 1057 (length of interval).  $\square$

**Lemma 1059 (length is homogeneous).**

We have  $\ell(\emptyset) = 0$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, for instance, we have  $\emptyset = (a, a)$ . Therefore, from Definition 1057 (length of interval), we have  $\ell(\emptyset) = 0$ .  $\square$

**Lemma 1060 (length of partition).** Let  $a, b, c \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have

$$(13.28) \quad \ell((a, b) \cap (c, \infty)) + \ell((a, b) \cap (c, \infty)^c) = \ell((a, b)).$$

*Proof.* Let  $I \stackrel{\text{def.}}{=} (a, b)$ , and  $E \stackrel{\text{def.}}{=} (c, \infty)$ . We have  $E^c = (-\infty, c]$ .

**Case  $c \leq a$ .** Then, we have  $I \cap E = I$  and  $I \cap E^c = \emptyset$ . Hence, from Lemma 1059 (length is homogeneous), we have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .

**Case  $a < c < b$ .** Then, we have  $I \cap E = (a, c)$  and  $I \cap E^c = [c, b)$ . Hence, from Definition 1057 (length of interval), and **abelian additive group properties of  $\mathbb{R}$** , we have

$$\ell(I \cap E) + \ell(I \cap E^c) = \ell((a, c)) + \ell([c, b)) = (c - a) + (b - c) = b - a = \ell(I).$$

**Case  $b \leq c$ .** Then, we have  $I \cap E = \emptyset$  and  $I \cap E^c = I$ . Hence, from Lemma 1059 (length is homogeneous), we have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .

Therefore, we always have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .  $\square$

**Definition 1061 (set of countable cover with bounded open intervals).**

Let  $A \subset \mathbb{R}$ . The *set of countable covers of  $A$  with bounded open intervals* is

$$(13.29) \quad C_A \stackrel{\text{def.}}{=} \left\{ ((a_n, b_n))_{n \in \mathbb{N}} \mid \left( \forall n \in \mathbb{N}, a_n, b_n \in \mathbb{R} \wedge a_n \leq b_n \right) \wedge A \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n) \right\}.$$

**Lemma 1062 (set of countable cover with bounded open intervals is nonempty).**

Let  $A \subset \mathbb{R}$ . Then, we have  $C_A \neq \emptyset$ .

*Proof.* For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} (-n, n)$ . Then, from **the Archimedean property of  $\mathbb{R}$** , we have  $\mathbb{R} \subset \bigcup_{n \in \mathbb{N}} I_n$ . Therefore, from Definition 1061 (set of countable cover with bounded open intervals), and **properties of inclusion**, we have  $(I_n)_{n \in \mathbb{N}} \in C_A$ .  $\square$

**Definition 1063 ( $\lambda^*$ , Lebesgue measure candidate).**

The *Lebesgue measure candidate* is the function  $\lambda^* : \mathcal{P}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}_+$  defined by

$$(13.30) \quad \forall A \subset \mathbb{R}, \quad \lambda^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \ell(I_n) \mid (I_n)_{n \in \mathbb{N}} \in C_A \right\}.$$

*Remark 1064.* Note that  $\lambda^*$  in the previous definition is not  $\sigma$ -additive on  $\mathcal{P}(\mathbb{R})$ , e.g. see [34, Ex. 2.28 pp. 86–87]. Thus, it is not a measure on  $\mathcal{P}(\mathbb{R})$ . In fact, there is no measure on  $\mathcal{P}(\mathbb{R})$  that generalizes the length of interval, e.g. see [34, Ex. 2.29 pp. 87–88].

**Lemma 1065** ( $\lambda^*$  is nonnegative). *Let  $A \subset \mathbb{R}$ . Then, we have  $\lambda^*(A) \geq 0$ .*

*Proof.* Direct consequence of Lemma 1058 (*length is nonnegative*), and Lemma 490 (*infimum is monotone function*).  $\square$

**Lemma 1066** ( $\lambda^*$  is homogeneous). *We have  $\lambda^*(\emptyset) = 0$ .*

*Proof.* From Lemma 1065 ( $\lambda^*$  is nonnegative), we have  $\lambda(\emptyset) \geq 0$ .

For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} (0, 0) = \emptyset$ . Let  $n \in \mathbb{N}$ . Then, from Definition 1057 (*length of interval*), and **field properties of  $\mathbb{R}$** , we have  $\ell(I_n) = 0 - 0 = 0$ . Thus, we have  $\sum_{n \in \mathbb{N}} \ell(I_n) = 0$ . Moreover, we have  $\emptyset \subset \bigcup_{n \in \mathbb{N}} I_n$ . Thus, we have  $(I_n)_{n \in \mathbb{N}} \in C_\emptyset$ . Hence, from Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*), and Definition 9 (*infimum, lower bound*), we have  $\lambda^*(\emptyset) \leq 0$ .

Therefore  $\lambda^*(\emptyset) = 0$ .  $\square$

**Lemma 1067** ( $\lambda^*$  is monotone). *Let  $A, B \subset \mathbb{R}$ . Assume that  $A \subset B$ . Then,  $\lambda^*(A) \leq \lambda^*(B)$ .*

*Proof.* Let  $(I_n)_{n \in \mathbb{N}} \in C_B$ . Then, from Definition 1061 (*set of countable cover with bounded open intervals*), and **transitivity of inclusion**, we have  $A \subset B \subset \bigcup_{n \in \mathbb{N}} I_n$ . Thus, from Definition 1061 (*set of countable cover with bounded open intervals*), we have  $(I_n)_{n \in \mathbb{N}} \in C_A$ . Hence, from Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*), and Definition 9 (*infimum, lower bound, for  $A$* ), we have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \ell(I_n)$ .

Therefore, from Definition 9 (*infimum, greatest lower bound, for  $B$* ), and Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*), we have  $\lambda^*(A) \leq \lambda^*(B)$ .  $\square$

**Lemma 1068** ( $\lambda^*$  is  $\sigma$ -subadditive). *Let  $(A_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ . Then,  $\lambda^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .*

*Proof.* Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ .

**Case  $\exists n_0 \in \mathbb{N}$  such that  $\lambda^*(A_{n_0}) = \infty$ .** Then, from Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rule 1 applies*), we have  $\lambda^*(A) \leq \infty = \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .

**Case  $\forall n \in \mathbb{N}$ ,  $\lambda^*(A_n) < \infty$ .** Let  $\varepsilon > 0$ . Let  $n \in \mathbb{N}$ . Then, from Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*), and Lemma 11 (*finite infimum*), there exists  $(I_{n,m})_{m \in \mathbb{N}} \in C_{A_n}$  such that  $\sum_{m \in \mathbb{N}} \ell(I_{n,m}) < \lambda^*(A_n) + \frac{\varepsilon}{2^{n+1}}$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}^2$  be a bijection. Then, from Definition 1061 (*set of countable cover with bounded open intervals*), Lemma 225 (*double countable union, with  $I = J \stackrel{\text{def.}}{=} \mathbb{N}$* ), Lemma 224 (*definition of double countable union, with  $I = J \stackrel{\text{def.}}{=} \mathbb{N}$* ), and **countability of  $\mathbb{N}^2$** , we have

$$A = \bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} I_{n,m} \right) = \bigcup_{n,m \in \mathbb{N}} I_{n,m} = \bigcup_{p \in \mathbb{N}} I_{\varphi(p)}.$$

Then, from Definition 1061 (*set of countable cover with bounded open intervals*),  $(I_{\varphi(n)})_{n \in \mathbb{N}}$  belongs to  $C_A$ , and thus, from Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*), Definition 9 (*infimum, lower bound*), Lemma 403 (*definition of double series in  $\overline{\mathbb{R}}_+$* ), Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ), and **additive properties of  $\mathbb{R}_+$** , we have

$$\lambda^*(A) \leq \sum_{p \in \mathbb{N}} \ell(I_{\varphi(p)}) = \sum_{n,m \in \mathbb{N}} \ell(I_{n,m}) = \sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{N}} \ell(I_{n,m}) \right) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n) + \varepsilon.$$

Hence, from Lemma 381 (*limit is monotone*), we have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .

Therefore, we always have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .  $\square$

**Lemma 1069** ( $\lambda^*$  generalizes length of interval).

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have  $\lambda^*([a, b]) = b - a$ .

*Proof.* Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ .

**(1). Closed interval.** Let  $\varepsilon > 0$ . Then, we have  $[a, b] \subset (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$ . Thus, from Definition 1061 (set of countable cover with bounded open intervals), Definition 1063 ( $\lambda^*$ , Lebesgue measure candidate), Definition 9 (infimum, lower bound), Lemma 1066 ( $\lambda^*$  is homogeneous), Definition 1057 (length of interval), and **ordered field properties of  $\mathbb{R}$** , the sequence  $((a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}), \emptyset, \dots)$  belongs to  $C_{[a, b]}$ , and  $\lambda^*([a, b]) \leq \ell(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) = b - a + \varepsilon$ . Hence, from Lemma 381 (limit is monotone), we have  $\lambda^*([a, b]) \leq b - a$ .

Let  $((a_n, b_n))_{n \in \mathbb{N}} \in C_{[a, b]}$  (with  $\forall n \in \mathbb{N}, a_n \leq b_n$ ). Then, from Definition 1061 (set of countable cover with bounded open intervals), and Lemma 299 (finite cover of compact interval), there exist  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b] \subset \bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a, b < b_{i_q}$ , and for all  $p \in [0..q-1]$ ,  $a_{i_{p+1}} < b_{i_p}$ . Then, from **ordered field properties of  $\mathbb{R}$** , Definition 1057 (length of interval), and Lemma 396 (addition in  $\overline{\mathbb{R}}_+$  is compatible with order), we have

$$\begin{aligned} b - a < b_{i_q} - a_{i_0} &\leq b_{i_q} + \sum_{p \in [0..q-1]} (-a_{i_{p+1}} + b_{i_p}) - a_{i_0} = \sum_{p \in [0..q]} (b_{i_p} - a_{i_p}) \\ &= \sum_{p \in [0..q]} \ell((a_{i_p}, b_{i_p})) \leq \sum_{n \in \mathbb{N}} \ell((a_n, b_n)). \end{aligned}$$

Hence, from Definition 1063 ( $\lambda^*$ , Lebesgue measure candidate), and Definition 9 (infimum, greatest lower bound), we have  $b - a \leq \lambda^*([a, b])$ .

Therefore, we have  $\lambda^*([a, b]) = b - a$ .

**(2). Open interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$\left[ a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2} \right] \subset (a, b) \subset [a, b].$$

Hence, from (1), and Lemma 1067 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*((a, b)) \leq b - a$ . Therefore, from Lemma 381 (limit is monotone), we have  $\lambda^*((a, b)) = b - a$ .

**(3). Left-open right-closed interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$[a + \varepsilon, b] \subset (a, b] \subset [a, b].$$

Hence, from (1), and Lemma 1067 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*((a, b]) \leq b - a$ . Therefore, from Lemma 381 (limit is monotone), we have  $\lambda^*((a, b]) = b - a$ .

**(4). Left-closed right-open interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$(a, b - \varepsilon] \subset [a, b) \subset [a, b].$$

Hence, from (1), and Lemma 1067 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*([a, b)) \leq b - a$ . Therefore, from Lemma 381 (limit is monotone), we have  $\lambda^*([a, b)) = b - a$ .  $\square$

*Remark 1070.*

Note that similar proofs using monotonicity provide for all  $a \in \mathbb{R}$ ,  $\lambda^*((-\infty, a]) = \lambda^*([a, \infty)) = \infty$ .

**Definition 1071** ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).

The set  $\mathcal{L}$  of subsets of  $\mathbb{R}$  defined by

$$(13.31) \quad \mathcal{L} \stackrel{\text{def.}}{=} \{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\}$$

is called the *Lebesgue  $\sigma$ -algebra*.

*Remark 1072.* The set  $\mathcal{L}$  is shown below to be a  $\sigma$ -algebra; hence its name.

**Lemma 1073 (equivalent definition of  $\mathcal{L}$ ).** *We have*

$$(13.32) \quad \mathcal{L} = \{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \leq \lambda^*(A)\}.$$

*Proof.* Direct consequence of Lemma 211 (*compatibility of pseudopartition with intersection, with  $\text{card}(I)$  equals 2*), Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*), Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total, anti-symmetry*), and Definition 1071 ( *$\mathcal{L}$ , Lebesgue  $\sigma$ -algebra*).  $\square$

**Lemma 1074 ( $\mathcal{L}$  is closed under complement).**  *$\mathcal{L}$  is closed under complement.*

*Proof.* Let  $E \in \mathcal{L}$ . Let  $A \subset \mathbb{R}$ . Then, from **the double complement law**, Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*), and Definition 1071 ( *$\mathcal{L}$ , Lebesgue  $\sigma$ -algebra*), we have

$$\lambda^*(A \cap E^c) + \lambda^*(A \cap (E^c)^c) = \lambda^*(A \cap E^c) + \lambda^*(A \cap E) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) = \lambda^*(A).$$

Therefore,  $E^c \in \mathcal{L}$ .  $\square$

**Lemma 1075 ( $\mathcal{L}$  is closed under finite union).**  *$\mathcal{L}$  is closed under finite union.*

*Proof.* For all  $n \in [2.. \infty)$ , let  $P(n)$  be the property:  $\forall (E_i)_{i \in [1..n]} \in \mathcal{L}, \bigcup_{i \in [1..n]} E_i \in \mathcal{L}$ .

**Induction:  $P(2)$ .** Let  $E_1, E_2 \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} E_1 \cup E_2$ .

Let  $A \subset \mathbb{R}$ . Then, from **properties of intersection, union and complement**, we have

$$A \cap E = A \cap (E_1 \uplus (E_1^c \cap E_2)) = (A \cap E_1) \cup (A \cap (E_1^c \cap E_2)).$$

Thus, from Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*), **associativity of intersection**, and Definition 1071 ( *$\mathcal{L}$ , Lebesgue  $\sigma$ -algebra,  $E_2$ , then  $E_1$  belong to  $\mathcal{L}$* ), we have

$$\begin{aligned} \lambda^*(A \cap E) + \lambda^*(A \cap E^c) &= \lambda^*((A \cap E_1) \cup (A \cap (E_1^c \cap E_2))) + \lambda^*(A \cap (E_1^c \cap E_2^c)) \\ &\leq \lambda^*(A \cap E_1) + \lambda^*((A \cap E_1^c) \cap E_2) + \lambda^*((A \cap E_1^c) \cap E_2^c) \\ &= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c) \\ &= \lambda^*(A). \end{aligned}$$

Hence, from Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ), we have  $E \in \mathcal{L}$ .

**Induction;  $P(n)$  implies  $P(n+1)$ .** Let  $n \in [2.. \infty)$ . Assume that  $P(n)$  holds.

Let  $(E_i)_{i \in [1..n+1]} \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} \bigcup_{i \in [1..n]} E_i$ . Then, from  $P(n)$ , we have  $E \in \mathcal{L}$ , and from **associativity of union**, and  $P(2)$ , we have  $\bigcup_{i \in [1..n+1]} E_i = E \cup E_{n+1} \in \mathcal{L}$ .

Therefore,  $P(n)$  holds for all  $n \in [2.. \infty)$ .  $\square$

**Lemma 1076 ( $\mathcal{L}$  is closed under finite intersection).**  *$\mathcal{L}$  is closed under finite intersection.*

*Proof.* Direct consequence of **De Morgan's laws**, Lemma 1074 ( *$\mathcal{L}$  is closed under complement*), and Lemma 1075 ( *$\mathcal{L}$  is closed under finite union*).  $\square$

**Lemma 1077 ( $\mathcal{L}$  is algebra of sets).**  *$\mathcal{L}$  is an algebra of sets on  $\mathbb{R}$ .*

*Proof.* Let  $A \subset \mathbb{R}$ . Then, from **properties of intersection**, and Lemma 1066 ( *$\lambda^*$  is homogeneous*), we have  $\lambda^*(A \cap \mathbb{R}) + \lambda^*(A \cap \mathbb{R}^c) = \lambda^*(A) + \lambda^*(\emptyset) = \lambda^*(A)$ . Hence,  $\mathbb{R} \in \mathcal{L}$ .

Therefore, from Lemma 1074 ( *$\mathcal{L}$  is closed under complement*), Lemma 1076 ( *$\mathcal{L}$  is closed under finite intersection*), Lemma 545 (*closedness under intersection and set difference,  $\mathcal{L}$  is closed under set difference*), Lemma 637 (*other equivalent definition of algebra of sets*),  $\mathcal{L}$  is an algebra of sets on  $\mathbb{R}$ .  $\square$

**Lemma 1078** ( $\lambda^*$  is additive on  $\mathcal{L}$ ). Let  $n \in [2..∞)$ . Let  $(E_i)_{i \in [1..n]} \in \mathcal{L}$ .  
Assume that the  $E_i$ 's are pairwise disjoint. Then,  $\lambda^*$  is additive on  $\mathcal{L}$ :

$$(13.33) \quad \forall A \subset \mathbb{R}, \quad \lambda^* \left( A \cap \bigsqcup_{i \in [1..n]} E_i \right) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i).$$

*Proof.* For all  $n \in [2..∞)$ , let  $P(n)$  be the property: for all  $(E_i)_{i \in [1..n]} \in \mathcal{L}$ ,

$$(\forall i, j \in [1..n], i \neq j \Rightarrow E_i \cap E_j = \emptyset) \implies \lambda^* \left( A \cap \bigsqcup_{i \in [1..n]} E_i \right) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i).$$

**Induction:  $P(2)$ .** Let  $E_1, E_2 \in \mathcal{L}$ . Assume that  $E_1 \cap E_2 = \emptyset$ . Let  $A \subset \mathbb{R}$ . Then, from **properties of intersection, union, and complement**, Lemma 211 (*compatibility of pseudopartition with intersection*, with  $\text{card}(I) = 2$ ), and Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*,  $E_1 \in \mathcal{L}$ ), we have

$$\begin{aligned} \lambda^*(A \cap (E_1 \uplus E_2)) &= \lambda^*((A \cap E_1) \uplus (A \cap E_2)) \\ &= \lambda^*([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1) + \lambda^*([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1^c) \\ &= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_2). \end{aligned}$$

**Induction:  $P(n)$  implies  $P(n+1)$ .** Let  $n \in [2..∞)$ . Assume that  $P(n)$  holds.

Let  $(E_i)_{i \in [1..n+1]} \in \mathcal{L}$ . Assume that for all  $i, j \in [1..n+1]$ ,  $i \neq j$  implies  $E_i \cap E_j = \emptyset$ . Let  $E$  be the disjoint union  $\bigsqcup_{i \in [1..n]} E_i$ . Then, from **properties of intersection, union, and complement**,  $P(2)$ , and  $P(n)$ , we have  $E \cap E_{n+1} = \emptyset$ , and

$$\lambda^*(A \cap (E \uplus E_{n+1})) = \lambda^*(A \cap E) + \lambda^*(A \cap E_{n+1}) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i) + \lambda^*(A \cap E_{n+1}).$$

Therefore,  $P(n)$  holds for all  $n \in [2..∞)$ . □

*Remark 1079.* The additive property of  $\lambda^*$  *per se* actually corresponds to the case  $A = \mathbb{R}$ .

**Lemma 1080** ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ).  $\lambda^*$  is  $\sigma$ -additive on  $(\mathbb{R}, \mathcal{L})$ .

*Proof.* Let  $(E_n)_{n \in \mathbb{N}} \in \mathcal{L}$ . Assume that for all  $i, j \in \mathbb{N}$ ,  $i \neq j$  implies  $E_i \cap E_j = \emptyset$ .

Let  $E \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} E_n$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 1078 ( $\lambda^*$  is additive on  $\mathcal{L}$ , with  $A \stackrel{\text{def.}}{=} \mathbb{R}$ ), and Lemma 1067 ( $\lambda^*$  is monotone), we have

$$\sum_{i \in [0..n]} \lambda^*(E_i) = \lambda^* \left( \bigcup_{i \in [0..n]} E_i \right) \leq \lambda^*(E).$$

Thus, from Lemma 381 (*limit is monotone*), we have

$$\sum_{n \in \mathbb{N}} \lambda^*(E_n) \leq \lambda^*(E).$$

Hence, from Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive), we have

$$\lambda^*(E) = \sum_{n \in \mathbb{N}} \lambda^*(E_n).$$

Therefore, from Definition 874 ( $\sigma$ -additivity),  $\lambda^*$  is  $\sigma$ -additive over  $(\mathbb{R}, \mathcal{L})$ . □

**Lemma 1081 (disjoint union of union in  $\mathcal{L}$ ).** Let  $I \subset \mathbb{N}$  be nonempty.  
 Let  $J \stackrel{\text{def.}}{=} [0..\text{card}(I))$ . Let  $\varphi : J \rightarrow I$  be the increasing numbering of  $I$ . Let  $(E_i)_{i \in I} \in \mathcal{L}$ .  
 Then, there exists  $(F_i)_{i \in I} \in \mathcal{L}$  pairwise disjoint such that for all  $i \in I$ ,  $F_i \subset E_i$ , and we have

$$(13.34) \quad \forall n \in J, \quad \bigcup_{j \in [0..n]} E_{\varphi(j)} = \biguplus_{j \in [0..n]} F_{\varphi(j)} \in \mathcal{L}.$$

*Proof.* Direct consequence of Lemma 1077 ( $\mathcal{L}$  is algebra of sets), and Lemma 652 (disjoint union of union in algebra of sets).  $\square$

**Lemma 1082 ( $\mathcal{L}$  is closed under countable union).**  $\mathcal{L}$  is closed under countable union.

*Proof.* Let  $I \subset \mathbb{N}$  be nonempty. Let  $(E_i)_{i \in I} \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} \bigcup_{i \in I} E_i$ .

**Case  $\text{card}(I) < \infty$ .** Then, from Lemma 1075 ( $\mathcal{L}$  is closed under finite union), we have  $E \in \mathcal{L}$ .

**Case  $\text{card}(I) = \infty$ .** Then, from Lemma 1081 (disjoint union of union in  $\mathcal{L}$ ), let  $\varphi : \mathbb{N} \rightarrow I$  be the increasing numbering of  $I$ , let  $(F_i)_{i \in I} \in \mathcal{L}$  pairwise disjoint such that for all  $n \in \mathbb{N}$ , we have  $\biguplus_{i \in [0..n]} F_{\varphi(i)} = \bigcup_{i \in [0..n]} E_{\varphi(i)}$ .

Let  $n \in \mathbb{N}$ . Then, from **properties of union**, and **monotonicity of complement**, we have  $E^c \subset F^c$  where  $F \stackrel{\text{def.}}{=} \biguplus_{i \in [0..n]} F_{\varphi(i)}$ . Let  $A \subset \mathbb{R}$ . Then, from Lemma 1067 ( $\lambda^*$  is monotone, with  $A \cap E^c \subset A \cap F^c$ ), Lemma 1078 ( $\lambda^*$  is additive on  $\mathcal{L}$ ), Lemma 1075 ( $\mathcal{L}$  is closed under finite union,  $F \in \mathcal{L}$ ), and Definition 1071 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra, with  $F$ ), we have

$$\begin{aligned} \sum_{i \in [0..n]} \lambda^*(A \cap F_{\varphi(i)}) + \lambda^*(A \cap E^c) &\leq \sum_{i \in [0..n]} \lambda^*(A \cap F_{\varphi(i)}) + \lambda^*(A \cap F^c) \\ &= \lambda^*(A \cap F) + \lambda^*(A \cap F^c) = \lambda^*(A). \end{aligned}$$

Thus, from Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive), **distributivity of intersection over union**, and Lemma 381 (limit is monotone), we have

$$\lambda^* \left( A \cap \bigcup_{n \in \mathbb{N}} F_{\varphi(n)} \right) + \lambda^*(A \cap E^c) \leq \sum_{n \in \mathbb{N}} \lambda^*(A \cap F_{\varphi(n)}) + \lambda^*(A \cap E^c) \leq \lambda^*(A).$$

Hence, since **the limit is a function**, we have  $\bigcup_{n \in \mathbb{N}} F_{\varphi(n)} = \bigcup_{n \in \mathbb{N}} E_{\varphi(n)} = E$ , and

$$\lambda^*(A \cap E) + \lambda^*(A \cap E^c) \leq \lambda^*(A).$$

Therefore, from Lemma 1073 (equivalent definition of  $\mathcal{L}$ ), we have  $E \in \mathcal{L}$ .  $\square$

**Lemma 1083 (rays are Lebesgue-measurable).** Let  $a \in \mathbb{R}$ . Then, we have

$$(13.35) \quad (a, \infty), (-\infty, a], (-\infty, a), [a, \infty), \in \mathcal{L}.$$

*Proof.* (1).  $(a, \infty) \in \mathcal{L}$ .

Let  $A \subset \mathbb{R}$ . Let  $\alpha \stackrel{\text{def.}}{=} \lambda^*(A \cap (a, \infty)) + \lambda^*(A \cap (-\infty, a])$ . Let us show that  $\alpha \leq \lambda^*(A)$ .

**Case  $\lambda^*(A) = \infty$ .** Trivial. **Case  $\lambda^*(A)$  finite.** Let  $\varepsilon > 0$ . From Definition 1063 ( $\lambda^*$ , Lebesgue measure candidate), and Lemma 11 (finite infimum), let  $(I_n)_{n \in \mathbb{N}} \in C_A$  such that

$$\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \ell(I_n) \leq \lambda^*(A) + \varepsilon.$$

For all  $n \in \mathbb{N}$ , let  $I'_n \stackrel{\text{def.}}{=} I_n \cap (a, \infty)$  and  $I''_n \stackrel{\text{def.}}{=} I_n \cap (-\infty, a]$ . Let  $n \in \mathbb{N}$ . From Lemma 235 (intervals are closed under finite intersection),  $I'_n$  and  $I''_n$  are intervals (possibly empty). Then, from Lemma 1069 ( $\lambda^*$  generalizes length of interval), and since  $I'_n$  and  $I''_n$  are contiguous, we



have  $\lambda^*(I'_n) + \lambda^*(I''_n) = \ell(I'_n) + \ell(I''_n) = \ell(I_n)$ . Hence, from Lemma 1067 ( $\lambda^*$  is monotone), and Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive), we have

$$\alpha \leq \lambda^* \left( \bigcup_{n \in \mathbb{N}} I'_n \right) + \lambda^* \left( \bigcup_{n \in \mathbb{N}} I''_n \right) \leq \sum_{n \in \mathbb{N}} (\lambda^*(I'_n) + \lambda^*(I''_n)) = \sum_{n \in \mathbb{N}} \ell(I_n) \leq \lambda^*(A) + \varepsilon.$$

Therefore, from Lemma 381 (*limit is monotone*), we have  $\alpha \leq \lambda^*(A)$ , and from Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ), we have  $(a, \infty) \in \mathcal{L}$ .

(2).  $(-\infty, a] \in \mathcal{L}$ .

Direct consequence of Lemma 1074 ( $\mathcal{L}$  is closed under complement), and (1).

(3).  $(-\infty, a) \in \mathcal{L}$ .

Direct consequence of Lemma 1082 ( $\mathcal{L}$  is closed under countable union), and (2) with

$$(-\infty, a) = \bigcup_{n \in \mathbb{N}} \left( -\infty, a - \frac{1}{n+1} \right].$$

(4).  $[a, \infty) \in \mathcal{L}$ .

Direct consequence of Lemma 1074 ( $\mathcal{L}$  is closed under complement), and (3). □

**Lemma 1084 (intervals are Lebesgue-measurable).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have

$$(13.36) \quad (a, b), [a, b], [a, b), (a, b] \in \mathcal{L}.$$

*Proof.* Direct consequence of Lemma 1076 ( $\mathcal{L}$  is closed under finite intersection), and Lemma 1083 (*rays are Lebesgue-measurable*) with

$$\begin{aligned} (a, b) &= (-\infty, b) \cap (a, \infty), & [a, b] &= (-\infty, b] \cap [a, \infty), \\ [a, b) &= (-\infty, b) \cap [a, \infty), & (a, b] &= (-\infty, b] \cap (a, \infty). \end{aligned}$$

□

**Lemma 1085 ( $\mathcal{L}$  is  $\sigma$ -algebra).**

$(\mathbb{R}, \mathcal{L})$  is a measurable space.

*Proof.* Direct consequence of Lemma 1077 ( $\mathcal{L}$  is algebra of sets), Definition 634 (*algebra of sets*), Lemma 1082 ( $\mathcal{L}$  is closed under countable union), and Lemma 696 ( $\sigma$ -algebra). □

**Lemma 1086 ( $\lambda^*$  is measure on  $\mathcal{L}$ ).**

$(\mathbb{R}, \mathcal{L}, \lambda^*_{|\mathcal{L}})$  is a measure space.

*Proof.* Direct consequence of Lemma 1085 ( $\mathcal{L}$  is  $\sigma$ -algebra), Lemma 1065 ( $\lambda^*$  is nonnegative), Lemma 1066 ( $\lambda^*$  is homogeneous), Lemma 1080 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ), and Lemma 913 (*measure*). □

**Lemma 1087 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ).**

We have  $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$ .

*Proof.* Direct consequence of Lemma 710 (*properties of  $\sigma$ -algebra generation, monotonicity*), Lemma 796 (*Borel subsets of  $\mathbb{R}$* ), and Lemma 1084 (*intervals are Lebesgue-measurable*). □

**Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ).**

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^*_{|\mathcal{B}(\mathbb{R})})$  is a measure space.

*Proof.* Direct consequence of Lemma 1086 ( $\lambda^*$  is measure on  $\mathcal{L}$ ), and Lemma 1087 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ). □

*Remark 1089.* See the sketch of next proof in Section 5.9.



**Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).**

There exists a unique measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  that generalizes the length of bounded open intervals.

This measure is denoted  $\lambda \stackrel{\text{def.}}{=} \lambda_{|\mathcal{B}(\mathbb{R})}^*$ ; it is called the (Borel-)Lebesgue measure on Borel subsets (of  $\mathbb{R}$ ).

*Proof. Existence.* Direct consequence of Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ), and Lemma 1069 ( $\lambda^*$  generalizes length of interval, for open intervals).

**Uniqueness.**

Let  $G \stackrel{\text{def.}}{=} \{[a, b]\}_{a < b}$ . Then, we have  $G \subset \mathcal{P}(\mathbb{R})$  and  $G \neq \emptyset$ . Hence, from Lemma 235 (intervals are closed under finite intersection), and Definition 602 ( $\pi$ -system),  $G$  is a  $\pi$ -system.

From Lemma 796 (Borel subsets of  $\mathbb{R}$ ,  $\{[a, b]\}_{a < b}$  and  $\{[a, b]\}_{a < b}$  generate  $\mathcal{B}(\mathbb{R})$ ), Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra, generators belong to  $\mathcal{B}(\mathbb{R})$ ), we have

$$(\forall a, b \in \mathbb{R}, \quad a < b \implies (a, b), [a, b] \in \mathcal{B}(\mathbb{R})), \quad \text{and} \quad \Sigma_{\mathbb{R}}(G) = \mathcal{B}(\mathbb{R}).$$

For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} [n, n + 1)$ . Then,  $(I_n)_{n \in \mathbb{N}}$  is obviously pairwise disjoint, and from **the Archimedean property of  $\mathbb{R}$** , we have  $\mathbb{R} = \bigsqcup_{n \in \mathbb{N}} I_n$ . Let  $n \in \mathbb{N}$ . Then,  $I_n \in G$ , and from Lemma 1069 ( $\lambda^*$  generalizes length of interval), we have  $\lambda^*(I_n) = 1 < \infty$ .

Let  $\mu$  be a measure on  $\mathcal{B}(\mathbb{R})$  such that, for all  $a, b \in \mathbb{R}$ ,  $a < b$  implies  $\mu((a, b)) = b - a$ . Let  $a, b \in \mathbb{R}$ . Assume that  $a < b$ . For all  $p \in \mathbb{N}$ , let  $A_p \stackrel{\text{def.}}{=} \left(a - \frac{1}{p+1}, b\right)$ . Then, from **the non-increasing property of  $\left(p \mapsto \frac{1}{p+1}\right)$  in  $\mathbb{R}_+$  and its limit 0 when  $p \rightarrow \infty$ , the definition of inclusion, additive group properties of  $\mathbb{R}$ , linearity of the limit in  $\mathbb{R}$** , and Lemma 927 (measure is continuous from above, with  $\mu_1$  and  $(A_p)_{p \in \mathbb{N}}$ ), the sequence  $(A_p)_{p \in \mathbb{N}}$  is nonincreasing, for all  $p \in \mathbb{N}$ , we have  $\mu_1(A_p) = b - a + \frac{1}{p+1} < \infty$ , and

$$\mu_1([a, b]) = \mu_1\left(\bigcap_{p \in \mathbb{N}} A_p\right) = \inf_{p \in \mathbb{N}} \mu_1(A_p) = \lim_{p \rightarrow \infty} \left(b - a + \frac{1}{p+1}\right) = b - a.$$

Hence, from Lemma 1069 ( $\lambda^*$  generalizes length of interval),  $\mu$  and  $\lambda_{|\mathcal{B}(\mathbb{R})}^*$  coincide on  $G$ .

Therefore, from Lemma 992 (uniqueness of measures extended from a  $\pi$ -system, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $\Sigma \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$ ,  $\mu_1 \stackrel{\text{def.}}{=} \lambda^*$ ,  $\mu_2 = \mu$ , and  $X_n \stackrel{\text{def.}}{=} I_n$ ), we have, for all  $A \in \mathcal{B}(\mathbb{R})$ ,  $\mu(A) = \lambda_{|\mathcal{B}(\mathbb{R})}^*(A)$ .  $\square$

*Remark 1091.* Note that  $\mathbf{N}(\mathbb{R}, \mathcal{L}, \lambda^*) = \mathbf{N}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \subset \mathcal{L}$ , e.g. see [34, Ex. 2.33 pp. 92–94].

**Lemma 1092 (Lebesgue measure generalizes length of interval).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have  $\lambda([a, b]) = b - a$ .

*Proof.* Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), and Lemma 1069 ( $\lambda^*$  generalizes length of interval).  $\square$

*Remark 1093.* Note that similarly to  $\lambda^*$ , we can prove for all  $a \in \mathbb{R}$ ,  $\lambda((-\infty, a]) = \lambda([a, \infty)) = \infty$ .

**Lemma 1094 (Lebesgue measure is  $\sigma$ -finite).** The measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is  $\sigma$ -finite.

*Proof.* Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), Lemma 1069 ( $\lambda^*$  generalizes length of interval, with  $b = -a \in \mathbb{N}$ ), **the Archimedean property of  $\mathbb{R}$** , and Definition 885 ( $\sigma$ -finiteness, with  $A_n \stackrel{\text{def.}}{=} (-n, n)$ ).  $\square$

**Lemma 1095 (Lebesgue measure is diffuse).** The measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is diffuse.

*Proof.* Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), Lemma 1069 ( $\lambda^*$  generalizes length of interval, with  $b = a$ ), and Definition 856 (diffuseness,  $\{a\} = [a, a]$ ).  $\square$

## Chapter 14

# Simple function

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*Remark 1096.* In this chapter, we study functions defined on a measurable space that take a finite number of values in a normed vector space over a valued field of characteristic 0 (namely  $\mathbb{R}$  or  $\mathbb{C}$ ).

## 14.1 Indicator function

*Remark 1097.* In this section, the measurable number space  $\mathbb{K}$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\overline{\mathbb{R}}$  (see Definition 748).

**Definition 1098** ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. The set of measurable indicator functions on  $\mathbb{K}$  is denoted  $\mathcal{IF}_{\mathbb{K}}(X, \Sigma)$  (or simply  $\mathcal{IF}_{\mathbb{K}}$ ); it is defined by  $\mathcal{IF}_{\mathbb{K}}(X, \Sigma) \stackrel{\text{def.}}{=} \{\mathbb{1}_A \mid A \in \Sigma\}$ .

When  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , it is simply denoted  $\mathcal{IF}(X, \Sigma)$  (or  $\mathcal{IF}$ ).

*Remark 1099.* Functions in  $\overline{\mathcal{IF}_{\mathbb{K}}}$  are shown below to be measurable; hence its name.

**Lemma 1100** (indicator and support are inverse). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $A \in \Sigma$ , let  $f \in \mathcal{IF}_{\mathbb{K}}$ . Then, we have

$$(14.1) \quad \mathbb{1}_A \in \mathcal{IF}_{\mathbb{K}}, \quad \{f \neq 0_{\mathbb{K}}\} \in \Sigma, \quad \{\mathbb{1}_A \neq 0_{\mathbb{K}}\} = A, \quad \text{and} \quad \mathbb{1}_{\{f \neq 0_{\mathbb{K}}\}} = f.$$

*Proof.* Direct consequence of Definition 748 (measurability and structure), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), the definition of the indicator function, and the definition of inverse image.  $\square$

**Lemma 1101** ( $\mathcal{IF}_{\mathbb{K}}$  is measurable). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Then, we have  $\mathcal{IF}_{\mathbb{K}} \subset \mathcal{M}_{\mathbb{K}}$ .

Moreover, when  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , we have  $\mathcal{IF} \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), Lemma 811 (measurability of indicator function), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), nonnegativeness of indicator function ( $\mathcal{IF} \subset \mathcal{M}_+$ ), and Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $\mathcal{M}_+ \subset \mathcal{M}$ ).  $\square$

**Lemma 1102** ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \in \Sigma$  be pairwise disjoint. Then, we have  $\sum_{i \in I} \mathbb{1}_{A_i} = \mathbb{1}_{\bigcup_{i \in I} A_i} \in \mathcal{IF}_{\mathbb{K}}$ .

*Proof.* Direct consequence of Definition 748 (measurability and structure), Lemma 525 (masking is  $\sigma$ -additive, with  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), Definition 741 (measurable space,  $\Sigma$  is a  $\sigma$ -algebra), and Lemma 696 ( $\sigma$ -algebra,  $P_{\Sigma} \Rightarrow Uc$ ).  $\square$

**Lemma 1103** ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $A, B \in \Sigma$ . Then, we have  $\mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B} \in \mathcal{IF}_{\mathbb{K}}$ .

*Proof.* Direct consequence of Definition 748 (measurability and structure), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), the formula for the indicator of intersection (product), and Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $P_{\Sigma} \Rightarrow I$ ).  $\square$

*Remark 1104.*

We recall the notation  $\mathbb{1}_U^V$  to denote the indicator function of  $U$  defined on  $V$  (when  $U \subset V$ ).

**Lemma 1105** ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero). Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \mathbb{K}$  and  $\hat{f} : X \rightarrow \mathbb{K}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A)$  iff  $\hat{f} \mathbb{1}_A \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$ .

If so, there exists  $B \in \Sigma$  such that  $B \subset A$ ,  $f|_A = \mathbb{1}_B^A$ , and  $\hat{f} \mathbb{1}_A = \mathbb{1}_B^X$ .

*Proof.* **“Left” implies “right”.** Direct consequence of Definition 748 (*measurability and structure*), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*, with  $X \stackrel{\text{def.}}{=} A$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma \cap A$ , thus  $f|_A = \mathbb{1}_B^A$  with  $B \in \Sigma \cap A$ ), Lemma 764 (*measurability of measurable subspace*, with  $Y \stackrel{\text{def.}}{=} A$  belongs to  $\Sigma$ , thus  $B \in \Sigma$  and  $B \subset A$ ), Lemma 524 (*restriction is masking*, with  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ), **the definition of the indicator function** ( $\hat{f} \mathbb{1}_A = \mathbb{1}_B^X$ ), and Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*,  $\hat{f} \mathbb{1}_A \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$ ).

**“Right” implies “left”.** Direct consequence of Definition 748 (*measurability and structure*), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*,  $\hat{f} \mathbb{1}_A = \mathbb{1}_B^X$  with  $B \in \Sigma$  such that  $B \subset A$ ), Lemma 764 (*measurability of measurable subspace*, with  $Y \stackrel{\text{def.}}{=} A \in \Sigma$ , thus  $B \in \Sigma \cap A$ ), Lemma 524 (*restriction is masking*, with  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ,  $f|_A = \mathbb{1}_B^A$ ), and Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*, with  $X \stackrel{\text{def.}}{=} A$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma \cap A$ , thus  $f|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A)$ ).

Therefore, we have the equivalence and both representations.  $\square$

**Lemma 1106** ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under restriction*).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $f : X \rightarrow \mathbb{K}$ . Then, we have  $f \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A)$ .

*Proof.* **“Left” implies “right”.** Direct consequence of Definition 748 (*measurability and structure*), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*,  $f = \mathbb{1}_B^X$  with  $B \in \Sigma$ ), **the rule for the restriction of an indicator function** ( $f|_A = \mathbb{1}_{B \cap A}^A$ ), Lemma 764 (*measurability of measurable subspace*,  $B \cap A \in \Sigma \cap A$ ), and Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*, with  $X \stackrel{\text{def.}}{=} A$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma \cap A$ , thus  $f|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A)$ ).

**“Right” implies “left”.** Direct consequence of Lemma 697 (*equivalent definition of  $\sigma$ -algebra*,  $A \stackrel{\text{def.}}{=} X \in \Sigma$ ), Lemma 764 (*measurability of measurable subspace*,  $\Sigma \cap X = \Sigma$ ), and **the definition of restriction of function** ( $f|_X = f$ , thus  $f \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$ ).

Therefore, we have the equivalence.  $\square$

**Lemma 1107** ( $\mathcal{IF}_{\mathbb{K}}$ , *masking is  $\sigma$ -additive*).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be a measurable number space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ . Let  $f : X \rightarrow \mathbb{K}$ . Then, we have  $f \mathbb{1}_A = \sum_{i \in I} f \mathbb{1}_{A_i}$ , and  $f \mathbb{1}_A \in \mathcal{IF}_{\mathbb{K}}$  iff for all  $i \in I$ ,  $f \mathbb{1}_{A_i} \in \mathcal{IF}_{\mathbb{K}}$ .

*Proof.* (1). **Identity.** Direct consequence of Definition 748 (*measurability and structure*), and Lemma 525 (*masking is  $\sigma$ -additive*).

**“Left” implies “right”.** Direct consequence of Definition 209 (*pseudopartition*,  $A_i \subset A$ ), Lemma 1100 (*indicator and support are inverse*,  $f \mathbb{1}_A = \mathbb{1}_B$  with  $B \in \Sigma$  such that  $B \subset A$ ), and Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under multiplication*,  $f \mathbb{1}_{A_i} = \mathbb{1}_{B \cap A_i} \in \mathcal{IF}_{\mathbb{K}}$ ).

**“Right” implies “left”.** Direct consequence of Definition 748 (*measurability and structure*), Lemma 1100 (*indicator and support are inverse*,  $f \mathbb{1}_{A_i} = \mathbb{1}_{B_i}$  with  $B_i \in \Sigma$  and  $B_i \subset A_i$ ), **compatibility of inclusion with disjoint union (the  $B_i$ 's are pairwise disjoint)**, Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  *is  $\sigma$ -additive*,  $\sum_{i \in I} \mathbb{1}_{B_i} \in \mathcal{IF}_{\mathbb{K}}$ ), and (1) ( $f \mathbb{1}_A = \sum_{i \in I} f \mathbb{1}_{A_i} \in \mathcal{IF}_{\mathbb{K}}$ ).

Therefore, we have the equivalence.  $\square$

## 14.2 Simple function

**Lemma 1108** ( $\mathcal{SF}_E$  simple representation). *Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . The vector subspace of finite vector linear combinations of indicator functions of measurable subsets is called vector space of (vector) simple functions (from  $X$ ) to  $E$ ; it is denoted  $\mathcal{SF}_E(X, \Sigma)$  (or simply  $\mathcal{SF}_E$ ); it is defined by*

$$(14.2) \quad \mathcal{SF}_E(X, \Sigma) \stackrel{\text{def.}}{=} \left\{ \sum_{i \in [0..n]} f_i u_i \mid n \in \mathbb{N} \wedge (f_i)_{i \in [0..n]} \in \mathcal{IF}_{\mathbb{K}} \wedge (u_i)_{i \in [0..n]} \in E \right\}.$$

*Let  $f \in \mathcal{SF}_E$ . A triple  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  such that  $f = \sum_{i \in [0..n]} \mathbb{1}_{A_i} u_i$  is called simple representation (of  $f$ ). If so, for all  $x \in X \setminus \bigcup_{i \in [0..n]} A_i$ , we have  $f(x) = 0$ .*

*Moreover, when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ , we have  $\mathcal{SF}_{\mathbb{K}} = \text{span}(\mathcal{IF}_{\mathbb{K}})$ , and when  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , it is simply denoted  $\mathcal{SF}(X, \Sigma)$  (or  $\mathcal{SF}$ ).*

*Proof.* Direct consequence of **the definition of the indicator function**, Lemma 71 (*zero times yields zero,  $0_{\mathbb{K}} u = 0_E$* ), Definition 61 (*vector space,  $(E, +_E)$  is an abelian group*), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*), and Lemma 1100 (*indicator and support are inverse*).  $\square$

*Remark 1109.* Note that in a simple representation, the values  $u_i$ 's may not be unique and may not be values of the function, and the supports  $A_i$ 's may be empty, may not be related to preimages of  $u_i$ 's, may overlap and thus may not form a partition of  $X$ .

*Remark 1110.* We recall that  $\{f = u\}$  denotes the subset  $f^{-1}(\{u\})$ .

**Lemma 1111** ( $\mathcal{SF}_E$  canonical representation).

*Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $f : X \rightarrow E$ . Then, we have  $f \in \mathcal{SF}_E$  iff  $f(X)$  is finite, and for all  $u \in f(X)$ ,  $\{f = u\}$  belongs to  $\Sigma$ .*

*If so, we have the following canonical representation:*

$$(14.3) \quad f = \sum_{u \in f(X)} \mathbb{1}_{\{f=u\}} u,$$

*i.e. there exists a triple  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  which is unique up to a permutation of  $[0..n]$  such that*

$$(14.4) \quad \forall i \in [0..n], \quad A_i = \{f = u_i\} \neq \emptyset,$$

$$(14.5) \quad (A_i)_{i \in [0..n]} \text{ forms a partition of } X,$$

$$(14.6) \quad f(X) = \{u_i \mid i \in [0..n]\},$$

$$(14.7) \quad f = \sum_{i \in [0..n]} \mathbb{1}_{A_i} u_i.$$

*Moreover, when  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , the triple is unique, for instance by specifying*

$$(14.8) \quad \forall i \in [0..n-1], \quad u_i < u_{i+1}.$$

*Proof. “Left” implies “right”.* Assume that  $f \in \mathcal{SF}_E$ . Then, from Lemma 1108 ( $\mathcal{SF}_E$  simple representation), let  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  be a simple representation of  $f$ . Thus, from **the definition of the indicator function**,  $f$  can only take the values  $\sum_{i \in [0..n]} \delta_i u_i$  where  $(\delta_i)_{i \in [0..n]} \in \{0, 1\}$ . Hence, the cardinality of  $f(X)$  is at most  $2^{n+1}$ .

Let  $(\delta_i)_{i \in [0..n]} \in \{0, 1\}$ . Let  $u \stackrel{\text{def.}}{=} \sum_{i \in [0..n]} \delta_i u_i \in f(X)$ . Let  $I_u \subset [0..n]$  such that  $i \in I_u$  iff  $\delta_i = 1$ . Then, we have<sup>1</sup>  $u = \sum_{i \in I_u} u_i$ . Unfortunately, several partial sums of  $u_i$ 's may lead to the same value  $u$ . Thus, from Lemma 697 (*equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection*), we have

$$\{f = u\} = \bigcup_{\substack{I_u \subset \mathcal{P}([0..n]) \\ \sum_{i \in I_u} u_i = u}} \left( \bigcap_{i \in I_u} A_i \right) \in \Sigma.$$

From **properties of inverse image**, the collection  $(\{f = u\})_{u \in f(X)}$  makes a finite partition of the whole set  $X$ . Let  $x \in X$ . Let  $u \stackrel{\text{def.}}{=} f(x) \in E$ . Then, from Definition 61 (*vector space*), we have  $f(x) = u = 1 u = \mathbf{1}_{\{f=u\}}(x) u$ . Hence, property (14.3) holds. And when  $E \stackrel{\text{def.}}{=} \mathbb{R}$ , property (14.7) is equivalent, up to a nondecreasing reordering of  $f(X)$ .

**“Right” implies “left”.** Let  $n \stackrel{\text{def.}}{=} \text{card}(f(X)) - 1 \in \mathbb{N}$ . Let  $(u_i)_{i \in [0..n]} \in E$  be the  $n + 1$  distinct values taken by the function  $f$ , i.e. such that  $f(X) = \{u_i \mid i \in [0..n]\}$ . For all  $i \in [0..n]$ , let

$$A_i \stackrel{\text{def.}}{=} \{f = u_i\} \in \Sigma.$$

Then, from **the definition and properties of image and inverse image**, and **the definition of partition**, the  $n$  subsets  $(A_i)_{i \in [0..n]}$  constitutes a partition of  $X$ . Thus, from **the definition of the indicator function**, we have  $f = \sum_{i \in [0..n]} \mathbf{1}_{A_i} u_i$ . Hence, from Lemma 1108 ( *$\mathcal{SF}_E$  simple representation*), we have  $f \in \mathcal{SF}_E$ .

**Uniqueness.** Direct consequence of (14.4)–(14.8) since  $\text{card}(f(X)) = n + 1$  (i.e.  $n$  is unique),  $f(X) = \{a_i \mid i \in [0..n]\}$  with  $a_i < a_{i+1}$  (i.e. the  $a_i$ 's are unique), and  $A_i = \{f = a_i\}$  (i.e. the  $A_i$ 's are unique).

Therefore, we have the equivalence, and the representation is unique when  $E \stackrel{\text{def.}}{=} \mathbb{R}$ . □

*Remark 1112.* To sum up, in a canonical representation, the set of  $u_i$  values is unique and is the set of values of the function, the supports  $A_i$ 's are the nonempty preimages of the  $u_i$ 's and form a partition of  $X$ . Hence, the canonical representation of the zero function is  $\mathbf{1}_X 0_E$ .

Note that properties (14.3) and (14.7) are identical, up to a permutation of  $f(X)$ .

**Lemma 1113 ( $\mathcal{SF}_E$  is measurable).** *Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a measurable normed vector space over  $\mathbb{K}$ . Then, we have  $\mathcal{SF}_E \subset \mathcal{M}_E$ .*

*Moreover, when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ , we have  $\mathcal{IF}_{\mathbb{K}} \subset \mathcal{SF}_{\mathbb{K}} \subset \mathcal{M}_{\mathbb{K}}$ , and when  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , we have  $\mathcal{IF} \subset \mathcal{SF} \subset \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ .*

*Proof.* Direct consequence of Definition 748 (*measurability and structure*), Lemma 1111 ( *$\mathcal{SF}_E$  canonical representation,  $f(X)$  is finite and for all  $u \in f(X)$ ,  $\{f = u\} \in \Sigma$* ), Lemma 810 ( *$\mathcal{M}_{\mathbb{K}}$  countable range*), Lemma 1108 ( *$\mathcal{SF}_E$  simple representation,  $\mathcal{IF}_{\mathbb{K}} \subset \mathcal{SF}_{\mathbb{K}}$* ), and Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$* ). □

<sup>1</sup>For instance, the case  $I_u = \emptyset$  may correspond to  $u \stackrel{\text{def.}}{=} 0_E$  when  $0_E \notin \{u_i \mid i \in [0..n]\}$ .

**Lemma 1114** ( $\mathcal{SF}_E$  disjoint representation).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $f : X \rightarrow E$ . Then, we have  $f \in \mathcal{SF}_E$  iff there exist  $n \in \mathbb{N}$ ,  $(A_i)_{i \in [0..n]} \in \Sigma$ , and  $(u_i)_{i \in [0..n]} \in E$  such that

$$(14.9) \quad \forall i \in [0..n], \quad A_i \subset \{f = u_i\},$$

$$(14.10) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(14.11) \quad f = \sum_{i \in [0..n]} \mathbb{1}_{A_i} u_i.$$

If so, the triple  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  is called disjoint representation (of  $f$ ).

*Proof.* Direct consequence of Lemma 1111 ( $\mathcal{SF}_E$  canonical representation, “left” implies “right”), and Lemma 1108 ( $\mathcal{SF}_E$  simple representation, “right” implies “left”).  $\square$

*Remark 1115.* In a disjoint representation, the values  $u_i$ ’s may not be unique ( $u_i = u_j$  with  $i \neq j$  is possible) and may not be the values of the function ( $A_i = \emptyset$  is possible). More precisely, the supports  $A_i$ ’s may be empty but are subsets of the associated preimage of  $u_i$  (thus the values that are not taken by the function are associated with an empty support). They form a pseudopartition of  $X$  (see Definition 209).

Note that the nonempty  $A_i$ ’s form a partition of  $(\{f = u\})_{u \in f(X)}$  (see next lemma).

**Lemma 1116** ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $f \in \mathcal{SF}_E$ . Let  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  be a disjoint representation of  $f$ , and  $(m, (B_j)_{j \in [0..m]}, (v_j)_{j \in [0..m]}) \in \mathbb{N} \times \Sigma \times E$  be its canonical representation.

For all  $j \in [0..m]$ , let  $I'_j \stackrel{\text{def.}}{=} \{i \in [0..n] \mid \emptyset \neq A_i \subset B_j\}$ . Let  $I' \stackrel{\text{def.}}{=} \{i \in [0..n] \mid A_i \neq \emptyset\}$ . Then, we have

$$(14.12) \quad \forall j \in [0..m], \quad (\forall i \in I'_j, u_i = v_j) \quad \wedge \quad B_j = \bigsqcup_{i \in I'_j} A_i,$$

$$(14.13) \quad (\forall p, q \in [0..m], p \neq q \Rightarrow I'_p \cap I'_q = \emptyset) \quad \wedge \quad \bigsqcup_{j \in [0..m]} I'_j = I'.$$

*Proof.* Let  $i \in [0..n]$  and  $j \in [0..m]$ . Assume that  $A_i \neq \emptyset$ .

**(0a)**  $A_i \subset B_j \Rightarrow u_i = v_j$ . Let  $x \in A_i \subset B_j$ . Then, from Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation,  $f(x) = u_i$ ), and Lemma 1111 ( $\mathcal{SF}_E$  canonical representation,  $f(x) = v_j$ ), we have  $u_i = v_j$ .

**(0b)**  $u_i = v_j \Rightarrow A_i \subset B_j$ . Then, from Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), and Lemma 1111 ( $\mathcal{SF}_E$  canonical representation), we have  $A_i \subset \{f = u_i\} = \{f = v_j\} = B_j$ .

**(1).** Let  $j \in [0..m]$ . From Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), the  $A_i$ ’s are pairwise disjoint. Let  $A'_j \stackrel{\text{def.}}{=} \bigsqcup_{i \in I'_j} A_i$ . Let  $x \in B_j$ . Then, from Lemma 1111 ( $\mathcal{SF}_E$  canonical representation,  $f(x) = v_j$ ), and Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation,  $X = \bigsqcup_{i \in [0..n]} A_i$  and  $A_i \subset \{f = u_i\}$ ), there exists  $i \in [0..n]$  such that  $x \in A_i$ , i.e.  $A_i \neq \emptyset$  and  $u_i = f(x) = v_j$ . Thus, from (0b), we have  $i \in I'_j$ , and  $B_j \subset A'_j$ . Hence, since the other inclusion is obvious, we have equality, and from (0a), property (14.12) holds.

**(2a).** Let  $p, q \in [0..m]$ . Assume that  $p \neq q$ .

Let  $i \in I'_p \cap I'_q$ . Then, from Lemma 1111 ( $\mathcal{SF}_E$  canonical representation, pairwise disjunction), we have  $\emptyset \neq A_i \subset B_p \cap B_q = \emptyset$ , which is impossible. Hence, we have  $I'_p \cap I'_q = \emptyset$ .

**(2b).** Let  $i \in I'$ , i.e.  $A_i \neq \emptyset$ . Then, from Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation,  $u_i \in f(X)$ ), and Lemma 1111 ( $\mathcal{SF}_E$  canonical representation,  $f(X) = \{v_j \mid j \in [0..m]\}$ ), there exists  $j \in [0..m]$



such that  $u_i = v_j$ . Thus, from (0b), we have  $i \in I'_j$ , and  $I' \subset \uplus_{j \in [0..m]} I'_j$ . Hence, since the other inclusion is obvious, we have equality, and from (2a), property (14.13) holds.

Therefore, both properties hold.  $\square$

**Lemma 1117 ( $\mathcal{SF}_E$  is vector space).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ .

Then,  $\mathcal{SF}_E$  is a subvector space of  $E^X$ . Moreover, when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ,  $\mathcal{SF}_{\mathbb{K}}$  is a subalgebra of  $\mathbb{K}^X$ .

Let  $f, g \in \mathcal{SF}_E$ . Let  $a \in \mathbb{K}$ . Let  $(n, (A_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]})$  and  $(m, (B_j)_{j \in [0..m]}, (v_j)_{j \in [0..m]})$  in  $\mathbb{N} \times \Sigma \times E$  be representations of  $f$  and  $g$ . Let  $N \stackrel{\text{def.}}{=} nm + n + m$ . Let  $\varphi : [0..N] \rightarrow [0..n] \times [0..m]$  be a bijection. For all  $(i, j) \in [0..n] \times [0..m]$ , let  $C_{i,j} \stackrel{\text{def.}}{=} A_i \cap B_j$ , and  $w_{i,j}^+ \stackrel{\text{def.}}{=} u_i + v_j$ .

Then, if both representations are disjoint,  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (w_{\varphi(k)}^+)_{k \in [0..N]}) \in \mathbb{N} \times \Sigma \times E$  is also a disjoint representation of  $f + g$ , and  $(n, (A_i)_{i \in [0..n]}, (au_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times E$  is a representation of  $af$ , of the same nature as that of  $f$  (when  $a \neq 0_{\mathbb{K}}$  for canonical representation).

Moreover, when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ , for all  $(i, j) \in [0..n] \times [0..m]$ , let  $w_{i,j}^* \stackrel{\text{def.}}{=} u_i * v_j$ . Then,  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (w_{\varphi(k)}^*)_{k \in [0..N]}) \in \mathbb{N} \times \Sigma \times \mathbb{K}$  is a representation of  $f * g$ , which is disjoint when both representations of  $f$  and  $g$  are.

*Proof.* Since  $N + 1 = (n + 1)(m + 1)$ , such a bijection  $\varphi$  exists.

(1). From Lemma 1108 ( $\mathcal{SF}_E$  simple representation), and Lemma 81 (closed under vector operations is subspace),  $\mathcal{SF}_E$  is a vector subspace of  $E^X$ . From **commutativity of multiplication in  $\mathbb{K}$** , and Definition 61 (vector space, compatibility of scalar multiplication with field multiplication),  $(n, (A_i)_{i \in [0..n]}, (au_i)_{i \in [0..n]})$  is a representation of  $af$ .

Moreover, when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ , from **distributivity of multiplication over addition and closedness of multiplication in  $\mathbb{K}$** , Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication), Lemma 1108 ( $\mathcal{SF}_E$  simple representation, with  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ), and Lemma 286 (vector subspace and closed under multiplication is subalgebra),  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (w_{\varphi(k)}^*)_{k \in [0..N]})$  is a representation of  $f * g$ , and  $\mathcal{SF}_{\mathbb{K}}$  is a subalgebra of  $\mathbb{K}^X$ .

Assume now that both representations of  $f$  and  $g$  are disjoint.

(2a). From Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), the representation of  $af$  is also disjoint.

(2b). From Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), **compatibility of intersection with pairwise disjunction, idempotent law for intersection**, and **left and right distributivity of intersection over union**, the  $A_i \cap B_j$ 's are pairwise disjoint, and we have

$$X = \biguplus_{i \in [0..n]} A_i \cap \biguplus_{j \in [0..m]} B_j = \biguplus_{k \in [0..N]} C_{\varphi(k)}.$$

Hence, from Definition 209 (pseudopartition),  $(C_{\varphi(k)})_{k \in [0..N]}$  forms a pseudopartition of  $X$ .

(2c). Let  $i \in [0..n]$ . Then, from **field properties of  $\mathbb{K}$ , the definition of the indicator function ( $\mathbf{1}_X \equiv \mathbf{1}_{\mathbb{K}}$ )**, Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation,  $X = \biguplus_{j \in [0..m]} B_j$ ), Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive, with  $I \stackrel{\text{def.}}{=} [0..m]$ ), **left distributivity of multiplication over addition in  $\mathbb{K}$** , and Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication), we have

$$\mathbf{1}_{A_i} = \mathbf{1}_{A_i} \mathbf{1}_{\biguplus_{j \in [0..m]} B_j} = \mathbf{1}_{A_i} \left( \sum_{j \in [0..m]} \mathbf{1}_{B_j} \right) = \sum_{j \in [0..m]} (\mathbf{1}_{A_i} \mathbf{1}_{B_j}) = \sum_{j \in [0..m]} \mathbf{1}_{C_{i,j}}.$$



Thus, from **distributivity of scalar multiplication over field addition in  $E$** , the **lower bound property of intersection** ( $C_{i,j} \subset A_i$ ), (2b), and Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), we have

$$f = \sum_{i \in [0..n]} \left( \sum_{j \in [0..m]} \mathbb{1}_{C_{i,j}} \right) u_i = \sum_{(i,j) \in [0..n] \times [0..m]} \mathbb{1}_{C_{i,j}} u_i,$$

and the representation is disjoint. Similarly, we obtain the disjoint representation

$$g = \sum_{(i,j) \in [0..n] \times [0..m]} \mathbb{1}_{C_{i,j}} v_j.$$

Hence, from **additive group properties of  $E$** , and **distributivity of scalar multiplication over field addition in  $E$** ,  $\left( N, (C_{\varphi(k)})_{k \in [0..N]}, (w_{\varphi(k)}^+)_{k \in [0..N]} \right)$  is a representation of  $f + g$ .

(2d). Let  $u, v \in E$ . Let  $\diamond$  be a binary operator over  $E$  (using infix notation). Then, from **the definition of inverse image**, we have  $(f \diamond g)(\{f = u\} \cap \{g = v\}) \subset \{u \diamond v\}$ . Hence, from **the identity**  $A \subset \psi^{-1}(B) \Leftrightarrow \psi(A) \subset B$  (with  $\psi \stackrel{\text{def.}}{=} f \diamond g$ ), we have

$$\{f = u\} \cap \{g = v\} \subset \{f \diamond g = u \diamond v\}.$$

(2e). Let  $k \in [0..N]$ . Let  $(i, j) \stackrel{\text{def.}}{=} \varphi(k) \in [0..n] \times [0..m]$ . Then, from Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation,  $A_i \subset \{f = u_i\}$  and  $B_j \subset \{g = v_j\}$ ), **monotonicity of intersection** ( $A_i \cap B_j$  is a subset of  $\{f = u_i\} \cap \{g = v_j\}$ ), and (2d) with the binary operators vector addition and field multiplication (when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ), we have  $C_{\varphi(k)} \subset \{f + g = c_{\varphi(k)}^+\}$  and  $C_{\varphi(k)} \subset \{f * g = c_{\varphi(k)}^*\}$ . Hence, from Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation), and (2b), both representations for  $f + g$  and  $f * g$  (when  $E \stackrel{\text{def.}}{=} \mathbb{K}$ ) are disjoint.

(2f). Assume finally that the representation of  $f$  is canonical, and that  $a \neq 0_{\mathbb{K}}$ . Then, from Lemma 1111 ( $\mathcal{SF}_E$  canonical representation), and **the bijective property of scalar multiplication by  $a$  in  $E$** , the  $au_i$ 's are pairwise distinct,  $A_i = \{af = au_i\} \neq \emptyset$ , and the representation of  $af$  is also canonical.  $\square$

*Remark 1118.* Note that in the previous lemma,  $\left( N, (C_{\varphi(k)})_{k \in [0..N]}, (w_{\varphi(k)}^+)_{k \in [0..N]} \right)$  is not a representation of  $f + g$  when there is nonempty overlapping in the representation of  $f$  or  $g$ . Indeed, the values in overlapped parts are counted too many times.

Note also that even though the representations of  $f$  and  $g$  are canonical, the given representations of  $f + g$  and  $f * g$  are only disjoint. Indeed, the  $u_i + v_j$ 's and  $u_i * v_j$ 's may not be pairwise distinct, and thus the  $A_i \cap B_j$ 's may no longer be their inverse images.

**Lemma 1119** ( $\mathcal{SF}_E$  is closed under extension by zero).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow E$  and  $\hat{f} : X \rightarrow E$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{SF}_E(A, \Sigma \overline{\cap} A)$  iff  $\mathbb{1}_A \hat{f} \in \mathcal{SF}_E(X, \Sigma)$ .

If so, there exists  $(n, (\hat{g}_i)_{i \in [0..n]}, (u_i)_{i \in [0..n]}) \in \mathbb{N} \times \mathcal{IF}_{\mathbb{K}}(X, \Sigma) \times E$  such that

$$(14.14) \quad \forall i \in [0..n], \mathbb{1}_A \hat{g}_i = \hat{g}_i, \quad f|_A = \sum_{i \in [0..n]} (\hat{g}_i)|_A u_i \quad \text{and} \quad \mathbb{1}_A \hat{f} = \sum_{i \in [0..n]} \hat{g}_i u_i.$$

*Proof.* “Left” implies “right”. From Lemma 1108 ( $\mathcal{SF}_E$  simple representation, with  $f \stackrel{\text{def.}}{=} f|_A$ ), and Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions, with  $(X, \Sigma) \stackrel{\text{def.}}{=} (A, \Sigma \overline{\cap} A)$ ), let  $n \in \mathbb{N}$ ,  $(B_i)_{i \in [0..n]} \in \Sigma \overline{\cap} A$  and  $(u_i)_{i \in [0..n]} \in E$  such that

$$(f_i)_{i \in [0..n]} \stackrel{\text{def.}}{=} (\mathbb{1}_{B_i}^A)_{i \in [0..n]} \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \overline{\cap} A) \quad \text{and} \quad f|_A = \sum_{i \in [0..n]} f_i u_i.$$

Then, from Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero, with  $f = g_i \stackrel{\text{def.}}{=} \mathbb{1}_{B_i}^Y$  and  $\hat{f} = \hat{g}_i$  set to  $\mathbb{1}_{B_i}^X$ ), we have  $f_i = (\hat{g}_i)|_A$  and  $\mathbb{1}_A \hat{g}_i = \hat{g}_i \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$ . Hence, from Lemma 524 (restriction is masking, with  $f \stackrel{\text{def.}}{=} \sum_{i \in [0..n]} g_i u_i$  and  $\hat{f} \stackrel{\text{def.}}{=} \sum_{i \in [0..n]} \hat{g}_i u_i$ ), and Lemma 1108 ( $\mathcal{SF}_E$  simple representation), we have  $\mathbb{1}_A \hat{f} = \sum_{i \in [0..n]} \hat{g}_i u_i \in \mathcal{SF}_E(X, \Sigma)$ .

**“Right” implies “left”.** From Lemma 1108 ( $\mathcal{SF}_E$  simple representation, with  $f \stackrel{\text{def.}}{=} \mathbb{1}_A \hat{f}$ ), and Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), let  $n \in \mathbb{N}$ ,  $(B_i)_{i \in [0..n]} \in \Sigma$  and  $(u_i)_{i \in [0..n]} \in E$  such that

$$(\hat{g}_i)_{i \in [0..n]} \stackrel{\text{def.}}{=} (\mathbb{1}_{B_i}^X)_{i \in [0..n]} \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma) \quad \text{and} \quad \mathbb{1}_A \hat{f} = \sum_{i \in [0..n]} \hat{g}_i u_i.$$

Then, from Lemma 524 (restriction is masking), and **linearity of restriction of function**,

$$f|_A = (\mathbb{1}_A \hat{f})|_A = \sum_{i \in [0..n]} (\hat{g}_i)|_A u_i.$$

Hence, from Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero, with  $Y = X$ ,  $\hat{f} \stackrel{\text{def.}}{=} \hat{g}_i$  and  $\mathbb{1}_A \hat{g}_i = \hat{g}_i \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$ ), and Lemma 1108 ( $\mathcal{SF}_E$  simple representation), we have

$$(\hat{g}_i)|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A) \quad \text{and} \quad f|_A \in \mathcal{SF}_E(A, \Sigma \cap A).$$

Therefore, we have the equivalence and both representations.  $\square$

**Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $f : X \rightarrow E$ . Then, we have  $f \in \mathcal{SF}_E(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A \in \mathcal{SF}_E(A, \Sigma \cap A)$ .

*Proof.* **“Left” implies “right”.**

Direct consequence of Lemma 1108 ( $\mathcal{SF}_E$  simple representation,  $f = \sum_{i \in [0..n]} f_i u_i$  with  $n \in \mathbb{N}$ ,  $(f_i)_{i \in [0..n]} \in \mathcal{IF}_{\mathbb{K}}(X, \Sigma)$  and  $(u_i)_{i \in [0..n]} \in E$ ), **compatibility of vector linear combination and restriction of function** ( $f|_A = \sum_{i \in [0..n]} f_i|_A u_i$ ), Lemma 1106 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under restriction,  $f_i|_A \in \mathcal{IF}_{\mathbb{K}}(A, \Sigma \cap A)$ ), and Lemma 1108 ( $\mathcal{SF}_E$  simple representation,  $f|_A$  belongs to  $\mathcal{SF}_E(A, \Sigma \cap A)$ ).

**“Right” implies “left”.** Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $A \stackrel{\text{def.}}{=} X \in \Sigma$ ), Lemma 764 (measurability of measurable subspace,  $\Sigma \cap X = \Sigma$ ), and **the definition of restriction of function** ( $f|_X = f$ , thus  $f \in \mathcal{SF}_E(X, \Sigma)$ ).

Therefore, we have the equivalence.  $\square$

**Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mathbb{K}$  be  $\mathbb{R}$ , or  $\mathbb{C}$ . Let  $E$  be a vector space over  $\mathbb{K}$ . Let  $I \subset \mathbb{N}$  and  $n \in \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $f : X \rightarrow E$ .

Then, we have  $\mathbb{1}_A f = \sum_{i \in I} \mathbb{1}_{A_i} f$ , and  $\mathbb{1}_A f \in \mathcal{SF}_E$  implies for all  $i \in I$ ,  $\mathbb{1}_{A_i} f \in \mathcal{SF}_E$ .

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$ , we have for all  $i \in [0..n]$ ,  $\mathbb{1}_{A_i} f \in \mathcal{SF}_E$  implies  $\mathbb{1}_A f \in \mathcal{SF}_E$ .

*Proof.* (1). **Identity.** Exactly Lemma 525 (masking is  $\sigma$ -additive).

**“Left” implies “right”.** Assume that  $\mathbb{1}_A f \in \mathcal{SF}_E$ . Then, from Lemma 1108 ( $\mathcal{SF}_E$  simple representation), let  $p \in \mathbb{N}$ ,  $(f_j)_{j \in [0..p]} \in \mathcal{IF}_{\mathbb{K}}$ , and  $(u_j)_{j \in [0..p]} \in E$  such that

$$\mathbb{1}_A f = \sum_{j \in [0..p]} f_j u_j.$$

Let  $i \in I$ . Let  $j \in [0..p]$ . Let  $g_j^i \stackrel{\text{def.}}{=} \mathbb{1}_{A_i} f_j$ . Then, from Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*,  $f_j = \mathbb{1}_B$  with  $B \in \Sigma$ ), and Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under multiplication*), we have  $g_j^i = \mathbb{1}_{A_i \cap B} \in \mathcal{IF}_{\mathbb{K}}$ . Moreover, since  $A_i \subset A$  (i.e.  $A_i = A \cap A_i$ ), from Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under multiplication*), and Definition 61 (*vector space*, compatibility of scalar multiplication with field multiplication, and distributivity of scalar multiplication wrt vector addition), we have

$$\mathbb{1}_{A_i} f = \mathbb{1}_{A \cap A_i} f = \mathbb{1}_{A_i} (\mathbb{1}_A f) = \sum_{j \in [0..p]} \mathbb{1}_{A_i} f_j u_j = \sum_{j \in [0..p]} g_j^i u_j.$$

Hence, from Lemma 1108 ( $\mathcal{SF}_E$  *simple representation*), we have  $\mathbb{1}_{A_i} f \in \mathcal{SF}_E$ .

**“Right” implies “left” when  $I \stackrel{\text{def.}}{=} [0..n]$ .** Assume that for all  $i \in [0..n]$ ,  $\mathbb{1}_{A_i} f \in \mathcal{SF}_E$ . Then, from Lemma 1108 ( $\mathcal{SF}_E$  *simple representation*), for all  $i \in [0..n]$ , let  $p^i \in \mathbb{N}$ ,  $(f_j^i)_{j \in [0..p^i]} \in \mathcal{IF}_{\mathbb{K}}$ , and  $(u_j^i)_{j \in [0..p^i]} \in E$  such that  $\mathbb{1}_{A_i} f = \sum_{j \in [0..p^i]} f_j^i u_j^i$ . Let  $p \stackrel{\text{def.}}{=} -1 + \sum_{i \in [0..n]} (1 + p^i) \in \mathbb{N}$ ,

$$(f_j)_{j \in [0..p]} \stackrel{\text{def.}}{=} ((\mathbb{1}_{A_i} f_j^i)_{j \in [0..p^i]})_{i \in [0..n]} \in \mathcal{IF}_{\mathbb{K}} \quad \text{and} \quad (u_j)_{j \in [0..p]} \stackrel{\text{def.}}{=} ((u_j^i)_{j \in [0..p^i]})_{i \in [0..n]} \in E.$$

Then, from Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  *is  $\sigma$ -additive, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), and Definition 61 (*vector space*, commutativity and associativity of vector addition in  $E$ ), we have*

$$\mathbb{1}_A f = \sum_{i \in [0..n]} \mathbb{1}_{A_i} f = \sum_{i \in [0..n]} \sum_{j \in [0..p^i]} f_j^i u_j^i = \sum_{j \in [0..p]} f_j u_j.$$

Hence, from Lemma 1108 ( $\mathcal{SF}_E$  *simple representation*), we have  $\mathbb{1}_A f \in \mathcal{SF}_E$ .

Therefore, we have both implications. □

### 14.3 Nonnegative simple function

*Remark 1122.* In this section, we consider the case where  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$  is equipped with the usual total order. We recall that the notation  $\mathcal{SF}$  denotes the vector space of simple functions to the field of real numbers  $\mathbb{R}$ , see Lemma 1108.

**Definition 1123** ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

Let  $(X, \Sigma)$  be a measurable space. The subset of nonnegative simple functions is denoted  $\mathcal{SF}_+(X, \Sigma)$  (or simply  $\mathcal{SF}_+$ ); it is defined by  $\mathcal{SF}_+ \stackrel{\text{def.}}{=} \{f \in \mathcal{SF} \mid f(X) \subset \mathbb{R}_+\}$ .

**Lemma 1124** ( $\mathcal{SF}_+$  is measurable).

Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathcal{IF} \subset \mathcal{SF}_+ \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), **nonnegativeness of the indicator function**, Lemma 1113 ( $\mathcal{SF}_E$  is measurable, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).  $\square$

**Lemma 1125** ( $\mathcal{SF}_+$  disjoint representation).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ .

Then,  $f \in \mathcal{SF}_+$  iff there exists a triple  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  such that

$$(14.15) \quad \forall i \in [0..n], \quad A_i \subset \{f = a_i\},$$

$$(14.16) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(14.17) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}.$$

*Proof.* “Left” implies “right”. Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $f(A_i) = \{a_i\}$  is included in  $\mathbb{R}_+$ ), and **commutativity of multiplication in  $\mathbb{R}_+$** .

“Right” implies “left”. Direct consequence of **commutativity of multiplication in  $\mathbb{R}_+$** , Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $f \in \mathcal{SF}$ ), **closedness of addition in  $\mathbb{R}_+$  ( $f \geq 0$ )**, and Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

Therefore, we have the equivalence.  $\square$

**Lemma 1126** ( $\mathcal{SF}_+$  canonical representation).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ .

Then,  $f \in \mathcal{SF}_+$  iff there exists a unique triple  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  such that

$$(14.18) \quad \forall i \in [0..n-1], \quad a_i < a_{i+1},$$

$$(14.19) \quad \forall i \in [0..n], \quad A_i = \{f = a_i\} \neq \emptyset,$$

$$(14.20) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(14.21) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$$

which can also be written (with  $\{f = y\} \in \Sigma$  for all  $y \in f(X)$ )

$$(14.22) \quad f = \sum_{y \in f(X)} y \mathbb{1}_{\{f=y\}}.$$

*Proof.* “Left” implies “right”. Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of non-negative simple functions), Lemma 1111 ( $\mathcal{SF}_E$  canonical representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , with  $a_i \in f(X) \subset \mathbb{R}_+$ ), and **commutativity of multiplication in  $\mathbb{R}_+$** .

“Right” implies “left”. Direct consequence of **commutativity of multiplication in  $\mathbb{R}_+$** , Lemma 1111 ( $\mathcal{SF}_E$  canonical representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $f \in \mathcal{SF}$ ), **closedness of addition in  $\mathbb{R}_+$  ( $f \geq 0$ )**, and Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

Therefore, we have the equivalence.  $\square$

**Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{SF}_+$ . Let  $(n, (A_i)_{i \in [0..m]}, (a_i)_{i \in [0..m]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be a disjoint representation of  $f$ , and  $(m, (B_j)_{j \in [0..m]}, (b_j)_{j \in [0..m]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be its canonical representation.

For all  $j \in [0..m]$ , let  $I'_j \stackrel{\text{def.}}{=} \{i \in [0..n] \mid \emptyset \neq A_i \subset B_j\}$ . Let  $I' \stackrel{\text{def.}}{=} \{i \in [0..n] \mid A_i \neq \emptyset\}$ . Then, we have

$$(14.23) \quad \forall j \in [0..m], \quad B_j = \bigsqcup_{i \in I'_j} A_i \quad \wedge \quad (\forall i \in I'_j, a_i = b_j),$$

$$(14.24) \quad (\forall p, q \in [0..m], p \neq q \Rightarrow I'_p \cap I'_q = \emptyset) \quad \wedge \quad \bigsqcup_{j \in [0..m]} I'_j = I'.$$

*Proof.* Direct consequence of Lemma 1116 ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation,  $a_i \geq 0$ ), and Lemma 1126 ( $\mathcal{SF}_+$  canonical representation,  $b_i \geq 0$ ).  $\square$

**Lemma 1128 ( $\mathcal{SF}_+$  simple representation).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \in \mathcal{SF}_+$  iff there exists a triple  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  such that  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ .

If so, for all  $x \in X \setminus \bigcup_{i \in [0..n]} A_i$ , we have  $f(x) = 0$ .

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1108 ( $\mathcal{SF}_E$  simple representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), **commutativity of multiplication in  $\mathbb{R}_+$** , and **closedness of addition in  $\mathbb{R}_+$  ( $\sum_{i \in [0..n]} a_i \mathbb{1}_{A_i} \geq 0$ )**.  $\square$

**Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Let  $\alpha \in \mathbb{R}_+$ . Then, we have  $f + g, \alpha f, fg \in \mathcal{SF}_+$ .

Let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]})$  and  $(m, (B_j)_{j \in [0..m]}, (b_j)_{j \in [0..m]})$  in  $\mathbb{N} \times \Sigma \times \mathbb{R}_+$  be representations of  $f$  and  $g$ . Let  $N \stackrel{\text{def.}}{=} nm + n + m$ . Let  $\varphi : [0..N] \rightarrow [0..n] \times [0..m]$  be a bijection.

For all  $(i, j) \in [0..n] \times [0..m]$ , let  $C_{i,j} \stackrel{\text{def.}}{=} A_i \cap B_j$ ,  $c_{i,j}^+ \stackrel{\text{def.}}{=} a_i + b_j$ , and  $c_{i,j}^* \stackrel{\text{def.}}{=} a_i b_j$ .

Then, if both representations are disjoint,  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (c_{\varphi(k)}^+)_{k \in [0..N]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  is also a disjoint representation of  $f + g$ ,  $(n, (A_i)_{i \in [0..n]}, (\alpha a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  is a representation of  $\alpha f$ , of the same nature as that of  $f$  (when  $\alpha \neq 0$  for canonical representation), and  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (c_{\varphi(k)}^*)_{k \in [0..N]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  is a representation of  $fg$ , which is disjoint when both representations of  $f$  and  $g$  are.

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1117 ( $\mathcal{SF}_E$  is vector space, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 277 (algebra over a field), **closedness of addition and multiplication in  $\mathbb{R}_+$** . Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation), and **associativity and commutativity of intersection**.  $\square$

**Lemma 1130** ( $\mathcal{SF}_+$  is closed under extension by zero). Let  $(X, \Sigma)$  be a measurable space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{SF}_+(A, \Sigma \cap A)$  iff  $\hat{f} \mathbf{1}_A \in \mathcal{SF}_+(X, \Sigma)$ .

If so, there exists  $(n, (\hat{g}_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \mathcal{IF}(X, \Sigma) \times \mathbb{R}_+$  such that

$$(14.25) \quad \forall i \in [0..n], \hat{g}_i \mathbf{1}_A = \hat{g}_i, \quad f|_A = \sum_{i \in [0..n]} a_i (\hat{g}_i)|_A \quad \text{and} \quad \hat{f} \mathbf{1}_A = \sum_{i \in [0..n]} a_i \hat{g}_i.$$

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions,  $\mathcal{SF}_+$  equals  $\mathcal{SF} \cap \mathbb{R}_+^X$ ), Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Lemma 524 (restriction is masking, with  $E \stackrel{\text{def.}}{=} \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), compatibility of restriction and masking with nonnegativeness, and commutativity of multiplication in  $\mathbb{R}_+$ .  $\square$

**Lemma 1131** ( $\mathcal{SF}_+$  is closed under restriction). Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \in \mathcal{SF}_+(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A \in \mathcal{SF}_+(A, \Sigma \cap A)$ .

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions,  $\mathcal{SF}_+$  equals  $\mathcal{SF} \cap \mathbb{R}_+^X$ ), Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and compatibility of restriction with nonnegativeness.  $\square$

**Lemma 1132** ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive). Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$  and  $n \in \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ . Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \mathbf{1}_A = \sum_{i \in I} f \mathbf{1}_{A_i}$ , and  $f \mathbf{1}_A \in \mathcal{SF}_+$  implies for all  $i \in I$ ,  $f \mathbf{1}_{A_i} \in \mathcal{SF}_+$ .

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$ , we have for all  $i \in [0..n]$ ,  $f \mathbf{1}_{A_i} \in \mathcal{SF}_+$  implies  $f \mathbf{1}_A \in \mathcal{SF}_+$ .

*Proof.* Direct consequence of Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions,  $\mathcal{SF}_+$  equals  $\mathcal{SF} \cap \mathbb{R}_+^X$ ), Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive, with  $E \stackrel{\text{def.}}{=} \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and commutativity of multiplication in  $\mathbb{R}_+$ .  $\square$

*Remark 1133.*

The next lemma implements the second step of the Lebesgue scheme (see Section 4.4).

Note that taking  $f, g \in \mathcal{IF}$  in property (14.27) would only allow to add two indicator functions, and not a finite number to reach all simple functions.

**Lemma 1134** (Lebesgue scheme step 2).

Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate on functions from  $X$  to  $\bar{\mathbb{R}}_+$ . Assume that  $P$  is compatible with positive linear combination of indicator functions:

$$(14.26) \quad \forall a \in \mathbb{R}_+, \forall f \in \mathcal{IF}, \quad P(f) \implies P(af),$$

$$(14.27) \quad \forall f, g \in \mathcal{SF}_+, \quad P(f) \wedge P(g) \implies P(f + g).$$

Then,  $P$  holds on  $\mathcal{IF}$  implies  $P$  holds on  $\mathcal{SF}_+$ .

*Proof.* Direct consequence of Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions), Lemma 1128 ( $\mathcal{SF}_+$  simple representation), and induction on the number of operands of the finite sum.  $\square$

## 14.4 Adapted sequence

### Definition 1135 (adapted sequence).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $(\varphi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$ . The sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is called *adapted sequence for  $f$*  iff it is nondecreasing and  $f = \lim_{n \in \mathbb{N}} \varphi_n = \sup_{n \in \mathbb{N}} \varphi_n$ .

### Lemma 1136 (adapted sequence in $\mathcal{M}_+$ ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . For all  $n \in \mathbb{N}$ , let  $\varphi_n : X \rightarrow \mathbb{R}_+$  defined by

$$(14.28) \quad \forall x \in X, \quad \varphi_n(x) \stackrel{\text{def.}}{=} \begin{cases} \frac{\lfloor 2^n f(x) \rfloor}{2^n} & \text{when } f(x) < n, \\ n & \text{otherwise.} \end{cases}$$

Then,  $(\varphi_n)_{n \in \mathbb{N}}$  is an adapted sequence for  $f$ .

*Proof.* Let  $n \in \mathbb{N}$ . Let  $x \in X$ . Assume that  $f(x) < n$ . Let  $i \stackrel{\text{def.}}{=} \lfloor 2^n f(x) \rfloor \in \mathbb{N}$ . Then, from **the definition of the floor function**, we have  $0 \leq i \leq 2^n f(x) < i + 1 \leq n 2^n$ . Thus, for all  $i \in \mathbb{N}$  such that  $0 \leq i < n 2^n$ , for all  $x \in X$  such that  $\frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}$ , we have  $\varphi_n(x) = \frac{i}{2^n}$ . Hence,

$$\varphi_n = \sum_{0 \leq i < n 2^n} \frac{i}{2^n} \mathbb{1}_{\{\frac{i}{2^n} \leq f < \frac{i+1}{2^n}\}} + n \mathbb{1}_{\{f \geq n\}}.$$

Moreover, from Lemma 820 (*measurability of numeric function*), and Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection (with  $\text{card}(I) = 2$ )), we have

$$\left\{ \frac{i}{2^n} \leq f < \frac{i+1}{2^n} \right\} = \left\{ \frac{i}{2^n} \leq f \right\} \cap \left\{ f < \frac{i+1}{2^n} \right\} \in \Sigma, \quad \text{and} \quad \{f \geq n\} \in \Sigma.$$

Hence, from Lemma 1108 ( *$\mathcal{SF}_E$  simple representation*, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and Definition 1123 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*), we have  $\varphi_n \in \mathcal{SF}_+$ .

Let  $n \in \mathbb{N}$ . Let  $x \in X$ . **Case  $n + 1 \leq f(x)$ .** Then, we have

$$\varphi_n(x) = n < n + 1 = \varphi_{n+1}(x).$$

**Case  $f(x) < n + 1$ .** Let  $i \stackrel{\text{def.}}{=} \lfloor 2^{n+1} f(x) \rfloor$ . Then, we have

$$\varphi_{n+1}(x) = \frac{i}{2^{n+1}} \quad \text{with } 0 \leq \frac{i}{2^{n+1}} \leq f(x) < \frac{i+1}{2^{n+1}}.$$

**Case  $\frac{i}{2^{n+1}} \geq n$ .** Then,  $f(x) \geq n$  and we have

$$\varphi_n(x) = n \leq \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

**Case  $\frac{i}{2^{n+1}} < n$  and  $i$  even.** Let  $j \stackrel{\text{def.}}{=} \frac{i}{2} \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have

$$\frac{j}{2^n} \leq f(x) < \frac{j}{2^n} + \frac{1}{2^{n+1}} < \frac{j+1}{2^n}.$$

Thus, from **the definition of the floor function**, we have  $j = \lfloor 2^n f(x) \rfloor$  and

$$\varphi_n(x) = \frac{j}{2^n} = \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

**Case  $\frac{i}{2^{n+1}} < n$  and  $i$  odd.** Let  $j \stackrel{\text{def.}}{=} \frac{i-1}{2} \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have

$$\frac{j}{2^n} \leq \frac{j}{2^n} + \frac{1}{2^{n+1}} \leq f(x) < \frac{j+1}{2^n}.$$



Thus, from **the definition of the floor function**, we have  $j = \lfloor 2^n f(x) \rfloor$  and

$$\varphi_n(x) = \frac{j}{2^n} = \frac{i-1}{2^{n+1}} < \frac{i}{2^{n+1}} = \varphi_{n+1}(x).$$

Thus, we always have  $\varphi_n(x) \leq \varphi_{n+1}(x)$ . Hence, the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is nondecreasing.

Let  $x \in X$ . We have  $f(x) \in \overline{\mathbb{R}}_+$ . **Case  $f(x) = \infty$ .** Then, for all  $n \in \mathbb{N}$ , we have  $\varphi_n(x) = n$ . Hence,  $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$ . **Case  $f(x) \in \mathbb{R}_+$ .** Then, from **the Archimedean property of  $\mathbb{R}$** , there exists  $N \in \mathbb{N}$  such that  $f(x) < N$ . Let  $n \in \mathbb{N}$  such that  $N \leq n$ . Then, from **the definition of the floor function**, we have

$$2^n \varphi_n(x) = \lfloor 2^n f(x) \rfloor \leq 2^n f(x) < \lfloor 2^n f(x) \rfloor + 1 = 2^n \varphi_n(x) + 1.$$

Thus, from **ordered field properties of  $\mathbb{R}$  (with  $2^n > 0$ )**, we have  $f(x) - \frac{1}{2^n} < \varphi_n(x) \leq f(x)$ . Hence, from **the squeeze theorem**, we have  $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$ .

Therefore, from Definition 1135 (*adapted sequence*),  $(\varphi_n)_{n \in \mathbb{N}}$  is an adapted sequence for  $f$ .  $\square$

**Lemma 1137** ( *$\mathcal{M}_+$  is closed under extension by zero*). Let  $(X, \Sigma)$  be a measurable space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$  iff  $\hat{f} \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ .  
If so, there exists  $(\hat{\varphi}_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$  such that

$$(14.29) \quad \forall n \in \mathbb{N}, \hat{\varphi}_n \mathbb{1}_A = \hat{\varphi}_n, \quad f|_A = \sup_{n \in \mathbb{N}} (\hat{\varphi}_n)|_A \quad \text{and} \quad \hat{f} \mathbb{1}_A = \sup_{n \in \mathbb{N}} \hat{\varphi}_n.$$

*Proof.* **“Left” implies “right”.** From Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* , with  $f \stackrel{\text{def.}}{=} f|_A$ ), and Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*, with  $(X, \Sigma) \stackrel{\text{def.}}{=} (A, \Sigma \cap A)$ ), for all  $p \in \mathbb{N}$ , let  $n^p \in \mathbb{N}$ ,  $(a_i^p)_{i \in [0..n^p]} \in \mathbb{R}_+$  and  $(B_i^p)_{i \in [0..n^p]} \in \Sigma \cap A$  such that

$$(\psi_p)_{p \in \mathbb{N}} \stackrel{\text{def.}}{=} \sum_{i \in [0..n^p]} a_i^p \mathbb{1}_{B_i^p} \in \mathcal{SF}_+(A, \Sigma \cap A) \quad \text{and} \quad f|_A = \sup_{p \in \mathbb{N}} \psi_p.$$

Then, from Lemma 1130 ( *$\mathcal{SF}_+$  is closed under extension by zero*, with  $f = \varphi_p \stackrel{\text{def.}}{=} \sum_{i \in [0..n^p]} a_i^p \mathbb{1}_{B_i^p}$  and  $\hat{f} = \hat{\varphi}_p \stackrel{\text{def.}}{=} \sum_{i \in [0..n^p]} a_i^p \mathbb{1}_{B_i^p}$ ), for all  $p \in \mathbb{N}$ , we have

$$\psi_p = (\hat{\varphi}_p)|_A \quad \text{and} \quad \hat{\varphi}_p \mathbb{1}_A = \hat{\varphi}_p \in \mathcal{SF}_+(X, \Sigma).$$

Hence, from Lemma 524 (*restriction is masking*, with  $f \stackrel{\text{def.}}{=} \sup_{p \in \mathbb{N}} \varphi_p$  and  $\hat{f} \stackrel{\text{def.}}{=} \sup_{p \in \mathbb{N}} \hat{\varphi}_p$ ), Lemma 1124 ( *$\mathcal{SF}_+$  is measurable*, thus  $\hat{\varphi}_p \in \mathcal{M}_+(X, \Sigma)$ ), and Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum*), we have  $\hat{f} \mathbb{1}_A = \sup_{p \in \mathbb{N}} \hat{\varphi}_p \in \mathcal{M}_+(X, \Sigma)$ .

**“Right” implies “left”.** From Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* , with  $f \stackrel{\text{def.}}{=} \hat{f} \mathbb{1}_A$ ), and Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*), for all  $p \in \mathbb{N}$ , let  $n^p \in \mathbb{N}$ ,  $(a_i^p)_{i \in [0..n^p]} \in \mathbb{R}_+$  and  $(B_i^p)_{i \in [0..n^p]} \in \Sigma$  such that

$$(\hat{\varphi}_p)_{p \in \mathbb{N}} \stackrel{\text{def.}}{=} \sum_{i \in [0..n^p]} a_i^p \mathbb{1}_{B_i^p} \in \mathcal{SF}_+(X, \Sigma) \quad \text{and} \quad \hat{f} \mathbb{1}_A = \sup_{p \in \mathbb{N}} \hat{\varphi}_p.$$

Then, from Lemma 524 (*restriction is masking*), and **compatibility of restriction of function with supremum**, we have  $f|_A = (\hat{f} \mathbb{1}_A)|_A = \sup_{p \in \mathbb{N}} \psi_p|_A$ . Hence, from Lemma 1130 ( *$\mathcal{SF}_+$  is closed under extension by zero*, with  $Y = X$ ,  $\hat{f} \stackrel{\text{def.}}{=} \hat{\varphi}_p$  and  $\hat{\varphi}_p \mathbb{1}_A = \hat{\varphi}_p \in \mathcal{SF}_+(X, \Sigma)$ ), Lemma 1124 ( *$\mathcal{SF}_+$  is measurable*, with  $(X, \Sigma) \stackrel{\text{def.}}{=} (A, \Sigma \cap A)$ ), and Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum*), we have  $(\hat{\varphi}_p)|_A \in \mathcal{SF}_+(A, \Sigma \cap A) \subset \mathcal{M}_+(A, \Sigma \cap A)$  and  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ .

Therefore, we always have the equivalence and the representations.  $\square$



**Lemma 1138** ( $\mathcal{M}_+$  is closed under restriction). *Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $f \in \mathcal{M}_+(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ .*

*Proof.* “Left” implies “right”. Direct consequence of Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$ ,  $f = \sup_{n \in \mathbb{N}} \varphi_n$  with  $(\varphi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$* ), **compatibility of restriction of function with supremum** ( $f|_A = \sup_{n \in \mathbb{N}} \varphi_n|_A$ ), Lemma 1131 ( *$\mathcal{SF}_+$  is closed under restriction,  $\varphi_n|_A$  belongs to  $\mathcal{SF}_+(A, \Sigma \cap A)$* ), Lemma 1124 ( *$\mathcal{SF}_+$  is measurable,  $\varphi_n|_A \in \mathcal{M}_+(A, \Sigma \cap A)$* ), and Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum,  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$* ).

“Right” implies “left”. Lemma 697 (*equivalent definition of  $\sigma$ -algebra,  $A \stackrel{\text{def.}}{=} X \in \Sigma$* ), Lemma 764 (*measurability of measurable subspace,  $\Sigma \cap X = \Sigma$* ), and **the definition of restriction of function** ( $f|_X = f$ , thus  $f \in \mathcal{M}_+(X, \Sigma)$ ).  $\square$

**Lemma 1139** ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive). *Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $f : X \rightarrow \bar{\mathbb{R}}$ . Then, we have  $f \mathbb{1}_A = \sum_{i \in I} f \mathbb{1}_{A_i}$ , and  $f \mathbb{1}_A \in \mathcal{M}_+$  iff for all  $i \in I$ ,  $f \mathbb{1}_{A_i} \in \mathcal{M}_+$ .*

*Proof.* (1). **Identity.** Exactly Lemma 525 (*masking is  $\sigma$ -additive, with  $E \stackrel{\text{def.}}{=} \bar{\mathbb{R}}$* ).

“Left” implies “right”. Direct consequence of Lemma 1103 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$  and  $\mathbb{1}_{A_i} = \mathbb{1}_{A \cap A_i} = \mathbb{1}_A \mathbb{1}_{A_i}$* ), **associativity of multiplication in  $\mathbb{R}$** , Lemma 1124 ( *$\mathcal{SF}_+$  is measurable,  $\mathcal{IF} \subset \mathcal{M}_+$* ), and Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*).

“Right” implies “left”.

Direct consequence of (1), and Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*).

Therefore, we have the equivalence.  $\square$

*Remark 1140.*

The next lemma implements the third step of the Lebesgue scheme (see Section 4.4).

**Lemma 1141** (*Lebesgue scheme step 3*).

*Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate on functions from  $X$  to  $\bar{\mathbb{R}}_+$ .*

*Assume that  $P$  is compatible with the limit of nondecreasing nonnegative simple functions:*

$$(14.30) \quad \forall (f_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+, \quad (\forall n \in \mathbb{N}, f_n \leq f_{n+1} \wedge P(f_n)) \implies P\left(\lim_{n \rightarrow \infty} f_n\right).$$

*Then,  $P$  holds on  $\mathcal{SF}_+$  implies  $P$  holds on  $\mathcal{M}_+$ .*

*Proof.* Direct consequence of Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ).  $\square$

## Chapter 15

# Lebesgue integration of nonnegative function

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*Remark* 1142.

The first three sections of this chapter follow steps 1 to 3 of the Lebesgue scheme (see Section 4.4).

## 15.1 Integration of indicator function

*Remark 1143.* In this section, functions take their values in  $\{0, 1\}$ , and the expressions involving integrals are taken in  $\overline{\mathbb{R}}_+$ . We recall that the notation  $\mathcal{IF}$  denotes the set of measurable indicator functions on the field of real numbers  $\mathbb{R}$ , see Definition 1098.

**Definition 1144 (integral in  $\mathcal{IF}$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{IF}$ . The integral of  $f$  (for the measure  $\mu$ ) is denoted  $\int f d\mu$ ; it is defined by

$$(15.1) \quad \int f d\mu \stackrel{\text{def.}}{=} \mu(\{f \neq 0\}) \in \overline{\mathbb{R}}_+.$$

**Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $\int \mathbf{1}_A d\mu = \mu(A)$ .

*Proof.* Direct consequence of Lemma 1100 (indicator and support are inverse, with  $\mathbb{K}$  set to  $\mathbb{R}$ ,  $\mathbf{1}_A \in \mathcal{IF}$  and  $\{\mathbf{1}_A \neq 0\} = A \in \Sigma$ ), and Definition 1144 (integral in  $\mathcal{IF}$ ).  $\square$

**Lemma 1146 (integral in  $\mathcal{IF}$  is  $\sigma$ -additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Then, we have

$$(15.2) \quad \int \sum_{i \in I} \mathbf{1}_{A_i} d\mu = \mu(A) = \sum_{i \in I} \int \mathbf{1}_{A_i} d\mu.$$

*Proof.* Direct consequence of Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ , with  $A$  and the  $A_i$ 's), Lemma 913 (measure,  $\sigma$ -additivity), Definition 874 ( $\sigma$ -additivity).  $\square$

**Lemma 1147 (integral in  $\mathcal{IF}$  is monotone).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A, B \in \Sigma$ . Assume that  $A \subset B$ . Then, we have  $\int \mathbf{1}_A d\mu \leq \int \mathbf{1}_B d\mu$ .

*Proof.* Direct consequence of Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ), and Lemma 919 (measure is monotone).  $\square$

**Lemma 1148 (integral in  $\mathcal{IF}$  over subset).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{IF}(A, \Sigma \cap A)$  iff  $\hat{f} \mathbf{1}_A \in \mathcal{IF}(X, \Sigma)$ .

If so, there exists  $B \in \Sigma$  such that  $B \subset A$ ,  $f|_A = \mathbf{1}_B^A$ ,  $\hat{f} \mathbf{1}_A = \mathbf{1}_B^X$ , and we have

$$(15.3) \quad \int f|_A d\mu_A = \mu(B) = \int \hat{f} \mathbf{1}_A d\mu.$$

This integral is denoted  $\int_A f d\mu$ ; it is called integral of  $f$  over  $A$ .

*Proof.* Direct consequence of Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 1144 (integral in  $\mathcal{IF}$ ), and Lemma 932 (trace measure is restricted measure, thus  $\mu_A(B)$  equals  $\mu(B)$ ).  $\square$

**Lemma 1149 (integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $Y$  be a set such that  $A \subset Y \subset X$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f} \mathbf{1}_A = \sum_{i \in I} \hat{f} \mathbf{1}_{A_i}$ , and  $\hat{f} \mathbf{1}_A \in \mathcal{IF}$  iff for all  $i \in I$ ,  $\hat{f} \mathbf{1}_{A_i} \in \mathcal{IF}$ . If so, we have

$$(15.4) \quad \int_A f d\mu = \sum_{i \in I} \int_{A_i} f d\mu.$$

*Proof.* Direct consequence of Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , *masking is  $\sigma$ -additive*, with  $\mathbb{K} \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and  $f \stackrel{\text{def.}}{=} \hat{f}$ ), Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*, with  $A$  and the  $A_i$ 's), and Lemma 1146 (*integral in  $\mathcal{IF}$  is  $\sigma$ -additive*).  $\square$

**Lemma 1150 (*integral in  $\mathcal{IF}$  over negligible subset*).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$  and  $\mu(A) = 0$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ , and either  $f|_A \in \mathcal{IF}(A, \Sigma \cap A)$ , or  $\hat{f} \mathbf{1}_A \in \mathcal{IF}(X, \Sigma)$ . Then, we have both, and

$$(15.5) \quad \int_A f \, d\mu = 0.$$

*Proof.*

Direct consequence of Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*, thus exists  $B \in \Sigma$  such that  $B \subset A$  and  $\int_A f \, d\mu = \mu(B)$ ), Lemma 913 (*measure, nonnegativeness*), Lemma 919 (*measure is monotone*, thus  $\mu(B) \leq \mu(A) = 0$ ), and **the definition of order (antisymmetry, thus  $\mu(B) = 0$ )**.  $\square$

*Remark 1151.* In the next lemma, as in the countable case, when  $Y$  is uncountable, the sum (of nonnegative values) is understood as the supremum for all finite subsets:

$$(15.6) \quad \sum_{y \in Y} f(y) \stackrel{\text{def.}}{=} \sup_{\substack{Z \subset Y \\ \text{card}(Z) < \infty}} \sum_{z \in Z} f(z).$$

**Lemma 1152 (*integral in  $\mathcal{IF}$  with counting measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{IF}$ . Then, we have

$$(15.7) \quad \int f \, d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 1100 (*indicator and support are inverse*), Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* , with  $A = \{f \neq 0\} \in \Sigma$ ), Lemma 935 (*counting measure*), **the definition of the cardinality**, and **the definition of the indicator function**.  $\square$

## 15.2 Integration of nonnegative simple function

*Remark 1153.* In this section, functions take their values in  $\mathbb{R}_+$ , and the expressions involving integrals are taken in  $\overline{\mathbb{R}}_+$ . We recall that the notation  $\mathcal{SF}_+$  denotes the set of nonnegative simple functions to the field of real numbers  $\mathbb{R}$ , see Definition 1123.

**Lemma 1154 (integral in  $\mathcal{SF}_+$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, we have  $f = \sum_{y \in f(X)} y \mathbb{1}_{\{f=y\}}$ , and the sum  $\sum_{y \in f(X)} y \mu(\{f=y\})$  is well-defined in  $\overline{\mathbb{R}}_+$ . The integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by*

$$(15.8) \quad \int f d\mu \stackrel{\text{def.}}{=} \sum_{y \in f(X)} y \int \mathbb{1}_{\{f=y\}} d\mu = \sum_{y \in f(X)} y \mu(\{f=y\}) \in \overline{\mathbb{R}}_+.$$

*Proof.* Direct consequence of Lemma 1126 ( $\mathcal{SF}_+$  canonical representation), Lemma 913 (measure, nonnegativeness), Lemma 425 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory),  $a_i \in \mathbb{R}_+ \subset \overline{\mathbb{R}}_+$ ), Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed), and Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ).  $\square$

*Remark 1155.* Keeping the same notation for the integrals is justified in the next lemma.

**Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{IF}$ . Then, the values of  $\int f d\mu$  provided by Definition 1144 (integral in  $\mathcal{IF}$ ), and Lemma 1154 (integral in  $\mathcal{SF}_+$ ) coincide. In other terms, for all  $A \in \Sigma$ , we have  $\mathbb{1}_A \in \mathcal{SF}_+$  and  $\int \mathbb{1}_A d\mu = \mu(A)$ .*

*Proof.* Direct consequence of Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $A \in \Sigma$ ), Lemma 1128 ( $\mathcal{SF}_+$  simple representation, with  $n=0$ ,  $A_0=A$ , and  $a_0=1$ ), Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ), and Lemma 1154 (integral in  $\mathcal{SF}_+$ , with  $f(X) = \{0, 1\}$ ,  $\{f=0\} = A^c$ , and  $\{f=1\} = A$ ).  $\square$

**Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be a disjoint representation of  $f$ . Then, we have*

$$(15.9) \quad \int f d\mu = \sum_{i \in [0..n]} a_i \mu(A_i).$$

*Proof.* From Lemma 1126 ( $\mathcal{SF}_+$  canonical representation), let  $(m, (B_j)_{j \in [0..m]}, (b_j)_{j \in [0..m]})$  be the canonical representation of  $f$  in  $\mathbb{N} \times \Sigma \times \mathbb{R}_+$ .

For all  $j \in [0..m]$ , let  $I'_j \stackrel{\text{def.}}{=} \{i \in [0..n] \mid \emptyset \neq A_i \subset B_j\}$ . Let  $I' \stackrel{\text{def.}}{=} \{i \in [0..n] \mid A_i \neq \emptyset\}$ .

Then, from Lemma 1154 (integral in  $\mathcal{SF}_+$ ), Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is sub-partition of canonical representation, both properties), Lemma 913 (measure,  $\sigma$ -additivity and homogeneity), Definition 874 ( $\sigma$ -additivity, with  $I \stackrel{\text{def.}}{=} I'_j$ ), and **left distributivity of multiplication over addition in  $\mathbb{R}$** , we have

$$\begin{aligned} \int f d\mu &= \sum_{j \in [0..m]} b_j \mu(B_j) = \sum_{j \in [0..m]} b_j \mu \left( \bigsqcup_{i \in I'_j} A_i \right) \\ &= \sum_{j \in [0..m]} \sum_{i \in I'_j} a_i \mu(A_i) = \sum_{i \in I'} a_i \mu(A_i) = \sum_{i \in [0..n]} a_i \mu(A_i). \end{aligned}$$

$\square$

**Lemma 1158 (integral in  $\mathcal{SF}_+$  of zero is zero).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $0 \in \mathcal{SF}_+$ , and we have  $\int 0 d\mu = 0$ .

*Proof.* Direct consequence of **the definition of the indicator function ( $1_\emptyset \equiv 0$ )**, Lemma 696 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ , with  $A \stackrel{\text{def.}}{=} \emptyset$ ), and Lemma 913 (measure,  $\mu(\emptyset) = 0$ ).  $\square$

**Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f, g \in \mathcal{SF}_+$ . Let  $a \in \mathbb{R}_+$ . Then,  $f + g, af \in \mathcal{SF}_+$ , and we have

$$(15.10) \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu, \quad \text{and} \quad \int af d\mu = a \int f d\mu.$$

*Proof.* From Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),  $f + g, af \in \mathcal{SF}_+$ .

**Addition.** From Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation), let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]})$  and  $(m, (B_j)_{j \in [0..m]}, (b_j)_{j \in [0..m]})$  in  $\mathbb{N} \times \Sigma \times \mathbb{R}_+$  be disjoint representations of  $f$  and  $g$ . In particular, we have  $X = \bigsqcup_{i \in [0..n]} A_i = \bigsqcup_{j \in [0..m]} B_j$ .

Let  $N \stackrel{\text{def.}}{=} nm + n + m$ . Let  $\varphi : [0..N] \rightarrow [0..n] \times [0..m]$  be a bijection (with  $N+1 = (n+1)(m+1)$ ). For all  $(i, j) \in [0..n] \times [0..m]$ , let  $C_{i,j} \stackrel{\text{def.}}{=} A_i \cap B_j$  and  $c_{i,j} \stackrel{\text{def.}}{=} a_i + b_j$ . Then, from Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, addition),  $(N, (C_{\varphi(k)})_{k \in [0..N]}, (c_{\varphi(k)})_{k \in [0..m]})$  is a disjoint representation of  $f + g$ . Thus, from Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)), **associativity and commutativity of addition in  $\mathbb{R}$  ( $\sum_k = \sum_{i,j}$ )**, Lemma 922 (measure over countable pseudopartition, first with  $A_i$  and the pseudopartition  $(B_j)_{j \in [0..m]}$ , then with  $B_j$  and the pseudopartition  $(A_i)_{i \in [0..n]}$ ), and **left distributivity of multiplication over addition in  $\mathbb{R}$** , we have

$$\begin{aligned} \int (f + g) d\mu &= \sum_{k \in [0..N]} c_{\varphi(k)} \mu(C_{\varphi(k)}) = \sum_{(i,j) \in [0..n] \times [0..m]} c_{i,j} \mu(C_{i,j}), \\ \int f d\mu &= \sum_{i \in [0..n]} a_i \mu(A_i) = \sum_{(i,j) \in [0..n] \times [0..m]} a_i \mu(C_{i,j}), \\ \int g d\mu &= \sum_{j \in [0..m]} b_j \mu(B_j) = \sum_{(i,j) \in [0..n] \times [0..m]} b_j \mu(C_{i,j}). \end{aligned}$$

Hence, from **left distributivity of multiplication over addition in  $\mathbb{R}$** , we have the equality.

**Nonnegative scalar multiplication. Case  $a = 0$ .**

Direct consequence of **field properties of  $\mathbb{R}$  ( $0f$  is the zero function)**, Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)), and Lemma 1158 (integral in  $\mathcal{SF}_+$  of zero is zero).

**Case  $a > 0$ .** Let  $y \in f(X)$ . Let  $z = ay \in (af)(X)$ . Then, from Definition 326 (multiplication in  $\overline{\mathbb{R}}$ , for all  $a \in \mathbb{R}_+$ ,  $\frac{a}{a} = 1$ ), we have  $\{af = z\} = \{f = y\}$ . Hence, from Lemma 1154 (integral in  $\mathcal{SF}_+$ ), we have

$$\int af d\mu = \sum_{z \in (af)(X)} z \mu(\{af = z\}) = \sum_{y \in f(X)} ay \mu(\{f = y\}) = a \int f d\mu.$$

Therefore, the integral in  $\mathcal{SF}_+$  is positive linear.  $\square$

**Lemma 1160 (equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be a simple representation of  $f$ . Then, we have

$$(15.11) \quad \int f d\mu = \sum_{i \in [0..n]} a_i \mu(A_i).$$

*Proof.* Direct consequence of Lemma 1128 ( $\mathcal{SF}_+$  simple representation, thus such representation exists), Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear), and Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ).  $\square$

**Lemma 1161 (integral in  $\mathcal{SF}_+$  is monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Then, we have

$$(15.12) \quad f \leq g \implies \int f \, d\mu \leq \int g \, d\mu.$$

*Proof.* Assume that  $f \leq g$ . Then, from Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1117 ( $\mathcal{SF}_E$  is vector space, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Definition 277 (algebra over a field,  $\mathcal{SF}$  is a vector space), and Definition 61 (vector space,  $(\mathcal{SF}, +)$  is an abelian group), we have  $g = f + (g - f)$  with  $f, g - f \in \mathcal{SF}_+$ . Therefore, from Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear), and Lemma 1154 (integral in  $\mathcal{SF}_+$ , nonnegativeness with  $g - f \in \mathcal{SF}_+$ ), we have

$$\int g \, d\mu = \int f \, d\mu + \int (g - f) \, d\mu \geq \int f \, d\mu.$$

$\square$

**Lemma 1162 (integral in  $\mathcal{SF}_+$  is continuous).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(15.13) \quad \int f \, d\mu = \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi \, d\mu.$$

*Proof.* From Definition 2 (supremum, upper bound), and since  $f \in \mathcal{SF}_+$ , we have

$$\int f \, d\mu \leq \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi \, d\mu.$$

Conversely, let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ . Then, from Lemma 1161 (integral in  $\mathcal{SF}_+$  is monotone), and Definition 2 (supremum, least upper bound), we have

$$\sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi \, d\mu \leq \int f \, d\mu.$$

Therefore, we have the equality.  $\square$

**Lemma 1163 (integral in  $\mathcal{SF}_+$  over subset).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ .

Then, we have  $f|_A \in \mathcal{SF}_+(A, \Sigma \cap A)$  iff  $\hat{f} \mathbf{1}_A \in \mathcal{SF}_+(X, \Sigma)$ .

If so, there exists  $(n, (\hat{g}_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \mathcal{IF}(X, \Sigma) \times \mathbb{R}_+$  such that

$$(15.14) \quad \forall i \in [0..n], \hat{g}_i \mathbf{1}_A = \hat{g}_i, \quad f|_A = \sum_{i \in [0..n]} a_i (\hat{g}_i)|_A, \quad \hat{f} \mathbf{1}_A = \sum_{i \in [0..n]} a_i \hat{g}_i,$$

and we have

$$(15.15) \quad \int f|_A \, d\mu_A = \sum_{i \in [0..n]} a_i \int_A \hat{g}_i \, d\mu = \int f \mathbf{1}_A \, d\mu.$$

This integral is still denoted  $\int_A f \, d\mu$ ; it is still called integral of  $f$  over  $A$ .

*Proof.* Direct consequence of Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero), Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear), Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ), and Lemma 1148 (integral in  $\mathcal{IF}$  over subset).  $\square$

**Lemma 1164 (integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  and  $n \in \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ .

Let  $Y$  be a set such that  $A \subset Y \subset X$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f} \mathbf{1}_A = \sum_{i \in I} \hat{f} \mathbf{1}_{A_i}$ , and  $\hat{f} \mathbf{1}_A \in \mathcal{SF}_+$  implies for all  $i \in I$ ,  $\hat{f} \mathbf{1}_{A_i} \in \mathcal{SF}_+$ . If so, we have

$$(15.16) \quad \int_A f \, d\mu = \sum_{i \in I} \int_{A_i} f \, d\mu.$$

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$ , we have for all  $i \in [0..n]$ ,  $\hat{f} \mathbf{1}_{A_i} \in \mathcal{SF}_+$  implies  $\hat{f} \mathbf{1}_A \in \mathcal{SF}_+$ .

*Proof.* Direct consequence of Lemma 1132 ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive), Lemma 1163 (integral in  $\mathcal{SF}_+$  over subset, with  $A$  and the  $A_i$ 's), Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear, additivity), and **induction on the number of operands**.  $\square$

**Lemma 1165 (integral in  $\mathcal{SF}_+$  over negligible subset).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$  and  $\mu(A) = 0$ . Let  $f : Y \rightarrow \mathbb{R}$  and  $\hat{f} : X \rightarrow \mathbb{R}$ . Assume that  $\hat{f}|_Y = f$ , and either  $f|_A \in \mathcal{SF}_+(A, \Sigma \cap A)$ , or  $\hat{f} \mathbf{1}_A \in \mathcal{SF}_+(X, \Sigma)$ . Then, we have both, and

$$(15.17) \quad \int_A f \, d\mu = 0.$$

*Proof.* Direct consequence of Lemma 1163 (integral in  $\mathcal{SF}_+$  over subset, thus exist  $n \in \mathbb{N}$ , scalars  $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$  and  $(\hat{g}_i)_{i \in [0..n]} \in \mathcal{IF}(X, \Sigma)$  such that  $\hat{g}_i \mathbf{1}_A = \hat{g}_i$  and  $\int_A f \, d\mu = \sum_{i \in [0..n]} a_i \int_A \hat{g}_i \, d\mu$ ), Lemma 1150 (integral in  $\mathcal{IF}$  over negligible subset, thus  $\int_A \hat{g}_i \, d\mu = 0$ ), and **field properties of  $\mathbb{R}$  (thus  $\sum_{i \in [0..n]} a_i \int_A \hat{g}_i \, d\mu = 0$ )**.  $\square$

**Lemma 1166 (density measure in  $\mathcal{SF}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $w \in \mathcal{SF}_+$ . Then, the function  $(B \in \Sigma \mapsto \int w \mathbf{1}_B \, d\mu)$  is a measure on  $(X, \Sigma)$ .

This function is denoted  $\mu_w$ ; it is called measure of density  $w$ , (or simply density measure).

*Proof.* Let  $B \in \Sigma$ . Then, from Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, multiplication), and Lemma 1154 (integral in  $\mathcal{SF}_+$ ), we have  $w \mathbf{1}_B \in \mathcal{SF}_+$ , and  $\int w \mathbf{1}_B \, d\mu \geq 0$ .

From Lemma 696 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), **the definition of the indicator function ( $\mathbf{1}_\emptyset \equiv \mathbf{0}$ )**, **field properties of  $\mathbb{R}$** , and Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), we have  $w \mathbf{1}_\emptyset = \mathbf{1}_\emptyset \in \mathcal{SF}_+$ . Hence, from Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ), and Lemma 913 (measure, homogeneity), we have  $\int w \mathbf{1}_\emptyset \, d\mu = \mu(\emptyset) = 0$ .

From Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation), let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be a disjoint representation of  $w$ , i.e. where the  $A_i$ 's are pairwise disjoint. Let  $J \subset \mathbb{N}$ . Let  $(B_j)_{j \in J}$  be pairwise disjoint in  $\Sigma$ , and let  $B \stackrel{\text{def.}}{=} \bigsqcup_{j \in J} B_j$ . Then, from Lemma 696 ( $\sigma$ -algebra,  $P_\Sigma \Rightarrow U_C$ ), Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, multiplication), and Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation), we have  $w \mathbf{1}_B \in \mathcal{SF}_+$ , and  $(n, (A_i \cap B)_{i \in [0..n]}, (a_i)_{i \in [0..n]})$  is a disjoint representation of  $w \mathbf{1}_B$ . Hence, from Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)), **distributivity of intersection over union**, Lemma 913 (measure,  $\sigma$ -additivity), Lemma 419 (multiplication in  $\mathbb{R}$  is distributive over addition (measure theory)), Lemma 394 (ad-



dition in  $\overline{\mathbb{R}}_+$  is associative), and Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative), we have

$$\begin{aligned} \int w \mathbb{1}_{\biguplus_{j \in J} B_j} d\mu &= \sum_{i \in [0..n]} a_i \mu \left( A_i \cap \biguplus_{j \in J} B_j \right) = \sum_{i \in [0..n]} a_i \mu \left( \biguplus_{j \in J} (A_i \cap B_j) \right) \\ &= \sum_{i \in [0..n]} a_i \left( \sum_{j \in J} \mu(A_i \cap B_j) \right) = \sum_{j \in J} \left( \sum_{i \in [0..n]} a_i \mu(A_i \cap B_j) \right) = \sum_{j \in J} \int w \mathbb{1}_{B_j} d\mu. \end{aligned}$$

Therefore, from Lemma 913 (measure),  $(A \mapsto \int w \mathbb{1}_A d\mu)$  is a measure on  $(X, \Sigma)$ . □

**Lemma 1167 (integral and density measure in  $\mathcal{SF}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]}) \in \mathbb{N} \times \Sigma \times \mathbb{R}_+$  be a disjoint representation of  $f$ . Then, we have

$$(15.18) \quad \forall i \in [0..n], \quad \mu_f(A_i) = a_i \mu(A_i),$$

$$(15.19) \quad \int f d\mu = \sum_{i \in [0..n]} \mu_f(A_i).$$

*Proof.* (1). Let  $i \in [0..n]$ . Then, from Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, multiplication), Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation, pairwise disjunction), the definition of the indicator function ( $\mathbb{1}_\emptyset \equiv 0$ ), and field properties of  $\mathbb{R}$ , we have

$$f \mathbb{1}_{A_i} = \left( \sum_{i' \in [0..n]} a_{i'} \mathbb{1}_{A_{i'}} \right) \mathbb{1}_{A_i} = \left( \sum_{i' \in [0..n]} a_{i'} \mathbb{1}_{A_{i'} \cap A_i} \right) = a_i \mathbb{1}_{A_i}.$$

Hence, from Lemma 1166 (density measure in  $\mathcal{SF}_+$ ), and Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint), with  $f \stackrel{\text{def.}}{=} a_i \mathbb{1}_{A_i}$ ), we have

$$\mu_f(A_i) = \int f \mathbb{1}_{A_i} d\mu = \int a_i \mathbb{1}_{A_i} d\mu = a_i \mu(A_i).$$

(2). From Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)), and (1), we have

$$\int f d\mu = \sum_{i \in [0..n]} a_i \mu(A_i) = \sum_{i \in [0..n]} \mu_f(A_i).$$

Therefore both properties hold. □

**Lemma 1168 (density integral in  $\mathcal{SF}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $w, f \in \mathcal{SF}_+$ . Then, we have

$$(15.20) \quad \int f d\mu_w = \int f w d\mu.$$

*Proof.* From Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation), let  $(n, (A_i)_{i \in [0..n]}, (a_i)_{i \in [0..n]})$  and  $(m, (B_j)_{j \in [0..m]}, (b_j)_{j \in [0..m]})$  in  $\mathbb{N} \times \Sigma \times \mathbb{R}_+$  be disjoint representations of  $f$  and  $w$ .

Then, from Lemma 1154 (integral in  $\mathcal{SF}_+$ , with  $f$ ), Lemma 1166 (density measure in  $\mathcal{SF}_+$ ), Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, multiplication with  $w$  and  $\mathbb{1}_{A_i}$ ), and Lemma 1154 (integral in  $\mathcal{SF}_+$ , with  $w \mathbb{1}_{A_i}$ ), we have

$$\int f d\mu_w = \sum_{i \in [0..n]} a_i \mu_w(A_i) = \sum_{i \in [0..n]} a_i \left( \int w \mathbb{1}_{A_i} d\mu \right) = \sum_{i \in [0..n]} a_i \left( \sum_{j \in [0..m]} b_j \mu(B_j \cap A_i) \right).$$

Moreover, from Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations, multiplication with  $f$  and  $w$ ), Lemma 1154 (integral in  $\mathcal{SF}_+$ ), and Lemma 419 (multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)), we have

$$\begin{aligned} \int f w d\mu &= \int \left( \sum_{(i,j) \in [0..n] \times [0..m]} a_i b_j \mathbb{1}_{A_i \cap B_j} \right) d\mu \\ &= \sum_{(i,j) \in [0..n] \times [0..m]} a_i b_j \mu(A_i \cap B_j) = \sum_{i \in [0..n]} a_i \left( \sum_{j \in [0..m]} b_j \mu(A_i \cap B_j) \right). \end{aligned}$$

Therefore, from **commutativity of intersection**, we have the equality.  $\square$

**Lemma 1169 (integral in  $\mathcal{SF}_+$  with trivial measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(15.21) \quad \int f d0 = 0.$$

*Proof.* Direct consequence of Lemma 1154 (integral in  $\mathcal{SF}_+$ ), Lemma 933 (trivial measure), and Lemma 318 (zero is identity element for addition in  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 1170 (integral in  $\mathcal{SF}_+$  with counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(15.22) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 1128 ( $\mathcal{SF}_+$  simple representation), Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{LF}$ ), Lemma 1152 (integral in  $\mathcal{LF}$  with counting measure), Lemma 394 (addition in  $\overline{\mathbb{R}}_+$  is associative), and Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative).  $\square$

### 15.3 Integration of nonnegative measurable function

*Remark 1171.* In this section, functions take their values in  $\overline{\mathbb{R}}_+$ , and the expressions involving integrals are also taken in  $\overline{\mathbb{R}}_+$ .

**Lemma 1172 (integral in  $\mathcal{M}_+$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\{\int \varphi d\mu \mid \varphi \in \mathcal{SF}_+ \wedge \varphi \leq f\}$  admits a supremum.*

*The integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by*

$$(15.23) \quad \int f d\mu \stackrel{\text{def.}}{=} \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi d\mu \in \overline{\mathbb{R}}_+.$$

*A function  $f : X \rightarrow \overline{\mathbb{R}}_+$  is said  $\mu$ -integrable (in  $\mathcal{M}_+$ ) iff  $f \in \mathcal{M}_+$  and  $\int f d\mu < \infty$ .*

*Proof.* Direct consequence of Lemma 1154 (integral in  $\mathcal{SF}_+$ ), and Definition 2 (supremum, with the Lebesgue integral in  $\mathcal{SF}_+$  over  $\{\varphi \in \mathcal{SF}_+ \mid \varphi \leq f\}$ ),  $\square$

*Remark 1173.* Keeping the same notation for the integrals is justified in the next lemma.

**Lemma 1174 (integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$ ).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, the values of  $\int f d\mu$  provided by Lemma 1154 (integral in  $\mathcal{SF}_+$ ), and Lemma 1172 (integral in  $\mathcal{M}_+$ ) coincide.*

*Proof.* Direct consequence of Lemma 1154 (integral in  $\mathcal{SF}_+$ ), Lemma 1162 (integral in  $\mathcal{SF}_+$  is continuous), and Lemma 1172 (integral in  $\mathcal{M}_+$ ).  $\square$

**Lemma 1175 (integral in  $\mathcal{M}_+$  of indicator function).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $\mathbb{1}_A \in \mathcal{M}_+$  and  $\int \mathbb{1}_A d\mu = \mu(A)$ .*

*Moreover, when  $\mu(A)$  is finite, then  $\mathbb{1}_A$  is  $\mu$ -integrable.*

*Proof.* Direct consequence of Lemma 1124 ( $\mathcal{SF}_+$  is measurable), Lemma 1174 (integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$ ), Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ), and Lemma 1172 (integral in  $\mathcal{M}_+$ ).  $\square$

**Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $a \in \mathbb{R}_+$ . Then,  $af \in \mathcal{M}_+$ , and we have*

$$(15.24) \quad \int af d\mu = a \int f d\mu.$$

*Proof.* From Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  $af \in \mathcal{M}_+$ .

**Case  $a = 0$ .** Then, from Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 1108 ( $\mathcal{SF}_E$  simple representation, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and Definition 61 (vector space,  $0 \in \mathcal{SF}$ ), we have

$$0f = 0 \in \mathcal{SF}_+ \subset \mathcal{M}_+.$$

Hence, from Lemma 1172 (integral in  $\mathcal{M}_+$ ), Lemma 1160 (equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)), Lemma 1154 (integral in  $\mathcal{SF}_+$ ), and Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)), we have

$$\int 0f d\mu = \int 0 d\mu = 0 = 0 \times \int f d\mu.$$

**Case  $a > 0$ .** Then, from Lemma 417 (multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)), and Definition 326 (multiplication in  $\overline{\mathbb{R}}$ , for all  $a \in \mathbb{R}_+^*$ ,  $\frac{a}{a} = 1$ ), we have  $(\star) \forall b \in \mathbb{R}_+, \frac{1}{a}(ab) = b$ .

Let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ . Then, from Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations), we have  $a\varphi \in \mathcal{SF}_+$ . Moreover, from Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)), property  $(\star)$ , Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear, positive homogeneity), Lemma 1172 (integral in  $\mathcal{M}_+$ ), and Definition 2 (supremum, upper bound), we have  $a\varphi \leq af$  and

$$\int \varphi d\mu = \frac{1}{a} \int a\varphi d\mu \leq \frac{1}{a} \int af d\mu.$$

Thus, from Lemma 1172 (integral in  $\mathcal{M}_+$ ), Definition 2 (supremum, least upper bound), and property  $(\star)$ , we have

$$a \int f d\mu = a \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi d\mu \leq a \frac{1}{a} \int af d\mu = \int af d\mu.$$

Moreover, the same result also holds for function  $af \in \mathcal{M}_+$  and number  $\frac{1}{a} > 0$ , and with property  $(\star)$ , we have  $\int af d\mu \leq a \int \frac{1}{a}(af) d\mu = a \int f d\mu$ . Hence, we have the equality.

Therefore, we always have the equality.  $\square$

**Lemma 1177 (integral in  $\mathcal{M}_+$  of zero is zero).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $0 \in \mathcal{M}_+$ , and we have  $\int 0 d\mu = 0$ .

*Proof.* Direct consequence of Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous, with  $a \stackrel{\text{def.}}{=} 0$ ), Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ), and Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)).  $\square$

**Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have

$$(15.25) \quad f \leq g \implies \int f d\mu \leq \int g d\mu.$$

*Proof.* Assume that  $f \leq g$ . Then,  $\{\varphi \in \mathcal{SF}_+ \mid \varphi \leq f\} \subset \{\varphi \in \mathcal{SF}_+ \mid \varphi \leq g\}$ . Therefore, from Lemma 1172 (integral in  $\mathcal{M}_+$ ), and Lemma 489 (supremum is monotone (set)), we have

$$\int f d\mu \leq \int g d\mu.$$

$\square$

*Remark 1179.* The next proof follows step 3 of the Lebesgue scheme (see Section 4.4).

See also the sketch of the proof in Section 5.8.

**Theorem 1180 (Beppo Levi, monotone convergence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ .

Assume that the sequence is pointwise nondecreasing. Then,  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have

$$(15.26) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \overline{\mathbb{R}}_+).$$

*Proof.* From Lemma 314 ( $\overline{\mathbb{R}}$  is complete), and Lemma 501 (supremum is limit), the existence of the pointwise limit is guaranteed in  $\overline{\mathbb{R}}$ , and we have  $f \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} f_n$ .

From Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent), we have  $f \in \mathcal{M}_+$ . Then, from Lemma 1172 (integral in  $\mathcal{M}_+$ ), the integral of  $f$  is well-defined. Moreover, from Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 314 ( $\overline{\mathbb{R}}$  is complete), the sequence  $(\int f_n d\mu)_{n \in \mathbb{N}}$  is nondecreasing, hence convergent in  $\overline{\mathbb{R}}$  towards

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

Thus, from Definition 2 (*supremum*, *upper bound*), and Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), for all  $n \in \mathbb{N}$ , we have

$$\int f_n d\mu \leq \int \sup_{n \in \mathbb{N}} f_n d\mu = \int f d\mu.$$

Hence, from Definition 2 (*supremum*, *least upper bound*), we have

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu \leq \int f d\mu.$$

Let  $a \in (0, 1)$ . Let  $\varphi \in \mathcal{SF}_+$ . Assume that  $\varphi \leq f$ . Let  $n \in \mathbb{N}$ . Let  $A_n \stackrel{\text{def.}}{=} \{a\varphi \leq f_n\} \subset X$ . Then, from Lemma 1108 ( *$\mathcal{SF}_E$  simple representation*, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $-a\varphi \in \mathcal{SF}$ ), Lemma 1113 ( *$\mathcal{SF}_E$  is measurable*, with  $E = \mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $-a\varphi \in \mathcal{M}$ ), and Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*,  $-a\varphi$  takes finite values), we have  $f_n - a\varphi \in \mathcal{M}$ . Thus, from Lemma 820 (*measurability of numeric function*, with  $f_n - a\varphi$ ), and Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*), we have  $A_n \in \Sigma$ , and  $\mathbb{1}_{A_n} \in \mathcal{M}_+$ . Moreover, from Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*), and Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*, *transitivity*), the sequence  $(A_n)_{n \in \mathbb{N}}$  is nondecreasing. Let  $x \in X$ . **Case  $\varphi(x) = 0$ .** Then, for all  $n \in \mathbb{N}$ ,  $x \in A_n$ . **Case  $\varphi(x) > 0$ .** Then, since  $0 < a < 1$ , we have  $a\varphi(x) < \varphi(x) \leq f(x)$ . Thus, from Lemma 381 (*limit is monotone*), there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a\varphi(x) \leq f_n(x)$ , i.e.  $x \in A_n$ . Hence, we have  $X = \bigcup_{n \in \mathbb{N}} A_n$ .

Let  $B \in \Sigma$ . Then, from Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, *closedness under countable intersection* (with  $\text{card}(I) = 2$ )), **monotonicity of intersection**, and **distributivity of intersection over union**, the sequence  $(B \cap A_n)_{n \in \mathbb{N}}$  belongs to  $\Sigma$ , is nondecreasing, and  $\bigcup_{n \in \mathbb{N}} (B \cap A_n) = B$ . Thus, from Lemma 925 (*measure is continuous from below*), and Definition 879 (*continuity from below*), we have  $\mu(B) = \mu(\bigcup_{n \in \mathbb{N}} (B \cap A_n)) = \lim_{n \rightarrow \infty} \mu(B \cap A_n)$ .

Then, from Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*), there exists  $k \in \mathbb{N}$ ,  $(a_i)_{i \in [0..k]} \in \mathbb{R}_+$ , and  $(B_i)_{i \in [0..k]} \in \Sigma$  such that

$$\varphi = \sum_{i \in [0..k]} a_i \mathbb{1}_{B_i}, \quad \text{and} \quad \int \varphi d\mu = \sum_{i \in [0..k]} a_i \mu(B_i).$$

Thus, from Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*), we have

$$\forall n \in \mathbb{N}, \quad \int \varphi \mathbb{1}_{A_n} d\mu = \int \sum_{i \in [0..k]} a_i \mathbb{1}_{B_i \cap A_n} d\mu = \sum_{i \in [0..k]} a_i \mu(B_i \cap A_n).$$

Hence, from the previous result on the measure  $\mu$ , **field properties of  $\mathbb{R}$  ( $\frac{a}{a} = 1$ )**, Lemma 414 (*limit in  $\overline{\mathbb{R}}_+$  is positive linear*, with  $a_i \in \mathbb{R}_+$  and  $a \in \mathbb{R}_+^*$ ), and Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*, with  $a \in \mathbb{R}_+^*$ ), we have

$$\int \varphi d\mu = \sum_{i \in [0..k]} a_i \lim_{n \rightarrow \infty} \mu(B_i \cap A_n) = \lim_{n \rightarrow \infty} \int \varphi \mathbb{1}_{A_n} d\mu = \frac{1}{a} \lim_{n \rightarrow \infty} \int a\varphi \mathbb{1}_{A_n} d\mu.$$

Let  $n \in \mathbb{N}$ . Then, from the definition of the  $A_n$ 's, and **the definition of the indicator function**, we have  $a\varphi \mathbb{1}_{A_n} \leq f_n \mathbb{1}_{A_n} \leq f_n$ . Thus, from Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), Lemma 381 (*limit is monotone*), and Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*), we have

$$\int \varphi d\mu \leq \frac{1}{a} \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Then, from **left continuity of the division at  $a = 1$** , and Lemma 381 (*limit is monotone*), we have

$$\forall \varphi \in \mathcal{SF}_+, \quad \varphi \leq f \Rightarrow \int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Hence, from Lemma 1172 (*integral in  $\mathcal{M}_+$* ), and Definition 2 (*supremum, least upper bound*),

$$\int f \, d\mu = \sup_{\substack{\varphi \in \mathcal{SF}_+ \\ \varphi \leq f}} \int \varphi \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

Therefore, we have  $\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$ .  $\square$

**Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\infty f \in \mathcal{M}_+$ , and we have

$$(15.27) \quad \int \infty f \, d\mu = \infty \int f \, d\mu.$$

*Proof.* From Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $f \geq 0$ ), Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*), Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*), and Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  $(nf)_{n \in \mathbb{N}}$  is a nondecreasing sequence in  $\mathcal{M}_+$  such that  $\lim_{n \rightarrow \infty} nf = \infty f \in \mathcal{M}_+$ . Therefore, from Theorem 1180 (*Beppo Levi, monotone convergence*), Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*), and Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*), we have

$$\begin{aligned} \int \infty f \, d\mu &= \int \left( \lim_{n \rightarrow \infty} n \right) f \, d\mu = \int \left( \lim_{n \rightarrow \infty} nf \right) \, d\mu \\ &= \lim_{n \rightarrow \infty} \left( \int nf \, d\mu \right) = \lim_{n \rightarrow \infty} \left( n \int f \, d\mu \right) = \left( \lim_{n \rightarrow \infty} n \right) \int f \, d\mu = \infty \int f \, d\mu. \end{aligned}$$

$\square$

**Lemma 1182 (*usage of adapted sequences*).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f \in \mathcal{M}_+$ . Let  $(\varphi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  be an adapted sequence of  $f$ . Then, we have

$$(15.28) \quad \int f \, d\mu = \int \lim_{n \rightarrow \infty} \varphi_n \, d\mu = \lim_{n \rightarrow \infty} \int \varphi_n \, d\mu.$$

*Proof.* Direct consequence of Definition 1135 (*adapted sequence*), Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ), Theorem 1180 (*Beppo Levi, monotone convergence*), and Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ).  $\square$

**Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then,  $f + g \in \mathcal{M}_+$ , and we have

$$(15.29) \quad \int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

*Proof.* From Lemma 843 ( $\mathcal{M}_+$  is closed under addition), we have  $f + g \in \mathcal{M}_+$ .

From Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ), let  $(\varphi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  be adapted sequences for  $f$  and  $g$ . Then, from Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear, additivity*), Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*), Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*), and Definition 1135 (*adapted sequence*),  $(\varphi_n + \psi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  is an adapted sequence for  $f + g$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear, additivity*), we have

$$\int (\varphi_n + \psi_n) \, d\mu = \int \varphi_n \, d\mu + \int \psi_n \, d\mu.$$

Therefore, from Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*), and Lemma 1182 (*usage of adapted sequences*), we have

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

$\square$

**Lemma 1184 (integral in  $\mathcal{M}_+$  is positive linear).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Let  $a \in \overline{\mathbb{R}}_+$ . Then,  $f + g, af \in \mathcal{M}_+$ , and we have

$$(15.30) \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu, \quad \text{and} \quad \int af d\mu = a \int f d\mu.$$

*Proof.* Direct consequence of Lemma 1183 (integral in  $\mathcal{M}_+$  is additive), Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous), and Lemma 1181 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ).  $\square$

**Lemma 1185 (integral in  $\mathcal{M}_+$  is  $\sigma$ -additive).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then,  $\sum_{n \in \mathbb{N}} f_n \in \mathcal{M}_+$ , and we have

$$(15.31) \quad \int \left( \sum_{n \in \mathbb{N}} f_n \right) d\mu = \sum_{n \in \mathbb{N}} \left( \int f_n d\mu \right).$$

*Proof.* Direct consequence of Lemma 850 ( $\mathcal{M}_+$  is closed under countable sum), and Theorem 1180 (Beppo Levi, monotone convergence, with nondecreasing sequence  $(\sum_{i \in [0, n]} f_i)_{n \in \mathbb{N}}$ ).  $\square$

**Lemma 1186 (integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts).** Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then, we have

$$(15.32) \quad \int |f| d\mu = \int f^+ d\mu + \int f^- d\mu.$$

*Proof.* Direct consequence of Lemma 842 ( $\mathcal{M}$  is closed under absolute value,  $|f|$  belongs to  $\mathcal{M}_+$ ), Lemma 836 (measurability of nonnegative and nonpositive parts,  $f^+, f^- \in \mathcal{M}_+$ ), Lemma 532 (decomposition into nonnegative and nonpositive parts,  $|f| = f^+ + f^-$ ), and Lemma 1183 (integral in  $\mathcal{M}_+$  is additive).  $\square$

**Lemma 1187 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts).** Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$  such that  $f + g$  is well-defined and belongs to  $\mathcal{M}$ . Then, we have in  $\overline{\mathbb{R}}_+$

$$(15.33) \quad \int (f + g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f + g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu.$$

*Proof.* Direct consequence of Lemma 836 (measurability of nonnegative and nonpositive parts,  $f^\pm, g^\pm, (f + g)^\pm \in \mathcal{M}_+$ ), Lemma 533 (compatibility of nonpositive and nonnegative parts with addition), Lemma 843 ( $\mathcal{M}_+$  is closed under addition), and Lemma 1183 (integral in  $\mathcal{M}_+$  is additive).  $\square$

**Lemma 1188 (integral in  $\mathcal{M}_+$  is almost definite).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then, we have

$$(15.34) \quad \int f d\mu = 0 \iff f \stackrel{\mu}{\text{a.e.}} 0.$$

*Proof.* Let  $A \stackrel{\text{def.}}{=} \{f > 0\} = f^{-1}(0, \infty]$ . Then, from Lemma 820 (measurability of numeric function,  $A \in \Sigma$ ), Lemma 1175 (integral in  $\mathcal{M}_+$  of indicator function,  $\mathbb{1}_A \in \mathcal{M}_+$  and  $\int \mathbb{1}_A d\mu = \mu(A)$ ), Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication,  $\infty f, \infty \mathbb{1}_A \in \mathcal{M}_+$ ), Lemma 1181 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ), and Lemma 427 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory),  $\infty f = \infty \mathbb{1}_A$ ), we have

$$\infty \int f d\mu = \int \infty f d\mu = \int \infty \mathbb{1}_A d\mu = \infty \int \mathbb{1}_A d\mu = \infty \mu(A).$$



Therefore, from Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*), multiplication by  $\infty$  is definite in  $\overline{\mathbb{R}}_+$ ), Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions,  $\{f = 0\}^c = A$* ), Lemma 948 (*negligibility of measurable subset*), and Definition 953 (*property almost satisfied*), we have

$$\begin{aligned} \int f \, d\mu = 0 &\iff \infty \int f \, d\mu = 0 \iff \infty \mu(A) = 0 \\ &\iff \mu(A) = 0 \iff \mu(\{f = 0\}^c) = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0. \end{aligned}$$

□

**Lemma 1189 (extended almost modus ponens for integral in  $\mathcal{M}_+$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. Let  $P$  and  $Q$  be predicates on  $\overline{\mathbb{R}}^I$ ,  $P$  being lifted into a predicate on  $(\overline{\mathbb{R}}^I)^X$ . Assume that  $P(0, \dots, 0)$  holds, and that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} \in \mathcal{M}_+$ , we have*

$$(15.35) \quad \forall x \in X, P(f(x)) \implies Q\left(\int f_i \, d\mu\right)_{i \in I}.$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I} \in \mathcal{M}_+$ , we have

$$(15.36) \quad P(g) \, \mu \text{ a.e.} \implies Q\left(\int g_i \, d\mu\right)_{i \in I}.$$

*Proof.* Let  $g = (g_i)_{i \in I} \in \mathcal{M}_+$ . Assume that  $P(g)$  holds  $\mu$ -almost everywhere.

From Lemma 955 (*equivalent definition of property almost satisfied*), let  $A \in \Sigma$  such that  $\mu(A^c) = 0$  and  $A \subset \{P(g)\}$ , i.e. such that  $P(g|_A)$  holds. Let  $f = (f_i)_{i \in I} \stackrel{\text{def.}}{=} (g_i \mathbb{1}_A)_{i \in I}$ . Then, from **the definition of the indicator function**, and since  $P(0, \dots, 0)$ , we have  $P(f)$  everywhere. Thus, from Lemma 913 (*measure*), Definition 741 (*measurable space,  $\Sigma$  is a  $\sigma$ -algebra*), Lemma 696 ( *$\sigma$ -algebra,  $A^c \in \Sigma$* ), Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function,  $\mathbb{1}_A, \mathbb{1}_{A^c} \in \mathcal{M}_+$* ), and Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*), for all  $i \in I$ , we have  $f_i, g_i \mathbb{1}_{A^c} \in \mathcal{M}_+$ . Hence, from assumption, the property  $Q\left(\int f_i \, d\mu\right)_{i \in I}$  holds.

Let  $i \in I$ . Then, from **the definition of the indicator function ( $\{\mathbb{1}_{A^c} = 0\}^c = A^c$ )**, Lemma 948 (*negligibility of measurable subset,  $\{\mathbb{1}_{A^c} = 0\}^c \in \mathbb{N}$* ), Definition 953 (*property almost satisfied,  $\mathbb{1}_{A^c} \stackrel{\mu \text{ a.e.}}{=} 0$* ), Lemma 985 (*compatibility of almost equality with operator, with the unary operator left multiplication by  $g_i$* ), and Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*), we have  $g_i \mathbb{1}_{A^c} \stackrel{\mu \text{ a.e.}}{=} g_i 0 = 0$ . Hence, from **properties of the indicator function**, Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*), Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*), Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*), and Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ), we have

$$\int g_i \, d\mu = \int g_i(\mathbb{1}_A + \mathbb{1}_{A^c}) \, d\mu = \int f_i \, d\mu.$$

Therefore,  $Q\left(\int g_i \, d\mu\right)_{i \in I}$  also holds. □

**Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have*

$$(15.37) \quad f \stackrel{\mu \text{ a.e.}}{=} g \implies \int f \, d\mu = \int g \, d\mu.$$

*Proof.* Direct consequence of **reflexivity of equality**, Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$ , with  $\text{card}(I) = 2$  and  $P = Q \stackrel{\text{def.}}{=} \text{equality}$* ), and Lemma 1172 (*integral in  $\mathcal{M}_+$ , integral is a function*). □



**Lemma 1191 (integral in  $\mathcal{M}_+$  is almost monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have

$$(15.38) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \implies \int f \, d\mu \leq \int g \, d\mu.$$

*Proof.* Direct consequence of **the definition of order (reflexivity)**, Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total*, thus  $0 \leq 0$ ), Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* , with  $\text{card}(I) = 2$  and  $P = Q \stackrel{\text{def.}}{\leq}$ ), and Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*).  $\square$

**Lemma 1192 (integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{\geq} 0$ . Then, we have  $f^- \stackrel{\mu \text{ a.e.}}{=} 0$ , and  $\int f^- \, d\mu = 0$ .

*Proof.* Direct consequence of Definition 528 (*nonnegative and nonpositive parts*,  $\{f^- \neq 0\}$  equals  $\{f < 0\}$ ), Definition 974 (*almost binary relation*,  $\{f < 0\} \in \mathbf{N}$ , thus  $f^- \stackrel{\mu \text{ a.e.}}{=} 0$ ), and Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*).  $\square$

**Lemma 1193 (integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then, we have  $f^\pm \stackrel{\mu \text{ a.e.}}{=} g^\pm$  in  $\mathcal{M}_+$ , and  $\int f^\pm \, d\mu = \int g^\pm \, d\mu$ .

*Proof.* Direct consequence of Definition 528 (*nonnegative and nonpositive parts*), Lemma 985 (*compatibility of almost equality with operator*, with the operators  $\max(\cdot, 0)$  and  $\max(-\cdot, 0)$ ), Lemma 836 (*measurability of nonnegative and nonpositive parts*,  $f^\pm, g^\pm \in \mathcal{M}_+$ ), and Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*).  $\square$

**Lemma 1194 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f, g \stackrel{\mu \text{ a.e.}}{\geq} 0$ . Then,  $f + g$  is well-defined almost everywhere, and we have  $f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ .

Moreover, there exist  $f', g' \in \mathcal{M}$  such that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ , and for all such functions,  $f' + g' \in \mathcal{M}$ ,  $f', g', f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \stackrel{\mu \text{ a.e.}}{\geq} 0$ ,  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ , and we have

$$(15.39) \quad \int (f \stackrel{\mu \text{ a.e.}}{+} g)^+ \, d\mu = \int (f' + g')^+ \, d\mu = \int (f')^+ \, d\mu + \int (g')^+ \, d\mu = \int f^+ \, d\mu + \int g^+ \, d\mu.$$

*Proof.* From Definition 974 (*almost binary relation*,  $\{f < 0\}, \{g < 0\} \in \mathbf{N}$ ), Lemma 951 (*subset of negligible is negligible*,  $\{f = \infty\} \cap \{g = -\infty\} \subset \{g < 0\}$  and  $\{f = -\infty\} \cap \{g = \infty\} \subset \{f < 0\}$  are in  $\mathbf{N}$ ), Lemma 952 ( *$\mathbf{N}$  is closed under countable union*, with  $\text{card}(I) = 2$ ), Definition 997 (*summability domain*), and Lemma 1000 (*negligibility and summability domain*),  $f + g$  is well-defined almost everywhere. Thus, from Lemma 1002 (*almost sum*, thus  $f'$  and  $g'$  exist), and Lemma 1003 (*compatibility of almost sum with almost equality*), we have  $f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ .

Hence, from Lemma 836 (*measurability of nonnegative and nonpositive parts*,  $(f')^\pm, (g')^\pm$  and  $(f' + g')^\pm$  belong to  $\mathcal{M}_+$ ), and Lemma 1193 (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*), we have

$$\int (f')^\pm \, d\mu = \int f^\pm \, d\mu, \quad \int (g')^\pm \, d\mu = \int g^\pm \, d\mu, \quad \int (f' + g')^\pm \, d\mu = \int (f \stackrel{\mu \text{ a.e.}}{+} g)^\pm \, d\mu.$$

Then, from Lemma 995 (*compatibility of almost nonnegativeness with almost equality*, thus  $f' \stackrel{\mu \text{ a.e.}}{\geq} 0$  and  $g' \stackrel{\mu \text{ a.e.}}{\geq} 0$ ), and Lemma 984 (*compatibility of almost binary relation with operator*, with the

binary operator addition), we have  $f' + g' \stackrel{\mu \text{ a.e.}}{\geq} 0$ . Hence, from Lemma 995 (*compatibility of almost nonnegativeness with almost equality*), we also have  $f \stackrel{\mu \text{ a.e.}}{+} g \stackrel{\mu \text{ a.e.}}{\geq} 0$ .

Therefore, from Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*, thus  $\int (f')^- d\mu = \int (g')^- d\mu = \int (f' + g')^- d\mu = 0$ ), Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*, with  $f \stackrel{\text{def.}}{=} f'$  and  $g \stackrel{\text{def.}}{=} g'$ ), and Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*), we have  $\int (f' + g')^+ d\mu$  equals  $\int (f')^+ d\mu + \int (g')^+ d\mu$ , and (15.39) holds.  $\square$

**Lemma 1195 (generalized Bienaymé–Chebyshev inequality).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}(X)$ . Let  $g \in \mathcal{M}_+(\overline{\mathbb{R}})$ . Let  $a \in \overline{\mathbb{R}}$ . Assume that  $g$  is nondecreasing. Then, we have  $\{f \geq a\} \in \Sigma$ ,  $g \circ f \in \mathcal{M}_+$ , and*

$$(15.40) \quad g(a) \mu(\{f \geq a\}) \leq \int g \circ f d\mu.$$

*Proof.* Let  $A \stackrel{\text{def.}}{=} \{f \geq a\}$ . Let  $x \in A$ . Then, from **the definition of nondecreasingness**, we have  $g(a) \leq g \circ f(x)$ . Let  $x \in A^c$ . Then, from Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*,  $g \geq 0$ ), we have  $0 \leq g \circ f(x)$ . Hence, from **the definition of the indicator function**, and Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*), we have

$$g(a) \mathbb{1}_A \leq g \circ f.$$

Therefore, from Lemma 820 (*measurability of numeric function*), Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* ), Lemma 757 (*compatibility of measurability with composition*,  $g \circ f \in \mathcal{M}$ ), Lemma 845 ( $\mathcal{M}_+$  *is closed under nonnegative scalar multiplication*), Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), and Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*), we have  $A \in \Sigma$ , and

$$g(a) \mu(A) = g(a) \int \mathbb{1}_A d\mu \leq \int g \circ f d\mu.$$

$\square$

**Lemma 1196 (Bienaymé–Chebyshev inequality).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $p \in (0, \infty)$ . Let  $a \in \overline{\mathbb{R}}_+$ . Then, we have  $\{|f| \geq a\} \in \Sigma$ ,  $|f|^p \in \mathcal{M}_+$ , and*

$$(15.41) \quad a^p \mu(\{|f| \geq a\}) \leq \int |f|^p d\mu.$$

*Proof.* Direct consequence of Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*, with  $|f|$  and  $g \stackrel{\text{def.}}{=} (x \mapsto |x|^p)$ ).  $\square$

**Lemma 1197 (integrable in  $\mathcal{M}_+$  is almost finite).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$  be  $\mu$ -integrable. Then, we have  $\mu(\{f = \infty\}) = 0$ , i.e.  $f \stackrel{\mu \text{ a.e.}}{<} \infty$ .*

*Proof.* Direct consequence of Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*,  $f \in \mathcal{M}$ ), Lemma 1196 (*Bienaymé–Chebyshev inequality*, with  $p \stackrel{\text{def.}}{=} 1$  and  $a \stackrel{\text{def.}}{=} \infty$ ), Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*,  $|f| = f$ ), Definition 312 (*extended real numbers*,  $\overline{\mathbb{R}}$ ,  $A \stackrel{\text{def.}}{=} \{|f| \geq \infty\} = \{f = \infty\}$ ), Lemma 1172 (*integral in  $\mathcal{M}_+$* ,  $\int f d\mu < \infty$ ), Lemma 424 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*,  $\mu(A) = 0$ ), Lemma 948 (*negligibility of measurable subset*,  $A \in \mathbf{N}$ ), and Definition 953 (*property almost satisfied*,  $A^c = \{f < \infty\}$ ).  $\square$

**Lemma 1198 (almost bounded by integrable in  $\mathcal{M}_+$  is integrable).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$  iff  $f \in \mathcal{M}_+$  and there exists  $g : X \rightarrow \overline{\mathbb{R}}_+$  such that  $g$  is  $\mu$ -integrable in  $\mathcal{M}_+$  and  $f \stackrel{\mu \text{ a.e.}}{\leq} g$ .*

*Proof.* “Left” implies “right”.

Direct consequence of Lemma 983 (*almost order is order relation*, reflexivity, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} g \stackrel{\text{def.}}{=} f$ ).

“Right” implies “left”. Direct consequence of Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*), Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*, *transitivity*), and Lemma 1172 (*integral in  $\mathcal{M}_+$ ,  $\mu$ -integrability*).  $\square$

**Lemma 1199** (*integral in  $\mathcal{M}_+$  over subset*). *Let  $(X, \Sigma, \mu)$  be a measure space.*

*Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ .*

*Then, we have  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$  iff  $\hat{f} \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ .*

*If so, there exists  $(\hat{\varphi}_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$  such that*

$$(15.42) \quad \forall n \in \mathbb{N}, \hat{\varphi}_n \mathbb{1}_A = \hat{\varphi}_n, \quad f|_A = \sup_{n \in \mathbb{N}} (\hat{\varphi}_n)|_A, \quad \hat{f} \mathbb{1}_A = \sup_{n \in \mathbb{N}} \hat{\varphi}_n,$$

*and we have*

$$(15.43) \quad \int f|_A d\mu_A = \sup_{n \in \mathbb{N}} \int_A \hat{\varphi}_n d\mu = \int \hat{f} \mathbb{1}_A d\mu.$$

*This integral is still denoted  $\int_A f d\mu$ ; it is still called integral of  $f$  over  $A$ .*

*Proof.* Direct consequence of Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*), Theorem 1180 (*Beppo Levi, monotone convergence*), Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ), and Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*).  $\square$

**Lemma 1200** (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*).

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ .*

*Let  $Y$  be a set such that  $A \subset Y \subset X$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ .*

*Then,  $\hat{f} \mathbb{1}_A = \sum_{i \in I} \hat{f} \mathbb{1}_{A_i}$ , and  $\hat{f} \mathbb{1}_A \in \mathcal{M}_+$  iff for all  $i \in I$ ,  $\hat{f} \mathbb{1}_{A_i} \in \mathcal{M}_+$ . If so, we have*

$$(15.44) \quad \int_A f d\mu = \sum_{i \in I} \int_{A_i} f d\mu.$$

*Proof.* Direct consequence of Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*), Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset, with  $A$  and the  $A_i$ 's*), and Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*).  $\square$

**Lemma 1201** (*integral in  $\mathcal{M}_+$  over negligible subset*). *Let  $(X, \Sigma, \mu)$  be a measure space.*

*Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$  and  $\mu(A) = 0$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ , and either  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ , or  $\hat{f} \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ . Then, we have both, and*

$$(15.45) \quad \int_A f d\mu = 0.$$

*Proof.* Direct consequence of Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset, thus exists nonnegative simple functions  $(\hat{\varphi}_n)_{n \in \mathbb{N}}$  such that  $\hat{\varphi}_n \mathbb{1}_A = \hat{\varphi}_n$  and  $\int_A f d\mu = \sup_{n \in \mathbb{N}} \int_A \hat{\varphi}_n d\mu$ ), Lemma 1165 (*integral in  $\mathcal{SF}_+$  over negligible subset, thus  $\int_A \hat{\varphi}_n d\mu = 0$ ), and Lemma 482 (*extrema of constant function, thus  $\sup_{n \in \mathbb{N}} \int_A \hat{\varphi}_n d\mu = 0$* ).  $\square$**

**Lemma 1202** (*integral in  $\mathcal{M}_+$  over singleton*).

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $a \in X$ . Assume that  $\{a\} \in \Sigma$ . Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ .*

*Then, we have  $f \mathbb{1}_{\{a\}} = f(a) \mathbb{1}_{\{a\}} \in \mathcal{M}_+$ , and*

$$(15.46) \quad \int_{\{a\}} f d\mu = f(a) \mu(\{a\}).$$

*Proof.* From **the definition of the indicator function**, Lemma 811 (*measurability of indicator function*), and Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication), we have  $f \mathbb{1}_{\{a\}} = f(a) \mathbb{1}_{\{a\}} \in \mathcal{M}_+$ . Then, from Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*, with  $A \stackrel{\text{def.}}{=} \{a\}$ ,  $Y \stackrel{\text{def.}}{=} X$  and  $\hat{f} \stackrel{\text{def.}}{=} f$ ), Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*, homogeneity in  $\overline{\mathbb{R}}_+$ ), and Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*), we have

$$\int_{\{a\}} f d\mu = \int f \mathbb{1}_{\{a\}} d\mu = f(a) \int \mathbb{1}_{\{a\}} d\mu = f(a) \mu(\{a\}).$$

□

**Lemma 1203** (*finite integral in  $\mathcal{M}_+$  is closed under restriction*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ . Assume that either  $f \in \mathcal{M}_+(X, \Sigma)$ , or for all  $A \in \Sigma$ ,  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ . Then, we have both, and

$$(15.47) \quad \int f d\mu < \infty \iff \forall A \in \Sigma, \int_A f d\mu < \infty.$$

*Proof.* From Lemma 1138 ( $\mathcal{M}_+$  is closed under restriction), and Lemma 1137 ( $\mathcal{M}_+$  is closed under extension by zero, with  $Y = X$  and  $\hat{f} = f$ ), we have

$$f \in \mathcal{M}_+(X, \Sigma), \quad \text{and} \quad \forall A \in \Sigma, \quad f|_A \in \mathcal{M}_+(A, \Sigma \cap A) \quad \wedge \quad f \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma).$$

**“Left” implies “right”.** Let  $A \in \Sigma$ . Then, from Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*, with  $Y = X$  and  $\hat{f} = f$ ), **the definition of the indicator function (thus  $f \mathbb{1}_A \leq f$ )**, Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), and **transitivity of order**, we have

$$\int_A f d\mu = \int f \mathbb{1}_A d\mu \leq \int f d\mu < \infty.$$

**“Right” implies “left”.** Direct consequence of Lemma 697 (*equivalent definition of  $\sigma$ -algebra*, thus  $X \in \Sigma$ ), **the definition of the indicator function (thus  $\mathbb{1}_X \equiv 1$ )**, and Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*, with  $A = Y = X$  and  $\hat{f} = f$ ).

Therefore, we have the equivalence. □

**Lemma 1204** (*integrable in  $\mathcal{M}_+$  is closed under restriction*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ .

Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A$  is  $\mu_A$ -integrable in  $\mathcal{M}_+(A, \Sigma \cap A)$ .

*Proof.* From Lemma 1172 (*integral in  $\mathcal{M}_+$ , integrability with  $(X, \Sigma, \mu)$ , then  $(A, \Sigma \cap A, \mu_A)$* ), Lemma 1138 ( $\mathcal{M}_+$  is closed under restriction), Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*), and Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*, with  $Y = X$  and  $\hat{f} = f$ ), we have

$$\begin{aligned} f \text{ } \mu\text{-integrable in } \mathcal{M}_+(X, \Sigma) &\iff f \in \mathcal{M}_+(X, \Sigma) \quad \wedge \quad \int f d\mu < \infty \\ &\iff (\forall A \in \Sigma, \quad f|_A \in \mathcal{M}_+(A, \Sigma \cap A) \\ &\quad \wedge \quad (\forall A \in \Sigma, \quad \int_A f d\mu < \infty) \\ &\iff \forall A \in \Sigma, \quad f|_A \in \mathcal{M}_+(A, \Sigma \cap A) \quad \wedge \quad \int f|_A d\mu_A < \infty \\ &\iff \forall A \in \Sigma, \quad f|_A \text{ } \mu_A\text{-integrable in } \mathcal{M}_+(A, \Sigma \cap A). \end{aligned}$$

□

**Lemma 1205 (finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ .

Assume that either  $f \mathbf{1}_A \in \mathcal{M}_+$ , or for all  $i \in I$ ,  $f \mathbf{1}_{A_i} \in \mathcal{M}_+$ . Then, we have both, and

$$(15.48) \quad \int_A f d\mu < \infty \implies \forall i \in I, \int_{A_i} f d\mu < \infty,$$

$$(15.49) \quad \forall i \in [0..n], \int_{A_i} f d\mu < \infty \implies \int_A f d\mu < \infty. \quad (\text{when } I \stackrel{\text{def.}}{=} [0..n] \text{ with } n \in \mathbb{N}).$$

*Proof.* From Lemma 1139 ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive), we have

$$f \mathbf{1}_A \in \mathcal{M}_+, \quad \text{and} \quad \forall i \in I, \quad f \mathbf{1}_{A_i} \in \mathcal{M}_+.$$

(15.48). Let  $i \in I$ . Then, from Lemma 1199 (integral in  $\mathcal{M}_+$  over subset, with  $Y = X$ ,  $\hat{f} = f$  and  $A \stackrel{\text{def.}}{=} A_i$ , then  $A$ ), Definition 209 (pseudopartition, thus  $A_i \subset A$ ), **the definition of the indicator function (thus  $f \mathbf{1}_{A_i} \leq f \mathbf{1}_A$ )**, Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), and **transitivity of order**, we have

$$\int_{A_i} f d\mu = \int f \mathbf{1}_{A_i} d\mu \leq \int f \mathbf{1}_A d\mu = \int_A f d\mu < \infty.$$

(15.49). Direct consequence of Lemma 1200 (integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive), and **closedness of finite addition in  $\mathbb{R}$** .

Therefore, we have both implications. □

**Lemma 1206 (integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ .

Then, we have  $f \mathbf{1}_A = \sum_{i \in I} f \mathbf{1}_{A_i}$ , and  $f \mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}_+$  implies for all  $i \in I$ ,  $f \mathbf{1}_{A_i}$  is  $\mu$ -integrable in  $\mathcal{M}_+$ .

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$  with  $n \in \mathbb{N}$ , we have for all  $i \in [0..n]$ ,  $f \mathbf{1}_{A_i}$  is  $\mu$ -integrable in  $\mathcal{M}_+$  implies  $f \mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 525 (masking is  $\sigma$ -additive, with  $E \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ , thus the identity holds), Lemma 1172 (integral in  $\mathcal{M}_+$ , integrability, with  $f \mathbf{1}_A$  and  $f \mathbf{1}_{A_i}$ ), Lemma 1139 ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive), and Lemma 1205 (finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive). □

*Remark 1207.* See the sketch of next proof in Section 5.7.

**Theorem 1208 (Fatou's lemma).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then,  $\liminf_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have

$$(15.50) \quad \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \overline{\mathbb{R}}_+).$$

*Proof.* From Lemma 518 (limit inferior bounded from below, with  $m \stackrel{\text{def.}}{=} 0$ ), and Lemma 830 ( $\mathcal{M}$  is closed under limit inferior), we have  $\liminf_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ .

Let  $n \in \mathbb{N}$ . Let  $g_n \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p}$ . Then, from Lemma 488 (infimum is monotone (set)), Lemma 492 (infimum of bounded is bounded, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$  and  $a \stackrel{\text{def.}}{=} 0$ ), and Lemma 828 ( $\mathcal{M}$  is closed under infimum),  $(g_n)_{n \in \mathbb{N}}$  is a nondecreasing sequence in  $\mathcal{M}_+$ . Thus, from Lemma 314 ( $\overline{\mathbb{R}}$  is complete, supremum exists), Lemma 501 (supremum is limit), Definition 516 (pointwise

convergence), Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent), and Lemma 504 (limit inferior),  $(g_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}_+$  towards

$$g \stackrel{\text{def.}}{=} \lim_{n \in \mathbb{N}} g_n = \sup_{n \in \mathbb{N}} g_n = \liminf_{n \rightarrow \infty} f_n \in \mathcal{M}_+.$$

Hence, from Theorem 1180 (Beppo Levi, monotone convergence, with  $(g_n)_{n \in \mathbb{N}}$ ),  $g$  is measurable and nonnegative, and we have

$$\int \liminf_{n \rightarrow \infty} f_n d\mu = \int g d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu.$$

Let  $n, p \in \mathbb{N}$ . Let  $x \in X$ . Then, from Definition 9 (infimum, lower bound), we have

$$g_n(x) \leq f_{n+p}(x).$$

Then, from Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), we have

$$\int g_n d\mu \leq \int f_{n+p} d\mu.$$

Hence, from Definition 9 (infimum, greatest lower bound), we have

$$\int g_n d\mu \leq \inf_{p \in \mathbb{N}} \int f_{n+p} d\mu.$$

Therefore, from Lemma 381 (limit is monotone), and Lemma 504 (limit inferior), we have

$$\int \liminf_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

□

**Lemma 1209 (integral in  $\mathcal{M}_+$  of pointwise convergent sequence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ .

Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $f_n \leq f$ . Then,  $f \in \mathcal{M}_+$ , and we have

$$(15.51) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* From Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent), we have

$$f = \lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+.$$

Moreover, from Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), we have

$$\forall n \in \mathbb{N}, \quad \int f_n d\mu \leq \int f d\mu.$$

Thus, from Definition 2 (supremum, least upper bound), Lemma 517 (limit inferior and limit superior of pointwise convergent), and Theorem 1208 (Fatou's lemma), we have

$$\limsup_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu = \int \lim_{n \rightarrow \infty} f_n d\mu = \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$$

Therefore, from Lemma 522 (limit inferior, limit superior and pointwise convergence), we have

$$\liminf_{n \rightarrow \infty} \int f_n d\mu = \limsup_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

□

**Lemma 1210** (*integral in  $\mathcal{M}_+$  with trivial measure*).

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}_+$ . Then, we have

$$(15.52) \quad \int f \, d0 = 0.$$

*Proof.* Direct consequence of Lemma 1182 (*usage of adapted sequences*), Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*), and Lemma 34 (*stationary sequence is convergent*).  $\square$

**Lemma 1211** (*integral in  $\mathcal{M}_+$  with counting measure*).

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{M}_+$ . Then, we have

$$(15.53) \quad \int f \, d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 1182 (*usage of adapted sequences*), Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*), Lemma 413 (*addition in  $\bar{\mathbb{R}}_+$  is continuous*), and Lemma 45 (*compatibility of limit with continuous functions*).  $\square$

**Lemma 1212** (*integral in  $\mathcal{M}_+$  with counting measure on  $\mathbb{N}$* ).

Let  $f : \mathbb{N} \rightarrow \bar{\mathbb{R}}_+$  be a nonnegative sequence. Then, we have

$$(15.54) \quad \int f \, d\delta_{\mathbb{N}} = \sum_{n \in \mathbb{N}} f(n).$$

*Proof.* Direct consequence of Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure, with  $Y = X$  set to  $\mathbb{N}$  and  $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$* ).  $\square$

*Remark 1213.* Note that the previous lemma makes the sums of nonnegative series be Lebesgue integrals with counting measure on natural numbers. Thus, the theory of nonnegative series can be derived from Lebesgue integration for nonnegative measurable functions.

**Lemma 1214** (*integral in  $\mathcal{M}_+$  with Dirac measure*).

Let  $(X, \Sigma)$  be a measurable space. Let  $\{a\} \in \Sigma$ . Let  $f \in \mathcal{M}_+$ . Then, we have

$$(15.55) \quad \int f \, d\delta_a = f(a).$$

*Proof.* Direct consequence of Definition 939 (*Dirac measure*), and Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure, with  $Y \stackrel{\text{def.}}{=} \{a\}$* ).  $\square$



## 15.4 Measure and integration over product space

*Remark 1215.* The “overline- $i$ ” notations from the next definition should help in writing expressions in the general case. But, for the sake of simplicity, we only present this section in the case of the product of two measure spaces. When  $i \in \{1, 2\}$ , the complement  $\{1, 2\} \setminus \{i\}$  is  $\{3 - i\}$ .

**Definition 1216** (*overline- $i$  notation to exclude term in product*).

Let  $I \subset \mathbb{N}$ . For all  $i \in \mathbb{N}$ , let  $\bar{I} \stackrel{\text{def.}}{=} I \setminus \{i\}$ , i.e.  $i \in I$  implies  $I = \bar{I} \uplus \{i\}$  (and  $\bar{I} = I$  when  $i \notin I$ ).

Let  $m \in [2.. \infty)$ . Let  $(X_i, \Sigma_i)_{i \in [1..m]}$  be  $m$  measurable spaces. Let

$$X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i, \quad \text{and} \quad \Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i.$$

Let  $i \in [1..m]$ . We define the following “overline- $i$ ” notations:

$$(15.56) \quad \bar{X}^i \stackrel{\text{def.}}{=} \prod_{j \in [1..m]^i} X_j, \quad \text{and} \quad \bar{\Sigma}^i \stackrel{\text{def.}}{=} \bigotimes_{j \in [1..m]^i} \Sigma_j.$$

### 15.4.1 Measure over product space

*Remark 1217.* The definition of sections and some of their properties are presented in Section 9.4.

**Lemma 1218** (*measure of section*). Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, the function  $F_i^A$  defined by

$$(15.57) \quad \forall x_i \in X_i, \quad F_i^A(x_i) \stackrel{\text{def.}}{=} \mu_j(s_i(x_i, A))$$

is well-defined on  $X_i$ , and takes its values in  $\bar{\mathbb{R}}_+$ . It is called *measure of  $i$ -th section of  $A$* .

*Proof.* Direct consequence of Lemma 786 (*measurability of section*), and Lemma 913 (*measure, nonnegativeness*).  $\square$

**Lemma 1219** (*measure of section of product*). Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, we have

$$(15.58) \quad F_i^{A_1 \times A_2} = \mu_j(A_j) \mathbb{1}_{A_i}.$$

*Proof.* Direct consequence of Lemma 775 (*product of measurable subsets is measurable,  $A_1 \times A_2$  belongs to  $\Sigma$* ), Lemma 1218 (*measure of section*), Lemma 784 (*section of product*), Lemma 913 (*measure, homogeneity and nonnegativeness with  $\mu_j$* ), Lemma 421 (*zero-product property in  $\bar{\mathbb{R}}$  (measure theory)*), and **the definition of the indicator function**.  $\square$

*Remark 1220.* The next proof follows the monotone class theorem scheme (see Section 4.2).

See also the sketch of the proof in Section 5.6.

**Lemma 1221** (*measurability of measure of section (finite)*).

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Assume that  $\mu_j$  is finite. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Then, we have  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ . From Definition 227 (*product of subset systems*), Definition 774 (*tensor product of  $\sigma$ -algebras*), and Definition 642 (*generated algebra of sets*), let  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$ ,  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ , and  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\bar{\Sigma})$ . Let  $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \in \Sigma \mid F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)\}$ .



(1).  $\bar{\Sigma} \subset \mathcal{S}_i$ . Let  $A \in \bar{\Sigma}$ . Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$  such that  $A = A_1 \times A_2$ .

From Lemma 1219 (*measure of section of product*,  $F_i^A = \mu_j(A_j) \mathbb{1}_{A_i}$ ), Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*,  $\mathbb{1}_{A_i} \in \mathcal{M}_+(X_i, \Sigma_i)$ ), Lemma 913 (*measure, nonnegativeness with  $\mu_j$* ), and Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*, with  $a \stackrel{\text{def.}}{=} \mu_j(A_j)$ ),  $F_i^A$  belongs to  $\mathcal{M}_+(X_i, \Sigma_i)$ . Thus, from Lemma 775 (*product of measurable subsets is measurable*,  $A \in \Sigma$ ), and the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ . Hence, we have  $\bar{\Sigma} \subset \mathcal{S}_i$ .

(2).  $\mathcal{A} \subset \mathcal{S}_i$ . Let  $A \in \mathcal{A}$ .

Then, from Definition 774 (*tensor product of  $\sigma$ -algebras*,  $\Sigma = \Sigma_X(\bar{\Sigma})$ ), Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*, with  $G \stackrel{\text{def.}}{=} \bar{\Sigma}$ , thus  $\mathcal{A} \subset \Sigma$ ), Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*), and Lemma 775 (*product of measurable subsets is measurable*,  $\bar{\Sigma} \subset \Sigma$ ), we have  $A \in \Sigma$ , and there exist  $n \in \mathbb{N}$ , and  $(A_p)_{p \in [0..n]}$  in  $\bar{\Sigma} \subset \Sigma$  such that, for all  $p, q \in [0..n]$ ,  $p \neq q \Rightarrow A_p \cap A_q = \emptyset$ , and  $A = \biguplus_{p \in [0..n]} A_p$ .

Let  $x_i \in X_i$ . Then, from Lemma 1218 (*measure of section*), Lemma 787 (*countable union of sections is measurable*, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), Lemma 785 (*compatibility of section with set operations*, the  $s_i(x_i, A_p)$ 's are pairwise disjoint), Lemma 913 (*measure,  $\sigma$ -additivity with  $\mu_j$* ), and Definition 874 ( *$\sigma$ -additivity*, with  $I \stackrel{\text{def.}}{=} [0..n]$  and  $\mu \stackrel{\text{def.}}{=} \mu_j$ ), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j \left( \biguplus_{p \in [0..n]} s_i(x_i, A_p) \right) = \sum_{p \in [0..n]} \mu_j(s_i(x_i, A_p)) = \sum_{p \in [0..n]} F_i^{A_p}(x_i).$$

Thus, from (1) ( $A_p \in \mathcal{S}_i$ ), the definition of  $\mathcal{S}_i$ , and Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), thus  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ ), we have  $A \in \mathcal{S}_i$ . Hence, we have  $\mathcal{A} \subset \mathcal{S}_i$ .

(3).  $\mathcal{S}_i$  is monotone class. Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}_i \subset \Sigma$ .

Let  $x_i \in X_i$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 786 (*measurability of section*), Lemma 787 (*countable union of sections is measurable*), and Lemma 788 (*countable intersection of sections is measurable*), we have

$$s_i(x_i, A_n), \quad s_i \left( x_i, \bigcup_{n \in \mathbb{N}} A_n \right) = \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n), \quad s_i \left( x_i, \bigcap_{n \in \mathbb{N}} A_n \right) = \bigcap_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j.$$

Assume first that, for all  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ . Let  $x_i \in X_i$ . Then, from Lemma 1218 (*measure of section*), the definition of  $A$ , Lemma 785 (*compatibility of section with set operations*,  $(s_i(x_i, A_n))_{n \in \mathbb{N}}$  is nondecreasing), and Lemma 925 (*measure is continuous from below*, with  $\mu_j$ ), we have

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j \left( s_i \left( x_i, \bigcup_{n \in \mathbb{N}} A_n \right) \right) \\ &= \mu_j \left( \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n) \right) = \sup_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sup_{n \in \mathbb{N}} F_i^{A_n}(x_i). \end{aligned}$$

Thus, from the definition of  $\mathcal{S}_i$ , and Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum*),  $F_i^A$  belongs to  $\mathcal{M}_+(X_i, \Sigma_i)$ . Hence, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ , i.e.  $\mathcal{S}_i$  is closed under nondecreasing union.

Assume now that, for all  $n \in \mathbb{N}$ ,  $A_n \supset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n$ . Let  $x_i \in X_i$ . Then, from Lemma 1218 (*measure of section*), the definition of  $A$ , Lemma 785 (*compatibility of section with set operations*,  $(s_i(x_i, A_n))_{n \in \mathbb{N}}$  is nonincreasing), Definition 857 (*finiteness*, with  $\mu_j$ ), Lemma 858 (*finite is bounded*,  $\mu_j(s_i(x_i, A_0))$  is finite), and Lemma 927 (*measure*

is continuous from above, with  $\mu_j$  and  $n_0 \stackrel{\text{def.}}{=} 0$ ), we have

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j\left(s_i\left(x_i, \bigcap_{n \in \mathbb{N}} A_n\right)\right) \\ &= \mu_j\left(\bigcap_{n \in \mathbb{N}} s_i(x_i, A_n)\right) = \inf_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \inf_{n \in \mathbb{N}} F_i^{A_n}(x_i). \end{aligned}$$

Thus, from the definition of  $\mathcal{S}_i$ , and Lemma 847 ( $\mathcal{M}_+$  is closed under infimum),  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ . Then, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ , i.e.  $\mathcal{S}_i$  is closed under nonincreasing intersection.

Hence, from Definition 664 (monotone class),  $\mathcal{S}_i$  is a monotone class.

Therefore, from Definition 774 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\bar{\Sigma})$ ), (2), (3), and Lemma 738 (usage of monotone class theorem, with  $G \stackrel{\text{def.}}{=} \bar{\Sigma}$ ), we have for all  $A \in \Sigma$ ,  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .  $\square$

*Remark 1222.* See the sketch of next proof in Section 5.6.

**Lemma 1223 (measurability of measure of section).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Then, we have  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

*Proof.* Let  $j \stackrel{\text{def.}}{=} 3-i$ . From Lemma 886 (equivalent definition of  $\sigma$ -finiteness, with  $\mu_j$ ), there exists  $(B_{j,n})_{n \in \mathbb{N}} \in \Sigma_j$  such that, for all  $n \in \mathbb{N}$ ,  $B_{j,n} \subset B_{j,n+1}$ ,  $\mu_j(B_{j,n}) < \infty$ , and  $X_j = \bigcup_{n \in \mathbb{N}} B_{j,n}$ .

For all  $n \in \mathbb{N}$ , for all  $A \in \Sigma_1 \otimes \Sigma_2$ , from Lemma 931 (restricted measure, with  $Y \stackrel{\text{def.}}{=} B_{j,n}$ ), and Lemma 1218 (measure of section, with  $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$ ), let  $\mu_{j,n}$  be the restricted measure, and  $F_{i,n}^A$  be the measure of  $i$ -th section of  $A$  defined by

$$\mu_{j,n} \stackrel{\text{def.}}{=} (\mu_j)_{B_{j,n}}' = (A_j \in \Sigma_j \mapsto \mu_j(A_j \cap B_{j,n})), \quad \text{and} \quad F_{i,n}^A \stackrel{\text{def.}}{=} (x_i \in X_i \mapsto \mu_{j,n}(s_i(x_i, A))).$$

Let  $n \in \mathbb{N}$ . Then, from Definition 857 (finiteness,  $\mu_{j,n}(X) = \mu_j(B_{j,n}) < \infty$ ),  $\mu_{j,n}$  is a finite measure on  $(X_j, \Sigma_j)$ . Moreover, from Lemma 1221 (measurability of measure of section (finite), with  $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$  finite), we have  $F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

Let  $x_i \in X_i$ . Then, from Lemma 786 (measurability of section), the definition of  $B_{j,n}$ , and **distributivity of intersection over union**, we have

$$\Sigma_j \ni s_i(x_i, A) = s_i(x_i, A) \cap \bigcup_{n \in \mathbb{N}} B_{j,n} = \bigcup_{n \in \mathbb{N}} (s_i(x_i, A) \cap B_{j,n}).$$

Thus, from Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $\text{card}(I) = 2$ )), we have  $s_i(x_i, A) \cap B_{j,n} \in \Sigma_j$ . Hence, from Lemma 1218 (measure of section), **monotonicity of intersection ( $(s_i(x_i, A) \cap B_{j,n})_{n \in \mathbb{N}}$  is nondecreasing)**, Lemma 925 (measure is continuous from below, with  $\mu_j$  and  $A_n \stackrel{\text{def.}}{=} s_i(x_i, A) \cap B_{j,n}$ ), and the definition of  $\mu_{j,n}$ ,

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j\left(s_i(x_i, A) \cap \bigcup_{n \in \mathbb{N}} B_{j,n}\right) = \mu_j\left(\bigcup_{n \in \mathbb{N}} (s_i(x_i, A) \cap B_{j,n})\right) \\ &= \sup_{n \in \mathbb{N}} \mu_j(s_i(x_i, A) \cap B_{j,n}) = \sup_{n \in \mathbb{N}} \mu_{j,n}(s_i(x_i, A)) = \sup_{n \in \mathbb{N}} F_{i,n}^A(x_i). \end{aligned}$$

Therefore, from Lemma 848 ( $\mathcal{M}_+$  is closed under supremum), we have

$$F_i^A = \sup_{n \in \mathbb{N}} F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i).$$

$\square$

**Definition 1224 (tensor product measure).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . A measure  $\mu$  on the product measurable space  $(X, \Sigma)$  is called *tensor product measure (on  $(X, \Sigma)$  relying on  $\mu_1$  and  $\mu_2$ )* iff

$$(15.59) \quad \forall A_1 \in \Sigma_1, \forall A_2 \in \Sigma_2, \quad \mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2).$$

**Definition 1225 (candidate tensor product measure).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $i \in \{1, 2\}$ . The function defined by

$$(15.60) \quad \forall A \in \Sigma_1 \otimes \Sigma_2, \quad (\mu_1 \otimes \mu_2)^i(A) \stackrel{\text{def.}}{=} \int F_i^A d\mu_i$$

is called *candidate tensor product measure*.

*Remark 1226.* See the sketch of next proof in Section 5.6.

**Lemma 1227 (candidate tensor product measure is tensor product measure).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $i \in \{1, 2\}$ . Then, the candidate tensor product measure  $(\mu_1 \otimes \mu_2)^i$  is a tensor product measure on  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ .

**Measure.** Let  $j \stackrel{\text{def.}}{=} 3 - i$ .

Let  $A \in \Sigma$ . Then, from Lemma 1218 (*measure of section*), Definition 1225 (*candidate tensor product measure*), Lemma 1223 (*measurability of measure of section*, with the  $\sigma$ -finite measure  $\mu_i$ ), Lemma 1172 (*integral in  $\mathcal{M}_+$* , nonnegativeness, with  $X \stackrel{\text{def.}}{=} X_i$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_i$ ), and Lemma 1218 (*measure of section*), the value  $(\mu_1 \otimes \mu_2)^i(A) = \int \mu_j(s_i(x_i, A)) d\mu_i$  is well-defined in  $\overline{\mathbb{R}}_+$ .

From Lemma 785 (*compatibility of section with set operations*,  $\emptyset$ ), Lemma 913 (*measure*, homogeneity with  $\mu_j$ ), and Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*), we have

$$(\mu_1 \otimes \mu_2)^i(\emptyset) = \int \mu_j(s_i(x_i, \emptyset)) d\mu_i = \int \mu_j(\emptyset) d\mu_i = \int 0 d\mu_i = 0.$$

Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma$ . Assume that the  $A_n$ 's are pairwise disjoint. Let  $A \stackrel{\text{def.}}{=} \bigsqcup_{n \in \mathbb{N}} A_n$ . Let  $x_i$  be in  $X_i$ . Let  $n \in \mathbb{N}$ . Let  $p, q \in \mathbb{N}$ . Assume that  $p \neq q$ . Then, from Lemma 786 (*measurability of section*), Lemma 787 (*countable union of sections is measurable*, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$ ), and Lemma 785 (*compatibility of section with set operations*, with intersection and  $\emptyset$ ),  $s_i(x_i, A_n) \in \Sigma_j$ , and

$$\begin{aligned} s_i(x_i, A) &= s_i\left(x_i, \bigcup_{n \in \mathbb{N}} A_n\right) = \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j, \\ s_i(x_i, A_p) \cap s_i(x_i, A_q) &= s_i(x_i, A_p \cap A_q) = s_i(x_i, \emptyset) = \emptyset. \end{aligned}$$

Thus, from Lemma 1218 (*measure of section*), Lemma 913 (*measure*,  $\sigma$ -additivity with  $\mu_j$ ), and Definition 874 ( *$\sigma$ -additivity*), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j\left(\bigsqcup_{n \in \mathbb{N}} s_i(x_i, A_n)\right) = \sum_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sum_{n \in \mathbb{N}} F_i^{A_n}(x_i).$$

Then, from Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*), and Definition 1225 (*candidate tensor product measure*), we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \sum_{n \in \mathbb{N}} F_i^{A_n} d\mu_i = \sum_{n \in \mathbb{N}} \int F_i^{A_n} d\mu_i = \sum_{n \in \mathbb{N}} (\mu_1 \otimes \mu_2)^i(A_n).$$

Hence, from Definition 874 ( $\sigma$ -additivity),  $(\mu_1 \otimes \mu_2)^i$  is  $\sigma$ -additive.

Therefore, from Lemma 913 (*measure*),  $(\mu_1 \otimes \mu_2)^i$  is a measure on  $(X, \Sigma)$ .

**Identity.** Let  $A_1 \in \Sigma_1$ . Let  $A_2 \in \Sigma_2$ . Let  $A \stackrel{\text{def.}}{=} A_1 \times A_2$ .

Then, from Lemma 775 (*product of measurable subsets is measurable*,  $A \in \Sigma$ ), Definition 1225 (*candidate tensor product measure*), Lemma 1219 (*measure of section of product*), Lemma 913 (*measure*, *nonnegativity with  $\mu_j$* ), Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*, with  $a \stackrel{\text{def.}}{=} \mu_j(A_j)$  in  $\bar{\mathbb{R}}_+$ ), Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*, with  $\mu \stackrel{\text{def.}}{=} \mu_i$  and  $A_i \in \Sigma_i$ ), Lemma 418 (*multiplication in  $\bar{\mathbb{R}}$  is commutative (measure theory)*), and since  $\{1, 2\} = \{i, j\}$ , we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \mu_j(A_j) \mathbb{1}_{A_i} d\mu_i = \mu_j(A_j) \mu_i(A_i) = \mu_1(A_1) \mu_2(A_2),$$

Therefore, from Definition 1224 (*tensor product measure*),  $(\mu_1 \otimes \mu_2)^i$  is a tensor product measure.  $\square$

**Lemma 1228 (tensor product of finite measures).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be finite measure spaces. Let  $\mu$  be a tensor product measure on  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$  relying on  $\mu_1$  and  $\mu_2$ . Then,  $\mu$  is finite.

*Proof.* Direct consequence of Lemma 913 (*measure*), Lemma 697 (*equivalent definition of  $\sigma$ -algebra*,  $X_1 \in \Sigma_1$ ,  $X_2 \in \Sigma_2$ , and  $X_1 \times X_2 \in \Sigma_1 \otimes \Sigma_2$ ), Definition 1224 (*tensor product measure*), Definition 857 (*finiteness*), and **closedness of multiplication in  $\mathbb{R}_+$** .  $\square$

**Lemma 1229 (tensor product of  $\sigma$ -finite measures).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $\mu$  be a tensor product measure on  $(X, \Sigma)$  relying on  $\mu_1$  and  $\mu_2$ . Then,  $\mu$  is  $\sigma$ -finite.

In particular, let  $i \in \{1, 2\}$ . Let  $(B_{i,n})_{n \in \mathbb{N}} \in \Sigma_i$  such that

$$(15.61) \quad (\forall n \in \mathbb{N}, \quad B_{i,n} \subset B_{i,n+1} \quad \wedge \quad \mu_i(B_{i,n}) < \infty) \quad \wedge \quad X_i = \bigcup_{n \in \mathbb{N}} B_{i,n}.$$

For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$ . Then, we have

$$(15.62) \quad (\forall n \in \mathbb{N}, \quad B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) < \infty) \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} B_n.$$

*Proof.* Existence of the  $B_{i,n}$ 's comes from Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*).

Let  $n \in \mathbb{N}$ . Then, from the definition of the  $B_n$ 's, Lemma 775 (*product of measurable subsets is measurable*), **monotonicity of Cartesian product**, Definition 1224 (*tensor product measure*), and **closedness of multiplication in  $\mathbb{R}_+$** , we have

$$B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) = \mu_1(B_{1,n}) \mu_2(B_{2,n}) < \infty.$$

Moreover, from **monotonicity of union**, we have  $\bigcup_{n \in \mathbb{N}} B_n \subset X$ . Conversely, let  $(x_1, x_2)$  be in  $X$ . Let  $i \in \{1, 2\}$ . Then, from Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*, with  $\mu_i$ ), there exists  $n_i \in \mathbb{N}$  such that  $x_i \in B_{i,n_i}$ . Thus, from monotonicity of the  $B_{i,n}$ 's,  $x_i$  belongs to  $B_{i, \max(n_1, n_2)}$ . Then, from the definition of the  $B_n$ 's, we have  $(x_1, x_2) \in B_{\max(n_1, n_2)}$ , i.e.  $X$  is included in  $\bigcup_{n \in \mathbb{N}} B_n$ . Hence, we have  $X = \bigcup_{n \in \mathbb{N}} B_n$ .

Therefore, from Definition 885 ( *$\sigma$ -finiteness*),  $\mu$  is  $\sigma$ -finite.  $\square$

*Remark 1230.* The uniqueness part of the next proof reproduces the schemes of the proof of Lemma 1221, it follows the monotone class theorem scheme (see Section 4.2).

See also the sketch of the proof in Section 5.6.

**Lemma 1231 (uniqueness of tensor product measure (finite)).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be finite measure spaces. Then, there exists a unique tensor product measure on the product measurable space  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ .

*Proof. Existence.* Direct consequence of Lemma 887 (finite is  $\sigma$ -finite, thus  $\mu_1$  and  $\mu_2$  are  $\sigma$ -finite), and Lemma 1227 (candidate tensor product measure is tensor product measure).

**Uniqueness.** Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $m$  and  $\tilde{m}$  be tensor product measures on  $(X, \Sigma)$  relying on  $\mu_1$  and  $\mu_2$ . Let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid m(A) = \tilde{m}(A)\}$ . Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\bar{\Sigma})$ .

(1).  $\bar{\Sigma} \subset \mathcal{S}$ . Direct consequence of Lemma 775 (product of measurable subsets is measurable), Definition 1224 (tensor product measure, with  $m$  and  $\tilde{m}$ ), and the definition of  $\mathcal{S}$ .

(2).  $\mathcal{A} \subset \mathcal{S}$ . Let  $A \in \mathcal{A}$ .

Then, from Definition 774 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\bar{\Sigma})$ ), Lemma 719 ( $\sigma$ -algebra contains algebra of sets, with  $G \stackrel{\text{def.}}{=} \bar{\Sigma}$ , thus  $\mathcal{A} \subset \Sigma$ ), Lemma 730 (algebra of sets generated by product of  $\sigma$ -algebras), and Lemma 775 (product of measurable subsets is measurable,  $\bar{\Sigma} \subset \Sigma$ ), we have  $A \in \Sigma$ , and there exist  $n \in \mathbb{N}$ , and  $(A_p)_{p \in [0..n]} \in \bar{\Sigma} \subset \Sigma$  such that, for all  $p, q \in [0..n]$ ,  $p \neq q$  implies emptiness of  $A_p \cap A_q$ , and  $A = \biguplus_{p \in [0..n]} A_p$ . Thus, from Lemma 926 (equivalent definition of measure (from below), additivity for  $m$  and  $\tilde{m}$ ), Definition 862 (additivity), (1) ( $A_p \in \mathcal{S}$ ), the definition of  $\mathcal{S}$ , and Definition 317 (addition in  $\bar{\mathbb{R}}$ , addition is a function (same arguments yield the same result)), we have

$$m(A) = m\left(\biguplus_{p \in [0..n]} A_p\right) = \sum_{p \in [0..n]} m(A_p) = \sum_{p \in [0..n]} \tilde{m}(A_p) = \tilde{m}\left(\biguplus_{p \in [0..n]} A_p\right) = \tilde{m}(A).$$

Then, from the definition of  $\mathcal{S}$ , we have  $A \in \mathcal{S}$ . Hence, we have  $\mathcal{A} \subset \mathcal{S}$ .

(3).  $\mathcal{S}$  is monotone class. Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S} \subset \Sigma$ .

Then, from Lemma 697 (equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection), we have  $\bigcup_{n \in \mathbb{N}} A_n, \bigcap_{n \in \mathbb{N}} A_n \in \Sigma$ .

Assume first that, for all  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ .

Then, from Lemma 925 (measure is continuous from below, with  $m$  and  $\tilde{m}$ ), the definition of  $\mathcal{S}$ , and Definition 2 (supremum, supremum is a function), we have

$$m(A) = m\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sup_{n \in \mathbb{N}} m(A_n) = \sup_{n \in \mathbb{N}} \tilde{m}(A_n) = \tilde{m}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \tilde{m}(A).$$

Hence, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}$ , i.e.  $\mathcal{S}$  is closed under nondecreasing union.

Assume now that, for all  $n \in \mathbb{N}$ ,  $A_n \supset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n$ .

Then, from Lemma 1228 (tensor product of finite measures, with  $m$  and  $\tilde{m}$ , both relying on finite measures  $\mu_1$  and  $\mu_2$ ), Lemma 858 (finite is bounded,  $m(A_0), \tilde{m}(A_0) < \infty$ ), Lemma 927 (measure is continuous from above, with  $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$  and  $n_0 \stackrel{\text{def.}}{=} 0$ ), the definition of  $\mathcal{S}$ , and Definition 9 (infimum, infimum is a function), we have

$$m(A) = m\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} m(A_n) = \inf_{n \in \mathbb{N}} \tilde{m}(A_n) = \tilde{m}\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \tilde{m}(A).$$

Thus, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}$ , i.e.  $\mathcal{S}$  is closed under nonincreasing intersection.

Hence, from Definition 664 (monotone class),  $\mathcal{S}$  is a monotone class.

Therefore, from Definition 774 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\bar{\Sigma})$ ), (2), (3), and Lemma 738 (usage of monotone class theorem, with  $G \stackrel{\text{def.}}{=} \bar{\Sigma}$ ), we have  $m = \tilde{m}$ .  $\square$

*Remark 1232.*

The uniqueness part of the next proof reproduces the schemes of the proof of Lemma 1223.

See also the sketch of the proof in Section 5.6.

**Lemma 1233 (uniqueness of tensor product measure).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Then, there exists a unique tensor product measure on  $(X, \Sigma)$ .

This measure is denoted  $\mu_1 \otimes \mu_2$ , and for all  $i \in \{1, 2\}$  with  $j \stackrel{\text{def.}}{=} 3 - i$ , we have

$$(15.63) \quad \forall A \in \Sigma, \quad (\mu_1 \otimes \mu_2)(A) = \int \mu_j(s_i(x_i, A)) d\mu_i.$$

It is called the (tensor) product measure of  $\mu_1$  and  $\mu_2$ , and  $(X, \Sigma, \mu_1 \otimes \mu_2)$  is called (tensor) product measure space.

*Proof. Existence.*

Direct consequence of Lemma 1227 (candidate tensor product measure is tensor product measure).

**Uniqueness.** Let  $m$  and  $\tilde{m}$  be tensor product measures on  $(X, \Sigma)$  relying on  $\mu_1$  and  $\mu_2$ .

From Lemma 886 (equivalent definition of  $\sigma$ -finiteness), for all  $i \in \{1, 2\}$ , there exists  $(B_{i,n})_{n \in \mathbb{N}}$  in  $\Sigma_i$  such that, for all  $n \in \mathbb{N}$ ,  $B_{i,n} \subset B_{i,n+1}$ ,  $\mu_i(B_{i,n}) < \infty$ , and  $X_i = \bigcup_{n \in \mathbb{N}} B_{i,n}$ . For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$ . Then, from Lemma 1229 (tensor product of  $\sigma$ -finite measures, with  $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$ ), we have for all  $n \in \mathbb{N}$ ,  $B_n \in \Sigma$ ,  $B_n \subset B_{n+1}$ ,  $m(B_n), \tilde{m}(B_n) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} B_n$ .

For all  $i \in \{1, 2\}$ , for all  $n \in \mathbb{N}$ , from Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $(A \in \Sigma$  implies  $A \cap B_n \in \Sigma)$ ), and Lemma 931 (restricted measure, with  $Y \stackrel{\text{def.}}{=} B_{i,n}$  and  $\mu \stackrel{\text{def.}}{=} \mu_i$ , then  $Y \stackrel{\text{def.}}{=} B_n$  and  $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$ ), let  $\mu_{i,n}, m_n$  and  $\tilde{m}_n$  be the restricted measures defined by

$$\begin{aligned} \mu_{i,n} &\stackrel{\text{def.}}{=} (\mu_i)'_{B_{i,n}} = (A_i \in \Sigma_i \mapsto \mu_i(A_i \cap B_{i,n})), \\ m_n &\stackrel{\text{def.}}{=} m'_{B_n} = (A \in \Sigma \mapsto m(A \cap B_n)), \quad \text{and} \quad \tilde{m}_n &\stackrel{\text{def.}}{=} (\tilde{m})'_{B_n} = (A \in \Sigma \mapsto \tilde{m}(A \cap B_n)). \end{aligned}$$

Let  $i \in \{1, 2\}$ . Let  $n \in \mathbb{N}$ . Then, from Definition 857 (finiteness, with  $\mu_{i,n}(X_i) = \mu_i(B_{i,n})$ ,  $m_n(X) = m(B_n)$  and  $\tilde{m}_n(X) = \tilde{m}(B_n)$  finite),  $\mu_{i,n}, m_n$  and  $\tilde{m}_n$  are finite measures.

Let  $n \in \mathbb{N}$ . Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ . Then, from Lemma 775 (product of measurable subsets is measurable,  $A_1 \times A_2 \in \Sigma$ ), the definitions of  $m_n, B_n, \mu_{1,n}, \mu_{2,n}$  and  $\tilde{m}_n$ , **compatibility of intersection with Cartesian product**, and Definition 1224 (tensor product measure, with  $\mu \stackrel{\text{def.}}{=} m, \tilde{m}$ ), we have

$$\begin{aligned} m_n(A_1 \times A_2) &= m((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = m((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) \\ &= \mu_1(A_1 \cap B_{1,n}) \mu_2(A_2 \cap B_{2,n}) = \mu_{1,n}(A_1) \mu_{2,n}(A_2) \\ &= \tilde{m}((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) = \tilde{m}((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = \tilde{m}_n(A_1 \times A_2). \end{aligned}$$

Thus, from Definition 1224 (tensor product measure, with  $\mu_1 \stackrel{\text{def.}}{=} \mu_{1,n}, \mu_2 \stackrel{\text{def.}}{=} \mu_{2,n}$  and  $\mu \stackrel{\text{def.}}{=} m_n$  and  $\tilde{m}_n$ ),  $m_n$  and  $\tilde{m}_n$  are tensor product measures on  $(X, \Sigma)$  both relying on finite measures  $\mu_{1,n}$  and  $\mu_{2,n}$ . Hence, from Lemma 1231 (uniqueness of tensor product measure (finite)), we have  $m_n = \tilde{m}_n$ .

Let  $A \in \Sigma$ . Then, from Lemma 697 (equivalent definition of  $\sigma$ -algebra,  $A \cap B_n \in \Sigma$ , then  $\bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \Sigma$ ), **distributivity of intersection over union, monotonicity of intersection** ( $(A \cap B_n)_{n \in \mathbb{N}}$  is nondecreasing), Lemma 925 (measure is continuous from below, with  $m, \tilde{m}$  and  $A_n \stackrel{\text{def.}}{=} A \cap B_n$ ), the definition of  $m_n$  and  $\tilde{m}_n$ , and Definition 2 (supremum, supremum is a



function), we have

$$\begin{aligned} m(A) &= m\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = m\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) \\ &= \sup_{n \in \mathbb{N}} m(A \cap B_n) = \sup_{n \in \mathbb{N}} m_n(A) = \sup_{n \in \mathbb{N}} \tilde{m}_n(A) = \sup_{n \in \mathbb{N}} \tilde{m}(A \cap B_n) \\ &= \tilde{m}\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) = \tilde{m}\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = \tilde{m}(A). \end{aligned}$$

Therefore, we have  $m = \tilde{m}$ .

**Identity.** Direct consequence of the uniqueness result above, Lemma 1227 (*candidate tensor product measure is tensor product measure*), Definition 1225 (*candidate tensor product measure*), and Lemma 1218 (*measure of section*).  $\square$

**Lemma 1234 (negligibility of measurable section).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ . Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then,  $A$  is  $\mu$ -negligible iff  $\mu(A) = 0$  iff for almost all  $x_i \in X_i$ ,  $\mu_j(s_i(x_i, A)) = 0$  iff for almost all  $x_i \in X_i$ ,  $s_i(x_i, A)$  is  $\mu_j$ -negligible.

*Proof.* Direct consequence of Lemma 786 (*measurability of section,  $s_i(x_i, A) \in \Sigma_j$* ), Lemma 948 (*negligibility of measurable subset, with  $A$ , then  $s_i(x_i, A)$* ), Lemma 1233 (*uniqueness of tensor product measure*), Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite, with  $\mu_i$  and  $(x_i \mapsto \mu_j(s_i(x_i, A)))$* ).  $\square$

## 15.4.2 Lebesgue measure over product space

**Lemma 1235 (Lebesgue measure on  $\mathbb{R}^2$ ).**

There is a unique measure on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$  that generalizes the area of bounded open boxes.

It is denoted  $\lambda^{\otimes 2} \stackrel{\text{def.}}{=} \lambda \otimes \lambda$ , and it is called the Lebesgue measure on (Borel subsets of)  $\mathbb{R}^2$ .

*Proof.* Direct consequence of Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ), Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*), and Lemma 1233 (*uniqueness of tensor product measure*).  $\square$

**Lemma 1236 (Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes).**

Let  $a_1, b_1, a_2, b_2 \in \mathbb{R}$ . Assume that  $a_1 \leq b_1$  and  $a_2 \leq b_2$ . Then, we have

$$(15.64) \quad \lambda^{\otimes 2}(\llbracket a_1, b_1 \rrbracket \times \llbracket a_2, b_2 \rrbracket) = (b_1 - a_1)(b_2 - a_2).$$

*Proof.* Direct consequence of Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ), Definition 1224 (*tensor product measure*), Lemma 1092 (*Lebesgue measure generalizes length of interval*), and **the definition of the area of boxes of  $\mathbb{R}^2$** .  $\square$

**Lemma 1237 (Lebesgue measure on  $\mathbb{R}^2$  is zero on lines).**

Let  $A \subset \mathbb{R}^2$  be a box (i.e. the Cartesian product of two intervals).

Then, we have  $\lambda^{\otimes 2}(A) = 0$  iff  $A$  is a line (i.e. at least one of the two intervals is a singleton).

*Proof.* Direct consequence of Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes, with  $a_1 = b_1$  or  $a_2 = b_2$* ), and Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*).  $\square$

**Lemma 1238 (Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite).**

The measure space  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$  is  $\sigma$ -finite.

*Proof.* Direct consequence of Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ), Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*), and Lemma 1229 (*tensor product of  $\sigma$ -finite measures*).  $\square$

**Lemma 1239 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*).**

*The measure space  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$  is diffuse.*

*Proof.* Direct consequence of Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes, with  $a_1 = b_1$  and  $a_2 = b_2$* ), and Definition 856 (*diffuseness,  $\{(a_1, a_2)\} = [a_1, a_1] \times [a_2, a_2]$* ).  $\square$

### 15.4.3 Integration over product space of nonnegative function

**Definition 1240 (*partial function of function from product space*).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $f : X_1 \times X_2 \rightarrow \overline{\mathbb{R}}_+$ .

Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $\psi \stackrel{\text{def.}}{=} ((x_i, x_j) \mapsto (x_1, x_2))$ .

For all  $x_i \in X_i$ ,  $f_{x_i}$  denotes the function  $(x_j \mapsto f \circ \psi(x_i, x_j))$  from  $X_j$  to  $\overline{\mathbb{R}}_+$ .

Moreover, when  $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$ ,  $I_{f,i}$  denotes the function  $(x_i \mapsto \int f_{x_i} d\mu_j)$  from  $X_i$  to  $\overline{\mathbb{R}}_+$ .

*Remark 1241.* The next proof follows steps 1 to 3 of the Lebesgue scheme (see Section 4.4).

See also the sketch of the proof in Section 5.5.

**Theorem 1242 (*Tonelli*).** *Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces.*

*Let  $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ .*

*Then, for all  $x_i \in X_i$ ,  $f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$ ,  $I_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have*

$$(15.65) \quad \int f d(\mu_1 \otimes \mu_2) = \int I_{f,i} d\mu_i \quad (\text{in } \overline{\mathbb{R}}_+).$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ , and  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ . Let  $x_i \in X_i$ .

Let  $P$  be the predicate defined on functions from  $X$  to  $\overline{\mathbb{R}}_+$  by

$$f_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j) \quad \wedge \quad I_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i) \quad \wedge \quad \int f d\mu = \int I_{f,i} d\mu_i.$$

(1).  $P$  holds on  $\mathcal{IF}(X, \Sigma)$ . Let  $f \in \mathcal{IF}(X, \Sigma)$ .

Let  $A \stackrel{\text{def.}}{=} \{f \neq 0\}$ . Then, from Lemma 1100 (*indicator and support are inverse*), we have

$A \in \Sigma$  and  $f = \mathbb{1}_A$ . Then, from Definition 1240 (*partial function of function from product space*), Lemma 789 (*indicator of section*), Lemma 786 (*measurability of section,  $s_i(x_i, A) \in \Sigma_j$* ),

and Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function, with  $A \stackrel{\text{def.}}{=} s_i(x_i, A)$  and  $\mu_j$* ), we have

$$f_{x_i} = \mathbb{1}_{s_i(x_i, A)} \in \mathcal{M}_+(X_j, \Sigma_j), \quad \text{and} \quad I_{f,i} = \int \mathbb{1}_{s_i(x_i, A)} d\mu_j = \mu_j(s_i(x_i, A)).$$

Hence, from Lemma 1218 (*measure of section*), and Lemma 1223 (*measurability of measure of section*), we have  $I_{f,i} = F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

Therefore, from Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function, with  $\mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$  and  $\mu$* ), and Lemma 1233 (*uniqueness of tensor product measure*), we have

$$\int f d\mu = \mu(A) = \int F_i^A d\mu_i = \int I_{f,i} d\mu_i.$$

(2).  $P$  holds on  $\mathcal{SF}_+(X, \Sigma)$ .

Let  $f \in \mathcal{IF}(X, \Sigma)$ ,  $g \in \mathcal{SF}_+(X, \Sigma)$  and  $a \in \mathbb{R}_+$ . Assume that  $P(f)$  and  $P(g)$  hold.

Then, from **left linearity of composition**, Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*,



with  $\mu_j$  and  $(f+g)_{x_i}$ , then  $(af)_{x_i}$ , Lemma 843 ( $\mathcal{M}_+$  is closed under addition, with  $\mu_i$ ), and Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication, with  $\mu_i$ ), we have

$$(f+g)_{x_i} = f_{x_i} + g_{x_i}, \quad (af)_{x_i} = af_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j), \\ I_{f+g,i} = I_{f,i} + I_{g,i}, \quad I_{af,i} = aI_{f,i} \in \mathcal{M}_+(X_i, \Sigma_i).$$

Hence, from Lemma 1128 ( $\mathcal{SF}_+$  simple representation,  $f, af \in \mathcal{SF}(X, \Sigma)$ ), Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations,  $f+g \in \mathcal{SF}_+(X, \Sigma)$ ), Lemma 1124 ( $\mathcal{SF}_+$  is measurable,  $f, g, f+g, af \in \mathcal{M}_+(X, \Sigma)$ ), and Lemma 1184 (integral in  $\mathcal{M}_+$  is positive linear, with  $\mu$  and  $\mu_i$ ),

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu = \int I_{f,i} d\mu_i + \int I_{g,i} d\mu_i = \int I_{f+g,i} d\mu_i, \\ \int af d\mu = a \int f d\mu = a \int I_{f,i} d\mu_i = \int I_{af,i} d\mu_i,$$

i.e.  $P$  is compatible with positive linear combination of indicator functions.

Therefore, from (1), and Lemma 1134 (Lebesgue scheme step 2),  $P$  holds on  $\mathcal{SF}_+(X, \Sigma)$ .

**(3).  $P$  holds on  $\mathcal{M}_+(X, \Sigma)$ .**

Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$ . Assume that  $(f_n)_{n \in \mathbb{N}}$  is nondecreasing and  $P(f_n)$  holds for all  $n \in \mathbb{N}$ . Then, from Lemma 381 (limit is monotone), **composition law for the limit**, Theorem 1180 (Beppo Levi, monotone convergence, with  $\mu_j$  and  $(\lim_{n \rightarrow \infty} f_n)_{x_i}$ ), and Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent, with  $\mu_i$ ), we have

$$((f_n)_{x_i})_{n \in \mathbb{N}} \text{ is nondecreasing, } \left( \lim_{n \rightarrow \infty} f_n \right)_{x_i} = \lim_{n \rightarrow \infty} (f_n)_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j), \\ I_{\lim_{n \rightarrow \infty} f_n, i} = \int \left( \lim_{n \rightarrow \infty} f_n \right)_{x_i} d\mu_j = \lim_{n \rightarrow \infty} \int (f_n)_{x_i} d\mu_j = \lim_{n \rightarrow \infty} I_{f_n, i} \in \mathcal{M}_+(X_i, \Sigma_i).$$

Hence, from Theorem 1180 (Beppo Levi, monotone convergence, with  $\mu$  and  $\mu_j$ ), we have

$$\int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int I_{f_n, i} d\mu_i = \int I_{\lim_{n \rightarrow \infty} f_n, i} d\mu_i,$$

i.e.  $P$  is compatible with the limit of nondecreasing nonnegative simple functions.

Therefore, from (2), and Lemma 1141 (Lebesgue scheme step 3),  $P$  holds on  $\mathcal{M}_+(X, \Sigma)$ .  $\square$

**Lemma 1243 (Tonelli over subset).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $A \subset Y \subset X_1 \times X_2$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \bar{\mathbb{R}}$ . Assume that  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ . Let  $i \in \{1, 2\}$  with  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $\psi \stackrel{\text{def.}}{=} ((x_i, x_j) \mapsto (x_1, x_2))$ . For all  $x_i \in X_i$ , let  $A_{x_i} \stackrel{\text{def.}}{=} s_i(x_i, A)$ , and  $f_{x_i}^A \stackrel{\text{def.}}{=} (x_j \in A_{x_i} \mapsto f \circ \psi(x_i, x_j))$ . Let  $I_{f,i}^A \stackrel{\text{def.}}{=} (x_i \in X_i \mapsto \int_{A_{x_i}} f_{x_i}^A d\mu_j)$ . Then, for all  $x_i \in X_i$ ,  $f_{x_i}^A \in \mathcal{M}_+(A_{x_i}, \Sigma_j \cap A_{x_i})$ ,  $I_{f,i}^A \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have

$$(15.66) \quad \int_A f d(\mu_1 \otimes \mu_2) = \int I_{f,i}^A d\mu_i.$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ . Let  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ . Let  $\hat{f} : X \rightarrow \bar{\mathbb{R}}_+$  such that  $\hat{f}|_Y = f$ . Then, from Lemma 1199 (integral in  $\mathcal{M}_+$  over subset), we have  $\hat{f} \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$  and  $\int_A f d\mu = \int \hat{f} \mathbb{1}_A d\mu$ . Hence, from Theorem 1242 (Tonelli, with  $f \stackrel{\text{def.}}{=} \hat{f} \mathbb{1}_A$ ), and Definition 1240 (partial function of function from product space), for all  $x_i \in X_i$ ,  $(\hat{f} \mathbb{1}_A)_{x_i} \in \mathcal{M}_+(X_j, \Sigma_j)$ ,  $I_{\hat{f} \mathbb{1}_A, i} \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have  $\int \hat{f} \mathbb{1}_A d\mu = \int I_{\hat{f} \mathbb{1}_A, i} d\mu_i$ .

Let  $x_i \in X_i$ . Then, by construction, and from Lemma 789 (indicator of section), we have  $f_{x_i}^A = (\hat{f} \mathbb{1}_A)_{x_i} = \hat{f}_{x_i} \mathbb{1}_{A_{x_i}}$ . Therefore, from Lemma 1199 (integral in  $\mathcal{M}_+$  over subset, with  $\mu_j$ ), we have  $f_{x_i}^A \in \mathcal{M}_+(A_{x_i}, \Sigma_j \cap A_{x_i})$ ,  $I_{f,i}^A = \int (\hat{f} \mathbb{1}_A)_{x_i} d\mu_j = I_{\hat{f} \mathbb{1}_A, i}$ , i.e.  $I_{f,i}^A \in \mathcal{M}_+(X_i, \Sigma_i)$  and  $\int_A f d\mu = \int I_{f,i}^A d\mu_i$ .  $\square$

**Lemma 1244 (Tonelli for tensor product).**

For all  $i \in \{1, 2\}$ , let  $(X_i, \Sigma_i, \mu_i)$  be a  $\sigma$ -finite measure space, and  $f_i \in \mathcal{M}_+(X_i, \Sigma_i)$ . Then,  $f_1 \otimes f_2 \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ , and we have

$$(15.67) \quad \int (f_1 \otimes f_2) d(\mu_1 \otimes \mu_2) = \left( \int f_1 d\mu_1 \right) \left( \int f_2 d\mu_2 \right).$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ ,  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ , and  $f \stackrel{\text{def.}}{=} f_1 \otimes f_2$ .

Then, from Definition 851 (*tensor product of numeric functions*), Lemma 853 (*measurability of tensor product of numeric functions, with  $\mathbb{K} \stackrel{\text{def.}}{=} \overline{\mathbb{R}}_+$* ), Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*), and Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*),  $f$  belongs to  $\mathcal{M}_+(X, \Sigma)$ .

Then, from Theorem 1242 (*Tonelli, with  $i \stackrel{\text{def.}}{=} 1$* ), Definition 1240 (*partial function of function from product space*), Definition 851 (*tensor product of numeric functions*), Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $f \stackrel{\text{def.}}{=} f_2$  and  $a \stackrel{\text{def.}}{=} f_1(x_1) \in \overline{\mathbb{R}}_+$* ), and Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*), we have

$$\int f d\mu = \int I_{f,1} d\mu_1 \quad \text{where } \forall x_1 \in X_1, I_{f,1}(x_1) \stackrel{\text{def.}}{=} \int f_1(x_1) f_2 d\mu_2 = \left( \int f_2 d\mu_2 \right) f_1(x_1).$$

Therefore, from Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $a \stackrel{\text{def.}}{=} \int f_2 d\mu_2$  belongs to  $\overline{\mathbb{R}}_+$  and  $f \stackrel{\text{def.}}{=} f_1$* ), and Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*), we have

$$\int f d\mu = \int \left( \int f_2 d\mu_2 \right) f_1 d\mu_1 = \left( \int f_1 d\mu_1 \right) \left( \int f_2 d\mu_2 \right).$$

□



## Chapter 16

# Lebesgue integration of real function

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*Remark 1245.* From now on, the expressions involving integrals are taken in  $\mathbb{R}$ , thus functions also take almost all their values in  $\mathbb{R}$ .

## 16.1 Definition of the integral

*Remark 1246.* This section follows step 4 of the Lebesgue scheme (see Section 4.4).

**Definition 1247 (integrability).** Let  $(X, \Sigma, \mu)$  be a measure space. A function  $f : X \rightarrow \overline{\mathbb{R}}$  is said  $\mu$ -integrable (in  $\mathcal{M}$ ) iff  $f^+$  and  $f^-$  are  $\mu$ -integrable in  $\mathcal{M}_+$ .  
Moreover, when  $f$  is finite and  $\mu$ -integrable, we may say that it is  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ .

**Lemma 1248 (integrable is measurable).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f$  is  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have  $f \in \mathcal{M}$ .

*Proof.* Direct consequence of Definition 1247 (integrability), Lemma 1172 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability with  $f^+$  and  $f^-$ ), and Lemma 836 (measurability of nonnegative and nonpositive parts).  $\square$

**Lemma 1249 (equivalent definition of integrability).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f \in \mathcal{M}$ , and  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$ , i.e.  $\int |f| d\mu < \infty$ .

*Proof.* “Left” implies “right”. Direct consequence of Definition 1247 (integrability), Lemma 1248 (integrable is measurable), Lemma 842 ( $\mathcal{M}$  is closed under absolute value), Lemma 1186 (integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts), Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed), and Lemma 1172 (integral in  $\mathcal{M}_+$ ,  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ).

“Right” implies “left”. Direct consequence of Lemma 836 (measurability of nonnegative and nonpositive parts), Lemma 1186 (integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts), Lemma 397 (infinity-sum property in  $\overline{\mathbb{R}}_+$ , contrapositive), Lemma 1172 (integral in  $\mathcal{M}_+$ ,  $f^+$  and  $f^-$  are  $\mu$ -integrable in  $\mathcal{M}_+$ ), and Definition 1247 (integrability).

Therefore, we have the equivalence.  $\square$

**Lemma 1250 (integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f$  is nonnegative. Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 1249 (equivalent definition of integrability), and Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative,  $|f| = f$ ).  $\square$

**Lemma 1251 (integrable is almost finite).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have  $\mu(\{f = \pm\infty\}) = 0$ , i.e.  $|f| \stackrel{\mu \text{ a.e.}}{<} \infty$ .

*Proof.* Direct consequence of Lemma 1249 (equivalent definition of integrability,  $|f|$  is integrable in  $\mathcal{M}_+$ ), Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even,  $\{f = \pm\infty\} = \{|f| = \infty\}$ ), and Lemma 1197 (integrable in  $\mathcal{M}_+$  is almost finite, with  $|f|$ ).  $\square$

**Lemma 1252 (almost bounded by integrable is integrable).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f \in \mathcal{M}$  and there exists  $g : X \rightarrow \overline{\mathbb{R}}_+$  such that  $g$  is  $\mu$ -integrable in  $\mathcal{M}_+$  and  $|f| \stackrel{\mu \text{ a.e.}}{\leq} g$ .

*Proof.* Direct consequence of Lemma 1249 (equivalent definition of integrability), Lemma 1198 (almost bounded by integrable in  $\mathcal{M}_+$  is integrable, with  $f \stackrel{\text{def.}}{=} |f|$ ), the tautology ( $P \Rightarrow Q$ ) implies ( $P \wedge Q \Leftrightarrow P$ ), and Lemma 842 ( $\mathcal{M}$  is closed under absolute value, thus  $f \in \mathcal{M} \wedge |f| \in \mathcal{M}_+$  is equivalent to  $f \in \mathcal{M}$ ).  $\square$

**Lemma 1253 (integrability is closed under extension by zero).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $f|_A$  is  $\mu_A$ -integrable in  $\mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f} \mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}(X, \Sigma)$ .

If so, the function  $f$  is said  $\mu$ -integrable over  $A$ . Moreover,  $(\hat{f}^\pm)|_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(A, \Sigma \cap A)$ ,  $\hat{f}^\pm \mathbf{1}_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(X, \Sigma)$ , and we have

$$(16.1) \quad \mathcal{D}^+((\hat{f}^+)|_A, -(\hat{f}^-)|_A) = \mathcal{D}^+(\hat{f}^+ \mathbf{1}_A, -\hat{f}^- \mathbf{1}_A) = X,$$

$$(16.2) \quad f|_A = (\hat{f}^+)|_A - (\hat{f}^-)|_A \quad \text{and} \quad \hat{f} \mathbf{1}_A = \hat{f}^+ \mathbf{1}_A - \hat{f}^- \mathbf{1}_A.$$

*Proof.* From Definition 1247 (integrability, with  $f|_A$ , then  $\hat{f} \mathbf{1}_A$ ), Lemma 1172 (integral in  $\mathcal{M}_+$ , integrability, with  $f|_A$ , then  $\hat{f} \mathbf{1}_A$ ), Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), Lemma 1199 (integral in  $\mathcal{M}_+$  over subset, with  $f \stackrel{\text{def.}}{=} f^\pm$  and  $\hat{f} \stackrel{\text{def.}}{=} \hat{f}^\pm$ ), and Lemma 534 (compatibility of nonpositive and nonnegative parts with mask), we have

$$\begin{aligned} f|_A \text{ } \mu_A\text{-integrable in } \mathcal{M}(A, \Sigma \cap A) &\iff (f|_A)^\pm \text{ } \mu_A\text{-integrable in } \mathcal{M}_+(A, \Sigma \cap A) \\ &\iff (f|_A)^\pm \in \mathcal{M}_+(A, \Sigma \cap A) \quad \wedge \quad \int (f|_A)^\pm d\mu < \infty \\ &\iff (f^\pm)|_A \in \mathcal{M}_+(A, \Sigma \cap A) \quad \wedge \quad \int (f^\pm)|_A d\mu < \infty \\ &\iff \hat{f}^\pm \mathbf{1}_A \in \mathcal{M}_+(X, \Sigma) \quad \wedge \quad \int \hat{f}^\pm \mathbf{1}_A d\mu < \infty \\ &\iff (\hat{f} \mathbf{1}_A)^\pm \in \mathcal{M}_+(X, \Sigma) \quad \wedge \quad \int (\hat{f} \mathbf{1}_A)^\pm d\mu < \infty \\ &\iff (\hat{f} \mathbf{1}_A)^\pm \text{ } \mu\text{-integrable in } \mathcal{M}_+(X, \Sigma) \\ &\iff \hat{f} \mathbf{1}_A \text{ } \mu\text{-integrable in } \mathcal{M}(X, \Sigma). \end{aligned}$$

Moreover, from Definition 1247 (integrability), **the definition of the indicator function (thus masking is idempotent)**, Lemma 531 (nonnegative and nonpositive parts are orthogonal, thus  $(\hat{f}^+)|_A$  and  $(\hat{f}^-)|_A$  on the one hand, and  $\hat{f}^+ \mathbf{1}_A$  and  $\hat{f}^- \mathbf{1}_A$  on the other, do not take infinite values at the same point), Lemma 998 (summability on summability domain, thus  $\mathcal{D}^+((\hat{f}^+)|_A, -(\hat{f}^-)|_A) = \mathcal{D}^+(\hat{f}^+ \mathbf{1}_A, -\hat{f}^- \mathbf{1}_A) = X$ ), Lemma 532 (decomposition into nonnegative and nonpositive parts), Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), and Lemma 534 (compatibility of nonpositive and nonnegative parts with mask),  $(\hat{f}^\pm)|_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(A, \Sigma \cap A)$ ,  $\hat{f}^\pm \mathbf{1}_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(X, \Sigma)$  and all satisfy (16.1) and (16.2).  $\square$

**Lemma 1254 (integrability is closed under restriction).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ .

Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}(X, \Sigma)$  iff for all  $A \in \Sigma$ ,  $f|_A$  is  $\mu_A$ -integrable in  $\mathcal{M}(A, \Sigma \cap A)$ .

*Proof.* From Definition 1247 (integrability, with  $(X, \Sigma, \mu)$ , then  $(A, \Sigma \cap A, \mu_A)$ ), Lemma 1204 (integrable in  $\mathcal{M}_+$  is closed under restriction), and Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), we have

$$\begin{aligned} f \text{ } \mu\text{-integrable in } \mathcal{M}(X, \Sigma) &\iff f^\pm \text{ } \mu\text{-integrable in } \mathcal{M}_+(X, \Sigma) \\ &\iff \forall A \in \Sigma, \quad f^\pm|_A \text{ } \mu_A\text{-integrable in } \mathcal{M}_+(A, \Sigma \cap A) \\ &\iff \forall A \in \Sigma, \quad f|_A \text{ } \mu_A\text{-integrable in } \mathcal{M}(A, \Sigma \cap A) \\ &\iff \forall A \in \Sigma, \quad f|_A \text{ } \mu_A\text{-integrable in } \mathcal{M}(A, \Sigma \cap A). \end{aligned}$$

$\square$

**Lemma 1255 (integrability, masking is  $\sigma$ -additive).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \biguplus_{i \in I} A_i$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f \mathbf{1}_A = \sum_{i \in I} f \mathbf{1}_{A_i}$ , and  $f \mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}$  implies for all  $i \in I$ ,  $f \mathbf{1}_{A_i}$  is  $\mu$ -integrable in  $\mathcal{M}$ .

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$  with  $n \in \mathbb{N}$ , we have for all  $i \in [0..n]$ ,  $f \mathbf{1}_{A_i}$  is  $\mu$ -integrable in  $\mathcal{M}$  implies  $f \mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}$ .

*Proof.* Direct consequence of Lemma 525 (masking is  $\sigma$ -additive, with  $E \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ , thus the identity holds), Definition 1247 (integrability, with  $f \mathbf{1}_A$  and  $f \mathbf{1}_{A_i}$ ), Lemma 1206 (integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive), and Lemma 534 (compatibility of nonpositive and nonnegative parts with mask).  $\square$

**Lemma 1256 (integral in  $\mathcal{M}$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then,  $\int f^+ d\mu - \int f^- d\mu$  is finite.

The Lebesgue integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by

$$(16.3) \quad \int f d\mu \stackrel{\text{def.}}{=} \int f^+ d\mu - \int f^- d\mu \in \mathbb{R}.$$

*Proof.* Direct consequence of Definition 1247 (integrability), Lemma 1172 (integral in  $\mathcal{M}_+$ ,  $f^+$  and  $f^-$  are  $\mu$ -integrable in  $\mathcal{M}_+$ , thus their integrals are finite).  $\square$

*Remark 1257.* Keeping the same notation for the integrals is justified in the next lemma.

**Lemma 1258 (integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Assume that  $f$  is nonnegative. Then, both Lemma 1172 (integral in  $\mathcal{M}_+$ ), and Lemma 1256 (integral in  $\mathcal{M}$ ) provide the same value for the integral of  $f$ .

*Proof.* Direct consequence of Definition 528 (nonnegative and nonpositive parts,  $f^+ = f$  and  $f^- = 0$ ), Definition 1247 (integrability), Lemma 1172 (integral in  $\mathcal{M}_+$ ,  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ), Lemma 1256 (integral in  $\mathcal{M}$ ), and Lemma 1177 (integral in  $\mathcal{M}_+$  of zero is zero).  $\square$

**Lemma 1259 (integral in  $\mathcal{M}$  of zero is zero).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then, 0 is  $\mu$ -integrable in  $\mathcal{M}$ , and  $\int 0 d\mu = 0$ .

*Proof.* Direct consequence of Lemma 1177 (integral in  $\mathcal{M}_+$  of zero is zero), Lemma 1172 (integral in  $\mathcal{M}_+$ , thus 0 is  $\mu$ -integrable in  $\mathcal{M}_+$ ), Lemma 1250 (integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$ ), and Lemma 1258 (integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$ ).  $\square$

**Lemma 1260 (integral in  $\mathcal{M}$  of indicator function).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Assume that  $\mu(A) < \infty$ . Then,  $\mathbf{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(16.4) \quad \int \mathbf{1}_A d\mu = \mu(A) \in \mathbb{R}.$$

*Proof.* Direct consequence of Lemma 1175 (integral in  $\mathcal{M}_+$  of indicator function), Lemma 1250 (integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$ ), and Lemma 1258 (integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$ ).  $\square$

**Definition 1261 (merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . The integral of  $f$  (for the measure  $\mu$ ) exists iff  $f$  is nonnegative (and  $\int f d\mu \in \mathbb{R}_+$ ), or  $f$  is  $\mu$ -integrable (and  $\int f d\mu \in \mathbb{R}$ ).

**Lemma 1262 (compatibility of integral with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then,  $g$  is  $\mu$ -integrable, and we have

$$(16.5) \quad \int f \, d\mu = \int g \, d\mu.$$

*Proof.* Direct consequence of Lemma 985 (compatibility of almost equality with operator, with the binary operator  $\max$  and the unary operator additive inverse,  $f^+ \stackrel{\mu \text{ a.e.}}{=} g^+$  and  $f^- \stackrel{\mu \text{ a.e.}}{=} g^-$ ), Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality), Definition 1247 (integrability), and Lemma 1256 (integral in  $\mathcal{M}$ ).  $\square$

**Lemma 1263 (finite representative of integrable in  $\mathcal{M}$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then, there exists  $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$  such that  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$  and  $\int f \, d\mu = \int \tilde{f} \, d\mu$ .

Moreover, if  $f$  is nonnegative (i.e. if  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ), then  $\tilde{f} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 1251 (integrable is almost finite,  $\{f = \pm\infty\}$  is  $\mu$ -negligible), Lemma 821 ( $\mathcal{M}$  is closed under finite part,  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_{f^{-1}(\mathbb{R})}$ , thus  $\{f \neq \tilde{f}\} = \{f = \pm\infty\}$ ), Lemma 1262 (compatibility of integral with almost equality), Lemma 811 (measurability of indicator function), and Lemma 844 ( $\mathcal{M}_+$  is closed under multiplication).  $\square$

**Lemma 1264 (extended almost modus ponens for integral in  $\mathcal{M}$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$  be nonempty. Let  $P$  be a predicate on  $\overline{\mathbb{R}}^I$ , lifted into a predicate on  $(\overline{\mathbb{R}}^I)^X$ . Let  $Q$  be a predicate on  $\mathbb{R}^I$ . Assume that  $P(0, \dots, 0)$  holds, and that for all  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I} \in \mathcal{M}_{\mathbb{R}}$  be  $\mu$ -integrable, we have

$$(16.6) \quad \forall x \in X, P(f(x)) \implies Q\left(\int f_i \, d\mu\right)_{i \in I}.$$

Then, for all  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I}$   $\mu$ -integrable in  $\mathcal{M}$ , we have

$$(16.7) \quad P(g) \, \mu \text{ a.e.} \implies Q\left(\int g_i \, d\mu\right)_{i \in I}.$$

*Proof.* Let  $g \stackrel{\text{def.}}{=} (g_i)_{i \in I}$  be  $\mu$ -integrable functions in  $\mathcal{M}$ . Assume that  $P(g) \, \mu$  a.e. holds. Let  $B \stackrel{\text{def.}}{=} \{P(g)\}$ , and  $C \stackrel{\text{def.}}{=} \bigcap_{i \in I} g_i^{-1}(\mathbb{R})$ . Then, from Lemma 1251 (integrable is almost finite, with  $g_i$ ), Lemma 960 (countable almost properties is almost property, with  $P_0 \stackrel{\text{def.}}{=} (P(g))$  and for all  $i \in I$ ,  $P_{i+1} \stackrel{\text{def.}}{=} (g_i \text{ finite})$ ), there exists  $A \subset B \cap C$  such that  $A, A^c \in \Sigma$ , and  $\mu(A^c) = 0$ .

For all  $i \in I$ , let  $f_i \stackrel{\text{def.}}{=} g_i \mathbf{1}_A$ , and let  $f \stackrel{\text{def.}}{=} (f_i)_{i \in I}$ . Then, from Lemma 1008 (masking almost nowhere, with  $g_i$ ), Lemma 833 (measurability and masking), **the definition of the indicator function**, and Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), for all  $i \in I$ ,  $f_i \stackrel{\mu \text{ a.e.}}{=} g_i$ ,  $f_i \in \mathcal{M}_{\mathbb{R}}$ , we have  $P(f)$  everywhere, and thus  $Q\left(\int f_i \, d\mu\right)_{i \in I}$  holds. Therefore, from Lemma 1262 (compatibility of integral with almost equality, with  $f \stackrel{\text{def.}}{=} f_i$  and  $g \stackrel{\text{def.}}{=} g_i$ ), we also have  $Q\left(\int g_i \, d\mu\right)_{i \in I}$ .  $\square$

*Remark 1265.* The previous lemma allows to extend a property on the integrals of real-valued  $\mu$ -integrable functions to the same property on extended real-valued  $\mu$ -integrable functions whereas the pointwise premises may be satisfied only  $\mu$ -almost everywhere.

**Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $a \in \mathbb{R}$ . Then,  $af$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(16.8) \quad \int af \, d\mu = a \int f \, d\mu.$$



*Proof.* From Lemma 1249 (equivalent definition of integrability, thus  $f \in \mathcal{M}$  and  $\int |f| d\mu < \infty$ ), Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication, thus  $af \in \mathcal{M}$ ), **the definition of the absolute value in  $\mathbb{R}$  (thus  $|a| \in \mathbb{R}_+$ )**, Lemma 842 ( $\mathcal{M}$  is closed under absolute value, thus  $|f| \in \mathcal{M}_+$ ), Lemma 348 (absolute value in  $\overline{\mathbb{R}}$  is multiplicative, thus  $|af| = |a||f|$ ), Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous), and **closedness of multiplication in  $\mathbb{R}_+$** ,  $\int |af| d\mu = |a| \int |f| d\mu < \infty$ . Hence, from Lemma 1249 (equivalent definition of integrability),  $af$  is  $\mu$ -integrable in  $\mathcal{M}$ .

From Lemma 836 (measurability of nonnegative and nonpositive parts), Definition 528 (nonnegative and nonpositive parts), and Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)), we have  $f^\pm \in \mathcal{M}_+$ ,  $(af)^\pm = af^\pm \in \mathcal{M}_+$  when  $a \geq 0$ , and  $(af)^\pm = -af^\mp$  belong to  $\mathcal{M}_+$  when  $a < 0$ . Therefore, from Lemma 1256 (integral in  $\mathcal{M}$ , with  $af$ , then  $f$ ), Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous, with  $af^\pm$  (when  $a \geq 0$ ) or  $-af^\mp$  (when  $a < 0$ )), and **field properties of  $\mathbb{R}$  (all integrals below are finite)**, we have in both cases

$$\int af d\mu = \int (af)^+ d\mu - \int (af)^- d\mu = a \int f^+ d\mu - a \int f^- d\mu = a \int f d\mu.$$

□

**Lemma 1267 (integral in  $\mathcal{M}$  of constant function).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $a \in \mathbb{R}$ . Assume that  $\mu(A) < \infty$ . Then,  $a\mathbb{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(16.9) \quad \int a\mathbb{1}_A d\mu = a\mu(A) \in \mathbb{R}.$$

*Proof.* Direct consequence of Lemma 1260 (integral in  $\mathcal{M}$  of indicator function), and Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous, with  $f \stackrel{\text{def.}}{=} \mathbb{1}_A$ ). □

**Lemma 1268 (Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g, f', g' \in \mathcal{M}$ . Assume that  $f + g$  is well-defined almost everywhere,  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f' + g', f' \stackrel{\mu \text{ a.e.}}{+} g' \in \mathcal{M}$ , and we have

$$(16.10) \quad \int |f' \stackrel{\mu \text{ a.e.}}{+} g'| d\mu = \int |f' + g'| d\mu \leq \int |f'| d\mu + \int |g'| d\mu = \int |f| d\mu + \int |g| d\mu.$$

*Proof.* From Lemma 1002 (almost sum, such  $f'$  and  $g'$  exist), and Lemma 1003 (compatibility of almost sum with almost equality), we have  $f' + g', f' \stackrel{\mu \text{ a.e.}}{+} g' \in \mathcal{M}$ , and  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f' \stackrel{\mu \text{ a.e.}}{+} g'$ . Hence, from Lemma 842 ( $\mathcal{M}$  is closed under absolute value), Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality), and Lemma 843 ( $\mathcal{M}_+$  is closed under addition), we have

$$|f|, |g|, |f'|, |g'|, |f' + g'|, |f'| + |g'|, |f' \stackrel{\mu \text{ a.e.}}{+} g'| \in \mathcal{M}_+, \\ \int |f'| d\mu = \int |f| d\mu, \quad \int |g'| d\mu = \int |g| d\mu, \quad \text{and} \quad \int |f' + g'| d\mu = \int |f' \stackrel{\mu \text{ a.e.}}{+} g'| d\mu.$$

Therefore, from Lemma 349 (absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality), Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 1183 (integral in  $\mathcal{M}_+$  is additive), we have

$$\int |f' + g'| d\mu \leq \int (|f'| + |g'|) d\mu = \int |f'| d\mu + \int |g'| d\mu.$$

□

**Lemma 1269 (integral in  $\mathcal{M}$  is additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $f', g' \in \mathcal{M}$ . Assume that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f + g$  is well-defined almost everywhere,  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f + g$  are  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(16.11) \quad \int (f' + g') d\mu = \int (f + g) d\mu = \int f' d\mu + \int g' d\mu = \int f d\mu + \int g d\mu.$$

Moreover, when  $\mathcal{D}^+(f, g) = X$ , e.g. when  $f, g$  are  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ ,  $f + g$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(16.12) \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

*Proof.* From Lemma 1249 (equivalent definition of integrability, thus  $f, g \in \mathcal{M}$  and  $\int |f| d\mu, \int |g| d\mu$  are finite), Lemma 1002 (almost sum, such  $f'$  and  $g'$  exist), and Lemma 1003 (compatibility of almost sum with almost equality),  $f + g$  is well-defined  $\mu$ -almost everywhere. Then, from Lemma 1268 (Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )), and **closedness of addition in  $\mathbb{R}_+$** ,  $f' + g', f' + g' \stackrel{\mu \text{ a.e.}}{=} f + g \in \mathcal{M}$ ,  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f + g$ , and we have  $\int |f' + g'| d\mu = \int |f + g| d\mu < \infty$ . Hence, from Lemma 1249 (equivalent definition of integrability),  $f' + g'$  and  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f + g$  are  $\mu$ -integrable in  $\mathcal{M}$ .

From Lemma 1262 (compatibility of integral with almost equality, with  $f'$  and  $g'$ ),  $f'$  and  $g'$  are also  $\mu$ -integrable in  $\mathcal{M}$ , and we have  $\int f' d\mu = \int f d\mu$  and  $\int g' d\mu = \int g d\mu$ . Then, from Lemma 836 (measurability of nonnegative and nonpositive parts),  $f'^{\pm}, g'^{\pm}$ , and  $(f' + g')^{\pm}$  belong to  $\mathcal{M}_+$ . Therefore, from Lemma 1256 (integral in  $\mathcal{M}$ , with  $f' + g'$ , then  $f'$  and  $g'$ ), Lemma 1187 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts, with  $f'$  and  $g'$ ), and **field properties of  $\mathbb{R}$  (all integrals below are finite)**, we have

$$\begin{aligned} \int (f' + g') d\mu &= \int (f' + g')^+ d\mu - \int (f' + g')^- d\mu \\ &= \int (f')^+ d\mu + \int (g')^+ d\mu - \left( \int (f')^- d\mu + \int (g')^- d\mu \right) \\ &= \left( \int (f')^+ d\mu - \int (f')^- d\mu \right) + \left( \int (g')^+ d\mu - \int (g')^- d\mu \right) \\ &= \int f' d\mu + \int g' d\mu. \end{aligned}$$

The specific case  $\mathcal{D}^+(f, g) = X$  is a direct consequence of Lemma 1004 (almost sum is sum).  $\square$

**Lemma 1270 (integral in  $\mathcal{M}_{\mathbb{R}}$  is linear).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then, the integral is linear on  $\mu$ -integrable functions in  $\mathcal{M}_{\mathbb{R}}$ : let  $a \in \mathbb{R}$ , and  $f, g$  be  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ , then  $af, f + g$  are  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ , and we have

$$(16.13) \quad \int af d\mu = a \int f d\mu, \quad \text{and} \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

*Proof.* Direct consequence of Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space, thus  $af, f + g \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ , thus  $f \in \mathcal{M}$ ), Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous), and Lemma 1269 (integral in  $\mathcal{M}$  is additive).  $\square$

**Lemma 1271 (integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g$  be  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ . Then, we have

$$(16.14) \quad f \leq g \implies \int f d\mu \leq \int g d\mu \quad (\text{in } \mathbb{R}).$$

*Proof.* From Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 61 (vector space), **additive group properties of  $\mathcal{M}_{\mathbb{R}}$  (thus  $g - f \in \mathcal{M}_{\mathbb{R}}$ )**, and Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), we have  $g - f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ . Hence, from Lemma 1268 (Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ ), with  $(f, g) \stackrel{\text{def.}}{=} (g, -f)$ ), Lemma 1004 (almost sum is sum, thus  $g \stackrel{\mu \text{ a.e.}}{+} (-f) = g - f$ ), Lemma 346 (absolute value in  $\overline{\mathbb{R}}$  is even, thus  $|-f| = |f|$ ), Lemma 1249 (equivalent definition of integrability, with  $f$  and  $g$ , then  $g - f$ ), and **additive group properties of  $\mathbb{R}$  (closedness under addition)**, we have

$$\int |g - f| d\mu \leq \int |g| d\mu + \int |f| d\mu < \infty, \quad \text{and } g - f \text{ is } \mu\text{-integrable.}$$

Therefore, from Lemma 1172 (integral in  $\mathcal{M}_+$ , nonnegativeness), Lemma 1269 (integral in  $\mathcal{M}$  is additive, with  $f, g - f \in \mathcal{M}_{\mathbb{R}}$ ), and **ordered field properties of  $\mathbb{R}$** , we have

$$\int f d\mu = \int f d\mu \leq \int f d\mu + \int (g - f) d\mu = \int g d\mu.$$

□

**Lemma 1272 (integral in  $\mathcal{M}$  is almost monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have

$$(16.15) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \implies \int f d\mu \leq \int g d\mu \quad (\text{in } \mathbb{R}).$$

*Proof.* Direct consequence of Lemma 313 (order in  $\overline{\mathbb{R}}$  is total, reflexivity (thus  $0 \leq 0$ )), Lemma 1271 (integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone), and Lemma 1264 (extended almost modus ponens for integral in  $\mathcal{M}$ , with  $\text{card}(I) = 2$ ,  $P \stackrel{\text{def.}}{\leq_{\mathbb{R}}}$  and  $Q \stackrel{\text{def.}}{\leq_{\mathbb{R}}}$ ). □

**Lemma 1273 (compatibility of integral in  $\mathcal{M}$  with absolute value).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have

$$(16.16) \quad \left| \int f d\mu \right| \leq \int |f| d\mu.$$

*Proof.* From Definition 1247 (integrability), and Lemma 1249 (equivalent definition of integrability),  $f^+$ ,  $f^-$  and  $|f|$  are  $\mu$ -integrable in  $\mathcal{M}_+$ , i.e. all integrals below are finite. Therefore, from Lemma 1256 (integral in  $\mathcal{M}$ , with  $f$ , then  $|f|$ ), **the triangle inequality for the absolute value in  $\mathbb{R}$** , Lemma 530 (nonnegative and nonpositive parts are nonnegative), Lemma 1172 (integral in  $\mathcal{M}_+$ , nonnegativeness), **valued field properties of  $\mathbb{R}$  ( $a \geq 0 \implies |a| = a$ )**, and Lemma 1186 (integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts), we have

$$\left| \int f d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right| \leq \int f^+ d\mu + \int f^- d\mu = \int |f| d\mu.$$

□

## 16.2 Notations for specific cases

### Lemma 1274 (integral over subset).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ , and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ , and  $f$  is  $\mu$ -integrable over  $A$ . Then,  $(\hat{f}^\pm)|_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(A, \Sigma \cap A)$ ,  $\hat{f}^\pm \mathbf{1}_A$  are  $\mu$ -integrable in  $\mathcal{M}_+(X, \Sigma)$ , and we have

$$(16.17) \quad \mathcal{D}^+((\hat{f}^+)|_A, -(\hat{f}^-)|_A) = \mathcal{D}^+(\hat{f}^+ \mathbf{1}_A, -\hat{f}^- \mathbf{1}_A) = X,$$

$$(16.18) \quad f|_A = (\hat{f}^+)|_A - (\hat{f}^-)|_A, \quad \hat{f} \mathbf{1}_A = \hat{f}^+ \mathbf{1}_A - \hat{f}^- \mathbf{1}_A,$$

$$(16.19) \quad \int f|_A d\mu_A = \int_A \hat{f}^+ d\mu - \int_A \hat{f}^- d\mu = \int \hat{f} \mathbf{1}_A d\mu.$$

This integral is still denoted  $\int_A f d\mu$ ; it is still called integral of  $f$  over  $A$ .

*Proof.* Direct consequence of Lemma 1253 (integrability is closed under extension by zero), Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous, with  $a \stackrel{\text{def.}}{=} -1$ ), Lemma 1269 (integral in  $\mathcal{M}$  is additive, with  $\mathcal{D}^+(\hat{g}^+, -\hat{g}^-) = \mathcal{D}^+((\hat{g}^+)|_A, -(\hat{g}^-)|_A) = X$ ), Lemma 1258 (integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$ ), and Lemma 1199 (integral in  $\mathcal{M}_+$  over subset).  $\square$

### Lemma 1275 (integral over subset is $\sigma$ -additive).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ . Let  $Y$  be a set such that  $A \subset Y \subset X$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f} \mathbf{1}_A = \sum_{i \in I} \hat{f} \mathbf{1}_{A_i}$ , and  $f$  is  $\mu$ -integrable over  $A$  implies for all  $i \in I$ ,  $f$  is  $\mu$ -integrable over  $A_i$ . If so, we have

$$(16.20) \quad \int_A f d\mu = \sum_{i \in I} \int_{A_i} f d\mu.$$

Moreover, when  $I \stackrel{\text{def.}}{=} [0..n]$  with  $n \in \mathbb{N}$ , we have for all  $i \in [0..n]$ ,  $f$  is  $\mu$ -integrable over  $A_i$  implies  $f$  is  $\mu$ -integrable over  $A$ .

*Proof.* From Lemma 1255 (integrability, masking is  $\sigma$ -additive), we have the first identity and both implications.

**Identity (16.20).** Assume that  $f$  is  $\mu$ -integrable over  $A$ , and over all the  $A_i$ 's. Then, from Lemma 1274 (integral over subset, with  $A$  and all the  $A_i$ 's), Lemma 1172 (integral in  $\mathcal{M}_+$ , integrability), Lemma 1200 (integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive, with  $A$  and all the  $A_i$ 's), and **commutativity and associativity of the sum of absolutely convergent series**,  $\hat{f}^\pm \mathbf{1}_A$  and for all  $i \in I$ ,  $\hat{f}^\pm \mathbf{1}_{A_i}$  are  $\mu$ -integrable in  $\mathcal{M}_+$ , i.e.  $\hat{f}^\pm \mathbf{1}_A, \hat{f}^\pm \mathbf{1}_{A_i} \in \mathcal{M}_+$  and  $\int_A \hat{f}^\pm d\mu, \int_{A_i} \hat{f}^\pm d\mu$  are finite, and we have

$$\begin{aligned} \int_A f d\mu &= \int_A \hat{f}^+ d\mu - \int_A \hat{f}^- d\mu = \sum_{i \in I} \int_{A_i} \hat{f}^+ d\mu - \sum_{i \in I} \int_{A_i} \hat{f}^- d\mu \\ &= \sum_{i \in I} \left( \int_{A_i} \hat{f}^+ d\mu - \int_{A_i} \hat{f}^- d\mu \right) = \sum_{i \in I} \int_{A_i} f d\mu. \end{aligned}$$

$\square$

### Lemma 1276 (integral over negligible subset).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset Y \subset X$ . Assume that  $A \in \Sigma$  and  $\mu(A) = 0$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ , and  $f$  is  $\mu$ -integrable over  $A$ . Then, we have

$$(16.21) \quad \int_A f d\mu = 0.$$

*Proof.* Direct consequence of Lemma 1274 (*integral over subset*, thus  $\hat{f}^\pm \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$  and  $\int_A f d\mu = \int_A \hat{f}^+ d\mu - \int_A \hat{f}^- d\mu$ ), Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*, thus  $\int_A \hat{f}^\pm d\mu$  are equal to 0), and **additive group properties of  $\mathbb{R}$  (thus  $0 - 0 = 0$ )**.  $\square$

**Lemma 1277 (*integral over singleton*).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $a \in X$ . Assume that  $\{a\} \in \Sigma$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable over  $\{a\}$  iff*

$$(16.22) \quad (f(a) \text{ and } \mu(\{a\}) \text{ are finite}) \quad \vee \quad f(a) = 0 \quad \vee \quad \mu(\{a\}) = 0.$$

*If so, we have*

$$(16.23) \quad \int_{\{a\}} f d\mu = f(a) \mu(\{a\}).$$

*Proof.* From Lemma 1253 (*integrability is closed under extension by zero*, with  $A \stackrel{\text{def.}}{=} \{a\}$ ,  $Y \stackrel{\text{def.}}{=} X$  and  $\hat{f} \stackrel{\text{def.}}{=} f$ ), Lemma 1249 (*equivalent definition of integrability*), Lemma 1172 (*integral in  $\mathcal{M}_+$ , integrability*), Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*, with  $|f|$ , thus  $|f \mathbb{1}_{\{a\}}| = |f(a)| \mathbb{1}_{\{a\}}$  belong to  $\mathcal{M}_+$  and  $\int |f \mathbb{1}_{\{a\}}| d\mu = |f(a)| \mu(\{a\})$ ), Lemma 428 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*), **closedness of the absolute value in  $\mathbb{R}$** , Definition 339 (*absolute value in  $\overline{\mathbb{R}}$ , absolute value is closed in  $\{\pm\infty\}$ ), and Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*), we have*

$$\begin{aligned} f \text{ is } \mu\text{-integrable over } \{a\} &\iff f \mathbb{1}_{\{a\}} \text{ is } \mu\text{-integrable in } \mathcal{M} \\ &\iff |f \mathbb{1}_{\{a\}}| \text{ is } \mu\text{-integrable in } \mathcal{M}_+ \\ &\iff |f \mathbb{1}_{\{a\}}| \in \mathcal{M}_+ \quad \wedge \quad \int |f \mathbb{1}_{\{a\}}| d\mu < \infty \\ &\iff |f(a)| \mu(\{a\}) < \infty \\ &\iff (f(a) \in \mathbb{R} \wedge \mu(\{a\}) \in \mathbb{R}_+) \quad \vee \quad f(a) = 0 \quad \vee \quad \mu(\{a\}) = 0. \end{aligned}$$

Assume that  $f$  is  $\mu$ -integrable over  $\{a\}$ . Then, from Lemma 1274 (*integral over subset*, with  $A \stackrel{\text{def.}}{=} \{a\}$ ,  $Y \stackrel{\text{def.}}{=} X$  and  $\hat{f} \stackrel{\text{def.}}{=} f$ ), Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*), **right distributivity of multiplication over addition in  $\mathbb{R}$** , and Lemma 532 (*decomposition into nonnegative and nonpositive parts*, for  $f$ ), we have

$$\int_{\{a\}} f d\mu = \int_{\{a\}} f^+ d\mu - \int_{\{a\}} f^- d\mu = f^+(a) \mu(\{a\}) - f^-(a) \mu(\{a\}) = f(a) \mu(\{a\}).$$

$\square$

**Lemma 1278 (*integral over interval*).**

*Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a measure space. Assume that  $\mu$  is diffuse. Let  $a, b \in \overline{\mathbb{R}}$  such that  $a \leq b$ . Let  $f$  be  $\mu$ -integrable over  $(a, b)$ . Then, the integral over the interval remains the same when the interval is closed at one or both of its finite extremities.*

*The integral of  $f$  over an interval with extremities  $a$  and  $b$  is called integral of  $f$  from  $a$  to  $b$ ; it is denoted  $\int_a^b f(x) d\mu(x)$ . When  $a < b$ , we assume the convention  $\int_b^a f(x) d\mu(x) \stackrel{\text{def.}}{=} - \int_a^b f(x) d\mu(x)$ .*

*Proof.* Direct consequence of Definition 856 (*diffuseness*), Lemma 1275 (*integral over subset is  $\sigma$ -additive*, on  $[a, b] = \{a\} \uplus (a, b) \uplus \{b\}$ ), Lemma 744 (*some Borel subsets*, singletons of  $\mathbb{R}$  are measurable), and Lemma 1277 (*integral over singleton*).  $\square$

**Lemma 1279 (*Chasles relation, integral over split intervals*).**

*Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a measure space. Assume that  $\mu$  is diffuse. Let  $a, b, c \in \overline{\mathbb{R}}$ . Let  $f$  be  $\mu$ -integrable over  $(\min(a, b, c), \max(a, b, c))$ . Then, we have*

$$(16.24) \quad \int_a^b f(x) d\mu(x) = \int_a^c f(x) d\mu(x) + \int_c^b f(x) d\mu(x).$$

*Proof.* Direct consequence of Lemma 1275 (*integral over subset is  $\sigma$ -additive*). Lemma 1278 (*integral over interval, convention for reverse bounds*), and **field properties of  $\mathbb{R}$** .

It is trivial when  $a \leq c \leq b$ . For instance, when  $c < b < a$ , we have

$$\int_c^a f(x) d\mu(x) = \int_c^b f(x) d\mu(x) + \int_b^a f(x) d\mu(x).$$

Hence, we have

$$\begin{aligned} \int_a^b f(x) d\mu(x) &= - \int_b^a f(x) d\mu(x) \\ &= - \int_c^a f(x) d\mu(x) + \int_c^b f(x) d\mu(x) \\ &= \int_a^c f(x) d\mu(x) + \int_c^b f(x) d\mu(x). \end{aligned}$$

□

**Lemma 1280 (*integral with trivial measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then,  $f$  is 0-integrable, and we have

$$(16.25) \quad \int f d0 = 0.$$

*Proof.* From Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*), Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and Lemma 1249 (*equivalent definition of integrability*),  $f$  is 0-integrable in  $\mathcal{M}$ . Then, from Lemma 1256 (*integral in  $\mathcal{M}$* ), Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*), and **additive group properties of  $\mathbb{R}$** , we have

$$\int f d0 = \int f^+ d0 - \int f^- d0 = 0.$$

□

**Lemma 1281 (*integral with counting measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{M}$ . Assume that  $\sum_{y \in Y} |f(y)|$  is finite. Then,  $f$  is  $\delta_Y$ -integrable, and we have

$$(16.26) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* From Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure,  $\int |f| d\delta_Y$  is finite*), and Lemma 1249 (*equivalent definition of integrability*),  $f$  is  $\delta_Y$ -integrable in  $\mathcal{M}$ . Then, from Lemma 1256 (*integral in  $\mathcal{M}$* ), Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*), and **associativity and commutativity of (possibly uncountable) addition for absolutely convergent sums**,

$$\int f d\delta_Y = \int f^+ d\delta_Y - \int f^- d\delta_Y = \sum_{y \in Y} f^+(y) - \sum_{y \in Y} f^-(y) = \sum_{y \in Y} f(y).$$

□

**Lemma 1282 (*integral with counting measure on  $\mathbb{N}$* ).** Let  $f : \mathbb{N} \rightarrow \overline{\mathbb{R}}$  be a sequence. Assume that  $\sum_{n \in \mathbb{N}} |f(n)|$  is finite. Then,  $f$  is  $\delta_{\mathbb{N}}$ -integrable in  $\mathcal{M}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ , and we have

$$(16.27) \quad \int f d\delta_{\mathbb{N}} = \sum_{n \in \mathbb{N}} f(n).$$

*Proof.* Direct consequence of Lemma 1281 (*integral with counting measure*, with  $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$  and  $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$ ).  $\square$

*Remark 1283.* Note that the previous lemma makes the sums of absolutely convergent series be Lebesgue integrals with counting measure on natural numbers. Thus, the theory of absolutely converging series can be derived from the theory of Lebesgue integration.

**Lemma 1284 (*integral with Dirac measure*).** *Let  $(X, \Sigma)$  be a measurable space. Let  $\{a\} \in \Sigma$ . Let  $f \in \mathcal{M}$ . Assume that  $f(a)$  is finite. Then,  $f$  is  $\delta_a$ -integrable, and we have*

$$(16.28) \quad \int f d\delta_a = f(a).$$

*Proof.* Direct consequence of Definition 939 (*Dirac measure*), and Lemma 1281 (*integral with counting measure*, with  $Y \stackrel{\text{def.}}{=} \{a\}$ ).  $\square$

**Definition 1285 (*integral for Lebesgue measure on  $\mathbb{R}$* ).**

Let  $a, b \in \overline{\mathbb{R}}$ . Let  $Y \stackrel{\text{def.}}{=} [\min(a, b), \max(a, b)]$ . Let  $f : Y \rightarrow \mathbb{R}$ . Assume that  $f$  is  $\lambda$ -integrable over  $Y$  (for the Lebesgue measure  $\lambda$ ). The integral of  $f$  from  $a$  to  $b$  is denoted  $\int_a^b f(x) dx$ .

## 16.3 The seminormed vector space $\mathcal{L}^p$

### 16.3.1 The $p$ -seminorm

**Lemma 1286 ( $p$ -seminorm).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $p \in (0, \infty)$ . Then,  $|f|^p, |f| \in \mathcal{M}_+$ , and  $\int |f|^p d\mu$  and  $\text{ess sup}^+ |f|$  are well-defined (in  $\overline{\mathbb{R}}_+$ ).

Let  $\tilde{N}_p \stackrel{\text{def.}}{=} (f \in \mathcal{M} \mapsto \int |f|^p d\mu \in \overline{\mathbb{R}}_+)$ . The function  $N_p \stackrel{\text{def.}}{=} \tilde{N}_p^{\frac{1}{p}}$  is called  $p$ -seminorm, or  $\mathcal{L}^p$ -seminorm. In particular, for  $p = 1$ , we have  $N_1 = \tilde{N}_1 = (f \in \mathcal{M} \mapsto \int |f| d\mu \in \overline{\mathbb{R}}_+)$ .

The function  $N_\infty \stackrel{\text{def.}}{=} (f \in \mathcal{M} \mapsto \text{ess sup}^+ |f| \in \overline{\mathbb{R}}_+)$  is called  $\infty$ -seminorm, or  $\mathcal{L}^\infty$ -seminorm.

*Proof.* Direct consequence of Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $p$  and 1, thus  $|f|^p, |f| \in \mathcal{M}_+$ ), Lemma 1172 (integral in  $\mathcal{M}_+$ ), Lemma 361 (power function in  $\mathbb{R}$ , with  $\frac{1}{p} \in \mathbb{R}_+^*$ ), and Lemma 1036 (equivalent definition of nonnegative essential supremum).  $\square$

*Remark 1287.* When  $p \in [1, \infty)$ , the function  $N_p$  is shown below to be a seminorm on the vector space  $\mathcal{L}^p$ , hence its name. When  $p \in (0, 1)$ , the function  $N_p$  is only a quasinorm, for which the triangle inequality is relaxed, and its specific study is out of the scope of this document.

**Lemma 1288 ( $N_p$  with trivial measure).**

Let  $(X, \Sigma, \mu)$  be a trivial measure space. Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{M}$ . Then, we have  $N_p(f) = 0$ .

*Proof.* **Case  $p \in (0, \infty)$ .** Direct consequence of **ordered field properties of  $\mathbb{R}$  ( $\frac{1}{p} \in (0, \infty)$ )**, Lemma 1210 (integral in  $\mathcal{M}_+$  with trivial measure), Lemma 432 (zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)), and Lemma 1286 ( $p$ -seminorm, thus  $N_p(f) = \tilde{N}_p(f) = 0$ ).

**Case  $p = \infty$ .** Direct consequence of Lemma 1286 ( $p$ -seminorm), and Lemma 1037 (nonnegative essential supremum with trivial measure).

Therefore,  $N_p$  is always zero.  $\square$

**Lemma 1289 ( $N_p$  of indicator function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $p \in (0, \infty)$ . Then,  $\mathbb{1}_A \in \mathcal{M}$ , and we have

$$(16.29) \quad N_p(\mathbb{1}_A) = (\mu(A))^{\frac{1}{p}}, \quad \text{and} \quad N_\infty(\mathbb{1}_A) = \begin{cases} 1 & \text{if } \mu(A) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* From Lemma 811 (measurability of indicator function), we have  $\mathbb{1}_A \in \mathcal{M}$ .

**Case  $p \in (0, \infty)$ .** Direct consequence of **properties of the power function in  $\mathbb{R}$  (thus  $|\mathbb{1}_A|^p = \mathbb{1}_A \geq 0$ )**, Lemma 1175 (integral in  $\mathcal{M}_+$  of indicator function, thus  $\int \mathbb{1}_A d\mu = \mu(A)$ ), and Lemma 1286 ( $p$ -seminorm).

**Case  $\mu$  trivial, and  $p = \infty$ .**

Direct consequence of Lemma 933 (trivial measure, thus  $\mu(A) > 0$  cannot occur), Lemma 1037 (nonnegative essential supremum with trivial measure), and Lemma 1286 ( $p$ -seminorm).

**Case  $\mu$  nontrivial, and  $p = \infty$ .**

Direct consequence of Lemma 1041 (nonnegative essential supremum for extended simple functions, with  $n \stackrel{\text{def.}}{=} 1$ ,  $A_0 \stackrel{\text{def.}}{=} A$ ,  $A_1 \stackrel{\text{def.}}{=} A^c$ ,  $a_0 \stackrel{\text{def.}}{=} 1$  and  $a_1 \stackrel{\text{def.}}{=} 0$ ), and Lemma 1286 ( $p$ -seminorm).

Therefore, we always have the equalities.  $\square$

**Lemma 1290 ( $N_p$  is nonnegative).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{M}$ . Then, we have  $N_p(f) \geq 0$ .



*Proof.* **Case  $p \in (0, \infty)$ .** Direct consequence of Lemma 1286 ( *$p$ -seminorm*, with  $p \in (0, \infty)$ , thus  $\tilde{N}_p(f) \geq 0$ ), and Lemma 364 (*power function in  $\bar{\mathbb{R}}$  is nonnegative*, with  $\frac{1}{p} \in (0, \infty)$ ).

**Case  $p = \infty$ .** Direct consequence of Lemma 1286 ( *$p$ -seminorm*,  $N_\infty(f) \in \bar{\mathbb{R}}_+$ ).

Therefore,  $N_p$  is always nonnegative.  $\square$

**Lemma 1291 (link between  $N_1$  and  $N_p$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty)$ . Let  $f \in \mathcal{M}$ , and  $g \in \mathcal{M}_+$ . Then,  $|f|^p, g^{\frac{1}{p}} \in \mathcal{M}_+ \subset \mathcal{M}$ , and we have in  $\bar{\mathbb{R}}_+$*

$$(16.30) \quad N_p^p(f) = \tilde{N}_p(f) = N_1(|f|^p), \quad N_p(f) = N_1^{\frac{1}{p}}(|f|^p),$$

$$(16.31) \quad N_1(g) = \int g \, d\mu = \int \left(g^{\frac{1}{p}}\right)^p \, d\mu = \tilde{N}_p\left(g^{\frac{1}{p}}\right) = N_p^p\left(g^{\frac{1}{p}}\right), \quad \text{and} \quad N_1^{\frac{1}{p}}(g) = N_p\left(g^{\frac{1}{p}}\right).$$

*Proof.* Direct consequence of Lemma 841 ( *$\mathcal{M}$  is closed under power function*, thus  $|f|^p, g^{\frac{1}{p}} \in \mathcal{M}_+$ , a subset of  $\mathcal{M}$ ), Lemma 443 (*compatibility of power function in  $\bar{\mathbb{R}}$  with absolute value (measure theory)*), Lemma 1286 ( *$p$ -seminorm*, with  $(p, f)$  and  $(1, |f|^p)$ , then  $(1, g)$  and  $(p, g^{\frac{1}{p}})$ ), Lemma 366 (*compatibility of power function in  $\bar{\mathbb{R}}$  with multiplication*, with  $\frac{1}{p}$  and  $p$ ), and **field properties of  $\mathbb{R}$  (thus  $\frac{1}{p} p = 1$ )**.  $\square$

**Lemma 1292 (integrable is finite  $N_1$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in X \rightarrow \bar{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f \in \mathcal{M}$  and  $N_1(f) < \infty$ .*

*Proof.* Direct consequence of Lemma 1249 (*equivalent definition of integrability*), and Lemma 1286 ( *$p$ -seminorm*, with  $p \stackrel{\text{def.}}{=} 1$ ).  $\square$

**Lemma 1293 ( $N_\infty$  bounded is almost bounded).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $a \in \bar{\mathbb{R}}_+$ . Then, we have*

$$(16.32) \quad N_\infty(f) \leq a \iff |f| \stackrel{\mu \text{ a.e.}}{\leq} a.$$

*Proof.* Direct consequence of Lemma 1286 ( *$p$ -seminorm*), Lemma 1039 (*nonnegative essential supremum is almost upper bound*,  $a \in C_{|f|}$ ), and Lemma 1046 (*nonnegative essential supremum bounded by constant*).  $\square$

**Lemma 1294 (compatibility of  $N_p$  with power function).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $\alpha \in [0, \infty)$ . Let  $f \in \mathcal{M}$ .*

*Then,  $|f|^\alpha \in \mathcal{M}_+ \subset \mathcal{M}$  and  $|f|^\alpha \in \mathcal{M}_\mathbb{R} \cap \mathcal{M}_+ \subset \mathcal{M}$  when  $f \in \mathcal{M}_\mathbb{R}$ , and we have*

$$(16.33) \quad \mu \text{ nontrivial} \quad \vee \quad \alpha \neq 0 \quad \implies \quad N_p^\alpha(f) = N_{\frac{p}{\alpha}}(|f|^\alpha).$$

*In particular, we have  $N_\infty^\alpha(f) = N_\infty(|f|^\alpha)$  when  $\alpha \neq 0$ , and  $N_p^0(f) = N_\infty(|f|^0) = 1$  when  $\mu$  is nontrivial.*

*Proof.* From Lemma 841 ( *$\mathcal{M}$  is closed under power function*, with  $p \stackrel{\text{def.}}{=} \alpha$  finite),  $|f|^\alpha$  belongs to  $\mathcal{M}_+ \subset \mathcal{M}$  when  $f \in \mathcal{M}$ , and to  $\mathcal{M}_\mathbb{R} \cap \mathcal{M}_+ \subset \mathcal{M}$  when  $f \in \mathcal{M}_\mathbb{R}$ .

**Case  $p, \alpha \in (0, \infty)$ .** From **ordered field properties of  $\mathbb{R}$  (thus  $\frac{1}{p}, \frac{p}{\alpha} \in (0, \infty)$ )**, Lemma 1286 ( *$p$ -seminorm*, with  $(p, f)$ , then  $(\frac{p}{\alpha}, |f|^\alpha)$ ), Lemma 366 (*compatibility of power function in  $\bar{\mathbb{R}}$  with multiplication*, with  $(p, q) \stackrel{\text{def.}}{=} (\frac{1}{p}, \alpha)$ , then  $(\alpha, \frac{p}{\alpha})$ ), and Lemma 443 (*compatibility of power function in  $\bar{\mathbb{R}}$  with absolute value (measure theory)*, with  $p \stackrel{\text{def.}}{=} \alpha$ ), we have

$$N_p^\alpha(f) = \left( \left( \int |f|^p \, d\mu \right)^{\frac{1}{p}} \right)^\alpha = \left( \int |f|^p \, d\mu \right)^{\frac{\alpha}{p}} = \left( \int ||f|^\alpha|^{\frac{p}{\alpha}} \, d\mu \right)^{\frac{\alpha}{p}} = N_{\frac{p}{\alpha}}(|f|^\alpha).$$

**Case  $p = \infty$  or  $\alpha = 0$ .** Then, from Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$ , thus  $\frac{1}{\alpha} = \infty$* ), Definition 326 (*multiplication in  $\overline{\mathbb{R}}$ , with  $p > 0$ , thus  $\frac{p}{\alpha} = \infty$* ), and Lemma 1286 ( *$p$ -seminorm, with  $p \stackrel{\text{def.}}{=} \infty$* ), we have  $N_{\frac{p}{\alpha}} = N_{\infty}$ . **Case  $p = \infty$  and  $\alpha \in (0, \infty)$ .** Direct consequence of Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*), and Lemma 1054 (*nonnegative essential supremum is compatible with power function, with any measure  $\mu$ , thus  $N_{\infty}^{\alpha}(f) = N_{\infty}(|f|^{\alpha})$* ). **Case  $\mu$  nontrivial, and  $\alpha = 0$ .** Direct consequence of Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory), thus  $N_p^0(f) = |f|^0 = 1$* ), and Lemma 1040 (*nonnegative essential supremum of constant function, thus  $N_{\infty}(|f|^0) = 1 = N_p^0(f)$* ).

Therefore, the equality always holds.  $\square$

**Lemma 1295 (compatibility of  $N_p$  with absolute value).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{M}$ . Then,  $|f| \in \mathcal{M}_+ \subset \mathcal{M}$  and we have  $N_p(|f|) = N_p(f)$ .

*Proof.* Direct consequence of Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$ , thus  $\frac{p}{1} = p$* ), Lemma 357 (*exponentiation in  $\overline{\mathbb{R}}$  of power 1, thus  $|f|^1 = |f|$  and  $N_p^1(f) = N_p(f)$* ), and Lemma 1294 (*compatibility of  $N_p$  with power function, with  $\alpha \stackrel{\text{def.}}{=} 1$* ).  $\square$

**Lemma 1296 ( $N_p$  is almost definite).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Then,  $N_p$  is almost definite:

$$(16.34) \quad \forall f \in \mathcal{M}, \quad N_p(f) = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* **Case  $p \in (0, \infty)$ .** Direct consequence of Lemma 1286 ( *$p$ -seminorm*), **ordered field properties of  $\mathbb{R}$  (thus  $\frac{1}{p} \in (0, \infty)$** ), Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite, with  $p \stackrel{\text{def.}}{=} \frac{1}{p}$* ), Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite, with  $f \stackrel{\text{def.}}{=} |f|^p$* ), and Lemma 1007 (*power function is almost definite, with  $\frac{1}{p}$  and  $p$* ).

**Case  $p = \infty$ .**

Direct consequence of Lemma 1286 ( *$p$ -seminorm, with  $p \stackrel{\text{def.}}{=} \infty$* ), Lemma 1050 (*nonnegative essential supremum is almost definite*), and Lemma 1005 (*absolute value is almost definite*).

Therefore,  $N_p$  is always almost definite.  $\square$

**Lemma 1297 (compatibility of  $N_p$  with almost equality).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f, g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then, we have  $N_p(f) = N_p(g)$ .

*Proof.* From Lemma 985 (*compatibility of almost equality with operator, with the absolute value and the power function of degree  $p$* ), we have  $|f| \stackrel{\mu \text{ a.e.}}{=} |g|$ , and  $|f|^p \stackrel{\mu \text{ a.e.}}{=} |g|^p$ .

**Case  $p \in (0, \infty)$ .** Direct consequence of Lemma 1262 (*compatibility of integral with almost equality*), Lemma 361 (*power function in  $\overline{\mathbb{R}}$ , with  $\frac{1}{p} > 0$* ), and Lemma 1286 ( *$p$ -seminorm*).

**Case  $p = \infty$ .** Direct consequence of Lemma 842 ( *$\mathcal{M}$  is closed under absolute value,  $|f|, |g| \in \mathcal{M}_+$* ), Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*), and Lemma 1286 ( *$p$ -seminorm*).

Therefore, the equality always holds.  $\square$

**Lemma 1298 ( $N_p$  is monotone for nonnegative functions).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f, g \in \mathcal{M}$ . Then, we have

$$(16.35) \quad 0 \leq f \leq g \implies N_p(f) \leq N_p(g),$$

$$(16.36) \quad |f| \leq g \implies N_p(f) \leq N_p(g).$$

Moreover, in both cases when  $p$  is finite,  $|g|^p$  is  $\mu$ -integrable implies  $|f|^p$  is  $\mu$ -integrable.

*Proof.* **Property (16.35).** From Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), we have  $f, g \in \mathcal{M}_+$ . **Case  $p \in (0, \infty)$ .** Direct consequence of Lemma 367 (power function in  $\bar{\mathbb{R}}$  is monotone, with  $f, g$  and  $p$ , then with  $\int |f|^p d\mu, \int |g|^p d\mu$  and  $\frac{1}{p}$ ), Lemma 841 ( $\mathcal{M}$  is closed under power function), Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 1286 ( $p$ -seminorm). **Case  $p = \infty$ .** Direct consequence of Lemma 958 (everywhere is almost everywhere, thus  $f \stackrel{\mu \text{ a.e.}}{\leq} g$ ), Lemma 1045 (nonnegative essential supremum is almost monotone), and Lemma 1286 ( $p$ -seminorm).

**Property (16.36).** Direct consequence of Lemma 342 (absolute value in  $\bar{\mathbb{R}}$  is nonnegative, thus  $|f| \geq 0$ ), Lemma 1295 (compatibility of  $N_p$  with absolute value, thus  $|f| \in \mathcal{M}$ , and  $N_p(|f|) = N_p(f)$ ), and Property (16.35) (with  $f \stackrel{\text{def.}}{=} |f|$ ).

**Integrability.** Direct consequence of Properties (16.35) and (16.36) (thus  $N_p(f) \leq N_p(g)$ ), Lemma 367 (power function in  $\bar{\mathbb{R}}$  is monotone, thus  $N_p^p(|f|^p) \leq N_p^p(|g|^p)$ ), Lemma 1291 (link between  $N_1$  and  $N_p$ , thus  $N_1(|f|^p) = N_p^p(f)$  and  $N_1(|g|^p) = N_p^p(g)$ ), Lemma 313 (order in  $\bar{\mathbb{R}}$  is total, transitivity), and Lemma 1292 (integrable is finite  $N_1$ , with  $g$  and  $f$ ).  $\square$

**Lemma 1299** ( $N_p$  is almost monotone for almost nonnegative functions).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f, g \in \mathcal{M}$ . Then, we have

$$(16.37) \quad 0 \stackrel{\mu \text{ a.e.}}{\leq} f \stackrel{\mu \text{ a.e.}}{\leq} g \implies N_p(f) \leq N_p(g),$$

$$(16.38) \quad |f| \stackrel{\mu \text{ a.e.}}{\leq} g \implies N_p(f) \leq N_p(g).$$

Moreover, in both cases when  $p$  is finite,  $|g|^p$  is  $\mu$ -integrable implies  $|f|^p$  is  $\mu$ -integrable.

*Proof.* Direct consequence of Lemma 1298 ( $N_p$  is monotone for nonnegative functions, the integrability property is unchanged), Lemma 960 (countable almost properties is almost property, with  $I \stackrel{\text{def.}}{=} \{0, 1\}$ ,  $P_0 \stackrel{\text{def.}}{=} (0 \leq f)$  and  $P_1 \stackrel{\text{def.}}{=} (f \leq g)$ ), Lemma 1297 (compatibility of  $N_p$  with almost equality, thus  $P_2(f, g) \stackrel{\text{def.}}{=} (N_p(f) \leq N_p(g))$  is compatible with almost equality), and Lemma 965 (extended almost modus ponens (simplified 0), with  $\text{card}(I) = 2$ ,  $P(f, g) \stackrel{\text{def.}}{=} P_0 \wedge P_1$  and  $S \stackrel{\text{def.}}{=} P_2$ , then  $P(f, g) \stackrel{\text{def.}}{=} (|f| \leq g)$  and the same  $S$ ).  $\square$

**Lemma 1300** ( $N_p$  is absolutely homogeneous).

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $p \in (0, \infty]$ . Then,  $N_p$  is absolutely homogeneous of degree 1:

$$(16.39) \quad \forall a \in \bar{\mathbb{R}}, \forall f \in \mathcal{M}, \quad af \in \mathcal{M} \quad \wedge \quad N_p(af) = |a|N_p(f).$$

*Proof.* From Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication), we have  $af \in \mathcal{M}$ .

**Case  $p \in (0, \infty)$ .** Direct consequence of Lemma 1286 ( $p$ -seminorm, thus  $|af|^p, |f|^p \in \mathcal{M}_+$ ), Lemma 446 (power function in  $\bar{\mathbb{R}}$  is multiplicative (measure theory)), Lemma 439 (power function in  $\bar{\mathbb{R}}$  is nonnegative (measure theory), thus  $|a|^p \geq 0$ ), Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous, for  $a$  finite, thus  $\tilde{N}_p$  is absolutely homogeneous of degree  $p$ ), Lemma 361 (power function in  $\bar{\mathbb{R}}$ , thus  $|\pm \infty|^p = \infty$ ), Lemma 1181 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ), Lemma 366 (compatibility of power function in  $\bar{\mathbb{R}}$  with multiplication, with  $q \stackrel{\text{def.}}{=} \frac{1}{p}$ ), and **field properties of  $\mathbb{R}$** .

**Case  $p = \infty$ .** Direct consequence of Lemma 1286 ( $p$ -seminorm, thus  $|af|, |f| \in \mathcal{M}_+$ ), Lemma 429 (absolute value in  $\bar{\mathbb{R}}$  is multiplicative (measure theory)), Lemma 342 (absolute value in  $\bar{\mathbb{R}}$  is nonnegative, thus  $|a| \geq 0$ ), Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication,  $|a||f| \in \mathcal{M}_+$ ), and Lemma 1044 (nonnegative essential supremum is positive homogeneous).

Therefore,  $N_p$  is always absolutely homogeneous.  $\square$

**Lemma 1301** ( $N_p$  of constant function). *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $a \in \overline{\mathbb{R}}$ . Let  $p \in (0, \infty)$ . Then,  $a\mathbf{1}_A \in \mathcal{M}$ , and we have*

$$(16.40) \quad N_p(a\mathbf{1}_A) = |a|(\mu(A))^{\frac{1}{p}}, \quad \text{and} \quad N_\infty(a\mathbf{1}_A) = \begin{cases} |a| & \text{if } \mu(A) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Direct consequence of Lemma 1289 ( $N_p$  of indicator function), Lemma 1300 ( $N_p$  is absolutely homogeneous), and Lemma 328 (one is identity element for multiplication in  $\overline{\mathbb{R}}$ ).  $\square$

### 16.3.2 Hölder inequality

**Lemma 1302 (generalized Hölder inequality (with  $\infty$ )).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f, g \in \mathcal{M}$ . Then,  $fg \in \mathcal{M}$ , and we have

$$(16.41) \quad N_p(fg) \leq N_\infty(f)N_p(g) \quad \wedge \quad N_p(fg) \leq N_p(f)N_\infty(g).$$

*Proof.* From Lemma 824 ( $\mathcal{M}$  is closed under multiplication), Lemma 1286 ( $p$ -seminorm, with  $p$  and  $\infty$ ), and Lemma 425 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)), we have  $fg \in \mathcal{M}$ , and  $N_p(fg)$ ,  $N_p(f)$ ,  $N_\infty(f)$ ,  $N_p(g)$ ,  $N_\infty(g)$ ,  $N_\infty(f)N_p(g)$  and  $N_p(f)N_\infty(g)$  exist and belong to  $\overline{\mathbb{R}}_+$ .

**Case  $p \in (0, \infty)$ .** From Lemma 1286 ( $p$ -seminorm), Lemma 1039 (nonnegative essential supremum is almost upper bound, with  $f \in \mathcal{M}$ ), we have  $|f| \stackrel{\mu \text{ a.e.}}{\leq} N_\infty(f)$ . Then, from Lemma 444 (power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)), Lemma 420 (multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)), Lemma 984 (compatibility of almost binary relation with operator, with unary operator power function, then the unary operator right multiplication by  $|g(x)|^p$ ), and Lemma 446 (power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)), we have  $|fg|^p \stackrel{\mu \text{ a.e.}}{\leq} N_\infty^p(f)|g|^p$ . Thus, from Lemma 1191 (integral in  $\mathcal{M}_+$  is almost monotone), Lemma 439 (power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory), thus  $N_\infty^p(f) \geq 0$ ), and Lemma 1184 (integral in  $\mathcal{M}_+$  is positive linear), we have

$$N_p^p(fg) = \int |fg|^p d\mu \leq N_\infty^p(f) \int |g|^p d\mu = N_\infty^p(f)N_p^p(g).$$

Hence, from Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone, with  $\frac{1}{p}$ ), Lemma 446 (power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)), Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication, with  $q \stackrel{\text{def.}}{=} \frac{1}{p}$ ), and **field properties of  $\mathbb{R}$** , we have  $N_p(fg) \leq N_\infty(f)N_p(g)$ . Then, by exchanging the roles of  $f$  and  $g$ , and from Lemma 418 (multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)), we also have  $N_p(fg) \leq N_p(f)N_\infty(g)$ .

**Case  $p = \infty$ .** Direct consequence of Lemma 1286 ( $p$ -seminorm), and Lemma 1053 (nonnegative essential supremum is submultiplicative).

Therefore, the inequalities always hold.  $\square$

*Remark 1303.* Note that from now on, we only consider  $p$ -seminorms with  $p \geq 1$ .

**Lemma 1304 (Hölder inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p, q \in [1, \infty]$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $f, g \in \mathcal{M}$ . Then,  $fg \in \mathcal{M}$ , and we have

$$(16.42) \quad N_1(fg) \leq N_p(f)N_q(g).$$

*Proof.* From Lemma 824 ( $\mathcal{M}$  is closed under multiplication), we have  $fg \in \mathcal{M}$ . Thus, from Definition 451 (Hölder conjugates), and Lemma 1286 ( $p$ -seminorm),  $N_1(fg)$ ,  $N_p(f)$ , and  $N_q(g)$  exist, and belong to  $\overline{\mathbb{R}}_+$ . Moreover, from Lemma 425 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)), we have  $N_p(f)N_q(g) \in \overline{\mathbb{R}}_+$ .

**Case  $N_p(f)N_q(g) = 0$ .** Direct consequence of Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory), thus  $N_p(f) = 0$  or  $N_q(g) = 0$ ), Lemma 1296 ( $N_p$  is almost definite, thus  $f \stackrel{\mu \text{ a.e.}}{=} 0$  or  $g \stackrel{\mu \text{ a.e.}}{=} 0$ ), Lemma 985 (compatibility of almost equality with operator, thus  $fg \stackrel{\mu \text{ a.e.}}{=} 0$ ), Lemma 1296 ( $N_p$  is almost definite, thus  $N_1(fg) = 0$ ), and **the definition of order (reflexivity, thus  $N_1(fg) = 0 \leq 0 = N_p(f)N_q(g)$ )**.

**Case  $N_p(f)N_q(g) = \infty$ .**

Direct consequence of Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $N_1(fg) \leq \infty = N_p(f)N_q(g)$ ).

**Case  $(p, q) = (1, \infty)$ , or  $(p, q) = (\infty, 1)$ .**

Direct consequence of Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*, with  $p = 1$ ).

**Case  $N_p(f)N_q(g) \in (0, \infty)$ , and  $p, q \in (1, \infty)$ .**

Then, from Lemma 1290 ( *$N_p$  is nonnegative, with  $p$  and  $q$* ), Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory), contrapositive*), and Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$ , contrapositive*), we have  $N_p(f), N_q(g) \in (0, \infty)$ . Thus, from **ordered field properties of  $\mathbb{R}$**  ( $1/N_p(f), 1/N_q(g) \in (0, \infty)$ ), and Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ), let  $\tilde{f} \stackrel{\text{def.}}{=} f/N_p(f)$  and  $\tilde{g} \stackrel{\text{def.}}{=} g/N_q(g)$ . Hence, from Lemma 1300 ( *$N_p$  is absolutely homogeneous, with  $p, q$  and 1*), Lemma 1290 ( *$N_p$  is nonnegative, with  $p$  and  $q$* ), and **valued field properties of  $\mathbb{R}$** , we have

$$N_p(\tilde{f}) = N_q(\tilde{g}) = 1, \quad \text{and} \quad N_1(\tilde{f}\tilde{g}) = \frac{N_1(fg)}{N_p(f)N_q(g)}.$$

Let  $x \in X$ . Then, from Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*), Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*), and Lemma 453 (*Young's inequality for products (measure theory)*), we have in  $\overline{\mathbb{R}}_+$

$$\left| \tilde{f}(x)\tilde{g}(x) \right| = |\tilde{f}(x)| |\tilde{g}(x)| \leq \frac{|\tilde{f}(x)|^p}{p} + \frac{|\tilde{g}(x)|^q}{q}.$$

Moreover, from Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*), Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*), Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*), Lemma 841 ( *$\mathcal{M}$  is closed under power function*), **ordered field properties of  $\mathbb{R}$**  (thus  $\frac{1}{p}, \frac{1}{q} \in (0, 1) \subset \overline{\mathbb{R}}_+$ ), Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*), and Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*), we have

$$\tilde{f}, \tilde{g}, \tilde{f}\tilde{g} \in \mathcal{M}, \quad \text{and} \quad |\tilde{f}|, |\tilde{g}|, |\tilde{f}\tilde{g}|, |\tilde{f}|^p, |\tilde{g}|^q, \frac{|\tilde{f}|^p}{p}, \frac{|\tilde{g}|^q}{q}, \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} \in \mathcal{M}_+.$$

Thus, from Lemma 1286 ( *$p$ -seminorm, with  $p \stackrel{\text{def.}}{=} 1$* ), Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*), Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*), and Definition 451 (*Hölder conjugates*), we have

$$\begin{aligned} \frac{N_1(fg)}{N_p(f)N_q(g)} &= N_1(\tilde{f}\tilde{g}) = \int |\tilde{f}\tilde{g}| d\mu \leq \int \left( \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} \right) d\mu \\ &= \frac{1}{p} \int |\tilde{f}|^p d\mu + \frac{1}{q} \int |\tilde{g}|^q d\mu = \frac{N_p(\tilde{f})}{p} + \frac{N_q(\tilde{g})}{q} = \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Hence, from **ordered field properties of  $\mathbb{R}$** , we have  $N_1(fg) \leq N_p(f)N_q(g)$ .

Therefore, the inequality always holds.  $\square$

**Lemma 1305 (generalized Hölder inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $p, p_1, p_2 \in [1, \infty]$  such that  $\frac{1}{p} \stackrel{\text{def.}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ . Let  $f_1, f_2 \in \mathcal{M}$ . Then,  $f_1 f_2 \in \mathcal{M}$ , and

$$(16.43) \quad N_p(f_1 f_2) \leq N_{p_1}(f_1) N_{p_2}(f_2).$$

*Proof.* From Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*), we have  $f_1 f_2 \in \mathcal{M}$ .

**Case  $p_1, p_2 \in [1, \infty)$ .** Then,  $p$  is finite. Let  $q_1 \stackrel{\text{def.}}{=} \frac{p_1 + p_2}{p_2}$ , and  $q_2 \stackrel{\text{def.}}{=} \frac{p_1 + p_2}{p_1}$ .

Then, from **ordered field properties of  $\mathbb{R}$** , and Definition 451 (*Hölder conjugates*),  $q_1$  and  $q_2$  are Hölder conjugates in  $(1, \infty)$ , and  $\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{p_1 p_2}{p_1 + p_2} = p$ . Let  $g_1 \stackrel{\text{def.}}{=} |f_1|^{p_1}$ , and  $g_2 \stackrel{\text{def.}}{=} |f_2|^{p_2}$ .

Let  $i \in \{1, 2\}$ . Then, from Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*), we have  $|g_i|^{q_i} = |f_i|^{p_i}$ .

Then, from Lemma 841 ( *$\mathcal{M}$  is closed under power function, thus  $g_1, g_2 \in \mathcal{M}_+ \subset \mathcal{M}$* ), Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication, thus  $g_1 g_2 \in \mathcal{M}_+$* ), Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory), thus  $g_1 g_2 = |f_1 f_2|^p$* ), Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory),  $|g_1 g_2| = g_1 g_2$* ), Lemma 1291 (*link between  $N_1$  and  $N_p$ , with  $(p, f_1 f_2)$ ,  $(q_i, g_i)$  and  $(p_i, f_i)$ , with  $i \in \{1, 2\}$* ), Lemma 1304 (*Hölder inequality, with  $q_1, q_2$  and  $g_1, g_2$* ), **field properties of  $\mathbb{R}$** , and Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone, with  $\frac{1}{p}$* ),

$$\begin{aligned} N_p^p(f_1 f_2) &= N_1(g_1 g_2) \leq N_{q_1}(g_1) N_{q_2}(g_2) \\ &= N_1^{\frac{1}{q_1}}(|g_1|^{q_1}) N_1^{\frac{1}{q_2}}(|g_2|^{q_2}) = N_1^{\frac{p_2}{p_1 + p_2}}(|f_1|^{p_1}) N_1^{\frac{p_1}{p_1 + p_2}}(|f_2|^{p_2}) \\ &= N_{\frac{p_1 p_2}{p_1 + p_2}}^{\frac{p_1 p_2}{p_1 + p_2}}(f_1) N_{\frac{p_1 p_2}{p_1 + p_2}}^{\frac{p_1 p_2}{p_1 + p_2}}(f_2) = N_{p_1}^p(f_1) N_{p_2}^p(f_2), \end{aligned}$$

and  $N_p(f_1 f_2) \leq N_{p_1}(f_1) N_{p_2}(f_2)$ .

**Case  $p_i = \infty$  with  $i \in \{1, 2\}$ .** Then, we have  $p_j = p$  with  $j = 3 - i$ .

Direct consequence of Lemma 1302 (*generalized Hölder inequality (with  $\infty$ ), with  $p_j, f_1$  and  $f_2$ , left inequality when  $i = 1$  and right inequality when  $i = 2$* ).

Therefore, the inequality always holds.  $\square$

**Lemma 1306 (compatibility of  $N_p$  with Hölder conjugates).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty)$ . Let  $q \in (1, \infty]$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $f \in \mathcal{M}$ . Then,  $|f|^{p-1}, |f|^0 \in \mathcal{M}$ , and we have

$$(16.44) \quad p > 1 \implies N_q(|f|^{p-1}) = N_p^{p-1}(f),$$

$$(16.45) \quad \mu \text{ nontrivial} \implies N_\infty(|f|^0) = N_1^0(f) = 1.$$

*Proof.* **Case  $p, q \in (1, \infty)$ .**

Then, from Definition 451 (*Hölder conjugates*), and **field properties of  $\mathbb{R}$  (with  $q \neq 0$ )**, we have  $pq = p + q$ ,  $(p - 1)q = p$ , and  $\frac{p}{q} = p - 1$ . Hence, from Lemma 1291 (*link between  $N_1$  and  $N_p$ , with  $(q, |f|^{p-1})$ , then  $(p, f)$* ), Lemma 1286 ( *$p$ -seminorm, with  $(q, |f|^{p-1})$ , then  $(p, f)$* ), and Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*), we have

$$\begin{aligned} \tilde{N}_q(|f|^{p-1}) &= N_1(|f|^{(p-1)q}) = N_1(|f|^p) = \tilde{N}_p(f), \quad \text{and} \\ N_q(|f|^{p-1}) &= \tilde{N}_p^{\frac{1}{q}}(f) = N_p^{\frac{p}{q}}(f) = N_p^{p-1}(f). \end{aligned}$$

**Case  $\mu$  nontrivial, and  $p = 1$ .**

Direct consequence of Definition 451 (*Hölder conjugates, thus  $q = \infty$* ), Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative, thus  $|f| \geq 0$* ), Lemma 1290 ( *$N_p$  is nonnegative, thus  $N_1(f) \geq 0$* ), Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory), thus  $|f|^0 = N_1^0(f) = 1$* ), and Lemma 1040 (*nonnegative essential supremum of constant function, with  $\mu$  nontrivial, thus  $N_\infty(|f|^0) = 1 = N_1^0(f)$* ).

Therefore, the equality always holds.  $\square$

*Remark 1307.* Note that the previous lemma does not hold when  $p = \infty$  (and  $q = 1$ ).

Indeed, let  $a, b \in \overline{\mathbb{R}}$  such that  $a < b$ , and let  $f = \mathbb{1}_{[a, b]}$ . Then, we have

$$\begin{aligned} N_q(f^{p-1}) &= N_1(\mathbb{1}_{[a, b]}^\infty) = N_1(\mathbb{1}_{[a, b]}) = \mu([a, b]), \\ N_p^{p-1}(f) &= N_\infty^\infty(\mathbb{1}_{[a, b]}) = 1^\infty = 1. \end{aligned}$$

The equality of both previous expressions is not compatible with the  $\sigma$ -additivity property of the measure  $\mu$ ; e.g. we would have

$$\mu([0, 2]) = \mu([0, 1]) = \mu([1, 2]) = 1, \quad \text{and} \quad \mu([0, 2]) = \mu([0, 1] \uplus [1, 2]) = 1 + 1 = 2.$$



### 16.3.3 Minkowski inequality

*Remark 1308.* In the next lemma, the summability domain  $\mathcal{D}^+(f, g)$  and the almost sum  $f \overset{\mu \text{ a.e.}}{+} g$  are respectively defined in Definition 997, and Lemma 1002.

**Lemma 1309 (Minkowski inequality in  $\mathcal{M}$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $f, g, f', g' \in \mathcal{M}$ . Assume that  $f + g$  is well-defined almost everywhere,  $f' \overset{\mu \text{ a.e.}}{=} f$ ,  $g' \overset{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f' + g', f \overset{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and we have*

$$(16.46) \quad N_p \left( f \overset{\mu \text{ a.e.}}{+} g \right) = N_p(f' + g') \leq N_p(f') + N_p(g') = N_p(f) + N_p(g).$$

Moreover, when  $\mathcal{D}^+(f, g) = X$ , e.g. when  $f, g \in \mathcal{M}_{\mathbb{R}}$ ,  $f + g \in \mathcal{M}$ , and we have

$$(16.47) \quad N_p(f + g) \leq N_p(f) + N_p(g).$$

*Proof.* From Lemma 1002 (*almost sum, such  $f'$  and  $g'$  exist*), and Lemma 1003 (*compatibility of almost sum with almost equality*), we have  $f' + g', f \overset{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and  $f' + g' \overset{\mu \text{ a.e.}}{=} f \overset{\mu \text{ a.e.}}{+} g$ . Hence, from Lemma 1286 (*p-seminorm*), and Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  $N_p(f), N_p(g), N_p(f'), N_p(g'), N_p(f' + g'), N_p \left( f \overset{\mu \text{ a.e.}}{+} g \right)$  are well-defined in  $\overline{\mathbb{R}}_+$ , and we have

$$\begin{aligned} N_p(f') &= N_p(f), & N_p(g') &= N_p(g) & (\text{thus } N_p(f') + N_p(g') &= N_p(f) + N_p(g)), \\ & & \text{and } N_p(f' + g') &= N_p \left( f \overset{\mu \text{ a.e.}}{+} g \right). \end{aligned}$$

Moreover, from Lemma 841 ( *$\mathcal{M}$  is closed under power function*), and Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*), for all  $p \in [1, \infty)$ , we have

$$|f|^p, |g|^p, |f'|^p, |g'|^p, |f' + g'|^p, |f'|^p + |g'|^p, \left| f \overset{\mu \text{ a.e.}}{+} g \right|^p \in \mathcal{M}_+ \subset \mathcal{M}.$$

**Case  $N_p(f) = \infty$  or  $N_p(g) = \infty$ .** Direct consequence of Lemma 1290 ( *$N_p$  is nonnegative*), Definition 317 (*addition in  $\overline{\mathbb{R}}$ ,  $N_p(f) + N_p(g) = \infty$* ), Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ).

**Case  $N_p(f' + g') = 0$ .** Direct consequence of Lemma 1290 ( *$N_p$  is nonnegative*), and Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed,  $N_p(f) + N_p(g) \geq 0$* ).

**Case  $p = 1$ .** Direct consequence of Lemma 1286 (*p-seminorm, with  $f' + g'$ , then  $f'$  and  $g'$* ), and Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*).

**Case  $p = \infty$ .** Direct consequence of Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality, with  $f' + g'$  well-defined everywhere*), Lemma 958 (*everywhere is almost everywhere*), Lemma 1045 (*nonnegative essential supremum is almost monotone, with  $f \stackrel{\text{def.}}{=} |f' + g'|, g \stackrel{\text{def.}}{=} |f'| + |g'| \in \mathcal{M}_+$* ), Lemma 1051 (*nonnegative essential supremum is subadditive, with  $f \stackrel{\text{def.}}{=} |f'|, g \stackrel{\text{def.}}{=} |g'| \in \mathcal{M}_+$* ), and Lemma 1286 (*p-seminorm*).

**Case  $p \in (1, \infty)$ ,  $N_p(f), N_p(g) < \infty$ , and  $N_p(f' + g') > 0$ .** From Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*), and Lemma 371 (*power function satisfies extended relaxed triangle inequality, with  $f' + g'$  well-defined everywhere*), we have

$$0 \leq |f' + g'|^p \leq 2^{p-1}(|f'|^p + |g'|^p).$$

Then, from Lemma 1286 (*p-seminorm*), Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear, additivity and positive homogeneity*), Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*), Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*), **compatibility of the power function in  $\mathbb{R}$  with multiplication ( $(2^{p-1})^{\frac{1}{p}} = 2^{1-\frac{1}{p}}$ )**, Lemma 1291



(link between  $N_1$  and  $N_p$ ), closedness of the power function in  $\mathbb{R}_+$  (with powers  $1 - \frac{1}{p}$ ,  $p$  and  $\frac{1}{p}$ , all in  $(0, \infty)$ ), closedness of addition in  $\mathbb{R}_+$ , and closedness of multiplication in  $\mathbb{R}_+$  (in short, all terms in the right-hand side are finite), we have

$$\begin{aligned}\tilde{N}_p(f' + g') &= \int |f' + g'|^p d\mu \leq 2^{p-1}(\tilde{N}_p(f') + \tilde{N}_p(g')), \quad \text{and} \\ N_p(f' + g') &= \tilde{N}_p^{\frac{1}{p}}(f' + g') \leq 2^{1-\frac{1}{p}}(N_p^p(f') + N_p^p(g'))^{\frac{1}{p}} < \infty.\end{aligned}$$

Moreover, from Lemma 365 (compatibility of power function in  $\bar{\mathbb{R}}$  with addition, with  $p \stackrel{\text{def.}}{=} 1$  and  $q \stackrel{\text{def.}}{=} p - 1 > 0$ ), Lemma 349 (absolute value in  $\bar{\mathbb{R}}$  satisfies triangle inequality, with  $f' + g'$  well-defined everywhere), Lemma 342 (absolute value in  $\bar{\mathbb{R}}$  is nonnegative), Lemma 439 (power function in  $\bar{\mathbb{R}}$  is nonnegative (measure theory), with  $p - 1 > 0$ ), Lemma 420 (multiplication in  $\bar{\mathbb{R}}$  is compatible with order (measure theory)), Lemma 419 (multiplication in  $\bar{\mathbb{R}}$  is distributive over addition (measure theory)), and Lemma 429 (absolute value in  $\bar{\mathbb{R}}$  is multiplicative (measure theory)),

$$|f' + g'|^p = |f' + g'| |f' + g'|^{p-1} \leq (|f'| + |g'|) |f' + g'|^{p-1} = |f'| |f' + g'|^{p-1} + |g'| |f' + g'|^{p-1}.$$

Hence, from Lemma 1286 ( $p$ -seminorm, with  $p$ , then 1), Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $p - 1 \in (0, \infty)$ ), Lemma 824 ( $\mathcal{M}$  is closed under multiplication,  $f' |f' + g'|^{p-1}$  and  $g' |f' + g'|^{p-1}$  belong to  $\mathcal{M}$ ), Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 1183 (integral in  $\mathcal{M}_+$  is additive), we have

$$\begin{aligned}\tilde{N}_p(f' + g') &= \int |f' + g'|^p d\mu \leq \int |f'| |f' + g'|^{p-1} d\mu + \int |g'| |f' + g'|^{p-1} d\mu \\ &= N_1(f' |f' + g'|^{p-1}) + N_1(g' |f' + g'|^{p-1}).\end{aligned}$$

Let  $q$  be the Hölder conjugate of  $p$ . Then, from Lemma 1291 (link between  $N_1$  and  $N_p$ ), Lemma 1304 (Hölder inequality, with  $(f, |f' + g'|^{p-1})$  and  $(g, |f' + g'|^{p-1})$ ), Lemma 419 (multiplication in  $\bar{\mathbb{R}}$  is distributive over addition (measure theory)), and Lemma 1306 (compatibility of  $N_p$  with Hölder conjugates, with  $p \in (1, \infty)$ ), we have

$$\begin{aligned}N_p^p(f' + g') &= \tilde{N}_p(f' + g') \leq (N_p(f') + N_p(g')) N_q(|f' + g'|^{p-1}) \\ &= (N_p(f') + N_p(g')) N_p^{p-1}(f' + g').\end{aligned}$$

Hence, from ordered field properties of  $\mathbb{R}$  (with  $N_p^{p-1}(f' + g') \in (0, \infty)$ ), we have

$$N_p(f' + g') \leq N_p(f') + N_p(g').$$

Therefore, the inequality always holds.

The specific case  $\mathcal{D}^+(f, g) = X$  is a direct consequence of Lemma 1004 (almost sum is sum).  $\square$

**Lemma 1310 (Minkowski equality (nonnegative functions)).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g, f', g' \in \mathcal{M}$ . Assume that  $f, g \stackrel{\mu \text{ a.e.}}{\geq} 0$ ,  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$  and  $\mathcal{D}^+(f', g') = X$ . Then,  $f + g$  is well-defined almost everywhere,  $f' + g', f' \stackrel{\mu \text{ a.e.}}{+} g' \in \mathcal{M}$ , and we have

$$(16.48) \quad N_1\left(f \stackrel{\mu \text{ a.e.}}{+} g\right) = N_1(f' + g') = N_1(f') + N_1(g') = N_1(f) + N_1(g).$$

Moreover, when  $\mathcal{D}^+(f, g) = X$ , e.g. when  $f, g \in \mathcal{M}_+$ ,  $f + g \in \mathcal{M}$ , and we have

$$(16.49) \quad N_1(f + g) = N_1(f) + N_1(g).$$

*Proof.* From Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*), and Lemma 1003 (*compatibility of almost sum with almost equality*),  $f + g$  is well-defined almost everywhere,  $f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ ,  $f', g', f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \stackrel{\mu \text{ a.e.}}{\geq} 0$ ,  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ , and we have

$$\int (f' + g')^+ d\mu = \int (f')^+ d\mu + \int (g')^+ d\mu.$$

Then, from Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*),

$$\int (f')^- d\mu = \int (g')^- d\mu = \int (f' + g')^- d\mu = 0.$$

Hence, from Lemma 1286 ( *$p$ -seminorm, with  $p \stackrel{\text{def.}}{=} 1$* ), and Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*), we have

$$\begin{aligned} N_1(f' + g') &= \int |f' + g'| d\mu = \int (f' + g')^+ d\mu + \int (f' + g')^- d\mu \\ &= \int (f')^+ d\mu + \int (g')^+ d\mu = \int (f')^+ d\mu + \int (f')^- d\mu + \int (g')^+ d\mu + \int (g')^- d\mu \\ &= \int |f'| d\mu + \int |g'| d\mu = N_1(f') + N_1(g'). \end{aligned}$$

Therefore, from Lemma 1297 (*compatibility of  $N_p$  with almost equality, with  $(f \stackrel{\mu \text{ a.e.}}{+} g, f' + g')$ , then  $(f', f)$ , and  $(g', g)$* ), we have the equalities.  $\square$

**Lemma 1311 (Minkowski equality (linearly dependent functions)).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $f, g \in \mathcal{M}$ . Let  $a \in \overline{\mathbb{R}}_+$ . Assume that  $f = ag$ . Then,  $f + g$  is well-defined, belongs to  $\mathcal{M}$ , and we have

$$(16.50) \quad N_p(f + g) = N_p(f) + N_p(g).$$

*Proof.* From Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication,  $ag \in \mathcal{M}$* ), Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*), Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*), thus  $g(x) \geq 0 \Rightarrow ag(x) \geq 0$ , and  $g(x) \leq 0 \Rightarrow ag(x) \leq 0$ , and Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined, thus  $g$  and  $ag$  never take opposite infinite values at the same point*),  $f + g$  is well-defined, and belongs to  $\mathcal{M}$ .

Therefore, from Definition 317 (*addition in  $\overline{\mathbb{R}}$ , rule 1*), Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ), Lemma 1300 ( *$N_p$  is absolutely homogeneous, with  $a + 1$ , then  $a$* ), and Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*), we have  $|a + 1| = a + 1 = |a| + 1$ , and

$$N_p(f + g) = (|a| + 1)N_p(g) = N_p(f) + N_p(g).$$

$\square$

**Lemma 1312 (Minkowski equality (almost linearly dependent functions)).** Let  $(X, \Sigma, \mu)$

be a measure space. Let  $p \in [1, \infty]$ . Let  $f, g \in \mathcal{M}$ . Let  $a \in \overline{\mathbb{R}}_+$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} ag$ .

Then,  $f + g$  is well-defined almost everywhere,  $f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and we have

$$(16.51) \quad N_p\left(f \stackrel{\mu \text{ a.e.}}{+} g\right) = N_p(f) + N_p(g).$$

*Proof.* From Lemma 955 (*equivalent definition of property almost satisfied*), let  $B \in \Sigma$  such that  $B \subset \{f = ag\}$  and  $\mu(B^c) = 0$ . Let  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_B$ , and  $\tilde{g} \stackrel{\text{def.}}{=} g \mathbf{1}_B$ . Then, from Lemma 833

(*measurability and masking*, with  $f$  and  $g$ ), and Lemma 1008 (*masking almost nowhere*, with  $f$  and  $g$ ), we have  $\tilde{f}, \tilde{g} \in \mathcal{M}$ ,  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$  and  $\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g$ .

Let  $x \in X$ . **Case  $x \in B$ .** Then, from **the definition of the indicator function**, we have  $\tilde{f}(x) = f(x) = ag(x) = a\tilde{g}(x)$ . **Case  $x \notin B$ .** Then, from **the definition of the indicator function**, and Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*), we have  $\tilde{f}(x) = 0 = a\tilde{g}(x)$ . Hence, we have  $\tilde{f} = a\tilde{g}$ , and from Lemma 1311 (*Minkowski equality (linearly dependent functions)*),  $\tilde{f} + \tilde{g}$  is well-defined, belongs to  $\mathcal{M}$ , and we have  $N_p(\tilde{f} + \tilde{g}) = N_p(\tilde{f}) + N_p(\tilde{g})$ .

Therefore, from Lemma 1003 (*compatibility of almost sum with almost equality*), and Lemma 1297 (*compatibility of  $N_p$  with almost equality*), we have  $\tilde{f} + \tilde{g} \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ ,  $f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ ,  $f + g$  is well-defined almost everywhere,  $N_p(f \stackrel{\mu \text{ a.e.}}{+} g) = N_p(\tilde{f} + \tilde{g})$ ,  $N_p(\tilde{f}) = N_p(f)$ ,  $N_p(\tilde{g}) = N_p(g)$ , and the equality is satisfied.  $\square$

### 16.3.4 The seminormed vector space $\mathcal{L}^p$

*Remark 1313.*

Note that in the beginning of this section, we consider again  $p$ -seminorms with  $p > 0$ .

**Definition 1314** ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . The vector space of  $p$ -powered integrable functions (for  $p < \infty$ ), or the vector space of essentially bounded measurable functions (for  $p = \infty$ ), is denoted  $\mathcal{L}^p(X, \Sigma, \mu)$  (or simply  $\mathcal{L}^p$ ); it is defined by

$$(16.52) \quad \mathcal{L}^p(X, \Sigma, \mu) \stackrel{\text{def.}}{=} \{f \in \mathcal{M}_{\mathbb{R}} \mid N_p(f) < \infty\}.$$

*Remark 1315.*

The set  $\mathcal{L}^p$  is shown below to be a (seminormed) vector space when  $p \geq 1$ , hence its name.

**Lemma 1316** (equivalent definition of  $\mathcal{L}^p$ ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $p \in (0, \infty)$ .

Then,  $f \in \mathcal{L}^p$  iff  $f \in \mathcal{M}$  is finite, and  $|f|^p$  is  $\mu$ -integrable in  $\mathcal{M}_+$ , i.e.  $\int |f|^p d\mu < \infty$ .

In particular,  $f \in \mathcal{L}^1$  iff  $f$  is  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$  iff  $f \in \mathcal{M}$  is finite and  $\int |f| d\mu < \infty$ .

*Proof.* Direct consequence of Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), Lemma 1286 ( $p$ -seminorm), Lemma 363 (finite power function in  $\overline{\mathbb{R}}$ ), and Lemma 841 ( $\mathcal{M}$  is closed under power function).  $\square$

**Lemma 1317** ( $\mathcal{L}^p$  with trivial measure).

Let  $(X, \Sigma)$  be a measurable space. Let  $p \in (0, \infty]$ . Then, we have  $\mathcal{L}^p(X, \Sigma, 0) = \mathcal{M}_{\mathbb{R}}$ .

*Proof.* Direct consequence of Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), Lemma 1288 ( $N_p$  with trivial measure), and Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ).  $\square$

**Lemma 1318** ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ ).

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{L}^p$ . Assume that  $f \geq 0$ . Then, we have  $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $f \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), and Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).  $\square$

**Lemma 1319** (integral is positive linear form on  $\mathcal{L}^1$ ). Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $\mathcal{I} : \mathcal{L}^1 \rightarrow \mathbb{R}$  be the function defined by: for all  $f \in \mathcal{L}^1$ ,  $\mathcal{I}(f) \stackrel{\text{def.}}{=} \int f d\mu$ .

Then,  $\mathcal{I}$  is a nondecreasing positive linear form on  $\mathcal{L}^1$ .

Moreover, for all  $f \in \mathcal{L}^1$ , we have  $|\mathcal{I}(f)| \leq \mathcal{I}(|f|) = N_1(f)$ .

*Proof.* (0). **Existence.** Direct consequence of Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , with  $p \stackrel{\text{def.}}{=} 1$ ), and Lemma 1256 (integral in  $\mathcal{M}$ , thus  $\mathcal{I}(f)$  exists and belongs to  $\mathbb{R}$ ).

(1). **Monotonicity.** Direct consequence of Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , thus  $f, g \in \mathcal{L}^1$  are  $\mu$ -integrable in  $\mathcal{M}_{\mathbb{R}}$ ), and Lemma 1271 (integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone).

(2). **Nonnegativeness.** Direct consequence of Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ , thus  $f \in \mathcal{L}^1$  and  $f \geq 0$  implies  $f \in \mathcal{M}_+$ ), and Lemma 1172 (integral in  $\mathcal{M}_+$ , integral is nonnegative).

(3). **Linearity.** Direct consequence of Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ), and Lemma 1270 (integral in  $\mathcal{M}_{\mathbb{R}}$  is linear).

Therefore, from Definition 64 (linear map), and Definition 66 (linear form),  $\mathcal{I}$  is a positive linear form on  $\mathcal{L}^1$ .

(4). **Inequality.** Direct consequence of Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ), Lemma 1273 (compatibility of integral in  $\mathcal{M}$  with absolute value), and Lemma 1286 ( $p$ -seminorm).  $\square$

**Lemma 1320** (*integral in  $\mathcal{L}^1$  is almost monotone*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{L}^1$ . Then, we have

$$(16.53) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \implies \int f d\mu \leq \int g d\mu \quad (\text{in } \mathbb{R}).$$

*Proof.* Direct consequence of Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* , with  $p \stackrel{\text{def.}}{=} 1$ , thus  $f, g$  are  $\mu$ -integrable), and Lemma 1272 (*integral in  $\mathcal{M}$  is almost monotone*).  $\square$

**Lemma 1321** (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ).

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $p \in (0, \infty)$ . Let  $f \in \mathcal{M}$ , and  $g \in \mathcal{M}_+$ . Then,  $|f|^p, g^{\frac{1}{p}} \in \mathcal{M}_+ \subset \mathcal{M}$ , we have in  $\overline{\mathbb{R}}_+$

$$(16.54) \quad N_p^p(f) = \tilde{N}_p(f) = N_1(|f|^p), \quad N_p(f) = N_1^{\frac{1}{p}}(|f|^p),$$

$$(16.55) \quad N_1(g) = \int g d\mu = \int \left(g^{\frac{1}{p}}\right)^p d\mu = \tilde{N}_p\left(g^{\frac{1}{p}}\right) = N_p^p\left(g^{\frac{1}{p}}\right), \quad N_1^{\frac{1}{p}}(g) = N_p\left(g^{\frac{1}{p}}\right),$$

and the equivalences

$$(16.56) \quad f \in \mathcal{L}^p \iff |f|^p \in \mathcal{L}^1 \iff |f|^p \text{ is finite and } \mu\text{-integrable in } \mathcal{M}_+,$$

$$(16.57) \quad g \in \mathcal{L}^1 \iff g^{\frac{1}{p}} \in \mathcal{L}^p \iff g^{\frac{1}{p}} \text{ is finite, and } g \text{ is } \mu\text{-integrable in } \mathcal{M}_+.$$

*Proof.* Direct consequence of Lemma 1291 (*link between  $N_1$  and  $N_p$* ), Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*), Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ), and Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*).  $\square$

**Lemma 1322** ( *$\mathcal{L}^p$  is closed under absolute value*).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{L}^p$ . Then, we have  $|f| \in \mathcal{L}^p \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*), Lemma 842 ( *$\mathcal{M}$  is closed under absolute value, thus  $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$* ), Lemma 1295 (*compatibility of  $N_p$  with absolute value, thus  $N_p(|f|) < \infty$* ).  $\square$

**Lemma 1323** (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in (0, \infty]$ . Let  $f \in \mathcal{M}$ . Then, we have the equivalence

$$(16.58) \quad f \in \mathcal{L}^p \iff \exists g \in \mathcal{L}^p, |f| \leq g.$$

*Proof.* “Left” implies “right”. Direct consequence of Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value,  $g \stackrel{\text{def.}}{=} |f| \in \mathcal{L}^p$* ), and **reflexivity of order in  $\mathbb{R}$** .

“Right” implies “left”. Let  $g \in \mathcal{L}^p$ . Assume that  $|f| \leq g$ . Then, from Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* , thus  $g \in \mathcal{M}$  is finite), Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*), Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*), Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total, transitivity, with  $0 \leq |f| \leq g = |g| < \infty$* ), Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions, with  $|f|$  and  $g$* ), Lemma 341 (*finite absolute value in  $\overline{\mathbb{R}}$ , thus  $f$  is finite*), and Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ , with  $|f|$  and  $|f|$* ), we have  $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ ,  $g = |g| \in \mathcal{M}_+$ , and  $f \in \mathcal{M}_{\mathbb{R}}$ .

**Case  $p \in (0, \infty)$ .** From Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone, with  $p$ , then  $\frac{1}{p}$ , both in  $(0, \infty)$* ), Lemma 841 ( *$\mathcal{M}$  is closed under power function,  $|f|^p, |g|^p \in \mathcal{M}_+$* ), and Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*), we have  $|f|^p \leq g^p = |g|^p$ , and  $(\int |f|^p d\mu)^{\frac{1}{p}} \leq (\int |g|^p d\mu)^{\frac{1}{p}}$ .

**Case  $p = \infty$ .** From Lemma 1045 (*nonnegative essential supremum is almost monotone*), and Lemma 958 (*everywhere is almost everywhere*), we have  $\text{ess sup}^+ |f| \leq \text{ess sup}^+ g = \text{ess sup}^+ |g|$ .

Hence, in both cases, from Lemma 1286 ( *$p$ -seminorm*), Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, with  $g$ , then  $f$* ), and we have  $N_p(f) \leq N_p(g) < \infty$ , and  $f \in \mathcal{L}^p$ .

Therefore, we have the equivalence.  $\square$

**Lemma 1324 (finite  $N_p$  has finite representative in  $\mathcal{L}^p$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $p \in (0, \infty]$ . Assume that  $N_p(f)$  is finite, i.e. that  $|f|^p$  is  $\mu$ -integrable in  $\mathcal{M}_+$  when  $p$  is finite, or that  $f$  is essentially bounded when  $p \stackrel{\text{def.}}{=} \infty$ . Then, there exists  $\tilde{f} \in \mathcal{L}^p$  such that  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$  and  $N_p(f) = N_p(\tilde{f}) < \infty$ .

*Proof.* Let  $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ , and  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbb{1}_A$ . Then, from Lemma 821 ( $\mathcal{M}$  is closed under finite part), we have  $A \in \Sigma$  and  $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ .

**Case  $p \in (0, \infty)$ .** Then, from Lemma 439 (power function in  $\bar{\mathbb{R}}$  is nonnegative (measure theory), thus  $|f|^p = \pm \infty \Leftrightarrow |f|^p = \infty$ ), Lemma 1251 (integrable is almost finite, thus  $\mu(\{|f|^p = \infty\}) = 0$ ), and Lemma 363 (finite power function in  $\bar{\mathbb{R}}$ , thus  $A^c = \{|f|^p = \infty\}$ ), we have  $\mu(A^c) = 0$ .

**Case  $p = \infty$ .** Then, from Lemma 1048 (essentially bounded is almost finite), we have  $\mu(A^c) = 0$ .

Therefore, in both cases, from Lemma 1008 (masking almost nowhere, thus  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ), Lemma 1297 (compatibility of  $N_p$  with almost equality), and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have  $N_p(\tilde{f}) = N_p(f) < \infty$ , and  $\tilde{f} \in \mathcal{L}^p$ .  $\square$

*Remark 1325.* From the previous lemma, we can restrict the study of integrable functions to the sole case of real-valued functions, and rely on the finite part, see Definition 526.

**Lemma 1326 (integrable and nonnegative has finite representative in  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Assume that  $\int f d\mu$  is finite

Then, there exists  $\tilde{f} \in \mathcal{L}^1 \cap \mathcal{M}_+$  such that  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$  and  $N_1(f) = N_1(\tilde{f}) < \infty$ .

*Proof.* From Lemma 1172 (integral in  $\mathcal{M}_+$ , thus  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ), Lemma 1263 (finite representative of integrable in  $\mathcal{M}$ ), Lemma 1286 ( $p$ -seminorm, with  $p \stackrel{\text{def.}}{=} 1$ ), **properties of the absolute value (thus  $|f| = f$ )**, and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), there exists  $\tilde{f} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$  such that

$$f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}, \quad N_1(\tilde{f}) = \int \tilde{f} d\mu = \int f d\mu < \infty, \quad \text{and} \quad \tilde{f} \in \mathcal{L}^1.$$

$\square$

**Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p$  in  $(0, \infty]$ .

Then,  $\mathcal{L}^p$  is closed under scalar multiplication, and  $N_p$  is absolutely homogeneous of degree 1.

*Proof.* Let  $a \in \mathbb{R}$ . Let  $f \in \mathcal{L}^p \subset \mathcal{M}_{\mathbb{R}}$ . Then, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have  $N_p(f) < \infty$ . Hence, from Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , thus  $af \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1300 ( $N_p$  is absolutely homogeneous), and **closedness of the absolute value and multiplication in  $\mathbb{R}$** , we have  $N_p(af) = |a|N_p(f) < \infty$ .

Therefore, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions),  $af \in \mathcal{L}^p$ , and  $N_p$  is absolutely homogeneous of degree 1.  $\square$

**Lemma 1328 (constant function is  $\mathcal{L}^p$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ .

Let  $a \in \mathbb{R}$ . Let  $p \in (0, \infty)$ . Then,  $a\mathbb{1}_A \in \mathcal{L}^\infty$ , and when  $\mu(A) < \infty$ ,  $a\mathbb{1}_A \in \mathcal{L}^p$ , and we have

$$(16.59) \quad \int a\mathbb{1}_A d\mu = a\mu(A) \in \mathbb{R}.$$

*Proof.* Direct consequence of Lemma 1301 ( $N_p$  of constant function, thus  $a\mathbb{1}_A \in \mathcal{M}$  and equalities on  $p$ -seminorms), **closedness of multiplication in  $\mathbb{R}$** , Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ , thus  $a\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}$ ), **closedness of the power function in  $\mathbb{R}$  (thus  $N_p(a\mathbb{1}_A), N_\infty(a\mathbb{1}_A) < \infty$ )**, Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, with  $p \in (0, \infty)$  and  $p \stackrel{\text{def.}}{=} \infty$ ), and Lemma 1267 (integral in  $\mathcal{M}$  of constant function, thus (16.59) holds).  $\square$

**Lemma 1329 (first mean value inequality).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $f \in \mathcal{M}$ . Let  $m, M \in \mathbb{R}$ . Let  $p \in (0, \infty)$ . Assume that  $m \leq f|_A \leq M$ . Then,  $f \mathbf{1}_A \in \mathcal{L}^\infty$ , and when  $\mu(A) < \infty$ ,  $f \mathbf{1}_A \in \mathcal{L}^p$ , and we have*

$$(16.60) \quad m \mu(A) \leq \int_A f d\mu \leq M \mu(A).$$

*For instance, this holds with  $m \stackrel{\text{def.}}{=} \inf(f(A))$  and  $M \stackrel{\text{def.}}{=} \sup(f(A))$  when they are both finite, and then, inequalities are strict iff  $f|_A$  is not equal to one of its bounds  $\mu_A$ -almost everywhere.*

*Proof.* From Lemma 913 (*measure, nonnegativeness*), we have  $\mu(A) \geq 0$ .

Let  $a \in \mathbb{R}$ . Then, from Lemma 1328 (*constant function is  $\mathcal{L}^p$* ), we have  $a \mathbf{1}_A \in \mathcal{L}^\infty$ , and when  $\mu(A) < \infty$ ,  $a \mathbf{1}_A \in \mathcal{L}^p$ , and  $\int a \mathbf{1}_A d\mu = a\mu(A) \in \mathbb{R}$ .

Thus, from **ordered field properties of  $\mathbb{R}$  (thus  $a \stackrel{\text{def.}}{=} \max(-m, M) \in \mathbb{R}$ )**, and the previous result, we have  $g \stackrel{\text{def.}}{=} \max(-m, M) \mathbf{1}_A \in \mathcal{L}^\infty$ ,  $g \in \mathcal{L}^p$  when  $\mu(A) < \infty$ , and  $\int g d\mu$  equals  $\max(-m, M)\mu(A) \in \mathbb{R}$ .

**Case  $0 \leq m \leq M$ .** Then, we have  $-m \leq M$  and  $|f \mathbf{1}_A| \leq M = \max(-m, M)$ .

**Case  $m < 0 \leq M$ .** Then, we have  $|f \mathbf{1}_A| \leq \max(-m, M)$ .

**Case  $m \leq M < 0$ .** Then, we have  $M \leq -m$  and  $|f \mathbf{1}_A| \leq -m = \max(-m, M)$ .

Hence, in all cases, from Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$ , with  $|f \mathbf{1}_A| \leq g$* ), we have  $f \mathbf{1}_A \in \mathcal{L}^\infty$ , and when  $\mu(A) < \infty$ ,  $f \in \mathcal{L}^p$ .

Therefore, from **the definition of restriction of function and masking (thus we have  $m \mathbf{1}_A \leq f \mathbf{1}_A \leq M \mathbf{1}_A$ )**, Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$ ,  $\mathcal{I}$  is nondecreasing*), the first result (with  $a \stackrel{\text{def.}}{=} m, M$  and  $\mu(A) < \infty$ ), and Lemma 1274 (*integral over subset, with  $Y \stackrel{\text{def.}}{=} X$  and  $\hat{f} \stackrel{\text{def.}}{=} f$* ), we have

$$m \mu(A) \leq \int f \mathbf{1}_A d\mu = \int_A f d\mu = \int f \mathbf{1}_A d\mu \leq M \mu(A).$$

Then, from Definition 9 (*infimum, lower bound, thus  $m \leq f|_A$* ), and Definition 2 (*supremum, upper bound, thus  $f|_A \leq M$* ), this holds when  $m \stackrel{\text{def.}}{=} \inf(f(A))$  and  $M \stackrel{\text{def.}}{=} \sup(f(A))$ . Moreover, from Lemma 1274 (*integral over subset*), the first result (with  $a \stackrel{\text{def.}}{=} m$  and  $\mu(A)$  finite), Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ), **additive group properties of  $\mathbb{R}$** , Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ , thus  $f|_A - m$  belongs to  $\mathcal{M}_+(A, \Sigma \cap A)$* ), Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ), Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*), and Lemma 985 (*compatibility of almost equality with operator, with the unary operator translation by  $m$* ), we have

$$\int_A f d\mu = m \mu(A) \iff \int (f|_A - m) d\mu_A = 0 \iff f|_A \stackrel{\mu_A \text{ a.e.}}{=} m.$$

Similarly, with  $M - f|_A \in \mathcal{L}^1(A, \Sigma \cap A) \cap \mathcal{M}_+(A, \Sigma \cap A)$ , we have  $\int_A f d\mu = M \mu(A) \iff f|_A \stackrel{\mu_A \text{ a.e.}}{=} M$ . Hence, from **the tautology  $(P \Leftrightarrow Q) \Leftrightarrow (\neg P \Leftrightarrow \neg Q)$** , we have strict inequalities in (16.61) iff  $f|_A$  is not equal to  $m$  nor to  $M$   $\mu_A$ -almost everywhere.  $\square$

*Remark 1330.*

Note that in the previous lemma, when the measure is trivial, strict inequalities and the restriction of the function being not equal almost everywhere to one of its bounds are both wrong!

**Lemma 1331 (first mean value theorem).** *Let  $(X, \Sigma, \mu)$  be a finite measure space. Let  $A \in \Sigma$  be nonempty. Let  $f \in \mathcal{M}$ . Let  $m \stackrel{\text{def.}}{=} \inf(f(A))$  and  $M \stackrel{\text{def.}}{=} \sup(f(A))$ . Let  $p \in (0, \infty]$ . Assume that  $m, M \in \mathbb{R}$ ,  $(m, M) \subset f(A)$ , and  $f|_A$  is not equal to  $m$  nor  $M$   $\mu_A$ -almost everywhere. Then,  $f \mathbf{1}_A \in \mathcal{L}^\infty$ , and when  $\mu(A) < \infty$ ,  $f \mathbf{1}_A \in \mathcal{L}^p$ , and we have*

$$(16.61) \quad \exists x \in A, \quad \int_A f d\mu = f(x) \mu(A).$$



*Proof.* Direct consequence of Lemma 1329 (*first mean value inequality*, with strict inequalities, thus  $\mu(A) = 0$  is impossible), Lemma 913 (*measure, nonnegativeness*, thus  $\mu(A) \in \mathbb{R}_+^*$ ), **ordered field properties of  $\mathbb{R}$** , and Definition 229 (*interval*, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$ , there exists  $y \in (m, M) \subset f(A)$  and  $x \in A$  such that  $\int_A f d\mu = y \mu(A) = f(x) \mu(A)$ ).  $\square$

*Remark 1332.* Note that the previous lemma holds for instance when  $f$  is continuous and bounded.

*Remark 1333.*

Note that from now on, we consider again only  $p$ -seminorms and  $\mathcal{L}^p$  spaces with  $p \geq 1$ .

**Lemma 1334 (generalized Hölder inequality in  $\mathcal{L}^p$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $k \in [1, \infty)$ . Let  $p, (p_i)_{i \in [1..k]} \in [1, \infty)$  such that  $\frac{1}{p} \stackrel{\text{def.}}{=} \sum_{i \in [1..k]} \frac{1}{p_i} \leq 1$ . For all  $i \in [1..k]$ , let  $f_i \in \mathcal{L}^{p_i}$ . Then, we have  $f \stackrel{\text{def.}}{=} \prod_{i \in [1..k]} f_i \in \mathcal{L}^p$ , and

$$(16.62) \quad N_p(f) \leq \prod_{i \in [1..k]} N_{p_i}(f_i).$$

*Proof.* From Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), and Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra), for all  $i \in [1..k]$ , we have  $f_i \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$  and  $N_{p_i}(f_i) < \infty$ , and thus  $f \in \mathcal{M}_{\mathbb{R}}$ .

Therefore, from Lemma 1305 (*generalized Hölder inequality*), **induction on  $k$** , and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have

$$N_p(f) \leq \prod_{i \in [1..k]} N_{p_i}(f_i) < \infty, \quad \text{and} \quad f \in \mathcal{L}^p.$$

$\square$

**Lemma 1335 (interpolation inequality in  $\mathcal{L}^p$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p, q \in [1, \infty)$ . Assume that  $p < q$ . Let  $r \in [p, q]$ . Let  $\alpha \stackrel{\text{def.}}{=} \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}$ . Let  $f \in \mathcal{L}^p \cap \mathcal{L}^q$ .

Then, we have  $f \in \mathcal{L}^r$ , and

$$(16.63) \quad N_r(f) \leq N_p^\alpha(f) N_q^{1-\alpha}(f).$$

*Proof.* From Lemma 455 (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* ), we have

$$\alpha \in [0, 1], \quad \text{and} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

**Case  $\mu$  trivial.** Direct consequence of Lemma 1317 ( $\mathcal{L}^p$  with trivial measure, thus  $\mathcal{L}^p = \mathcal{L}^q = \mathcal{L}^r$  equal  $\mathcal{M}_{\mathbb{R}}$ ), Lemma 1288 ( $N_p$  with trivial measure, thus  $N_p(f) = N_q(f) = N_r(f) = 0$ ), Lemma 431 (*one-exponentiation property in  $\mathbb{R}$  (measure theory)*, thus  $0^0 = 0^{1-1} = 1$ ), Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*, thus  $0^1 = 0^{1-0} = 0^\alpha = 0$  when  $\alpha \in (0, 1)$ ), Lemma 318 (*zero is identity element for addition in  $\mathbb{R}$* ), and Lemma 313 (*order in  $\mathbb{R}$  is total, reflexivity*).

**Case  $\mu$  nontrivial.** From Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, with  $p$  and  $q$ ), **closedness of the absolute value in  $\mathbb{R}$** , and **compatibility of the power function in  $\mathbb{R}$  with addition**, we have  $f \in \mathcal{M}_{\mathbb{R}}$ ,  $N_p(f), N_q(f) < \infty$ , and  $|f| = |f|^\alpha |f|^{1-\alpha}$ . Then, from **ordered field properties of  $\mathbb{R}$  ( $\alpha, 1-\alpha \in [0, 1] \subset \mathbb{R}_+$ )**, Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $p \stackrel{\text{def.}}{=} \alpha$  or  $1-\alpha$ ), Lemma 1294 (*compatibility of  $N_p$  with power function*), Lemma 435 (*finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*, with  $a \stackrel{\text{def.}}{=} N_p(f)$  or  $N_q(f)$ )



and  $b \stackrel{\text{def.}}{=} \alpha$  or  $1 - \alpha$ , all finite), and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have

$$\begin{aligned} |f|^\alpha &\in \mathcal{M}_{\mathbb{R}}, & N_{\frac{p}{\alpha}}(|f|^\alpha) &= N_p^\alpha(f) < \infty, & |f|^\alpha &\in \mathcal{L}^{\frac{p}{\alpha}}, \quad \text{and} \\ |f|^{1-\alpha} &\in \mathcal{M}_{\mathbb{R}}, & N_{\frac{q}{1-\alpha}}(|f|^{1-\alpha}) &= N_q^{1-\alpha}(f) < \infty, & |f|^{1-\alpha} &\in \mathcal{L}^{\frac{q}{1-\alpha}}. \end{aligned}$$

Hence, from Lemma 1295 (compatibility of  $N_p$  with absolute value), Lemma 1334 (generalized Hölder inequality in  $\mathcal{L}^p$ , with  $(p, p_1, p_2) \stackrel{\text{def.}}{=} (r, \frac{p}{\alpha}, \frac{q}{1-\alpha}) \in [1, \infty]$ ,  $f_1 \stackrel{\text{def.}}{=} |f|^\alpha$  and  $f_2 \stackrel{\text{def.}}{=} |f|^{1-\alpha}$ ), and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have

$$N_r(f) = N_r(|f|) \leq N_{\frac{p}{\alpha}}(|f|^\alpha) N_{\frac{q}{1-\alpha}}(|f|^{1-\alpha}) = N_p^\alpha(f) N_q^{1-\alpha}(f) < \infty, \quad \text{and} \quad f \in \mathcal{L}^r.$$

Therefore, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we always have  $f \in \mathcal{L}^r$  and the inequality.  $\square$

**Lemma 1336 (Minkowski inequality in  $\mathcal{L}^p$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $f, g \in \mathcal{L}^p$ . Then, we have  $f + g \in \mathcal{L}^p$ , and  $N_p(f + g) \leq N_p(f) + N_p(g)$ . Moreover, let  $n \in \mathbb{N}$ , and  $(f_i)_{i \in [0..n]} \in \mathcal{L}^p$ . Then,  $\sum_{i \in [0..n]} f_i \in \mathcal{L}^p$ , and we have*

$$(16.64) \quad N_p \left( \sum_{i \in [0..n]} f_i \right) \leq \sum_{i \in [0..n]} N_p(f_i).$$

*Proof.* From Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), and Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space), we have  $f, g \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ,  $N_p(f)$  and  $N_p(g)$  are finite, and  $f + g \in \mathcal{M}_{\mathbb{R}}$  is well-defined everywhere.

Therefore, from Lemma 1309 (Minkowski inequality in  $\mathcal{M}$ ), and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have

$$N_p(f + g) \leq N_p(f) + N_p(g) < \infty, \quad \text{and} \quad f + g \in \mathcal{L}^p.$$

Then, the finite-sum version follows from **induction on  $n$** .  $\square$

**Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p$  in  $[1, \infty]$ . Then,  $(\mathcal{L}^p, N_p)$  is a seminormed vector space. In particular,  $\mathcal{L}^p$  is closed under scalar multiplication and addition.*

*Proof.* From Lemma 347 (absolute value in  $\overline{\mathbb{R}}$  is definite, thus  $|0| = 0$ ), Lemma 1300 ( $N_p$  is absolutely homogeneous, with  $a \stackrel{\text{def.}}{=} 0$ , thus  $N_p(0) = 0$  is finite), Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $\mathcal{L}^p \subset \mathcal{M}_{\mathbb{R}}$  and  $0 \in \mathcal{L}^p$ ), and Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$  and  $0 : X \rightarrow \mathbb{R}$  as zero),  $\mathcal{L}^p$  is a subset of the vector space  $\mathcal{M}_{\mathbb{R}}$  containing the zero, and we have  $N_p(\mathcal{L}^p) \subset \mathbb{R}$ .

Let  $f, g \in \mathcal{L}^p \subset \mathcal{M}_{\mathbb{R}}$ . Then, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have  $N_p(f), N_p(g) < \infty$ . Thus, from Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ , thus  $f + g \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1336 (Minkowski inequality in  $\mathcal{L}^p$ ), and **closedness of addition in  $\mathbb{R}$** , we have  $N_p(f + g) \leq N_p(f) + N_p(g) < \infty$ . Hence, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions),  $f + g \in \mathcal{L}^p$ , and  $N_p$  satisfies the triangle inequality.

Therefore, from Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication), Lemma 81 (closed under vector operations is subspace,  $\mathcal{L}^p$  is a vector subspace of  $\mathcal{M}_{\mathbb{R}}$ ), Definition 77 (subspace,  $\mathcal{L}^p$  is a vector space), and Definition 288 (seminorm,  $N_p$  is a seminorm over  $\mathcal{L}^p$ ),  $(\mathcal{L}^p, N_p)$  is a seminormed vector space.  $\square$

*Remark 1338.* In the next lemmas, the belonging (of  $f_n$ 's,  $f$  and  $\tilde{f}$ ) to  $\mathcal{L}^p$  could be relaxed into a belonging only from some rank, and/or into equality almost everywhere to functions in  $\mathcal{L}^p$ .

**Lemma 1339 (convergence in  $\mathcal{L}^p$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^p$ . Then, for all  $n \in \mathbb{N}$ ,  $N_p(f_n - f)$  is well-defined. The sequence  $(f_n)_{n \in \mathbb{N}}$  is said convergent towards  $f$  in  $\mathcal{L}^p$  iff  $\lim_{n \rightarrow \infty} N_p(f_n - f) = 0$ ; it is denoted  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ .

*Proof.* Direct consequence of Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, thus  $f_n - f \in \mathcal{L}^p$ ), Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , thus  $f_n - f \in \mathcal{M}$ ), and Lemma 1286 ( $p$ -seminorm).  $\square$

**Lemma 1340 (equivalent definition of convergence in  $\mathcal{L}^p$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty)$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^p$ . Then, we have

$$(16.65) \quad f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f \iff \lim_{n \rightarrow \infty} \tilde{N}_p(f_n - f) = 0 \iff \lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0.$$

*Proof.* Direct consequence of Lemma 1339 (convergence in  $\mathcal{L}^p$ ), Lemma 1286 ( $p$ -seminorm, definition of  $\tilde{N}_p$  and  $N_p$  for  $p \in [1, \infty)$ ), Lemma 369 (power function in  $\bar{\mathbb{R}}$  is definite, with  $\frac{1}{p} \in (0, \infty)$ , thus  $0 = 0^{\frac{1}{p}}$ ), Lemma 390 (power function in  $\bar{\mathbb{R}}$  is continuous), and Lemma 45 (compatibility of limit with continuous functions, with the power function and  $p \in (0, \infty)$ ).  $\square$

**Lemma 1341 (limits in  $\mathcal{L}^p$  are almost equal).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p$  be in  $[1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}}, f, \tilde{f} \in \mathcal{L}^p$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$  and  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} \tilde{f}$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ .

*Proof.* Let  $\varepsilon > 0$ . Then, from Lemma 1339 (convergence in  $\mathcal{L}^p$ , with  $f$  and  $\tilde{f}$ ), Definition 27 (convergent sequence), let  $M_1, M_2 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ ,

$$n \geq M_1 \Rightarrow N_p(f_n - f) \leq \frac{\varepsilon}{2}, \quad \text{and} \quad n \geq M_2 \Rightarrow N_p(f_n - \tilde{f}) \leq \frac{\varepsilon}{2}.$$

Let  $M \stackrel{\text{def.}}{=} \max(M_1, M_2)$ . Let  $n \in \mathbb{N}$ . Assume that  $n \geq M$ . Then, from Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication, with the scalar  $-1$ ), Lemma 1336 (Minkowski inequality in  $\mathcal{L}^p$ ), and **ordered field properties of  $\mathbb{R}$** , we have  $N_p(f - \tilde{f}) \leq N_p(f - f_n) + N_p(f_n - \tilde{f}) \leq \varepsilon$ .

Therefore, from Lemma 264 (pseudo-point separation), Lemma 1296 ( $N_p$  is almost definite), and Lemma 985 (compatibility of almost equality with operator, with the unary operator translation by  $\tilde{f}$ ), we have  $N_p(f - \tilde{f}) = 0$  and  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ .  $\square$

*Remark 1342.*

The next theorem is different from Theorem 1180 in two aspects. First, the sequence is not supposed to be nonnegative. But, as it is monotone and in  $\mathcal{L}^1$  (and thus finite), one can subtract the first item to recover a nonnegative sequence. Second, it is assumed that  $(\int f_n d\mu)_{n \in \mathbb{N}}$  is bounded. Hence, the pointwise limit is finite almost everywhere, and we obtain the pointwise convergence almost everywhere towards a finite function, which thus belongs to  $\mathcal{L}^1$ .

**Theorem 1343 (Beppo Levi, monotone convergence in  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}^1$ . Assume that the sequence is pointwise nondecreasing, and that  $\sup_{n \in \mathbb{N}} \int f_n d\mu < \infty$ . Then, there exists  $f \in \mathcal{L}^1$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ,  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} f$ ,  $f_0 \leq f$ , for all  $n \in \mathbb{N}$ ,  $f_{n+1} \stackrel{\mu \text{ a.e.}}{\leq} f$ , and we have

$$(16.66) \quad \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \mathbb{R}).$$

*Proof. 1. Swap integral and limit.* For all  $n \in \mathbb{N}$ , let  $g_n \stackrel{\text{def.}}{=} f_n - f_0$ .

Then, from Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, with  $p \stackrel{\text{def.}}{=} 1$ , thus  $g_n \in \mathcal{L}^1$ ), Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , thus  $f_n$  is finite), **ordered field properties of  $\mathbb{R}$**  (thus

$0 \leq g_n \leq g_{n+1}$ ), and Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ ), the sequence  $(g_n)_{n \in \mathbb{N}}$  is point-wise nondecreasing in  $\mathcal{M}_+$ . Moreover, from Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$ , monotonicity*), and Lemma 501 (*supremum is limit*), the sequence  $(\int f_n d\mu)_{n \in \mathbb{N}}$  is nondecreasing in  $\mathbb{R}$ , and for all  $n \in \mathbb{N}$ , we have

$$\int f_n d\mu \leq \sup_{n \in \mathbb{N}} \int f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu < \infty.$$

Thus, from Theorem 1180 (*Beppo Levi, monotone convergence*), Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$ , linearity*), **linearity of the limit in  $\mathbb{R}$** , and Lemma 34 (*stationary sequence is convergent*),  $g \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} g_n \in \mathcal{M}_+$ , and we have

$$\int g d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu - \int f_0 d\mu < \infty.$$

Hence, from Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ), there exists  $\tilde{g} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$  such that we have

$$\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g, \quad N_1(\tilde{g}) = \int \tilde{g} d\mu = \int g d\mu < \infty, \quad \text{and} \quad \tilde{g} \in \mathcal{L}^1.$$

Let  $f \stackrel{\text{def.}}{=} \tilde{g} + f_0$ . Then, from Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, thus  $f \in \mathcal{L}^1$ ), Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$ , thus  $f_0, \tilde{g}$  and  $f$  are finite*), Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions, thus  $\tilde{g} \geq 0$* ), **ordered field properties of  $\mathbb{R}$** , Lemma 985 (*compatibility of almost equality with operator, with the unary operator translation by  $f_0$* ), and Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*), we have

$$f \geq f_0, \quad \text{and} \quad f \stackrel{\mu \text{ a.e.}}{=} g + f_0 = \lim_{n \rightarrow \infty} g_n + f_0 = \lim_{n \rightarrow \infty} f_n.$$

Hence, from Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$ , linearity*), and **field properties of  $\mathbb{R}$  (all integrals below are finite)**, we have

$$\int f d\mu = \int \tilde{g} d\mu + \int f_0 d\mu = \int g d\mu + \int f_0 d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Let  $n \in \mathbb{N}$ . Then, from Definition 2 (*supremum, upper bound*), Lemma 501 (*supremum is limit, with  $(f_n)_{n \in \mathbb{N}}$  nondecreasing*), and Lemma 994 (*transitivity of almost inequality and equality*), we have  $f_n \leq \sup_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} f_n \stackrel{\mu \text{ a.e.}}{=} f$ , and thus  $f_n \stackrel{\mu \text{ a.e.}}{\leq} f$ .

**2. Convergence in  $\mathcal{L}^1$ .** Let  $n \in \mathbb{N}$ .

Then, from Lemma 984 (*compatibility of almost binary relation with operator, with the unary operator translation by  $f_n$ , thus  $f - f_n \stackrel{\mu \text{ a.e.}}{\geq} 0$* ), Lemma 1006 (*absolute value is almost equal to almost nonnegative function, thus  $|f_n - f| \stackrel{\mu \text{ a.e.}}{=} f - f_n$* ), Lemma 1286 ( *$p$ -seminorm, with  $p \stackrel{\text{def.}}{=} 1$* ), Lemma 1262 (*compatibility of integral with almost equality*), and Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$ , linearity*), we have  $N_1(f_n - f) = \int (f - f_n) d\mu = \int f d\mu - \int f_n d\mu$ . Hence, from **linearity of the limit in  $\mathbb{R}$** , and Lemma 34 (*stationary sequence is convergent*), we have

$$\lim_{n \rightarrow \infty} N_1(f_n - f) = \int f d\mu - \lim_{n \rightarrow \infty} \int f_n d\mu = 0.$$

Therefore, from Lemma 1339 (*convergence in  $\mathcal{L}^p$ , with  $p \stackrel{\text{def.}}{=} 1$* ), we have  $f_n \xrightarrow[\mathcal{L}^1]{n \rightarrow \infty} f$ . □

*Remark 1344.* The next theorem is different from Theorem 1208 in two aspects. First, the sequence is only supposed to be *almost nonnegative*. Second, it is assumed that the  $f_n$ 's are in  $\mathcal{L}^1$  and that  $(\int f_n d\mu)_{n \in \mathbb{N}}$  is bounded. Thus, the integrals remain finite and the lim inf remains almost finite. Hence, it is almost equal to a function in  $\mathcal{L}^1$ .

**Theorem 1345 (Fatou's lemma in  $\mathcal{L}^1$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}^1$ . Assume that the sequence is almost nonnegative, and that  $\sup_{n \in \mathbb{N}} \int f_n d\mu < \infty$ . Then, there exists  $f \in \mathcal{L}^1$  such that  $f \stackrel{\mu \text{ a.e.}}{=} \liminf_{n \rightarrow \infty} f_n$ , and we have

$$(16.67) \quad \int f d\mu = \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu \quad (\text{in } \mathbb{R}_+).$$

*Proof.* From Lemma 960 (*countable almost properties is almost property*, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$  and  $P_n$  set to  $(f_n \geq 0)$ ), there exists  $A \in \Sigma$  such that  $A \subset \bigcap_{n \in \mathbb{N}} \{f_n \geq 0\}$  and  $\mu(A^c) = 0$ .

For all  $n \in \mathbb{N}$ , let  $\tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbf{1}_A$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 1008 (*masking almost nowhere*, with  $f_n$ ), Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* , with  $p \stackrel{\text{def.}}{=} 1$ , thus  $f_n \in \mathcal{M}$  is finite), Lemma 833 (*measurability and masking*), **the definition of the indicator function**, Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), and Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), we have  $\tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n$ , and  $\tilde{f}_n \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

Let  $\tilde{f} \stackrel{\text{def.}}{=} \liminf_{n \in \mathbb{N}} \tilde{f}_n$ . Then, from Lemma 985 (*compatibility of almost equality with operator*, with the infinitary operator  $\liminf$ , thus  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} \liminf_{n \in \mathbb{N}} f_n$ ), Lemma 1262 (*compatibility of integral with almost equality*, with  $\tilde{f}$ , then  $\tilde{f}_n$ ), Theorem 1208 (*Fatou's lemma*, with  $\tilde{f}_n$ ), Definition 2 (*supremum*, upper bound), and Lemma 519 (*limit inferior bounded from above*, with  $M \stackrel{\text{def.}}{=} \sup_{n \in \mathbb{N}} \int f_n d\mu$ ), we have  $\tilde{f} \in \mathcal{M}_+$ , and

$$\begin{aligned} \int \liminf_{n \rightarrow \infty} f_n d\mu &= \int \tilde{f} d\mu = \int \liminf_{n \rightarrow \infty} \tilde{f}_n d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int \tilde{f}_n d\mu = \liminf_{n \rightarrow \infty} \int f_n d\mu \leq \sup_{n \in \mathbb{N}} \int f_n d\mu < \infty. \end{aligned}$$

Therefore, from Lemma 1290 ( $N_p$  is nonnegative, with  $p \stackrel{\text{def.}}{=} 1$ ), and Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ), there exists  $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$  such that we have

$$f \stackrel{\mu \text{ a.e.}}{=} \tilde{f} = \liminf_{n \in \mathbb{N}} f_n, \quad 0 \leq N_1(f) = \int f d\mu = \int \tilde{f} d\mu < \infty, \quad \text{and} \quad f \in \mathcal{L}^1.$$

□

### 16.3.5 Lebesgue's dominated convergence theorem

*Remark 1346.* See the sketch of next proof in Section 5.4.

**Theorem 1347 (Lebesgue, dominated convergence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $g \in \mathcal{L}^1$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then,  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^1$ ,  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} f$ , and we have

$$(16.68) \quad \int f \, d\mu = \int \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (\text{in } \mathbb{R}).$$

*Proof.* From Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent), Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous), Lemma 381 (limit is monotone), and Lemma 1323 (bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$ , with  $p \stackrel{\text{def.}}{=} 1$ ), we have  $f \in \mathcal{M}$ ,  $|f| \leq g$ , and  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^1$ . Thus, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), **nonnegativeness of the absolute value in  $\mathbb{R}$** , **transitivity of order in  $\mathbb{R}$** , **properties of the absolute value in  $\mathbb{R}$  (thus  $|g| = g$ )**, and Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , with  $p \stackrel{\text{def.}}{=} 1$ ),  $(f_n)_{n \in \mathbb{N}}, f$  and  $g$  are finite,  $g$  is nonnegative, and  $(\int f_n \, d\mu)_{n \in \mathbb{N}}, \int f \, d\mu$  and  $\int g \, d\mu$  are finite.

Let  $n \in \mathbb{N}$ . Let  $g_n \stackrel{\text{def.}}{=} |f_n - f|$ . Then, from Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, with  $p \stackrel{\text{def.}}{=} 1$ ), Lemma 1322 ( $\mathcal{L}^p$  is closed under absolute value), Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous), **linearity of the limit in  $\mathbb{R}$** , Lemma 34 (stationary sequence is convergent), **the definition of the absolute value in  $\mathbb{R}$  (thus  $|0| = 0$ )**, and Definition 516 (pointwise convergence), we have  $g_n \in \mathcal{L}^1 \cap \mathcal{M}_+$ , and  $\lim_{n \rightarrow \infty} g_n = |\lim_{n \rightarrow \infty} f_n - f| = 0$ . Moreover, from Lemma 349 (absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality, thus  $g_n \leq |f_n| + |f| \leq 2g$ ), Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, with  $p \stackrel{\text{def.}}{=} 1$ ), Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ , with  $2g - g_n \geq 0$ ), Lemma 517 (limit inferior and limit superior of pointwise convergent), and **linearity of the limit in  $\mathbb{R}$** , we have  $2g - g_n \in \mathcal{L}^1 \cap \mathcal{M}_+$ , and  $\liminf_{n \rightarrow \infty} (2g - g_n) = 2g - \lim_{n \rightarrow \infty} g_n = 2g$ .

Then, from Theorem 1208 (Fatou's lemma, with  $f_n \stackrel{\text{def.}}{=} 2g - g_n$ ), Lemma 1319 (integral is positive linear form on  $\mathcal{L}^1$ ), and Lemma 510 (duality limit inferior-limit superior), we have

$$\begin{aligned} 2 \int g \, d\mu &= \int \liminf_{n \rightarrow \infty} (2g - g_n) \, d\mu \leq \liminf_{n \rightarrow \infty} \int (2g - g_n) \, d\mu \\ &= \liminf_{n \rightarrow \infty} \left( 2 \int g \, d\mu - \int g_n \, d\mu \right) = 2 \int g \, d\mu - \limsup_{n \rightarrow \infty} \int g_n \, d\mu. \end{aligned}$$

Hence, from **ordered field properties of  $\mathbb{R}$  (with  $\int g \, d\mu$  finite)**, we have  $\limsup_{n \rightarrow \infty} \int g_n \, d\mu \leq 0$ . Therefore, from Lemma 1319 (integral is positive linear form on  $\mathcal{L}^1$ ,  $\int g_n \, d\mu \geq 0$ ), Lemma 518 (limit inferior bounded from below,  $\liminf_{n \rightarrow \infty} \int g_n \, d\mu \geq 0$ ), Lemma 522 (limit inferior, limit superior and pointwise convergence, with  $\limsup \leq 0 \leq \liminf$ ), Lemma 1286 ( $p$ -seminorm), and Lemma 1339 (convergence in  $\mathcal{L}^p$ ), we have

$$0 = \liminf_{n \rightarrow \infty} \int g_n \, d\mu = \limsup_{n \rightarrow \infty} \int g_n \, d\mu = \lim_{n \rightarrow \infty} \int g_n \, d\mu = \lim_{n \rightarrow \infty} N_1(f_n - f),$$

i.e. the sequence  $(f_n)_{n \in \mathbb{N}}$  is convergent towards  $f$  in  $\mathcal{L}^1$ .

Moreover, from **nonnegativeness of the absolute value in  $\mathbb{R}$** , and Lemma 1319 (integral is positive linear form on  $\mathcal{L}^1$ ), we have

$$0 \leq \left| \int f_n \, d\mu - \int f \, d\mu \right| = \left| \int (f_n - f) \, d\mu \right| \leq \int |f_n - f| \, d\mu = \int g_n \, d\mu.$$

Hence, from **the squeeze theorem**, and **linearity of the limit in  $\mathbb{R}$** , we have in  $\mathbb{R}$

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim_{n \rightarrow \infty} f_n \, d\mu.$$

□

*Remark 1348.* See the sketch of next proof in Section 5.3.

**Theorem 1349 (Lebesgue, extended dominated convergence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ,  $g$  is  $\mu$ -integrable, and for all  $n \in \mathbb{N}$ ,  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then,  $(f_n)_{n \in \mathbb{N}}$  and  $f$  are  $\mu$ -integrable, and there exist  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^1$  such that, for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n$ ,  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ,  $\tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} \tilde{f}$ , and we have

$$(16.69) \quad \int f \, d\mu = \int \tilde{f} \, d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (\text{in } \mathbb{R}).$$

*Proof.* Let  $B \stackrel{\text{def.}}{=} \{f = \liminf_{n \rightarrow \infty} f_n\} \cap \{f = \limsup_{n \rightarrow \infty} f_n\}$ , and  $\forall n \in \mathbb{N}$ ,  $C_n \stackrel{\text{def.}}{=} \{|f_n| \leq g\}$ . Then, from Definition 974 (*almost binary relation, with inequality*), Lemma 517 (*limit inferior and limit superior of pointwise convergent*), Definition 953 (*property almost satisfied, thus*  $B^c, (C_n)_{n \in \mathbb{N}} \in \mathbf{N}$ ), Lemma 960 (*countable almost properties is almost property, with*  $I \stackrel{\text{def.}}{=} \mathbb{N}$ ,  $P_0 \stackrel{\text{def.}}{=} (f = \liminf_{n \rightarrow \infty} f_n)$ ,  $P_1 \stackrel{\text{def.}}{=} (f = \limsup_{n \rightarrow \infty} f_n)$  and for all  $n \in \mathbb{N}$ ,  $P_{n+2} \stackrel{\text{def.}}{=} (|f_n| \leq g)$ ), and Lemma 948 (*negligibility of measurable subset*), there exists  $D \subset B \cap \bigcap_{n \in \mathbb{N}} C_n$  such that  $D$  and  $D^c$  are measurable, and  $\mu(D^c) = 0$ , i.e.  $D^c \in \mathbf{N}$ .

Let  $A \stackrel{\text{def.}}{=} D \cap g^{-1}(\mathbb{R}_+)$ , for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbf{1}_A$ ,  $\tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_A$  and  $\tilde{g} \stackrel{\text{def.}}{=} g \mathbf{1}_A$ . Then, from Lemma 1251 (*integrable is almost finite, with*  $g$ ), Lemma 342 (*absolute value in  $\mathbb{R}$  is nonnegative, thus*  $|f_n| \geq 0$ ), Lemma 313 (*order in  $\mathbb{R}$  is total, transitivity, thus*  $D \subset \{g \geq 0\}$ ), Lemma 951 (*subset of negligible is negligible, thus*  $\{g \geq 0\}^c \in \mathbf{N}$ ), and Definition 953 (*property almost satisfied, with*  $P \stackrel{\text{def.}}{=} (g \geq 0)$ ),  $g$  is  $\mu$ -almost everywhere finite and nonnegative. Then, from Lemma 1009 (*finite nonnegative part, with*  $(f, A) \stackrel{\text{def.}}{=} (g, D)$ ), we have  $A \in \Sigma$ ,  $\tilde{g} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ ,  $\mu(A^c) = 0$ , and  $g \stackrel{\mu \text{ a.e.}}{=} \tilde{g}$ . Hence, from Lemma 1297 (*compatibility of  $N_p$  with almost equality*), Lemma 1286 (*p-seminorm*), Lemma 1249 (*equivalent definition of integrability*), and Definition 1314 ( $\mathcal{L}^p$ , *vector space of p-powered integrable functions, with*  $p \stackrel{\text{def.}}{=} 1$ ), we have  $N_1(\tilde{g}) = N_1(g) = \int |g| \, d\mu < \infty$ , i.e.  $\tilde{g} \in \mathcal{L}^1$ . Moreover, from Lemma 833 (*measurability and masking, with*  $f_n$  and  $f$ ), and Lemma 1008 (*masking almost nowhere, with*  $f_n$  and  $f$ ), we have  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{M}$ , for all  $n \in \mathbb{N}$ ,  $f_n \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_n$ , and  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ .

Let  $x \in X$ . Let  $n \in \mathbb{N}$ . **Case  $x \in A$ .** Then, from **the definition of the indicator function**, and since  $A \subset B \cap C_n$ , we have

$$\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x) = \tilde{f}(x), \quad \text{and} \quad |\tilde{f}_n(x)| = |f_n(x)| \leq g(x) = \tilde{g}(x).$$

**Case  $x \notin A$ .** Then, from **the definition of the indicator function (thus  $\tilde{f}_n(x)$ ,  $\tilde{f}(x)$  and  $\tilde{g}(x)$  are all equal to 0)**, Lemma 34 (*stationary sequence is convergent*), and Lemma 347 (*absolute value in  $\mathbb{R}$  is definite*), we have  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = 0 = \tilde{f}(x)$ , and  $|\tilde{f}_n(x)| = 0 \leq 0 = \tilde{g}(x)$ . Hence, in all cases, from Theorem 1347 (*Lebesgue, dominated convergence, with*  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f}$  and  $\tilde{g}$ ), and Lemma 1337 ( $\mathcal{L}^p$  *is seminormed vector space, with*  $p \stackrel{\text{def.}}{=} 1$ ), we have  $\tilde{f}_n, \tilde{f} \in \mathcal{L}^1$ ,  $\tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^1} \tilde{f}$ ,  $\int \tilde{f} \, d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n \, d\mu$ , and  $\tilde{f}_n - \tilde{f}$  is well-defined.

Therefore, from Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$ , with*  $p \stackrel{\text{def.}}{=} 1$ ), Lemma 1249 (*equivalent definition of integrability, thus*  $\tilde{f}$  and  $\tilde{f}_n$  are  $\mu$ -integrable), and Lemma 1262 (*compatibility of integral with almost equality, with*  $(f, g) \stackrel{\text{def.}}{=} (\tilde{f}, \tilde{f})$ , then  $(\tilde{f}_n, f_n)$ ), for all  $n \in \mathbb{N}$ ,  $f$  and  $f_n$  are  $\mu$ -integrable, and we have  $\int f \, d\mu = \int \tilde{f} \, d\mu$ , and  $\int \tilde{f}_n \, d\mu = \int f_n \, d\mu$ .  $\square$

*Remark 1350.* See the sketch of next proof in Section 5.2.



**Theorem 1351 (Lebesgue, extended dominated convergence in  $\mathcal{L}^p$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty)$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{M}$ . Let  $g \in \mathcal{L}^p$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, there exists  $(\tilde{f}_n)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^p$  such that we have

$$(16.70) \quad \forall n \in \mathbb{N}, \quad \tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n \quad \wedge \quad N_p(\tilde{f}_n) = N_p(f_n) \leq N_p(g) < \infty,$$

$$(16.71) \quad \tilde{f} \stackrel{\mu \text{ a.e.}}{=} f \quad \wedge \quad N_p(\tilde{f}) = N_p(f) \leq N_p(g) < \infty,$$

$$(16.72) \quad \tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} \tilde{f},$$

$$(16.73) \quad \int |f|^p d\mu = \int |\tilde{f}|^p d\mu = \int \lim_{n \rightarrow \infty} |\tilde{f}_n|^p d\mu = \lim_{n \rightarrow \infty} \int |\tilde{f}_n|^p d\mu = \lim_{n \rightarrow \infty} \int |f_n|^p d\mu \quad (\text{in } \mathbb{R}).$$

*Proof. 1. Swap integral and limit.* For all  $n \in \mathbb{N}$ , let  $h_n \stackrel{\text{def.}}{=} |f_n|^p$ , and  $h \stackrel{\text{def.}}{=} |f|^p$ .

Then, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $g \in \mathcal{M}_{\mathbb{R}}$  and  $N_p(g) < \infty$ ), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $f_n, f$  and  $g$ ), and Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , with  $g$ ),  $(h_n)_{n \in \mathbb{N}}$ ,  $h$  and  $|g|^p$  belong to  $\mathcal{M}_+ \subset \mathcal{M}$ ,  $N_p(g) < \infty$ , and  $|g|^p$  is  $\mu$ -integrable.

Let  $n \in \mathbb{N}$ . Then, from Lemma 1299 ( $N_p$  is almost monotone for almost nonnegative functions), we have  $N_p(f_n) \leq N_p(g) < \infty$ . Moreover, from Lemma 439 (power function in  $\mathbb{R}$  is nonnegative (measure theory)), Lemma 343 (absolute value in  $\mathbb{R}$  is equal to nonnegative, thus  $|h_n| = h_n$ ), Lemma 367 (power function in  $\mathbb{R}$  is monotone), and Lemma 984 (compatibility of almost binary relation with operator, with the unary operator power function), we have  $|h_n| \stackrel{\mu \text{ a.e.}}{\leq} |g|^p$ .

Then, from Lemma 1012 (compatibility of almost limit with power function, with  $g \stackrel{\text{def.}}{=} 0$ , thus  $h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} h$ ), and Theorem 1349 (Lebesgue, extended dominated convergence, with  $f_n \stackrel{\text{def.}}{=} h_n, f \stackrel{\text{def.}}{=} h$  and  $g \stackrel{\text{def.}}{=} |g|^p$ ),  $(h_n)_{n \in \mathbb{N}}$  and  $h$  are  $\mu$ -integrable, and we have

$$\int |f|^p d\mu = \int h d\mu = \lim_{n \rightarrow \infty} \int h_n d\mu = \lim_{n \rightarrow \infty} \int |f_n|^p d\mu < \infty.$$

Moreover, from Lemma 1324 (finite  $N_p$  has finite representative in  $\mathcal{L}^p$ , with  $f_n$  and  $f$ ), there exists  $\tilde{f}, (\tilde{f}_n)_{n \in \mathbb{N}} \in \mathcal{L}^p$  such that  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ,  $N_p(\tilde{f}) = N_p(f) < \infty$ , and for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n \stackrel{\mu \text{ a.e.}}{=} f_n$  and  $N_p(\tilde{f}_n) = N_p(f_n) \leq N_p(g) < \infty$ . Hence, from Lemma 1014 (compatibility of almost limit with almost equality, with  $g_n \stackrel{\text{def.}}{=} \tilde{f}_n$  and  $g = \tilde{f}$ ), Lemma 962 (almost everywhere for almost the same, with  $\text{card}(I) = 1, f \stackrel{\text{def.}}{=} f_n$  and  $P(f) \stackrel{\text{def.}}{=}} (|f_n| \leq g)$ ), Lemma 1015 (compatibility of almost limit with almost inequality, with  $f_n \stackrel{\text{def.}}{=} \tilde{f}_n$  and  $f \stackrel{\text{def.}}{=} \tilde{f}$ ), and Lemma 1299 ( $N_p$  is almost monotone for almost nonnegative functions, with  $f \stackrel{\text{def.}}{=} \tilde{f}$ ), we have

$$\tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} \tilde{f}, \quad (|\tilde{f}_n|)_{n \in \mathbb{N}}, |\tilde{f}| \stackrel{\mu \text{ a.e.}}{\leq} g, \quad \text{and} \quad N_p(\tilde{f}) = N_p(f) \leq N_p(g) < \infty.$$

**2. Convergence in  $\mathcal{L}^p$ .** For all  $n \in \mathbb{N}$ , let  $k_n \stackrel{\text{def.}}{=} |\tilde{f}_n - \tilde{f}|^p$ .

Then, from Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, thus  $\tilde{f}_n - \tilde{f}, 2g \in \mathcal{L}^p$ ), Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $\tilde{f}_n - \tilde{f}, 2g \in \mathcal{M}_{\mathbb{R}}$  and  $N_p(2g) < \infty$ ), Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), Lemma 841 ( $\mathcal{M}$  is closed under power function, with  $\tilde{f}_n - \tilde{f}$  and  $2g$ ), Lemma 754 (constant function is measurable, with constant 0), Lemma 429 (absolute value in  $\mathbb{R}$  is multiplicative (measure theory), with  $a \stackrel{\text{def.}}{=} 2 = |2|$ ), and Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ , with  $2g$ ),  $(k_n)_{n \in \mathbb{N}}, 0, 2^p|g|^p \in \mathcal{M}_+ \subset \mathcal{M}$ ,  $N_p(2g) < \infty$ , and  $2^p|g|^p$  is  $\mu$ -integrable. Moreover, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1012 (compatibility of almost limit with power function, with  $(f_n)_{n \in \mathbb{N}} \stackrel{\text{def.}}{=}} (\tilde{f}_n)_{n \in \mathbb{N}}$  and  $f, g \stackrel{\text{def.}}{=} \tilde{f}$ ), and

Lemma 369 (*power function in  $\bar{\mathbb{R}}$  is definite*), for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n - \tilde{f}$  is well-defined, and we have  $k_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} |\tilde{f} - \tilde{f}|^p = 0$ .

Let  $n \in \mathbb{N}$ . Then, from Lemma 371 (*power function satisfies extended relaxed triangle inequality, with  $\tilde{f}_n - \tilde{f}$  well-defined*), and Lemma 984 (*compatibility of almost binary relation with operator, with the unary operator power function, then the binary operator addition*), we have  $k_n \leq 2^{p-1}(|\tilde{f}_n|^p + |\tilde{f}|^p) \stackrel{\mu \text{ a.e.}}{\leq} 2^p |g|^p$ .

Therefore, from Theorem 1349 (*Lebesgue, extended dominated convergence, with  $f_n \stackrel{\text{def.}}{=} k_n$ ,  $f \stackrel{\text{def.}}{=} k$  and  $g \stackrel{\text{def.}}{=} 2^p |g|^p$ ), Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*), and Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  $(k_n)_{n \in \mathbb{N}}$  are  $\mu$ -integrable, and we have*

$$0 = \int 0 \, d\mu = \lim_{n \rightarrow \infty} \int k_n \, d\mu = \lim_{n \rightarrow \infty} \int |\tilde{f}_n - \tilde{f}|^p \, d\mu, \quad \text{and} \quad \tilde{f}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} \tilde{f}.$$

□

**Theorem 1352 (Beppo Levi, monotone convergence in  $\mathcal{L}^p$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty)$ . Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}^p \cap \mathcal{M}_+$ .

Assume that the sequence is pointwise nondecreasing, and that  $\sup_{n \in \mathbb{N}} N_p(f_n) < \infty$ .

Then, there exists  $f \in \mathcal{L}^p \cap \mathcal{M}_+$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ ,  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ , for all  $n \in \mathbb{N}$ ,  $f_n \stackrel{\mu \text{ a.e.}}{\leq} f$ , and

$$(16.74) \quad \int f^p \, d\mu = \lim_{n \rightarrow \infty} \int f_n^p \, d\mu \quad (\text{in } \mathbb{R}_+).$$

*Proof. 1. Swap integral and limit.* For all  $n \in \mathbb{N}$ , let  $g_n \stackrel{\text{def.}}{=} |f_n|^p$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$ , thus  $f_n$  is finite*), Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions, thus  $f_n \geq 0$* ), **properties of the power function in  $\mathbb{R}$** , and Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$ , with  $f \stackrel{\text{def.}}{=} f_n$* ),  $g_n = f_n^p$  is finite and nonnegative,  $(g_n)_{n \in \mathbb{N}}$  is nondecreasing,  $g_n \in \mathcal{L}^1$ , and we have  $\sup_{n \in \mathbb{N}} \int g_n \, d\mu < \infty$ . Hence, from Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$ , with  $f_n \stackrel{\text{def.}}{=} g_n$* ), Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$ , with  $p \stackrel{\text{def.}}{=} 1$* ), Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), and Lemma 1172 (*integral in  $\mathcal{M}_+$ , nonnegativeness*), there exists  $g \in \mathcal{L}^1 \cap \mathcal{M}_+$  such that  $g_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} g$ ,  $0 \leq g_0 \leq g$ , for all  $n \in \mathbb{N}$ ,  $0 \leq g_{n+1} \stackrel{\mu \text{ a.e.}}{\leq} g$ , and we have in  $\mathbb{R}_+$

$$\int g \, d\mu = \lim_{n \rightarrow \infty} \int g_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n^p \, d\mu.$$

Let  $f \stackrel{\text{def.}}{=} g^{\frac{1}{p}}$ . Then, from Definition 1314 ( $\mathcal{L}^p$ , *vector space of  $p$ -powered integrable functions, thus  $g \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$* ), Lemma 841 ( $\mathcal{M}$  *is closed under power function, with  $\frac{1}{p}$  and  $g$* ), and Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$ , with  $g$* ), we have

$$f \in \mathcal{L}^p \cap \mathcal{M}_+, \quad \int f^p \, d\mu = \tilde{N}_p(f) = N_1(g) = \int g \, d\mu, \quad \text{and} \quad (16.74) \text{ holds.}$$

Moreover, from **monotonicity of the power function in  $\mathbb{R}_+$** , and Lemma 984 (*compatibility of almost binary relation with operator, with the unary operator power function*), Lemma 390 (*power function in  $\bar{\mathbb{R}}$  is continuous, with  $\frac{1}{p}$* ), Lemma 1011 (*compatibility of almost limit with continuous function, with  $(g_n)_{n \in \mathbb{N}}$  and  $g_x \stackrel{\text{def.}}{=} (y \mapsto |y|^{\frac{1}{p}})$* ), we have for all  $n \in \mathbb{N}$ ,  $f_n \stackrel{\mu \text{ a.e.}}{\leq} f$ , and  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

**2. Convergence in  $\mathcal{L}^p$ .** For all  $n \in \mathbb{N}$ , let  $h_n \stackrel{\text{def.}}{=} |f_n - f|$ .

Then, from Definition 1314 ( $\mathcal{L}^p$ , *vector space of  $p$ -powered integrable functions, thus  $f_n, f \in \mathcal{M}_{\mathbb{R}}$* ), Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$ , thus  $f_n - f, 0 \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$* ),



Lemma 842 ( $\mathcal{M}$  is closed under absolute value), Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, closedness under scalar multiplication), Lemma 1012 (compatibility of almost limit with power function, with  $p \stackrel{\text{def.}}{=} 1$  and  $g \stackrel{\text{def.}}{=} f$ ), **properties of the absolute value in  $\mathbb{R}$  (definiteness, identity function on nonnegative values, and the triangle inequality)**, Lemma 984 (compatibility of almost binary relation with operator, with the unary operator translation by  $f$ ), we have  $h_n, 0 \in \mathcal{M}$ ,  $2f \in \mathcal{L}^p$ ,  $h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} |f - f| = 0$ , and  $|h_n| \leq |f_n| + |f| \xrightarrow{\mu \text{ a.e.}} 2f$ .

Therefore, from Lemma 1259 (integral in  $\mathcal{M}$  of zero is zero), **definiteness of the power function in  $\mathbb{R}$** , Theorem 1351 (Lebesgue, extended dominated convergence in  $\mathcal{L}^p$ , with  $f_n \stackrel{\text{def.}}{=} h_n$ ,  $f \stackrel{\text{def.}}{=} 0$  and  $g \stackrel{\text{def.}}{=} 2f$ ), and Lemma 1340 (equivalent definition of convergence in  $\mathcal{L}^p$ ), we have

$$0 = \int |0|^p d\mu = \lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu, \quad \text{and} \quad f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f.$$

□

### 16.3.6 The Riesz–Fischer theorem

*Remark 1353.* In the next theorem, the notion of Cauchy sequence is meant for the pseudometric induced by the seminorm  $N_p$ .

See the sketch of the proof in Section 5.1.

**Theorem 1354 (Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{L}^p$  be a Cauchy sequence.

Then, there exists  $f \in \mathcal{L}^p$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ .

Moreover, there exists  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  increasing and  $h \in \mathcal{L}^p$  such that for all  $n \in \mathbb{N}$ ,  $|f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h$ , and  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ . When  $p = \infty$ , one can take  $k_n \stackrel{\text{def.}}{=} n$ , for all  $n \in \mathbb{N}$ .

*Proof.* Let  $l, m \in \mathbb{N}$ . Then, from Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space), Lemma 1322 ( $\mathcal{L}^p$  is closed under absolute value), and Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions), we have  $f_l, f_l - f_m \in \mathcal{L}^p \subset \mathcal{M}_{\mathbb{R}}$ , and  $|f_l|, |f_l - f_m| \in \mathcal{L}^p \cap \mathcal{M}_+ \subset \mathcal{M}_{\mathbb{R}}$ .

**1. Case  $p \in [1, \infty)$ .** Let  $i \in \mathbb{N}$ . Then, from Definition 35 (Cauchy sequence, with  $\varepsilon \stackrel{\text{def.}}{=} \frac{1}{2^i}$ ), let  $n_i \in \mathbb{N}$  such that for all  $l, m \in \mathbb{N}$ ,  $l, m \geq n_i$  implies  $N_p(f_l - f_m) \leq \frac{1}{2^i}$ .

Let  $j \in \mathbb{N}$ . Let  $k_0 \stackrel{\text{def.}}{=} n_0$ . For all  $i \in \mathbb{N}$ , let  $k_{i+1} \stackrel{\text{def.}}{=} \max(n_{i+1}, k_i + 1)$ . Thus, from **ordered field properties of  $\mathbb{R}$** ,  $(k_i)_{i \in \mathbb{N}}$  is an increasing sequence, and we have  $n_i \leq k_i \leq k_i + j$ , and  $N_p(f_{k_{i+j}} - f_{k_i}) \leq \frac{1}{2^i}$ . For instance, this is valid for  $j \stackrel{\text{def.}}{=} k_{i+1} - k_i$ .

**1.1. Use monotone convergence in  $\mathcal{L}^p$ .** For all  $n \in \mathbb{N}$ , let  $S_n \stackrel{\text{def.}}{=} \sum_{i \in [0..n)} |f_{k_{i+1}} - f_{k_i}|$ .

Then, from **nonnegativeness of the absolute value in  $\mathbb{R}$** ,  $(S_n)_{n \in \mathbb{N}}$  is nonnegative and non-decreasing. Let  $n \in \mathbb{N}$ . Then, from Lemma 1336 (Minkowski inequality in  $\mathcal{L}^p$ , finite version), Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ ), Lemma 1295 (compatibility of  $N_p$  with absolute value), and **ordered field properties of  $\mathbb{R}$** ,  $S_n \in \mathcal{L}^p \cap \mathcal{M}_+$ , and we have

$$N_p(S_n) \leq \sum_{i \in [0..n)} N_p(f_{k_{i+1}} - f_{k_i}) \leq \sum_{i \in [0..n)} \frac{1}{2^i} \leq \sum_{i \in \mathbb{N}} \frac{1}{2^i} = 2 < \infty.$$

Hence, from Lemma 493 (supremum of bounded is bounded, thus  $\sup_{n \in \mathbb{N}} N_p(S_n) < \infty$ ), and Theorem 1352 (Beppo Levi, monotone convergence in  $\mathcal{L}^p$ , with  $f_n \stackrel{\text{def.}}{=} S_n$ ), there exists  $S \in \mathcal{L}^p \cap \mathcal{M}_+$  such that  $S_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} S$ , and for all  $n \in \mathbb{N}$ ,  $S_n \stackrel{\mu \text{ a.e.}}{\leq} S$ .

For all  $n \in \mathbb{N}$ , let  $B \stackrel{\text{def.}}{=} \{S = \liminf_{n \rightarrow \infty} S_n\} \cap \{S = \limsup_{n \rightarrow \infty} S_n\}$  and  $C_n \stackrel{\text{def.}}{=} \{S_n \leq S\}$ . Then, from Lemma 960 (countable almost properties is almost property, with  $I \stackrel{\text{def.}}{=} \mathbb{N}$ ,  $P_0$  set to  $(S = \liminf_{n \rightarrow \infty} S_n)$ ,  $P_1 \stackrel{\text{def.}}{=} (S = \limsup_{n \rightarrow \infty} S_n)$  and  $P_{n+2} \stackrel{\text{def.}}{=} (S_n \leq S)$ ), there exists  $A \in \Sigma$  such that  $A \subset B \cap \bigcap_{n \in \mathbb{N}} C_n$  and  $\mu(A^c) = 0$ .

**1.2. Use extended dominated convergence in  $\mathcal{L}^p$ .** Let  $n, q \in \mathbb{N}$ . Let  $x \in A$ . Let  $\varepsilon > 0$ . Then, from **additive group properties of  $\mathbb{R}$** , and **the triangle inequality for the absolute value in  $\mathbb{R}$** , we have

$$|f_{k_{n+q}}(x) - f_{k_n}(x)| \leq \sum_{i \in [0..q)} |f_{k_{n+i+1}}(x) - f_{k_{n+i}}(x)| = S_{n+q}(x) - S_n(x) \leq S(x) - S_n(x) \leq S(x).$$

Thus, from Definition 27 (convergent sequence), Definition 36 (equivalent definition of Cauchy sequence), and **the completeness of  $\mathbb{R}$** , there exists  $M \in \mathbb{N}$  such that  $|S(x) - S_n(x)| \leq \varepsilon$  for all  $n \geq M$ ,  $(f_{k_n}(x))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ , and thus there exists  $\tilde{f}(x)$  in  $\mathbb{R}$  such that  $f_{k_n}(x) \xrightarrow[n \rightarrow \infty]{} \tilde{f}(x)$ . Hence, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $f_{k_n} \in \mathcal{M}_{\mathbb{R}}$ ), and Lemma 1016 (compatibility of almost limit with measurability), there exists  $f \in \mathcal{M}_{\mathbb{R}}$  such that  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

For all  $n, q \in \mathbb{N}$ , let  $h_n \stackrel{\text{def.}}{=} |f_{k_n} - f| = |f - f_{k_n}|$ , and  $g_q^n \stackrel{\text{def.}}{=} |f_{k_{n+q}} - f_{k_n}| \in \mathcal{L}^p$ . Then, from **properties of the absolute value in  $\mathbb{R}$  (thus  $|h_n| = h_n$ )**, Lemma 1012 (*compatibility of almost limit with power function*, first with  $f_n \stackrel{\text{def.}}{=} f_{k_n}$  and  $g \stackrel{\text{def.}}{=} f$ , thus  $h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} 0$ , then with  $f_q \stackrel{\text{def.}}{=} f_{k_{n+q}}$  and  $g \stackrel{\text{def.}}{=} f_{k_n}$ , thus  $g_q^n \xrightarrow[q \rightarrow \infty]{\mu \text{ a.e.}} h_n$ ), Lemma 1015 (*compatibility of almost limit with almost inequality*, with  $f_q \stackrel{\text{def.}}{=} g_q^n$ ,  $f \stackrel{\text{def.}}{=} h_n$  and  $g \stackrel{\text{def.}}{=} S$ , thus  $h_n \stackrel{\mu \text{ a.e.}}{\leq} S$ ), Lemma 1001 (*absolute value and almost inequality*, with  $g \stackrel{\text{def.}}{=} f_{k_n}$  finite and  $h \stackrel{\text{def.}}{=} S$ , thus  $|f| \stackrel{\mu \text{ a.e.}}{\leq} S + |f_{k_n}|$ ), Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, thus  $S + |f_{k_n}| \in \mathcal{L}^p$ ), and Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ), we have  $f \in \mathcal{L}^p$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 1001 (*absolute value and almost inequality*, with  $f \stackrel{\text{def.}}{=} f_{k_n}$ ,  $g \stackrel{\text{def.}}{=} f$  finite and  $h \stackrel{\text{def.}}{=} S$ ), Lemma 1322 ( $\mathcal{L}^p$  is closed under absolute value), and Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space), we have  $|f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h$  with  $h \stackrel{\text{def.}}{=} S + |f| \in \mathcal{L}^p$ .

Therefore, from Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*), **definiteness of the power function in  $\mathbb{R}$** , and Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$ , with  $f_n$  set to  $h_n$ ,  $f \stackrel{\text{def.}}{=} 0$  and  $g \stackrel{\text{def.}}{=} S$* ), we have for all  $n \in \mathbb{N}$ ,  $N_p(h_n) \leq N_p(S) < \infty$  (i.e.  $|f_{k_n} - f|^p$  is  $\mu$ -integrable), and

$$0 = \int |0|^p d\mu = \lim_{n \rightarrow \infty} \int |f_{k_n} - f|^p d\mu.$$

Hence, from Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$ , thus  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$* ), and Lemma 269 (*convergent subsequence of Cauchy sequence, with the pseudometric induced by the seminorm  $N_p$* ), we have  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ .

**2. Case  $p = \infty$ .** From Lemma 268 (*equivalent definition of Cauchy sequence (discrete criterion)*), and Lemma 1293 ( *$N_\infty$  bounded is almost bounded*), we have

$$\forall k \in \mathbb{N}, \exists K_k \in \mathbb{N}, \forall l, m \in \mathbb{N}, \quad l, m \geq K_k \implies |f_l - f_m| \stackrel{\mu \text{ a.e.}}{\leq} \frac{1}{k+1} \leq 1$$

where the rank  $K_k$  is independent of the variable of the  $f_n$ 's. Then, from **countability of  $\mathbb{N}^2$  (used twice)**, Lemma 960 (*countable almost properties is almost property, with  $P_{l,m}^k$  set to  $(|f_l - f_m| \leq \frac{1}{k+1})$  for  $l, m \geq K_k$* ), there exists  $A \in \Sigma$  such that

$$A \subset \bigcap_{k \in \mathbb{N}} \bigcap_{l, m \geq K_k} \{|f_l - f_m| \leq \frac{1}{k+1}\}, \quad \text{and} \quad \mu(A^c) = 0.$$

Let  $x \in A$ . Then, from Definition 268 (*equivalent definition of Cauchy sequence (discrete criterion)*), **completeness of  $\mathbb{R}$** , and Definition 953 (*property almost satisfied*),  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence, and there exists  $\tilde{f}(x) \in \mathbb{R}$  such that  $f_n(x) \xrightarrow[n \rightarrow \infty]{} \tilde{f}(x)$ , i.e.  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} \tilde{f}$ . Hence, from Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions, thus  $f_n \in \mathcal{M}_{\mathbb{R}}$ ), and Lemma 1016 (*compatibility of almost limit with measurability*), there exists  $f \in \mathcal{M}_{\mathbb{R}}$  such that  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .

For all  $n, q \in \mathbb{N}$ , let  $h_n \stackrel{\text{def.}}{=} |f_n - f| = |f - f_n|$ , and  $g_q^n \stackrel{\text{def.}}{=} |f_{K_0+q} - f_{K_0+n}| \in \mathcal{L}^\infty$ . Then, from **properties of the absolute value in  $\mathbb{R}$  (thus  $|h_n| = h_n$ )**, Lemma 1012 (*compatibility of almost limit with power function*, first with  $g \stackrel{\text{def.}}{=} f$ , thus  $h_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} 0$ , then with  $f_q \stackrel{\text{def.}}{=} f_{K_0+q}$  and  $g \stackrel{\text{def.}}{=} f_{K_0+n}$ , thus  $g_q^n \xrightarrow[q \rightarrow \infty]{\mu \text{ a.e.}} h_{K_0+n}$ ), Lemma 1015 (*compatibility of almost limit with almost inequality*, with  $f_q \stackrel{\text{def.}}{=} g_q^n$ ,  $f \stackrel{\text{def.}}{=} h_{K_0+n}$  and  $g \stackrel{\text{def.}}{=} 1$ , thus  $h_{K_0+n} \stackrel{\mu \text{ a.e.}}{\leq} 1$ ), Lemma 1001 (*absolute value and almost inequality*, with  $g \stackrel{\text{def.}}{=} f_{K_0+n}$  finite and  $h \stackrel{\text{def.}}{=} 1$ , thus  $|f| \stackrel{\mu \text{ a.e.}}{\leq} 1 + |f_{K_0+n}|$ ), Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space, thus  $1 + |f_{K_0+n}| \in \mathcal{L}^\infty$ ), and Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ), we have  $f \in \mathcal{L}^\infty$ .

Therefore, from Lemma 1339 (*convergence in  $\mathcal{L}^p$* ), Definition 27 (*convergent sequence*), and Lemma 1293 ( *$N_\infty$  bounded is almost bounded*), we have  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^\infty} f$ .

Let  $n \in \mathbb{N}$ . Then, from Lemma 1001 (*absolute value and almost inequality*), with  $f \stackrel{\text{def.}}{=} f_{K_0+n}$ ,  $g \stackrel{\text{def.}}{=} f$  finite and  $h \stackrel{\text{def.}}{=} 1$ , thus  $|f_{K_0+n}| \stackrel{\mu \text{ a.e.}}{\leq} 1 + |f|$ , Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*), Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*), and Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*), we have  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} h$  with  $h \stackrel{\text{def.}}{=} \max_{k < K_0} \{|f_k|, 1 + |f|\} \in \mathcal{L}^\infty$ .  $\square$

**Lemma 1355** (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^p$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ . Then, there exist  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  increasing and  $h \in \mathcal{L}^p$  such that  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , and for all  $n \in \mathbb{N}$ ,  $|f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h$ . When  $p = \infty$ , one can take  $k_n \stackrel{\text{def.}}{=} n$ , for all  $n \in \mathbb{N}$ .

*Proof.* Direct consequence of Lemma 37 (*convergent sequence is Cauchy*), Lemma 1354 (*Riesz-Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ), thus there exists  $\tilde{f}, h \in \mathcal{L}^p$  and  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  increasing such that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} \tilde{f}$ , for all  $n \in \mathbb{N}$ ,  $|f_{k_n}| \stackrel{\mu \text{ a.e.}}{\leq} h$  and  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} \tilde{f}$ , Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*), thus  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ , and Lemma 1014 (*compatibility of almost limit with almost equality*), with  $f_n, g_n \stackrel{\text{def.}}{=} f_{k_n}$ ,  $f \stackrel{\text{def.}}{=} \tilde{f}$  and  $g \stackrel{\text{def.}}{=} f$ , thus  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ .  $\square$

**Lemma 1356** (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $p \in [1, \infty]$ . Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^p$ . Let  $g \in \mathcal{M}$ . Assume that  $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^p} f$ , and  $f_n \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} g$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{=} g$ .

*Proof.* Direct consequence of Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ), thus there exists  $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$  increasing such that  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} f$ , Lemma 1010 (*compatibility of almost limit with subsequence*), thus  $f_{k_n} \xrightarrow[n \rightarrow \infty]{\mu \text{ a.e.}} g$ , and Lemma 1013 (*almost limits are almost equal*), thus  $f \stackrel{\mu \text{ a.e.}}{=} g$ .  $\square$



## Chapter 17

# Conclusions, perspectives

We have presented very detailed proofs of the main basic results in measure theory and Lebesgue integration such as the Beppo Levi (monotone convergence) theorem, Fatou's lemma, the Tonelli theorem, Lebesgue's dominated convergence theorem, and the Riesz–Fischer theorem.

The short-term purpose of this work was to help the formalization in a formal proof assistant such as Coq of the basic concepts in measure theory and Lebesgue integration and of the proofs of their main properties. First milestones towards this are dedicated to the integral of nonnegative measurable functions [10] where special attention was paid to the formalization of  $\sigma$ -algebras and of simple functions, and to the Tonelli theorem [11].

Our mean-term purpose is now to continue up to the formalization of  $L^p$  Lebesgue spaces and of  $W^{m,p}$  Sobolev spaces as Banach spaces, and in particular  $L^2$  and  $H^m \stackrel{\text{def.}}{=} W^{m,2}$  as Hilbert spaces. This will include parts of the distribution theory.

The long-term purpose of these studies is the formal proof of programs implementing the Finite Element Method. As a consequence, after having addressed the formalization of the Lax–Milgram theorem [20, 9], we will also have to write very detailed pen-and-paper proofs for the concepts and results of the interpolation and approximation theory to define the Finite Element Method itself.



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# Appendix A

## Lists of statements

This appendix collects the references (name and number) for all statements present in Part II. These are split into definitions, lemmas, and theorems.

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## Appendix B

# The proof cites explicitly...

This appendix gathers the explicit citations of the statements listed in Appendix A that appear in the proof of each result (lemmas and theorems). Statements from [20] are anonymized.

Printing is not advised!

**The proof of Lemma 211** (*compatibility of pseudopartition with intersection*)

cites explicitly:

Definition 209 (*pseudopartition*).

**The proof of Lemma 212** (*countable union is monotone*)

has no explicit citation.

**The proof of Lemma 213** (*equality of countable union*)

cites explicitly:

Lemma 212 (*countable union is monotone*).

**The proof of Lemma 215** (*disjoint union of union*)

cites explicitly:

Lemma 213 (*equality of countable union*).

**The proof of Lemma 217** (*monotone union of union*)

cites explicitly:

Lemma 213 (*equality of countable union*).

**The proof of Lemma 219** (*empty monotone intersection of union*)

has no explicit citation.

**The proof of Lemma 221** (*monotone union of monotone intersection*)

has no explicit citation.

**The proof of Lemma 222** (*technical inclusion for countable union*)

has no explicit citation.

**The proof of Lemma 223** (*order is meaningless in countable union*)

cites explicitly:

Lemma 222 (*technical inclusion for countable union*).

**The proof of Lemma 224** (*definition of double countable union*)

cites explicitly:

Lemma 223 (*order is meaningless in countable union*).

**The proof of Lemma 225** (*double countable union*)

cites explicitly:

Lemma 224 (*definition of double countable union*).

**The proof of Lemma 231 (*equivalent definition of interval*)**

cites explicitly:  
Statement(s) from [20],  
Definition 229 (*interval*).

**The proof of Lemma 232 (*empty open interval*)**

cites explicitly:  
Definition 229 (*interval*).

**The proof of Lemma 235 (*intervals are closed under finite intersection*)**

cites explicitly:  
Definition 229 (*interval*).

**The proof of Lemma 236 (*empty intersection of open intervals*)**

cites explicitly:  
Lemma 232 (*empty open interval*),  
Lemma 235 (*intervals are closed under finite intersection*).

**The proof of Lemma 238 (*trace interval on a subset*)**

cites explicitly:  
Definition 226 (*trace of subset system*),  
Definition 229 (*interval*).

**The proof of Lemma 241 (*intersection of topologies*)**

cites explicitly:  
Definition 240 (*topological space*).

**The proof of Lemma 243 (*generated topology is minimum*)**

cites explicitly:  
Lemma 241 (*intersection of topologies*),  
Definition 242 (*generated topology*).

**The proof of Lemma 244 (*equivalent definition of generated topology*)**

cites explicitly:  
Definition 240 (*topological space*),  
Lemma 243 (*generated topology is minimum*).

**The proof of Lemma 245 (*topology generation is monotone*)**

cites explicitly:  
Lemma 243 (*generated topology is minimum*).

**The proof of Lemma 247 (*equivalent definition of topological basis*)**

cites explicitly:  
Definition 240 (*topological space*),  
Lemma 243 (*generated topology is minimum*),  
Lemma 244 (*equivalent definition of generated topology*),  
Definition 246 (*topological basis*).

**The proof of Lemma 248 (*augmented topological basis*)**

cites explicitly:  
Definition 246 (*topological basis*).

**The proof of Lemma 249 (*topological basis is generator*)**

cites explicitly:  
Lemma 244 (*equivalent definition of generated topology*),  
Definition 246 (*topological basis*).

**The proof of Lemma 250 (*topological subbasis*)**

cites explicitly:

Definition 246 (*topological basis*),

Lemma 249 (*topological basis is generator*).

**The proof of Lemma 252 (*trace topology on subset*)**

cites explicitly:

Definition 226 (*trace of subset system*),

Definition 240 (*topological space*),

Definition 246 (*topological basis*).

**The proof of Lemma 254 (*box topology on Cartesian product*)**

cites explicitly:

Definition 227 (*product of subset systems*),

Definition 240 (*topological space*),

Definition 246 (*topological basis*),

Lemma 247 (*equivalent definition of topological basis*).

**The proof of Lemma 258 (*topological basis of order topology*)**

cites explicitly:

Lemma 235 (*intervals are closed under finite intersection*),

Lemma 244 (*equivalent definition of generated topology*),

Definition 246 (*topological basis*),

Definition 256 (*order topology*).

**The proof of Lemma 259 (*trace of order topology*)**

cites explicitly:

Lemma 238 (*trace interval on a subset*),

Lemma 252 (*trace topology on subset*),

Lemma 258 (*topological basis of order topology*).

**The proof of Lemma 261 (*compatibility of second-countability with Cartesian product*)**

cites explicitly:

Definition 227 (*product of subset systems*),

Lemma 254 (*box topology on Cartesian product*),

Definition 260 (*second-countability*).

**The proof of Lemma 264 (*pseudo-point separation*)**

cites explicitly:

Definition 262 (*pseudometric*).

**The proof of Lemma 265 (*equivalent definition of convergent sequence*)**

cites explicitly:

Statement(s) from [20].

**The proof of Lemma 266 (*compatibility of subsequence with convergence*)**

cites explicitly:

Statement(s) from [20].

**The proof of Lemma 268 (*equivalent definition of Cauchy sequence (discrete criterion)*)**

cites explicitly:

Statement(s) from [20],

Lemma 265 (*equivalent definition of convergent sequence*).

- The proof of Lemma 269** (*convergent subsequence of Cauchy sequence*)  
cites explicitly:  
Statement(s) from [20],  
Definition 262 (*pseudometric*).
- The proof of Lemma 271** (*quotient vector operations*)  
cites explicitly:  
Definition 270 (*relation compatible with vector operations*).
- The proof of Lemma 272** (*quotient vector space, equivalence relation*)  
cites explicitly:  
Statement(s) from [20],  
Lemma 271 (*quotient vector operations*).
- The proof of Lemma 273** (*quotient vector space*)  
cites explicitly:  
Statement(s) from [20],  
Definition 270 (*relation compatible with vector operations*),  
Lemma 272 (*quotient vector space, equivalence relation*).
- The proof of Lemma 274** (*linear map on quotient vector space*)  
cites explicitly:  
Statement(s) from [20],  
Lemma 271 (*quotient vector operations*),  
Lemma 273 (*quotient vector space*).
- The proof of Lemma 279** ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*)  
cites explicitly:  
Definition 277 (*algebra over a field*).
- The proof of Lemma 282** (*algebra of functions to algebra*)  
cites explicitly:  
Statement(s) from [20],  
Definition 277 (*algebra over a field*),  
Definition 280 (*inherited algebra operations*).
- The proof of Lemma 283** ( *$\mathbb{K}^X$  is algebra*)  
cites explicitly:  
Lemma 279 ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*),  
Lemma 282 (*algebra of functions to algebra*).
- The proof of Lemma 286** (*vector subspace and closed under multiplication is subalgebra*)  
cites explicitly:  
Statement(s) from [20],  
Definition 277 (*algebra over a field*),  
Definition 284 (*subalgebra*).
- The proof of Lemma 287** (*closed under algebra operations is subalgebra*)  
cites explicitly:  
Statement(s) from [20],  
Lemma 286 (*vector subspace and closed under multiplication is subalgebra*).
- The proof of Lemma 290** (*seminorm is homogeneous*)  
cites explicitly:  
Statement(s) from [20],  
Definition 288 (*seminorm*).

**The proof of Lemma 291 (*seminorm is nonnegative*)**

cites explicitly:

Statement(s) from [20],

Definition 288 (*seminorm*),

Lemma 290 (*seminorm is homogeneous*).

**The proof of Lemma 292 (*seminormed vector space is pseudometric space*)**

cites explicitly:

Statement(s) from [20],

Definition 262 (*pseudometric*),

Definition 288 (*seminorm*),

Lemma 290 (*seminorm is homogeneous*),

Lemma 291 (*seminorm is nonnegative*).

**The proof of Lemma 293 (*definite seminorm is norm*)**

cites explicitly:

Statement(s) from [20],

Definition 288 (*seminorm*).

**The proof of Lemma 298 (*definite semi-inner product is inner product*)**

cites explicitly:

Statement(s) from [20],

Definition 296 (*semi-inner product*).

**The proof of Lemma 299 (*finite cover of compact interval*)**

has no explicit citation.

**The proof of Lemma 300 (*power function satisfies relaxed triangle inequality*)**

has no explicit citation.

**The proof of Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* )**

cites explicitly:

Definition 302 (*Hölder conjugates in  $\mathbb{R}$* ).

**The proof of Lemma 304 (*Young's inequality for products in  $\mathbb{R}$* )**

cites explicitly:

Definition 302 (*Hölder conjugates in  $\mathbb{R}$* ).

**The proof of Lemma 305 (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* )**

cites explicitly:

Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),

Lemma 304 (*Young's inequality for products in  $\mathbb{R}$* ).

**The proof of Lemma 307 (*topology on  $\mathbb{R}$* )**

cites explicitly:

Lemma 249 (*topological basis is generator*),

Lemma 258 (*topological basis of order topology*).

**The proof of Lemma 308 (*trace topology on subset of  $\mathbb{R}$* )**

cites explicitly:

Lemma 252 (*trace topology on subset*),

Lemma 307 (*topology on  $\mathbb{R}$* ).

**The proof of Lemma 310 (*topology on  $\mathbb{R}^m$* )**

cites explicitly:

Definition 227 (*product of subset systems*),

Lemma 250 (*topological subbasis*),

Lemma 254 (*box topology on Cartesian product*),

Lemma 307 (*topology on  $\mathbb{R}$* ).



**The proof of Lemma 311** (*trace topology on subset of  $\mathbb{R}^m$* )

cites explicitly:

Lemma 252 (*trace topology on subset*),

Lemma 310 (*topology on  $\mathbb{R}^m$* ).

**The proof of Lemma 313** (*order in  $\overline{\mathbb{R}}$  is total*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 314** ( *$\overline{\mathbb{R}}$  is complete*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 318** (*zero is identity element for addition in  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 319** (*addition in  $\overline{\mathbb{R}}$  is associative when defined*)

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 320** (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*)

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 321** (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 322** (*infinity-sum property in  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 323** (*finite-sum property in  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 324** (*additive inverse in  $\overline{\mathbb{R}}$  is involution*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 325** (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 328** (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 329** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*)

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 330** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*)

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 331** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*)

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 332** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*)

cites explicitly:

Lemma 330 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),

Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).

**The proof of Lemma 333** (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*)

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 334** (*zero-product property in  $\overline{\mathbb{R}}$* )

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**The proof of Lemma 335** (*one-product property in  $\overline{\mathbb{R}}$* )

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**The proof of Lemma 338** (*multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined*)

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Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 340** (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* )

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Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 341** (*finite absolute value in  $\overline{\mathbb{R}}$* )

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**The proof of Lemma 342** (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*)

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**The proof of Lemma 343** (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*)

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**The proof of Lemma 344** (*inverse of absolute value in  $\overline{\mathbb{R}}$* )

cites explicitly:

Lemma 324 (*additive inverse in  $\overline{\mathbb{R}}$  is involution*),

Lemma 325 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),

Lemma 340 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 345 (*absolute value in  $\overline{\mathbb{R}}$  is idempotent*)**

cites explicitly:

Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*).

**The proof of Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*)**

cites explicitly:

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*)**

cites explicitly:

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*)**

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*).

**The proof of Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*).

**The proof of Lemma 351 (*absolute value in  $\overline{\mathbb{R}}$  satisfies countable triangle inequality*)**

cites explicitly:

Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*).

**The proof of Lemma 353 (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*)**

cites explicitly:

Definition 352 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* )**

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),

Definition 352 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ),

Definition 354 (*exponentiation in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 357 (*exponentiation in  $\overline{\mathbb{R}}$  of power 1*)**

cites explicitly:

Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 358 (*zero-exponentiation property in  $\overline{\mathbb{R}}$* )**

cites explicitly:

Lemma 336 (*infinity-product property in  $\overline{\mathbb{R}}$* ),

Definition 352 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ),

Definition 354 (*exponentiation in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 359 (one-exponentiation property in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Lemma 334 (zero-product property in  $\overline{\mathbb{R}}$ ),

Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ),

Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 360 (infinity-exponentiation property in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Lemma 336 (infinity-product property in  $\overline{\mathbb{R}}$ ),

Definition 352 (exponential and logarithm in  $\overline{\mathbb{R}}$ ),

Definition 354 (exponentiation in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 361 (power function in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Definition 339 (absolute value in  $\overline{\mathbb{R}}$ ),

Lemma 355 (exponentiation in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 362 (power function in  $\overline{\mathbb{R}}$  is absolute value)**

cites explicitly:

Lemma 357 (exponentiation in  $\overline{\mathbb{R}}$  of power 1),

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 363 (finite power function in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 364 (power function in  $\overline{\mathbb{R}}$  is nonnegative)**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 365 (compatibility of power function in  $\overline{\mathbb{R}}$  with addition)**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication)**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone)**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 368 (inverse of power function in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative),

Lemma 344 (inverse of absolute value in  $\overline{\mathbb{R}}$ ),

Lemma 364 (power function in  $\overline{\mathbb{R}}$  is nonnegative),

Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication),

Lemma 367 (power function in  $\overline{\mathbb{R}}$  is monotone).

**The proof of Lemma 369 (power function in  $\overline{\mathbb{R}}$  is definite)**

cites explicitly:

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 370 (power function in  $\overline{\mathbb{R}}$  is multiplicative)**

cites explicitly:

Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ),

Lemma 361 (power function in  $\overline{\mathbb{R}}$ ),

Lemma 369 (power function in  $\overline{\mathbb{R}}$  is definite).

**The proof of Lemma 371** (*power function satisfies extended relaxed triangle inequality*)

cites explicitly:

Lemma 300 (*power function satisfies relaxed triangle inequality*),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),

Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 372** (*topology on  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 256 (*order topology*),

Lemma 258 (*topological basis of order topology*),

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 373** (*trace topology on subset of  $\overline{\mathbb{R}}$* )

cites explicitly:

Definition 226 (*trace of subset system*),

Lemma 252 (*trace topology on subset*),

Lemma 259 (*trace of order topology*),

Lemma 307 (*topology on  $\mathbb{R}$* ).

**The proof of Lemma 375** (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*)

cites explicitly:

Lemma 373 (*trace topology on subset of  $\overline{\mathbb{R}}$* ),

Definition 374 (*tangent and arctangent in  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 377** (*metric on  $\overline{\mathbb{R}}$* )

cites explicitly:

Statement(s) from [20],

Definition 374 (*tangent and arctangent in  $\overline{\mathbb{R}}$* ),

Lemma 375 (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*).

**The proof of Lemma 378** (*equivalent topologies on  $\overline{\mathbb{R}}$* )

cites explicitly:

Lemma 249 (*topological basis is generator*),

Lemma 372 (*topology on  $\overline{\mathbb{R}}$* ),

Lemma 375 (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*).

**The proof of Lemma 379** (*convergence towards  $\pm\infty$* )

cites explicitly:

Lemma 372 (*topology on  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 380** (*convergence towards  $\pm\infty$  (discrete)*)

cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 379 (*convergence towards  $\pm\infty$* ).

**The proof of Lemma 381** (*limit is monotone*)

cites explicitly:

Statement(s) from [20],

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*).

**The proof of Lemma 382** (*cluster point at  $\pm\infty$  (discrete)*)

cites explicitly:

Definition 267 (*cluster point*),

Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*).

The proof of Lemma 383 (*limit is invariant by translation*)  
has no explicit citation.

The proof of Lemma 385 (*topology on  $\overline{\mathbb{R}}^m$* )  
cites explicitly:  
Lemma 254 (*box topology on Cartesian product*),  
Lemma 378 (*equivalent topologies on  $\overline{\mathbb{R}}$* ).

The proof of Lemma 386 (*trace topology on subset of  $\overline{\mathbb{R}}^m$* )  
cites explicitly:  
Definition 226 (*trace of subset system*),  
Lemma 252 (*trace topology on subset*),  
Lemma 310 (*topology on  $\mathbb{R}^m$* ),  
Lemma 385 (*topology on  $\overline{\mathbb{R}}^m$* ).

The proof of Lemma 387 (*addition in  $\overline{\mathbb{R}}$  is continuous*)  
cites explicitly:  
Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

The proof of Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*)  
cites explicitly:  
Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ).

The proof of Lemma 389 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*)  
cites explicitly:  
Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ).

The proof of Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous*)  
cites explicitly:  
Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ).

The proof of Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*)  
cites explicitly:  
Statement(s) from [20],  
Lemma 387 (*addition in  $\overline{\mathbb{R}}$  is continuous*),  
Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*).

The proof of Lemma 392 ( *$\overline{\mathbb{R}}_+$  is complete*)  
cites explicitly:  
Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*),  
Lemma 372 (*topology on  $\overline{\mathbb{R}}$* ).

The proof of Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*)  
cites explicitly:  
Definition 317 (*addition in  $\overline{\mathbb{R}}$* ).

The proof of Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*)  
cites explicitly:  
Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
Lemma 319 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*).

The proof of Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*)  
cites explicitly:  
Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
Lemma 320 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*).

**The proof of Lemma 396 (addition in  $\overline{\mathbb{R}}_+$  is compatible with order)**

cites explicitly:

Definition 317 (addition in  $\overline{\mathbb{R}}$ ),

Lemma 321 (addition in  $\overline{\mathbb{R}}$  is compatible with order when defined).

**The proof of Lemma 397 (infinity-sum property in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Lemma 322 (infinity-sum property in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Lemma 392 ( $\overline{\mathbb{R}}_+$  is complete),

Lemma 393 (addition in  $\overline{\mathbb{R}}_+$  is closed).

**The proof of Lemma 399 (series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition)**

cites explicitly:

Statement(s) from [20],

Definition 209 (pseudopartition),

Lemma 394 (addition in  $\overline{\mathbb{R}}_+$  is associative),

Lemma 395 (addition in  $\overline{\mathbb{R}}_+$  is commutative),

Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ).

**The proof of Lemma 400 (series in  $\overline{\mathbb{R}}_+$  is monotone)**

cites explicitly:

Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ),

Lemma 399 (series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition).

**The proof of Lemma 401 (subseries in  $\overline{\mathbb{R}}_+$  are lower bounds)**

cites explicitly:

Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ),

Lemma 400 (series in  $\overline{\mathbb{R}}_+$  is monotone).

**The proof of Lemma 402 (order is meaningless in series in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Lemma 401 (subseries in  $\overline{\mathbb{R}}_+$  are lower bounds).

**The proof of Lemma 403 (definition of double series in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Lemma 402 (order is meaningless in series in  $\overline{\mathbb{R}}_+$ ).

**The proof of Lemma 404 (double series in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Statement(s) from [20],

Lemma 381 (limit is monotone),

Lemma 396 (addition in  $\overline{\mathbb{R}}_+$  is compatible with order),

Lemma 398 (definition of series in  $\overline{\mathbb{R}}_+$ ),

Lemma 403 (definition of double series in  $\overline{\mathbb{R}}_+$ ).

**The proof of Lemma 407 (multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined)**

cites explicitly:

Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ),

Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ ).

**The proof of Lemma 408 (zero-product property in  $\overline{\mathbb{R}}_+$ )**

cites explicitly:

Lemma 334 (zero-product property in  $\overline{\mathbb{R}}$ ),

Definition 405 (multiplication in  $\overline{\mathbb{R}}_+$ ).

**The proof of Lemma 409 (*one-product property in  $\overline{\mathbb{R}}_+$* )**

cites explicitly:

Lemma 335 (*one-product property in  $\overline{\mathbb{R}}$* ),Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ).**The proof of Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* )**

cites explicitly:

Lemma 336 (*infinity-product property in  $\overline{\mathbb{R}}$* ),Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ).**The proof of Lemma 411 (*finite-product property in  $\overline{\mathbb{R}}_+$* )**

cites explicitly:

Lemma 337 (*finite-product property in  $\overline{\mathbb{R}}$* ),Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ).**The proof of Lemma 412 (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone*)**

cites explicitly:

Lemma 338 (*multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined*),Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ).**The proof of Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),Lemma 387 (*addition in  $\overline{\mathbb{R}}$  is continuous*).**The proof of Lemma 414 (*limit in  $\overline{\mathbb{R}}_+$  is positive linear*)**

cites explicitly:

Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*).**The proof of Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*)**

cites explicitly:

Lemma 329 (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*),Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).**The proof of Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*)**

cites explicitly:

Lemma 330 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).**The proof of Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*)**

cites explicitly:

Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),Lemma 332 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).**The proof of Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*)**

cites explicitly:

Lemma 333 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*),Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).**The proof of Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**

cites explicitly:

Lemma 334 (*zero-product property in  $\overline{\mathbb{R}}$* ),Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).



- The proof of Lemma 422 (*one-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 cites explicitly:  
 Lemma 335 (*one-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 423 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 cites explicitly:  
 Lemma 336 (*infinity-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 424 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 cites explicitly:  
 Lemma 423 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*)**  
 cites explicitly:  
 Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**  
 cites explicitly:  
 Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 428 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**  
 cites explicitly:  
 Lemma 424 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*)**  
 cites explicitly:  
 Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 cites explicitly:  
 Definition 354 (*exponentiation in  $\overline{\mathbb{R}}$* ),  
 Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 cites explicitly:  
 Lemma 359 (*one-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*).
- The proof of Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**  
 cites explicitly:  
 Lemma 358 (*zero-exponentiation property in  $\overline{\mathbb{R}}$* ).
- The proof of Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*)**  
 cites explicitly:  
 Lemma 360 (*infinity-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*).
- The proof of Lemma 434 (*finite-positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**  
 cites explicitly:  
 Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 435 (*finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

cites explicitly:

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 436 (*positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

cites explicitly:

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 437 (*less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

cites explicitly:

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 438 (*great-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

cites explicitly:

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*)**

cites explicitly:

Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),

Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),

Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*).

**The proof of Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*)**

cites explicitly:

Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),

Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ),

Lemma 365 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition*),

Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ),

Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 436 (*positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*)**

cites explicitly:

Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),

Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 437 (*less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 438 (*great-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*)**

cites explicitly:

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),

Lemma 345 (*absolute value in  $\overline{\mathbb{R}}$  is idempotent*),

Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*).

**The proof of Lemma 444 (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*)**

cites explicitly:

Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*).

**The proof of Lemma 445 (*power function in  $\overline{\mathbb{R}}$  is even (measure theory)*)**  
 cites explicitly:  
 Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*).

**The proof of Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*)**  
 cites explicitly:  
 Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 370 (*power function in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*).

**The proof of Lemma 448 (*multiplication in  $\overline{\mathbb{R}}_+$  is continuous (measure theory)*)**  
 cites explicitly:  
 Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**The proof of Lemma 449 (*power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)*)**  
 cites explicitly:  
 Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*).

**The proof of Lemma 453 (*Young's inequality for products (measure theory)*)**  
 cites explicitly:  
 Lemma 304 (*Young's inequality for products in  $\mathbb{R}$* ),  
 Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),  
 Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**The proof of Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*)**  
 cites explicitly:  
 Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),  
 Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*),  
 Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*),  
 Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 453 (*Young's inequality for products (measure theory)*).

**The proof of Lemma 455 (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* )**  
 cites explicitly:  
 Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 412 (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone*).

**The proof of Lemma 457 (*empty connected component*)**  
 cites explicitly:  
 Definition 456 (*connected component in  $\mathbb{R}$* ).

**The proof of Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*)**  
 cites explicitly:  
 Statement(s) from [20],

Definition 229 (*interval*),  
 Definition 240 (*topological space*),  
 Definition 456 (*connected component in  $\mathbb{R}$* ).

**The proof of Lemma 459 (*connected component of open subset of  $\mathbb{R}$  is maximal*)**  
 cites explicitly:  
 Definition 456 (*connected component in  $\mathbb{R}$* ),  
 Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*).

**The proof of Lemma 460 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*)**  
 cites explicitly:  
 Lemma 459 (*connected component of open subset of  $\mathbb{R}$  is maximal*).

**The proof of Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* )**  
 cites explicitly:  
 Statement(s) from [20],  
 Definition 229 (*interval*),  
 Definition 456 (*connected component in  $\mathbb{R}$* ),  
 Lemma 460 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*).

**The proof of Lemma 462 (*rational approximation of lower bound of open interval*)**  
 has no explicit citation.

**The proof of Lemma 463 (*rational approximation of upper bound of open interval*)**  
 cites explicitly:  
 Lemma 462 (*rational approximation of lower bound of open interval*).

**The proof of Lemma 464 (*open intervals with rational bounds cover open interval*)**  
 cites explicitly:  
 Statement(s) from [20],  
 Definition 240 (*topological space*),  
 Lemma 462 (*rational approximation of lower bound of open interval*),  
 Lemma 463 (*rational approximation of upper bound of open interval*).

**The proof of Theorem 465 ( *$\mathbb{R}$  is second-countable*)**  
 cites explicitly:  
 Lemma 250 (*topological subbasis*),  
 Definition 260 (*second-countability*),  
 Lemma 307 (*topology on  $\mathbb{R}$* ),  
 Lemma 464 (*open intervals with rational bounds cover open interval*).

**The proof of Lemma 466 ( *$\mathbb{R}^m$  is second-countable*)**  
 cites explicitly:  
 Lemma 261 (*compatibility of second-countability with Cartesian product*),  
 Lemma 310 (*topology on  $\mathbb{R}^m$* ),  
 Theorem 465 ( *$\mathbb{R}$  is second-countable*).

**The proof of Lemma 469 (*compatibility between projection and injections in  $\overline{\mathbb{R}}$* )**  
 cites explicitly:  
 Definition 468 (*projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 471 (*compatibility of projection and injections in  $\overline{\mathbb{R}}$  with subset operations*)**  
 cites explicitly:  
 Definition 468 (*projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 474 (open down)**

cites explicitly:

Lemma 373 (trace topology on subset of  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 475 (open up)**

cites explicitly:

Lemma 373 (trace topology on subset of  $\overline{\mathbb{R}}$ ),

Lemma 469 (compatibility between projection and injections in  $\overline{\mathbb{R}}$ ),

Lemma 474 (open down).

**The proof of Lemma 476 (open subsets of  $\overline{\mathbb{R}}$ )**

cites explicitly:

Definition 246 (topological basis),

Lemma 372 (topology on  $\overline{\mathbb{R}}$ ),

Definition 468 (projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ ),

Lemma 474 (open down),

Lemma 475 (open up).

**The proof of Lemma 477 (countable connected components of open subsets of  $\overline{\mathbb{R}}$ )**

cites explicitly:

Theorem 461 (countable connected components of open subsets of  $\mathbb{R}$ ),

Lemma 476 (open subsets of  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 478 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ )**

cites explicitly:

Lemma 462 (rational approximation of lower bound of open interval),

Lemma 463 (rational approximation of upper bound of open interval),

Lemma 464 (open intervals with rational bounds cover open interval).

**The proof of Lemma 479 ( $\overline{\mathbb{R}}$  is second-countable)**

cites explicitly:

Lemma 250 (topological subbasis),

Definition 260 (second-countability),

Lemma 372 (topology on  $\overline{\mathbb{R}}$ ),

Lemma 477 (countable connected components of open subsets of  $\overline{\mathbb{R}}$ ),

Lemma 478 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 480 ( $\overline{\mathbb{R}^m}$  is second-countable)**

cites explicitly:

Lemma 261 (compatibility of second-countability with Cartesian product),

Lemma 385 (topology on  $\overline{\mathbb{R}^m}$ ),

Lemma 479 ( $\overline{\mathbb{R}}$  is second-countable).

**The proof of Lemma 482 (extrema of constant function)**

cites explicitly:

Statement(s) from [20].

**The proof of Lemma 483 (compatibility of extrema with affine transformation)**

cites explicitly:

Statement(s) from [20],

Lemma 321 (addition in  $\overline{\mathbb{R}}$  is compatible with order when defined),

Lemma 331 (multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined).

**The proof of Lemma 484 (equivalent definition of finite infimum)**

cites explicitly:

Statement(s) from [20],

Lemma 265 (equivalent definition of convergent sequence).

**The proof of Lemma 485 (equivalent definition of finite infimum in  $\overline{\mathbb{R}}$ )**

cites explicitly:

Statement(s) from [20],

Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total),

Lemma 482 (extrema of constant function),

Lemma 484 (equivalent definition of finite infimum).

**The proof of Lemma 486 (equivalent definition of infimum)**

cites explicitly:

Statement(s) from [20],

Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total),

Lemma 380 (convergence towards  $\pm\infty$  (discrete)),

Lemma 485 (equivalent definition of finite infimum in  $\overline{\mathbb{R}}$ ).

**The proof of Lemma 487 (infimum is smaller than supremum)**

cites explicitly:

Statement(s) from [20],

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total).

**The proof of Lemma 488 (infimum is monotone (set))**

cites explicitly:

Statement(s) from [20],

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total).

**The proof of Lemma 489 (supremum is monotone (set))**

cites explicitly:

Statement(s) from [20],

Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone),

Lemma 488 (infimum is monotone (set)).

**The proof of Lemma 490 (infimum is monotone (function))**

cites explicitly:

Statement(s) from [20],

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total).

**The proof of Lemma 491 (supremum is monotone (function))**

cites explicitly:

Statement(s) from [20],

Lemma 325 (additive inverse in  $\overline{\mathbb{R}}$  is monotone),

Lemma 490 (infimum is monotone (function)).

**The proof of Lemma 492 (infimum of bounded is bounded)**

cites explicitly:

Lemma 482 (extrema of constant function),

Lemma 490 (infimum is monotone (function)).

**The proof of Lemma 493 (supremum of bounded is bounded)**

cites explicitly:

Lemma 482 (extrema of constant function),

Lemma 491 (supremum is monotone (function)).

**The proof of Lemma 494 (compatibility of infimum with absolute value)**

cites explicitly:

Statement(s) from [20],

Lemma 340 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ),

Lemma 487 (*infimum is smaller than supremum*),  
 Lemma 490 (*infimum is monotone (function)*),  
 Lemma 491 (*supremum is monotone (function)*).

**The proof of Lemma 495 (*compatibility of supremum with absolute value*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 346 (*absolute value in  $\bar{\mathbb{R}}$  is even*),  
 Lemma 494 (*compatibility of infimum with absolute value*).

**The proof of Lemma 496 (*compatibility of translation with infimum of sequence*)**

cites explicitly:

Lemma 488 (*infimum is monotone (set)*).

**The proof of Lemma 497 (*compatibility of translation with supremum of sequence*)**

cites explicitly:

Lemma 489 (*supremum is monotone (set)*).

**The proof of Lemma 498 (*infimum of nonincreasing sequence*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 496 (*compatibility of translation with infimum of sequence*).

**The proof of Lemma 499 (*supremum of nondecreasing sequence*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 498 (*infimum of nonincreasing sequence*).

**The proof of Lemma 500 (*infimum is limit*)**

cites explicitly:

Statement(s) from [20],  
 Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ),  
 Lemma 379 (*convergence towards  $\pm\infty$* ),  
 Lemma 482 (*extrema of constant function*).

**The proof of Lemma 501 (*supremum is limit*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 388 (*multiplication in  $\bar{\mathbb{R}}$  is continuous*),  
 Lemma 500 (*infimum is limit*).

**The proof of Lemma 502 (*series in  $\bar{\mathbb{R}}_+$  is supremum*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 398 (*definition of series in  $\bar{\mathbb{R}}_+$* ),  
 Lemma 488 (*infimum is monotone (set)*).

**The proof of Lemma 503 (*constant sum of monotone nonnegative functions*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 413 (*addition in  $\bar{\mathbb{R}}_+$  is continuous*),  
 Lemma 500 (*infimum is limit*),  
 Lemma 501 (*supremum is limit*).

**The proof of Lemma 504 (*limit inferior*)**

cites explicitly:

Lemma 314 ( *$\bar{\mathbb{R}}$  is complete*),  
 Lemma 488 (*infimum is monotone (set)*).

**The proof of Lemma 505 (*limit inferior is  $\infty$* )**

cites explicitly:

Statement(s) from [20],

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Lemma 504 (*limit inferior*).

**The proof of Lemma 506 (*equivalent definition of the limit inferior*)**

cites explicitly:

Statement(s) from [20],

Definition 267 (*cluster point*),

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Lemma 381 (*limit is monotone*),

Lemma 382 (*cluster point at  $\pm\infty$  (discrete)*),

Lemma 504 (*limit inferior*),

Lemma 505 (*limit inferior is  $\infty$* ).

**The proof of Lemma 507 (*limit inferior is invariant by translation*)**

cites explicitly:

Definition 267 (*cluster point*),

Lemma 383 (*limit is invariant by translation*),

Lemma 506 (*equivalent definition of the limit inferior*).

**The proof of Lemma 508 (*limit inferior is monotone*)**

cites explicitly:

Lemma 490 (*infimum is monotone (function)*),

Lemma 491 (*supremum is monotone (function)*),

Lemma 504 (*limit inferior*),

Lemma 507 (*limit inferior is invariant by translation*).

**The proof of Lemma 509 (*limit superior*)**

cites explicitly:

Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*).

**The proof of Lemma 510 (*duality limit inferior-limit superior*)**

cites explicitly:

Statement(s) from [20],

Lemma 504 (*limit inferior*),

Lemma 509 (*limit superior*).

**The proof of Lemma 511 (*equivalent definition of limit superior*)**

cites explicitly:

Lemma 325 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),

Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*),

Lemma 506 (*equivalent definition of the limit inferior*),

Lemma 510 (*duality limit inferior-limit superior*).

**The proof of Lemma 512 (*limit inferior is smaller than limit superior*)**

cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 506 (*equivalent definition of the limit inferior*),

Lemma 511 (*equivalent definition of limit superior*).

**The proof of Lemma 513 (*limit superior is monotone*)**

cites explicitly:

Lemma 325 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),

Lemma 508 (*limit inferior is monotone*),

Lemma 510 (*duality limit inferior-limit superior*).



- The proof of Lemma 514 (*compatibility of limit inferior with absolute value*)**  
 cites explicitly:  
 Lemma 381 (*limit is monotone*),  
 Lemma 389 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 494 (*compatibility of infimum with absolute value*),  
 Lemma 504 (*limit inferior*),  
 Lemma 509 (*limit superior*).
- The proof of Lemma 515 (*compatibility of limit superior with absolute value*)**  
 cites explicitly:  
 Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Lemma 514 (*compatibility of limit inferior with absolute value*).
- The proof of Lemma 517 (*limit inferior and limit superior of pointwise convergent*)**  
 cites explicitly:  
 Lemma 506 (*equivalent definition of the limit inferior*),  
 Lemma 511 (*equivalent definition of limit superior*),  
 Definition 516 (*pointwise convergence*).
- The proof of Lemma 518 (*limit inferior bounded from below*)**  
 cites explicitly:  
 Statement(s) from [20],  
 Lemma 508 (*limit inferior is monotone*),  
 Lemma 517 (*limit inferior and limit superior of pointwise convergent*).
- The proof of Lemma 519 (*limit inferior bounded from above*)**  
 cites explicitly:  
 Statement(s) from [20],  
 Lemma 508 (*limit inferior is monotone*),  
 Lemma 517 (*limit inferior and limit superior of pointwise convergent*).
- The proof of Lemma 520 (*limit superior bounded from below*)**  
 cites explicitly:  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Lemma 519 (*limit inferior bounded from above*).
- The proof of Lemma 521 (*limit superior bounded from above*)**  
 cites explicitly:  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Lemma 518 (*limit inferior bounded from below*).
- The proof of Lemma 522 (*limit inferior, limit superior and pointwise convergence*)**  
 cites explicitly:  
 Statement(s) from [20],  
 Lemma 505 (*limit inferior is  $\infty$* ),  
 Lemma 506 (*equivalent definition of the limit inferior*),  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Lemma 511 (*equivalent definition of limit superior*),  
 Lemma 512 (*limit inferior is smaller than limit superior*).
- The proof of Lemma 524 (*restriction is masking*)**  
 has no explicit citation.
- The proof of Lemma 525 (*masking is  $\sigma$ -additive*)**  
 cites explicitly:  
 Definition 209 (*pseudopartition*).

**The proof of Lemma 527 (*finite part is finite*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 526 (*finite part*).

**The proof of Lemma 529 (*equivalent definition of nonnegative and nonpositive parts*)**

cites explicitly:

Statement(s) from [20],

Definition 528 (*nonnegative and nonpositive parts*).

**The proof of Lemma 530 (*nonnegative and nonpositive parts are nonnegative*)**

cites explicitly:

Statement(s) from [20],

Definition 528 (*nonnegative and nonpositive parts*).

**The proof of Lemma 531 (*nonnegative and nonpositive parts are orthogonal*)**

cites explicitly:

Statement(s) from [20],

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 532 (*decomposition into nonnegative and nonpositive parts*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 531 (*nonnegative and nonpositive parts are orthogonal*).

**The proof of Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),

Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),

Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ),

Definition 528 (*nonnegative and nonpositive parts*),

Lemma 530 (*nonnegative and nonpositive parts are nonnegative*),

Lemma 531 (*nonnegative and nonpositive parts are orthogonal*),

Lemma 532 (*decomposition into nonnegative and nonpositive parts*).

**The proof of Lemma 534 (*compatibility of nonpositive and nonnegative parts with mask*)**

cites explicitly:

Definition 528 (*nonnegative and nonpositive parts*).

**The proof of Lemma 535 (*compatibility of nonpositive and nonnegative parts with restriction*)**

cites explicitly:

Definition 528 (*nonnegative and nonpositive parts*).

**The proof of Lemma 538 (*nonempty and with empty or full*)**

has no explicit citation.

**The proof of Lemma 539 (*with empty and full*)**

has no explicit citation.

**The proof of Lemma 541 (*closedness under local complement and complement*)**

has no explicit citation.

- The proof of Lemma 542 (*closedness under disjoint union and local complement*) has no explicit citation.
- The proof of Lemma 543 (*closedness under set difference and local complement*) has no explicit citation.
- The proof of Lemma 544 (*closedness under symmetric difference, local complement, and disjoint union*) has no explicit citation.
- The proof of Lemma 545 (*closedness under intersection and set difference*) has no explicit citation.
- The proof of Lemma 546 (*closedness under union and disjoint union*) has no explicit citation.
- The proof of Lemma 547 (*closedness under union and intersection*) has no explicit citation.
- The proof of Lemma 548 (*closedness under union and set difference*) cites explicitly:  
Lemma 545 (*closedness under intersection and set difference*),  
Lemma 547 (*closedness under union and intersection*).
- The proof of Lemma 550 (*closedness under symmetric difference and set difference*) has no explicit citation.
- The proof of Lemma 551 (*closedness under symmetric difference and intersection*) has no explicit citation.
- The proof of Lemma 552 (*nonempty is with empty*) has no explicit citation.
- The proof of Lemma 553 (*with full is with empty*) has no explicit citation.
- The proof of Lemma 554 (*disjoint decomposition*) has no explicit citation.
- The proof of Lemma 555 (*disjoint decomposition of complement and set difference*) has no explicit citation.
- The proof of Lemma 556 (*disjoint decomposition of local complement and complement*) has no explicit citation.
- The proof of Lemma 557 (*disjoint decomposition of set difference and local complement*) has no explicit citation.
- The proof of Lemma 558 (*closedness under finite operations*) has no explicit citation.
- The proof of Lemma 559 (*closedness under finite union and intersection*) cites explicitly:  
Lemma 547 (*closedness under union and intersection*),  
Lemma 558 (*closedness under finite operations*).

- The proof of Lemma 561 (*closedness under intersection and finite disjoint union closure*)  
has no explicit citation.
- The proof of Lemma 562 (*finite disjoint union closure is closed under finite disjoint union*)  
has no explicit citation.
- The proof of Lemma 563 (*closedness under set difference and finite disjoint union closure*)  
cites explicitly:  
Lemma 561 (*closedness under intersection and finite disjoint union closure*),  
Lemma 562 (*finite disjoint union closure is closed under finite disjoint union*).
- The proof of Lemma 566 (*closedness under countable disjoint union and local complement*)  
cites explicitly:  
Lemma 542 (*closedness under disjoint union and local complement*).
- The proof of Lemma 567 (*closedness under countable union and intersection*)  
has no explicit citation.
- The proof of Lemma 568 (*closedness under countable disjoint and monotone union*)  
cites explicitly:  
Lemma 215 (*disjoint union of union*).
- The proof of Lemma 569 (*closedness under countable monotone and disjoint union*)  
cites explicitly:  
Lemma 217 (*monotone union of union*).
- The proof of Lemma 570 (*closedness under countable disjoint union and countable union*)  
cites explicitly:  
Lemma 215 (*disjoint union of union*).
- The proof of Lemma 571 (*closedness under countable monotone union and countable union*)  
cites explicitly:  
Lemma 569 (*closedness under countable monotone and disjoint union*),  
Lemma 570 (*closedness under countable disjoint union and countable union*).
- The proof of Lemma 572 (*disjoint union of union in subset system*)  
cites explicitly:  
Lemma 215 (*disjoint union of union*),  
Lemma 558 (*closedness under finite operations*).
- The proof of Lemma 574 (*disjoint union of union in subset system (variant)*)  
cites explicitly:  
Lemma 215 (*disjoint union of union*),  
Lemma 552 (*nonempty is with empty*).
- The proof of Lemma 575 (*monotone union of union in subset system*)  
cites explicitly:  
Lemma 217 (*monotone union of union*),  
Lemma 558 (*closedness under finite operations*).
- The proof of Lemma 576 (*empty monotone intersection of union in subset system*)  
cites explicitly:  
Lemma 219 (*empty monotone intersection of union*).

**The proof of Lemma 577 (*monotone union of monotone intersection in subset system*)**

cites explicitly:

Lemma 221 (*monotone union of monotone intersection*).

**The proof of Lemma 582 (*intersection of subset system classes*)**

cites explicitly:

Definition 580 (*subset system class*).

**The proof of Lemma 583 (*subset system classes are monotone*)**

cites explicitly:

Definition 580 (*subset system class*).

**The proof of Lemma 585 (*subset system generation preserves class*)**

cites explicitly:

Definition 578 (*subset system*),

Definition 580 (*subset system class*),

Definition 584 (*generated subset system*).

**The proof of Lemma 586 (*properties of subset system generation*)**

cites explicitly:

Definition 578 (*subset system*),

Definition 580 (*subset system class*),

Definition 584 (*generated subset system*),

Lemma 585 (*subset system generation preserves class*).

**The proof of Lemma 587 (*other properties of subset system generation*)**

cites explicitly:

Lemma 585 (*subset system generation preserves class*),

Lemma 586 (*properties of subset system generation*).

**The proof of Lemma 588 (*generated subset systems are monotone*)**

cites explicitly:

Lemma 583 (*subset system classes are monotone*),

Definition 584 (*generated subset system*).

**The proof of Lemma 590 (*upgrade generated subset system*)**

cites explicitly:

Definition 580 (*subset system class*),

Lemma 585 (*subset system generation preserves class*),

Lemma 586 (*properties of subset system generation*).

**The proof of Lemma 591 (*cascading generated subset systems*)**

cites explicitly:

Lemma 583 (*subset system classes are monotone*),

Lemma 585 (*subset system generation preserves class*),

Lemma 586 (*properties of subset system generation*).

**The proof of Lemma 592 (*double cascading generated subset systems*)**

has no explicit citation.

**The proof of Lemma 594 (*class of nonempty subset systems*)**

cites explicitly:

Definition 580 (*subset system class*),

Lemma 582 (*intersection of subset system classes*).

**The proof of Lemma 595 (*generated nonempty subset systems*)**

cites explicitly:

Definition 584 (*generated subset system*),

Lemma 594 (*class of nonempty subset systems*).

**The proof of Lemma 596 (*subset system generation preserves class (with nonemptiness)*)**

cites explicitly:

Lemma 585 (*subset system generation preserves class*),

Lemma 594 (*class of nonempty subset systems*),

Lemma 595 (*generated nonempty subset systems*).

**The proof of Lemma 597 (*properties of generated subset system (with nonemptiness)*)**

cites explicitly:

Lemma 586 (*properties of subset system generation*),

Lemma 594 (*class of nonempty subset systems*),

Lemma 595 (*generated nonempty subset systems*).

**The proof of Lemma 598 (*other properties of subset system generation (with nonemptiness)*)**

cites explicitly:

Lemma 587 (*other properties of subset system generation*),

Lemma 595 (*generated nonempty subset systems*).

**The proof of Lemma 599 (*generated subset systems are monotone (with nonemptiness)*)**

cites explicitly:

Lemma 588 (*generated subset systems are monotone*),

Lemma 595 (*generated nonempty subset systems*).

**The proof of Lemma 600 (*upgrade generated subset system (with nonemptiness)*)**

cites explicitly:

Lemma 590 (*upgrade generated subset system*),

Lemma 595 (*generated nonempty subset systems*),

Lemma 596 (*subset system generation preserves class (with nonemptiness)*).

**The proof of Lemma 601 (*cascading generated subset systems (with nonemptiness)*)**

cites explicitly:

Lemma 591 (*cascading generated subset systems*),

Lemma 595 (*generated nonempty subset systems*).

**The proof of Lemma 604 (*equivalent definition of  $\pi$ -system*)**

cites explicitly:

Lemma 558 (*closedness under finite operations*),

Definition 602 ( *$\pi$ -system*).

**The proof of Lemma 605 (*intersection of  $\pi$ -systems*)**

cites explicitly:

Lemma 604 (*equivalent definition of  $\pi$ -system*).

**The proof of Lemma 607 (*generated  $\pi$ -system is  $\pi$ -system*)**

cites explicitly:

Lemma 596 (*subset system generation preserves class (with nonemptiness)*),

Lemma 605 (*intersection of  $\pi$ -systems*),

Definition 606 (*generated  $\pi$ -system*).

**The proof of Lemma 608 (*properties of  $\pi$ -system generation*)**

cites explicitly:

Lemma 597 (*properties of generated subset system (with nonemptiness)*),  
 Lemma 605 (*intersection of  $\pi$ -systems*),  
 Definition 606 (*generated  $\pi$ -system*).

**The proof of Lemma 609 (*other properties of  $\pi$ -system generation*)**

cites explicitly:

Lemma 598 (*other properties of subset system generation (with nonemptiness)*),  
 Lemma 605 (*intersection of  $\pi$ -systems*),  
 Definition 606 (*generated  $\pi$ -system*).

**The proof of Lemma 612 (*equivalent definition of ring of sets*)**

cites explicitly:

Lemma 538 (*nonempty and with empty or full*),  
 Lemma 544 (*closedness under symmetric difference, local complement, and disjoint union*),  
 Lemma 545 (*closedness under intersection and set difference*),  
 Lemma 546 (*closedness under union and disjoint union*),  
 Lemma 550 (*closedness under symmetric difference and set difference*),  
 Lemma 551 (*closedness under symmetric difference and intersection*),  
 Lemma 558 (*closedness under finite operations*),  
 Definition 611 (*ring of sets*).

**The proof of Lemma 614 (*ring of sets is  $\pi$ -system*)**

cites explicitly:

Lemma 604 (*equivalent definition of  $\pi$ -system*),  
 Lemma 612 (*equivalent definition of ring of sets*).

**The proof of Lemma 615 (*other properties of ring of sets*)**

cites explicitly:

Lemma 558 (*closedness under finite operations*),  
 Definition 602 ( *$\pi$ -system*),  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 614 (*ring of sets is  $\pi$ -system*).

**The proof of Lemma 616 (*intersection of rings of sets*)**

cites explicitly:

Lemma 612 (*equivalent definition of ring of sets*).

**The proof of Lemma 618 (*generated ring of sets is ring of sets*)**

cites explicitly:

Lemma 585 (*subset system generation preserves class*),  
 Lemma 616 (*intersection of rings of sets*),  
 Definition 617 (*generated ring of sets*).

**The proof of Lemma 619 (*properties of ring of sets generation*)**

cites explicitly:

Lemma 586 (*properties of subset system generation*),  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 616 (*intersection of rings of sets*),  
 Definition 617 (*generated ring of sets*).

**The proof of Lemma 620 (*other properties of ring of sets generation*)**

cites explicitly:

Lemma 587 (*other properties of subset system generation*),  
 Lemma 616 (*intersection of rings of sets*),  
 Definition 617 (*generated ring of sets*).

**The proof of Lemma 621 (ring of sets contains  $\pi$ -system)**

cites explicitly:

Lemma 599 (generated subset systems are monotone (with nonemptiness)),  
Definition 606 (generated  $\pi$ -system),  
Lemma 614 (ring of sets is  $\pi$ -system),  
Definition 617 (generated ring of sets).

**The proof of Lemma 622 (upgrade  $\pi$ -system to ring of sets)**

cites explicitly:

Lemma 600 (upgrade generated subset system (with nonemptiness)),  
Lemma 604 (equivalent definition of  $\pi$ -system),  
Lemma 605 (intersection of  $\pi$ -systems),  
Definition 606 (generated  $\pi$ -system),  
Lemma 612 (equivalent definition of ring of sets),  
Lemma 616 (intersection of rings of sets),  
Definition 617 (generated ring of sets),  
Lemma 621 (ring of sets contains  $\pi$ -system).

**The proof of Lemma 623 (ring of sets generated by  $\pi$ -system)**

cites explicitly:

Lemma 601 (cascading generated subset systems (with nonemptiness)),  
Lemma 605 (intersection of  $\pi$ -systems),  
Definition 606 (generated  $\pi$ -system),  
Lemma 614 (ring of sets is  $\pi$ -system),  
Lemma 616 (intersection of rings of sets),  
Definition 617 (generated ring of sets).

**The proof of Lemma 624 (disjoint union of union in ring of sets)**

cites explicitly:

Lemma 572 (disjoint union of union in subset system),  
Lemma 612 (equivalent definition of ring of sets).

**The proof of Lemma 627 (equivalent definition of semiring of sets)**

cites explicitly:

Lemma 538 (nonempty and with empty or full),  
Lemma 552 (nonempty is with empty),  
Definition 626 (semiring of sets).

**The proof of Lemma 629 (semiring of sets is  $\pi$ -system)**

cites explicitly:

Lemma 604 (equivalent definition of  $\pi$ -system),  
Definition 626 (semiring of sets).

**The proof of Lemma 630 (other properties of semiring of sets)**

cites explicitly:

Lemma 557 (disjoint decomposition of set difference and local complement),  
Definition 602 ( $\pi$ -system),  
Definition 626 (semiring of sets),  
Lemma 629 (semiring of sets is  $\pi$ -system).

**The proof of Lemma 631 (ring of sets is semiring of sets)**

cites explicitly:

Lemma 554 (disjoint decomposition),  
Lemma 612 (equivalent definition of ring of sets),  
Definition 626 (semiring of sets).



**The proof of Lemma 632 (*explicit ring of sets*)**

cites explicitly:

- Lemma 562 (*finite disjoint union closure is closed under finite disjoint union*),
- Lemma 563 (*closedness under set difference and finite disjoint union closure*),
- Definition 611 (*ring of sets*),
- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 618 (*generated ring of sets is ring of sets*),
- Lemma 619 (*properties of ring of sets generation*),
- Definition 626 (*semiring of sets*),
- Lemma 627 (*equivalent definition of semiring of sets*).

**The proof of Lemma 635 (*equivalent definition of algebra of sets*)**

cites explicitly:

- Lemma 538 (*nonempty and with empty or full*),
- Lemma 539 (*with empty and full*),
- Lemma 547 (*closedness under union and intersection*),
- Lemma 558 (*closedness under finite operations*),
- Definition 634 (*algebra of sets*).

**The proof of Lemma 637 (*other equivalent definition of algebra of sets*)**

cites explicitly:

- Lemma 541 (*closedness under local complement and complement*),
- Lemma 543 (*closedness under set difference and local complement*),
- Lemma 545 (*closedness under intersection and set difference*),
- Lemma 635 (*equivalent definition of algebra of sets*).

**The proof of Lemma 638 (*algebra of sets is ring of sets*)**

cites explicitly:

- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 635 (*equivalent definition of algebra of sets*),
- Lemma 637 (*other equivalent definition of algebra of sets*).

**The proof of Lemma 639 (*other properties of algebra of sets*)**

cites explicitly:

- Lemma 546 (*closedness under union and disjoint union*),
- Lemma 558 (*closedness under finite operations*),
- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 615 (*other properties of ring of sets*),
- Lemma 635 (*equivalent definition of algebra of sets*),
- Lemma 638 (*algebra of sets is ring of sets*).

**The proof of Lemma 640 (*algebra of sets is  $\pi$ -system*)**

cites explicitly:

- Lemma 614 (*ring of sets is  $\pi$ -system*),
- Lemma 638 (*algebra of sets is ring of sets*).

**The proof of Lemma 641 (*intersection of algebras of sets*)**

cites explicitly:

- Definition 634 (*algebra of sets*).

**The proof of Lemma 643 (*generated algebra of sets is algebra of sets*)**

cites explicitly:

- Lemma 585 (*subset system generation preserves class*),
- Lemma 641 (*intersection of algebras of sets*),
- Definition 642 (*generated algebra of sets*).

**The proof of Lemma 644 (*properties of algebra of sets generation*)**

cites explicitly:

Lemma 586 (*properties of subset system generation*),  
 Lemma 641 (*intersection of algebras of sets*),  
 Definition 642 (*generated algebra of sets*).

**The proof of Lemma 645 (*other properties of algebra of sets generation*)**

cites explicitly:

Lemma 587 (*other properties of subset system generation*),  
 Lemma 641 (*intersection of algebras of sets*),  
 Definition 642 (*generated algebra of sets*).

**The proof of Lemma 646 (*algebra of sets contains ring of sets*)**

cites explicitly:

Lemma 588 (*generated subset systems are monotone*),  
 Definition 617 (*generated ring of sets*),  
 Lemma 638 (*algebra of sets is ring of sets*),  
 Definition 642 (*generated algebra of sets*).

**The proof of Lemma 647 (*upgrade ring of sets to algebra of sets*)**

cites explicitly:

Lemma 590 (*upgrade generated subset system*),  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 616 (*intersection of rings of sets*),  
 Definition 617 (*generated ring of sets*),  
 Lemma 635 (*equivalent definition of algebra of sets*),  
 Lemma 637 (*other equivalent definition of algebra of sets*),  
 Lemma 638 (*algebra of sets is ring of sets*),  
 Lemma 641 (*intersection of algebras of sets*),  
 Definition 642 (*generated algebra of sets*),  
 Lemma 646 (*algebra of sets contains ring of sets*).

**The proof of Lemma 648 (*algebra of sets generated by ring of sets*)**

cites explicitly:

Lemma 591 (*cascading generated subset systems*),  
 Lemma 616 (*intersection of rings of sets*),  
 Definition 617 (*generated ring of sets*),  
 Lemma 638 (*algebra of sets is ring of sets*),  
 Lemma 641 (*intersection of algebras of sets*),  
 Definition 642 (*generated algebra of sets*).

**The proof of Lemma 649 (*algebra of sets contains  $\pi$ -system*)**

cites explicitly:

Lemma 621 (*ring of sets contains  $\pi$ -system*),  
 Lemma 646 (*algebra of sets contains ring of sets*).

**The proof of Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*)**

cites explicitly:

Lemma 545 (*closedness under intersection and set difference*),  
 Lemma 547 (*closedness under union and intersection*),  
 Lemma 550 (*closedness under symmetric difference and set difference*),  
 Lemma 604 (*equivalent definition of  $\pi$ -system*),  
 Lemma 614 (*ring of sets is  $\pi$ -system*),  
 Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*).

**The proof of Lemma 651 (*algebra of sets generated by  $\pi$ -system*)**

cites explicitly:

- Lemma 592 (*double cascading generated subset systems*),
- Definition 606 (*generated  $\pi$ -system*),
- Definition 617 (*generated ring of sets*),
- Lemma 623 (*ring of sets generated by  $\pi$ -system*),
- Definition 642 (*generated algebra of sets*),
- Lemma 648 (*algebra of sets generated by ring of sets*).

**The proof of Lemma 652 (*disjoint union of union in algebra of sets*)**

cites explicitly:

- Lemma 624 (*disjoint union of union in ring of sets*),
- Lemma 638 (*algebra of sets is ring of sets*).

**The proof of Lemma 655 (*equivalent definition of semi-algebra of sets*)**

cites explicitly:

- Lemma 538 (*nonempty and with empty or full*),
- Lemma 555 (*disjoint decomposition of complement and set difference*),
- Lemma 556 (*disjoint decomposition of local complement and complement*),
- Lemma 557 (*disjoint decomposition of set difference and local complement*),
- Definition 626 (*semiring of sets*),
- Definition 654 (*semi-algebra of sets*).

**The proof of Lemma 657 (*semi-algebra of sets is semiring of sets*)**

cites explicitly:

- Definition 654 (*semi-algebra of sets*).

**The proof of Lemma 658 (*other properties of semi-algebra of sets*)**

cites explicitly:

- Lemma 630 (*other properties of semiring of sets*),
- Lemma 657 (*semi-algebra of sets is semiring of sets*).

**The proof of Lemma 659 (*semi-algebra of sets is  $\pi$ -system*)**

cites explicitly:

- Lemma 629 (*semiring of sets is  $\pi$ -system*),
- Lemma 657 (*semi-algebra of sets is semiring of sets*).

**The proof of Lemma 660 (*algebra of sets is semi-algebra of sets*)**

cites explicitly:

- Lemma 554 (*disjoint decomposition*),
- Lemma 635 (*equivalent definition of algebra of sets*),
- Lemma 655 (*equivalent definition of semi-algebra of sets*).

**The proof of Lemma 661 (*algebra of sets is semiring of sets*)**

cites explicitly:

- Lemma 657 (*semi-algebra of sets is semiring of sets*),
- Lemma 660 (*algebra of sets is semi-algebra of sets*).

**The proof of Lemma 662 (*explicit algebra of sets*)**

cites explicitly:

- Lemma 618 (*generated ring of sets is ring of sets*),
- Lemma 632 (*explicit ring of sets*),
- Lemma 647 (*upgrade ring of sets to algebra of sets*),
- Definition 654 (*semi-algebra of sets*).

**The proof of Lemma 665 (*intersection of monotone classes*)**

cites explicitly:

- Definition 664 (*monotone class*).

**The proof of Lemma 667 (*generated monotone class is monotone class*)**

cites explicitly:

Lemma 585 (*subset system generation preserves class*),

Lemma 665 (*intersection of monotone classes*),

Definition 666 (*generated monotone class*).

**The proof of Lemma 668 (*properties of monotone class generation*)**

cites explicitly:

Lemma 586 (*properties of subset system generation*),

Lemma 665 (*intersection of monotone classes*),

Definition 666 (*generated monotone class*).

**The proof of Lemma 669 (*other properties of monotone class generation*)**

cites explicitly:

Lemma 587 (*other properties of subset system generation*),

Lemma 665 (*intersection of monotone classes*),

Definition 666 (*generated monotone class*).

**The proof of Lemma 671 ( *$\mathcal{C}^{\setminus}$  is symmetric*)**

cites explicitly:

Definition 670 (*monotone class and symmetric set difference*).

**The proof of Lemma 672 ( *$\mathcal{C}^{\setminus}$  is monotone class*)**

cites explicitly:

Definition 664 (*monotone class*),

Definition 670 (*monotone class and symmetric set difference*).

**The proof of Lemma 674 (*monotone class is closed under set difference*)**

cites explicitly:

Lemma 667 (*generated monotone class is monotone class*),

Lemma 668 (*properties of monotone class generation*),

Definition 670 (*monotone class and symmetric set difference*),

Lemma 671 ( *$\mathcal{C}^{\setminus}$  is symmetric*),

Lemma 672 ( *$\mathcal{C}^{\setminus}$  is monotone class*).

**The proof of Lemma 675 (*monotone class generated by algebra of sets*)**

cites explicitly:

Lemma 637 (*other equivalent definition of algebra of sets*),

Lemma 667 (*generated monotone class is monotone class*),

Lemma 674 (*monotone class is closed under set difference*).

**The proof of Lemma 678 (*equivalent definition of  $\lambda$ -system*)**

cites explicitly:

Lemma 541 (*closedness under local complement and complement*),

Lemma 542 (*closedness under disjoint union and local complement*),

Lemma 566 (*closedness under countable disjoint union and local complement*),

Lemma 568 (*closedness under countable disjoint and monotone union*),

Lemma 569 (*closedness under countable monotone and disjoint union*),

Definition 677 ( *$\lambda$ -system*).

**The proof of Lemma 679 ( *$\lambda$ -system is monotone class*)**

cites explicitly:

Lemma 567 (*closedness under countable union and intersection*),

Definition 664 (*monotone class*),

Definition 677 ( *$\lambda$ -system*),

Lemma 678 (*equivalent definition of  $\lambda$ -system*).

**The proof of Lemma 680 (*other properties of  $\lambda$ -system*)**

cites explicitly:

- Lemma 538 (*nonempty and with empty or full*),
- Lemma 539 (*with empty and full*),
- Lemma 558 (*closedness under finite operations*),
- Definition 664 (*monotone class*),
- Definition 677 ( *$\lambda$ -system*),
- Lemma 679 ( *$\lambda$ -system is monotone class*).

**The proof of Lemma 681 (*intersection of  $\lambda$ -systems*)**

cites explicitly:

- Definition 677 ( *$\lambda$ -system*).

**The proof of Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*)**

cites explicitly:

- Lemma 585 (*subset system generation preserves class*),
- Lemma 681 (*intersection of  $\lambda$ -systems*),
- Definition 682 (*generated  $\lambda$ -system*).

**The proof of Lemma 684 (*properties of  $\lambda$ -system generation*)**

cites explicitly:

- Lemma 586 (*properties of subset system generation*),
- Lemma 681 (*intersection of  $\lambda$ -systems*),
- Definition 682 (*generated  $\lambda$ -system*).

**The proof of Lemma 685 (*other properties of  $\lambda$ -system generation*)**

cites explicitly:

- Lemma 587 (*other properties of subset system generation*),
- Lemma 681 (*intersection of  $\lambda$ -systems*),
- Definition 682 (*generated  $\lambda$ -system*).

**The proof of Lemma 686 ( *$\lambda$ -system contains monotone class*)**

cites explicitly:

- Lemma 599 (*generated subset systems are monotone (with nonemptiness)*),
- Definition 666 (*generated monotone class*),
- Lemma 679 ( *$\lambda$ -system is monotone class*),
- Definition 682 (*generated  $\lambda$ -system*).

**The proof of Lemma 687 (*upgrade monotone class to  $\lambda$ -system*)**

cites explicitly:

- Lemma 541 (*closedness under local complement and complement*),
- Lemma 567 (*closedness under countable union and intersection*),
- Lemma 590 (*upgrade generated subset system*),
- Definition 664 (*monotone class*),
- Lemma 665 (*intersection of monotone classes*),
- Definition 666 (*generated monotone class*),
- Lemma 678 (*equivalent definition of  $\lambda$ -system*),
- Lemma 681 (*intersection of  $\lambda$ -systems*),
- Definition 682 (*generated  $\lambda$ -system*),
- Lemma 686 ( *$\lambda$ -system contains monotone class*).

**The proof of Lemma 688 ( *$\lambda$ -system generated by monotone class*)**

cites explicitly:

- Lemma 591 (*cascading generated subset systems*),
- Lemma 665 (*intersection of monotone classes*),
- Definition 666 (*generated monotone class*),

Lemma 679 ( *$\lambda$ -system is monotone class*),  
 Lemma 681 (*intersection of  $\lambda$ -systems*),  
 Definition 682 (*generated  $\lambda$ -system*).

**The proof of Lemma 690 ( *$\Lambda^\cap$  is symmetric*)**

cites explicitly:  
 Definition 689 ( *$\lambda$ -system and intersection*).

**The proof of Lemma 691 ( *$\Lambda^\cap$  is  $\lambda$ -system*)**

cites explicitly:  
 Definition 677 ( *$\lambda$ -system*),  
 Lemma 678 (*equivalent definition of  $\lambda$ -system*).

**The proof of Lemma 692 ( *$\lambda$ -system with intersection*)**

cites explicitly:  
 Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*),  
 Lemma 684 (*properties of  $\lambda$ -system generation*),  
 Definition 689 ( *$\lambda$ -system and intersection*),  
 Lemma 690 ( *$\Lambda^\cap$  is symmetric*),  
 Lemma 691 ( *$\Lambda^\cap$  is  $\lambda$ -system*).

**The proof of Lemma 693 ( *$\lambda$ -system is closed under intersection*)**

cites explicitly:  
 Definition 689 ( *$\lambda$ -system and intersection*),  
 Lemma 692 ( *$\lambda$ -system with intersection*).

**The proof of Lemma 694 ( *$\lambda$ -system generated by  $\pi$ -system*)**

cites explicitly:  
 Lemma 604 (*equivalent definition of  $\pi$ -system*),  
 Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*),  
 Lemma 693 ( *$\lambda$ -system is closed under intersection*).

**The proof of Lemma 696 ( *$\sigma$ -algebra*)**

cites explicitly:  
 Definition 634 (*algebra of sets*).

**The proof of Lemma 697 (*equivalent definition of  $\sigma$ -algebra*)**

cites explicitly:  
 Lemma 538 (*nonempty and with empty or full*),  
 Lemma 539 (*with empty and full*),  
 Lemma 567 (*closedness under countable union and intersection*),  
 Lemma 696 ( *$\sigma$ -algebra*).

**The proof of Lemma 699 ( *$\sigma$ -algebra is algebra of sets*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*).

**The proof of Lemma 700 (*other properties of  $\sigma$ -algebra*)**

cites explicitly:  
 Definition 634 (*algebra of sets*),  
 Lemma 635 (*equivalent definition of algebra of sets*),  
 Lemma 637 (*other equivalent definition of algebra of sets*),  
 Lemma 639 (*other properties of algebra of sets*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 699 ( *$\sigma$ -algebra is algebra of sets*).

**The proof of Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*)**

cites explicitly:

Definition 677 ( *$\lambda$ -system*),

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 700 (*other properties of  $\sigma$ -algebra*).

**The proof of Lemma 702 ( *$\sigma$ -algebra is monotone class*)**

cites explicitly:

Lemma 679 ( *$\lambda$ -system is monotone class*),

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*).

**The proof of Lemma 703 ( *$\sigma$ -algebra is ring of sets*)**

cites explicitly:

Lemma 638 (*algebra of sets is ring of sets*),

Lemma 699 ( *$\sigma$ -algebra is algebra of sets*).

**The proof of Lemma 704 ( *$\sigma$ -algebra is  $\pi$ -system*)**

cites explicitly:

Lemma 640 (*algebra of sets is  $\pi$ -system*),

Lemma 699 ( *$\sigma$ -algebra is algebra of sets*).

**The proof of Lemma 705 ( *$\sigma$ -algebra is semi-algebra of sets*)**

cites explicitly:

Lemma 660 (*algebra of sets is semi-algebra of sets*),

Lemma 699 ( *$\sigma$ -algebra is algebra of sets*).

**The proof of Lemma 706 ( *$\sigma$ -algebra is semiring of sets*)**

cites explicitly:

Lemma 661 (*algebra of sets is semiring of sets*),

Lemma 699 ( *$\sigma$ -algebra is algebra of sets*).

**The proof of Lemma 707 (*intersection of  $\sigma$ -algebras*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*).

**The proof of Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*)**

cites explicitly:

Lemma 585 (*subset system generation preserves class*),

Lemma 707 (*intersection of  $\sigma$ -algebras*),

Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 710 (*properties of  $\sigma$ -algebra generation*)**

cites explicitly:

Lemma 586 (*properties of subset system generation*),

Lemma 707 (*intersection of  $\sigma$ -algebras*),

Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 711 (*other properties of  $\sigma$ -algebra generation*)**

cites explicitly:

Lemma 587 (*other properties of subset system generation*),

Lemma 707 (*intersection of  $\sigma$ -algebras*),

Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 712 (*countable union  $\sigma$ -algebra generator*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),

Lemma 711 (*other properties of  $\sigma$ -algebra generation*).

**The proof of Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*)**

cites explicitly:

Lemma 588 (*generated subset systems are monotone*),

Definition 682 (*generated  $\lambda$ -system*),

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*)**

cites explicitly:

Lemma 545 (*closedness under intersection and set difference*),

Lemma 570 (*closedness under countable disjoint union and countable union*),

Lemma 590 (*upgrade generated subset system*),

Definition 677 ( *$\lambda$ -system*),

Lemma 681 (*intersection of  $\lambda$ -systems*),

Definition 682 (*generated  $\lambda$ -system*),

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 700 (*other properties of  $\sigma$ -algebra*),

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Lemma 707 (*intersection of  $\sigma$ -algebras*),

Definition 708 (*generated  $\sigma$ -algebra*),

Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*).

**The proof of Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*)**

cites explicitly:

Lemma 591 (*cascading generated subset systems*),

Lemma 681 (*intersection of  $\lambda$ -systems*),

Definition 682 (*generated  $\lambda$ -system*),

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Lemma 707 (*intersection of  $\sigma$ -algebras*),

Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 716 ( *$\sigma$ -algebra contains monotone class*)**

cites explicitly:

Lemma 686 ( *$\lambda$ -system contains monotone class*),

Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*).

**The proof of Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*)**

cites explicitly:

Lemma 539 (*with empty and full*),

Lemma 542 (*closedness under disjoint union and local complement*),

Lemma 546 (*closedness under union and disjoint union*),

Lemma 547 (*closedness under union and intersection*),

Definition 677 ( *$\lambda$ -system*),

Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),

Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*).

**The proof of Lemma 718 ( *$\sigma$ -algebra generated by monotone class*)**

cites explicitly:

Lemma 592 (*double cascading generated subset systems*),

Definition 666 (*generated monotone class*),

Definition 682 (*generated  $\lambda$ -system*),

Lemma 688 ( *$\lambda$ -system generated by monotone class*),

Definition 708 (*generated  $\sigma$ -algebra*),

Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*).



**The proof of Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*)**

cites explicitly:

- Lemma 588 (*generated subset systems are monotone*),
- Definition 642 (*generated algebra of sets*),
- Lemma 699 ( *$\sigma$ -algebra is algebra of sets*),
- Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*)**

cites explicitly:

- Lemma 570 (*closedness under countable disjoint union and countable union*),
- Lemma 571 (*closedness under countable monotone union and countable union*),
- Lemma 590 (*upgrade generated subset system*),
- Lemma 637 (*other equivalent definition of algebra of sets*),
- Lemma 639 (*other properties of algebra of sets*),
- Lemma 641 (*intersection of algebras of sets*),
- Definition 642 (*generated algebra of sets*),
- Lemma 696 ( *$\sigma$ -algebra*),
- Lemma 699 ( *$\sigma$ -algebra is algebra of sets*),
- Lemma 700 (*other properties of  $\sigma$ -algebra*),
- Lemma 707 (*intersection of  $\sigma$ -algebras*),
- Definition 708 (*generated  $\sigma$ -algebra*),
- Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*).

**The proof of Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*)**

cites explicitly:

- Lemma 591 (*cascading generated subset systems*),
- Lemma 641 (*intersection of algebras of sets*),
- Definition 642 (*generated algebra of sets*),
- Lemma 699 ( *$\sigma$ -algebra is algebra of sets*),
- Lemma 707 (*intersection of  $\sigma$ -algebras*),
- Definition 708 (*generated  $\sigma$ -algebra*).

**The proof of Lemma 722 ( *$\sigma$ -algebra contains ring of sets*)**

cites explicitly:

- Lemma 646 (*algebra of sets contains ring of sets*),
- Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*).

**The proof of Lemma 723 (*upgrade ring of sets to  $\sigma$ -algebra*)**

cites explicitly:

- Lemma 647 (*upgrade ring of sets to algebra of sets*),
- Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*).

**The proof of Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*)**

cites explicitly:

- Lemma 592 (*double cascading generated subset systems*),
- Definition 617 (*generated ring of sets*),
- Definition 642 (*generated algebra of sets*),
- Lemma 648 (*algebra of sets generated by ring of sets*),
- Definition 708 (*generated  $\sigma$ -algebra*),
- Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*).

**The proof of Lemma 725 ( *$\sigma$ -algebra contains  $\pi$ -system*)**

cites explicitly:

- Lemma 621 (*ring of sets contains  $\pi$ -system*),
- Lemma 722 ( *$\sigma$ -algebra contains ring of sets*).

**The proof of Lemma 726 (upgrade  $\pi$ -system to  $\sigma$ -algebra)**

cites explicitly:

Lemma 650 (upgrade  $\pi$ -system to algebra of sets),

Lemma 720 (upgrade algebra of sets to  $\sigma$ -algebra).

**The proof of Lemma 727 ( $\sigma$ -algebra generated by  $\pi$ -system)**

cites explicitly:

Lemma 592 (double cascading generated subset systems),

Definition 606 (generated  $\pi$ -system),

Definition 617 (generated ring of sets),

Lemma 623 (ring of sets generated by  $\pi$ -system),

Definition 708 (generated  $\sigma$ -algebra),

Lemma 724 ( $\sigma$ -algebra generated by ring of sets).

**The proof of Lemma 728 (disjoint union of union in  $\sigma$ -algebra)**

cites explicitly:

Lemma 572 (disjoint union of union in subset system),

Lemma 700 (other properties of  $\sigma$ -algebra).

**The proof of Lemma 730 (algebra of sets generated by product of  $\sigma$ -algebras)**

cites explicitly:

Definition 227 (product of subset systems),

Lemma 655 (equivalent definition of semi-algebra of sets),

Lemma 662 (explicit algebra of sets),

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Lemma 700 (other properties of  $\sigma$ -algebra).

**The proof of Theorem 732 (Dynkin  $\pi$ - $\lambda$  theorem)**

cites explicitly:

Lemma 604 (equivalent definition of  $\pi$ -system),

Lemma 694 ( $\lambda$ -system generated by  $\pi$ -system),

Lemma 701 ( $\sigma$ -algebra is  $\lambda$ -system),

Lemma 704 ( $\sigma$ -algebra is  $\pi$ -system),

Lemma 714 (upgrade  $\lambda$ -system to  $\sigma$ -algebra).

**The proof of Lemma 734 (usage of Dynkin  $\pi$ - $\lambda$  theorem)**

cites explicitly:

Lemma 607 (generated  $\pi$ -system is  $\pi$ -system),

Lemma 684 (properties of  $\lambda$ -system generation),

Definition 708 (generated  $\sigma$ -algebra),

Lemma 710 (properties of  $\sigma$ -algebra generation),

Theorem 732 (Dynkin  $\pi$ - $\lambda$  theorem).

**The proof of Theorem 736 (monotone class)**

cites explicitly:

Lemma 635 (equivalent definition of algebra of sets),

Lemma 675 (monotone class generated by algebra of sets),

Lemma 699 ( $\sigma$ -algebra is algebra of sets),

Lemma 702 ( $\sigma$ -algebra is monotone class),

Lemma 717 (upgrade monotone class to  $\sigma$ -algebra).

**The proof of Lemma 738 (usage of monotone class theorem)**

cites explicitly:

Lemma 643 (generated algebra of sets is algebra of sets),

Lemma 668 (properties of monotone class generation),

Definition 708 (generated  $\sigma$ -algebra),

Lemma 710 (*properties of  $\sigma$ -algebra generation*),  
 Theorem 736 (*monotone class*).

**The proof of Lemma 740 (*properties of premeasurable space*)**

cites explicitly:

Lemma 554 (*disjoint decomposition*),  
 Definition 611 (*ring of sets*),  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 614 (*ring of sets is  $\pi$ -system*),  
 Lemma 615 (*other properties of ring of sets*),  
 Lemma 631 (*ring of sets is semiring of sets*),  
 Definition 739 (*premeasurable space*).

**The proof of Lemma 742 (*properties of measurable space*)**

cites explicitly:

Lemma 554 (*disjoint decomposition*),  
 Lemma 631 (*ring of sets is semiring of sets*),  
 Lemma 660 (*algebra of sets is semi-algebra of sets*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 699 ( *$\sigma$ -algebra is algebra of sets*),  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),  
 Lemma 702 ( *$\sigma$ -algebra is monotone class*),  
 Lemma 703 ( *$\sigma$ -algebra is ring of sets*),  
 Lemma 704 ( *$\sigma$ -algebra is  $\pi$ -system*),  
 Definition 739 (*premeasurable space*),  
 Definition 741 (*measurable space*).

**The proof of Lemma 744 (*some Borel subsets*)**

cites explicitly:

Definition 240 (*topological space*),  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Definition 743 (*Borel  $\sigma$ -algebra*).

**The proof of Lemma 745 (*countable union Borel  $\sigma$ -algebra generator*)**

cites explicitly:

Lemma 712 (*countable union  $\sigma$ -algebra generator*),  
 Definition 743 (*Borel  $\sigma$ -algebra*).

**The proof of Lemma 750 (*inverse  $\sigma$ -algebra*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 749 (*measurable function*).

**The proof of Lemma 751 (*image  $\sigma$ -algebra*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 749 (*measurable function*).

**The proof of Lemma 752 (*identity function is measurable*)**

cites explicitly:

Definition 749 (*measurable function*).

**The proof of Lemma 753 (*countable range is measurable*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
Lemma 700 (*other properties of  $\sigma$ -algebra*),  
Definition 741 (*measurable space*),  
Definition 749 (*measurable function*).

**The proof of Lemma 754 (*constant function is measurable*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
Definition 741 (*measurable space*),  
Lemma 753 (*countable range is measurable*).

**The proof of Lemma 755 (*inverse image of generating family*)**

cites explicitly:

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
Lemma 710 (*properties of  $\sigma$ -algebra generation*),  
Lemma 750 (*inverse  $\sigma$ -algebra*),  
Lemma 751 (*image  $\sigma$ -algebra*).

**The proof of Lemma 756 (*equivalent definition of measurable function*)**

cites explicitly:

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
Lemma 710 (*properties of  $\sigma$ -algebra generation*),  
Definition 749 (*measurable function*),  
Lemma 755 (*inverse image of generating family*).

**The proof of Lemma 757 (*compatibility of measurability with composition*)**

cites explicitly:

Definition 749 (*measurable function*).

**The proof of Lemma 759 ( *$\mathcal{M}_Y$  inverse image of countable is measurable*)**

cites explicitly:

Lemma 744 (*some Borel subsets*),  
Definition 749 (*measurable function*),  
Definition 758 ( *$\mathcal{M}_Y$ , set of measurable functions to  $Y$* ).

**The proof of Lemma 760 ( *$\mathcal{M}_Y$  countable range*)**

cites explicitly:

Lemma 753 (*countable range is measurable*),  
Lemma 759 ( *$\mathcal{M}_Y$  inverse image of countable is measurable*).

**The proof of Lemma 761 (*continuous is measurable*)**

cites explicitly:

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
Definition 743 (*Borel  $\sigma$ -algebra*),  
Definition 749 (*measurable function*),  
Lemma 756 (*equivalent definition of measurable function*).

**The proof of Lemma 763 (*trace  $\sigma$ -algebra*)**

cites explicitly:

Definition 226 (*trace of subset system*),  
Lemma 696 ( *$\sigma$ -algebra*),  
Definition 741 (*measurable space*),  
Definition 749 (*measurable function*).

**The proof of Lemma 764 (*measurability of measurable subspace*)**

cites explicitly:

Definition 226 (*trace of subset system*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 763 (*trace  $\sigma$ -algebra*).

**The proof of Lemma 765 (*generating measurable subspace*)**

cites explicitly:

Definition 226 (*trace of subset system*),  
 Lemma 755 (*inverse image of generating family*),  
 Lemma 764 (*measurability of measurable subspace*).

**The proof of Lemma 766 (*generating on a partition*)**

cites explicitly:

Lemma 211 (*compatibility of pseudopartition with intersection*),  
 Definition 226 (*trace of subset system*),  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 765 (*generating measurable subspace*).

**The proof of Lemma 768 (*Borel sub- $\sigma$ -algebra*)**

cites explicitly:

Lemma 252 (*trace topology on subset*),  
 Definition 743 (*Borel  $\sigma$ -algebra*),  
 Lemma 765 (*generating measurable subspace*).

**The proof of Lemma 769 (*Borel subsets of partition*)**

cites explicitly:

Lemma 766 (*generating on a partition*).

**The proof of Lemma 770 (*source restriction of measurable function*)**

cites explicitly:

Definition 226 (*trace of subset system*),  
 Definition 749 (*measurable function*),  
 Lemma 757 (*compatibility of measurability with composition*),  
 Lemma 763 (*trace  $\sigma$ -algebra*).

**The proof of Lemma 771 (*destination restriction of measurable function*)**

cites explicitly:

Definition 749 (*measurable function*).

**The proof of Lemma 772 (*measurability of function defined on a pseudopartition*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 749 (*measurable function*).

**The proof of Lemma 775 (*product of measurable subsets is measurable*)**

cites explicitly:

Definition 227 (*product of subset systems*),  
 Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Definition 774 (*tensor product of  $\sigma$ -algebras*).

**The proof of Lemma 776 (*measurability of function to product space*)**

cites explicitly:

Definition 227 (*product of subset systems*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 749 (*measurable function*),  
 Lemma 756 (*equivalent definition of measurable function*),

Definition 774 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 775 (*product of measurable subsets is measurable*).

**The proof of Lemma 777 (*canonical projection is measurable*)**  
 cites explicitly:  
 Definition 749 (*measurable function*),  
 Lemma 776 (*measurability of function to product space*).

**The proof of Lemma 778 (*permutation is measurable*)**  
 cites explicitly:  
 Lemma 776 (*measurability of function to product space*),  
 Lemma 777 (*canonical projection is measurable*).

**The proof of Lemma 779 (*generating product measurable space*)**  
 cites explicitly:  
 Definition 227 (*product of subset systems*),  
 Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Lemma 710 (*properties of  $\sigma$ -algebra generation*),  
 Definition 749 (*measurable function*),  
 Lemma 755 (*inverse image of generating family*),  
 Definition 774 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 777 (*canonical projection is measurable*).

**The proof of Lemma 780 (*Borel subsets in product measurable subspace*)**  
 cites explicitly:  
 Lemma 252 (*trace topology on subset*),  
 Lemma 254 (*box topology on Cartesian product*),  
 Definition 743 (*Borel  $\sigma$ -algebra*),  
 Lemma 765 (*generating measurable subspace*),  
 Lemma 779 (*generating product measurable space*).

**The proof of Lemma 784 (*section of product*)**  
 cites explicitly:  
 Definition 783 (*section in Cartesian product*).

**The proof of Lemma 785 (*compatibility of section with set operations*)**  
 cites explicitly:  
 Definition 783 (*section in Cartesian product*).

**The proof of Lemma 786 (*measurability of section*)**  
 cites explicitly:  
 Definition 227 (*product of subset systems*),  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 774 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 784 (*section of product*),  
 Lemma 785 (*compatibility of section with set operations*).

**The proof of Lemma 787 (*countable union of sections is measurable*)**  
 cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Lemma 785 (*compatibility of section with set operations*),  
 Lemma 786 (*measurability of section*).

**The proof of Lemma 788** (*countable intersection of sections is measurable*)

cites explicitly:

- Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),
- Lemma 785 (*compatibility of section with set operations*),
- Lemma 786 (*measurability of section*).

**The proof of Lemma 789** (*indicator of section*)

cites explicitly:

- Definition 783 (*section in Cartesian product*).

**The proof of Lemma 790** (*measurability of function from product space*)

cites explicitly:

- Definition 749 (*measurable function*),
- Lemma 786 (*measurability of section*).

**The proof of Lemma 794** (*some Borel subsets of numbers*)

cites explicitly:

- Lemma 744 (*some Borel subsets*).

**The proof of Lemma 796** (*Borel subsets of  $\mathbb{R}$* )

cites explicitly:

- Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Lemma 696 ( *$\sigma$ -algebra*),
- Lemma 700 (*other properties of  $\sigma$ -algebra*),
- Definition 708 (*generated  $\sigma$ -algebra*),
- Lemma 711 (*other properties of  $\sigma$ -algebra generation*),
- Lemma 745 (*countable union Borel  $\sigma$ -algebra generator*),
- Lemma 794 (*some Borel subsets of numbers*).

**The proof of Lemma 797** (*countable generator of Borel subsets of  $\mathbb{R}$* )

cites explicitly:

- Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Theorem 465 ( *$\mathbb{R}$  is second-countable*),
- Lemma 796 (*Borel subsets of  $\mathbb{R}$* ).

**The proof of Lemma 798** (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* )

cites explicitly:

- Lemma 696 ( *$\sigma$ -algebra*),
- Definition 743 (*Borel  $\sigma$ -algebra*),
- Lemma 768 (*Borel sub- $\sigma$ -algebra*),
- Lemma 769 (*Borel subsets of partition*).

**The proof of Lemma 800** (*generators of Borel subsets of  $\overline{\mathbb{R}}$* )

cites explicitly:

- Definition 708 (*generated  $\sigma$ -algebra*),
- Lemma 766 (*generating on a partition*),
- Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ).

**The proof of Lemma 802** (*Borel subsets of  $\overline{\mathbb{R}}$* )

cites explicitly:

- Definition 226 (*trace of subset system*),
- Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),
- Lemma 800 (*generators of Borel subsets of  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 803** (*Borel subsets of  $\mathbb{R}_+$* )

cites explicitly:

- Lemma 765 (*generating measurable subspace*),

Lemma 768 (*Borel sub- $\sigma$ -algebra*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ).

**The proof of Lemma 804 (*Borel subsets of  $\overline{\mathbb{R}}_+$* )**

cites explicitly:  
 Lemma 765 (*generating measurable subspace*),  
 Lemma 768 (*Borel sub- $\sigma$ -algebra*),  
 Lemma 802 (*Borel subsets of  $\overline{\mathbb{R}}$* ).

**The proof of Lemma 806 (*Borel subsets of  $\mathbb{C}$* )**

cites explicitly:  
 Lemma 780 (*Borel subsets in product measurable subspace*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ).

**The proof of Lemma 808 (*inverse image of countable is measurable*)**

cites explicitly:  
 Lemma 759 ( *$\mathcal{M}_Y$  inverse image of countable is measurable*).

**The proof of Lemma 809 (*inverse image is measurable*)**

cites explicitly:  
 Lemma 808 (*inverse image of countable is measurable*).

**The proof of Lemma 810 ( *$\mathcal{M}_{\mathbb{K}}$  countable range*)**

cites explicitly:  
 Lemma 760 ( *$\mathcal{M}_Y$  countable range*).

**The proof of Lemma 811 (*measurability of indicator function*)**

cites explicitly:  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 748 (*measurability and structure*),  
 Lemma 810 ( *$\mathcal{M}_{\mathbb{K}}$  countable range*).

**The proof of Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*)**

cites explicitly:  
 Lemma 279 ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*),  
 Lemma 282 (*algebra of functions to algebra*),  
 Lemma 287 (*closed under algebra operations is subalgebra*),  
 Lemma 754 (*constant function is measurable*),  
 Lemma 757 (*compatibility of measurability with composition*),  
 Definition 758 ( *$\mathcal{M}_Y$ , set of measurable functions to  $Y$* ),  
 Lemma 761 (*continuous is measurable*),  
 Lemma 776 (*measurability of function to product space*),  
 Lemma 780 (*Borel subsets in product measurable subspace*).

**The proof of Lemma 814 ( *$\mathcal{M}_{\mathbb{K}}$  is vector space*)**

cites explicitly:  
 Statement(s) from [20],  
 Definition 277 (*algebra over a field*),  
 Definition 284 (*subalgebra*),  
 Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*).

**The proof of Lemma 815 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* )**

cites explicitly:  
 Lemma 814 ( *$\mathcal{M}_{\mathbb{K}}$  is vector space*).



**The proof of Lemma 816 (measurability of numeric function to  $\mathbb{R}$ )**

cites explicitly:

- Definition 743 (Borel  $\sigma$ -algebra),
- Definition 749 (measurable function),
- Lemma 756 (equivalent definition of measurable function),
- Lemma 794 (some Borel subsets of numbers),
- Lemma 796 (Borel subsets of  $\mathbb{R}$ ),
- Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ).

**The proof of Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ )**

cites explicitly:

- Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),
- Definition 749 (measurable function),
- Definition 758 ( $\mathcal{M}_Y$ , set of measurable functions to  $Y$ ),
- Lemma 798 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ),
- Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ),
- Definition 817 ( $\mathcal{M}$ , set of measurable numeric functions).

**The proof of Lemma 820 (measurability of numeric function)**

cites explicitly:

- Lemma 372 (topology on  $\overline{\mathbb{R}}$ ),
- Definition 743 (Borel  $\sigma$ -algebra),
- Definition 749 (measurable function),
- Lemma 756 (equivalent definition of measurable function),
- Lemma 794 (some Borel subsets of numbers),
- Lemma 802 (Borel subsets of  $\overline{\mathbb{R}}$ ),
- Definition 817 ( $\mathcal{M}$ , set of measurable numeric functions).

**The proof of Lemma 821 ( $\mathcal{M}$  is closed under finite part)**

cites explicitly:

- Definition 209 (pseudopartition),
- Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),
- Definition 526 (finite part),
- Lemma 527 (finite part is finite),
- Lemma 696 ( $\sigma$ -algebra),
- Lemma 754 (constant function is measurable),
- Lemma 772 (measurability of function defined on a pseudopartition),
- Lemma 808 (inverse image of countable is measurable),
- Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ).

**The proof of Lemma 822 ( $\mathcal{M}$  is closed under addition when defined)**

cites explicitly:

- Definition 209 (pseudopartition),
- Definition 317 (addition in  $\overline{\mathbb{R}}$ ),
- Definition 526 (finite part),
- Lemma 697 (equivalent definition of  $\sigma$ -algebra),
- Definition 743 (Borel  $\sigma$ -algebra),
- Definition 749 (measurable function),
- Lemma 754 (constant function is measurable),
- Lemma 772 (measurability of function defined on a pseudopartition),
- Lemma 809 (inverse image is measurable),
- Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra),
- Definition 817 ( $\mathcal{M}$ , set of measurable numeric functions),
- Lemma 821 ( $\mathcal{M}$  is closed under finite part).

**The proof of Lemma 823 ( $\mathcal{M}$  is closed under finite sum when defined)**

cites explicitly:

Lemma 822 ( $\mathcal{M}$  is closed under addition when defined).

**The proof of Lemma 824 ( $\mathcal{M}$  is closed under multiplication)**

cites explicitly:

Definition 326 (multiplication in  $\overline{\mathbb{R}}$ ),

Lemma 410 (infinity-product property in  $\overline{\mathbb{R}}_+$ ),

Definition 415 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)),

Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)),

Definition 526 (finite part),

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Definition 743 (Borel  $\sigma$ -algebra),

Definition 749 (measurable function),

Lemma 754 (constant function is measurable),

Lemma 772 (measurability of function defined on a pseudopartition),

Lemma 809 (inverse image is measurable),

Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra),

Definition 817 ( $\mathcal{M}$ , set of measurable numeric functions),

Lemma 820 (measurability of numeric function),

Lemma 821 ( $\mathcal{M}$  is closed under finite part).

**The proof of Lemma 825 ( $\mathcal{M}$  is closed under finite product)**

cites explicitly:

Lemma 824 ( $\mathcal{M}$  is closed under multiplication).

**The proof of Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication)**

cites explicitly:

Lemma 754 (constant function is measurable),

Lemma 824 ( $\mathcal{M}$  is closed under multiplication).

**The proof of Lemma 828 ( $\mathcal{M}$  is closed under infimum)**

cites explicitly:

Statement(s) from [20],

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Lemma 820 (measurability of numeric function).

**The proof of Lemma 829 ( $\mathcal{M}$  is closed under supremum)**

cites explicitly:

Statement(s) from [20],

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Lemma 820 (measurability of numeric function).

**The proof of Lemma 830 ( $\mathcal{M}$  is closed under limit inferior)**

cites explicitly:

Lemma 504 (limit inferior),

Lemma 828 ( $\mathcal{M}$  is closed under infimum),

Lemma 829 ( $\mathcal{M}$  is closed under supremum).

**The proof of Lemma 831 ( $\mathcal{M}$  is closed under limit superior)**

cites explicitly:

Lemma 509 (limit superior),

Lemma 828 ( $\mathcal{M}$  is closed under infimum),

Lemma 829 ( $\mathcal{M}$  is closed under supremum).

**The proof of Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent)**

cites explicitly:

Lemma 522 (*limit inferior, limit superior and pointwise convergence*),  
 Lemma 830 ( *$\mathcal{M}$  is closed under limit inferior*),  
 Lemma 831 ( *$\mathcal{M}$  is closed under limit superior*).

**The proof of Lemma 833 (*measurability and masking*)**

cites explicitly:  
 Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Lemma 811 (*measurability of indicator function*),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*).

**The proof of Lemma 834 (*measurability of restriction*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 749 (*measurable function*),  
 Definition 817 ( *$\mathcal{M}$ , set of measurable numeric functions*).

**The proof of Lemma 836 (*measurability of nonnegative and nonpositive parts*)**

cites explicitly:  
 Definition 528 (*nonnegative and nonpositive parts*),  
 Lemma 530 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 532 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 754 (*constant function is measurable*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 829 ( *$\mathcal{M}$  is closed under supremum*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**The proof of Lemma 837 ( *$\mathcal{M}_+$  is closed under masking*)**

cites explicitly:  
 Lemma 833 (*measurability and masking*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**The proof of Lemma 838 ( *$\mathcal{M}_+$  is closed under finite part*)**

cites explicitly:  
 Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**The proof of Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*)**

cites explicitly:  
 Definition 526 (*finite part*),  
 Lemma 529 (*equivalent definition of nonnegative and nonpositive parts*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Lemma 820 (*measurability of numeric function*),  
 Lemma 833 (*measurability and masking*),  
 Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 838 ( *$\mathcal{M}_+$  is closed under finite part*).

**The proof of Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*)**

cites explicitly:  
 Lemma 340 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 449 (*power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)*),  
 Lemma 753 (*countable range is measurable*),  
 Lemma 761 (*continuous is measurable*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),  
 Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**The proof of Lemma 841 ( $\mathcal{M}$  is closed under power function)**

cites explicitly:

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),  
 Lemma 431 (*one-exponentiation property in  $\mathbb{R}$  (measure theory)*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 757 (*compatibility of measurability with composition*),  
 Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ),  
 Definition 817 ( $\mathcal{M}$ , *set of measurable numeric functions*),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*),  
 Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**The proof of Lemma 842 ( $\mathcal{M}$  is closed under absolute value)**

cites explicitly:

Lemma 362 (*power function in  $\overline{\mathbb{R}}$  is absolute value*),  
 Lemma 841 ( $\mathcal{M}$  is closed under power function).

**The proof of Lemma 843 ( $\mathcal{M}_+$  is closed under addition)**

cites explicitly:

Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 822 ( $\mathcal{M}$  is closed under addition when defined),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**The proof of Lemma 844 ( $\mathcal{M}_+$  is closed under multiplication)**

cites explicitly:

Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 824 ( $\mathcal{M}$  is closed under multiplication),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**The proof of Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication)**

cites explicitly:

Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**The proof of Lemma 847 ( $\mathcal{M}_+$  is closed under infimum)**

cites explicitly:

Lemma 492 (*infimum of bounded is bounded*),  
 Lemma 828 ( $\mathcal{M}$  is closed under infimum).

**The proof of Lemma 848 ( $\mathcal{M}_+$  is closed under supremum)**

cites explicitly:

Lemma 493 (*supremum of bounded is bounded*),  
 Lemma 829 ( $\mathcal{M}$  is closed under supremum).

**The proof of Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent)**

cites explicitly:

Lemma 392 ( $\overline{\mathbb{R}}_+$  is complete),

Lemma 832 ( *$\mathcal{M}$  is closed under limit when pointwise convergent*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**The proof of Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*)**

cites explicitly:  
 Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*).

**The proof of Lemma 853 (*measurability of tensor product of numeric functions*)**

cites explicitly:  
 Definition 748 (*measurability and structure*),  
 Lemma 757 (*compatibility of measurability with composition*),  
 Definition 758 ( *$\mathcal{M}_Y$ , set of measurable functions to  $Y$* ),  
 Lemma 777 (*canonical projection is measurable*),  
 Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*),  
 Lemma 825 ( *$\mathcal{M}$  is closed under finite product*),  
 Definition 851 (*tensor product of numeric functions*).

**The proof of Lemma 858 (*finite is bounded*)**

cites explicitly:  
 Definition 857 (*finiteness*).

**The proof of Lemma 863 (*decomposition over finite pseudopartition*)**

cites explicitly:  
 Lemma 211 (*compatibility of pseudopartition with intersection*),  
 Definition 862 (*additivity*).

**The proof of Lemma 864 (*binary inclusion-exclusion principle*)**

cites explicitly:  
 Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),  
 Definition 862 (*additivity*).

**The proof of Lemma 866 (*subtractivity*)**

cites explicitly:  
 Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 862 (*additivity*),  
 Definition 865 (*subtractivity*).

**The proof of Lemma 867 (*monotonicity*)**

cites explicitly:  
 Definition 862 (*additivity*).

**The proof of Lemma 869 (*subadditivity*)**

cites explicitly:  
 Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),  
 Lemma 557 (*disjoint decomposition of set difference and local complement*),  
 Lemma 574 (*disjoint union of union in subset system (variant)*),  
 Definition 859 (*subadditivity*),  
 Definition 862 (*additivity*),  
 Lemma 867 (*monotonicity*).

**The proof of Lemma 871 ( *$\sigma$ -subadditive is subadditive*)**

cites explicitly:  
 Definition 859 (*subadditivity*),  
 Definition 870 ( *$\sigma$ -subadditivity*).

**The proof of Lemma 873 ( *$\sigma$ -superadditive is superadditive*)**

cites explicitly:

Definition 861 (*superadditivity*),Definition 872 ( *$\sigma$ -superadditivity*).**The proof of Lemma 875 ( *$\sigma$ -additive is additive*)**

cites explicitly:

Definition 862 (*additivity*),Definition 874 ( *$\sigma$ -additivity*).**The proof of Lemma 876 (*decomposition over countable pseudopartition*)**

cites explicitly:

Lemma 211 (*compatibility of pseudopartition with intersection*),Definition 874 ( *$\sigma$ -additivity*).**The proof of Lemma 878 ( *$\sigma$ -subadditivity*)**

cites explicitly:

Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),Lemma 557 (*disjoint decomposition of set difference and local complement*),Lemma 574 (*disjoint union of union in subset system (variant)*),Lemma 867 (*monotonicity*),Definition 870 ( *$\sigma$ -subadditivity*),Definition 874 ( *$\sigma$ -additivity*),Lemma 875 ( *$\sigma$ -additive is additive*).**The proof of Lemma 880 (*continuity from below*)**

cites explicitly:

Lemma 501 (*supremum is limit*),Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),Lemma 554 (*disjoint decomposition*),Lemma 572 (*disjoint union of union in subset system*),Definition 862 (*additivity*),Lemma 867 (*monotonicity*),Definition 874 ( *$\sigma$ -additivity*),Lemma 875 ( *$\sigma$ -additive is additive*),Definition 879 (*continuity from below*).**The proof of Lemma 881 ( *$\sigma$ -additivity (from below)*)**

cites explicitly:

Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),Lemma 575 (*monotone union of union in subset system*),Definition 862 (*additivity*),Definition 874 ( *$\sigma$ -additivity*),Lemma 875 ( *$\sigma$ -additive is additive*),Definition 879 (*continuity from below*),Lemma 880 (*continuity from below*).**The proof of Lemma 883 (*continuity from above*)**

cites explicitly:

Statement(s) from [20],

Lemma 483 (*compatibility of extrema with affine transformation*),Lemma 498 (*infimum of nonincreasing sequence*),Lemma 500 (*infimum is limit*),Lemma 554 (*disjoint decomposition*),Lemma 577 (*monotone union of monotone intersection in subset system*),Definition 865 (*subtractivity*),

Lemma 866 (*subtractivity*),  
 Lemma 867 (*monotonicity*),  
 Lemma 875 ( *$\sigma$ -additive is additive*),  
 Definition 879 (*continuity from below*),  
 Lemma 880 (*continuity from below*),  
 Definition 882 (*continuity from above*).

**The proof of Lemma 884 ( *$\sigma$ -additivity (from above)*)**

cites explicitly:

Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),  
 Lemma 503 (*constant sum of monotone nonnegative functions*),  
 Lemma 576 (*empty monotone intersection of union in subset system*),  
 Definition 862 (*additivity*),  
 Definition 874 ( *$\sigma$ -additivity*),  
 Lemma 875 ( *$\sigma$ -additive is additive*),  
 Definition 882 (*continuity from above*),  
 Lemma 883 (*continuity from above*).

**The proof of Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*)**

cites explicitly:

Lemma 575 (*monotone union of union in subset system*),  
 Definition 859 (*subadditivity*),  
 Definition 885 ( *$\sigma$ -finiteness*).

**The proof of Lemma 887 (*finite is  $\sigma$ -finite*)**

cites explicitly:

Definition 857 (*finiteness*),  
 Definition 885 ( *$\sigma$ -finiteness*).

**The proof of Lemma 890 (*content over finite pseudopartition*)**

cites explicitly:

Definition 602 ( *$\pi$ -system*),  
 Lemma 863 (*decomposition over finite pseudopartition*),  
 Definition 888 (*content*).

**The proof of Lemma 891 (*content satisfies the binary inclusion-exclusion principle*)**

cites explicitly:

Definition 626 (*semiring of sets*),  
 Lemma 864 (*binary inclusion-exclusion principle*),  
 Definition 888 (*content*).

**The proof of Lemma 892 (*content is subtractive*)**

cites explicitly:

Lemma 612 (*equivalent definition of ring of sets*),  
 Definition 865 (*subtractivity*),  
 Lemma 866 (*subtractivity*),  
 Definition 888 (*content*).

**The proof of Lemma 893 (*content is monotone*)**

cites explicitly:

Lemma 630 (*other properties of semiring of sets*),  
 Lemma 867 (*monotonicity*),  
 Definition 888 (*content*).

**The proof of Lemma 894 (*content satisfies the finite Boole inequality*)**

cites explicitly:

Definition 626 (*semiring of sets*),

Lemma 869 (*subadditivity*),  
Definition 888 (*content*).

**The proof of Lemma 895 (*finite content is bounded*)**

cites explicitly:  
Definition 654 (*semi-algebra of sets*),  
Lemma 658 (*other properties of semi-algebra of sets*),  
Lemma 858 (*finite is bounded*),  
Definition 888 (*content*),  
Lemma 893 (*content is monotone*).

**The proof of Lemma 896 ( *$\sigma$ -finite content*)**

cites explicitly:  
Lemma 612 (*equivalent definition of ring of sets*),  
Lemma 631 (*ring of sets is semiring of sets*),  
Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*),  
Lemma 894 (*content satisfies the finite Boole inequality*).

**The proof of Lemma 897 (*premeasure*)**

cites explicitly:  
Definition 739 (*premeasurable space*),  
Lemma 875 ( *$\sigma$ -additive is additive*),  
Definition 888 (*content*).

**The proof of Lemma 899 (*premeasure is additive*)**

cites explicitly:  
Definition 888 (*content*),  
Lemma 897 (*premeasure*).

**The proof of Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*)**

cites explicitly:  
Lemma 740 (*properties of premeasurable space*),  
Lemma 891 (*content satisfies the binary inclusion-exclusion principle*),  
Lemma 897 (*premeasure*).

**The proof of Lemma 901 (*premeasure is subtractive*)**

cites explicitly:  
Lemma 740 (*properties of premeasurable space*),  
Lemma 892 (*content is subtractive*),  
Lemma 897 (*premeasure*).

**The proof of Lemma 902 (*premeasure is monotone*)**

cites explicitly:  
Lemma 554 (*disjoint decomposition*),  
Lemma 740 (*properties of premeasurable space*),  
Lemma 893 (*content is monotone*),  
Lemma 897 (*premeasure*).

**The proof of Lemma 903 (*finite premeasure is bounded*)**

cites explicitly:  
Lemma 660 (*algebra of sets is semi-algebra of sets*),  
Lemma 742 (*properties of measurable space*),  
Lemma 895 (*finite content is bounded*),  
Lemma 897 (*premeasure*).



**The proof of Lemma 904 ( *$\sigma$ -finite premeasure*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 896 ( *$\sigma$ -finite content*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 905 (*premeasure over countable pseudopartition*)**

cites explicitly:

Definition 602 ( *$\pi$ -system*),  
 Lemma 740 (*properties of premeasurable space*),  
 Lemma 876 (*decomposition over countable pseudopartition*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 906 (*premeasure satisfies the Boole inequality*)**

cites explicitly:

Definition 626 (*semiring of sets*),  
 Lemma 740 (*properties of premeasurable space*),  
 Lemma 871 ( *$\sigma$ -subadditive is subadditive*),  
 Lemma 878 ( *$\sigma$ -subadditivity*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 907 (*compatibility of null premeasure with countable union*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 740 (*properties of premeasurable space*),  
 Lemma 897 (*premeasure*),  
 Lemma 906 (*premeasure satisfies the Boole inequality*).

**The proof of Lemma 908 (*premeasure is continuous from below*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 880 (*continuity from below*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 909 (*equivalent definition of premeasure (from below)*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 881 ( *$\sigma$ -additivity (from below)*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 910 (*premeasure is continuous from above*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 883 (*continuity from above*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 911 (*equivalent definition of premeasure (from above)*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 884 ( *$\sigma$ -additivity (from above)*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 912 (*properties of premeasure space*)**

cites explicitly:

Lemma 740 (*properties of premeasurable space*),  
 Lemma 897 (*premeasure*),  
 Lemma 899 (*premeasure is additive*),

Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*),  
 Lemma 901 (*premeasure is subtractive*),  
 Lemma 902 (*premeasure is monotone*),  
 Lemma 904 ( *$\sigma$ -finite premeasure*),  
 Lemma 905 (*premeasure over countable pseudopartition*),  
 Lemma 906 (*premeasure satisfies the Boole inequality*),  
 Lemma 908 (*premeasure is continuous from below*),  
 Lemma 910 (*premeasure is continuous from above*).

**The proof of Lemma 913 (*measure*)**

cites explicitly:  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Definition 874 ( *$\sigma$ -additivity*),  
 Lemma 897 (*premeasure*).

**The proof of Lemma 915 (*measure space is premeasure space*)**

cites explicitly:  
 Lemma 703 ( *$\sigma$ -algebra is ring of sets*),  
 Definition 741 (*measurable space*),  
 Lemma 897 (*premeasure*),  
 Lemma 913 (*measure*).

**The proof of Lemma 916 (*measure is additive*)**

cites explicitly:  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Definition 862 (*additivity*),  
 Lemma 899 (*premeasure is additive*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*)**

cites explicitly:  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 864 (*binary inclusion-exclusion principle*),  
 Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 918 (*measure is subtractive*)**

cites explicitly:  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Definition 865 (*subtractivity*),  
 Lemma 901 (*premeasure is subtractive*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 919 (*measure is monotone*)**

cites explicitly:  
 Lemma 902 (*premeasure is monotone*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 920 (*finite measure is bounded*)**

cites explicitly:  
 Lemma 903 (*finite premeasure is bounded*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 921 ( *$\sigma$ -finite measure*)**

cites explicitly:

Lemma 904 ( *$\sigma$ -finite premeasure*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 922 (*measure over countable pseudopartition*)**

cites explicitly:  
 Lemma 905 (*premeasure over countable pseudopartition*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 923 (*measure satisfies the Boole inequality*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 870 ( *$\sigma$ -subadditivity*),  
 Lemma 906 (*premeasure satisfies the Boole inequality*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 924 (*compatibility of null measure with countable union*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 907 (*compatibility of null premeasure with countable union*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 925 (*measure is continuous from below*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 908 (*premeasure is continuous from below*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 926 (*equivalent definition of measure (from below)*)**

cites explicitly:  
 Lemma 909 (*equivalent definition of premeasure (from below)*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 927 (*measure is continuous from above*)**

cites explicitly:  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 910 (*premeasure is continuous from above*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 928 (*equivalent definition of measure (from above)*)**

cites explicitly:  
 Lemma 911 (*equivalent definition of premeasure (from above)*),  
 Lemma 915 (*measure space is premeasure space*).

**The proof of Lemma 929 (*properties of measure space*)**

cites explicitly:  
 Lemma 742 (*properties of measurable space*),  
 Lemma 912 (*properties of premeasure space*),  
 Lemma 913 (*measure*).

**The proof of Lemma 930 (*trace measure*)**

cites explicitly:  
 Lemma 763 (*trace  $\sigma$ -algebra*),  
 Lemma 764 (*measurability of measurable subspace*),  
 Lemma 913 (*measure*).

**The proof of Lemma 931 (*restricted measure*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
Lemma 913 (*measure*).

**The proof of Lemma 932 (*trace measure is restricted measure*)**

cites explicitly:

Lemma 764 (*measurability of measurable subspace*),  
Lemma 930 (*trace measure*),  
Lemma 931 (*restricted measure*).

**The proof of Lemma 933 (*trivial measure*)**

cites explicitly:

Lemma 913 (*measure*).

**The proof of Lemma 934 (*equivalent definition of trivial measure*)**

cites explicitly:

Lemma 913 (*measure*),  
Lemma 919 (*measure is monotone*),  
Lemma 933 (*trivial measure*).

**The proof of Lemma 935 (*counting measure*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
Lemma 696 ( *$\sigma$ -algebra*),  
Definition 741 (*measurable space*),  
Definition 874 ( *$\sigma$ -additivity*),  
Lemma 913 (*measure*).

**The proof of Lemma 936 (*finiteness of counting measure*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
Definition 857 (*finiteness*),  
Lemma 935 (*counting measure*).

**The proof of Lemma 937 ( *$\sigma$ -finite counting measure*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
Definition 741 (*measurable space*),  
Definition 885 ( *$\sigma$ -finiteness*),  
Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*),  
Lemma 913 (*measure*),  
Lemma 935 (*counting measure*).

**The proof of Lemma 940 (*equivalent definition of Dirac measure*)**

cites explicitly:

Lemma 935 (*counting measure*),  
Definition 939 (*Dirac measure*).

**The proof of Lemma 941 (*Dirac measure is finite*)**

cites explicitly:

Lemma 935 (*counting measure*),  
Lemma 936 (*finiteness of counting measure*),  
Definition 939 (*Dirac measure*).

**The proof of Lemma 946 (*equivalent definition of considerable subset*)**

cites explicitly:

Lemma 913 (*measure*),  
Definition 942 (*negligible subset*),  
Definition 945 (*considerable subset*).

**The proof of Lemma 947 (*negligibility with trivial measure*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),Lemma 933 (*trivial measure*),Definition 942 (*negligible subset*).**The proof of Lemma 948 (*negligibility of measurable subset*)**

cites explicitly:

Lemma 913 (*measure*),Lemma 919 (*measure is monotone*),Definition 942 (*negligible subset*).**The proof of Lemma 949 (*empty set is negligible*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),Definition 741 (*measurable space*),Lemma 913 (*measure*),Lemma 948 (*negligibility of measurable subset*).**The proof of Lemma 951 (*subset of negligible is negligible*)**

cites explicitly:

Definition 942 (*negligible subset*).**The proof of Lemma 952 ( *$\mathbf{N}$  is closed under countable union*)**

cites explicitly:

Lemma 924 (*compatibility of null measure with countable union*),Definition 942 (*negligible subset*).**The proof of Lemma 955 (*equivalent definition of property almost satisfied*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),Definition 741 (*measurable space*),Lemma 913 (*measure*),Definition 942 (*negligible subset*),Definition 953 (*property almost satisfied*).**The proof of Lemma 956 (*property almost satisfied with trivial measure*)**

cites explicitly:

Lemma 947 (*negligibility with trivial measure*),Definition 953 (*property almost satisfied*).**The proof of Lemma 957 (*measurable property almost satisfied*)**

cites explicitly:

Lemma 948 (*negligibility of measurable subset*),Definition 953 (*property almost satisfied*).**The proof of Lemma 958 (*everywhere is almost everywhere*)**

cites explicitly:

Lemma 949 (*empty set is negligible*),Definition 953 (*property almost satisfied*).**The proof of Lemma 959 (*almost modus ponens*)**

cites explicitly:

Lemma 951 (*subset of negligible is negligible*),Definition 953 (*property almost satisfied*).

**The proof of Lemma 960 (countable almost properties is almost property)**

cites explicitly:

Lemma 951 (subset of negligible is negligible),

Lemma 952 ( $\mathbf{N}$  is closed under countable union),

Definition 953 (property almost satisfied),

Lemma 955 (equivalent definition of property almost satisfied).

**The proof of Lemma 962 (almost everywhere for almost the same)**

cites explicitly:

Lemma 958 (everywhere is almost everywhere),

Lemma 959 (almost modus ponens),

Lemma 960 (countable almost properties is almost property).

**The proof of Lemma 963 (extended almost modus ponens)**

cites explicitly:

Lemma 955 (equivalent definition of property almost satisfied),

Lemma 962 (almost everywhere for almost the same).

**The proof of Lemma 965 (extended almost modus ponens (simplified 0))**

cites explicitly:

Lemma 963 (extended almost modus ponens).

**The proof of Lemma 966 (extended almost modus ponens (simplified 1))**

cites explicitly:

Lemma 963 (extended almost modus ponens).

**The proof of Lemma 967 (extended almost modus ponens (simplified 2))**

cites explicitly:

Lemma 958 (everywhere is almost everywhere),

Lemma 966 (extended almost modus ponens (simplified 1)).

**The proof of Lemma 970 (extension to almost defined)**

cites explicitly:

Definition 953 (property almost satisfied),

Lemma 959 (almost modus ponens),

Lemma 960 (countable almost properties is almost property).

**The proof of Lemma 972 (generalized almost modus ponens)**

cites explicitly:

Lemma 959 (almost modus ponens),

Lemma 960 (countable almost properties is almost property),

Lemma 970 (extension to almost defined).

**The proof of Lemma 975 (compatibility of almost binary relation with reflexivity)**

cites explicitly:

Lemma 970 (extension to almost defined),

Definition 974 (almost binary relation).

**The proof of Lemma 977 (compatibility of almost binary relation with extended reflexivity)**

cites explicitly:

Lemma 960 (countable almost properties is almost property),

Lemma 972 (generalized almost modus ponens),

Definition 974 (almost binary relation).

**The proof of Lemma 978 (compatibility of almost binary relation with symmetry)**

cites explicitly:

Lemma 972 (*generalized almost modus ponens*),  
 Definition 974 (*almost binary relation*).

**The proof of Lemma 979 (*compatibility of almost binary relation with extended anti-symmetry*)**

cites explicitly:

Lemma 960 (*countable almost properties is almost property*),  
 Lemma 972 (*generalized almost modus ponens*),  
 Definition 974 (*almost binary relation*).

**The proof of Lemma 980 (*compatibility of almost binary relation with transitivity*)**

cites explicitly:

Lemma 960 (*countable almost properties is almost property*),  
 Lemma 972 (*generalized almost modus ponens*),  
 Definition 974 (*almost binary relation*).

**The proof of Lemma 981 (*almost equivalence is equivalence relation*)**

cites explicitly:

Lemma 975 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 978 (*compatibility of almost binary relation with symmetry*),  
 Lemma 980 (*compatibility of almost binary relation with transitivity*).

**The proof of Lemma 982 (*almost equality is equivalence relation*)**

cites explicitly:

Lemma 981 (*almost equivalence is equivalence relation*).

**The proof of Lemma 983 (*almost order is order relation*)**

cites explicitly:

Lemma 975 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 979 (*compatibility of almost binary relation with extended antisymmetry*),  
 Lemma 980 (*compatibility of almost binary relation with transitivity*).

**The proof of Lemma 984 (*compatibility of almost binary relation with operator*)**

cites explicitly:

Lemma 960 (*countable almost properties is almost property*),  
 Lemma 972 (*generalized almost modus ponens*),  
 Definition 974 (*almost binary relation*).

**The proof of Lemma 985 (*compatibility of almost equality with operator*)**

cites explicitly:

Lemma 984 (*compatibility of almost binary relation with operator*).

**The proof of Lemma 988 (*definiteness is almost definiteness*)**

cites explicitly:

Lemma 972 (*generalized almost modus ponens*).

**The proof of Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*)**

cites explicitly:

Definition 209 (*pseudopartition*),  
 Definition 602 ( *$\pi$ -system*),  
 Lemma 608 (*properties of  $\pi$ -system generation*),  
 Lemma 678 (*equivalent definition of  $\lambda$ -system*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),  
 Definition 874 ( *$\sigma$ -additivity*),  
 Definition 879 (*continuity from below*),

Lemma 913 (*measure*),  
 Lemma 919 (*measure is monotone*),  
 Lemma 925 (*measure is continuous from below*).

**The proof of Lemma 993 (*transitivity of almost nonnegativeness*)**

cites explicitly:

Lemma 980 (*compatibility of almost binary relation with transitivity*).

**The proof of Lemma 994 (*transitivity of almost inequality and equality*)**

cites explicitly:

Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),  
 Lemma 980 (*compatibility of almost binary relation with transitivity*).

**The proof of Lemma 995 (*compatibility of almost nonnegativeness with almost equality*)**

cites explicitly:

Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),  
 Lemma 993 (*transitivity of almost nonnegativeness*).

**The proof of Lemma 996 (*almost finite*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Lemma 341 (*finite absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 808 (*inverse image of countable is measurable*),  
 Lemma 957 (*measurable property almost satisfied*).

**The proof of Lemma 998 (*summability on summability domain*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 997 (*summability domain*).

**The proof of Lemma 999 (*measurability of summability domain*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 809 (*inverse image is measurable*).

**The proof of Lemma 1000 (*negligibility and summability domain*)**

cites explicitly:

Definition 953 (*property almost satisfied*),  
 Lemma 998 (*summability on summability domain*).

**The proof of Lemma 1001 (*absolute value and almost inequality*)**

cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*),  
 Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),  
 Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Lemma 972 (*generalized almost modus ponens*),  
 Lemma 983 (*almost order is order relation*),  
 Lemma 1000 (*negligibility and summability domain*).

**The proof of Lemma 1002 (*almost sum*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),



Lemma 811 (*measurability of indicator function*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Definition 953 (*property almost satisfied*),  
 Lemma 999 (*measurability of summability domain*),  
 Lemma 1000 (*negligibility and summability domain*).

**The proof of Lemma 1003 (*compatibility of almost sum with almost equality*)**

cites explicitly:

Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 962 (*almost everywhere for almost the same*),  
 Lemma 982 (*almost equality is equivalence relation*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 998 (*summability on summability domain*),  
 Lemma 1002 (*almost sum*).

**The proof of Lemma 1004 (*almost sum is sum*)**

cites explicitly:

Lemma 815 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ),  
 Definition 997 (*summability domain*),  
 Lemma 998 (*summability on summability domain*),  
 Lemma 1002 (*almost sum*).

**The proof of Lemma 1005 (*absolute value is almost definite*)**

cites explicitly:

Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 988 (*definiteness is almost definiteness*).

**The proof of Lemma 1006 (*absolute value is almost equal to almost nonnegative function*)**

cites explicitly:

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),  
 Lemma 972 (*generalized almost modus ponens*).

**The proof of Lemma 1007 (*power function is almost definite*)**

cites explicitly:

Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 988 (*definiteness is almost definiteness*).

**The proof of Lemma 1008 (*masking almost nowhere*)**

cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Lemma 913 (*measure*),  
 Definition 942 (*negligible subset*),  
 Definition 953 (*property almost satisfied*).

**The proof of Lemma 1009 (*finite nonnegative part*)**

cites explicitly:

Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*),  
 Lemma 955 (*equivalent definition of property almost satisfied*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Lemma 966 (*extended almost modus ponens (simplified 1)*).

**The proof of Lemma 1010 (compatibility of almost limit with subsequence)**

cites explicitly:

Lemma 266 (compatibility of subsequence with convergence),

Lemma 972 (generalized almost modus ponens).

**The proof of Lemma 1011 (compatibility of almost limit with continuous function)**

cites explicitly:

Statement(s) from [20],

Lemma 517 (limit inferior and limit superior of pointwise convergent),

Lemma 958 (everywhere is almost everywhere),

Lemma 960 (countable almost properties is almost property),

Lemma 972 (generalized almost modus ponens).

**The proof of Lemma 1012 (compatibility of almost limit with power function)**

cites explicitly:

Definition 317 (addition in  $\overline{\mathbb{R}}$ ),

Lemma 449 (power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)),

Lemma 1011 (compatibility of almost limit with continuous function).

**The proof of Lemma 1013 (almost limits are almost equal)**

cites explicitly:

Lemma 517 (limit inferior and limit superior of pointwise convergent),

Lemma 960 (countable almost properties is almost property),

Lemma 967 (extended almost modus ponens (simplified 2)).

**The proof of Lemma 1014 (compatibility of almost limit with almost equality)**

cites explicitly:

Lemma 958 (everywhere is almost everywhere),

Lemma 960 (countable almost properties is almost property),

Lemma 962 (almost everywhere for almost the same).

**The proof of Lemma 1015 (compatibility of almost limit with almost inequality)**

cites explicitly:

Statement(s) from [20],

Lemma 381 (limit is monotone),

Lemma 389 (absolute value in  $\overline{\mathbb{R}}$  is continuous),

Lemma 960 (countable almost properties is almost property),

Lemma 967 (extended almost modus ponens (simplified 2)).

**The proof of Lemma 1016 (compatibility of almost limit with measurability)**

cites explicitly:

Statement(s) from [20],

Lemma 517 (limit inferior and limit superior of pointwise convergent),

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent),

Lemma 833 (measurability and masking),

Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions),

Definition 953 (property almost satisfied),

Lemma 960 (countable almost properties is almost property),

Lemma 1008 (masking almost nowhere),

Lemma 1014 (compatibility of almost limit with almost equality).

**The proof of Lemma 1017 (essential supremum)**

cites explicitly:

Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),

Lemma 314 ( $\overline{\mathbb{R}}$  is complete),

Lemma 820 (*measurability of numeric function*),  
 Lemma 913 (*measure*),  
 Lemma 919 (*measure is monotone*).

**The proof of Lemma 1019 (*essential supremum with trivial measure*)**

cites explicitly:  
 Lemma 933 (*trivial measure*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1020 (*essential supremum is bounded by supremum*)**

cites explicitly:  
 Statement(s) from [20],  
 Lemma 913 (*measure*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1021 (*essential supremum is minimum*)**

cites explicitly:  
 Statement(s) from [20],  
 Lemma 486 (*equivalent definition of infimum*),  
 Lemma 924 (*compatibility of null measure with countable union*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1022 (*essential supremum is almost upper bound*)**

cites explicitly:  
 Lemma 948 (*negligibility of measurable subset*),  
 Definition 974 (*almost binary relation*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1021 (*essential supremum is minimum*).

**The proof of Lemma 1023 (*essential supremum of constant function*)**

cites explicitly:  
 Lemma 754 (*constant function is measurable*),  
 Lemma 913 (*measure*),  
 Lemma 934 (*equivalent definition of trivial measure*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1024 (*essential supremum of extended simple functions*)**

cites explicitly:  
 Definition 209 (*pseudopartition*),  
 Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 811 (*measurability of indicator function*),  
 Lemma 823 ( *$\mathcal{M}$  is closed under finite sum when defined*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Definition 862 (*additivity*),  
 Lemma 913 (*measure*),  
 Lemma 916 (*measure is additive*),  
 Lemma 919 (*measure is monotone*),  
 Lemma 934 (*equivalent definition of trivial measure*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1026 (*essential supremum of infinite times function*)**

cites explicitly:  
 Definition 209 (*pseudopartition*),  
 Lemma 809 (*inverse image is measurable*),  
 Lemma 820 (*measurability of numeric function*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),

Lemma 933 (*trivial measure*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1019 (*essential supremum with trivial measure*),  
 Lemma 1024 (*essential supremum of extended simple functions*).

**The proof of Lemma 1027 (*essential supremum is positive homogeneous*)**

cites explicitly:

Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1021 (*essential supremum is minimum*),  
 Lemma 1023 (*essential supremum of constant function*).

**The proof of Lemma 1029 (*essential supremum is almost monotone*)**

cites explicitly:

Lemma 820 (*measurability of numeric function*),  
 Lemma 948 (*negligibility of measurable subset*),  
 Lemma 951 (*subset of negligible is negligible*),  
 Lemma 952 ( *$\mathbf{N}$  is closed under countable union*),  
 Definition 974 (*almost binary relation*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1021 (*essential supremum is minimum*).

**The proof of Lemma 1030 (*essential supremum bounded by constant*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Lemma 983 (*almost order is order relation*),  
 Lemma 1019 (*essential supremum with trivial measure*),  
 Lemma 1022 (*essential supremum is almost upper bound*),  
 Lemma 1023 (*essential supremum of constant function*),  
 Lemma 1029 (*essential supremum is almost monotone*).

**The proof of Lemma 1031 (*essential supremum of absolute value is nonnegative*)**

cites explicitly:

Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 1030 (*essential supremum bounded by constant*).

**The proof of Lemma 1032 (*compatibility of essential supremum with almost equality*)**

cites explicitly:

Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),  
 Lemma 1029 (*essential supremum is almost monotone*).

**The proof of Lemma 1036 (*equivalent definition of nonnegative essential supremum*)**

cites explicitly:

Statement(s) from [20],  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 913 (*measure*),  
 Lemma 933 (*trivial measure*),  
 Lemma 934 (*equivalent definition of trivial measure*),  
 Lemma 1017 (*essential supremum*).

**The proof of Lemma 1037 (*nonnegative essential supremum with trivial measure*)**

cites explicitly:

Lemma 933 (*trivial measure*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1036 (*equivalent definition of nonnegative essential supremum*).

**The proof of Lemma 1038 (*nonnegative essential supremum is minimum*)**

cites explicitly:  
 Lemma 1021 (*essential supremum is minimum*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**The proof of Lemma 1039 (*nonnegative essential supremum is almost upper bound*)**

cites explicitly:  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 956 (*property almost satisfied with trivial measure*),  
 Lemma 1022 (*essential supremum is almost upper bound*),  
 Definition 1034 (*nonnegative essential supremum*).

**The proof of Lemma 1040 (*nonnegative essential supremum of constant function*)**

cites explicitly:  
 Lemma 754 (*constant function is measurable*),  
 Lemma 1023 (*essential supremum of constant function*),  
 Definition 1034 (*nonnegative essential supremum*).

**The proof of Lemma 1041 (*nonnegative essential supremum for extended simple functions*)**

cites explicitly:  
 Lemma 1024 (*essential supremum of extended simple functions*),  
 Definition 1034 (*nonnegative essential supremum*).

**The proof of Lemma 1043 (*nonnegative essential supremum of infinite times function*)**

cites explicitly:  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 933 (*trivial measure*),  
 Lemma 934 (*equivalent definition of trivial measure*),  
 Lemma 1026 (*essential supremum of infinite times function*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**The proof of Lemma 1044 (*nonnegative essential supremum is positive homogeneous*)**

cites explicitly:  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 1027 (*essential supremum is positive homogeneous*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1036 (*equivalent definition of nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*),  
 Lemma 1038 (*nonnegative essential supremum is minimum*),  
 Lemma 1043 (*nonnegative essential supremum of infinite times function*).

**The proof of Lemma 1045 (*nonnegative essential supremum is almost monotone*)**

cites explicitly:  
 Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 1029 (*essential supremum is almost monotone*),

Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**The proof of Lemma 1046 (*nonnegative essential supremum bounded by constant*)**  
 cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 956 (*property almost satisfied with trivial measure*),  
 Lemma 1030 (*essential supremum bounded by constant*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**The proof of Lemma 1048 (*essentially bounded is almost finite*)**  
 cites explicitly:

Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 983 (*almost order is order relation*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*),  
 Definition 1047 (*essentially bounded*).

**The proof of Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*)**

cites explicitly:  
 Lemma 1032 (*compatibility of essential supremum with almost equality*),  
 Definition 1034 (*nonnegative essential supremum*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*).

**The proof of Lemma 1050 (*nonnegative essential supremum is almost definite*)**  
 cites explicitly:

Definition 209 (*pseudopartition*),  
 Lemma 809 (*inverse image is measurable*),  
 Lemma 820 (*measurability of numeric function*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 948 (*negligibility of measurable subset*),  
 Definition 953 (*property almost satisfied*),  
 Lemma 956 (*property almost satisfied with trivial measure*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*),  
 Lemma 1038 (*nonnegative essential supremum is minimum*),  
 Lemma 1040 (*nonnegative essential supremum of constant function*),  
 Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*).

**The proof of Lemma 1051 (*nonnegative essential supremum is subadditive*)**  
 cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*),  
 Lemma 1039 (*nonnegative essential supremum is almost upper bound*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*).

**The proof of Lemma 1053 (*nonnegative essential supremum is submultiplicative*)**  
 cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),

Lemma 1037 (*nonnegative essential supremum with trivial measure*),  
 Lemma 1039 (*nonnegative essential supremum is almost upper bound*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*).

**The proof of Lemma 1054 (*nonnegative essential supremum is compatible with power function*)**

cites explicitly:

Lemma 422 (*one-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),  
 Lemma 444 (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*),  
 Definition 835 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*),  
 Lemma 841 ( $\mathcal{M}$  *is closed under power function*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 1037 (*nonnegative essential supremum with trivial measure*),  
 Lemma 1039 (*nonnegative essential supremum is almost upper bound*),  
 Lemma 1040 (*nonnegative essential supremum of constant function*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*).

**The proof of Lemma 1058 (*length is nonnegative*)**

cites explicitly:

Definition 1057 (*length of interval*).

**The proof of Lemma 1059 (*length is homogeneous*)**

cites explicitly:

Definition 1057 (*length of interval*).

**The proof of Lemma 1060 (*length of partition*)**

cites explicitly:

Definition 1057 (*length of interval*),  
 Lemma 1059 (*length is homogeneous*).

**The proof of Lemma 1062 (*set of countable cover with bounded open intervals is nonempty*)**

cites explicitly:

Definition 1061 (*set of countable cover with bounded open intervals*).

**The proof of Lemma 1065 ( $\lambda^*$  *is nonnegative*)**

cites explicitly:

Lemma 490 (*infimum is monotone (function)*),  
 Lemma 1058 (*length is nonnegative*).

**The proof of Lemma 1066 ( $\lambda^*$  *is homogeneous*)**

cites explicitly:

Statement(s) from [20],  
 Definition 1057 (*length of interval*),  
 Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*),  
 Lemma 1065 ( $\lambda^*$  *is nonnegative*).

**The proof of Lemma 1067 ( $\lambda^*$  *is monotone*)**

cites explicitly:

Statement(s) from [20],  
 Definition 1061 (*set of countable cover with bounded open intervals*),  
 Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*).



**The proof of Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive)**

cites explicitly:

Statement(s) from [20],

Lemma 224 (*definition of double countable union*),

Lemma 225 (*double countable union*),

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),

Lemma 381 (*limit is monotone*),

Lemma 403 (*definition of double series in  $\overline{\mathbb{R}}_+$* ),

Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ),

Definition 1061 (*set of countable cover with bounded open intervals*),

Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*).

**The proof of Lemma 1069 ( $\lambda^*$  generalizes length of interval)**

cites explicitly:

Statement(s) from [20],

Lemma 299 (*finite cover of compact interval*),

Lemma 381 (*limit is monotone*),

Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),

Definition 1057 (*length of interval*),

Definition 1061 (*set of countable cover with bounded open intervals*),

Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*),

Lemma 1066 ( $\lambda^*$  is homogeneous),

Lemma 1067 ( $\lambda^*$  is monotone).

**The proof of Lemma 1073 (equivalent definition of  $\mathcal{L}$ )**

cites explicitly:

Lemma 211 (*compatibility of pseudopartition with intersection*),

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive),

Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*).

**The proof of Lemma 1074 ( $\mathcal{L}$  is closed under complement)**

cites explicitly:

Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),

Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*).

**The proof of Lemma 1075 ( $\mathcal{L}$  is closed under finite union)**

cites explicitly:

Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive),

Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*),

Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ).

**The proof of Lemma 1076 ( $\mathcal{L}$  is closed under finite intersection)**

cites explicitly:

Lemma 1074 ( $\mathcal{L}$  is closed under complement),

Lemma 1075 ( $\mathcal{L}$  is closed under finite union).

**The proof of Lemma 1077 ( $\mathcal{L}$  is algebra of sets)**

cites explicitly:

Lemma 545 (*closedness under intersection and set difference*),

Lemma 637 (*other equivalent definition of algebra of sets*),

Lemma 1066 ( $\lambda^*$  is homogeneous),

Lemma 1074 ( $\mathcal{L}$  is closed under complement),

Lemma 1076 ( $\mathcal{L}$  is closed under finite intersection).

**The proof of Lemma 1078 ( $\lambda^*$  is additive on  $\mathcal{L}$ )**

cites explicitly:



Lemma 211 (*compatibility of pseudopartition with intersection*),  
 Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*).

**The proof of Lemma 1080 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ )**

cites explicitly:  
 Lemma 381 (*limit is monotone*),  
 Definition 874 ( $\sigma$ -*additivity*),  
 Lemma 1067 ( $\lambda^*$  is *monotone*),  
 Lemma 1068 ( $\lambda^*$  is  $\sigma$ -*subadditive*),  
 Lemma 1078 ( $\lambda^*$  is *additive on  $\mathcal{L}$* ).

**The proof of Lemma 1081 (*disjoint union of union in  $\mathcal{L}$* )**

cites explicitly:  
 Lemma 652 (*disjoint union of union in algebra of sets*),  
 Lemma 1077 ( $\mathcal{L}$  is *algebra of sets*).

**The proof of Lemma 1082 ( $\mathcal{L}$  is closed under countable union)**

cites explicitly:  
 Lemma 381 (*limit is monotone*),  
 Lemma 1067 ( $\lambda^*$  is *monotone*),  
 Lemma 1068 ( $\lambda^*$  is  $\sigma$ -*subadditive*),  
 Definition 1071 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*),  
 Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),  
 Lemma 1075 ( $\mathcal{L}$  is closed under *finite union*),  
 Lemma 1078 ( $\lambda^*$  is *additive on  $\mathcal{L}$* ),  
 Lemma 1081 (*disjoint union of union in  $\mathcal{L}$* ).

**The proof of Lemma 1083 (*rays are Lebesgue-measurable*)**

cites explicitly:  
 Statement(s) from [20],  
 Lemma 235 (*intervals are closed under finite intersection*),  
 Lemma 381 (*limit is monotone*),  
 Definition 1063 ( $\lambda^*$ , *Lebesgue measure candidate*),  
 Lemma 1067 ( $\lambda^*$  is *monotone*),  
 Lemma 1068 ( $\lambda^*$  is  $\sigma$ -*subadditive*),  
 Lemma 1069 ( $\lambda^*$  *generalizes length of interval*),  
 Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),  
 Lemma 1074 ( $\mathcal{L}$  is closed under *complement*),  
 Lemma 1082 ( $\mathcal{L}$  is closed under *countable union*).

**The proof of Lemma 1084 (*intervals are Lebesgue-measurable*)**

cites explicitly:  
 Lemma 1076 ( $\mathcal{L}$  is closed under *finite intersection*),  
 Lemma 1083 (*rays are Lebesgue-measurable*).

**The proof of Lemma 1085 ( $\mathcal{L}$  is  $\sigma$ -algebra)**

cites explicitly:  
 Definition 634 (*algebra of sets*),  
 Lemma 696 ( $\sigma$ -*algebra*),  
 Lemma 1077 ( $\mathcal{L}$  is *algebra of sets*),  
 Lemma 1082 ( $\mathcal{L}$  is closed under *countable union*).

**The proof of Lemma 1086 ( $\lambda^*$  is *measure on  $\mathcal{L}$* )**

cites explicitly:  
 Lemma 913 (*measure*),  
 Lemma 1065 ( $\lambda^*$  is *nonnegative*),

Lemma 1066 ( $\lambda^*$  is homogeneous),  
 Lemma 1080 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ),  
 Lemma 1085 ( $\mathcal{L}$  is  $\sigma$ -algebra).

**The proof of Lemma 1087 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ )**

cites explicitly:  
 Lemma 710 (properties of  $\sigma$ -algebra generation),  
 Lemma 796 (Borel subsets of  $\mathbb{R}$ ),  
 Lemma 1084 (intervals are Lebesgue-measurable).

**The proof of Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ )**

cites explicitly:  
 Lemma 1086 ( $\lambda^*$  is measure on  $\mathcal{L}$ ),  
 Lemma 1087 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ).

**The proof of Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ )**

cites explicitly:  
 Lemma 235 (intervals are closed under finite intersection),  
 Definition 602 ( $\pi$ -system),  
 Lemma 709 (generated  $\sigma$ -algebra is  $\sigma$ -algebra),  
 Lemma 796 (Borel subsets of  $\mathbb{R}$ ),  
 Lemma 927 (measure is continuous from above),  
 Lemma 992 (uniqueness of measures extended from a  $\pi$ -system),  
 Lemma 1069 ( $\lambda^*$  generalizes length of interval),  
 Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ).

**The proof of Lemma 1092 (Lebesgue measure generalizes length of interval)**

cites explicitly:  
 Lemma 1069 ( $\lambda^*$  generalizes length of interval),  
 Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).

**The proof of Lemma 1094 (Lebesgue measure is  $\sigma$ -finite)**

cites explicitly:  
 Definition 885 ( $\sigma$ -finiteness),  
 Lemma 1069 ( $\lambda^*$  generalizes length of interval),  
 Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).

**The proof of Lemma 1095 (Lebesgue measure is diffuse)**

cites explicitly:  
 Definition 856 (diffuseness),  
 Lemma 1069 ( $\lambda^*$  generalizes length of interval),  
 Theorem 1090 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).

**The proof of Lemma 1100 (indicator and support are inverse)**

cites explicitly:  
 Definition 748 (measurability and structure),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions).

**The proof of Lemma 1101 ( $\mathcal{IF}_{\mathbb{K}}$  is measurable)**

cites explicitly:  
 Lemma 811 (measurability of indicator function),  
 Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),  
 Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions).

**The proof of Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive)**

cites explicitly:

Lemma 525 (*masking is  $\sigma$ -additive*),  
 Lemma 696 ( *$\sigma$ -algebra*),  
 Definition 741 (*measurable space*),  
 Definition 748 (*measurability and structure*),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*).

**The proof of Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication)**

cites explicitly:  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 748 (*measurability and structure*),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*).

**The proof of Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero)**

cites explicitly:  
 Lemma 524 (*restriction is masking*),  
 Definition 748 (*measurability and structure*),  
 Lemma 764 (*measurability of measurable subspace*),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*).

**The proof of Lemma 1106 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under restriction)**

cites explicitly:  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 748 (*measurability and structure*),  
 Lemma 764 (*measurability of measurable subspace*),  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*).

**The proof of Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive)**

cites explicitly:  
 Definition 209 (*pseudopartition*),  
 Lemma 525 (*masking is  $\sigma$ -additive*),  
 Definition 748 (*measurability and structure*),  
 Lemma 1100 (*indicator and support are inverse*),  
 Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive),  
 Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication).

**The proof of Lemma 1108 ( $\mathcal{SF}_E$  simple representation)**

cites explicitly:  
 Statement(s) from [20],  
 Definition 1098 ( $\mathcal{IF}_{\mathbb{K}}$ , *set of measurable indicator functions*),  
 Lemma 1100 (*indicator and support are inverse*).

**The proof of Lemma 1111 ( $\mathcal{SF}_E$  canonical representation)**

cites explicitly:  
 Statement(s) from [20],  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation).

**The proof of Lemma 1113 ( $\mathcal{SF}_E$  is measurable)**

cites explicitly:  
 Definition 748 (*measurability and structure*),  
 Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$  countable range),  
 Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Lemma 1111 ( $\mathcal{SF}_E$  canonical representation).

**The proof of Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation)**

cites explicitly:

Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Lemma 1111 ( $\mathcal{SF}_E$  canonical representation).

**The proof of Lemma 1116 ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation)**

cites explicitly:

Lemma 1111 ( $\mathcal{SF}_E$  canonical representation),  
 Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation).

**The proof of Lemma 1117 ( $\mathcal{SF}_E$  is vector space)**

cites explicitly:

Statement(s) from [20],  
 Definition 209 (pseudopartition),  
 Lemma 286 (vector subspace and closed under multiplication is subalgebra),  
 Lemma 1102 ( $\mathcal{IF}_\mathbb{K}$  is  $\sigma$ -additive),  
 Lemma 1103 ( $\mathcal{IF}_\mathbb{K}$  is closed under multiplication),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Lemma 1111 ( $\mathcal{SF}_E$  canonical representation),  
 Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation).

**The proof of Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero)**

cites explicitly:

Lemma 524 (restriction is masking),  
 Definition 1098 ( $\mathcal{IF}_\mathbb{K}$ , set of measurable indicator functions),  
 Lemma 1105 ( $\mathcal{IF}_\mathbb{K}$  is closed under extension by zero),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation).

**The proof of Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction)**

cites explicitly:

Lemma 697 (equivalent definition of  $\sigma$ -algebra),  
 Lemma 764 (measurability of measurable subspace),  
 Lemma 1106 ( $\mathcal{IF}_\mathbb{K}$  is closed under restriction),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation).

**The proof of Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive)**

cites explicitly:

Statement(s) from [20],  
 Lemma 525 (masking is  $\sigma$ -additive),  
 Definition 1098 ( $\mathcal{IF}_\mathbb{K}$ , set of measurable indicator functions),  
 Lemma 1102 ( $\mathcal{IF}_\mathbb{K}$  is  $\sigma$ -additive),  
 Lemma 1103 ( $\mathcal{IF}_\mathbb{K}$  is closed under multiplication),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation).

**The proof of Lemma 1124 ( $\mathcal{SF}_+$  is measurable)**

cites explicitly:

Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions),  
 Definition 1098 ( $\mathcal{IF}_\mathbb{K}$ , set of measurable indicator functions),  
 Lemma 1113 ( $\mathcal{SF}_E$  is measurable),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation)**

cites explicitly:

Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1126 ( $\mathcal{SF}_+$  canonical representation)**

cites explicitly:

Lemma 1111 ( $\mathcal{SF}_E$  canonical representation),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation)**

cites explicitly:

Lemma 1116 ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions),  
 Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation),  
 Lemma 1126 ( $\mathcal{SF}_+$  canonical representation).

**The proof of Lemma 1128 ( $\mathcal{SF}_+$  simple representation)**

cites explicitly:

Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations)**

cites explicitly:

Definition 277 (algebra over a field),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions),  
 Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation).

**The proof of Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero)**

cites explicitly:

Lemma 524 (restriction is masking),  
 Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1131 ( $\mathcal{SF}_+$  is closed under restriction)**

cites explicitly:

Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1132 ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive)**

cites explicitly:

Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

**The proof of Lemma 1134 (Lebesgue scheme step 2)**

cites explicitly:

Definition 1098 ( $\mathcal{IF}_K$ , set of measurable indicator functions),  
 Lemma 1128 ( $\mathcal{SF}_+$  simple representation).

**The proof of Lemma 1136 (adapted sequence in  $\mathcal{M}_+$ )**

cites explicitly:

Lemma 697 (equivalent definition of  $\sigma$ -algebra),  
 Lemma 820 (measurability of numeric function),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions),  
 Definition 1135 (adapted sequence).

**The proof of Lemma 1137 ( $\mathcal{M}_+$  is closed under extension by zero)**

cites explicitly:

Lemma 524 (restriction is masking),  
 Lemma 848 ( $\mathcal{M}_+$  is closed under supremum),  
 Lemma 1124 ( $\mathcal{SF}_+$  is measurable),

Lemma 1128 ( $\mathcal{SF}_+$  simple representation),  
 Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero),  
 Lemma 1136 (adapted sequence in  $\mathcal{M}_+$ ).

**The proof of Lemma 1138 ( $\mathcal{M}_+$  is closed under restriction)**

cites explicitly:

Lemma 697 (equivalent definition of  $\sigma$ -algebra),  
 Lemma 764 (measurability of measurable subspace),  
 Lemma 848 ( $\mathcal{M}_+$  is closed under supremum),  
 Lemma 1124 ( $\mathcal{SF}_+$  is measurable),  
 Lemma 1131 ( $\mathcal{SF}_+$  is closed under restriction),  
 Lemma 1136 (adapted sequence in  $\mathcal{M}_+$ ).

**The proof of Lemma 1139 ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive)**

cites explicitly:

Lemma 525 (masking is  $\sigma$ -additive),  
 Lemma 844 ( $\mathcal{M}_+$  is closed under multiplication),  
 Lemma 850 ( $\mathcal{M}_+$  is closed under countable sum),  
 Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication),  
 Lemma 1124 ( $\mathcal{SF}_+$  is measurable).

**The proof of Lemma 1141 (Lebesgue scheme step 3)**

cites explicitly:

Lemma 1136 (adapted sequence in  $\mathcal{M}_+$ ).

**The proof of Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ )**

cites explicitly:

Lemma 1100 (indicator and support are inverse),  
 Definition 1144 (integral in  $\mathcal{IF}$ ).

**The proof of Lemma 1146 (integral in  $\mathcal{IF}$  is  $\sigma$ -additive)**

cites explicitly:

Definition 874 ( $\sigma$ -additivity),  
 Lemma 913 (measure),  
 Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive),  
 Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ).

**The proof of Lemma 1147 (integral in  $\mathcal{IF}$  is monotone)**

cites explicitly:

Lemma 919 (measure is monotone),  
 Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ).

**The proof of Lemma 1148 (integral in  $\mathcal{IF}$  over subset)**

cites explicitly:

Lemma 932 (trace measure is restricted measure),  
 Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero),  
 Definition 1144 (integral in  $\mathcal{IF}$ ).

**The proof of Lemma 1149 (integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive)**

cites explicitly:

Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive),  
 Lemma 1146 (integral in  $\mathcal{IF}$  is  $\sigma$ -additive),  
 Lemma 1148 (integral in  $\mathcal{IF}$  over subset).

**The proof of Lemma 1150 (integral in  $\mathcal{IF}$  over negligible subset)**

cites explicitly:

Lemma 913 (measure),

Lemma 919 (*measure is monotone*),  
 Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*).

**The proof of Lemma 1152 (*integral in  $\mathcal{IF}$  with counting measure*)**  
 cites explicitly:

Lemma 935 (*counting measure*),  
 Lemma 1100 (*indicator and support are inverse*),  
 Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* ).

**The proof of Lemma 1154 (*integral in  $\mathcal{SF}_+$* )**  
 cites explicitly:

Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 913 (*measure*),  
 Lemma 1126 ( *$\mathcal{SF}_+$  canonical representation*),  
 Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* ).

**The proof of Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* )**  
 cites explicitly:

Definition 1098 ( *$\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions*),  
 Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* ),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*)**  
 cites explicitly:

Definition 874 ( *$\sigma$ -additivity*),  
 Lemma 913 (*measure*),  
 Lemma 1126 ( *$\mathcal{SF}_+$  canonical representation*),  
 Lemma 1127 ( *$\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation*),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1158 (*integral in  $\mathcal{SF}_+$  of zero is zero*)**  
 cites explicitly:

Lemma 696 ( *$\sigma$ -algebra*),  
 Lemma 913 (*measure*),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**The proof of Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*)**  
 cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 922 (*measure over countable pseudopartition*),  
 Lemma 1125 ( *$\mathcal{SF}_+$  disjoint representation*),  
 Lemma 1129 ( *$\mathcal{SF}_+$  is closed under positive algebra operations*),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*),  
 Lemma 1158 (*integral in  $\mathcal{SF}_+$  of zero is zero*).

**The proof of Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*)**  
 cites explicitly:

Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**The proof of Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*)**  
 cites explicitly:

Statement(s) from [20],  
 Definition 277 (*algebra over a field*),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**The proof of Lemma 1162 (*integral in  $\mathcal{SF}_+$  is continuous*)**

cites explicitly:  
 Statement(s) from [20],  
 Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*).

**The proof of Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*)**

cites explicitly:  
 Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero),  
 Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**The proof of Lemma 1164 (*integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive*)**

cites explicitly:  
 Lemma 1132 ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*).

**The proof of Lemma 1165 (*integral in  $\mathcal{SF}_+$  over negligible subset*)**

cites explicitly:  
 Lemma 1150 (*integral in  $\mathcal{IF}$  over negligible subset*),  
 Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*).

**The proof of Lemma 1166 (*density measure in  $\mathcal{SF}_+$* )**

cites explicitly:  
 Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),  
 Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),  
 Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),  
 Lemma 696 ( $\sigma$ -algebra),  
 Lemma 913 (*measure*),  
 Definition 1123 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions),  
 Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation),  
 Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*).

**The proof of Lemma 1167 (*integral and density measure in  $\mathcal{SF}_+$* )**

cites explicitly:  
 Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation),  
 Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),  
 Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*),  
 Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1168 (*density integral in  $\mathcal{SF}_+$* )**

cites explicitly:  
 Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),  
 Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation),  
 Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),



Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*)**  
 cites explicitly:

Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 933 (*trivial measure*),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*)**  
 cites explicitly:

Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),  
 Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),  
 Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 1152 (*integral in  $\mathcal{IF}$  with counting measure*),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**The proof of Lemma 1172 (*integral in  $\mathcal{M}_+$* )**  
 cites explicitly:

Statement(s) from [20],  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* )**  
 cites explicitly:

Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1162 (*integral in  $\mathcal{SF}_+$  is continuous*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ).

**The proof of Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*)**  
 cites explicitly:

Lemma 1124 ( *$\mathcal{SF}_+$  is measurable*),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ).

**The proof of Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*)**  
 cites explicitly:

Statement(s) from [20],  
 Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*),  
 Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 1108 ( *$\mathcal{SF}_E$  simple representation*),  
 Definition 1123 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*),  
 Lemma 1129 ( *$\mathcal{SF}_+$  is closed under positive algebra operations*),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ).

**The proof of Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*)**  
 cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**The proof of Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*)**

cites explicitly:

Lemma 489 (*supremum is monotone (set)*),

Lemma 1172 (*integral in  $\mathcal{M}_+$* ).

**The proof of Theorem 1180 (*Beppo Levi, monotone convergence*)**

cites explicitly:

Statement(s) from [20],

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*),

Lemma 381 (*limit is monotone*),

Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),

Lemma 414 (*limit in  $\overline{\mathbb{R}}_+$  is positive linear*),

Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),

Lemma 501 (*supremum is limit*),

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 820 (*measurability of numeric function*),

Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),

Definition 879 (*continuity from below*),

Lemma 925 (*measure is continuous from below*),

Lemma 1108 ( *$\mathcal{SF}_E$  simple representation*),

Lemma 1113 ( *$\mathcal{SF}_E$  is measurable*),

Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),

Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*),

Lemma 1172 (*integral in  $\mathcal{M}_+$* ),

Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),

Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*).

**The proof of Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* )**

cites explicitly:

Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*),

Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),

Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),

Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Theorem 1180 (*Beppo Levi, monotone convergence*).

**The proof of Lemma 1182 (*usage of adapted sequences*)**

cites explicitly:

Definition 1135 (*adapted sequence*),

Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),

Theorem 1180 (*Beppo Levi, monotone convergence*).

**The proof of Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*)**

cites explicitly:

Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),

Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*),

Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),

Definition 1135 (*adapted sequence*),

Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),

Lemma 1182 (*usage of adapted sequences*).

**The proof of Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*)**

cites explicitly:

- Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),
- Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**The proof of Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*)**

cites explicitly:

- Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*),
- Theorem 1180 (*Beppo Levi, monotone convergence*).

**The proof of Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*)**

cites explicitly:

- Lemma 532 (*decomposition into nonnegative and nonpositive parts*),
- Lemma 836 (*measurability of nonnegative and nonpositive parts*),
- Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),
- Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**The proof of Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*)**

cites explicitly:

- Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),
- Lemma 836 (*measurability of nonnegative and nonpositive parts*),
- Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),
- Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**The proof of Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*)**

cites explicitly:

- Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),
- Lemma 820 (*measurability of numeric function*),
- Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),
- Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),
- Lemma 948 (*negligibility of measurable subset*),
- Definition 953 (*property almost satisfied*),
- Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),
- Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).

**The proof of Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* )**

cites explicitly:

- Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),
- Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),
- Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 696 ( *$\sigma$ -algebra*),
- Definition 741 (*measurable space*),
- Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*),
- Lemma 913 (*measure*),
- Lemma 948 (*negligibility of measurable subset*),
- Definition 953 (*property almost satisfied*),
- Lemma 955 (*equivalent definition of property almost satisfied*),
- Lemma 985 (*compatibility of almost equality with operator*),
- Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),
- Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),
- Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*).

**The proof of Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality)**

cites explicitly:

Lemma 1172 (integral in  $\mathcal{M}_+$ ),Lemma 1189 (extended almost modus ponens for integral in  $\mathcal{M}_+$ ).**The proof of Lemma 1191 (integral in  $\mathcal{M}_+$  is almost monotone)**

cites explicitly:

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total),Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone),Lemma 1189 (extended almost modus ponens for integral in  $\mathcal{M}_+$ ).**The proof of Lemma 1192 (integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function)**

cites explicitly:

Definition 528 (nonnegative and nonpositive parts),

Definition 974 (almost binary relation),

Lemma 1188 (integral in  $\mathcal{M}_+$  is almost definite).**The proof of Lemma 1193 (integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions)**

cites explicitly:

Definition 528 (nonnegative and nonpositive parts),

Lemma 836 (measurability of nonnegative and nonpositive parts),

Lemma 985 (compatibility of almost equality with operator),

Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality).**The proof of Lemma 1194 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions)**

cites explicitly:

Lemma 836 (measurability of nonnegative and nonpositive parts),

Lemma 951 (subset of negligible is negligible),

Lemma 952 ( $\mathbf{N}$  is closed under countable union),

Definition 974 (almost binary relation),

Lemma 984 (compatibility of almost binary relation with operator),

Lemma 995 (compatibility of almost nonnegativeness with almost equality),

Definition 997 (summability domain),

Lemma 1000 (negligibility and summability domain),

Lemma 1002 (almost sum),

Lemma 1003 (compatibility of almost sum with almost equality),

Lemma 1187 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts),Lemma 1190 (compatibility of integral in  $\mathcal{M}_+$  with almost equality),Lemma 1192 (integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function),Lemma 1193 (integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions).**The proof of Lemma 1195 (generalized Bienaymé–Chebyshev inequality)**

cites explicitly:

Lemma 421 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)),

Lemma 757 (compatibility of measurability with composition),

Lemma 820 (measurability of numeric function),

Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions),Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ),Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous),Lemma 1178 (integral in  $\mathcal{M}_+$  is monotone).

**The proof of Lemma 1196 (*Bienaymé–Chebyshev inequality*)**

cites explicitly:

Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*).**The proof of Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\bar{\mathbb{R}}$* ),Lemma 343 (*absolute value in  $\bar{\mathbb{R}}$  is equal to nonnegative*),Lemma 424 (*finite-product property in  $\bar{\mathbb{R}}$  (measure theory)*),Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),Lemma 948 (*negligibility of measurable subset*),Definition 953 (*property almost satisfied*),Lemma 1172 (*integral in  $\mathcal{M}_+$* ),Lemma 1196 (*Bienaymé–Chebyshev inequality*).**The proof of Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*)**

cites explicitly:

Lemma 313 (*order in  $\bar{\mathbb{R}}$  is total*),Lemma 983 (*almost order is order relation*),Lemma 1172 (*integral in  $\mathcal{M}_+$* ),Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*).**The proof of Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*)**

cites explicitly:

Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*),Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*),Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),Theorem 1180 (*Beppo Levi, monotone convergence*).**The proof of Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*)**

cites explicitly:

Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).**The proof of Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*)**

cites explicitly:

Lemma 482 (*extrema of constant function*),Lemma 1165 (*integral in  $\mathcal{SF}_+$  over negligible subset*),Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).**The proof of Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*)**

cites explicitly:

Lemma 811 (*measurability of indicator function*),Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).**The proof of Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*)**

cites explicitly:

Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*),Lemma 1138 ( *$\mathcal{M}_+$  is closed under restriction*),Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).

**The proof of Lemma 1204** (*integrable in  $\mathcal{M}_+$  is closed under restriction*)

cites explicitly:

- Lemma 1138 ( *$\mathcal{M}_+$  is closed under restriction*),
- Lemma 1172 (*integral in  $\mathcal{M}_+$* ),
- Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),
- Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*).

**The proof of Lemma 1205** (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*)

cites explicitly:

- Definition 209 (*pseudopartition*),
- Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),
- Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),
- Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*).

**The proof of Lemma 1206** (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*)

cites explicitly:

- Lemma 525 (*masking is  $\sigma$ -additive*),
- Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1172 (*integral in  $\mathcal{M}_+$* ),
- Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*).

**The proof of Theorem 1208** (*Fatou's lemma*)

cites explicitly:

- Statement(s) from [20],
- Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*),
- Lemma 381 (*limit is monotone*),
- Lemma 488 (*infimum is monotone (set)*),
- Lemma 492 (*infimum of bounded is bounded*),
- Lemma 501 (*supremum is limit*),
- Lemma 504 (*limit inferior*),
- Definition 516 (*pointwise convergence*),
- Lemma 518 (*limit inferior bounded from below*),
- Lemma 828 ( *$\mathcal{M}$  is closed under infimum*),
- Lemma 830 ( *$\mathcal{M}$  is closed under limit inferior*),
- Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),
- Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),
- Theorem 1180 (*Beppo Levi, monotone convergence*).

**The proof of Lemma 1209** (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*)

cites explicitly:

- Statement(s) from [20],
- Lemma 517 (*limit inferior and limit superior of pointwise convergent*),
- Lemma 522 (*limit inferior, limit superior and pointwise convergence*),
- Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),
- Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),
- Theorem 1208 (*Fatou's lemma*).

**The proof of Lemma 1210** (*integral in  $\mathcal{M}_+$  with trivial measure*)

cites explicitly:

- Statement(s) from [20],
- Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*),
- Lemma 1182 (*usage of adapted sequences*).

**The proof of Lemma 1211** (*integral in  $\mathcal{M}_+$  with counting measure*)

cites explicitly:

Statement(s) from [20],  
 Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*),  
 Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*),  
 Lemma 1182 (*usage of adapted sequences*).

**The proof of Lemma 1212** (*integral in  $\mathcal{M}_+$  with counting measure on  $\mathbb{N}$* )  
 cites explicitly:  
 Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*).

**The proof of Lemma 1214** (*integral in  $\mathcal{M}_+$  with Dirac measure*)  
 cites explicitly:  
 Definition 939 (*Dirac measure*),  
 Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*).

**The proof of Lemma 1218** (*measure of section*)  
 cites explicitly:  
 Lemma 786 (*measurability of section*),  
 Lemma 913 (*measure*).

**The proof of Lemma 1219** (*measure of section of product*)  
 cites explicitly:  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 775 (*product of measurable subsets is measurable*),  
 Lemma 784 (*section of product*),  
 Lemma 913 (*measure*),  
 Lemma 1218 (*measure of section*).

**The proof of Lemma 1221** (*measurability of measure of section (finite)*)  
 cites explicitly:  
 Definition 227 (*product of subset systems*),  
 Definition 642 (*generated algebra of sets*),  
 Definition 664 (*monotone class*),  
 Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*),  
 Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*),  
 Lemma 738 (*usage of monotone class theorem*),  
 Definition 774 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 775 (*product of measurable subsets is measurable*),  
 Lemma 785 (*compatibility of section with set operations*),  
 Lemma 786 (*measurability of section*),  
 Lemma 787 (*countable union of sections is measurable*),  
 Lemma 788 (*countable intersection of sections is measurable*),  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 847 ( *$\mathcal{M}_+$  is closed under infimum*),  
 Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum*),  
 Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*),  
 Definition 857 (*finiteness*),  
 Lemma 858 (*finite is bounded*),  
 Definition 874 ( *$\sigma$ -additivity*),  
 Lemma 913 (*measure*),  
 Lemma 925 (*measure is continuous from below*),  
 Lemma 927 (*measure is continuous from above*),  
 Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 1218 (*measure of section*),  
 Lemma 1219 (*measure of section of product*).



**The proof of Lemma 1223 (measurability of measure of section)**

cites explicitly:

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Lemma 786 (measurability of section),

Lemma 848 ( $\mathcal{M}_+$  is closed under supremum),

Definition 857 (finiteness),

Lemma 886 (equivalent definition of  $\sigma$ -finiteness),

Lemma 925 (measure is continuous from below),

Lemma 931 (restricted measure),

Lemma 1218 (measure of section),

Lemma 1221 (measurability of measure of section (finite)).

**The proof of Lemma 1227 (candidate tensor product measure is tensor product measure)**

cites explicitly:

Lemma 418 (multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)),

Lemma 775 (product of measurable subsets is measurable),

Lemma 785 (compatibility of section with set operations),

Lemma 786 (measurability of section),

Lemma 787 (countable union of sections is measurable),

Definition 874 ( $\sigma$ -additivity),

Lemma 913 (measure),

Lemma 1172 (integral in  $\mathcal{M}_+$ ),

Lemma 1175 (integral in  $\mathcal{M}_+$  of indicator function),

Lemma 1177 (integral in  $\mathcal{M}_+$  of zero is zero),

Lemma 1184 (integral in  $\mathcal{M}_+$  is positive linear),

Lemma 1185 (integral in  $\mathcal{M}_+$  is  $\sigma$ -additive),

Lemma 1218 (measure of section),

Lemma 1219 (measure of section of product),

Lemma 1223 (measurability of measure of section),

Definition 1224 (tensor product measure),

Definition 1225 (candidate tensor product measure).

**The proof of Lemma 1228 (tensor product of finite measures)**

cites explicitly:

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Definition 857 (finiteness),

Lemma 913 (measure),

Definition 1224 (tensor product measure).

**The proof of Lemma 1229 (tensor product of  $\sigma$ -finite measures)**

cites explicitly:

Lemma 775 (product of measurable subsets is measurable),

Definition 885 ( $\sigma$ -finiteness),

Lemma 886 (equivalent definition of  $\sigma$ -finiteness),

Definition 1224 (tensor product measure).

**The proof of Lemma 1231 (uniqueness of tensor product measure (finite))**

cites explicitly:

Statement(s) from [20],

Definition 317 (addition in  $\overline{\mathbb{R}}$ ),

Definition 664 (monotone class),

Lemma 697 (equivalent definition of  $\sigma$ -algebra),

Lemma 719 ( $\sigma$ -algebra contains algebra of sets),

Lemma 730 (algebra of sets generated by product of  $\sigma$ -algebras),



Lemma 738 (*usage of monotone class theorem*),  
 Definition 774 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 775 (*product of measurable subsets is measurable*),  
 Lemma 858 (*finite is bounded*),  
 Definition 862 (*additivity*),  
 Lemma 887 (*finite is  $\sigma$ -finite*),  
 Lemma 925 (*measure is continuous from below*),  
 Lemma 926 (*equivalent definition of measure (from below)*),  
 Lemma 927 (*measure is continuous from above*),  
 Definition 1224 (*tensor product measure*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Lemma 1228 (*tensor product of finite measures*).

**The proof of Lemma 1233 (*uniqueness of tensor product measure*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 775 (*product of measurable subsets is measurable*),  
 Definition 857 (*finiteness*),  
 Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*),  
 Lemma 925 (*measure is continuous from below*),  
 Lemma 931 (*restricted measure*),  
 Lemma 1218 (*measure of section*),  
 Definition 1224 (*tensor product measure*),  
 Definition 1225 (*candidate tensor product measure*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Lemma 1229 (*tensor product of  $\sigma$ -finite measures*),  
 Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**The proof of Lemma 1234 (*negligibility of measurable section*)**

cites explicitly:

Lemma 786 (*measurability of section*),  
 Lemma 948 (*negligibility of measurable subset*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1233 (*uniqueness of tensor product measure*).

**The proof of Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* )**

cites explicitly:

Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),  
 Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*),  
 Lemma 1233 (*uniqueness of tensor product measure*).

**The proof of Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*)**

cites explicitly:

Lemma 1092 (*Lebesgue measure generalizes length of interval*),  
 Definition 1224 (*tensor product measure*),  
 Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**The proof of Lemma 1237 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*)**

cites explicitly:

Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**The proof of Lemma 1238 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*)**

cites explicitly:

Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*),

Lemma 1229 (*tensor product of  $\sigma$ -finite measures*),  
 Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**The proof of Lemma 1239 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*)**

cites explicitly:

Definition 856 (*diffuseness*),  
 Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**The proof of Theorem 1242 (*Tonelli*)**

cites explicitly:

Lemma 381 (*limit is monotone*),  
 Lemma 786 (*measurability of section*),  
 Lemma 789 (*indicator of section*),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),  
 Lemma 1100 (*indicator and support are inverse*),  
 Lemma 1124 ( *$\mathcal{SF}_+$  is measurable*),  
 Lemma 1128 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 1129 ( *$\mathcal{SF}_+$  is closed under positive algebra operations*),  
 Lemma 1134 (*Lebesgue scheme step 2*),  
 Lemma 1141 (*Lebesgue scheme step 3*),  
 Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 1218 (*measure of section*),  
 Lemma 1223 (*measurability of measure of section*),  
 Lemma 1233 (*uniqueness of tensor product measure*),  
 Definition 1240 (*partial function of function from product space*).

**The proof of Lemma 1243 (*Tonelli over subset*)**

cites explicitly:

Lemma 789 (*indicator of section*),  
 Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),  
 Definition 1240 (*partial function of function from product space*),  
 Theorem 1242 (*Tonelli*).

**The proof of Lemma 1244 (*Tonelli for tensor product*)**

cites explicitly:

Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*),  
 Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Definition 851 (*tensor product of numeric functions*),  
 Lemma 853 (*measurability of tensor product of numeric functions*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Definition 1240 (*partial function of function from product space*),  
 Theorem 1242 (*Tonelli*).

**The proof of Lemma 1248 (*integrable is measurable*)**

cites explicitly:

Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Definition 1247 (*integrability*).

**The proof of Lemma 1249 (*equivalent definition of integrability*)**

cites explicitly:

Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),  
 Definition 1247 (*integrability*),  
 Lemma 1248 (*integrable is measurable*).

**The proof of Lemma 1250 (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* )**  
 cites explicitly:

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),  
 Lemma 1249 (*equivalent definition of integrability*).

**The proof of Lemma 1251 (*integrable is almost finite*)**  
 cites explicitly:

Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
 Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),  
 Lemma 1249 (*equivalent definition of integrability*).

**The proof of Lemma 1252 (*almost bounded by integrable is integrable*)**  
 cites explicitly:

Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*),  
 Lemma 1249 (*equivalent definition of integrability*).

**The proof of Lemma 1253 (*integrability is closed under extension by zero*)**  
 cites explicitly:

Lemma 531 (*nonnegative and nonpositive parts are orthogonal*),  
 Lemma 532 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 534 (*compatibility of nonpositive and nonnegative parts with mask*),  
 Lemma 535 (*compatibility of nonpositive and nonnegative parts with restriction*),  
 Lemma 998 (*summability on summability domain*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),  
 Definition 1247 (*integrability*).

**The proof of Lemma 1254 (*integrability is closed under restriction*)**  
 cites explicitly:

Lemma 535 (*compatibility of nonpositive and nonnegative parts with restriction*),  
 Lemma 1204 (*integrable in  $\mathcal{M}_+$  is closed under restriction*),  
 Definition 1247 (*integrability*).

**The proof of Lemma 1255 (*integrability, masking is  $\sigma$ -additive*)**  
 cites explicitly:

Lemma 525 (*masking is  $\sigma$ -additive*),  
 Lemma 534 (*compatibility of nonpositive and nonnegative parts with mask*),  
 Lemma 1206 (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),  
 Definition 1247 (*integrability*).

**The proof of Lemma 1256 (*integral in  $\mathcal{M}$* )**  
 cites explicitly:

Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Definition 1247 (*integrability*).

**The proof of Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* )**  
 cites explicitly:

Definition 528 (*nonnegative and nonpositive parts*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Definition 1247 (*integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*)**

cites explicitly:

Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 1250 (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* ),  
 Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ).

**The proof of Lemma 1260 (*integral in  $\mathcal{M}$  of indicator function*)**

cites explicitly:

Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 1250 (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* ),  
 Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ).

**The proof of Lemma 1262 (*compatibility of integral with almost equality*)**

cites explicitly:

Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),  
 Definition 1247 (*integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* )**

cites explicitly:

Lemma 811 (*measurability of indicator function*),  
 Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),  
 Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*),  
 Lemma 1251 (*integrable is almost finite*),  
 Lemma 1262 (*compatibility of integral with almost equality*).

**The proof of Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* )**

cites explicitly:

Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 833 (*measurability and masking*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Lemma 1008 (*masking almost nowhere*),  
 Lemma 1251 (*integrable is almost finite*),  
 Lemma 1262 (*compatibility of integral with almost equality*).

**The proof of Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*)**

cites explicitly:

Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),  
 Definition 528 (*nonnegative and nonpositive parts*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1267 (*integral in  $\mathcal{M}$  of constant function*)**

cites explicitly:

Lemma 1260 (*integral in  $\mathcal{M}$  of indicator function*),  
 Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*).

**The proof of Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*)**  
 cites explicitly:

Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 1002 (*almost sum*),  
 Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*).

**The proof of Lemma 1269 (*integral in  $\mathcal{M}$  is additive*)**  
 cites explicitly:

Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 1002 (*almost sum*),  
 Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Lemma 1004 (*almost sum is sum*),  
 Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ),  
 Lemma 1262 (*compatibility of integral with almost equality*),  
 Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*).

**The proof of Lemma 1270 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*)**  
 cites explicitly:

Lemma 814 ( *$\mathcal{M}_{\mathbb{K}}$  is vector space*),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),  
 Lemma 1269 (*integral in  $\mathcal{M}$  is additive*).

**The proof of Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*)**  
 cites explicitly:

Statement(s) from [20],  
 Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
 Lemma 815 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 1004 (*almost sum is sum*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),  
 Lemma 1269 (*integral in  $\mathcal{M}$  is additive*).

**The proof of Lemma 1272 (*integral in  $\mathcal{M}$  is almost monotone*)**  
 cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),  
 Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*).

**The proof of Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*)**  
 cites explicitly:

Lemma 530 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),

Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),  
 Definition 1247 (*integrability*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1274 (*integral over subset*)**

cites explicitly:

Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),  
 Lemma 1253 (*integrability is closed under extension by zero*),  
 Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),  
 Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),  
 Lemma 1269 (*integral in  $\mathcal{M}$  is additive*).

**The proof of Lemma 1275 (*integral over subset is  $\sigma$ -additive*)**

cites explicitly:

Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),  
 Lemma 1255 (*integrability, masking is  $\sigma$ -additive*),  
 Lemma 1274 (*integral over subset*).

**The proof of Lemma 1276 (*integral over negligible subset*)**

cites explicitly:

Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*),  
 Lemma 1274 (*integral over subset*).

**The proof of Lemma 1277 (*integral over singleton*)**

cites explicitly:

Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 428 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 532 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1253 (*integrability is closed under extension by zero*),  
 Lemma 1274 (*integral over subset*).

**The proof of Lemma 1278 (*integral over interval*)**

cites explicitly:

Lemma 744 (*some Borel subsets*),  
 Definition 856 (*diffuseness*),  
 Lemma 1275 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 1277 (*integral over singleton*).

**The proof of Lemma 1279 (*Chasles relation, integral over split intervals*)**

cites explicitly:

Lemma 1275 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 1278 (*integral over interval*).

**The proof of Lemma 1280 (*integral with trivial measure*)**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1281 (*integral with counting measure*)**

cites explicitly:

- Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*),
- Lemma 1249 (*equivalent definition of integrability*),
- Lemma 1256 (*integral in  $\mathcal{M}$* ).

**The proof of Lemma 1282 (*integral with counting measure on  $\mathbb{N}$* )**

cites explicitly:

- Lemma 1281 (*integral with counting measure*).

**The proof of Lemma 1284 (*integral with Dirac measure*)**

cites explicitly:

- Definition 939 (*Dirac measure*),
- Lemma 1281 (*integral with counting measure*).

**The proof of Lemma 1286 (*p-seminorm*)**

cites explicitly:

- Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),
- Lemma 841 ( *$\mathcal{M}$  is closed under power function*),
- Lemma 1036 (*equivalent definition of nonnegative essential supremum*),
- Lemma 1172 (*integral in  $\mathcal{M}_+$* ).

**The proof of Lemma 1288 ( *$N_p$  with trivial measure*)**

cites explicitly:

- Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),
- Lemma 1037 (*nonnegative essential supremum with trivial measure*),
- Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*),
- Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1289 ( *$N_p$  of indicator function*)**

cites explicitly:

- Lemma 811 (*measurability of indicator function*),
- Lemma 933 (*trivial measure*),
- Lemma 1037 (*nonnegative essential supremum with trivial measure*),
- Lemma 1041 (*nonnegative essential supremum for extended simple functions*),
- Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),
- Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1290 ( *$N_p$  is nonnegative*)**

cites explicitly:

- Lemma 364 (*power function in  $\overline{\mathbb{R}}$  is nonnegative*),
- Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1291 (*link between  $N_1$  and  $N_p$* )**

cites explicitly:

- Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),
- Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*),
- Lemma 841 ( *$\mathcal{M}$  is closed under power function*),
- Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1292 (*integrable is finite  $N_1$* )**

cites explicitly:

- Lemma 1249 (*equivalent definition of integrability*),
- Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1293 ( *$N_\infty$  bounded is almost bounded*)**

cites explicitly:



Lemma 1039 (*nonnegative essential supremum is almost upper bound*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*),  
 Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1294 (*compatibility of  $N_p$  with power function*)**

cites explicitly:

Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),  
 Definition 405 (*multiplication in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 1040 (*nonnegative essential supremum of constant function*),  
 Lemma 1054 (*nonnegative essential supremum is compatible with power function*),  
 Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1295 (*compatibility of  $N_p$  with absolute value*)**

cites explicitly:

Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 357 (*exponentiation in  $\overline{\mathbb{R}}$  of power 1*),  
 Lemma 1294 (*compatibility of  $N_p$  with power function*).

**The proof of Lemma 1296 ( *$N_p$  is almost definite*)**

cites explicitly:

Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 1005 (*absolute value is almost definite*),  
 Lemma 1007 (*power function is almost definite*),  
 Lemma 1050 (*nonnegative essential supremum is almost definite*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1297 (*compatibility of  $N_p$  with almost equality*)**

cites explicitly:

Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*),  
 Lemma 1262 (*compatibility of integral with almost equality*),  
 Lemma 1286 (*p-seminorm*).

**The proof of Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*)**

cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 1045 (*nonnegative essential supremum is almost monotone*),  
 Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 1286 (*p-seminorm*),  
 Lemma 1291 (*link between  $N_1$  and  $N_p$* ),  
 Lemma 1292 (*integrable is finite  $N_1$* ),  
 Lemma 1295 (*compatibility of  $N_p$  with absolute value*).



**The proof of Lemma 1299 ( $N_p$  is almost monotone for almost nonnegative functions)**

cites explicitly:

- Lemma 960 (*countable almost properties is almost property*),
- Lemma 965 (*extended almost modus ponens (simplified 0)*),
- Lemma 1297 (*compatibility of  $N_p$  with almost equality*),
- Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*).

**The proof of Lemma 1300 ( $N_p$  is absolutely homogeneous)**

cites explicitly:

- Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),
- Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),
- Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),
- Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),
- Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),
- Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),
- Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),
- Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),
- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),
- Lemma 1286 ( *$p$ -seminorm*).

**The proof of Lemma 1301 ( $N_p$  of constant function)**

cites explicitly:

- Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ),
- Lemma 1289 ( *$N_p$  of indicator function*),
- Lemma 1300 ( *$N_p$  is absolutely homogeneous*).

**The proof of Lemma 1302 (generalized Hölder inequality (with  $\infty$ ))**

cites explicitly:

- Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),
- Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),
- Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*),
- Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),
- Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),
- Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),
- Lemma 444 (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*),
- Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 984 (*compatibility of almost binary relation with operator*),
- Lemma 1039 (*nonnegative essential supremum is almost upper bound*),
- Lemma 1053 (*nonnegative essential supremum is submultiplicative*),
- Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),
- Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*),
- Lemma 1286 ( *$p$ -seminorm*).

**The proof of Lemma 1304 (Hölder inequality)**

cites explicitly:

- Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),
- Definition 326 (*multiplication in  $\overline{\mathbb{R}}$* ),
- Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),
- Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),
- Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),

Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Definition 451 (*Hölder conjugates*),  
 Lemma 453 (*Young's inequality for products (measure theory)*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 1286 ( *$p$ -seminorm*),  
 Lemma 1290 ( *$N_p$  is nonnegative*),  
 Lemma 1296 ( *$N_p$  is almost definite*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*),  
 Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**The proof of Lemma 1305 (*generalized Hölder inequality*)**

cites explicitly:

Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),  
 Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*),  
 Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Definition 451 (*Hölder conjugates*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*),  
 Lemma 1291 (*link between  $N_1$  and  $N_p$* ),  
 Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),  
 Lemma 1304 (*Hölder inequality*).

**The proof of Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*)**

cites explicitly:

Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),  
 Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Definition 451 (*Hölder conjugates*),  
 Lemma 1040 (*nonnegative essential supremum of constant function*),  
 Lemma 1286 ( *$p$ -seminorm*),  
 Lemma 1290 ( *$N_p$  is nonnegative*),  
 Lemma 1291 (*link between  $N_1$  and  $N_p$* ).

**The proof of Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* )**

cites explicitly:

Definition 312 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),  
 Lemma 365 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition*),  
 Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 371 (*power function satisfies extended relaxed triangle inequality*),  
 Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),

Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),  
 Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 1002 (*almost sum*),  
 Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Lemma 1004 (*almost sum is sum*),  
 Lemma 1045 (*nonnegative essential supremum is almost monotone*),  
 Lemma 1051 (*nonnegative essential supremum is subadditive*),  
 Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),  
 Lemma 1286 ( *$p$ -seminorm*),  
 Lemma 1290 ( *$N_p$  is nonnegative*),  
 Lemma 1291 (*link between  $N_1$  and  $N_p$* ),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  
 Lemma 1304 (*Hölder inequality*),  
 Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*).

**The proof of Lemma 1310 (*Minkowski equality (nonnegative functions)*)**

cites explicitly:

Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),  
 Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*),  
 Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),  
 Lemma 1286 ( *$p$ -seminorm*),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*).

**The proof of Lemma 1311 (*Minkowski equality (linearly dependent functions)*)**

cites explicitly:

Definition 317 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 339 (*absolute value in  $\overline{\mathbb{R}}$* ),  
 Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),  
 Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*).

**The proof of Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*)**

cites explicitly:

Definition 415 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 833 (*measurability and masking*),  
 Lemma 955 (*equivalent definition of property almost satisfied*),  
 Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Lemma 1008 (*masking almost nowhere*),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  
 Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**The proof of Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 363 (finite power function in  $\overline{\mathbb{R}}$ ),

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 841 ( $\mathcal{M}$  is closed under power function),

Lemma 1286 ( $p$ -seminorm),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1317 ( $\mathcal{L}^p$  with trivial measure)**

cites explicitly:

Definition 312 (extended real numbers,  $\overline{\mathbb{R}}$ ),

Lemma 1288 ( $N_p$  with trivial measure),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ )**

cites explicitly:

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Definition 835 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1319 (integral is positive linear form on  $\mathcal{L}^1$ )**

cites explicitly:

Statement(s) from [20],

Lemma 1172 (integral in  $\mathcal{M}_+$ ),

Lemma 1256 (integral in  $\mathcal{M}$ ),

Lemma 1270 (integral in  $\mathcal{M}_{\mathbb{R}}$  is linear),

Lemma 1271 (integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone),

Lemma 1273 (compatibility of integral in  $\mathcal{M}$  with absolute value),

Lemma 1286 ( $p$ -seminorm),

Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ),

Lemma 1318 ( $\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$ ).

**The proof of Lemma 1320 (integral in  $\mathcal{L}^1$  is almost monotone)**

cites explicitly:

Lemma 1272 (integral in  $\mathcal{M}$  is almost monotone),

Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ).

**The proof of Lemma 1321 (link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 366 (compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication),

Lemma 1291 (link between  $N_1$  and  $N_p$ ),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions),

Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ).

**The proof of Lemma 1322 ( $\mathcal{L}^p$  is closed under absolute value)**

cites explicitly:

Lemma 842 ( $\mathcal{M}$  is closed under absolute value),

Lemma 1295 (compatibility of  $N_p$  with absolute value),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1323 (bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 313 (order in  $\overline{\mathbb{R}}$  is total),

Lemma 341 (finite absolute value in  $\overline{\mathbb{R}}$ ),

Lemma 342 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative),

Lemma 343 (absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative),

Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 958 (*everywhere is almost everywhere*),  
 Lemma 1045 (*nonnegative essential supremum is almost monotone*),  
 Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 1286 (*p-seminorm*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*).

**The proof of Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* )**

cites explicitly:

Lemma 363 (*finite power function in  $\overline{\mathbb{R}}$* ),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),  
 Lemma 1008 (*masking almost nowhere*),  
 Lemma 1048 (*essentially bounded is almost finite*),  
 Lemma 1251 (*integrable is almost finite*),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*).

**The proof of Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* )**

cites explicitly:

Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),  
 Lemma 1286 (*p-seminorm*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*).

**The proof of Lemma 1327 ( *$\mathcal{L}^p$  is closed under scalar multiplication*)**

cites explicitly:

Lemma 814 ( *$\mathcal{M}_{\mathbb{K}}$  is vector space*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*).

**The proof of Lemma 1328 (*constant function is  $\mathcal{L}^p$* )**

cites explicitly:

Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 1267 (*integral in  $\mathcal{M}$  of constant function*),  
 Lemma 1301 ( *$N_p$  of constant function*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*).

**The proof of Lemma 1329 (*first mean value inequality*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 913 (*measure*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),  
 Lemma 1274 (*integral over subset*),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1328 (*constant function is  $\mathcal{L}^p$* ).

**The proof of Lemma 1331 (first mean value theorem)**

cites explicitly:

Definition 229 (*interval*),

Lemma 913 (*measure*),

Lemma 1329 (*first mean value inequality*).

**The proof of Lemma 1334 (generalized Hölder inequality in  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra),

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 1305 (*generalized Hölder inequality*),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1335 (interpolation inequality in  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),

Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 435 (*finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 455 (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* ),

Lemma 841 ( $\mathcal{M}$  is closed under power function),

Lemma 1288 ( $N_p$  with trivial measure),

Lemma 1294 (*compatibility of  $N_p$  with power function*),

Lemma 1295 (*compatibility of  $N_p$  with absolute value*),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions),

Lemma 1317 ( $\mathcal{L}^p$  with trivial measure),

Lemma 1334 (*generalized Hölder inequality in  $\mathcal{L}^p$* ).

**The proof of Lemma 1336 (Minkowski inequality in  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space),

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions).

**The proof of Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space)**

cites explicitly:

Statement(s) from [20],

Definition 288 (*seminorm*),

Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space),

Lemma 1300 ( $N_p$  is absolutely homogeneous),

Definition 1314 ( $\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions),

Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication),

Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ).

**The proof of Lemma 1339 (convergence in  $\mathcal{L}^p$ )**

cites explicitly:

Lemma 1286 ( $p$ -seminorm),

Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),

Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space).

**The proof of Lemma 1340 (equivalent definition of convergence in  $\mathcal{L}^p$ )**

cites explicitly:

Statement(s) from [20],

Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 1286 (*p-seminorm*),  
 Lemma 1339 (*convergence in  $\mathcal{L}^p$* ).

**The proof of Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 264 (*pseudo-point separation*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1296 ( *$N_p$  is almost definite*),  
 Lemma 1327 ( *$\mathcal{L}^p$  is closed under scalar multiplication*),  
 Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ),  
 Lemma 1339 (*convergence in  $\mathcal{L}^p$* ).

**The proof of Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* )**

cites explicitly:

Statement(s) from [20],  
 Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*),  
 Lemma 501 (*supremum is limit*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 994 (*transitivity of almost inequality and equality*),  
 Lemma 1006 (*absolute value is almost equal to almost nonnegative function*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1262 (*compatibility of integral with almost equality*),  
 Lemma 1286 (*p-seminorm*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),  
 Lemma 1339 (*convergence in  $\mathcal{L}^p$* ).

**The proof of Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* )**

cites explicitly:

Statement(s) from [20],  
 Lemma 519 (*limit inferior bounded from above*),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 833 (*measurability and masking*),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1008 (*masking almost nowhere*),  
 Theorem 1208 (*Fatou's lemma*),  
 Lemma 1262 (*compatibility of integral with almost equality*),  
 Lemma 1290 ( *$N_p$  is nonnegative*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ).

**The proof of Theorem 1347 (*Lebesgue, dominated convergence*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),



Lemma 381 (*limit is monotone*),  
 Lemma 389 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Definition 516 (*pointwise convergence*),  
 Lemma 517 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 518 (*limit inferior bounded from below*),  
 Lemma 522 (*limit inferior, limit superior and pointwise convergence*),  
 Lemma 832 ( *$\mathcal{M}$  is closed under limit when pointwise convergent*),  
 Theorem 1208 (*Fatou's lemma*),  
 Lemma 1286 (*p-seminorm*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),  
 Lemma 1339 (*convergence in  $\mathcal{L}^p$* ).

**The proof of Theorem 1349 (*Lebesgue, extended dominated convergence*)**

cites explicitly:

Statement(s) from [20],  
 Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 517 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 833 (*measurability and masking*),  
 Lemma 948 (*negligibility of measurable subset*),  
 Lemma 951 (*subset of negligible is negligible*),  
 Definition 953 (*property almost satisfied*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Definition 974 (*almost binary relation*),  
 Lemma 1008 (*masking almost nowhere*),  
 Lemma 1009 (*finite nonnegative part*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1251 (*integrable is almost finite*),  
 Lemma 1262 (*compatibility of integral with almost equality*),  
 Lemma 1286 (*p-seminorm*),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of p-powered integrable functions*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),  
 Theorem 1347 (*Lebesgue, dominated convergence*).

**The proof of Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* )**

cites explicitly:

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),  
 Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 371 (*power function satisfies extended relaxed triangle inequality*),  
 Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 754 (*constant function is measurable*),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),



Lemma 962 (*almost everywhere for almost the same*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 1012 (*compatibility of almost limit with power function*),  
 Lemma 1014 (*compatibility of almost limit with almost equality*),  
 Lemma 1015 (*compatibility of almost limit with almost inequality*),  
 Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),  
 Lemma 1299 ( *$N_p$  is almost monotone for almost nonnegative functions*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),  
 Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*).

**The proof of Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* )**  
 cites explicitly:

Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 815 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ),  
 Definition 835 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 1011 (*compatibility of almost limit with continuous function*),  
 Lemma 1012 (*compatibility of almost limit with power function*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),  
 Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**The proof of Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* )**  
 cites explicitly:

Statement(s) from [20],  
 Lemma 268 (*equivalent definition of Cauchy sequence (discrete criterion)*),  
 Lemma 269 (*convergent subsequence of Cauchy sequence*),  
 Lemma 493 (*supremum of bounded is bounded*),  
 Definition 953 (*property almost satisfied*),  
 Lemma 960 (*countable almost properties is almost property*),  
 Lemma 1001 (*absolute value and almost inequality*),  
 Lemma 1012 (*compatibility of almost limit with power function*),  
 Lemma 1015 (*compatibility of almost limit with almost inequality*),  
 Lemma 1016 (*compatibility of almost limit with measurability*),  
 Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),  
 Lemma 1293 ( *$N_{\infty}$  bounded is almost bounded*),  
 Lemma 1295 (*compatibility of  $N_p$  with absolute value*),  
 Definition 1314 ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),

Lemma 1339 (*convergence in  $\mathcal{L}^p$* ),  
Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  
Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),  
Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**The proof of Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* )**

cites explicitly:

Statement(s) from [20],

Lemma 1014 (*compatibility of almost limit with almost equality*),

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**The proof of Lemma 1356 (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* )**

cites explicitly:

Lemma 1010 (*compatibility of almost limit with subsequence*),

Lemma 1013 (*almost limits are almost equal*),

Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).



## Appendix C

# Is explicitly cited in the proof of...

This appendix gathers the explicit citations that appear in the proof of results (lemmas and theorems) for each statement listed in Appendix A. Statements from [20] are anonymized.

Printing is not advised!

### Statement(s) from [20]

are explicitly cited in the proof of:

- Lemma 231 (*equivalent definition of interval*),
- Lemma 265 (*equivalent definition of convergent sequence*),
- Lemma 266 (*compatibility of subsequence with convergence*),
- Lemma 268 (*equivalent definition of Cauchy sequence (discrete criterion)*),
- Lemma 269 (*convergent subsequence of Cauchy sequence*),
- Lemma 272 (*quotient vector space, equivalence relation*),
- Lemma 273 (*quotient vector space*),
- Lemma 274 (*linear map on quotient vector space*),
- Lemma 282 (*algebra of functions to algebra*),
- Lemma 286 (*vector subspace and closed under multiplication is subalgebra*),
- Lemma 287 (*closed under algebra operations is subalgebra*),
- Lemma 290 (*seminorm is homogeneous*),
- Lemma 291 (*seminorm is nonnegative*),
- Lemma 292 (*seminormed vector space is pseudometric space*),
- Lemma 293 (*definite seminorm is norm*),
- Lemma 298 (*definite semi-inner product is inner product*),
- Lemma 377 (*metric on  $\overline{\mathbb{R}}$* ),
- Lemma 381 (*limit is monotone*),
- Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*),
- Lemma 399 (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition*),
- Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ),
- Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*),
- Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Lemma 464 (*open intervals with rational bounds cover open interval*),
- Lemma 482 (*extrema of constant function*),
- Lemma 483 (*compatibility of extrema with affine transformation*),
- Lemma 484 (*equivalent definition of finite infimum*),
- Lemma 485 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ),
- Lemma 486 (*equivalent definition of infimum*),
- Lemma 487 (*infimum is smaller than supremum*),
- Lemma 488 (*infimum is monotone (set)*),
- Lemma 489 (*supremum is monotone (set)*),

- Lemma 490 (*infimum is monotone (function)*),  
 Lemma 491 (*supremum is monotone (function)*),  
 Lemma 494 (*compatibility of infimum with absolute value*),  
 Lemma 495 (*compatibility of supremum with absolute value*),  
 Lemma 498 (*infimum of nonincreasing sequence*),  
 Lemma 499 (*supremum of nondecreasing sequence*),  
 Lemma 500 (*infimum is limit*),  
 Lemma 501 (*supremum is limit*),  
 Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),  
 Lemma 503 (*constant sum of monotone nonnegative functions*),  
 Lemma 505 (*limit inferior is  $\infty$* ),  
 Lemma 506 (*equivalent definition of the limit inferior*),  
 Lemma 510 (*duality limit inferior-limit superior*),  
 Lemma 518 (*limit inferior bounded from below*),  
 Lemma 519 (*limit inferior bounded from above*),  
 Lemma 522 (*limit inferior, limit superior and pointwise convergence*),  
 Lemma 529 (*equivalent definition of nonnegative and nonpositive parts*),  
 Lemma 530 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 531 (*nonnegative and nonpositive parts are orthogonal*),  
 Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space),  
 Lemma 828 ( $\mathcal{M}$  is closed under infimum),  
 Lemma 829 ( $\mathcal{M}$  is closed under supremum),  
 Lemma 883 (*continuity from above*),  
 Lemma 907 (*compatibility of null premeasure with countable union*),  
 Lemma 1011 (*compatibility of almost limit with continuous function*),  
 Lemma 1015 (*compatibility of almost limit with almost inequality*),  
 Lemma 1016 (*compatibility of almost limit with measurability*),  
 Lemma 1020 (*essential supremum is bounded by supremum*),  
 Lemma 1021 (*essential supremum is minimum*),  
 Lemma 1036 (*equivalent definition of nonnegative essential supremum*),  
 Lemma 1066 ( $\lambda^*$  is homogeneous),  
 Lemma 1067 ( $\lambda^*$  is monotone),  
 Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive),  
 Lemma 1069 ( $\lambda^*$  generalizes length of interval),  
 Lemma 1083 (*rays are Lebesgue-measurable*),  
 Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Lemma 1111 ( $\mathcal{SF}_E$  canonical representation),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),  
 Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*),  
 Lemma 1162 (*integral in  $\mathcal{SF}_+$  is continuous*),  
 Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Theorem 1208 (*Fatou's lemma*),  
 Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
 Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*),  
 Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*),  
 Lemma 1231 (*uniqueness of tensor product measure (finite)*),  
 Lemma 1233 (*uniqueness of tensor product measure*),  
 Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),  
 Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1329 (*first mean value inequality*),  
 Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space),

Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  
 Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),  
 Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ),  
 Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).

**Definition 209 (*pseudopartition*)**

is explicitly cited in the proof of:

Lemma 211 (*compatibility of pseudopartition with intersection*),  
 Lemma 399 (*series in  $\mathbb{R}_+$  are compatible with pseudopartition*),  
 Lemma 525 (*masking is  $\sigma$ -additive*),  
 Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 1024 (*essential supremum of extended simple functions*),  
 Lemma 1026 (*essential supremum of infinite times function*),  
 Lemma 1050 (*nonnegative essential supremum is almost definite*),  
 Lemma 1107 ( *$\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive*),  
 Lemma 1117 ( *$\mathcal{SF}_E$  is vector space*),  
 Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*).

**Lemma 211 (*compatibility of pseudopartition with intersection*)**

is explicitly cited in the proof of:

Lemma 766 (*generating on a partition*),  
 Lemma 863 (*decomposition over finite pseudopartition*),  
 Lemma 876 (*decomposition over countable pseudopartition*),  
 Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),  
 Lemma 1078 ( *$\lambda^*$  is additive on  $\mathcal{L}$* ).

**Lemma 212 (*countable union is monotone*)**

is explicitly cited in the proof of:

Lemma 213 (*equality of countable union*).

**Lemma 213 (*equality of countable union*)**

is explicitly cited in the proof of:

Lemma 215 (*disjoint union of union*),  
 Lemma 217 (*monotone union of union*).

**Lemma 215 (*disjoint union of union*)**

is explicitly cited in the proof of:

Lemma 568 (*closedness under countable disjoint and monotone union*),  
 Lemma 570 (*closedness under countable disjoint union and countable union*),  
 Lemma 572 (*disjoint union of union in subset system*),  
 Lemma 574 (*disjoint union of union in subset system (variant)*).

**Lemma 217 (*monotone union of union*)**

is explicitly cited in the proof of:

Lemma 569 (*closedness under countable monotone and disjoint union*),  
 Lemma 575 (*monotone union of union in subset system*).

**Lemma 219 (*empty monotone intersection of union*)**

is explicitly cited in the proof of:

Lemma 576 (*empty monotone intersection of union in subset system*).

**Lemma 221** (*monotone union of monotone intersection*)

is explicitly cited in the proof of:

Lemma 577 (*monotone union of monotone intersection in subset system*).

**Lemma 222** (*technical inclusion for countable union*)

is explicitly cited in the proof of:

Lemma 223 (*order is meaningless in countable union*).

**Lemma 223** (*order is meaningless in countable union*)

is explicitly cited in the proof of:

Lemma 224 (*definition of double countable union*).

**Lemma 224** (*definition of double countable union*)

is explicitly cited in the proof of:

Lemma 225 (*double countable union*),

Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive).

**Lemma 225** (*double countable union*)

is explicitly cited in the proof of:

Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive).

**Definition 226** (*trace of subset system*)

is explicitly cited in the proof of:

Lemma 238 (*trace interval on a subset*),

Lemma 252 (*trace topology on subset*),

Lemma 373 (*trace topology on subset of  $\mathbb{R}$* ),

Lemma 386 (*trace topology on subset of  $\mathbb{R}^m$* ),

Lemma 763 (*trace  $\sigma$ -algebra*),

Lemma 764 (*measurability of measurable subspace*),

Lemma 765 (*generating measurable subspace*),

Lemma 766 (*generating on a partition*),

Lemma 770 (*source restriction of measurable function*),

Lemma 802 (*Borel subsets of  $\mathbb{R}$* ).

**Definition 227** (*product of subset systems*)

is explicitly cited in the proof of:

Lemma 254 (*box topology on Cartesian product*),

Lemma 261 (*compatibility of second-countability with Cartesian product*),

Lemma 310 (*topology on  $\mathbb{R}^m$* ),

Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*),

Lemma 775 (*product of measurable subsets is measurable*),

Lemma 776 (*measurability of function to product space*),

Lemma 779 (*generating product measurable space*),

Lemma 786 (*measurability of section*),

Lemma 1221 (*measurability of measure of section (finite)*).

**Definition 229** (*interval*)

is explicitly cited in the proof of:

Lemma 231 (*equivalent definition of interval*),

Lemma 232 (*empty open interval*),

Lemma 235 (*intervals are closed under finite intersection*),

Lemma 238 (*trace interval on a subset*),

Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*),

Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ),

Lemma 1331 (*first mean value theorem*).

**Lemma 231** (*equivalent definition of interval*)

is not yet used.

**Lemma 232** (*empty open interval*)

is explicitly cited in the proof of:

Lemma 236 (*empty intersection of open intervals*).

**Lemma 235** (*intervals are closed under finite intersection*)

is explicitly cited in the proof of:

Lemma 236 (*empty intersection of open intervals*),

Lemma 258 (*topological basis of order topology*),

Lemma 1083 (*rays are Lebesgue-measurable*),

Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 236** (*empty intersection of open intervals*)

is not yet used.

**Lemma 238** (*trace interval on a subset*)

is explicitly cited in the proof of:

Lemma 259 (*trace of order topology*).

**Definition 240** (*topological space*)

is explicitly cited in the proof of:

Lemma 241 (*intersection of topologies*),

Lemma 244 (*equivalent definition of generated topology*),

Lemma 247 (*equivalent definition of topological basis*),

Lemma 252 (*trace topology on subset*),

Lemma 254 (*box topology on Cartesian product*),

Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*),

Lemma 464 (*open intervals with rational bounds cover open interval*),

Lemma 744 (*some Borel subsets*).

**Lemma 241** (*intersection of topologies*)

is explicitly cited in the proof of:

Lemma 243 (*generated topology is minimum*).

**Definition 242** (*generated topology*)

is explicitly cited in the proof of:

Lemma 243 (*generated topology is minimum*).

**Lemma 243** (*generated topology is minimum*)

is explicitly cited in the proof of:

Lemma 244 (*equivalent definition of generated topology*),

Lemma 245 (*topology generation is monotone*),

Lemma 247 (*equivalent definition of topological basis*).

**Lemma 244** (*equivalent definition of generated topology*)

is explicitly cited in the proof of:

Lemma 247 (*equivalent definition of topological basis*),

Lemma 249 (*topological basis is generator*),

Lemma 258 (*topological basis of order topology*).

**Lemma 245** (*topology generation is monotone*)

is not yet used.

**Definition 246** (*topological basis*)

is explicitly cited in the proof of:

Lemma 247 (*equivalent definition of topological basis*),



Lemma 248 (*augmented topological basis*),  
 Lemma 249 (*topological basis is generator*),  
 Lemma 250 (*topological subbasis*),  
 Lemma 252 (*trace topology on subset*),  
 Lemma 254 (*box topology on Cartesian product*),  
 Lemma 258 (*topological basis of order topology*),  
 Lemma 476 (*open subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 247** (*equivalent definition of topological basis*)

is explicitly cited in the proof of:

Lemma 254 (*box topology on Cartesian product*).

**Lemma 248** (*augmented topological basis*)

is not yet used.

**Lemma 249** (*topological basis is generator*)

is explicitly cited in the proof of:

Lemma 250 (*topological subbasis*),

Lemma 307 (*topology on  $\mathbb{R}$* ),

Lemma 378 (*equivalent topologies on  $\overline{\mathbb{R}}$* ).

**Lemma 250** (*topological subbasis*)

is explicitly cited in the proof of:

Lemma 310 (*topology on  $\mathbb{R}^m$* ),

Theorem 465 ( *$\mathbb{R}$  is second-countable*),

Lemma 479 ( *$\mathbb{R}$  is second-countable*).

**Lemma 252** (*trace topology on subset*)

is explicitly cited in the proof of:

Lemma 259 (*trace of order topology*),

Lemma 308 (*trace topology on subset of  $\mathbb{R}$* ),

Lemma 311 (*trace topology on subset of  $\mathbb{R}^m$* ),

Lemma 373 (*trace topology on subset of  $\overline{\mathbb{R}}$* ),

Lemma 386 (*trace topology on subset of  $\overline{\mathbb{R}^m}$* ),

Lemma 768 (*Borel sub- $\sigma$ -algebra*),

Lemma 780 (*Borel subsets in product measurable subspace*).

**Lemma 254** (*box topology on Cartesian product*)

is explicitly cited in the proof of:

Lemma 261 (*compatibility of second-countability with Cartesian product*),

Lemma 310 (*topology on  $\mathbb{R}^m$* ),

Lemma 385 (*topology on  $\overline{\mathbb{R}^m}$* ),

Lemma 780 (*Borel subsets in product measurable subspace*).

**Definition 256** (*order topology*)

is explicitly cited in the proof of:

Lemma 258 (*topological basis of order topology*),

Lemma 372 (*topology on  $\mathbb{R}$* ).

**Lemma 258** (*topological basis of order topology*)

is explicitly cited in the proof of:

Lemma 259 (*trace of order topology*),

Lemma 307 (*topology on  $\mathbb{R}$* ),

Lemma 372 (*topology on  $\overline{\mathbb{R}}$* ).

**Lemma 259** (*trace of order topology*)

is explicitly cited in the proof of:

Lemma 373 (*trace topology on subset of  $\overline{\mathbb{R}}$* ).

**Definition 260** (*second-countability*)

is explicitly cited in the proof of:

Lemma 261 (*compatibility of second-countability with Cartesian product*),

Theorem 465 ( $\mathbb{R}$  is second-countable),

Lemma 479 ( $\overline{\mathbb{R}}$  is second-countable).

**Lemma 261** (*compatibility of second-countability with Cartesian product*)

is explicitly cited in the proof of:

Lemma 466 ( $\mathbb{R}^m$  is second-countable),

Lemma 480 ( $\overline{\mathbb{R}^m}$  is second-countable).

**Definition 262** (*pseudometric*)

is explicitly cited in the proof of:

Lemma 264 (*pseudo-point separation*),

Lemma 269 (*convergent subsequence of Cauchy sequence*),

Lemma 292 (*seminormed vector space is pseudometric space*).

**Lemma 264** (*pseudo-point separation*)

is explicitly cited in the proof of:

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*).

**Lemma 265** (*equivalent definition of convergent sequence*)

is explicitly cited in the proof of:

Lemma 268 (*equivalent definition of Cauchy sequence (discrete criterion)*),

Lemma 484 (*equivalent definition of finite infimum*).

**Lemma 266** (*compatibility of subsequence with convergence*)

is explicitly cited in the proof of:

Lemma 1010 (*compatibility of almost limit with subsequence*).

**Definition 267** (*cluster point*)

is explicitly cited in the proof of:

Lemma 382 (*cluster point at  $\pm\infty$  (discrete)*),

Lemma 506 (*equivalent definition of the limit inferior*),

Lemma 507 (*limit inferior is invariant by translation*).

**Lemma 268** (*equivalent definition of Cauchy sequence (discrete criterion)*)

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 269** (*convergent subsequence of Cauchy sequence*)

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Definition 270** (*relation compatible with vector operations*)

is explicitly cited in the proof of:

Lemma 271 (*quotient vector operations*),

Lemma 273 (*quotient vector space*).

**Lemma 271** (*quotient vector operations*)

is explicitly cited in the proof of:

Lemma 272 (*quotient vector space, equivalence relation*),

Lemma 274 (*linear map on quotient vector space*).

**Lemma 272** (*quotient vector space, equivalence relation*)

is explicitly cited in the proof of:

Lemma 273 (*quotient vector space*).

**Lemma 273** (*quotient vector space*)

is explicitly cited in the proof of:

Lemma 274 (*linear map on quotient vector space*).

**Lemma 274** (*linear map on quotient vector space*)

is not yet used.

**Definition 277** (*algebra over a field*)

is explicitly cited in the proof of:

Lemma 279 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra),

Lemma 282 (*algebra of functions to algebra*),

Lemma 286 (*vector subspace and closed under multiplication is subalgebra*),

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space),

Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),

Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*).

**Lemma 279** ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra)

is explicitly cited in the proof of:

Lemma 283 ( $\mathbb{K}^X$  is algebra),

Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra).

**Definition 280** (*inherited algebra operations*)

is explicitly cited in the proof of:

Lemma 282 (*algebra of functions to algebra*).

**Lemma 282** (*algebra of functions to algebra*)

is explicitly cited in the proof of:

Lemma 283 ( $\mathbb{K}^X$  is algebra),

Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra).

**Lemma 283** ( $\mathbb{K}^X$  is algebra)

is not yet used.

**Definition 284** (*subalgebra*)

is explicitly cited in the proof of:

Lemma 286 (*vector subspace and closed under multiplication is subalgebra*),

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space).

**Lemma 286** (*vector subspace and closed under multiplication is subalgebra*)

is explicitly cited in the proof of:

Lemma 287 (*closed under algebra operations is subalgebra*),

Lemma 1117 ( $\mathcal{SF}_E$  is vector space).

**Lemma 287** (*closed under algebra operations is subalgebra*)

is explicitly cited in the proof of:

Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  is algebra).

**Definition 288** (*seminorm*)

is explicitly cited in the proof of:

Lemma 290 (*seminorm is homogeneous*),

Lemma 291 (*seminorm is nonnegative*),

Lemma 292 (*seminormed vector space is pseudometric space*),

Lemma 293 (*definite seminorm is norm*),

Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space).

**Lemma 290** (*seminorm is homogeneous*)

is explicitly cited in the proof of:

Lemma 291 (*seminorm is nonnegative*),

Lemma 292 (*seminormed vector space is pseudometric space*).

**Lemma 291** (*seminorm is nonnegative*)

is explicitly cited in the proof of:

Lemma 292 (*seminormed vector space is pseudometric space*).

**Lemma 292** (*seminormed vector space is pseudometric space*)

is not yet used.

**Lemma 293** (*definite seminorm is norm*)

is not yet used.

**Definition 295** (*Banach space*)

is not yet used.

**Definition 296** (*semi-inner product*)

is explicitly cited in the proof of:

Lemma 298 (*definite semi-inner product is inner product*).

**Lemma 298** (*definite semi-inner product is inner product*)

is not yet used.

**Lemma 299** (*finite cover of compact interval*)

is explicitly cited in the proof of:

Lemma 1069 ( $\lambda^*$  generalizes length of interval).

**Lemma 300** (*power function satisfies relaxed triangle inequality*)

is explicitly cited in the proof of:

Lemma 371 (*power function satisfies extended relaxed triangle inequality*).

**Definition 302** (*Hölder conjugates in  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 303 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),

Lemma 304 (*Young's inequality for products in  $\mathbb{R}$* ).

**Lemma 303** (*2 is self-Hölder conjugate in  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 305 (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* ),

Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*).

**Lemma 304** (*Young's inequality for products in  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 305 (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* ),

Lemma 453 (*Young's inequality for products (measure theory)*).

**Lemma 305** (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* )

is not yet used.

**Lemma 307** (*topology on  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 308 (*trace topology on subset of  $\mathbb{R}$* ),

Lemma 310 (*topology on  $\mathbb{R}^m$* ),

Lemma 373 (*trace topology on subset of  $\overline{\mathbb{R}}$* ),

Theorem 465 ( *$\mathbb{R}$  is second-countable*).

**Lemma 308** (*trace topology on subset of  $\mathbb{R}$* )

is not yet used.

**Lemma 310** (*topology on  $\mathbb{R}^m$* )

is explicitly cited in the proof of:

- Lemma 311 (*trace topology on subset of  $\mathbb{R}^m$* ),
- Lemma 386 (*trace topology on subset of  $\overline{\mathbb{R}^m}$* ),
- Lemma 466 ( *$\mathbb{R}^m$  is second-countable*).

**Lemma 311** (*trace topology on subset of  $\mathbb{R}^m$* )

is not yet used.

**Definition 312** (*extended real numbers,  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 313 (*order in  $\overline{\mathbb{R}}$  is total*),
- Lemma 314 ( *$\overline{\mathbb{R}}$  is complete*),
- Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*),
- Lemma 324 (*additive inverse in  $\overline{\mathbb{R}}$  is involution*),
- Lemma 325 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),
- Lemma 333 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*),
- Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),
- Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*),
- Lemma 372 (*topology on  $\overline{\mathbb{R}}$* ),
- Lemma 381 (*limit is monotone*),
- Lemma 485 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ),
- Lemma 486 (*equivalent definition of infimum*),
- Lemma 500 (*infimum is limit*),
- Lemma 505 (*limit inferior is  $\infty$* ),
- Lemma 506 (*equivalent definition of the limit inferior*),
- Lemma 527 (*finite part is finite*),
- Lemma 531 (*nonnegative and nonpositive parts are orthogonal*),
- Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 935 (*counting measure*),
- Lemma 996 (*almost finite*),
- Lemma 1017 (*essential supremum*),
- Lemma 1030 (*essential supremum bounded by constant*),
- Lemma 1046 (*nonnegative essential supremum bounded by constant*),
- Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),
- Lemma 1280 (*integral with trivial measure*),
- Lemma 1304 (*Hölder inequality*),
- Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),
- Lemma 1317 ( *$\mathcal{L}^p$  with trivial measure*).

**Lemma 313** (*order in  $\overline{\mathbb{R}}$  is total*)

is explicitly cited in the proof of:

- Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*),
- Lemma 485 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ),
- Lemma 486 (*equivalent definition of infimum*),
- Lemma 487 (*infimum is smaller than supremum*),
- Lemma 488 (*infimum is monotone (set)*),
- Lemma 490 (*infimum is monotone (function)*),
- Lemma 512 (*limit inferior is smaller than limit superior*),
- Lemma 1001 (*absolute value and almost inequality*),
- Lemma 1045 (*nonnegative essential supremum is almost monotone*),
- Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*),

Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*),  
 Lemma 1272 (*integral in  $\mathcal{M}$  is almost monotone*),  
 Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 314** ( $\overline{\mathbb{R}}$  is complete)

is explicitly cited in the proof of:

Lemma 392 ( $\mathbb{R}_+$  is complete),  
 Lemma 504 (*limit inferior*),  
 Lemma 509 (*limit superior*),  
 Lemma 1017 (*essential supremum*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Theorem 1208 (*Fatou's lemma*).

**Definition 317** (addition in  $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:

Lemma 318 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 319 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*),  
 Lemma 320 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*),  
 Lemma 321 (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*),  
 Lemma 322 (*infinity-sum property in  $\overline{\mathbb{R}}$* ),  
 Lemma 323 (*finite-sum property in  $\overline{\mathbb{R}}$* ),  
 Lemma 324 (*additive inverse in  $\overline{\mathbb{R}}$  is involution*),  
 Lemma 325 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),  
 Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),  
 Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*),  
 Lemma 371 (*power function satisfies extended relaxed triangle inequality*),  
 Lemma 387 (*addition in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 393 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),  
 Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),  
 Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),  
 Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*),  
 Lemma 453 (*Young's inequality for products (measure theory)*),  
 Lemma 532 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 866 (*subtractivity*),  
 Lemma 998 (*summability on summability domain*),  
 Lemma 1002 (*almost sum*),  
 Lemma 1012 (*compatibility of almost limit with power function*),  
 Lemma 1024 (*essential supremum of extended simple functions*),  
 Lemma 1051 (*nonnegative essential supremum is subadditive*),  
 Lemma 1068 ( $\lambda^*$  is  $\sigma$ -subadditive),  
 Lemma 1231 (*uniqueness of tensor product measure (finite)*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**Lemma 318** (zero is identity element for addition in  $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:

Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),  
 Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*),

Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 319** (*addition in  $\overline{\mathbb{R}}$  is associative when defined*)

is explicitly cited in the proof of:  
 Lemma 394 (*addition in  $\overline{\mathbb{R}}_+$  is associative*).

**Lemma 320** (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*)

is explicitly cited in the proof of:  
 Lemma 395 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*).

**Lemma 321** (*addition in  $\overline{\mathbb{R}}$  is compatible with order when defined*)

is explicitly cited in the proof of:  
 Lemma 396 (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*),  
 Lemma 483 (*compatibility of extrema with affine transformation*),  
 Lemma 1001 (*absolute value and almost inequality*).

**Lemma 322** (*infinity-sum property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 397 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ).

**Lemma 323** (*finite-sum property in  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 324** (*additive inverse in  $\overline{\mathbb{R}}$  is involution*)

is explicitly cited in the proof of:  
 Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* ).

**Lemma 325** (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*)

is explicitly cited in the proof of:  
 Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 489 (*supremum is monotone (set)*),  
 Lemma 491 (*supremum is monotone (function)*),  
 Lemma 511 (*equivalent definition of limit superior*),  
 Lemma 513 (*limit superior is monotone*).

**Definition 326** (*multiplication in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 328 (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 329 (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*),  
 Lemma 330 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),  
 Lemma 331 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),  
 Lemma 333 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*),  
 Lemma 334 (*zero-product property in  $\overline{\mathbb{R}}$* ),  
 Lemma 335 (*one-product property in  $\overline{\mathbb{R}}$* ),  
 Lemma 336 (*infinity-product property in  $\overline{\mathbb{R}}$* ),  
 Lemma 337 (*finite-product property in  $\overline{\mathbb{R}}$* ),  
 Lemma 338 (*multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined*),  
 Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),  
 Lemma 370 (*power function in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Lemma 371 (*power function satisfies extended relaxed triangle inequality*),  
 Lemma 388 (*multiplication in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 407 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*),  
 Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 453 (*Young's inequality for products (measure theory)*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),



Lemma 833 (*measurability and masking*),  
 Lemma 1053 (*nonnegative essential supremum is submultiplicative*),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 1294 (*compatibility of  $N_p$  with power function*),  
 Lemma 1304 (*Hölder inequality*).

**Lemma 328** (*one is identity element for multiplication in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),  
 Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Lemma 1295 (*compatibility of  $N_p$  with absolute value*),  
 Lemma 1301 ( *$N_p$  of constant function*).

**Lemma 329** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*)

is explicitly cited in the proof of:

Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*).

**Lemma 330** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*)

is explicitly cited in the proof of:

Lemma 332 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),  
 Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*).

**Lemma 331** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*)

is explicitly cited in the proof of:

Lemma 332 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),  
 Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),  
 Lemma 483 (*compatibility of extrema with affine transformation*).

**Lemma 332** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*)

is explicitly cited in the proof of:

Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*).

**Lemma 333** (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order when defined*)

is explicitly cited in the proof of:

Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*).

**Lemma 334** (*zero-product property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 359 (*one-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 408 (*zero-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 335** (*one-product property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 409 (*one-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 422 (*one-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 336** (*infinity-product property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 358 (*zero-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 360 (*infinity-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 423 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*).



**Lemma 337** (*finite-product property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 411 (*finite-product property in  $\overline{\mathbb{R}}_+$* ).

**Lemma 338** (*multiplicative inverse in  $\overline{\mathbb{R}}$  is monotone when defined*)

is explicitly cited in the proof of:

Lemma 412 (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone*).

**Definition 339** (*absolute value in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 340 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 341 (*finite absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 342 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),

Lemma 343 (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*),

Lemma 346 (*absolute value in  $\overline{\mathbb{R}}$  is even*),

Lemma 347 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),

Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*),

Lemma 349 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),

Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*),

Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),

Lemma 389 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),

Lemma 532 (*decomposition into nonnegative and nonpositive parts*),

Lemma 1277 (*integral over singleton*),

Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**Lemma 340** (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 344 (*inverse of absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 494 (*compatibility of infimum with absolute value*),

Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**Lemma 341** (*finite absolute value in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 996 (*almost finite*),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ).

**Lemma 342** (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*)

is explicitly cited in the proof of:

Lemma 345 (*absolute value in  $\overline{\mathbb{R}}$  is idempotent*),

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),

Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1304 (*Hölder inequality*),

Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 343** (*absolute value in  $\overline{\mathbb{R}}$  is equal to nonnegative*)

is explicitly cited in the proof of:

Lemma 345 (*absolute value in  $\overline{\mathbb{R}}$  is idempotent*),

Lemma 368 (*inverse of power function in  $\overline{\mathbb{R}}$* ),

Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*),

Lemma 841 ( *$\mathcal{M}$  is closed under power function*),

Lemma 1006 (*absolute value is almost equal to almost nonnegative function*),

Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),

Lemma 1250 (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* ),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 344** (*inverse of absolute value in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 368 (*inverse of power function in  $\overline{\mathbb{R}}$* ),

Lemma 1001 (*absolute value and almost inequality*).

**Lemma 345** (*absolute value in  $\overline{\mathbb{R}}$  is idempotent*)

is explicitly cited in the proof of:

Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*).

**Lemma 346** (*absolute value in  $\overline{\mathbb{R}}$  is even*)

is explicitly cited in the proof of:

Lemma 350 (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*),

Lemma 445 (*power function in  $\overline{\mathbb{R}}$  is even (measure theory)*),

Lemma 495 (*compatibility of supremum with absolute value*),

Lemma 515 (*compatibility of limit superior with absolute value*),

Lemma 1251 (*integrable is almost finite*),

Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*).

**Lemma 347** (*absolute value in  $\overline{\mathbb{R}}$  is definite*)

is explicitly cited in the proof of:

Lemma 348 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*),

Lemma 1005 (*absolute value is almost definite*),

Lemma 1277 (*integral over singleton*),

Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 348** (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative*)

is explicitly cited in the proof of:

Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),

Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*).

**Lemma 349** (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*)

is explicitly cited in the proof of:

Lemma 351 (*absolute value in  $\overline{\mathbb{R}}$  satisfies countable triangle inequality*),

Lemma 1001 (*absolute value and almost inequality*),

Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 350** (*absolute value in  $\overline{\mathbb{R}}$  satisfies reverse triangle inequality*)

is explicitly cited in the proof of:

Lemma 1001 (*absolute value and almost inequality*).

**Lemma 351** (*absolute value in  $\overline{\mathbb{R}}$  satisfies countable triangle inequality*)

is not yet used.

**Definition 352** (*exponential and logarithm in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 353 (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*),

Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),

Lemma 358 (*zero-exponentiation property in  $\overline{\mathbb{R}}$* ),

Lemma 359 (*one-exponentiation property in  $\overline{\mathbb{R}}$* ),

Lemma 360 (*infinity-exponentiation property in  $\overline{\mathbb{R}}$* ).

**Lemma 353** (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*)  
is not yet used.

**Definition 354** (*exponentiation in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 355 (*exponentiation in  $\overline{\mathbb{R}}$* ),  
 Lemma 358 (*zero-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 359 (*one-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 360 (*infinity-exponentiation property in  $\overline{\mathbb{R}}$* ),  
 Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 355** (*exponentiation in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 357 (*exponentiation in  $\overline{\mathbb{R}}$  of power 1*),  
 Lemma 361 (*power function in  $\overline{\mathbb{R}}$* ),  
 Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 453 (*Young's inequality for products (measure theory)*).

**Lemma 357** (*exponentiation in  $\overline{\mathbb{R}}$  of power 1*)

is explicitly cited in the proof of:  
 Lemma 362 (*power function in  $\overline{\mathbb{R}}$  is absolute value*),  
 Lemma 1295 (*compatibility of  $N_p$  with absolute value*).

**Lemma 358** (*zero-exponentiation property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**Lemma 359** (*one-exponentiation property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 360** (*infinity-exponentiation property in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**Lemma 361** (*power function in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:  
 Lemma 362 (*power function in  $\overline{\mathbb{R}}$  is absolute value*),  
 Lemma 363 (*finite power function in  $\overline{\mathbb{R}}$* ),  
 Lemma 364 (*power function in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 365 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition*),  
 Lemma 366 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*),  
 Lemma 367 (*power function in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 369 (*power function in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 370 (*power function in  $\overline{\mathbb{R}}$  is multiplicative*),  
 Lemma 371 (*power function satisfies extended relaxed triangle inequality*),  
 Lemma 390 (*power function in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),  
 Lemma 1286 (*p-seminorm*),  
 Lemma 1297 (*compatibility of  $N_p$  with almost equality*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*).

**Lemma 362** (*power function in  $\overline{\mathbb{R}}$  is absolute value*)

is explicitly cited in the proof of:  
 Lemma 842 ( *$\mathcal{M}$  is closed under absolute value*).

**Lemma 363** (*finite power function in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),

Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ).

**Lemma 364** (*power function in  $\overline{\mathbb{R}}$  is nonnegative*)

is explicitly cited in the proof of:

Lemma 368 (*inverse of power function in  $\overline{\mathbb{R}}$* ),

Lemma 1290 ( *$N_p$  is nonnegative*).

**Lemma 365** (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition*)

is explicitly cited in the proof of:

Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 366** (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication*)

is explicitly cited in the proof of:

Lemma 368 (*inverse of power function in  $\overline{\mathbb{R}}$* ),

Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),

Lemma 1291 (*link between  $N_1$  and  $N_p$* ),

Lemma 1294 (*compatibility of  $N_p$  with power function*),

Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),

Lemma 1305 (*generalized Hölder inequality*),

Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*),

Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ).

**Lemma 367** (*power function in  $\overline{\mathbb{R}}$  is monotone*)

is explicitly cited in the proof of:

Lemma 368 (*inverse of power function in  $\overline{\mathbb{R}}$* ),

Lemma 444 (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*),

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),

Lemma 1305 (*generalized Hölder inequality*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 368** (*inverse of power function in  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 369** (*power function in  $\overline{\mathbb{R}}$  is definite*)

is explicitly cited in the proof of:

Lemma 370 (*power function in  $\overline{\mathbb{R}}$  is multiplicative*),

Lemma 1007 (*power function is almost definite*),

Lemma 1296 ( *$N_p$  is almost definite*),

Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 370** (*power function in  $\overline{\mathbb{R}}$  is multiplicative*)

is explicitly cited in the proof of:

Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*).

**Lemma 371** (*power function satisfies extended relaxed triangle inequality*)

is explicitly cited in the proof of:

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 372** (*topology on  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 378 (*equivalent topologies on  $\overline{\mathbb{R}}$* ),
- Lemma 379 (*convergence towards  $\pm\infty$* ),
- Lemma 392 ( *$\overline{\mathbb{R}}_+$  is complete*),
- Lemma 476 (*open subsets of  $\overline{\mathbb{R}}$* ),
- Lemma 479 ( *$\overline{\mathbb{R}}$  is second-countable*),
- Lemma 820 (*measurability of numeric function*).

**Lemma 373** (*trace topology on subset of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 375 (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*),
- Lemma 474 (*open down*),
- Lemma 475 (*open up*).

**Definition 374** (*tangent and arctangent in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 375 (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*),
- Lemma 377 (*metric on  $\overline{\mathbb{R}}$* ).

**Lemma 375** (*tangent and arctangent in  $\overline{\mathbb{R}}$  are homeomorphisms*)

is explicitly cited in the proof of:

- Lemma 377 (*metric on  $\overline{\mathbb{R}}$* ),
- Lemma 378 (*equivalent topologies on  $\overline{\mathbb{R}}$* ).

**Lemma 377** (*metric on  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 378** (*equivalent topologies on  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 385 (*topology on  $\overline{\mathbb{R}}^m$* ).

**Lemma 379** (*convergence towards  $\pm\infty$* )

is explicitly cited in the proof of:

- Lemma 380 (*convergence towards  $\pm\infty$  (discrete)*),
- Lemma 500 (*infimum is limit*).

**Lemma 380** (*convergence towards  $\pm\infty$  (discrete)*)

is explicitly cited in the proof of:

- Lemma 381 (*limit is monotone*),
- Lemma 382 (*cluster point at  $\pm\infty$  (discrete)*),
- Lemma 486 (*equivalent definition of infimum*).

**Lemma 381** (*limit is monotone*)

is explicitly cited in the proof of:

- Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ),
- Lemma 506 (*equivalent definition of the limit inferior*),
- Lemma 514 (*compatibility of limit inferior with absolute value*),
- Lemma 1015 (*compatibility of almost limit with almost inequality*),
- Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*),
- Lemma 1069 ( *$\lambda^*$  generalizes length of interval*),
- Lemma 1080 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),
- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*),
- Lemma 1083 (*rays are Lebesgue-measurable*),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Theorem 1208 (*Fatou's lemma*),

Theorem 1242 (*Tonelli*),  
 Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 382** (*cluster point at  $\pm\infty$  (discrete)*)  
 is explicitly cited in the proof of:  
 Lemma 506 (*equivalent definition of the limit inferior*).

**Lemma 383** (*limit is invariant by translation*)  
 is explicitly cited in the proof of:  
 Lemma 507 (*limit inferior is invariant by translation*).

**Lemma 385** (*topology on  $\overline{\mathbb{R}}^m$* )  
 is explicitly cited in the proof of:  
 Lemma 386 (*trace topology on subset of  $\overline{\mathbb{R}}^m$* ),  
 Lemma 480 ( *$\overline{\mathbb{R}}^m$  is second-countable*).

**Lemma 386** (*trace topology on subset of  $\overline{\mathbb{R}}^m$* )  
 is not yet used.

**Lemma 387** (*addition in  $\overline{\mathbb{R}}$  is continuous*)  
 is explicitly cited in the proof of:  
 Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*),  
 Lemma 413 (*addition in  $\overline{\mathbb{R}}_+$  is continuous*).

**Lemma 388** (*multiplication in  $\overline{\mathbb{R}}$  is continuous*)  
 is explicitly cited in the proof of:  
 Lemma 391 (*limit in  $\overline{\mathbb{R}}$  is linear*),  
 Lemma 448 (*multiplication in  $\overline{\mathbb{R}}_+$  is continuous (measure theory)*),  
 Lemma 501 (*supremum is limit*),  
 Lemma 511 (*equivalent definition of limit superior*),  
 Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).

**Lemma 389** (*absolute value in  $\overline{\mathbb{R}}$  is continuous*)  
 is explicitly cited in the proof of:  
 Lemma 514 (*compatibility of limit inferior with absolute value*),  
 Lemma 1015 (*compatibility of almost limit with almost inequality*),  
 Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 390** (*power function in  $\overline{\mathbb{R}}$  is continuous*)  
 is explicitly cited in the proof of:  
 Lemma 449 (*power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)*),  
 Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 391** (*limit in  $\overline{\mathbb{R}}$  is linear*)  
 is explicitly cited in the proof of:  
 Lemma 414 (*limit in  $\overline{\mathbb{R}}_+$  is positive linear*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ).

**Lemma 392** ( *$\overline{\mathbb{R}}_+$  is complete*)  
 is explicitly cited in the proof of:  
 Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 849 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*).

**Lemma 393** (*addition in  $\overline{\mathbb{R}}_+$  is closed*)  
 is explicitly cited in the proof of:  
 Lemma 398 (*definition of series in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 453 (*Young's inequality for products (measure theory)*),

- Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),
- Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),
- Lemma 1249 (*equivalent definition of integrability*),
- Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 394** (*addition in  $\overline{\mathbb{R}}_+$  is associative*)

is explicitly cited in the proof of:

- Lemma 399 (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition*),
- Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),
- Lemma 864 (*binary inclusion-exclusion principle*),
- Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),
- Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*).

**Lemma 395** (*addition in  $\overline{\mathbb{R}}_+$  is commutative*)

is explicitly cited in the proof of:

- Lemma 399 (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition*),
- Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),
- Lemma 1074 ( *$\mathcal{L}$  is closed under complement*),
- Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),
- Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*).

**Lemma 396** (*addition in  $\overline{\mathbb{R}}_+$  is compatible with order*)

is explicitly cited in the proof of:

- Lemma 404 (*double series in  $\mathbb{R}_+$* ),
- Lemma 869 (*subadditivity*),
- Lemma 878 ( *$\sigma$ -subadditivity*),
- Lemma 1069 ( *$\lambda^*$  generalizes length of interval*),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**Lemma 397** (*infinity-sum property in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

- Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),
- Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),
- Lemma 1249 (*equivalent definition of integrability*).

**Lemma 398** (*definition of series in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

- Lemma 399 (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition*),
- Lemma 400 (*series in  $\overline{\mathbb{R}}_+$  is monotone*),
- Lemma 401 (*subseries in  $\overline{\mathbb{R}}_+$  are lower bounds*),
- Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ),
- Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),
- Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*).

**Lemma 399** (*series in  $\overline{\mathbb{R}}_+$  are compatible with pseudopartition*)

is explicitly cited in the proof of:

- Lemma 400 (*series in  $\overline{\mathbb{R}}_+$  is monotone*).

**Lemma 400** (*series in  $\overline{\mathbb{R}}_+$  is monotone*)

is explicitly cited in the proof of:

- Lemma 401 (*subseries in  $\overline{\mathbb{R}}_+$  are lower bounds*).

**Lemma 401** (*subseries in  $\overline{\mathbb{R}}_+$  are lower bounds*)

is explicitly cited in the proof of:

- Lemma 402 (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* ).



**Lemma 402** (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

Lemma 403 (*definition of double series in  $\overline{\mathbb{R}}_+$* ).

**Lemma 403** (*definition of double series in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

Lemma 404 (*double series in  $\overline{\mathbb{R}}_+$* ),

Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Lemma 404** (*double series in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Definition 405** (*multiplication in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

Lemma 407 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*),

Lemma 408 (*zero-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 409 (*one-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 410 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 411 (*finite-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 412 (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone*),

Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),

Lemma 455 (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* ),

Lemma 1294 (*compatibility of  $N_p$  with power function*).

**Lemma 407** (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*)

is not yet used.

**Lemma 408** (*zero-product property in  $\overline{\mathbb{R}}_+$* )

is not yet used.

**Lemma 409** (*one-product property in  $\overline{\mathbb{R}}_+$* )

is not yet used.

**Lemma 410** (*infinity-product property in  $\overline{\mathbb{R}}_+$* )

is explicitly cited in the proof of:

Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),

Lemma 453 (*Young's inequality for products (measure theory)*),

Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),

Lemma 1304 (*Hölder inequality*).

**Lemma 411** (*finite-product property in  $\overline{\mathbb{R}}_+$* )

is not yet used.

**Lemma 412** (*multiplicative inverse in  $\overline{\mathbb{R}}_+$  is monotone*)

is explicitly cited in the proof of:

Lemma 455 (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* ).

**Lemma 413** (*addition in  $\overline{\mathbb{R}}_+$  is continuous*)

is explicitly cited in the proof of:

Lemma 503 (*constant sum of monotone nonnegative functions*),

Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),

Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*).

**Lemma 414** (*limit in  $\overline{\mathbb{R}}_+$  is positive linear*)

is explicitly cited in the proof of:

Theorem 1180 (*Beppo Levi, monotone convergence*).



**Definition 415** (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

- Lemma 417 (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*),
- Lemma 418 (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*),
- Lemma 419 (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*),
- Lemma 420 (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*),
- Lemma 421 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 422 (*one-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 423 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 425 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),
- Lemma 427 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),
- Lemma 429 (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),
- Lemma 430 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 448 (*multiplication in  $\overline{\mathbb{R}}_+$  is continuous (measure theory)*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1311 (*Minkowski equality (linearly dependent functions)*),
- Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*).

**Lemma 417** (*multiplication in  $\overline{\mathbb{R}}$  is associative (measure theory)*)

is explicitly cited in the proof of:

- Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*),
- Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 418** (*multiplication in  $\overline{\mathbb{R}}$  is commutative (measure theory)*)

is explicitly cited in the proof of:

- Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*),
- Lemma 1227 (*candidate tensor product measure is tensor product measure*),
- Lemma 1244 (*Tonelli for tensor product*),
- Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**Lemma 419** (*multiplication in  $\overline{\mathbb{R}}$  is distributive over addition (measure theory)*)

is explicitly cited in the proof of:

- Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),
- Lemma 1168 (*density integral in  $\mathcal{SF}_+$* ),
- Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),
- Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),
- Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**Lemma 420** (*multiplication in  $\overline{\mathbb{R}}$  is compatible with order (measure theory)*)

is explicitly cited in the proof of:

- Lemma 1027 (*essential supremum is positive homogeneous*),
- Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),
- Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),
- Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),
- Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),
- Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**Lemma 421** (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

- Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),
- Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),
- Lemma 453 (*Young's inequality for products (measure theory)*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),

Lemma 1027 (*essential supremum is positive homogeneous*),  
 Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*),  
 Lemma 1219 (*measure of section of product*),  
 Lemma 1237 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*),  
 Lemma 1304 (*Hölder inequality*).

**Lemma 422** (*one-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 1054 (*nonnegative essential supremum is compatible with power function*).

**Lemma 423** (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 424 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 424** (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 428 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*).

**Lemma 425** (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*)

is explicitly cited in the proof of:

Lemma 453 (*Young's inequality for products (measure theory)*),

Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*),

Lemma 844 ( *$\mathcal{M}_+$  is closed under multiplication*),

Lemma 845 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),

Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),

Lemma 1244 (*Tonelli for tensor product*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),

Lemma 1304 (*Hölder inequality*).

**Lemma 427** (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),

Lemma 453 (*Young's inequality for products (measure theory)*),

Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),

Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*).

**Lemma 428** (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 1277 (*integral over singleton*).

**Lemma 429** (*absolute value in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*)

is explicitly cited in the proof of:

Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1304 (*Hölder inequality*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 430** (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*)

is explicitly cited in the proof of:

Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*),

Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*),

- Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*),  
 Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),  
 Lemma 444 (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*),  
 Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*).

**Lemma 431 (*one-exponentiation property in  $\overline{\mathbb{R}}$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 446 (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*),  
 Lemma 449 (*power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 1054 (*nonnegative essential supremum is compatible with power function*),  
 Lemma 1294 (*compatibility of  $N_p$  with power function*),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 432 (*zero-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 1054 (*nonnegative essential supremum is compatible with power function*),  
 Lemma 1288 ( *$N_p$  with trivial measure*),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 433 (*values of the exponentiation in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 434 (*finite-positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 435 (*finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 436 (*positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 437 (*less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 438 (*great-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*),  
 Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**Lemma 434 (*finite-positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is not yet used.

**Lemma 435 (*finite-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 436 (*positive-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 441 (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*).

**Lemma 437 (*less-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*).

**Lemma 438 (*great-than-one-exponentiation property in  $\overline{\mathbb{R}}_+$  (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 442 (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*).

**Lemma 439 (*power function in  $\overline{\mathbb{R}}$  is nonnegative (measure theory)*)**

is explicitly cited in the proof of:

- Lemma 443 (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*),  
 Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*),  
 Lemma 841 ( *$\mathcal{M}$  is closed under power function*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 441** (*compatibility of power function in  $\overline{\mathbb{R}}$  with addition (measure theory)*)  
 is not yet used.

**Lemma 442** (*compatibility of power function in  $\overline{\mathbb{R}}$  with multiplication (measure theory)*)

is explicitly cited in the proof of:

Lemma 1054 (*nonnegative essential supremum is compatible with power function*).

**Lemma 443** (*compatibility of power function in  $\overline{\mathbb{R}}$  with absolute value (measure theory)*)

is explicitly cited in the proof of:

Lemma 1291 (*link between  $N_1$  and  $N_p$* ),

Lemma 1294 (*compatibility of  $N_p$  with power function*),

Lemma 1305 (*generalized Hölder inequality*).

**Lemma 444** (*power function in  $\overline{\mathbb{R}}$  is monotone (measure theory)*)

is explicitly cited in the proof of:

Lemma 1054 (*nonnegative essential supremum is compatible with power function*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**Lemma 445** (*power function in  $\overline{\mathbb{R}}$  is even (measure theory)*)

is not yet used.

**Lemma 446** (*power function in  $\overline{\mathbb{R}}$  is multiplicative (measure theory)*)

is explicitly cited in the proof of:

Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),

Lemma 1305 (*generalized Hölder inequality*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 448** (*multiplication in  $\overline{\mathbb{R}}_+$  is continuous (measure theory)*)

is not yet used.

**Lemma 449** (*power function in  $\overline{\mathbb{R}}$  is continuous (measure theory)*)

is explicitly cited in the proof of:

Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*),

Lemma 1012 (*compatibility of almost limit with power function*).

**Definition 451** (*Hölder conjugates*)

is explicitly cited in the proof of:

Lemma 1304 (*Hölder inequality*),

Lemma 1305 (*generalized Hölder inequality*),

Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*).

**Lemma 453** (*Young's inequality for products (measure theory)*)

is explicitly cited in the proof of:

Lemma 454 (*Young's inequality for products, case  $p = 2$  (measure theory)*),

Lemma 1304 (*Hölder inequality*).

**Lemma 454** (*Young's inequality for products, case  $p = 2$  (measure theory)*)

is not yet used.

**Lemma 455** (*interpolation of multiplicative inverse in segment of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Definition 456** (*connected component in  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 457 (*empty connected component*),

Lemma 458 (*connected component of open subset of  $\mathbb{R}$  is open interval*),

Lemma 459 (*connected component of open subset of  $\mathbb{R}$  is maximal*),

Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ).

**Lemma 457** (*empty connected component*)

is not yet used.

**Lemma 458** (*connected component of open subset of  $\mathbb{R}$  is open interval*)

is explicitly cited in the proof of:

Lemma 459 (*connected component of open subset of  $\mathbb{R}$  is maximal*).

**Lemma 459** (*connected component of open subset of  $\mathbb{R}$  is maximal*)

is explicitly cited in the proof of:

Lemma 460 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*).

**Lemma 460** (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*)

is explicitly cited in the proof of:

Theorem 461 (*countable connected components of open subsets of  $\mathbb{R}$* ).

**Theorem 461** (*countable connected components of open subsets of  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 477 (*countable connected components of open subsets of  $\overline{\mathbb{R}}$* ),

Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),

Lemma 797 (*countable generator of Borel subsets of  $\mathbb{R}$* ).

**Lemma 462** (*rational approximation of lower bound of open interval*)

is explicitly cited in the proof of:

Lemma 463 (*rational approximation of upper bound of open interval*),

Lemma 464 (*open intervals with rational bounds cover open interval*),

Lemma 478 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Lemma 463** (*rational approximation of upper bound of open interval*)

is explicitly cited in the proof of:

Lemma 464 (*open intervals with rational bounds cover open interval*),

Lemma 478 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Lemma 464** (*open intervals with rational bounds cover open interval*)

is explicitly cited in the proof of:

Theorem 465 ( *$\mathbb{R}$  is second-countable*),

Lemma 478 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Theorem 465** ( *$\mathbb{R}$  is second-countable*)

is explicitly cited in the proof of:

Lemma 466 ( *$\mathbb{R}^m$  is second-countable*),

Lemma 797 (*countable generator of Borel subsets of  $\mathbb{R}$* ).

**Lemma 466** ( *$\mathbb{R}^m$  is second-countable*)

is not yet used.

**Definition 468** (*projection and injections between  $\mathbb{R}$  and  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 469 (*compatibility between projection and injections in  $\overline{\mathbb{R}}$* ),

Lemma 471 (*compatibility of projection and injections in  $\overline{\mathbb{R}}$  with subset operations*),

Lemma 476 (*open subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 469** (*compatibility between projection and injections in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 475 (*open up*).

**Lemma 471** (*compatibility of projection and injections in  $\overline{\mathbb{R}}$  with subset operations*)

is not yet used.

**Lemma 474** (*open down*)

is explicitly cited in the proof of:

Lemma 475 (*open up*),

Lemma 476 (*open subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 475** (*open up*)

is explicitly cited in the proof of:

Lemma 476 (*open subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 476** (*open subsets of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 477 (*countable connected components of open subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 477** (*countable connected components of open subsets of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 479 ( *$\overline{\mathbb{R}}$  is second-countable*).

**Lemma 478** (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 479 ( *$\overline{\mathbb{R}}$  is second-countable*).

**Lemma 479** ( *$\overline{\mathbb{R}}$  is second-countable*)

is explicitly cited in the proof of:

Lemma 480 ( *$\overline{\mathbb{R}}^m$  is second-countable*).

**Lemma 480** ( *$\overline{\mathbb{R}}^m$  is second-countable*)

is not yet used.

**Lemma 482** (*extrema of constant function*)

is explicitly cited in the proof of:

Lemma 485 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ),

Lemma 492 (*infimum of bounded is bounded*),

Lemma 493 (*supremum of bounded is bounded*),

Lemma 500 (*infimum is limit*),

Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*).

**Lemma 483** (*compatibility of extrema with affine transformation*)

is explicitly cited in the proof of:

Lemma 883 (*continuity from above*).

**Lemma 484** (*equivalent definition of finite infimum*)

is explicitly cited in the proof of:

Lemma 485 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ).

**Lemma 485** (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

Lemma 486 (*equivalent definition of infimum*).

**Lemma 486** (*equivalent definition of infimum*)

is explicitly cited in the proof of:

Lemma 1021 (*essential supremum is minimum*).

**Lemma 487** (*infimum is smaller than supremum*)

is explicitly cited in the proof of:

Lemma 494 (*compatibility of infimum with absolute value*).

**Lemma 488** (*infimum is monotone (set)*)

is explicitly cited in the proof of:

Lemma 489 (*supremum is monotone (set)*),

Lemma 496 (*compatibility of translation with infimum of sequence*),

Lemma 502 (*series in  $\overline{\mathbb{R}}_+$  is supremum*),

Lemma 504 (*limit inferior*),

Theorem 1208 (*Fatou's lemma*).

**Lemma 489** (*supremum is monotone (set)*)

is explicitly cited in the proof of:

Lemma 497 (*compatibility of translation with supremum of sequence*),

Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*).

**Lemma 490** (*infimum is monotone (function)*)

is explicitly cited in the proof of:

Lemma 491 (*supremum is monotone (function)*),

Lemma 492 (*infimum of bounded is bounded*),

Lemma 494 (*compatibility of infimum with absolute value*),

Lemma 508 (*limit inferior is monotone*),

Lemma 1065 ( *$\lambda^*$  is nonnegative*).

**Lemma 491** (*supremum is monotone (function)*)

is explicitly cited in the proof of:

Lemma 493 (*supremum of bounded is bounded*),

Lemma 494 (*compatibility of infimum with absolute value*),

Lemma 508 (*limit inferior is monotone*).

**Lemma 492** (*infimum of bounded is bounded*)

is explicitly cited in the proof of:

Lemma 847 ( *$\mathcal{M}_+$  is closed under infimum*),

Theorem 1208 (*Fatou's lemma*).

**Lemma 493** (*supremum of bounded is bounded*)

is explicitly cited in the proof of:

Lemma 848 ( *$\mathcal{M}_+$  is closed under supremum*),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 494** (*compatibility of infimum with absolute value*)

is explicitly cited in the proof of:

Lemma 495 (*compatibility of supremum with absolute value*),

Lemma 514 (*compatibility of limit inferior with absolute value*).

**Lemma 495** (*compatibility of supremum with absolute value*)

is not yet used.



**Lemma 496** (*compatibility of translation with infimum of sequence*)

is explicitly cited in the proof of:

Lemma 498 (*infimum of nonincreasing sequence*).

**Lemma 497** (*compatibility of translation with supremum of sequence*)

is not yet used.

**Lemma 498** (*infimum of nonincreasing sequence*)

is explicitly cited in the proof of:

Lemma 499 (*supremum of nondecreasing sequence*),

Lemma 883 (*continuity from above*).

**Lemma 499** (*supremum of nondecreasing sequence*)

is not yet used.

**Lemma 500** (*infimum is limit*)

is explicitly cited in the proof of:

Lemma 501 (*supremum is limit*),

Lemma 503 (*constant sum of monotone nonnegative functions*),

Lemma 883 (*continuity from above*).

**Lemma 501** (*supremum is limit*)

is explicitly cited in the proof of:

Lemma 503 (*constant sum of monotone nonnegative functions*),

Lemma 880 (*continuity from below*),

Theorem 1180 (*Beppo Levi, monotone convergence*),

Theorem 1208 (*Fatou's lemma*),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ).

**Lemma 502** (*series in  $\overline{\mathbb{R}}_+$  is supremum*)

is explicitly cited in the proof of:

Lemma 880 (*continuity from below*),

Lemma 881 ( *$\sigma$ -additivity (from below)*),

Lemma 884 ( *$\sigma$ -additivity (from above)*).

**Lemma 503** (*constant sum of monotone nonnegative functions*)

is explicitly cited in the proof of:

Lemma 884 ( *$\sigma$ -additivity (from above)*).

**Lemma 504** (*limit inferior*)

is explicitly cited in the proof of:

Lemma 505 (*limit inferior is  $\infty$* ),

Lemma 506 (*equivalent definition of the limit inferior*),

Lemma 508 (*limit inferior is monotone*),

Lemma 510 (*duality limit inferior-limit superior*),

Lemma 514 (*compatibility of limit inferior with absolute value*),

Lemma 830 ( *$\mathcal{M}$  is closed under limit inferior*),

Theorem 1208 (*Fatou's lemma*).

**Lemma 505** (*limit inferior is  $\infty$* )

is explicitly cited in the proof of:

Lemma 506 (*equivalent definition of the limit inferior*),

Lemma 522 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 506** (*equivalent definition of the limit inferior*)

is explicitly cited in the proof of:

Lemma 507 (*limit inferior is invariant by translation*),



Lemma 511 (*equivalent definition of limit superior*),  
 Lemma 512 (*limit inferior is smaller than limit superior*),  
 Lemma 517 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 522 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 507** (*limit inferior is invariant by translation*)

is explicitly cited in the proof of:

Lemma 508 (*limit inferior is monotone*).

**Lemma 508** (*limit inferior is monotone*)

is explicitly cited in the proof of:

Lemma 513 (*limit superior is monotone*),

Lemma 518 (*limit inferior bounded from below*),

Lemma 519 (*limit inferior bounded from above*).

**Lemma 509** (*limit superior*)

is explicitly cited in the proof of:

Lemma 510 (*duality limit inferior-limit superior*),

Lemma 514 (*compatibility of limit inferior with absolute value*),

Lemma 831 ( *$\mathcal{M}$  is closed under limit superior*).

**Lemma 510** (*duality limit inferior-limit superior*)

is explicitly cited in the proof of:

Lemma 511 (*equivalent definition of limit superior*),

Lemma 513 (*limit superior is monotone*),

Lemma 515 (*compatibility of limit superior with absolute value*),

Lemma 520 (*limit superior bounded from below*),

Lemma 521 (*limit superior bounded from above*),

Lemma 522 (*limit inferior, limit superior and pointwise convergence*),

Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 511** (*equivalent definition of limit superior*)

is explicitly cited in the proof of:

Lemma 512 (*limit inferior is smaller than limit superior*),

Lemma 517 (*limit inferior and limit superior of pointwise convergent*),

Lemma 522 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 512** (*limit inferior is smaller than limit superior*)

is explicitly cited in the proof of:

Lemma 522 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 513** (*limit superior is monotone*)

is not yet used.

**Lemma 514** (*compatibility of limit inferior with absolute value*)

is explicitly cited in the proof of:

Lemma 515 (*compatibility of limit superior with absolute value*).

**Lemma 515** (*compatibility of limit superior with absolute value*)

is not yet used.

**Definition 516** (*pointwise convergence*)

is explicitly cited in the proof of:

Lemma 517 (*limit inferior and limit superior of pointwise convergent*),

Theorem 1208 (*Fatou's lemma*),

Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 517** (*limit inferior and limit superior of pointwise convergent*)

is explicitly cited in the proof of:

- Lemma 518 (*limit inferior bounded from below*),
- Lemma 519 (*limit inferior bounded from above*),
- Lemma 1011 (*compatibility of almost limit with continuous function*),
- Lemma 1013 (*almost limits are almost equal*),
- Lemma 1016 (*compatibility of almost limit with measurability*),
- Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),
- Theorem 1347 (*Lebesgue, dominated convergence*),
- Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 518** (*limit inferior bounded from below*)

is explicitly cited in the proof of:

- Lemma 521 (*limit superior bounded from above*),
- Theorem 1208 (*Fatou's lemma*),
- Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 519** (*limit inferior bounded from above*)

is explicitly cited in the proof of:

- Lemma 520 (*limit superior bounded from below*),
- Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ).

**Lemma 520** (*limit superior bounded from below*)

is not yet used.

**Lemma 521** (*limit superior bounded from above*)

is not yet used.

**Lemma 522** (*limit inferior, limit superior and pointwise convergence*)

is explicitly cited in the proof of:

- Lemma 832 ( *$\mathcal{M}$  is closed under limit when pointwise convergent*),
- Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),
- Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 524** (*restriction is masking*)

is explicitly cited in the proof of:

- Lemma 1105 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero*),
- Lemma 1119 ( *$\mathcal{SF}_E$  is closed under extension by zero*),
- Lemma 1130 ( *$\mathcal{SF}_+$  is closed under extension by zero*),
- Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*).

**Lemma 525** (*masking is  $\sigma$ -additive*)

is explicitly cited in the proof of:

- Lemma 1102 ( *$\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive*),
- Lemma 1107 ( *$\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive*),
- Lemma 1121 ( *$\mathcal{SF}_E$ , masking is  $\sigma$ -additive*),
- Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1206 (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1255 (*integrability, masking is  $\sigma$ -additive*).

**Definition 526** (*finite part*)

is explicitly cited in the proof of:

- Lemma 527 (*finite part is finite*),
- Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*).

**Lemma 527** (*finite part is finite*)

is explicitly cited in the proof of:

Lemma 821 ( *$\mathcal{M}$  is closed under finite part*).

**Definition 528** (*nonnegative and nonpositive parts*)

is explicitly cited in the proof of:

Lemma 529 (*equivalent definition of nonnegative and nonpositive parts*),

Lemma 530 (*nonnegative and nonpositive parts are nonnegative*),

Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 534 (*compatibility of nonpositive and nonnegative parts with mask*),

Lemma 535 (*compatibility of nonpositive and nonnegative parts with restriction*),

Lemma 836 (*measurability of nonnegative and nonpositive parts*),

Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*),

Lemma 1193 (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*),

Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),

Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*).

**Lemma 529** (*equivalent definition of nonnegative and nonpositive parts*)

is explicitly cited in the proof of:

Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*).

**Lemma 530** (*nonnegative and nonpositive parts are nonnegative*)

is explicitly cited in the proof of:

Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 836 (*measurability of nonnegative and nonpositive parts*),

Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*).

**Lemma 531** (*nonnegative and nonpositive parts are orthogonal*)

is explicitly cited in the proof of:

Lemma 532 (*decomposition into nonnegative and nonpositive parts*),

Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 1253 (*integrability is closed under extension by zero*).

**Lemma 532** (*decomposition into nonnegative and nonpositive parts*)

is explicitly cited in the proof of:

Lemma 533 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 836 (*measurability of nonnegative and nonpositive parts*),

Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),

Lemma 1253 (*integrability is closed under extension by zero*),

Lemma 1277 (*integral over singleton*).

**Lemma 533** (*compatibility of nonpositive and nonnegative parts with addition*)

is explicitly cited in the proof of:

Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*).

**Lemma 534** (*compatibility of nonpositive and nonnegative parts with mask*)

is explicitly cited in the proof of:

Lemma 1253 (*integrability is closed under extension by zero*),

Lemma 1255 (*integrability, masking is  $\sigma$ -additive*).

**Lemma 535** (*compatibility of nonpositive and nonnegative parts with restriction*)

is explicitly cited in the proof of:

Lemma 1253 (*integrability is closed under extension by zero*),

Lemma 1254 (*integrability is closed under restriction*).

**Lemma 538** (*nonempty and with empty or full*)

is explicitly cited in the proof of:

- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 627 (*equivalent definition of semiring of sets*),
- Lemma 635 (*equivalent definition of algebra of sets*),
- Lemma 655 (*equivalent definition of semi-algebra of sets*),
- Lemma 680 (*other properties of  $\lambda$ -system*),
- Lemma 697 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 539** (*with empty and full*)

is explicitly cited in the proof of:

- Lemma 635 (*equivalent definition of algebra of sets*),
- Lemma 680 (*other properties of  $\lambda$ -system*),
- Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),
- Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 541** (*closedness under local complement and complement*)

is explicitly cited in the proof of:

- Lemma 637 (*other equivalent definition of algebra of sets*),
- Lemma 678 (*equivalent definition of  $\lambda$ -system*),
- Lemma 687 (*upgrade monotone class to  $\lambda$ -system*).

**Lemma 542** (*closedness under disjoint union and local complement*)

is explicitly cited in the proof of:

- Lemma 566 (*closedness under countable disjoint union and local complement*),
- Lemma 678 (*equivalent definition of  $\lambda$ -system*),
- Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 543** (*closedness under set difference and local complement*)

is explicitly cited in the proof of:

- Lemma 637 (*other equivalent definition of algebra of sets*).

**Lemma 544** (*closedness under symmetric difference, local complement, and disjoint union*)

is explicitly cited in the proof of:

- Lemma 612 (*equivalent definition of ring of sets*).

**Lemma 545** (*closedness under intersection and set difference*)

is explicitly cited in the proof of:

- Lemma 548 (*closedness under union and set difference*),
- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 637 (*other equivalent definition of algebra of sets*),
- Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*),
- Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),
- Lemma 1077 ( *$\mathcal{L}$  is algebra of sets*).

**Lemma 546** (*closedness under union and disjoint union*)

is explicitly cited in the proof of:

- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 639 (*other properties of algebra of sets*),
- Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 547** (*closedness under union and intersection*)

is explicitly cited in the proof of:

- Lemma 548 (*closedness under union and set difference*),
- Lemma 559 (*closedness under finite union and intersection*),
- Lemma 635 (*equivalent definition of algebra of sets*),

Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*),  
 Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 548** (*closedness under union and set difference*)  
 is not yet used.

**Lemma 550** (*closedness under symmetric difference and set difference*)  
 is explicitly cited in the proof of:  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*).

**Lemma 551** (*closedness under symmetric difference and intersection*)  
 is explicitly cited in the proof of:  
 Lemma 612 (*equivalent definition of ring of sets*).

**Lemma 552** (*nonempty is with empty*)  
 is explicitly cited in the proof of:  
 Lemma 574 (*disjoint union of union in subset system (variant)*),  
 Lemma 627 (*equivalent definition of semiring of sets*).

**Lemma 553** (*with full is with empty*)  
 is not yet used.

**Lemma 554** (*disjoint decomposition*)  
 is explicitly cited in the proof of:  
 Lemma 631 (*ring of sets is semiring of sets*),  
 Lemma 660 (*algebra of sets is semi-algebra of sets*),  
 Lemma 740 (*properties of premeasurable space*),  
 Lemma 742 (*properties of measurable space*),  
 Lemma 880 (*continuity from below*),  
 Lemma 883 (*continuity from above*),  
 Lemma 902 (*premeasure is monotone*).

**Lemma 555** (*disjoint decomposition of complement and set difference*)  
 is explicitly cited in the proof of:  
 Lemma 655 (*equivalent definition of semi-algebra of sets*).

**Lemma 556** (*disjoint decomposition of local complement and complement*)  
 is explicitly cited in the proof of:  
 Lemma 655 (*equivalent definition of semi-algebra of sets*).

**Lemma 557** (*disjoint decomposition of set difference and local complement*)  
 is explicitly cited in the proof of:  
 Lemma 630 (*other properties of semiring of sets*),  
 Lemma 655 (*equivalent definition of semi-algebra of sets*),  
 Lemma 869 (*subadditivity*),  
 Lemma 878 ( *$\sigma$ -subadditivity*).

**Lemma 558** (*closedness under finite operations*)  
 is explicitly cited in the proof of:  
 Lemma 559 (*closedness under finite union and intersection*),  
 Lemma 572 (*disjoint union of union in subset system*),  
 Lemma 575 (*monotone union of union in subset system*),  
 Lemma 604 (*equivalent definition of  $\pi$ -system*),  
 Lemma 612 (*equivalent definition of ring of sets*),  
 Lemma 615 (*other properties of ring of sets*),  
 Lemma 635 (*equivalent definition of algebra of sets*),

Lemma 639 (*other properties of algebra of sets*),  
 Lemma 680 (*other properties of  $\lambda$ -system*).

**Lemma 559** (*closedness under finite union and intersection*)  
 is not yet used.

**Lemma 561** (*closedness under intersection and finite disjoint union closure*)  
 is explicitly cited in the proof of:  
 Lemma 563 (*closedness under set difference and finite disjoint union closure*).

**Lemma 562** (*finite disjoint union closure is closed under finite disjoint union*)  
 is explicitly cited in the proof of:  
 Lemma 563 (*closedness under set difference and finite disjoint union closure*),  
 Lemma 632 (*explicit ring of sets*).

**Lemma 563** (*closedness under set difference and finite disjoint union closure*)  
 is explicitly cited in the proof of:  
 Lemma 632 (*explicit ring of sets*).

**Lemma 566** (*closedness under countable disjoint union and local complement*)  
 is explicitly cited in the proof of:  
 Lemma 678 (*equivalent definition of  $\lambda$ -system*).

**Lemma 567** (*closedness under countable union and intersection*)  
 is explicitly cited in the proof of:  
 Lemma 679 ( *$\lambda$ -system is monotone class*),  
 Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 568** (*closedness under countable disjoint and monotone union*)  
 is explicitly cited in the proof of:  
 Lemma 678 (*equivalent definition of  $\lambda$ -system*).

**Lemma 569** (*closedness under countable monotone and disjoint union*)  
 is explicitly cited in the proof of:  
 Lemma 571 (*closedness under countable monotone union and countable union*),  
 Lemma 678 (*equivalent definition of  $\lambda$ -system*).

**Lemma 570** (*closedness under countable disjoint union and countable union*)  
 is explicitly cited in the proof of:  
 Lemma 571 (*closedness under countable monotone union and countable union*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*).

**Lemma 571** (*closedness under countable monotone union and countable union*)  
 is explicitly cited in the proof of:  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*).

**Lemma 572** (*disjoint union of union in subset system*)  
 is explicitly cited in the proof of:  
 Lemma 624 (*disjoint union of union in ring of sets*),  
 Lemma 728 (*disjoint union of union in  $\sigma$ -algebra*),  
 Lemma 880 (*continuity from below*).

**Lemma 574** (*disjoint union of union in subset system (variant)*)  
 is explicitly cited in the proof of:  
 Lemma 869 (*subadditivity*),  
 Lemma 878 ( *$\sigma$ -subadditivity*).

**Lemma 575** (*monotone union of union in subset system*)

is explicitly cited in the proof of:

- Lemma 881 ( *$\sigma$ -additivity (from below)*),
- Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*).

**Lemma 576** (*empty monotone intersection of union in subset system*)

is explicitly cited in the proof of:

- Lemma 884 ( *$\sigma$ -additivity (from above)*).

**Lemma 577** (*monotone union of monotone intersection in subset system*)

is explicitly cited in the proof of:

- Lemma 883 (*continuity from above*).

**Definition 578** (*subset system*)

is explicitly cited in the proof of:

- Lemma 585 (*subset system generation preserves class*),
- Lemma 586 (*properties of subset system generation*).

**Definition 580** (*subset system class*)

is explicitly cited in the proof of:

- Lemma 582 (*intersection of subset system classes*),
- Lemma 583 (*subset system classes are monotone*),
- Lemma 585 (*subset system generation preserves class*),
- Lemma 586 (*properties of subset system generation*),
- Lemma 590 (*upgrade generated subset system*),
- Lemma 594 (*class of nonempty subset systems*).

**Lemma 582** (*intersection of subset system classes*)

is explicitly cited in the proof of:

- Lemma 594 (*class of nonempty subset systems*).

**Lemma 583** (*subset system classes are monotone*)

is explicitly cited in the proof of:

- Lemma 588 (*generated subset systems are monotone*),
- Lemma 591 (*cascading generated subset systems*).

**Definition 584** (*generated subset system*)

is explicitly cited in the proof of:

- Lemma 585 (*subset system generation preserves class*),
- Lemma 586 (*properties of subset system generation*),
- Lemma 588 (*generated subset systems are monotone*),
- Lemma 595 (*generated nonempty subset systems*).

**Lemma 585** (*subset system generation preserves class*)

is explicitly cited in the proof of:

- Lemma 586 (*properties of subset system generation*),
- Lemma 587 (*other properties of subset system generation*),
- Lemma 590 (*upgrade generated subset system*),
- Lemma 591 (*cascading generated subset systems*),
- Lemma 596 (*subset system generation preserves class (with nonemptiness)*),
- Lemma 618 (*generated ring of sets is ring of sets*),
- Lemma 643 (*generated algebra of sets is algebra of sets*),
- Lemma 667 (*generated monotone class is monotone class*),
- Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*),
- Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*).



**Lemma 586** (*properties of subset system generation*)

is explicitly cited in the proof of:

- Lemma 587 (*other properties of subset system generation*),
- Lemma 590 (*upgrade generated subset system*),
- Lemma 591 (*cascading generated subset systems*),
- Lemma 597 (*properties of generated subset system (with nonemptiness)*),
- Lemma 619 (*properties of ring of sets generation*),
- Lemma 644 (*properties of algebra of sets generation*),
- Lemma 668 (*properties of monotone class generation*),
- Lemma 684 (*properties of  $\lambda$ -system generation*),
- Lemma 710 (*properties of  $\sigma$ -algebra generation*).

**Lemma 587** (*other properties of subset system generation*)

is explicitly cited in the proof of:

- Lemma 598 (*other properties of subset system generation (with nonemptiness)*),
- Lemma 620 (*other properties of ring of sets generation*),
- Lemma 645 (*other properties of algebra of sets generation*),
- Lemma 669 (*other properties of monotone class generation*),
- Lemma 685 (*other properties of  $\lambda$ -system generation*),
- Lemma 711 (*other properties of  $\sigma$ -algebra generation*).

**Lemma 588** (*generated subset systems are monotone*)

is explicitly cited in the proof of:

- Lemma 599 (*generated subset systems are monotone (with nonemptiness)*),
- Lemma 646 (*algebra of sets contains ring of sets*),
- Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*),
- Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*).

**Lemma 590** (*upgrade generated subset system*)

is explicitly cited in the proof of:

- Lemma 600 (*upgrade generated subset system (with nonemptiness)*),
- Lemma 647 (*upgrade ring of sets to algebra of sets*),
- Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),
- Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),
- Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*).

**Lemma 591** (*cascading generated subset systems*)

is explicitly cited in the proof of:

- Lemma 601 (*cascading generated subset systems (with nonemptiness)*),
- Lemma 648 (*algebra of sets generated by ring of sets*),
- Lemma 688 ( *$\lambda$ -system generated by monotone class*),
- Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),
- Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*).

**Lemma 592** (*double cascading generated subset systems*)

is explicitly cited in the proof of:

- Lemma 651 (*algebra of sets generated by  $\pi$ -system*),
- Lemma 718 ( *$\sigma$ -algebra generated by monotone class*),
- Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*),
- Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).

**Lemma 594** (*class of nonempty subset systems*)

is explicitly cited in the proof of:

- Lemma 595 (*generated nonempty subset systems*),
- Lemma 596 (*subset system generation preserves class (with nonemptiness)*),
- Lemma 597 (*properties of generated subset system (with nonemptiness)*).



**Lemma 595** (*generated nonempty subset systems*)

is explicitly cited in the proof of:

- Lemma 596 (*subset system generation preserves class (with nonemptiness)*),
- Lemma 597 (*properties of generated subset system (with nonemptiness)*),
- Lemma 598 (*other properties of subset system generation (with nonemptiness)*),
- Lemma 599 (*generated subset systems are monotone (with nonemptiness)*),
- Lemma 600 (*upgrade generated subset system (with nonemptiness)*),
- Lemma 601 (*cascading generated subset systems (with nonemptiness)*).

**Lemma 596** (*subset system generation preserves class (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 600 (*upgrade generated subset system (with nonemptiness)*),
- Lemma 607 (*generated  $\pi$ -system is  $\pi$ -system*).

**Lemma 597** (*properties of generated subset system (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 608 (*properties of  $\pi$ -system generation*).

**Lemma 598** (*other properties of subset system generation (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 609 (*other properties of  $\pi$ -system generation*).

**Lemma 599** (*generated subset systems are monotone (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 621 (*ring of sets contains  $\pi$ -system*),
- Lemma 686 ( *$\lambda$ -system contains monotone class*).

**Lemma 600** (*upgrade generated subset system (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 622 (*upgrade  $\pi$ -system to ring of sets*).

**Lemma 601** (*cascading generated subset systems (with nonemptiness)*)

is explicitly cited in the proof of:

- Lemma 623 (*ring of sets generated by  $\pi$ -system*).

**Definition 602** ( *$\pi$ -system*)

is explicitly cited in the proof of:

- Lemma 604 (*equivalent definition of  $\pi$ -system*),
- Lemma 615 (*other properties of ring of sets*),
- Lemma 630 (*other properties of semiring of sets*),
- Lemma 890 (*content over finite pseudopartition*),
- Lemma 905 (*premeasure over countable pseudopartition*),
- Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),
- Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 604** (*equivalent definition of  $\pi$ -system*)

is explicitly cited in the proof of:

- Lemma 605 (*intersection of  $\pi$ -systems*),
- Lemma 614 (*ring of sets is  $\pi$ -system*),
- Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),
- Lemma 629 (*semiring of sets is  $\pi$ -system*),
- Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*),
- Lemma 694 ( *$\lambda$ -system generated by  $\pi$ -system*),
- Theorem 732 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 605** (*intersection of  $\pi$ -systems*)

is explicitly cited in the proof of:

- Lemma 607 (*generated  $\pi$ -system is  $\pi$ -system*),
- Lemma 608 (*properties of  $\pi$ -system generation*),
- Lemma 609 (*other properties of  $\pi$ -system generation*),
- Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),
- Lemma 623 (*ring of sets generated by  $\pi$ -system*).

**Definition 606 (*generated  $\pi$ -system*)**

is explicitly cited in the proof of:

- Lemma 607 (*generated  $\pi$ -system is  $\pi$ -system*),
- Lemma 608 (*properties of  $\pi$ -system generation*),
- Lemma 609 (*other properties of  $\pi$ -system generation*),
- Lemma 621 (*ring of sets contains  $\pi$ -system*),
- Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),
- Lemma 623 (*ring of sets generated by  $\pi$ -system*),
- Lemma 651 (*algebra of sets generated by  $\pi$ -system*),
- Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).

**Lemma 607 (*generated  $\pi$ -system is  $\pi$ -system*)**

is explicitly cited in the proof of:

- Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 608 (*properties of  $\pi$ -system generation*)**

is explicitly cited in the proof of:

- Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*).

**Lemma 609 (*other properties of  $\pi$ -system generation*)**

is not yet used.

**Definition 611 (*ring of sets*)**

is explicitly cited in the proof of:

- Lemma 612 (*equivalent definition of ring of sets*),
- Lemma 632 (*explicit ring of sets*),
- Lemma 740 (*properties of premeasurable space*).

**Lemma 612 (*equivalent definition of ring of sets*)**

is explicitly cited in the proof of:

- Lemma 614 (*ring of sets is  $\pi$ -system*),
- Lemma 615 (*other properties of ring of sets*),
- Lemma 616 (*intersection of rings of sets*),
- Lemma 619 (*properties of ring of sets generation*),
- Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),
- Lemma 624 (*disjoint union of union in ring of sets*),
- Lemma 631 (*ring of sets is semiring of sets*),
- Lemma 632 (*explicit ring of sets*),
- Lemma 638 (*algebra of sets is ring of sets*),
- Lemma 639 (*other properties of algebra of sets*),
- Lemma 647 (*upgrade ring of sets to algebra of sets*),
- Lemma 740 (*properties of premeasurable space*),
- Lemma 892 (*content is subtractive*),
- Lemma 896 ( *$\sigma$ -finite content*).

**Lemma 614 (*ring of sets is  $\pi$ -system*)**

is explicitly cited in the proof of:

- Lemma 615 (*other properties of ring of sets*),
- Lemma 621 (*ring of sets contains  $\pi$ -system*),
- Lemma 623 (*ring of sets generated by  $\pi$ -system*),

Lemma 640 (*algebra of sets is  $\pi$ -system*),  
 Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*),  
 Lemma 740 (*properties of premeasurable space*).

**Lemma 615 (*other properties of ring of sets*)**

is explicitly cited in the proof of:  
 Lemma 639 (*other properties of algebra of sets*),  
 Lemma 740 (*properties of premeasurable space*).

**Lemma 616 (*intersection of rings of sets*)**

is explicitly cited in the proof of:  
 Lemma 618 (*generated ring of sets is ring of sets*),  
 Lemma 619 (*properties of ring of sets generation*),  
 Lemma 620 (*other properties of ring of sets generation*),  
 Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),  
 Lemma 623 (*ring of sets generated by  $\pi$ -system*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 648 (*algebra of sets generated by ring of sets*).

**Definition 617 (*generated ring of sets*)**

is explicitly cited in the proof of:  
 Lemma 618 (*generated ring of sets is ring of sets*),  
 Lemma 619 (*properties of ring of sets generation*),  
 Lemma 620 (*other properties of ring of sets generation*),  
 Lemma 621 (*ring of sets contains  $\pi$ -system*),  
 Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),  
 Lemma 623 (*ring of sets generated by  $\pi$ -system*),  
 Lemma 646 (*algebra of sets contains ring of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 648 (*algebra of sets generated by ring of sets*),  
 Lemma 651 (*algebra of sets generated by  $\pi$ -system*),  
 Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*),  
 Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).

**Lemma 618 (*generated ring of sets is ring of sets*)**

is explicitly cited in the proof of:  
 Lemma 632 (*explicit ring of sets*),  
 Lemma 662 (*explicit algebra of sets*).

**Lemma 619 (*properties of ring of sets generation*)**

is explicitly cited in the proof of:  
 Lemma 632 (*explicit ring of sets*).

**Lemma 620 (*other properties of ring of sets generation*)**

is not yet used.

**Lemma 621 (*ring of sets contains  $\pi$ -system*)**

is explicitly cited in the proof of:  
 Lemma 622 (*upgrade  $\pi$ -system to ring of sets*),  
 Lemma 649 (*algebra of sets contains  $\pi$ -system*),  
 Lemma 725 ( *$\sigma$ -algebra contains  $\pi$ -system*).

**Lemma 622 (*upgrade  $\pi$ -system to ring of sets*)**

is explicitly cited in the proof of:  
 Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*).

**Lemma 623** (*ring of sets generated by  $\pi$ -system*)

is explicitly cited in the proof of:

Lemma 651 (*algebra of sets generated by  $\pi$ -system*),

Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).

**Lemma 624** (*disjoint union of union in ring of sets*)

is explicitly cited in the proof of:

Lemma 652 (*disjoint union of union in algebra of sets*).

**Definition 626** (*semiring of sets*)

is explicitly cited in the proof of:

Lemma 627 (*equivalent definition of semiring of sets*),

Lemma 629 (*semiring of sets is  $\pi$ -system*),

Lemma 630 (*other properties of semiring of sets*),

Lemma 631 (*ring of sets is semiring of sets*),

Lemma 632 (*explicit ring of sets*),

Lemma 655 (*equivalent definition of semi-algebra of sets*),

Lemma 891 (*content satisfies the binary inclusion-exclusion principle*),

Lemma 894 (*content satisfies the finite Boole inequality*),

Lemma 906 (*premeasure satisfies the Boole inequality*).

**Lemma 627** (*equivalent definition of semiring of sets*)

is explicitly cited in the proof of:

Lemma 632 (*explicit ring of sets*).

**Lemma 629** (*semiring of sets is  $\pi$ -system*)

is explicitly cited in the proof of:

Lemma 630 (*other properties of semiring of sets*),

Lemma 659 (*semi-algebra of sets is  $\pi$ -system*).

**Lemma 630** (*other properties of semiring of sets*)

is explicitly cited in the proof of:

Lemma 658 (*other properties of semi-algebra of sets*),

Lemma 893 (*content is monotone*).

**Lemma 631** (*ring of sets is semiring of sets*)

is explicitly cited in the proof of:

Lemma 740 (*properties of premeasurable space*),

Lemma 742 (*properties of measurable space*),

Lemma 896 ( *$\sigma$ -finite content*).

**Lemma 632** (*explicit ring of sets*)

is explicitly cited in the proof of:

Lemma 662 (*explicit algebra of sets*).

**Definition 634** (*algebra of sets*)

is explicitly cited in the proof of:

Lemma 635 (*equivalent definition of algebra of sets*),

Lemma 641 (*intersection of algebras of sets*),

Lemma 696 ( *$\sigma$ -algebra*),

Lemma 700 (*other properties of  $\sigma$ -algebra*),

Lemma 1085 ( *$\mathcal{L}$  is  $\sigma$ -algebra*).

**Lemma 635** (*equivalent definition of algebra of sets*)

is explicitly cited in the proof of:

Lemma 637 (*other equivalent definition of algebra of sets*),

Lemma 638 (*algebra of sets is ring of sets*),

Lemma 639 (*other properties of algebra of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 660 (*algebra of sets is semi-algebra of sets*),  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Theorem 736 (*monotone class*).

**Lemma 637** (*other equivalent definition of algebra of sets*)

is explicitly cited in the proof of:

Lemma 638 (*algebra of sets is ring of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 675 (*monotone class generated by algebra of sets*),  
 Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 1077 ( *$\mathcal{L}$  is algebra of sets*).

**Lemma 638** (*algebra of sets is ring of sets*)

is explicitly cited in the proof of:

Lemma 639 (*other properties of algebra of sets*),  
 Lemma 640 (*algebra of sets is  $\pi$ -system*),  
 Lemma 646 (*algebra of sets contains ring of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 648 (*algebra of sets generated by ring of sets*),  
 Lemma 652 (*disjoint union of union in algebra of sets*),  
 Lemma 703 ( *$\sigma$ -algebra is ring of sets*).

**Lemma 639** (*other properties of algebra of sets*)

is explicitly cited in the proof of:

Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*).

**Lemma 640** (*algebra of sets is  $\pi$ -system*)

is explicitly cited in the proof of:

Lemma 704 ( *$\sigma$ -algebra is  $\pi$ -system*).

**Lemma 641** (*intersection of algebras of sets*)

is explicitly cited in the proof of:

Lemma 643 (*generated algebra of sets is algebra of sets*),  
 Lemma 644 (*properties of algebra of sets generation*),  
 Lemma 645 (*other properties of algebra of sets generation*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 648 (*algebra of sets generated by ring of sets*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*).

**Definition 642** (*generated algebra of sets*)

is explicitly cited in the proof of:

Lemma 643 (*generated algebra of sets is algebra of sets*),  
 Lemma 644 (*properties of algebra of sets generation*),  
 Lemma 645 (*other properties of algebra of sets generation*),  
 Lemma 646 (*algebra of sets contains ring of sets*),  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 648 (*algebra of sets generated by ring of sets*),  
 Lemma 651 (*algebra of sets generated by  $\pi$ -system*),  
 Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*),

Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*),  
 Lemma 1221 (*measurability of measure of section (finite)*).

**Lemma 643** (*generated algebra of sets is algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 738 (*usage of monotone class theorem*).

**Lemma 644** (*properties of algebra of sets generation*)  
 is not yet used.

**Lemma 645** (*other properties of algebra of sets generation*)  
 is not yet used.

**Lemma 646** (*algebra of sets contains ring of sets*)  
 is explicitly cited in the proof of:  
 Lemma 647 (*upgrade ring of sets to algebra of sets*),  
 Lemma 649 (*algebra of sets contains  $\pi$ -system*),  
 Lemma 722 ( *$\sigma$ -algebra contains ring of sets*).

**Lemma 647** (*upgrade ring of sets to algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 650 (*upgrade  $\pi$ -system to algebra of sets*),  
 Lemma 662 (*explicit algebra of sets*),  
 Lemma 723 (*upgrade ring of sets to  $\sigma$ -algebra*).

**Lemma 648** (*algebra of sets generated by ring of sets*)  
 is explicitly cited in the proof of:  
 Lemma 651 (*algebra of sets generated by  $\pi$ -system*),  
 Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*).

**Lemma 649** (*algebra of sets contains  $\pi$ -system*)  
 is not yet used.

**Lemma 650** (*upgrade  $\pi$ -system to algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 726 (*upgrade  $\pi$ -system to  $\sigma$ -algebra*).

**Lemma 651** (*algebra of sets generated by  $\pi$ -system*)  
 is not yet used.

**Lemma 652** (*disjoint union of union in algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 1081 (*disjoint union of union in  $\mathcal{L}$* ).

**Definition 654** (*semi-algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 655 (*equivalent definition of semi-algebra of sets*),  
 Lemma 657 (*semi-algebra of sets is semiring of sets*),  
 Lemma 662 (*explicit algebra of sets*),  
 Lemma 895 (*finite content is bounded*).

**Lemma 655** (*equivalent definition of semi-algebra of sets*)  
 is explicitly cited in the proof of:  
 Lemma 660 (*algebra of sets is semi-algebra of sets*),  
 Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*).

**Lemma 657** (*semi-algebra of sets is semiring of sets*)

is explicitly cited in the proof of:

- Lemma 658 (*other properties of semi-algebra of sets*),
- Lemma 659 (*semi-algebra of sets is  $\pi$ -system*),
- Lemma 661 (*algebra of sets is semiring of sets*).

**Lemma 658** (*other properties of semi-algebra of sets*)

is explicitly cited in the proof of:

- Lemma 895 (*finite content is bounded*).

**Lemma 659** (*semi-algebra of sets is  $\pi$ -system*)

is not yet used.

**Lemma 660** (*algebra of sets is semi-algebra of sets*)

is explicitly cited in the proof of:

- Lemma 661 (*algebra of sets is semiring of sets*),
- Lemma 705 ( *$\sigma$ -algebra is semi-algebra of sets*),
- Lemma 742 (*properties of measurable space*),
- Lemma 903 (*finite premeasure is bounded*).

**Lemma 661** (*algebra of sets is semiring of sets*)

is explicitly cited in the proof of:

- Lemma 706 ( *$\sigma$ -algebra is semiring of sets*).

**Lemma 662** (*explicit algebra of sets*)

is explicitly cited in the proof of:

- Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*).

**Definition 664** (*monotone class*)

is explicitly cited in the proof of:

- Lemma 665 (*intersection of monotone classes*),
- Lemma 672 ( *$\mathcal{C}^\lambda$  is monotone class*),
- Lemma 679 ( *$\lambda$ -system is monotone class*),
- Lemma 680 (*other properties of  $\lambda$ -system*),
- Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 665** (*intersection of monotone classes*)

is explicitly cited in the proof of:

- Lemma 667 (*generated monotone class is monotone class*),
- Lemma 668 (*properties of monotone class generation*),
- Lemma 669 (*other properties of monotone class generation*),
- Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),
- Lemma 688 ( *$\lambda$ -system generated by monotone class*).

**Definition 666** (*generated monotone class*)

is explicitly cited in the proof of:

- Lemma 667 (*generated monotone class is monotone class*),
- Lemma 668 (*properties of monotone class generation*),
- Lemma 669 (*other properties of monotone class generation*),
- Lemma 686 ( *$\lambda$ -system contains monotone class*),
- Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),
- Lemma 688 ( *$\lambda$ -system generated by monotone class*),
- Lemma 718 ( *$\sigma$ -algebra generated by monotone class*).

**Lemma 667** (*generated monotone class is monotone class*)

is explicitly cited in the proof of:

Lemma 674 (*monotone class is closed under set difference*),

Lemma 675 (*monotone class generated by algebra of sets*).

**Lemma 668** (*properties of monotone class generation*)

is explicitly cited in the proof of:

Lemma 674 (*monotone class is closed under set difference*),

Lemma 738 (*usage of monotone class theorem*).

**Lemma 669** (*other properties of monotone class generation*)

is not yet used.

**Definition 670** (*monotone class and symmetric set difference*)

is explicitly cited in the proof of:

Lemma 671 ( $\mathcal{C}^{\setminus}$  *is symmetric*),

Lemma 672 ( $\mathcal{C}^{\setminus}$  *is monotone class*),

Lemma 674 (*monotone class is closed under set difference*).

**Lemma 671** ( $\mathcal{C}^{\setminus}$  *is symmetric*)

is explicitly cited in the proof of:

Lemma 674 (*monotone class is closed under set difference*).

**Lemma 672** ( $\mathcal{C}^{\setminus}$  *is monotone class*)

is explicitly cited in the proof of:

Lemma 674 (*monotone class is closed under set difference*).

**Lemma 674** (*monotone class is closed under set difference*)

is explicitly cited in the proof of:

Lemma 675 (*monotone class generated by algebra of sets*).

**Lemma 675** (*monotone class generated by algebra of sets*)

is explicitly cited in the proof of:

Theorem 736 (*monotone class*).

**Definition 677** ( $\lambda$ -*system*)

is explicitly cited in the proof of:

Lemma 678 (*equivalent definition of  $\lambda$ -system*),

Lemma 679 ( *$\lambda$ -system is monotone class*),

Lemma 680 (*other properties of  $\lambda$ -system*),

Lemma 681 (*intersection of  $\lambda$ -systems*),

Lemma 691 ( $\Lambda^{\cap}$  *is  $\lambda$ -system*),

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),

Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 678** (*equivalent definition of  $\lambda$ -system*)

is explicitly cited in the proof of:

Lemma 679 ( *$\lambda$ -system is monotone class*),

Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),

Lemma 691 ( $\Lambda^{\cap}$  *is  $\lambda$ -system*),

Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*).

**Lemma 679** ( $\lambda$ -*system is monotone class*)

is explicitly cited in the proof of:

Lemma 680 (*other properties of  $\lambda$ -system*),

Lemma 686 ( *$\lambda$ -system contains monotone class*),



Lemma 688 ( *$\lambda$ -system generated by monotone class*),  
 Lemma 702 ( *$\sigma$ -algebra is monotone class*).

**Lemma 680** (*other properties of  $\lambda$ -system*)  
 is not yet used.

**Lemma 681** (*intersection of  $\lambda$ -systems*)  
 is explicitly cited in the proof of:  
 Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*),  
 Lemma 684 (*properties of  $\lambda$ -system generation*),  
 Lemma 685 (*other properties of  $\lambda$ -system generation*),  
 Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),  
 Lemma 688 ( *$\lambda$ -system generated by monotone class*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*).

**Definition 682** (*generated  $\lambda$ -system*)  
 is explicitly cited in the proof of:  
 Lemma 683 (*generated  $\lambda$ -system is  $\lambda$ -system*),  
 Lemma 684 (*properties of  $\lambda$ -system generation*),  
 Lemma 685 (*other properties of  $\lambda$ -system generation*),  
 Lemma 686 ( *$\lambda$ -system contains monotone class*),  
 Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),  
 Lemma 688 ( *$\lambda$ -system generated by monotone class*),  
 Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),  
 Lemma 718 ( *$\sigma$ -algebra generated by monotone class*).

**Lemma 683** (*generated  $\lambda$ -system is  $\lambda$ -system*)  
 is explicitly cited in the proof of:  
 Lemma 692 ( *$\lambda$ -system with intersection*),  
 Lemma 694 ( *$\lambda$ -system generated by  $\pi$ -system*).

**Lemma 684** (*properties of  $\lambda$ -system generation*)  
 is explicitly cited in the proof of:  
 Lemma 692 ( *$\lambda$ -system with intersection*),  
 Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 685** (*other properties of  $\lambda$ -system generation*)  
 is not yet used.

**Lemma 686** ( *$\lambda$ -system contains monotone class*)  
 is explicitly cited in the proof of:  
 Lemma 687 (*upgrade monotone class to  $\lambda$ -system*),  
 Lemma 716 ( *$\sigma$ -algebra contains monotone class*).

**Lemma 687** (*upgrade monotone class to  $\lambda$ -system*)  
 is explicitly cited in the proof of:  
 Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*).

**Lemma 688** ( *$\lambda$ -system generated by monotone class*)  
 is explicitly cited in the proof of:  
 Lemma 718 ( *$\sigma$ -algebra generated by monotone class*).

**Definition 689** ( *$\lambda$ -system and intersection*)  
 is explicitly cited in the proof of:

Lemma 690 ( *$\Lambda^\cap$  is symmetric*),  
 Lemma 692 ( *$\lambda$ -system with intersection*),  
 Lemma 693 ( *$\lambda$ -system is closed under intersection*).

**Lemma 690 ( *$\Lambda^\cap$  is symmetric*)**

is explicitly cited in the proof of:  
 Lemma 692 ( *$\lambda$ -system with intersection*).

**Lemma 691 ( *$\Lambda^\cap$  is  $\lambda$ -system*)**

is explicitly cited in the proof of:  
 Lemma 692 ( *$\lambda$ -system with intersection*).

**Lemma 692 ( *$\lambda$ -system with intersection*)**

is explicitly cited in the proof of:  
 Lemma 693 ( *$\lambda$ -system is closed under intersection*).

**Lemma 693 ( *$\lambda$ -system is closed under intersection*)**

is explicitly cited in the proof of:  
 Lemma 694 ( *$\lambda$ -system generated by  $\pi$ -system*).

**Lemma 694 ( *$\lambda$ -system generated by  $\pi$ -system*)**

is explicitly cited in the proof of:  
 Theorem 732 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 696 ( *$\sigma$ -algebra*)**

is explicitly cited in the proof of:  
 Lemma 697 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 699 ( *$\sigma$ -algebra is algebra of sets*),  
 Lemma 707 (*intersection of  $\sigma$ -algebras*),  
 Lemma 712 (*countable union  $\sigma$ -algebra generator*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 744 (*some Borel subsets*),  
 Lemma 750 (*inverse  $\sigma$ -algebra*),  
 Lemma 751 (*image  $\sigma$ -algebra*),  
 Lemma 753 (*countable range is measurable*),  
 Lemma 763 (*trace  $\sigma$ -algebra*),  
 Lemma 766 (*generating on a partition*),  
 Lemma 786 (*measurability of section*),  
 Lemma 787 (*countable union of sections is measurable*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),  
 Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),  
 Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),  
 Lemma 833 (*measurability and masking*),  
 Lemma 834 (*measurability of restriction*),  
 Lemma 923 (*measure satisfies the Boole inequality*),  
 Lemma 924 (*compatibility of null measure with countable union*),  
 Lemma 925 (*measure is continuous from below*),  
 Lemma 927 (*measure is continuous from above*),  
 Lemma 935 (*counting measure*),  
 Lemma 937 ( *$\sigma$ -finite counting measure*),  
 Lemma 947 (*negligibility with trivial measure*),  
 Lemma 949 (*empty set is negligible*),  
 Lemma 955 (*equivalent definition of property almost satisfied*),  
 Lemma 1008 (*masking almost nowhere*),  
 Lemma 1085 ( *$\mathcal{L}$  is  $\sigma$ -algebra*),  
 Lemma 1102 ( *$\mathcal{IF}_\mathbb{K}$  is  $\sigma$ -additive*),

- Lemma 1158 (*integral in  $\mathcal{SF}_+$  of zero is zero*),  
 Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),  
 Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ).

**Lemma 697 (*equivalent definition of  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

- Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*),  
 Lemma 742 (*properties of measurable space*),  
 Lemma 754 (*constant function is measurable*),  
 Lemma 764 (*measurability of measurable subspace*),  
 Lemma 772 (*measurability of function defined on a pseudopartition*),  
 Lemma 776 (*measurability of function to product space*),  
 Lemma 788 (*countable intersection of sections is measurable*),  
 Lemma 811 (*measurability of indicator function*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 828 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 829 ( *$\mathcal{M}$  is closed under supremum*),  
 Lemma 834 (*measurability of restriction*),  
 Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*),  
 Lemma 931 (*restricted measure*),  
 Lemma 936 (*finiteness of counting measure*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 999 (*measurability of summability domain*),  
 Lemma 1103 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication*),  
 Lemma 1106 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under restriction*),  
 Lemma 1111 ( *$\mathcal{SF}_E$  canonical representation*),  
 Lemma 1120 ( *$\mathcal{SF}_E$  is closed under restriction*),  
 Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Lemma 1138 ( *$\mathcal{M}_+$  is closed under restriction*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*),  
 Lemma 1223 (*measurability of measure of section*),  
 Lemma 1228 (*tensor product of finite measures*),  
 Lemma 1231 (*uniqueness of tensor product measure (finite)*),  
 Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 699 ( *$\sigma$ -algebra is algebra of sets*)**

is explicitly cited in the proof of:

- Lemma 700 (*other properties of  $\sigma$ -algebra*),  
 Lemma 703 ( *$\sigma$ -algebra is ring of sets*),  
 Lemma 704 ( *$\sigma$ -algebra is  $\pi$ -system*),  
 Lemma 705 ( *$\sigma$ -algebra is semi-algebra of sets*),  
 Lemma 706 ( *$\sigma$ -algebra is semiring of sets*),  
 Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*),  
 Theorem 736 (*monotone class*),  
 Lemma 742 (*properties of measurable space*).

**Lemma 700 (*other properties of  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 728 (*disjoint union of union in  $\sigma$ -algebra*),  
 Lemma 730 (*algebra of sets generated by product of  $\sigma$ -algebras*),  
 Lemma 742 (*properties of measurable space*),  
 Lemma 753 (*countable range is measurable*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),  
 Lemma 913 (*measure*),  
 Lemma 916 (*measure is additive*),  
 Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*),  
 Lemma 918 (*measure is subtractive*).

**Lemma 701 ( *$\sigma$ -algebra is  $\lambda$ -system*)**

is explicitly cited in the proof of:

Lemma 702 ( *$\sigma$ -algebra is monotone class*),  
 Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),  
 Theorem 732 (*Dynkin  $\pi$ - $\lambda$  theorem*),  
 Lemma 742 (*properties of measurable space*).

**Lemma 702 ( *$\sigma$ -algebra is monotone class*)**

is explicitly cited in the proof of:

Theorem 736 (*monotone class*),  
 Lemma 742 (*properties of measurable space*).

**Lemma 703 ( *$\sigma$ -algebra is ring of sets*)**

is explicitly cited in the proof of:

Lemma 742 (*properties of measurable space*),  
 Lemma 915 (*measure space is premeasure space*).

**Lemma 704 ( *$\sigma$ -algebra is  $\pi$ -system*)**

is explicitly cited in the proof of:

Theorem 732 (*Dynkin  $\pi$ - $\lambda$  theorem*),  
 Lemma 742 (*properties of measurable space*).

**Lemma 705 ( *$\sigma$ -algebra is semi-algebra of sets*)**

is not yet used.

**Lemma 706 ( *$\sigma$ -algebra is semiring of sets*)**

is not yet used.

**Lemma 707 (*intersection of  $\sigma$ -algebras*)**

is explicitly cited in the proof of:

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Lemma 710 (*properties of  $\sigma$ -algebra generation*),  
 Lemma 711 (*other properties of  $\sigma$ -algebra generation*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*).

**Definition 708 (*generated  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*),  
 Lemma 710 (*properties of  $\sigma$ -algebra generation*),

Lemma 711 (*other properties of  $\sigma$ -algebra generation*),  
 Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*),  
 Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 715 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),  
 Lemma 718 ( *$\sigma$ -algebra generated by monotone class*),  
 Lemma 719 ( *$\sigma$ -algebra contains algebra of sets*),  
 Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
 Lemma 721 ( *$\sigma$ -algebra generated by algebra of sets*),  
 Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*),  
 Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*),  
 Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),  
 Lemma 738 (*usage of monotone class theorem*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),  
 Lemma 800 (*generators of Borel subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 709 (*generated  $\sigma$ -algebra is  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 712 (*countable union  $\sigma$ -algebra generator*),  
 Lemma 744 (*some Borel subsets*),  
 Lemma 755 (*inverse image of generating family*),  
 Lemma 756 (*equivalent definition of measurable function*),  
 Lemma 761 (*continuous is measurable*),  
 Lemma 775 (*product of measurable subsets is measurable*),  
 Lemma 779 (*generating product measurable space*),  
 Lemma 786 (*measurability of section*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 710 (*properties of  $\sigma$ -algebra generation*)**

is explicitly cited in the proof of:

Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),  
 Lemma 738 (*usage of monotone class theorem*),  
 Lemma 755 (*inverse image of generating family*),  
 Lemma 756 (*equivalent definition of measurable function*),  
 Lemma 779 (*generating product measurable space*),  
 Lemma 1087 ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* ).

**Lemma 711 (*other properties of  $\sigma$ -algebra generation*)**

is explicitly cited in the proof of:

Lemma 712 (*countable union  $\sigma$ -algebra generator*),  
 Lemma 796 (*Borel subsets of  $\mathbb{R}$* ).

**Lemma 712 (*countable union  $\sigma$ -algebra generator*)**

is explicitly cited in the proof of:

Lemma 745 (*countable union Borel  $\sigma$ -algebra generator*).

**Lemma 713 ( *$\sigma$ -algebra contains  $\lambda$ -system*)**

is explicitly cited in the proof of:

Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*),  
 Lemma 716 ( *$\sigma$ -algebra contains monotone class*).

**Lemma 714 (*upgrade  $\lambda$ -system to  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 717 (*upgrade monotone class to  $\sigma$ -algebra*),  
 Theorem 732 (*Dynkin  $\pi$ - $\lambda$  theorem*).

- Lemma 715** ( *$\sigma$ -algebra generated by  $\lambda$ -system*)  
is explicitly cited in the proof of:  
Lemma 718 ( *$\sigma$ -algebra generated by monotone class*).
- Lemma 716** ( *$\sigma$ -algebra contains monotone class*)  
is not yet used.
- Lemma 717** (*upgrade monotone class to  $\sigma$ -algebra*)  
is explicitly cited in the proof of:  
Theorem 736 (*monotone class*).
- Lemma 718** ( *$\sigma$ -algebra generated by monotone class*)  
is not yet used.
- Lemma 719** ( *$\sigma$ -algebra contains algebra of sets*)  
is explicitly cited in the proof of:  
Lemma 720 (*upgrade algebra of sets to  $\sigma$ -algebra*),  
Lemma 722 ( *$\sigma$ -algebra contains ring of sets*),  
Lemma 1221 (*measurability of measure of section (finite)*),  
Lemma 1231 (*uniqueness of tensor product measure (finite)*).
- Lemma 720** (*upgrade algebra of sets to  $\sigma$ -algebra*)  
is explicitly cited in the proof of:  
Lemma 723 (*upgrade ring of sets to  $\sigma$ -algebra*),  
Lemma 726 (*upgrade  $\pi$ -system to  $\sigma$ -algebra*).
- Lemma 721** ( *$\sigma$ -algebra generated by algebra of sets*)  
is explicitly cited in the proof of:  
Lemma 724 ( *$\sigma$ -algebra generated by ring of sets*).
- Lemma 722** ( *$\sigma$ -algebra contains ring of sets*)  
is explicitly cited in the proof of:  
Lemma 725 ( *$\sigma$ -algebra contains  $\pi$ -system*).
- Lemma 723** (*upgrade ring of sets to  $\sigma$ -algebra*)  
is not yet used.
- Lemma 724** ( *$\sigma$ -algebra generated by ring of sets*)  
is explicitly cited in the proof of:  
Lemma 727 ( *$\sigma$ -algebra generated by  $\pi$ -system*).
- Lemma 725** ( *$\sigma$ -algebra contains  $\pi$ -system*)  
is not yet used.
- Lemma 726** (*upgrade  $\pi$ -system to  $\sigma$ -algebra*)  
is not yet used.
- Lemma 727** ( *$\sigma$ -algebra generated by  $\pi$ -system*)  
is not yet used.
- Lemma 728** (*disjoint union of union in  $\sigma$ -algebra*)  
is not yet used.
- Lemma 730** (*algebra of sets generated by product of  $\sigma$ -algebras*)  
is explicitly cited in the proof of:  
Lemma 1221 (*measurability of measure of section (finite)*),  
Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Theorem 732** (*Dynkin  $\pi$ - $\lambda$  theorem*)

is explicitly cited in the proof of:

Lemma 734 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 734** (*usage of Dynkin  $\pi$ - $\lambda$  theorem*)

is explicitly cited in the proof of:

Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*).

**Theorem 736** (*monotone class*)

is explicitly cited in the proof of:

Lemma 738 (*usage of monotone class theorem*).

**Lemma 738** (*usage of monotone class theorem*)

is explicitly cited in the proof of:

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Definition 739** (*premeasurable space*)

is explicitly cited in the proof of:

Lemma 740 (*properties of premeasurable space*),

Lemma 742 (*properties of measurable space*),

Lemma 897 (*premeasure*).

**Lemma 740** (*properties of premeasurable space*)

is explicitly cited in the proof of:

Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*),

Lemma 901 (*premeasure is subtractive*),

Lemma 902 (*premeasure is monotone*),

Lemma 904 ( *$\sigma$ -finite premeasure*),

Lemma 905 (*premeasure over countable pseudopartition*),

Lemma 906 (*premeasure satisfies the Boole inequality*),

Lemma 907 (*compatibility of null premeasure with countable union*),

Lemma 908 (*premeasure is continuous from below*),

Lemma 909 (*equivalent definition of premeasure (from below)*),

Lemma 910 (*premeasure is continuous from above*),

Lemma 911 (*equivalent definition of premeasure (from above)*),

Lemma 912 (*properties of premeasure space*).

**Definition 741** (*measurable space*)

is explicitly cited in the proof of:

Lemma 742 (*properties of measurable space*),

Lemma 750 (*inverse  $\sigma$ -algebra*),

Lemma 751 (*image  $\sigma$ -algebra*),

Lemma 753 (*countable range is measurable*),

Lemma 754 (*constant function is measurable*),

Lemma 763 (*trace  $\sigma$ -algebra*),

Lemma 786 (*measurability of section*),

Lemma 787 (*countable union of sections is measurable*),

Lemma 811 (*measurability of indicator function*),

Lemma 833 (*measurability and masking*),

Lemma 834 (*measurability of restriction*),

Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*),

Lemma 915 (*measure space is premeasure space*),

Lemma 935 (*counting measure*),

Lemma 937 ( *$\sigma$ -finite counting measure*),

Lemma 949 (*empty set is negligible*),



- Lemma 955 (*equivalent definition of property almost satisfied*),
- Lemma 1008 (*masking almost nowhere*),
- Lemma 1102 ( *$\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive*),
- Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ).

**Lemma 742 (*properties of measurable space*)**

is explicitly cited in the proof of:

- Lemma 903 (*finite premeasure is bounded*),
- Lemma 929 (*properties of measure space*).

**Definition 743 (*Borel  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

- Lemma 744 (*some Borel subsets*),
- Lemma 745 (*countable union Borel  $\sigma$ -algebra generator*),
- Lemma 761 (*continuous is measurable*),
- Lemma 768 (*Borel sub- $\sigma$ -algebra*),
- Lemma 780 (*Borel subsets in product measurable subspace*),
- Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),
- Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),
- Lemma 820 (*measurability of numeric function*),
- Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 744 (*some Borel subsets*)**

is explicitly cited in the proof of:

- Lemma 759 ( *$\mathcal{M}_Y$  inverse image of countable is measurable*),
- Lemma 794 (*some Borel subsets of numbers*),
- Lemma 1278 (*integral over interval*).

**Lemma 745 (*countable union Borel  $\sigma$ -algebra generator*)**

is explicitly cited in the proof of:

- Lemma 796 (*Borel subsets of  $\mathbb{R}$* ).

**Definition 748 (*measurability and structure*)**

is explicitly cited in the proof of:

- Lemma 811 (*measurability of indicator function*),
- Lemma 853 (*measurability of tensor product of numeric functions*),
- Lemma 1100 (*indicator and support are inverse*),
- Lemma 1102 ( *$\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive*),
- Lemma 1103 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication*),
- Lemma 1105 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero*),
- Lemma 1106 ( *$\mathcal{IF}_{\mathbb{K}}$  is closed under restriction*),
- Lemma 1107 ( *$\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive*),
- Lemma 1113 ( *$\mathcal{SF}_E$  is measurable*).

**Definition 749 (*measurable function*)**

is explicitly cited in the proof of:

- Lemma 750 (*inverse  $\sigma$ -algebra*),
- Lemma 751 (*image  $\sigma$ -algebra*),
- Lemma 752 (*identity function is measurable*),
- Lemma 753 (*countable range is measurable*),
- Lemma 756 (*equivalent definition of measurable function*),
- Lemma 757 (*compatibility of measurability with composition*),
- Lemma 759 ( *$\mathcal{M}_Y$  inverse image of countable is measurable*),
- Lemma 761 (*continuous is measurable*),
- Lemma 763 (*trace  $\sigma$ -algebra*),



Lemma 770 (*source restriction of measurable function*),  
 Lemma 771 (*destination restriction of measurable function*),  
 Lemma 772 (*measurability of function defined on a pseudopartition*),  
 Lemma 776 (*measurability of function to product space*),  
 Lemma 777 (*canonical projection is measurable*),  
 Lemma 779 (*generating product measurable space*),  
 Lemma 790 (*measurability of function from product space*),  
 Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),  
 Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 820 (*measurability of numeric function*),  
 Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 834 (*measurability of restriction*).

**Lemma 750 (*inverse  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 755 (*inverse image of generating family*).

**Lemma 751 (*image  $\sigma$ -algebra*)**

is explicitly cited in the proof of:

Lemma 755 (*inverse image of generating family*).

**Lemma 752 (*identity function is measurable*)**

is not yet used.

**Lemma 753 (*countable range is measurable*)**

is explicitly cited in the proof of:

Lemma 754 (*constant function is measurable*),

Lemma 760 ( *$\mathcal{M}_Y$  countable range*),

Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**Lemma 754 (*constant function is measurable*)**

is explicitly cited in the proof of:

Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*),

Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),

Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),

Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),

Lemma 836 (*measurability of nonnegative and nonpositive parts*),

Lemma 1023 (*essential supremum of constant function*),

Lemma 1040 (*nonnegative essential supremum of constant function*),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 755 (*inverse image of generating family*)**

is explicitly cited in the proof of:

Lemma 756 (*equivalent definition of measurable function*),

Lemma 765 (*generating measurable subspace*),

Lemma 779 (*generating product measurable space*).

**Lemma 756 (*equivalent definition of measurable function*)**

is explicitly cited in the proof of:

Lemma 761 (*continuous is measurable*),

Lemma 776 (*measurability of function to product space*),

Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),

Lemma 820 (*measurability of numeric function*).

**Lemma 757** (*compatibility of measurability with composition*)

is explicitly cited in the proof of:

- Lemma 770 (*source restriction of measurable function*),
- Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  *is algebra*),
- Lemma 841 ( $\mathcal{M}$  *is closed under power function*),
- Lemma 853 (*measurability of tensor product of numeric functions*),
- Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*).

**Definition 758** ( $\mathcal{M}_Y$ , *set of measurable functions to  $Y$* )

is explicitly cited in the proof of:

- Lemma 759 ( $\mathcal{M}_Y$  *inverse image of countable is measurable*),
- Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  *is algebra*),
- Lemma 819 ( $\mathcal{M}$  *and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 853 (*measurability of tensor product of numeric functions*).

**Lemma 759** ( $\mathcal{M}_Y$  *inverse image of countable is measurable*)

is explicitly cited in the proof of:

- Lemma 760 ( $\mathcal{M}_Y$  *countable range*),
- Lemma 808 (*inverse image of countable is measurable*).

**Lemma 760** ( $\mathcal{M}_Y$  *countable range*)

is explicitly cited in the proof of:

- Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$  *countable range*).

**Lemma 761** (*continuous is measurable*)

is explicitly cited in the proof of:

- Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$  *is algebra*),
- Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**Lemma 763** (*trace  $\sigma$ -algebra*)

is explicitly cited in the proof of:

- Lemma 764 (*measurability of measurable subspace*),
- Lemma 770 (*source restriction of measurable function*),
- Lemma 930 (*trace measure*).

**Lemma 764** (*measurability of measurable subspace*)

is explicitly cited in the proof of:

- Lemma 765 (*generating measurable subspace*),
- Lemma 930 (*trace measure*),
- Lemma 932 (*trace measure is restricted measure*),
- Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under extension by zero*),
- Lemma 1106 ( $\mathcal{IF}_{\mathbb{K}}$  *is closed under restriction*),
- Lemma 1120 ( $\mathcal{SF}_E$  *is closed under restriction*),
- Lemma 1138 ( $\mathcal{M}_+$  *is closed under restriction*).

**Lemma 765** (*generating measurable subspace*)

is explicitly cited in the proof of:

- Lemma 766 (*generating on a partition*),
- Lemma 768 (*Borel sub- $\sigma$ -algebra*),
- Lemma 780 (*Borel subsets in product measurable subspace*),
- Lemma 803 (*Borel subsets of  $\overline{\mathbb{R}}_+$* ),
- Lemma 804 (*Borel subsets of  $\overline{\mathbb{R}}_+$* ).

**Lemma 766** (*generating on a partition*)

is explicitly cited in the proof of:

- Lemma 769 (*Borel subsets of partition*),
- Lemma 800 (*generators of Borel subsets of  $\overline{\mathbb{R}}$* ).

**Lemma 768** (*Borel sub- $\sigma$ -algebra*)

is explicitly cited in the proof of:

- Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),
- Lemma 803 (*Borel subsets of  $\mathbb{R}_+$* ),
- Lemma 804 (*Borel subsets of  $\overline{\mathbb{R}}_+$* ).

**Lemma 769** (*Borel subsets of partition*)

is explicitly cited in the proof of:

- Lemma 798 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ).

**Lemma 770** (*source restriction of measurable function*)

is not yet used.

**Lemma 771** (*destination restriction of measurable function*)

is not yet used.

**Lemma 772** (*measurability of function defined on a pseudopartition*)

is explicitly cited in the proof of:

- Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*).

**Definition 774** (*tensor product of  $\sigma$ -algebras*)

is explicitly cited in the proof of:

- Lemma 775 (*product of measurable subsets is measurable*),
- Lemma 776 (*measurability of function to product space*),
- Lemma 779 (*generating product measurable space*),
- Lemma 786 (*measurability of section*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 775** (*product of measurable subsets is measurable*)

is explicitly cited in the proof of:

- Lemma 776 (*measurability of function to product space*),
- Lemma 1219 (*measure of section of product*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1227 (*candidate tensor product measure is tensor product measure*),
- Lemma 1229 (*tensor product of  $\sigma$ -finite measures*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*),
- Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 776** (*measurability of function to product space*)

is explicitly cited in the proof of:

- Lemma 777 (*canonical projection is measurable*),
- Lemma 778 (*permutation is measurable*),
- Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*).

**Lemma 777** (*canonical projection is measurable*)

is explicitly cited in the proof of:

- Lemma 778 (*permutation is measurable*),
- Lemma 779 (*generating product measurable space*),
- Lemma 853 (*measurability of tensor product of numeric functions*).

**Lemma 778** (*permutation is measurable*)

is not yet used.

**Lemma 779** (*generating product measurable space*)

is explicitly cited in the proof of:

Lemma 780 (*Borel subsets in product measurable subspace*).

**Lemma 780** (*Borel subsets in product measurable subspace*)

is explicitly cited in the proof of:

Lemma 806 (*Borel subsets of  $\mathbb{C}$* ),

Lemma 813 ( *$\mathcal{M}_{\mathbb{K}}$  is algebra*).

**Definition 783** (*section in Cartesian product*)

is explicitly cited in the proof of:

Lemma 784 (*section of product*),

Lemma 785 (*compatibility of section with set operations*),

Lemma 789 (*indicator of section*).

**Lemma 784** (*section of product*)

is explicitly cited in the proof of:

Lemma 786 (*measurability of section*),

Lemma 1219 (*measure of section of product*).

**Lemma 785** (*compatibility of section with set operations*)

is explicitly cited in the proof of:

Lemma 786 (*measurability of section*),

Lemma 787 (*countable union of sections is measurable*),

Lemma 788 (*countable intersection of sections is measurable*),

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*).

**Lemma 786** (*measurability of section*)

is explicitly cited in the proof of:

Lemma 787 (*countable union of sections is measurable*),

Lemma 788 (*countable intersection of sections is measurable*),

Lemma 790 (*measurability of function from product space*),

Lemma 1218 (*measure of section*),

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1223 (*measurability of measure of section*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Lemma 1234 (*negligibility of measurable section*),

Theorem 1242 (*Tonelli*).

**Lemma 787** (*countable union of sections is measurable*)

is explicitly cited in the proof of:

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*).

**Lemma 788** (*countable intersection of sections is measurable*)

is explicitly cited in the proof of:

Lemma 1221 (*measurability of measure of section (finite)*).

**Lemma 789** (*indicator of section*)

is explicitly cited in the proof of:

Theorem 1242 (*Tonelli*),

Lemma 1243 (*Tonelli over subset*).

**Lemma 790** (*measurability of function from product space*)

is not yet used.

**Lemma 794** (*some Borel subsets of numbers*)

is explicitly cited in the proof of:

- Lemma 796 (*Borel subsets of  $\mathbb{R}$* ),
- Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),
- Lemma 820 (*measurability of numeric function*).

**Lemma 796** (*Borel subsets of  $\mathbb{R}$* )

is explicitly cited in the proof of:

- Lemma 797 (*countable generator of Borel subsets of  $\mathbb{R}$* ),
- Lemma 802 (*Borel subsets of  $\mathbb{R}$* ),
- Lemma 803 (*Borel subsets of  $\mathbb{R}_+$* ),
- Lemma 806 (*Borel subsets of  $\mathbb{C}$* ),
- Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),
- Lemma 840 (*power function in  $\mathbb{R}$  is measurable*),
- Lemma 1087 ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* ),
- Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 797** (*countable generator of Borel subsets of  $\mathbb{R}$* )

is not yet used.

**Lemma 798** (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* )

is explicitly cited in the proof of:

- Lemma 800 (*generators of Borel subsets of  $\overline{\mathbb{R}}$* ),
- Lemma 819 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*).

**Lemma 800** (*generators of Borel subsets of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 802 (*Borel subsets of  $\mathbb{R}$* ).

**Lemma 802** (*Borel subsets of  $\overline{\mathbb{R}}$* )

is explicitly cited in the proof of:

- Lemma 804 (*Borel subsets of  $\overline{\mathbb{R}}_+$* ),
- Lemma 820 (*measurability of numeric function*).

**Lemma 803** (*Borel subsets of  $\mathbb{R}_+$* )

is not yet used.

**Lemma 804** (*Borel subsets of  $\overline{\mathbb{R}}_+$* )

is not yet used.

**Lemma 806** (*Borel subsets of  $\mathbb{C}$* )

is not yet used.

**Lemma 808** (*inverse image of countable is measurable*)

is explicitly cited in the proof of:

- Lemma 809 (*inverse image is measurable*),
- Lemma 821 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 996 (*almost finite*).

**Lemma 809** (*inverse image is measurable*)

is explicitly cited in the proof of:

- Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 999 (*measurability of summability domain*),
- Lemma 1026 (*essential supremum of infinite times function*),
- Lemma 1050 (*nonnegative essential supremum is almost definite*).

**Lemma 810** ( $\mathcal{M}_{\mathbb{K}}$  countable range)

is explicitly cited in the proof of:

Lemma 811 (*measurability of indicator function*),

Lemma 1113 ( $\mathcal{SF}_E$  is measurable).

**Lemma 811** (*measurability of indicator function*)

is explicitly cited in the proof of:

Lemma 833 (*measurability and masking*),

Lemma 1002 (*almost sum*),

Lemma 1024 (*essential supremum of extended simple functions*),

Lemma 1101 ( $\mathcal{IF}_{\mathbb{K}}$  is measurable),

Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),

Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),

Lemma 1289 ( $N_p$  of indicator function).

**Lemma 813** ( $\mathcal{M}_{\mathbb{K}}$  is algebra)

is explicitly cited in the proof of:

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$  is vector space),

Lemma 822 ( $\mathcal{M}$  is closed under addition when defined),

Lemma 824 ( $\mathcal{M}$  is closed under multiplication),

Lemma 853 (*measurability of tensor product of numeric functions*),

Lemma 1334 (*generalized Hölder inequality in  $\mathcal{L}^p$* ).

**Lemma 814** ( $\mathcal{M}_{\mathbb{K}}$  is vector space)

is explicitly cited in the proof of:

Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ),

Lemma 1270 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*),

Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication),

Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ),

Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space).

**Lemma 815** ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ )

is explicitly cited in the proof of:

Lemma 816 (*measurability of numeric function to  $\mathbb{R}$* ),

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 841 ( $\mathcal{M}$  is closed under power function),

Lemma 1004 (*almost sum is sum*),

Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 816** (*measurability of numeric function to  $\mathbb{R}$* )

is not yet used.

**Definition 817** ( $\mathcal{M}$ , set of measurable numeric functions)

is explicitly cited in the proof of:

Lemma 819 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ),

Lemma 820 (*measurability of numeric function*),

Lemma 822 ( $\mathcal{M}$  is closed under addition when defined),

Lemma 824 ( $\mathcal{M}$  is closed under multiplication),

Lemma 834 (*measurability of restriction*),

Lemma 841 ( $\mathcal{M}$  is closed under power function).

**Lemma 819** ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ )

is explicitly cited in the proof of:

Lemma 821 ( $\mathcal{M}$  is closed under finite part),

Lemma 833 (*measurability and masking*),

Lemma 1016 (*compatibility of almost limit with measurability*),  
 Lemma 1101 ( *$\mathcal{IF}_{\mathbb{K}}$  is measurable*),  
 Lemma 1113 ( *$\mathcal{SF}_E$  is measurable*),  
 Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),  
 Lemma 1270 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*),  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1328 (*constant function is  $\mathcal{L}^p$* ),  
 Lemma 1334 (*generalized Hölder inequality in  $\mathcal{L}^p$* ),  
 Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 820 (*measurability of numeric function*)**

is explicitly cited in the proof of:

Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 828 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 829 ( *$\mathcal{M}$  is closed under supremum*),  
 Lemma 839 ( *$\mathcal{M}$  is closed under nonnegative finite part*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1026 (*essential supremum of infinite times function*),  
 Lemma 1029 (*essential supremum is almost monotone*),  
 Lemma 1050 (*nonnegative essential supremum is almost definite*),  
 Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*).

**Lemma 821 ( *$\mathcal{M}$  is closed under finite part*)**

is explicitly cited in the proof of:

Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 838 ( *$\mathcal{M}_+$  is closed under finite part*),  
 Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),  
 Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ).

**Lemma 822 ( *$\mathcal{M}$  is closed under addition when defined*)**

is explicitly cited in the proof of:

Lemma 823 ( *$\mathcal{M}$  is closed under finite sum when defined*),  
 Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 843 ( *$\mathcal{M}_+$  is closed under addition*),  
 Lemma 1002 (*almost sum*),  
 Lemma 1003 (*compatibility of almost sum with almost equality*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1311 (*Minkowski equality (linearly dependent functions)*).

**Lemma 823 ( *$\mathcal{M}$  is closed under finite sum when defined*)**

is explicitly cited in the proof of:

Lemma 1024 (*essential supremum of extended simple functions*).

**Lemma 824 ( *$\mathcal{M}$  is closed under multiplication*)**

is explicitly cited in the proof of:

Lemma 825 ( *$\mathcal{M}$  is closed under finite product*),  
 Lemma 826 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 833 (*measurability and masking*),



Lemma 844 ( $\mathcal{M}_+$  is closed under multiplication),  
 Lemma 1002 (almost sum),  
 Lemma 1302 (generalized Hölder inequality (with  $\infty$ )),  
 Lemma 1304 (Hölder inequality),  
 Lemma 1305 (generalized Hölder inequality),  
 Lemma 1309 (Minkowski inequality in  $\mathcal{M}$ ).

**Lemma 825 ( $\mathcal{M}$  is closed under finite product)**

is explicitly cited in the proof of:

Lemma 853 (measurability of tensor product of numeric functions).

**Lemma 826 ( $\mathcal{M}$  is closed under scalar multiplication)**

is explicitly cited in the proof of:

Lemma 836 (measurability of nonnegative and nonpositive parts),  
 Lemma 845 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  
 Lemma 1024 (essential supremum of extended simple functions),  
 Lemma 1026 (essential supremum of infinite times function),  
 Lemma 1027 (essential supremum is positive homogeneous),  
 Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous),  
 Lemma 1300 ( $N_p$  is absolutely homogeneous),  
 Lemma 1304 (Hölder inequality),  
 Lemma 1311 (Minkowski equality (linearly dependent functions)).

**Lemma 828 ( $\mathcal{M}$  is closed under infimum)**

is explicitly cited in the proof of:

Lemma 830 ( $\mathcal{M}$  is closed under limit inferior),  
 Lemma 831 ( $\mathcal{M}$  is closed under limit superior),  
 Lemma 847 ( $\mathcal{M}_+$  is closed under infimum),  
 Theorem 1208 (Fatou's lemma).

**Lemma 829 ( $\mathcal{M}$  is closed under supremum)**

is explicitly cited in the proof of:

Lemma 830 ( $\mathcal{M}$  is closed under limit inferior),  
 Lemma 831 ( $\mathcal{M}$  is closed under limit superior),  
 Lemma 836 (measurability of nonnegative and nonpositive parts),  
 Lemma 848 ( $\mathcal{M}_+$  is closed under supremum).

**Lemma 830 ( $\mathcal{M}$  is closed under limit inferior)**

is explicitly cited in the proof of:

Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent),  
 Theorem 1208 (Fatou's lemma).

**Lemma 831 ( $\mathcal{M}$  is closed under limit superior)**

is explicitly cited in the proof of:

Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent).

**Lemma 832 ( $\mathcal{M}$  is closed under limit when pointwise convergent)**

is explicitly cited in the proof of:

Lemma 849 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent),  
 Lemma 1016 (compatibility of almost limit with measurability),  
 Theorem 1347 (Lebesgue, dominated convergence).

**Lemma 833 (measurability and masking)**

is explicitly cited in the proof of:

Lemma 837 ( $\mathcal{M}_+$  is closed under masking),  
 Lemma 839 ( $\mathcal{M}$  is closed under nonnegative finite part),  
 Lemma 1016 (compatibility of almost limit with measurability),



- Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),  
 Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 834** (*measurability of restriction*)  
 is not yet used.

**Definition 835** ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*)  
 is explicitly cited in the proof of:

- Lemma 836 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 837 ( $\mathcal{M}_+$  *is closed under masking*),  
 Lemma 838 ( $\mathcal{M}_+$  *is closed under finite part*),  
 Lemma 840 (*power function in  $\overline{\mathbb{R}}$  is measurable*),  
 Lemma 841 ( $\mathcal{M}$  *is closed under power function*),  
 Lemma 843 ( $\mathcal{M}_+$  *is closed under addition*),  
 Lemma 844 ( $\mathcal{M}_+$  *is closed under multiplication*),  
 Lemma 845 ( $\mathcal{M}_+$  *is closed under nonnegative scalar multiplication*),  
 Lemma 849 ( $\mathcal{M}_+$  *is closed under limit when pointwise convergent*),  
 Lemma 1016 (*compatibility of almost limit with measurability*),  
 Lemma 1036 (*equivalent definition of nonnegative essential supremum*),  
 Lemma 1050 (*nonnegative essential supremum is almost definite*),  
 Lemma 1053 (*nonnegative essential supremum is submultiplicative*),  
 Lemma 1054 (*nonnegative essential supremum is compatible with power function*),  
 Lemma 1101 ( $\mathcal{IF}_{\mathbb{K}}$  *is measurable*),  
 Lemma 1124 ( $\mathcal{SF}_+$  *is measurable*),  
 Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*),  
 Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),  
 Lemma 1244 (*Tonelli for tensor product*),  
 Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),  
 Lemma 1298 ( $N_p$  *is monotone for nonnegative functions*),  
 Lemma 1318 ( $\mathcal{L}^p$  *and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 836** (*measurability of nonnegative and nonpositive parts*)  
 is explicitly cited in the proof of:

- Lemma 839 ( $\mathcal{M}$  *is closed under nonnegative finite part*),  
 Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),  
 Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*),  
 Lemma 1193 (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*),  
 Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),  
 Lemma 1248 (*integrable is measurable*),  
 Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),  
 Lemma 1269 (*integral in  $\mathcal{M}$  is additive*).

**Lemma 837** ( $\mathcal{M}_+$  *is closed under masking*)  
 is not yet used.

**Lemma 838** ( $\mathcal{M}_+$  is closed under finite part)

is explicitly cited in the proof of:

Lemma 839 ( $\mathcal{M}$  is closed under nonnegative finite part).

**Lemma 839** ( $\mathcal{M}$  is closed under nonnegative finite part)

is explicitly cited in the proof of:

Lemma 1009 (finite nonnegative part).

**Lemma 840** (power function in  $\overline{\mathbb{R}}$  is measurable)

is explicitly cited in the proof of:

Lemma 841 ( $\mathcal{M}$  is closed under power function).

**Lemma 841** ( $\mathcal{M}$  is closed under power function)

is explicitly cited in the proof of:

Lemma 842 ( $\mathcal{M}$  is closed under absolute value),

Lemma 1054 (nonnegative essential supremum is compatible with power function),

Lemma 1286 ( $p$ -seminorm),

Lemma 1291 (link between  $N_1$  and  $N_p$ ),

Lemma 1294 (compatibility of  $N_p$  with power function),

Lemma 1298 ( $N_p$  is monotone for nonnegative functions),

Lemma 1304 (Hölder inequality),

Lemma 1305 (generalized Hölder inequality),

Lemma 1309 (Minkowski inequality in  $\mathcal{M}$ ),

Lemma 1316 (equivalent definition of  $\mathcal{L}^p$ ),

Lemma 1323 (bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$ ),

Lemma 1335 (interpolation inequality in  $\mathcal{L}^p$ ),

Theorem 1351 (Lebesgue, extended dominated convergence in  $\mathcal{L}^p$ ),

Theorem 1352 (Beppo Levi, monotone convergence in  $\mathcal{L}^p$ ).

**Lemma 842** ( $\mathcal{M}$  is closed under absolute value)

is explicitly cited in the proof of:

Lemma 1031 (essential supremum of absolute value is nonnegative),

Lemma 1039 (nonnegative essential supremum is almost upper bound),

Lemma 1046 (nonnegative essential supremum bounded by constant),

Lemma 1050 (nonnegative essential supremum is almost definite),

Lemma 1186 (integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts),

Lemma 1249 (equivalent definition of integrability),

Lemma 1252 (almost bounded by integrable is integrable),

Lemma 1266 (integral in  $\mathcal{M}$  is homogeneous),

Lemma 1268 (Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )),

Lemma 1297 (compatibility of  $N_p$  with almost equality),

Lemma 1304 (Hölder inequality),

Lemma 1322 ( $\mathcal{L}^p$  is closed under absolute value),

Theorem 1352 (Beppo Levi, monotone convergence in  $\mathcal{L}^p$ ).

**Lemma 843** ( $\mathcal{M}_+$  is closed under addition)

is explicitly cited in the proof of:

Lemma 850 ( $\mathcal{M}_+$  is closed under countable sum),

Lemma 1051 (nonnegative essential supremum is subadditive),

Lemma 1183 (integral in  $\mathcal{M}_+$  is additive),

Lemma 1187 (compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts),

Theorem 1242 (Tonelli),

Lemma 1268 (Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )),

Lemma 1304 (Hölder inequality),

Lemma 1309 (Minkowski inequality in  $\mathcal{M}$ ).

**Lemma 844** ( *$\mathcal{M}_+$  is closed under multiplication*)

is explicitly cited in the proof of:

- Lemma 1053 (*nonnegative essential supremum is submultiplicative*),
- Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),
- Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),
- Lemma 1305 (*generalized Hölder inequality*).

**Lemma 845** ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*)

is explicitly cited in the proof of:

- Lemma 1043 (*nonnegative essential supremum of infinite times function*),
- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),
- Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*),
- Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Theorem 1242 (*Tonelli*),
- Lemma 1300 ( *$N_p$  is absolutely homogeneous*),
- Lemma 1304 (*Hölder inequality*).

**Lemma 847** ( *$\mathcal{M}_+$  is closed under infimum*)

is explicitly cited in the proof of:

- Lemma 1221 (*measurability of measure of section (finite)*).

**Lemma 848** ( *$\mathcal{M}_+$  is closed under supremum*)

is explicitly cited in the proof of:

- Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*),
- Lemma 1138 ( *$\mathcal{M}_+$  is closed under restriction*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1223 (*measurability of measure of section*).

**Lemma 849** ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*)

is explicitly cited in the proof of:

- Lemma 850 ( *$\mathcal{M}_+$  is closed under countable sum*),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Theorem 1208 (*Fatou's lemma*),
- Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),
- Theorem 1242 (*Tonelli*).

**Lemma 850** ( *$\mathcal{M}_+$  is closed under countable sum*)

is explicitly cited in the proof of:

- Lemma 1139 ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*),
- Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),
- Lemma 1221 (*measurability of measure of section (finite)*).

**Definition 851** (*tensor product of numeric functions*)

is explicitly cited in the proof of:

- Lemma 853 (*measurability of tensor product of numeric functions*),
- Lemma 1244 (*Tonelli for tensor product*).

**Lemma 853** (*measurability of tensor product of numeric functions*)

is explicitly cited in the proof of:

- Lemma 1244 (*Tonelli for tensor product*).

**Definition 855** (*homogeneity*)

is not yet used.

**Definition 856** (*diffuseness*)

is explicitly cited in the proof of:

- Lemma 1095 (*Lebesgue measure is diffuse*),
- Lemma 1239 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*),
- Lemma 1278 (*integral over interval*).

**Definition 857** (*finiteness*)

is explicitly cited in the proof of:

- Lemma 858 (*finite is bounded*),
- Lemma 887 (*finite is  $\sigma$ -finite*),
- Lemma 936 (*finiteness of counting measure*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1223 (*measurability of measure of section*),
- Lemma 1228 (*tensor product of finite measures*),
- Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 858** (*finite is bounded*)

is explicitly cited in the proof of:

- Lemma 895 (*finite content is bounded*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Definition 859** (*subadditivity*)

is explicitly cited in the proof of:

- Lemma 869 (*subadditivity*),
- Lemma 871 ( *$\sigma$ -subadditive is subadditive*),
- Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*).

**Definition 861** (*superadditivity*)

is explicitly cited in the proof of:

- Lemma 873 ( *$\sigma$ -superadditive is superadditive*).

**Definition 862** (*additivity*)

is explicitly cited in the proof of:

- Lemma 863 (*decomposition over finite pseudopartition*),
- Lemma 864 (*binary inclusion-exclusion principle*),
- Lemma 866 (*subtractivity*),
- Lemma 867 (*monotonicity*),
- Lemma 869 (*subadditivity*),
- Lemma 875 ( *$\sigma$ -additive is additive*),
- Lemma 880 (*continuity from below*),
- Lemma 881 ( *$\sigma$ -additivity (from below)*),
- Lemma 884 ( *$\sigma$ -additivity (from above)*),
- Lemma 916 (*measure is additive*),
- Lemma 1024 (*essential supremum of extended simple functions*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 863** (*decomposition over finite pseudopartition*)

is explicitly cited in the proof of:

- Lemma 890 (*content over finite pseudopartition*).

**Lemma 864** (*binary inclusion-exclusion principle*)

is explicitly cited in the proof of:

Lemma 891 (*content satisfies the binary inclusion-exclusion principle*),  
 Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*).

**Definition 865 (*subtractivity*)**

is explicitly cited in the proof of:

Lemma 866 (*subtractivity*),  
 Lemma 883 (*continuity from above*),  
 Lemma 892 (*content is subtractive*),  
 Lemma 918 (*measure is subtractive*).

**Lemma 866 (*subtractivity*)**

is explicitly cited in the proof of:

Lemma 883 (*continuity from above*),  
 Lemma 892 (*content is subtractive*).

**Lemma 867 (*monotonicity*)**

is explicitly cited in the proof of:

Lemma 869 (*subadditivity*),  
 Lemma 878 ( *$\sigma$ -subadditivity*),  
 Lemma 880 (*continuity from below*),  
 Lemma 883 (*continuity from above*),  
 Lemma 893 (*content is monotone*).

**Lemma 869 (*subadditivity*)**

is explicitly cited in the proof of:

Lemma 894 (*content satisfies the finite Boole inequality*).

**Definition 870 ( *$\sigma$ -subadditivity*)**

is explicitly cited in the proof of:

Lemma 871 ( *$\sigma$ -subadditive is subadditive*),  
 Lemma 878 ( *$\sigma$ -subadditivity*),  
 Lemma 923 (*measure satisfies the Boole inequality*).

**Lemma 871 ( *$\sigma$ -subadditive is subadditive*)**

is explicitly cited in the proof of:

Lemma 906 (*premeasure satisfies the Boole inequality*).

**Definition 872 ( *$\sigma$ -superadditivity*)**

is explicitly cited in the proof of:

Lemma 873 ( *$\sigma$ -superadditive is superadditive*).

**Lemma 873 ( *$\sigma$ -superadditive is superadditive*)**

is not yet used.

**Definition 874 ( *$\sigma$ -additivity*)**

is explicitly cited in the proof of:

Lemma 875 ( *$\sigma$ -additive is additive*),  
 Lemma 876 (*decomposition over countable pseudopartition*),  
 Lemma 878 ( *$\sigma$ -subadditivity*),  
 Lemma 880 (*continuity from below*),  
 Lemma 881 ( *$\sigma$ -additivity (from below)*),  
 Lemma 884 ( *$\sigma$ -additivity (from above)*),  
 Lemma 913 (*measure*),  
 Lemma 935 (*counting measure*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 1080 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),  
 Lemma 1146 (*integral in  $\mathcal{IF}$  is  $\sigma$ -additive*),

Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*),  
 Lemma 1221 (*measurability of measure of section (finite)*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*).

**Lemma 875 ( $\sigma$ -additive is additive)**

is explicitly cited in the proof of:

Lemma 878 ( $\sigma$ -subadditivity),  
 Lemma 880 (*continuity from below*),  
 Lemma 881 ( $\sigma$ -additivity (from below)),  
 Lemma 883 (*continuity from above*),  
 Lemma 884 ( $\sigma$ -additivity (from above)),  
 Lemma 897 (*premeasure*).

**Lemma 876 (*decomposition over countable pseudopartition*)**

is explicitly cited in the proof of:

Lemma 905 (*premeasure over countable pseudopartition*).

**Lemma 878 ( $\sigma$ -subadditivity)**

is explicitly cited in the proof of:

Lemma 906 (*premeasure satisfies the Boole inequality*).

**Definition 879 (*continuity from below*)**

is explicitly cited in the proof of:

Lemma 880 (*continuity from below*),  
 Lemma 881 ( $\sigma$ -additivity (from below)),  
 Lemma 883 (*continuity from above*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Theorem 1180 (*Beppo Levi, monotone convergence*).

**Lemma 880 (*continuity from below*)**

is explicitly cited in the proof of:

Lemma 881 ( $\sigma$ -additivity (from below)),  
 Lemma 883 (*continuity from above*),  
 Lemma 908 (*premeasure is continuous from below*).

**Lemma 881 ( $\sigma$ -additivity (from below))**

is explicitly cited in the proof of:

Lemma 909 (*equivalent definition of premeasure (from below)*).

**Definition 882 (*continuity from above*)**

is explicitly cited in the proof of:

Lemma 883 (*continuity from above*),  
 Lemma 884 ( $\sigma$ -additivity (from above)).

**Lemma 883 (*continuity from above*)**

is explicitly cited in the proof of:

Lemma 884 ( $\sigma$ -additivity (from above)),  
 Lemma 910 (*premeasure is continuous from above*).

**Lemma 884 ( $\sigma$ -additivity (from above))**

is explicitly cited in the proof of:

Lemma 911 (*equivalent definition of premeasure (from above)*).

**Definition 885 ( $\sigma$ -finiteness)**

is explicitly cited in the proof of:

Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*),  
 Lemma 887 (*finite is  $\sigma$ -finite*),

Lemma 937 ( *$\sigma$ -finite counting measure*),  
 Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*),  
 Lemma 1229 (*tensor product of  $\sigma$ -finite measures*).

**Lemma 886 (*equivalent definition of  $\sigma$ -finiteness*)**

is explicitly cited in the proof of:

Lemma 896 ( *$\sigma$ -finite content*),  
 Lemma 937 ( *$\sigma$ -finite counting measure*),  
 Lemma 1223 (*measurability of measure of section*),  
 Lemma 1229 (*tensor product of  $\sigma$ -finite measures*),  
 Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 887 (*finite is  $\sigma$ -finite*)**

is explicitly cited in the proof of:

Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Definition 888 (*content*)**

is explicitly cited in the proof of:

Lemma 890 (*content over finite pseudopartition*),  
 Lemma 891 (*content satisfies the binary inclusion-exclusion principle*),  
 Lemma 892 (*content is subtractive*),  
 Lemma 893 (*content is monotone*),  
 Lemma 894 (*content satisfies the finite Boole inequality*),  
 Lemma 895 (*finite content is bounded*),  
 Lemma 897 (*premeasure*),  
 Lemma 899 (*premeasure is additive*).

**Lemma 890 (*content over finite pseudopartition*)**

is not yet used.

**Lemma 891 (*content satisfies the binary inclusion-exclusion principle*)**

is explicitly cited in the proof of:

Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*).

**Lemma 892 (*content is subtractive*)**

is explicitly cited in the proof of:

Lemma 901 (*premeasure is subtractive*).

**Lemma 893 (*content is monotone*)**

is explicitly cited in the proof of:

Lemma 895 (*finite content is bounded*),  
 Lemma 902 (*premeasure is monotone*).

**Lemma 894 (*content satisfies the finite Boole inequality*)**

is explicitly cited in the proof of:

Lemma 896 ( *$\sigma$ -finite content*).

**Lemma 895 (*finite content is bounded*)**

is explicitly cited in the proof of:

Lemma 903 (*finite premeasure is bounded*).

**Lemma 896 ( *$\sigma$ -finite content*)**

is explicitly cited in the proof of:

Lemma 904 ( *$\sigma$ -finite premeasure*).

**Lemma 897 (*premeasure*)**

is explicitly cited in the proof of:

Lemma 899 (*premeasure is additive*),



Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*),  
 Lemma 901 (*premeasure is subtractive*),  
 Lemma 902 (*premeasure is monotone*),  
 Lemma 903 (*finite premeasure is bounded*),  
 Lemma 904 ( *$\sigma$ -finite premeasure*),  
 Lemma 905 (*premeasure over countable pseudopartition*),  
 Lemma 906 (*premeasure satisfies the Boole inequality*),  
 Lemma 907 (*compatibility of null premeasure with countable union*),  
 Lemma 908 (*premeasure is continuous from below*),  
 Lemma 909 (*equivalent definition of premeasure (from below)*),  
 Lemma 910 (*premeasure is continuous from above*),  
 Lemma 911 (*equivalent definition of premeasure (from above)*),  
 Lemma 912 (*properties of premeasure space*),  
 Lemma 913 (*measure*),  
 Lemma 915 (*measure space is premeasure space*).

**Lemma 899 (*premeasure is additive*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 916 (*measure is additive*).

**Lemma 900 (*premeasure satisfies the binary inclusion-exclusion principle*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*).

**Lemma 901 (*premeasure is subtractive*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 918 (*measure is subtractive*).

**Lemma 902 (*premeasure is monotone*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 919 (*measure is monotone*).

**Lemma 903 (*finite premeasure is bounded*)**

is explicitly cited in the proof of:

Lemma 920 (*finite measure is bounded*).

**Lemma 904 ( *$\sigma$ -finite premeasure*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 921 ( *$\sigma$ -finite measure*).

**Lemma 905 (*premeasure over countable pseudopartition*)**

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),  
 Lemma 922 (*measure over countable pseudopartition*).

**Lemma 906 (*premeasure satisfies the Boole inequality*)**

is explicitly cited in the proof of:

Lemma 907 (*compatibility of null premeasure with countable union*),  
 Lemma 912 (*properties of premeasure space*),  
 Lemma 923 (*measure satisfies the Boole inequality*).



**Lemma 907** (*compatibility of null premeasure with countable union*)

is explicitly cited in the proof of:

Lemma 924 (*compatibility of null measure with countable union*).

**Lemma 908** (*premeasure is continuous from below*)

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),

Lemma 925 (*measure is continuous from below*).

**Lemma 909** (*equivalent definition of premeasure (from below)*)

is explicitly cited in the proof of:

Lemma 926 (*equivalent definition of measure (from below)*).

**Lemma 910** (*premeasure is continuous from above*)

is explicitly cited in the proof of:

Lemma 912 (*properties of premeasure space*),

Lemma 927 (*measure is continuous from above*).

**Lemma 911** (*equivalent definition of premeasure (from above)*)

is explicitly cited in the proof of:

Lemma 928 (*equivalent definition of measure (from above)*).

**Lemma 912** (*properties of premeasure space*)

is explicitly cited in the proof of:

Lemma 929 (*properties of measure space*).

**Lemma 913** (*measure*)

is explicitly cited in the proof of:

Lemma 915 (*measure space is premeasure space*),

Lemma 929 (*properties of measure space*),

Lemma 930 (*trace measure*),

Lemma 931 (*restricted measure*),

Lemma 933 (*trivial measure*),

Lemma 934 (*equivalent definition of trivial measure*),

Lemma 935 (*counting measure*),

Lemma 937 ( *$\sigma$ -finite counting measure*),

Lemma 946 (*equivalent definition of considerable subset*),

Lemma 948 (*negligibility of measurable subset*),

Lemma 949 (*empty set is negligible*),

Lemma 955 (*equivalent definition of property almost satisfied*),

Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),

Lemma 1008 (*masking almost nowhere*),

Lemma 1017 (*essential supremum*),

Lemma 1020 (*essential supremum is bounded by supremum*),

Lemma 1023 (*essential supremum of constant function*),

Lemma 1024 (*essential supremum of extended simple functions*),

Lemma 1036 (*equivalent definition of nonnegative essential supremum*),

Lemma 1086 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ),

Lemma 1146 (*integral in  $\mathcal{IF}$  is  $\sigma$ -additive*),

Lemma 1150 (*integral in  $\mathcal{IF}$  over negligible subset*),

Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),

Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*),

Lemma 1158 (*integral in  $\mathcal{SF}_+$  of zero is zero*),

Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),

Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),

Lemma 1218 (*measure of section*),

Lemma 1219 (*measure of section of product*),  
 Lemma 1221 (*measurability of measure of section (finite)*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Lemma 1228 (*tensor product of finite measures*),  
 Lemma 1329 (*first mean value inequality*),  
 Lemma 1331 (*first mean value theorem*).

**Lemma 915 (*measure space is premeasure space*)**

is explicitly cited in the proof of:

Lemma 916 (*measure is additive*),  
 Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*),  
 Lemma 918 (*measure is subtractive*),  
 Lemma 919 (*measure is monotone*),  
 Lemma 920 (*finite measure is bounded*),  
 Lemma 921 ( *$\sigma$ -finite measure*),  
 Lemma 922 (*measure over countable pseudopartition*),  
 Lemma 923 (*measure satisfies the Boole inequality*),  
 Lemma 924 (*compatibility of null measure with countable union*),  
 Lemma 925 (*measure is continuous from below*),  
 Lemma 926 (*equivalent definition of measure (from below)*),  
 Lemma 927 (*measure is continuous from above*),  
 Lemma 928 (*equivalent definition of measure (from above)*).

**Lemma 916 (*measure is additive*)**

is explicitly cited in the proof of:

Lemma 1024 (*essential supremum of extended simple functions*).

**Lemma 917 (*measure satisfies the binary inclusion-exclusion principle*)**

is not yet used.

**Lemma 918 (*measure is subtractive*)**

is not yet used.

**Lemma 919 (*measure is monotone*)**

is explicitly cited in the proof of:

Lemma 934 (*equivalent definition of trivial measure*),  
 Lemma 948 (*negligibility of measurable subset*),  
 Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 1017 (*essential supremum*),  
 Lemma 1024 (*essential supremum of extended simple functions*),  
 Lemma 1147 (*integral in  $\mathcal{IF}$  is monotone*),  
 Lemma 1150 (*integral in  $\mathcal{IF}$  over negligible subset*).

**Lemma 920 (*finite measure is bounded*)**

is not yet used.

**Lemma 921 ( *$\sigma$ -finite measure*)**

is not yet used.

**Lemma 922 (*measure over countable pseudopartition*)**

is explicitly cited in the proof of:

Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 923 (*measure satisfies the Boole inequality*)**

is not yet used.

**Lemma 924** (*compatibility of null measure with countable union*)

is explicitly cited in the proof of:

- Lemma 952 ( *$\mathbf{N}$  is closed under countable union*),
- Lemma 1021 (*essential supremum is minimum*).

**Lemma 925** (*measure is continuous from below*)

is explicitly cited in the proof of:

- Lemma 992 (*uniqueness of measures extended from a  $\pi$ -system*),
- Theorem 1180 (*Beppo Levi, monotone convergence*),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1223 (*measurability of measure of section*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*),
- Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 926** (*equivalent definition of measure (from below)*)

is explicitly cited in the proof of:

- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 927** (*measure is continuous from above*)

is explicitly cited in the proof of:

- Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),
- Lemma 1221 (*measurability of measure of section (finite)*),
- Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 928** (*equivalent definition of measure (from above)*)

is not yet used.

**Lemma 929** (*properties of measure space*)

is not yet used.

**Lemma 930** (*trace measure*)

is explicitly cited in the proof of:

- Lemma 932 (*trace measure is restricted measure*).

**Lemma 931** (*restricted measure*)

is explicitly cited in the proof of:

- Lemma 932 (*trace measure is restricted measure*),
- Lemma 1223 (*measurability of measure of section*),
- Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 932** (*trace measure is restricted measure*)

is explicitly cited in the proof of:

- Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*).

**Lemma 933** (*trivial measure*)

is explicitly cited in the proof of:

- Lemma 934 (*equivalent definition of trivial measure*),
- Lemma 947 (*negligibility with trivial measure*),
- Lemma 1019 (*essential supremum with trivial measure*),
- Lemma 1026 (*essential supremum of infinite times function*),
- Lemma 1036 (*equivalent definition of nonnegative essential supremum*),
- Lemma 1037 (*nonnegative essential supremum with trivial measure*),
- Lemma 1043 (*nonnegative essential supremum of infinite times function*),
- Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*),
- Lemma 1289 ( *$N_p$  of indicator function*).

**Lemma 934** (*equivalent definition of trivial measure*)

is explicitly cited in the proof of:

- Lemma 1023 (*essential supremum of constant function*),
- Lemma 1024 (*essential supremum of extended simple functions*),
- Lemma 1036 (*equivalent definition of nonnegative essential supremum*),
- Lemma 1043 (*nonnegative essential supremum of infinite times function*).

**Lemma 935** (*counting measure*)

is explicitly cited in the proof of:

- Lemma 936 (*finiteness of counting measure*),
- Lemma 937 ( *$\sigma$ -finite counting measure*),
- Lemma 940 (*equivalent definition of Dirac measure*),
- Lemma 941 (*Dirac measure is finite*),
- Lemma 1152 (*integral in  $\mathcal{IF}$  with counting measure*).

**Lemma 936** (*finiteness of counting measure*)

is explicitly cited in the proof of:

- Lemma 941 (*Dirac measure is finite*).

**Lemma 937** ( *$\sigma$ -finite counting measure*)

is not yet used.

**Definition 939** (*Dirac measure*)

is explicitly cited in the proof of:

- Lemma 940 (*equivalent definition of Dirac measure*),
- Lemma 941 (*Dirac measure is finite*),
- Lemma 1214 (*integral in  $\mathcal{M}_+$  with Dirac measure*),
- Lemma 1284 (*integral with Dirac measure*).

**Lemma 940** (*equivalent definition of Dirac measure*)

is not yet used.

**Lemma 941** (*Dirac measure is finite*)

is not yet used.

**Definition 942** (*negligible subset*)

is explicitly cited in the proof of:

- Lemma 946 (*equivalent definition of considerable subset*),
- Lemma 947 (*negligibility with trivial measure*),
- Lemma 948 (*negligibility of measurable subset*),
- Lemma 951 (*subset of negligible is negligible*),
- Lemma 952 ( *$\mathbf{N}$  is closed under countable union*),
- Lemma 955 (*equivalent definition of property almost satisfied*),
- Lemma 1008 (*masking almost nowhere*).

**Definition 943** (*complete measure*)

is not yet used.

**Definition 945** (*considerable subset*)

is explicitly cited in the proof of:

- Lemma 946 (*equivalent definition of considerable subset*).

**Lemma 946** (*equivalent definition of considerable subset*)

is not yet used.

**Lemma 947** (*negligibility with trivial measure*)

is explicitly cited in the proof of:

- Lemma 956 (*property almost satisfied with trivial measure*).

**Lemma 948** (*negligibility of measurable subset*)

is explicitly cited in the proof of:

- Lemma 949 (*empty set is negligible*),
- Lemma 957 (*measurable property almost satisfied*),
- Lemma 1022 (*essential supremum is almost upper bound*),
- Lemma 1029 (*essential supremum is almost monotone*),
- Lemma 1050 (*nonnegative essential supremum is almost definite*),
- Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),
- Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),
- Lemma 1234 (*negligibility of measurable section*),
- Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 949** (*empty set is negligible*)

is explicitly cited in the proof of:

- Lemma 958 (*everywhere is almost everywhere*).

**Lemma 951** (*subset of negligible is negligible*)

is explicitly cited in the proof of:

- Lemma 959 (*almost modus ponens*),
- Lemma 960 (*countable almost properties is almost property*),
- Lemma 1029 (*essential supremum is almost monotone*),
- Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),
- Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 952** (**N is closed under countable union**)

is explicitly cited in the proof of:

- Lemma 960 (*countable almost properties is almost property*),
- Lemma 1029 (*essential supremum is almost monotone*),
- Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*).

**Definition 953** (*property almost satisfied*)

is explicitly cited in the proof of:

- Lemma 955 (*equivalent definition of property almost satisfied*),
- Lemma 956 (*property almost satisfied with trivial measure*),
- Lemma 957 (*measurable property almost satisfied*),
- Lemma 958 (*everywhere is almost everywhere*),
- Lemma 959 (*almost modus ponens*),
- Lemma 960 (*countable almost properties is almost property*),
- Lemma 970 (*extension to almost defined*),
- Lemma 1000 (*negligibility and summability domain*),
- Lemma 1002 (*almost sum*),
- Lemma 1008 (*masking almost nowhere*),
- Lemma 1016 (*compatibility of almost limit with measurability*),
- Lemma 1050 (*nonnegative essential supremum is almost definite*),
- Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),
- Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),
- Theorem 1349 (*Lebesgue, extended dominated convergence*),
- Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 955** (*equivalent definition of property almost satisfied*)

is explicitly cited in the proof of:

- Lemma 960 (*countable almost properties is almost property*),

Lemma 963 (*extended almost modus ponens*),  
 Lemma 1009 (*finite nonnegative part*),  
 Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),  
 Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*).

**Lemma 956 (*property almost satisfied with trivial measure*)**

is explicitly cited in the proof of:

Lemma 1039 (*nonnegative essential supremum is almost upper bound*),  
 Lemma 1046 (*nonnegative essential supremum bounded by constant*),  
 Lemma 1050 (*nonnegative essential supremum is almost definite*).

**Lemma 957 (*measurable property almost satisfied*)**

is explicitly cited in the proof of:

Lemma 996 (*almost finite*).

**Lemma 958 (*everywhere is almost everywhere*)**

is explicitly cited in the proof of:

Lemma 962 (*almost everywhere for almost the same*),  
 Lemma 967 (*extended almost modus ponens (simplified 2)*),  
 Lemma 994 (*transitivity of almost inequality and equality*),  
 Lemma 1011 (*compatibility of almost limit with continuous function*),  
 Lemma 1014 (*compatibility of almost limit with almost equality*),  
 Lemma 1031 (*essential supremum of absolute value is nonnegative*),  
 Lemma 1048 (*essentially bounded is almost finite*),  
 Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ).

**Lemma 959 (*almost modus ponens*)**

is explicitly cited in the proof of:

Lemma 962 (*almost everywhere for almost the same*),  
 Lemma 970 (*extension to almost defined*),  
 Lemma 972 (*generalized almost modus ponens*).

**Lemma 960 (*countable almost properties is almost property*)**

is explicitly cited in the proof of:

Lemma 962 (*almost everywhere for almost the same*),  
 Lemma 970 (*extension to almost defined*),  
 Lemma 972 (*generalized almost modus ponens*),  
 Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),  
 Lemma 979 (*compatibility of almost binary relation with extended antisymmetry*),  
 Lemma 980 (*compatibility of almost binary relation with transitivity*),  
 Lemma 984 (*compatibility of almost binary relation with operator*),  
 Lemma 1001 (*absolute value and almost inequality*),  
 Lemma 1009 (*finite nonnegative part*),  
 Lemma 1011 (*compatibility of almost limit with continuous function*),  
 Lemma 1013 (*almost limits are almost equal*),  
 Lemma 1014 (*compatibility of almost limit with almost equality*),  
 Lemma 1015 (*compatibility of almost limit with almost inequality*),  
 Lemma 1016 (*compatibility of almost limit with measurability*),  
 Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),  
 Lemma 1299 ( *$N_p$  is almost monotone for almost nonnegative functions*),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 962** (*almost everywhere for almost the same*)

is explicitly cited in the proof of:

- Lemma 963 (*extended almost modus ponens*),
- Lemma 1003 (*compatibility of almost sum with almost equality*),
- Lemma 1014 (*compatibility of almost limit with almost equality*),
- Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 963** (*extended almost modus ponens*)

is explicitly cited in the proof of:

- Lemma 965 (*extended almost modus ponens (simplified 0)*),
- Lemma 966 (*extended almost modus ponens (simplified 1)*).

**Lemma 965** (*extended almost modus ponens (simplified 0)*)

is explicitly cited in the proof of:

- Lemma 1299 ( *$N_p$  is almost monotone for almost nonnegative functions*).

**Lemma 966** (*extended almost modus ponens (simplified 1)*)

is explicitly cited in the proof of:

- Lemma 967 (*extended almost modus ponens (simplified 2)*),
- Lemma 1009 (*finite nonnegative part*).

**Lemma 967** (*extended almost modus ponens (simplified 2)*)

is explicitly cited in the proof of:

- Lemma 1013 (*almost limits are almost equal*),
- Lemma 1015 (*compatibility of almost limit with almost inequality*).

**Definition 969** (*almost definition*)

is not yet used.

**Lemma 970** (*extension to almost defined*)

is explicitly cited in the proof of:

- Lemma 972 (*generalized almost modus ponens*),
- Lemma 975 (*compatibility of almost binary relation with reflexivity*).

**Lemma 972** (*generalized almost modus ponens*)

is explicitly cited in the proof of:

- Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),
- Lemma 978 (*compatibility of almost binary relation with symmetry*),
- Lemma 979 (*compatibility of almost binary relation with extended antisymmetry*),
- Lemma 980 (*compatibility of almost binary relation with transitivity*),
- Lemma 984 (*compatibility of almost binary relation with operator*),
- Lemma 988 (*definiteness is almost definiteness*),
- Lemma 1001 (*absolute value and almost inequality*),
- Lemma 1006 (*absolute value is almost equal to almost nonnegative function*),
- Lemma 1010 (*compatibility of almost limit with subsequence*),
- Lemma 1011 (*compatibility of almost limit with continuous function*).

**Definition 974** (*almost binary relation*)

is explicitly cited in the proof of:

- Lemma 975 (*compatibility of almost binary relation with reflexivity*),
- Lemma 977 (*compatibility of almost binary relation with extended reflexivity*),
- Lemma 978 (*compatibility of almost binary relation with symmetry*),
- Lemma 979 (*compatibility of almost binary relation with extended antisymmetry*),
- Lemma 980 (*compatibility of almost binary relation with transitivity*),
- Lemma 984 (*compatibility of almost binary relation with operator*),
- Lemma 1022 (*essential supremum is almost upper bound*),
- Lemma 1029 (*essential supremum is almost monotone*),



Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*),  
 Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 975** (*compatibility of almost binary relation with reflexivity*)

is explicitly cited in the proof of:

Lemma 981 (*almost equivalence is equivalence relation*),  
 Lemma 983 (*almost order is order relation*).

**Lemma 977** (*compatibility of almost binary relation with extended reflexivity*)

is explicitly cited in the proof of:

Lemma 994 (*transitivity of almost inequality and equality*),  
 Lemma 995 (*compatibility of almost nonnegativeness with almost equality*),  
 Lemma 1032 (*compatibility of essential supremum with almost equality*).

**Lemma 978** (*compatibility of almost binary relation with symmetry*)

is explicitly cited in the proof of:

Lemma 981 (*almost equivalence is equivalence relation*).

**Lemma 979** (*compatibility of almost binary relation with extended antisymmetry*)

is explicitly cited in the proof of:

Lemma 983 (*almost order is order relation*).

**Lemma 980** (*compatibility of almost binary relation with transitivity*)

is explicitly cited in the proof of:

Lemma 981 (*almost equivalence is equivalence relation*),  
 Lemma 983 (*almost order is order relation*),  
 Lemma 993 (*transitivity of almost nonnegativeness*),  
 Lemma 994 (*transitivity of almost inequality and equality*).

**Lemma 981** (*almost equivalence is equivalence relation*)

is explicitly cited in the proof of:

Lemma 982 (*almost equality is equivalence relation*).

**Lemma 982** (*almost equality is equivalence relation*)

is explicitly cited in the proof of:

Lemma 1003 (*compatibility of almost sum with almost equality*).

**Lemma 983** (*almost order is order relation*)

is explicitly cited in the proof of:

Lemma 1001 (*absolute value and almost inequality*),  
 Lemma 1030 (*essential supremum bounded by constant*),  
 Lemma 1048 (*essentially bounded is almost finite*),  
 Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*).

**Lemma 984** (*compatibility of almost binary relation with operator*)

is explicitly cited in the proof of:

Lemma 985 (*compatibility of almost equality with operator*),  
 Lemma 1051 (*nonnegative essential supremum is subadditive*),  
 Lemma 1053 (*nonnegative essential supremum is submultiplicative*),  
 Lemma 1054 (*nonnegative essential supremum is compatible with power function*),  
 Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),  
 Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).



**Lemma 985** (*compatibility of almost equality with operator*)

is explicitly cited in the proof of:

Lemma 1003 (*compatibility of almost sum with almost equality*),

Lemma 1005 (*absolute value is almost definite*),

Lemma 1007 (*power function is almost definite*),

Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),

Lemma 1193 (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*),

Lemma 1262 (*compatibility of integral with almost equality*),

Lemma 1297 (*compatibility of  $N_p$  with almost equality*),

Lemma 1304 (*Hölder inequality*),

Lemma 1329 (*first mean value inequality*),

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ).

**Lemma 988** (*definiteness is almost definiteness*)

is explicitly cited in the proof of:

Lemma 1005 (*absolute value is almost definite*),

Lemma 1007 (*power function is almost definite*).

**Lemma 992** (*uniqueness of measures extended from a  $\pi$ -system*)

is explicitly cited in the proof of:

Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 993** (*transitivity of almost nonnegativeness*)

is explicitly cited in the proof of:

Lemma 995 (*compatibility of almost nonnegativeness with almost equality*).

**Lemma 994** (*transitivity of almost inequality and equality*)

is explicitly cited in the proof of:

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ).

**Lemma 995** (*compatibility of almost nonnegativeness with almost equality*)

is explicitly cited in the proof of:

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*).

**Lemma 996** (*almost finite*)

is not yet used.

**Definition 997** (*summability domain*)

is explicitly cited in the proof of:

Lemma 998 (*summability on summability domain*),

Lemma 1004 (*almost sum is sum*),

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*).

**Lemma 998** (*summability on summability domain*)

is explicitly cited in the proof of:

Lemma 1000 (*negligibility and summability domain*),

Lemma 1003 (*compatibility of almost sum with almost equality*),

Lemma 1004 (*almost sum is sum*),

Lemma 1253 (*integrability is closed under extension by zero*).

**Lemma 999** (*measurability of summability domain*)

is explicitly cited in the proof of:

Lemma 1002 (*almost sum*).

**Lemma 1000** (*negligibility and summability domain*)

is explicitly cited in the proof of:

Lemma 1001 (*absolute value and almost inequality*),

Lemma 1002 (*almost sum*),

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*).

**Lemma 1001** (*absolute value and almost inequality*)

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1002** (*almost sum*)

is explicitly cited in the proof of:

Lemma 1003 (*compatibility of almost sum with almost equality*),

Lemma 1004 (*almost sum is sum*),

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),

Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1003** (*compatibility of almost sum with almost equality*)

is explicitly cited in the proof of:

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),

Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Lemma 1310 (*Minkowski equality (nonnegative functions)*),

Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*).

**Lemma 1004** (*almost sum is sum*)

is explicitly cited in the proof of:

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1005** (*absolute value is almost definite*)

is explicitly cited in the proof of:

Lemma 1296 ( *$N_p$  is almost definite*).

**Lemma 1006** (*absolute value is almost equal to almost nonnegative function*)

is explicitly cited in the proof of:

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ).

**Lemma 1007** (*power function is almost definite*)

is explicitly cited in the proof of:

Lemma 1296 ( *$N_p$  is almost definite*).

**Lemma 1008** (*masking almost nowhere*)

is explicitly cited in the proof of:

Lemma 1016 (*compatibility of almost limit with measurability*),

Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),

Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*),

Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),

Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1009** (*finite nonnegative part*)

is explicitly cited in the proof of:

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1010** (*compatibility of almost limit with subsequence*)

is explicitly cited in the proof of:

Lemma 1356 (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* ).

**Lemma 1011** (*compatibility of almost limit with continuous function*)

is explicitly cited in the proof of:

Lemma 1012 (*compatibility of almost limit with power function*),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 1012** (*compatibility of almost limit with power function*)

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1013** (*almost limits are almost equal*)

is explicitly cited in the proof of:

Lemma 1356 (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* ).

**Lemma 1014** (*compatibility of almost limit with almost equality*)

is explicitly cited in the proof of:

Lemma 1016 (*compatibility of almost limit with measurability*),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).

**Lemma 1015** (*compatibility of almost limit with almost inequality*)

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1016** (*compatibility of almost limit with measurability*)

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1017** (*essential supremum*)

is explicitly cited in the proof of:

Lemma 1019 (*essential supremum with trivial measure*),

Lemma 1020 (*essential supremum is bounded by supremum*),

Lemma 1021 (*essential supremum is minimum*),

Lemma 1022 (*essential supremum is almost upper bound*),

Lemma 1023 (*essential supremum of constant function*),

Lemma 1024 (*essential supremum of extended simple functions*),

Lemma 1026 (*essential supremum of infinite times function*),

Lemma 1027 (*essential supremum is positive homogeneous*),

Lemma 1029 (*essential supremum is almost monotone*),

Lemma 1036 (*equivalent definition of nonnegative essential supremum*).

**Lemma 1019** (*essential supremum with trivial measure*)

is explicitly cited in the proof of:

Lemma 1026 (*essential supremum of infinite times function*),

Lemma 1030 (*essential supremum bounded by constant*).

**Lemma 1020** (*essential supremum is bounded by supremum*)

is not yet used.

**Lemma 1021** (*essential supremum is minimum*)

is explicitly cited in the proof of:

- Lemma 1022 (*essential supremum is almost upper bound*),
- Lemma 1027 (*essential supremum is positive homogeneous*),
- Lemma 1029 (*essential supremum is almost monotone*),
- Lemma 1038 (*nonnegative essential supremum is minimum*).

**Lemma 1022** (*essential supremum is almost upper bound*)

is explicitly cited in the proof of:

- Lemma 1030 (*essential supremum bounded by constant*),
- Lemma 1039 (*nonnegative essential supremum is almost upper bound*).

**Lemma 1023** (*essential supremum of constant function*)

is explicitly cited in the proof of:

- Lemma 1027 (*essential supremum is positive homogeneous*),
- Lemma 1030 (*essential supremum bounded by constant*),
- Lemma 1040 (*nonnegative essential supremum of constant function*).

**Lemma 1024** (*essential supremum of extended simple functions*)

is explicitly cited in the proof of:

- Lemma 1026 (*essential supremum of infinite times function*),
- Lemma 1041 (*nonnegative essential supremum for extended simple functions*).

**Lemma 1026** (*essential supremum of infinite times function*)

is explicitly cited in the proof of:

- Lemma 1043 (*nonnegative essential supremum of infinite times function*).

**Lemma 1027** (*essential supremum is positive homogeneous*)

is explicitly cited in the proof of:

- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*).

**Lemma 1029** (*essential supremum is almost monotone*)

is explicitly cited in the proof of:

- Lemma 1030 (*essential supremum bounded by constant*),
- Lemma 1032 (*compatibility of essential supremum with almost equality*),
- Lemma 1045 (*nonnegative essential supremum is almost monotone*).

**Lemma 1030** (*essential supremum bounded by constant*)

is explicitly cited in the proof of:

- Lemma 1031 (*essential supremum of absolute value is nonnegative*),
- Lemma 1046 (*nonnegative essential supremum bounded by constant*).

**Lemma 1031** (*essential supremum of absolute value is nonnegative*)

is not yet used.

**Lemma 1032** (*compatibility of essential supremum with almost equality*)

is explicitly cited in the proof of:

- Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*).

**Definition 1034** (*nonnegative essential supremum*)

is explicitly cited in the proof of:

- Lemma 1037 (*nonnegative essential supremum with trivial measure*),
- Lemma 1038 (*nonnegative essential supremum is minimum*),
- Lemma 1039 (*nonnegative essential supremum is almost upper bound*),
- Lemma 1040 (*nonnegative essential supremum of constant function*),
- Lemma 1041 (*nonnegative essential supremum for extended simple functions*),
- Lemma 1043 (*nonnegative essential supremum of infinite times function*),

- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1045 (*nonnegative essential supremum is almost monotone*),
- Lemma 1046 (*nonnegative essential supremum bounded by constant*),
- Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*).

**Lemma 1036** (*equivalent definition of nonnegative essential supremum*)

is explicitly cited in the proof of:

- Lemma 1037 (*nonnegative essential supremum with trivial measure*),
- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1286 (*p-seminorm*).

**Lemma 1037** (*nonnegative essential supremum with trivial measure*)

is explicitly cited in the proof of:

- Lemma 1038 (*nonnegative essential supremum is minimum*),
- Lemma 1043 (*nonnegative essential supremum of infinite times function*),
- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1045 (*nonnegative essential supremum is almost monotone*),
- Lemma 1046 (*nonnegative essential supremum bounded by constant*),
- Lemma 1049 (*compatibility of nonnegative essential supremum with almost equality*),
- Lemma 1050 (*nonnegative essential supremum is almost definite*),
- Lemma 1051 (*nonnegative essential supremum is subadditive*),
- Lemma 1053 (*nonnegative essential supremum is submultiplicative*),
- Lemma 1054 (*nonnegative essential supremum is compatible with power function*),
- Lemma 1288 ( $N_p$  with trivial measure),
- Lemma 1289 ( $N_p$  of indicator function).

**Lemma 1038** (*nonnegative essential supremum is minimum*)

is explicitly cited in the proof of:

- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*),
- Lemma 1050 (*nonnegative essential supremum is almost definite*).

**Lemma 1039** (*nonnegative essential supremum is almost upper bound*)

is explicitly cited in the proof of:

- Lemma 1051 (*nonnegative essential supremum is subadditive*),
- Lemma 1053 (*nonnegative essential supremum is submultiplicative*),
- Lemma 1054 (*nonnegative essential supremum is compatible with power function*),
- Lemma 1293 ( $N_\infty$  bounded is almost bounded),
- Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**Lemma 1040** (*nonnegative essential supremum of constant function*)

is explicitly cited in the proof of:

- Lemma 1050 (*nonnegative essential supremum is almost definite*),
- Lemma 1054 (*nonnegative essential supremum is compatible with power function*),
- Lemma 1294 (*compatibility of  $N_p$  with power function*),
- Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*).

**Lemma 1041** (*nonnegative essential supremum for extended simple functions*)

is explicitly cited in the proof of:

- Lemma 1289 ( $N_p$  of indicator function).

**Lemma 1043** (*nonnegative essential supremum of infinite times function*)

is explicitly cited in the proof of:

- Lemma 1044 (*nonnegative essential supremum is positive homogeneous*).

**Lemma 1044** (*nonnegative essential supremum is positive homogeneous*)

is explicitly cited in the proof of:

- Lemma 1300 ( $N_p$  is absolutely homogeneous).

**Lemma 1045** (*nonnegative essential supremum is almost monotone*)

is explicitly cited in the proof of:

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ).

**Lemma 1046** (*nonnegative essential supremum bounded by constant*)

is explicitly cited in the proof of:

Lemma 1048 (*essentially bounded is almost finite*),

Lemma 1051 (*nonnegative essential supremum is subadditive*),

Lemma 1053 (*nonnegative essential supremum is submultiplicative*),

Lemma 1054 (*nonnegative essential supremum is compatible with power function*),

Lemma 1293 ( *$N_\infty$  bounded is almost bounded*).

**Definition 1047** (*essentially bounded*)

is explicitly cited in the proof of:

Lemma 1048 (*essentially bounded is almost finite*).

**Lemma 1048** (*essentially bounded is almost finite*)

is explicitly cited in the proof of:

Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ).

**Lemma 1049** (*compatibility of nonnegative essential supremum with almost equality*)

is explicitly cited in the proof of:

Lemma 1050 (*nonnegative essential supremum is almost definite*),

Lemma 1297 (*compatibility of  $N_p$  with almost equality*).

**Lemma 1050** (*nonnegative essential supremum is almost definite*)

is explicitly cited in the proof of:

Lemma 1296 ( *$N_p$  is almost definite*).

**Lemma 1051** (*nonnegative essential supremum is subadditive*)

is explicitly cited in the proof of:

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1053** (*nonnegative essential supremum is submultiplicative*)

is explicitly cited in the proof of:

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**Lemma 1054** (*nonnegative essential supremum is compatible with power function*)

is explicitly cited in the proof of:

Lemma 1294 (*compatibility of  $N_p$  with power function*).

**Definition 1057** (*length of interval*)

is explicitly cited in the proof of:

Lemma 1058 (*length is nonnegative*),

Lemma 1059 (*length is homogeneous*),

Lemma 1060 (*length of partition*),

Lemma 1066 ( *$\lambda^*$  is homogeneous*),

Lemma 1069 ( *$\lambda^*$  generalizes length of interval*).

**Lemma 1058** (*length is nonnegative*)

is explicitly cited in the proof of:

Lemma 1065 ( *$\lambda^*$  is nonnegative*).

**Lemma 1059** (*length is homogeneous*)

is explicitly cited in the proof of:

Lemma 1060 (*length of partition*).

**Lemma 1060** (*length of partition*)

is not yet used.

**Definition 1061** (*set of countable cover with bounded open intervals*)

is explicitly cited in the proof of:

Lemma 1062 (*set of countable cover with bounded open intervals is nonempty*),

Lemma 1067 ( *$\lambda^*$  is monotone*),

Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*),

Lemma 1069 ( *$\lambda^*$  generalizes length of interval*).

**Lemma 1062** (*set of countable cover with bounded open intervals is nonempty*)

is not yet used.

**Definition 1063** ( *$\lambda^*$ , Lebesgue measure candidate*)

is explicitly cited in the proof of:

Lemma 1066 ( *$\lambda^*$  is homogeneous*),

Lemma 1067 ( *$\lambda^*$  is monotone*),

Lemma 1068 ( *$\lambda^*$  is  $\sigma$ -subadditive*),

Lemma 1069 ( *$\lambda^*$  generalizes length of interval*),

Lemma 1083 (*rays are Lebesgue-measurable*).

**Lemma 1065** ( *$\lambda^*$  is nonnegative*)

is explicitly cited in the proof of:

Lemma 1066 ( *$\lambda^*$  is homogeneous*),

Lemma 1086 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ).

**Lemma 1066** ( *$\lambda^*$  is homogeneous*)

is explicitly cited in the proof of:

Lemma 1069 ( *$\lambda^*$  generalizes length of interval*),

Lemma 1077 ( *$\mathcal{L}$  is algebra of sets*),

Lemma 1086 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ).

**Lemma 1067** ( *$\lambda^*$  is monotone*)

is explicitly cited in the proof of:

Lemma 1069 ( *$\lambda^*$  generalizes length of interval*),

Lemma 1080 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),

Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*),

Lemma 1083 (*rays are Lebesgue-measurable*).

**Lemma 1068** ( *$\lambda^*$  is  $\sigma$ -subadditive*)

is explicitly cited in the proof of:

Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),

Lemma 1075 ( *$\mathcal{L}$  is closed under finite union*),

Lemma 1080 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),

Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*),

Lemma 1083 (*rays are Lebesgue-measurable*).

**Lemma 1069** ( *$\lambda^*$  generalizes length of interval*)

is explicitly cited in the proof of:

Lemma 1083 (*rays are Lebesgue-measurable*),

Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),

Lemma 1092 (*Lebesgue measure generalizes length of interval*),

Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*),

Lemma 1095 (*Lebesgue measure is diffuse*).



**Definition 1071** ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*)

is explicitly cited in the proof of:

- Lemma 1073 (*equivalent definition of  $\mathcal{L}$* ),
- Lemma 1074 ( *$\mathcal{L}$  is closed under complement*),
- Lemma 1075 ( *$\mathcal{L}$  is closed under finite union*),
- Lemma 1078 ( *$\lambda^*$  is additive on  $\mathcal{L}$* ),
- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 1073** (*equivalent definition of  $\mathcal{L}$* )

is explicitly cited in the proof of:

- Lemma 1075 ( *$\mathcal{L}$  is closed under finite union*),
- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*),
- Lemma 1083 (*rays are Lebesgue-measurable*).

**Lemma 1074** ( *$\mathcal{L}$  is closed under complement*)

is explicitly cited in the proof of:

- Lemma 1076 ( *$\mathcal{L}$  is closed under finite intersection*),
- Lemma 1077 ( *$\mathcal{L}$  is algebra of sets*),
- Lemma 1083 (*rays are Lebesgue-measurable*).

**Lemma 1075** ( *$\mathcal{L}$  is closed under finite union*)

is explicitly cited in the proof of:

- Lemma 1076 ( *$\mathcal{L}$  is closed under finite intersection*),
- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 1076** ( *$\mathcal{L}$  is closed under finite intersection*)

is explicitly cited in the proof of:

- Lemma 1077 ( *$\mathcal{L}$  is algebra of sets*),
- Lemma 1084 (*intervals are Lebesgue-measurable*).

**Lemma 1077** ( *$\mathcal{L}$  is algebra of sets*)

is explicitly cited in the proof of:

- Lemma 1081 (*disjoint union of union in  $\mathcal{L}$* ),
- Lemma 1085 ( *$\mathcal{L}$  is  $\sigma$ -algebra*).

**Lemma 1078** ( *$\lambda^*$  is additive on  $\mathcal{L}$* )

is explicitly cited in the proof of:

- Lemma 1080 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),
- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 1080** ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* )

is explicitly cited in the proof of:

- Lemma 1086 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ).

**Lemma 1081** (*disjoint union of union in  $\mathcal{L}$* )

is explicitly cited in the proof of:

- Lemma 1082 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 1082** ( *$\mathcal{L}$  is closed under countable union*)

is explicitly cited in the proof of:

- Lemma 1083 (*rays are Lebesgue-measurable*),
- Lemma 1085 ( *$\mathcal{L}$  is  $\sigma$ -algebra*).

**Lemma 1083** (*rays are Lebesgue-measurable*)

is explicitly cited in the proof of:

- Lemma 1084 (*intervals are Lebesgue-measurable*).



**Lemma 1084** (*intervals are Lebesgue-measurable*)

is explicitly cited in the proof of:

Lemma 1087 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ).

**Lemma 1085** ( $\mathcal{L}$  is  $\sigma$ -algebra)

is explicitly cited in the proof of:

Lemma 1086 ( $\lambda^*$  is measure on  $\mathcal{L}$ ).

**Lemma 1086** ( $\lambda^*$  is measure on  $\mathcal{L}$ )

is explicitly cited in the proof of:

Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ).

**Lemma 1087** ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ )

is explicitly cited in the proof of:

Lemma 1088 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ).

**Lemma 1088** ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ )

is explicitly cited in the proof of:

Theorem 1090 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Theorem 1090** (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* )

is explicitly cited in the proof of:

Lemma 1092 (*Lebesgue measure generalizes length of interval*),

Lemma 1094 (*Lebesgue measure is  $\sigma$ -finite*),

Lemma 1095 (*Lebesgue measure is diffuse*),

Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**Lemma 1092** (*Lebesgue measure generalizes length of interval*)

is explicitly cited in the proof of:

Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Lemma 1094** (*Lebesgue measure is  $\sigma$ -finite*)

is explicitly cited in the proof of:

Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ),

Lemma 1238 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 1095** (*Lebesgue measure is diffuse*)

is not yet used.

**Definition 1098** ( $\mathcal{IF}_{\mathbb{K}}$ , set of measurable indicator functions)

is explicitly cited in the proof of:

Lemma 1100 (*indicator and support are inverse*),

Lemma 1101 ( $\mathcal{IF}_{\mathbb{K}}$  is measurable),

Lemma 1102 ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive),

Lemma 1103 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication),

Lemma 1105 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero),

Lemma 1106 ( $\mathcal{IF}_{\mathbb{K}}$  is closed under restriction),

Lemma 1108 ( $\mathcal{SF}_E$  simple representation),

Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero),

Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),

Lemma 1124 ( $\mathcal{SF}_+$  is measurable),

Lemma 1134 (*Lebesgue scheme step 2*),

Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**Lemma 1100** (*indicator and support are inverse*)

is explicitly cited in the proof of:

Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive),

Lemma 1108 ( $\mathcal{SF}_E$  simple representation),  
 Lemma 1145 (equivalent definition of integral in  $\mathcal{IF}$ ),  
 Lemma 1152 (integral in  $\mathcal{IF}$  with counting measure),  
 Theorem 1242 (Tonelli).

**Lemma 1101** ( $\mathcal{IF}_{\mathbb{K}}$  is measurable)  
 is not yet used.

**Lemma 1102** ( $\mathcal{IF}_{\mathbb{K}}$  is  $\sigma$ -additive)  
 is explicitly cited in the proof of:  
 Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),  
 Lemma 1146 (integral in  $\mathcal{IF}$  is  $\sigma$ -additive).

**Lemma 1103** ( $\mathcal{IF}_{\mathbb{K}}$  is closed under multiplication)  
 is explicitly cited in the proof of:  
 Lemma 1107 ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),  
 Lemma 1139 ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive).

**Lemma 1105** ( $\mathcal{IF}_{\mathbb{K}}$  is closed under extension by zero)  
 is explicitly cited in the proof of:  
 Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero),  
 Lemma 1148 (integral in  $\mathcal{IF}$  over subset).

**Lemma 1106** ( $\mathcal{IF}_{\mathbb{K}}$  is closed under restriction)  
 is explicitly cited in the proof of:  
 Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction).

**Lemma 1107** ( $\mathcal{IF}_{\mathbb{K}}$ , masking is  $\sigma$ -additive)  
 is explicitly cited in the proof of:  
 Lemma 1149 (integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive).

**Lemma 1108** ( $\mathcal{SF}_E$  simple representation)  
 is explicitly cited in the proof of:  
 Lemma 1111 ( $\mathcal{SF}_E$  canonical representation),  
 Lemma 1113 ( $\mathcal{SF}_E$  is measurable),  
 Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Lemma 1119 ( $\mathcal{SF}_E$  is closed under extension by zero),  
 Lemma 1120 ( $\mathcal{SF}_E$  is closed under restriction),  
 Lemma 1121 ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive),  
 Lemma 1128 ( $\mathcal{SF}_+$  simple representation),  
 Lemma 1136 (adapted sequence in  $\mathcal{M}_+$ ),  
 Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous),  
 Theorem 1180 (Beppo Levi, monotone convergence).

**Lemma 1111** ( $\mathcal{SF}_E$  canonical representation)  
 is explicitly cited in the proof of:  
 Lemma 1113 ( $\mathcal{SF}_E$  is measurable),  
 Lemma 1114 ( $\mathcal{SF}_E$  disjoint representation),  
 Lemma 1116 ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation),  
 Lemma 1117 ( $\mathcal{SF}_E$  is vector space),  
 Lemma 1126 ( $\mathcal{SF}_+$  canonical representation).

**Lemma 1113** ( $\mathcal{SF}_E$  is measurable)

is explicitly cited in the proof of:

Lemma 1124 ( $\mathcal{SF}_+$  is measurable),

Theorem 1180 (*Beppo Levi, monotone convergence*).

**Lemma 1114** ( $\mathcal{SF}_E$  disjoint representation)

is explicitly cited in the proof of:

Lemma 1116 ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation),

Lemma 1117 ( $\mathcal{SF}_E$  is vector space),

Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation).

**Lemma 1116** ( $\mathcal{SF}_E$  disjoint representation is subpartition of canonical representation)

is explicitly cited in the proof of:

Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation).

**Lemma 1117** ( $\mathcal{SF}_E$  is vector space)

is explicitly cited in the proof of:

Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),

Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*).

**Lemma 1119** ( $\mathcal{SF}_E$  is closed under extension by zero)

is explicitly cited in the proof of:

Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero).

**Lemma 1120** ( $\mathcal{SF}_E$  is closed under restriction)

is explicitly cited in the proof of:

Lemma 1131 ( $\mathcal{SF}_+$  is closed under restriction).

**Lemma 1121** ( $\mathcal{SF}_E$ , masking is  $\sigma$ -additive)

is explicitly cited in the proof of:

Lemma 1132 ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive).

**Definition 1123** ( $\mathcal{SF}_+$ , subset of nonnegative simple functions)

is explicitly cited in the proof of:

Lemma 1124 ( $\mathcal{SF}_+$  is measurable),

Lemma 1125 ( $\mathcal{SF}_+$  disjoint representation),

Lemma 1126 ( $\mathcal{SF}_+$  canonical representation),

Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation),

Lemma 1128 ( $\mathcal{SF}_+$  simple representation),

Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),

Lemma 1130 ( $\mathcal{SF}_+$  is closed under extension by zero),

Lemma 1131 ( $\mathcal{SF}_+$  is closed under restriction),

Lemma 1132 ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive),

Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*),

Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),

Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 1124** ( $\mathcal{SF}_+$  is measurable)

is explicitly cited in the proof of:

Lemma 1137 ( $\mathcal{M}_+$  is closed under extension by zero),

Lemma 1138 ( $\mathcal{M}_+$  is closed under restriction),

Lemma 1139 ( $\mathcal{M}_+$ , masking is  $\sigma$ -additive),

Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),

Theorem 1242 (*Tonelli*).

**Lemma 1125** ( $\mathcal{SF}_+$  disjoint representation)

is explicitly cited in the proof of:

- Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation),
- Lemma 1129 ( $\mathcal{SF}_+$  is closed under positive algebra operations),
- Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear),
- Lemma 1166 (density measure in  $\mathcal{SF}_+$ ),
- Lemma 1167 (integral and density measure in  $\mathcal{SF}_+$ ),
- Lemma 1168 (density integral in  $\mathcal{SF}_+$ ).

**Lemma 1126** ( $\mathcal{SF}_+$  canonical representation)

is explicitly cited in the proof of:

- Lemma 1127 ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation),
- Lemma 1154 (integral in  $\mathcal{SF}_+$ ),
- Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)).

**Lemma 1127** ( $\mathcal{SF}_+$  disjoint representation is subpartition of canonical representation)

is explicitly cited in the proof of:

- Lemma 1157 (equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)).

**Lemma 1128** ( $\mathcal{SF}_+$  simple representation)

is explicitly cited in the proof of:

- Lemma 1134 (Lebesgue scheme step 2),
- Lemma 1137 ( $\mathcal{M}_+$  is closed under extension by zero),
- Lemma 1156 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ),
- Lemma 1160 (equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)),
- Lemma 1170 (integral in  $\mathcal{SF}_+$  with counting measure),
- Theorem 1242 (Tonelli).

**Lemma 1129** ( $\mathcal{SF}_+$  is closed under positive algebra operations)

is explicitly cited in the proof of:

- Lemma 1159 (integral in  $\mathcal{SF}_+$  is positive linear),
- Lemma 1166 (density measure in  $\mathcal{SF}_+$ ),
- Lemma 1167 (integral and density measure in  $\mathcal{SF}_+$ ),
- Lemma 1168 (density integral in  $\mathcal{SF}_+$ ),
- Lemma 1176 (integral in  $\mathcal{M}_+$  is positive homogeneous),
- Theorem 1242 (Tonelli).

**Lemma 1130** ( $\mathcal{SF}_+$  is closed under extension by zero)

is explicitly cited in the proof of:

- Lemma 1137 ( $\mathcal{M}_+$  is closed under extension by zero),
- Lemma 1163 (integral in  $\mathcal{SF}_+$  over subset).

**Lemma 1131** ( $\mathcal{SF}_+$  is closed under restriction)

is explicitly cited in the proof of:

- Lemma 1138 ( $\mathcal{M}_+$  is closed under restriction).

**Lemma 1132** ( $\mathcal{SF}_+$ , masking is  $\sigma$ -additive)

is explicitly cited in the proof of:

- Lemma 1164 (integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive).

**Lemma 1134** (Lebesgue scheme step 2)

is explicitly cited in the proof of:

- Theorem 1242 (Tonelli).

**Definition 1135** (adapted sequence)

is explicitly cited in the proof of:

Lemma 1136 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Lemma 1182 (*usage of adapted sequences*),  
 Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**Lemma 1136** (*adapted sequence in  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

Lemma 1137 ( *$\mathcal{M}_+$  is closed under extension by zero*),  
 Lemma 1138 ( *$\mathcal{M}_+$  is closed under restriction*),  
 Lemma 1141 (*Lebesgue scheme step 3*),  
 Lemma 1182 (*usage of adapted sequences*),  
 Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**Lemma 1137** ( *$\mathcal{M}_+$  is closed under extension by zero*)

is explicitly cited in the proof of:

Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),  
 Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*).

**Lemma 1138** ( *$\mathcal{M}_+$  is closed under restriction*)

is explicitly cited in the proof of:

Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*),  
 Lemma 1204 (*integrable in  $\mathcal{M}_+$  is closed under restriction*).

**Lemma 1139** ( *$\mathcal{M}_+$ , masking is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),  
 Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),  
 Lemma 1206 (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*).

**Lemma 1141** (*Lebesgue scheme step 3*)

is explicitly cited in the proof of:

Theorem 1242 (*Tonelli*).

**Definition 1144** (*integral in  $\mathcal{IF}$* )

is explicitly cited in the proof of:

Lemma 1145 (*equivalent definition of integral in  $\mathcal{IF}$* ),  
 Lemma 1148 (*integral in  $\mathcal{IF}$  over subset*).

**Lemma 1145** (*equivalent definition of integral in  $\mathcal{IF}$* )

is explicitly cited in the proof of:

Lemma 1146 (*integral in  $\mathcal{IF}$  is  $\sigma$ -additive*),  
 Lemma 1147 (*integral in  $\mathcal{IF}$  is monotone*),  
 Lemma 1152 (*integral in  $\mathcal{IF}$  with counting measure*),  
 Lemma 1154 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*).

**Lemma 1146** (*integral in  $\mathcal{IF}$  is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1149 (*integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive*).

**Lemma 1147** (*integral in  $\mathcal{IF}$  is monotone*)

is not yet used.

**Lemma 1148** (*integral in  $\mathcal{IF}$  over subset*)

is explicitly cited in the proof of:

Lemma 1149 (*integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive*),  
 Lemma 1150 (*integral in  $\mathcal{IF}$  over negligible subset*),  
 Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*).

**Lemma 1149** (*integral in  $\mathcal{IF}$  over subset is  $\sigma$ -additive*)  
is not yet used.

**Lemma 1150** (*integral in  $\mathcal{IF}$  over negligible subset*)  
is explicitly cited in the proof of:  
Lemma 1165 (*integral in  $\mathcal{SF}_+$  over negligible subset*).

**Lemma 1152** (*integral in  $\mathcal{IF}$  with counting measure*)  
is explicitly cited in the proof of:  
Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*).

**Lemma 1154** (*integral in  $\mathcal{SF}_+$* )  
is explicitly cited in the proof of:  
Lemma 1156 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
Lemma 1157 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*),  
Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*),  
Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),  
Lemma 1168 (*density integral in  $\mathcal{SF}_+$* ),  
Lemma 1169 (*integral in  $\mathcal{SF}_+$  with trivial measure*),  
Lemma 1172 (*integral in  $\mathcal{M}_+$* ),  
Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),  
Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 1156** (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* )  
is explicitly cited in the proof of:  
Lemma 1158 (*integral in  $\mathcal{SF}_+$  of zero is zero*),  
Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*),  
Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*),  
Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),  
Lemma 1170 (*integral in  $\mathcal{SF}_+$  with counting measure*),  
Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*).

**Lemma 1157** (*equivalent definition of the integral in  $\mathcal{SF}_+$  (disjoint)*)  
is explicitly cited in the proof of:  
Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
Lemma 1166 (*density measure in  $\mathcal{SF}_+$* ),  
Lemma 1167 (*integral and density measure in  $\mathcal{SF}_+$* ).

**Lemma 1158** (*integral in  $\mathcal{SF}_+$  of zero is zero*)  
is explicitly cited in the proof of:  
Lemma 1159 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 1159** (*integral in  $\mathcal{SF}_+$  is positive linear*)  
is explicitly cited in the proof of:  
Lemma 1160 (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*),  
Lemma 1161 (*integral in  $\mathcal{SF}_+$  is monotone*),  
Lemma 1163 (*integral in  $\mathcal{SF}_+$  over subset*),  
Lemma 1164 (*integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive*),  
Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
Theorem 1180 (*Beppo Levi, monotone convergence*),  
Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*).

**Lemma 1160** (*equivalent definition of the integral in  $\mathcal{SF}_+$  (simple)*)  
is explicitly cited in the proof of:  
Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
Theorem 1180 (*Beppo Levi, monotone convergence*).

**Lemma 1161** (*integral in  $\mathcal{SF}_+$  is monotone*)

is explicitly cited in the proof of:

Lemma 1162 (*integral in  $\mathcal{SF}_+$  is continuous*).

**Lemma 1162** (*integral in  $\mathcal{SF}_+$  is continuous*)

is explicitly cited in the proof of:

Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ).

**Lemma 1163** (*integral in  $\mathcal{SF}_+$  over subset*)

is explicitly cited in the proof of:

Lemma 1164 (*integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive*),

Lemma 1165 (*integral in  $\mathcal{SF}_+$  over negligible subset*),

Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).

**Lemma 1164** (*integral in  $\mathcal{SF}_+$  over subset is  $\sigma$ -additive*)

is not yet used.

**Lemma 1165** (*integral in  $\mathcal{SF}_+$  over negligible subset*)

is explicitly cited in the proof of:

Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*).

**Lemma 1166** (*density measure in  $\mathcal{SF}_+$* )

is explicitly cited in the proof of:

Lemma 1167 (*integral and density measure in  $\mathcal{SF}_+$* ),

Lemma 1168 (*density integral in  $\mathcal{SF}_+$* ).

**Lemma 1167** (*integral and density measure in  $\mathcal{SF}_+$* )

is not yet used.

**Lemma 1168** (*density integral in  $\mathcal{SF}_+$* )

is not yet used.

**Lemma 1169** (*integral in  $\mathcal{SF}_+$  with trivial measure*)

is explicitly cited in the proof of:

Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*).

**Lemma 1170** (*integral in  $\mathcal{SF}_+$  with counting measure*)

is explicitly cited in the proof of:

Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*).

**Lemma 1172** (*integral in  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

Lemma 1174 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),

Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),

Lemma 1176 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Lemma 1178 (*integral in  $\mathcal{M}_+$  is monotone*),

Theorem 1180 (*Beppo Levi, monotone convergence*),

Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),

Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*),

Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*),

Lemma 1204 (*integrable in  $\mathcal{M}_+$  is closed under restriction*),

Lemma 1206 (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Lemma 1248 (*integrable is measurable*),

Lemma 1249 (*equivalent definition of integrability*),

Lemma 1253 (*integrability is closed under extension by zero*),

Lemma 1256 (*integral in  $\mathcal{M}$* ),



Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),  
 Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),  
 Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),  
 Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*),  
 Lemma 1275 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 1277 (*integral over singleton*),  
 Lemma 1286 ( *$p$ -seminorm*),  
 Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 1174** (*integral in  $\mathcal{M}_+$  generalizes integral in  $SF_+$* )

is explicitly cited in the proof of:

Lemma 1175 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 1182 (*usage of adapted sequences*),  
 Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*).

**Lemma 1175** (*integral in  $\mathcal{M}_+$  of indicator function*)

is explicitly cited in the proof of:

Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),  
 Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Lemma 1221 (*measurability of measure of section (finite)*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Theorem 1242 (*Tonelli*),  
 Lemma 1260 (*integral in  $\mathcal{M}$  of indicator function*),  
 Lemma 1289 ( *$N_p$  of indicator function*).

**Lemma 1176** (*integral in  $\mathcal{M}_+$  is positive homogeneous*)

is explicitly cited in the proof of:

Lemma 1177 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*),  
 Lemma 1244 (*Tonelli for tensor product*),  
 Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*),  
 Lemma 1304 (*Hölder inequality*).

**Lemma 1177** (*integral in  $\mathcal{M}_+$  of zero is zero*)

is explicitly cited in the proof of:

Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),  
 Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*).

**Lemma 1178** (*integral in  $\mathcal{M}_+$  is monotone*)

is explicitly cited in the proof of:

Theorem 1180 (*Beppo Levi, monotone convergence*),  
 Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*),  
 Lemma 1195 (*generalized Bienaymé–Chebyshev inequality*),  
 Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*),  
 Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),  
 Theorem 1208 (*Fatou's lemma*),  
 Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
 Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),



Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),  
 Lemma 1304 (*Hölder inequality*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ).

**Theorem 1180 (*Beppo Levi, monotone convergence*)**

is explicitly cited in the proof of:

Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 1182 (*usage of adapted sequences*),  
 Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Lemma 1199 (*integral in  $\mathcal{M}_+$  over subset*),  
 Theorem 1208 (*Fatou's lemma*),  
 Theorem 1242 (*Tonelli*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ).

**Lemma 1181 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* )**

is explicitly cited in the proof of:

Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 1188 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 1300 ( *$N_p$  is absolutely homogeneous*).

**Lemma 1182 (*usage of adapted sequences*)**

is explicitly cited in the proof of:

Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 1210 (*integral in  $\mathcal{M}_+$  with trivial measure*),  
 Lemma 1211 (*integral in  $\mathcal{M}_+$  with counting measure*).

**Lemma 1183 (*integral in  $\mathcal{M}_+$  is additive*)**

is explicitly cited in the proof of:

Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*),  
 Lemma 1187 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*),  
 Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),  
 Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*),  
 Lemma 1304 (*Hölder inequality*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1184 (*integral in  $\mathcal{M}_+$  is positive linear*)**

is explicitly cited in the proof of:

Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*),  
 Theorem 1242 (*Tonelli*),  
 Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1185 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*)**

is explicitly cited in the proof of:

Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),  
 Lemma 1227 (*candidate tensor product measure is tensor product measure*).

**Lemma 1186 (*integral in  $\mathcal{M}_+$  of decomposition into nonnegative and nonpositive parts*)**

is explicitly cited in the proof of:

Lemma 1249 (*equivalent definition of integrability*),  
 Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*),  
 Lemma 1310 (*Minkowski equality (nonnegative functions)*).

**Lemma 1187** (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonpositive and nonnegative parts*)

is explicitly cited in the proof of:

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*).

**Lemma 1188** (*integral in  $\mathcal{M}_+$  is almost definite*)

is explicitly cited in the proof of:

Lemma 1189 (*extended almost modus ponens for integral in  $\mathcal{M}_+$* ),

Lemma 1192 (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*),

Lemma 1234 (*negligibility of measurable section*),

Lemma 1296 ( *$N_p$  is almost definite*),

Lemma 1329 (*first mean value inequality*).

**Lemma 1189** (*extended almost modus ponens for integral in  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

Lemma 1190 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),

Lemma 1191 (*integral in  $\mathcal{M}_+$  is almost monotone*).

**Lemma 1190** (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*)

is explicitly cited in the proof of:

Lemma 1193 (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*),

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),

Lemma 1262 (*compatibility of integral with almost equality*),

Lemma 1268 (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*).

**Lemma 1191** (*integral in  $\mathcal{M}_+$  is almost monotone*)

is explicitly cited in the proof of:

Lemma 1198 (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*).

**Lemma 1192** (*integral in  $\mathcal{M}_+$  of nonpositive part of almost nonnegative function*)

is explicitly cited in the proof of:

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*),

Lemma 1310 (*Minkowski equality (nonnegative functions)*).

**Lemma 1193** (*integral in  $\mathcal{M}_+$  of nonnegative and nonpositive parts of almost equal functions*)

is explicitly cited in the proof of:

Lemma 1194 (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*).

**Lemma 1194** (*compatibility of integral in  $\mathcal{M}_+$  with addition of nonnegative parts for almost nonnegative functions*)

is explicitly cited in the proof of:

Lemma 1310 (*Minkowski equality (nonnegative functions)*).

**Lemma 1195** (*generalized Bienaymé–Chebyshev inequality*)

is explicitly cited in the proof of:

Lemma 1196 (*Bienaymé–Chebyshev inequality*).

**Lemma 1196** (*Bienaymé–Chebyshev inequality*)

is explicitly cited in the proof of:

Lemma 1197 (*integrable in  $\mathcal{M}_+$  is almost finite*).

**Lemma 1197** (*integrable in  $\mathcal{M}_+$  is almost finite*)

is explicitly cited in the proof of:

Lemma 1251 (*integrable is almost finite*).

**Lemma 1198** (*almost bounded by integrable in  $\mathcal{M}_+$  is integrable*)

is explicitly cited in the proof of:

Lemma 1252 (*almost bounded by integrable is integrable*).

**Lemma 1199** (*integral in  $\mathcal{M}_+$  over subset*)

is explicitly cited in the proof of:

Lemma 1200 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),

Lemma 1201 (*integral in  $\mathcal{M}_+$  over negligible subset*),

Lemma 1202 (*integral in  $\mathcal{M}_+$  over singleton*),

Lemma 1203 (*finite integral in  $\mathcal{M}_+$  is closed under restriction*),

Lemma 1204 (*integrable in  $\mathcal{M}_+$  is closed under restriction*),

Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),

Lemma 1243 (*Tonelli over subset*),

Lemma 1253 (*integrability is closed under extension by zero*),

Lemma 1274 (*integral over subset*).

**Lemma 1200** (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1205 (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*),

Lemma 1275 (*integral over subset is  $\sigma$ -additive*).

**Lemma 1201** (*integral in  $\mathcal{M}_+$  over negligible subset*)

is explicitly cited in the proof of:

Lemma 1276 (*integral over negligible subset*).

**Lemma 1202** (*integral in  $\mathcal{M}_+$  over singleton*)

is explicitly cited in the proof of:

Lemma 1277 (*integral over singleton*).

**Lemma 1203** (*finite integral in  $\mathcal{M}_+$  is closed under restriction*)

is explicitly cited in the proof of:

Lemma 1204 (*integrable in  $\mathcal{M}_+$  is closed under restriction*).

**Lemma 1204** (*integrable in  $\mathcal{M}_+$  is closed under restriction*)

is explicitly cited in the proof of:

Lemma 1254 (*integrability is closed under restriction*).

**Lemma 1205** (*finite integral in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1206 (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*).

**Lemma 1206** (*integrable in  $\mathcal{M}_+$ , masking is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1255 (*integrability, masking is  $\sigma$ -additive*).

**Theorem 1208** (*Fatou's lemma*)

is explicitly cited in the proof of:

Lemma 1209 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),

Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),

Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 1209** (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*)

is not yet used.

**Lemma 1210** (*integral in  $\mathcal{M}_+$  with trivial measure*)

is explicitly cited in the proof of:

Lemma 1280 (*integral with trivial measure*),

Lemma 1288 ( *$N_p$  with trivial measure*).

**Lemma 1211** (*integral in  $\mathcal{M}_+$  with counting measure*)

is explicitly cited in the proof of:

Lemma 1212 (*integral in  $\mathcal{M}_+$  with counting measure on  $\mathbb{N}$* ),

Lemma 1214 (*integral in  $\mathcal{M}_+$  with Dirac measure*),

Lemma 1281 (*integral with counting measure*).

**Lemma 1212** (*integral in  $\mathcal{M}_+$  with counting measure on  $\mathbb{N}$* )

is not yet used.

**Lemma 1214** (*integral in  $\mathcal{M}_+$  with Dirac measure*)

is not yet used.

**Definition 1216** (*overline- $i$  notation to exclude term in product*)

is not yet used.

**Lemma 1218** (*measure of section*)

is explicitly cited in the proof of:

Lemma 1219 (*measure of section of product*),

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1223 (*measurability of measure of section*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Lemma 1233 (*uniqueness of tensor product measure*),

Theorem 1242 (*Tonelli*).

**Lemma 1219** (*measure of section of product*)

is explicitly cited in the proof of:

Lemma 1221 (*measurability of measure of section (finite)*),

Lemma 1227 (*candidate tensor product measure is tensor product measure*).

**Lemma 1221** (*measurability of measure of section (finite)*)

is explicitly cited in the proof of:

Lemma 1223 (*measurability of measure of section*).

**Lemma 1223** (*measurability of measure of section*)

is explicitly cited in the proof of:

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Theorem 1242 (*Tonelli*).

**Definition 1224** (*tensor product measure*)

is explicitly cited in the proof of:

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Lemma 1228 (*tensor product of finite measures*),

Lemma 1229 (*tensor product of  $\sigma$ -finite measures*),

Lemma 1231 (*uniqueness of tensor product measure (finite)*),

Lemma 1233 (*uniqueness of tensor product measure*),

Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Definition 1225** (*candidate tensor product measure*)

is explicitly cited in the proof of:

Lemma 1227 (*candidate tensor product measure is tensor product measure*),

Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 1227** (*candidate tensor product measure is tensor product measure*)

is explicitly cited in the proof of:

Lemma 1231 (*uniqueness of tensor product measure (finite)*),

Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 1228** (*tensor product of finite measures*)

is explicitly cited in the proof of:

Lemma 1231 (*uniqueness of tensor product measure (finite)*).

**Lemma 1229** (*tensor product of  $\sigma$ -finite measures*)

is explicitly cited in the proof of:

Lemma 1233 (*uniqueness of tensor product measure*),

Lemma 1238 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 1231** (*uniqueness of tensor product measure (finite)*)

is explicitly cited in the proof of:

Lemma 1233 (*uniqueness of tensor product measure*).

**Lemma 1233** (*uniqueness of tensor product measure*)

is explicitly cited in the proof of:

Lemma 1234 (*negligibility of measurable section*),

Lemma 1235 (*Lebesgue measure on  $\mathbb{R}^2$* ),

Theorem 1242 (*Tonelli*).

**Lemma 1234** (*negligibility of measurable section*)

is not yet used.

**Lemma 1235** (*Lebesgue measure on  $\mathbb{R}^2$* )

is explicitly cited in the proof of:

Lemma 1236 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*),

Lemma 1238 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 1236** (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*)

is explicitly cited in the proof of:

Lemma 1237 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*),

Lemma 1239 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*).

**Lemma 1237** (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*)

is not yet used.

**Lemma 1238** (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*)

is not yet used.

**Lemma 1239** (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*)

is not yet used.

**Definition 1240** (*partial function of function from product space*)

is explicitly cited in the proof of:

Theorem 1242 (*Tonelli*),

Lemma 1243 (*Tonelli over subset*),

Lemma 1244 (*Tonelli for tensor product*).

**Theorem 1242** (*Tonelli*)

is explicitly cited in the proof of:

Lemma 1243 (*Tonelli over subset*),

Lemma 1244 (*Tonelli for tensor product*).

**Lemma 1243** (*Tonelli over subset*)

is not yet used.

**Lemma 1244** (*Tonelli for tensor product*)

is not yet used.

**Definition 1247** (*integrability*)

is explicitly cited in the proof of:

- Lemma 1248 (*integrable is measurable*),
- Lemma 1249 (*equivalent definition of integrability*),
- Lemma 1253 (*integrability is closed under extension by zero*),
- Lemma 1254 (*integrability is closed under restriction*),
- Lemma 1255 (*integrability, masking is  $\sigma$ -additive*),
- Lemma 1256 (*integral in  $\mathcal{M}$* ),
- Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),
- Lemma 1262 (*compatibility of integral with almost equality*),
- Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*).

**Lemma 1248** (*integrable is measurable*)

is explicitly cited in the proof of:

- Lemma 1249 (*equivalent definition of integrability*).

**Lemma 1249** (*equivalent definition of integrability*)

is explicitly cited in the proof of:

- Lemma 1250 (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* ),
- Lemma 1251 (*integrable is almost finite*),
- Lemma 1252 (*almost bounded by integrable is integrable*),
- Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),
- Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),
- Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),
- Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*),
- Lemma 1277 (*integral over singleton*),
- Lemma 1280 (*integral with trivial measure*),
- Lemma 1281 (*integral with counting measure*),
- Lemma 1292 (*integrable is finite  $N_1$* ),
- Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1250** (*integrability in  $\mathcal{M}$  generalizes integrability in  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

- Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),
- Lemma 1260 (*integral in  $\mathcal{M}$  of indicator function*).

**Lemma 1251** (*integrable is almost finite*)

is explicitly cited in the proof of:

- Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),
- Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),
- Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),
- Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1252** (*almost bounded by integrable is integrable*)

is not yet used.

**Lemma 1253** (*integrability is closed under extension by zero*)

is explicitly cited in the proof of:

- Lemma 1274 (*integral over subset*),
- Lemma 1277 (*integral over singleton*).

**Lemma 1254** (*integrability is closed under restriction*)

is not yet used.

**Lemma 1255** (*integrability, masking is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1275 (*integral over subset is  $\sigma$ -additive*).

**Lemma 1256** (*integral in  $\mathcal{M}$* )

is explicitly cited in the proof of:

Lemma 1258 (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* ),

Lemma 1262 (*compatibility of integral with almost equality*),

Lemma 1266 (*integral in  $\mathcal{M}$  is homogeneous*),

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1273 (*compatibility of integral in  $\mathcal{M}$  with absolute value*),

Lemma 1280 (*integral with trivial measure*),

Lemma 1281 (*integral with counting measure*),

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ).

**Lemma 1258** (*integral in  $\mathcal{M}$  generalizes integral in  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

Lemma 1259 (*integral in  $\mathcal{M}$  of zero is zero*),

Lemma 1260 (*integral in  $\mathcal{M}$  of indicator function*),

Lemma 1274 (*integral over subset*),

Lemma 1329 (*first mean value inequality*).

**Lemma 1259** (*integral in  $\mathcal{M}$  of zero is zero*)

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1260** (*integral in  $\mathcal{M}$  of indicator function*)

is explicitly cited in the proof of:

Lemma 1267 (*integral in  $\mathcal{M}$  of constant function*).

**Definition 1261** (*merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* )

is not yet used.

**Lemma 1262** (*compatibility of integral with almost equality*)

is explicitly cited in the proof of:

Lemma 1263 (*finite representative of integrable in  $\mathcal{M}$* ),

Lemma 1264 (*extended almost modus ponens for integral in  $\mathcal{M}$* ),

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1297 (*compatibility of  $N_p$  with almost equality*),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ),

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1263** (*finite representative of integrable in  $\mathcal{M}$* )

is explicitly cited in the proof of:

Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ).

**Lemma 1264** (*extended almost modus ponens for integral in  $\mathcal{M}$* )

is explicitly cited in the proof of:

Lemma 1272 (*integral in  $\mathcal{M}$  is almost monotone*).

**Lemma 1266** (*integral in  $\mathcal{M}$  is homogeneous*)

is explicitly cited in the proof of:

Lemma 1267 (*integral in  $\mathcal{M}$  of constant function*),

Lemma 1270 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*),

Lemma 1274 (*integral over subset*).



**Lemma 1267** (*integral in  $\mathcal{M}$  of constant function*)

is explicitly cited in the proof of:

Lemma 1328 (*constant function is  $\mathcal{L}^p$* ).

**Lemma 1268** (*Minkowski inequality in  $\mathcal{M}$  ( $p = 1$ )*)

is explicitly cited in the proof of:

Lemma 1269 (*integral in  $\mathcal{M}$  is additive*),

Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1269** (*integral in  $\mathcal{M}$  is additive*)

is explicitly cited in the proof of:

Lemma 1270 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*),

Lemma 1271 (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*),

Lemma 1274 (*integral over subset*).

**Lemma 1270** (*integral in  $\mathcal{M}_{\mathbb{R}}$  is linear*)

is explicitly cited in the proof of:

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ).

**Lemma 1271** (*integral in  $\mathcal{M}_{\mathbb{R}}$  is monotone*)

is explicitly cited in the proof of:

Lemma 1272 (*integral in  $\mathcal{M}$  is almost monotone*),

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ).

**Lemma 1272** (*integral in  $\mathcal{M}$  is almost monotone*)

is explicitly cited in the proof of:

Lemma 1320 (*integral in  $\mathcal{L}^1$  is almost monotone*).

**Lemma 1273** (*compatibility of integral in  $\mathcal{M}$  with absolute value*)

is explicitly cited in the proof of:

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ).

**Lemma 1274** (*integral over subset*)

is explicitly cited in the proof of:

Lemma 1275 (*integral over subset is  $\sigma$ -additive*),

Lemma 1276 (*integral over negligible subset*),

Lemma 1277 (*integral over singleton*),

Lemma 1329 (*first mean value inequality*).

**Lemma 1275** (*integral over subset is  $\sigma$ -additive*)

is explicitly cited in the proof of:

Lemma 1278 (*integral over interval*),

Lemma 1279 (*Chasles relation, integral over split intervals*).

**Lemma 1276** (*integral over negligible subset*)

is not yet used.

**Lemma 1277** (*integral over singleton*)

is explicitly cited in the proof of:

Lemma 1278 (*integral over interval*).

**Lemma 1278** (*integral over interval*)

is explicitly cited in the proof of:

Lemma 1279 (*Chasles relation, integral over split intervals*).

**Lemma 1279** (*Chasles relation, integral over split intervals*)

is not yet used.



**Lemma 1280** (*integral with trivial measure*)

is not yet used.

**Lemma 1281** (*integral with counting measure*)

is explicitly cited in the proof of:

Lemma 1282 (*integral with counting measure on  $\mathbb{N}$* ),

Lemma 1284 (*integral with Dirac measure*).

**Lemma 1282** (*integral with counting measure on  $\mathbb{N}$* )

is not yet used.

**Lemma 1284** (*integral with Dirac measure*)

is not yet used.

**Definition 1285** (*integral for Lebesgue measure on  $\mathbb{R}$* )

is not yet used.

**Lemma 1286** ( *$p$ -seminorm*)

is explicitly cited in the proof of:

Lemma 1288 ( *$N_p$  with trivial measure*),

Lemma 1289 ( *$N_p$  of indicator function*),

Lemma 1290 ( *$N_p$  is nonnegative*),

Lemma 1291 (*link between  $N_1$  and  $N_p$* ),

Lemma 1292 (*integrable is finite  $N_1$* ),

Lemma 1293 ( *$N_\infty$  bounded is almost bounded*),

Lemma 1294 (*compatibility of  $N_p$  with power function*),

Lemma 1296 ( *$N_p$  is almost definite*),

Lemma 1297 (*compatibility of  $N_p$  with almost equality*),

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),

Lemma 1300 ( *$N_p$  is absolutely homogeneous*),

Lemma 1302 (*generalized Hölder inequality (with  $\infty$ )*),

Lemma 1304 (*Hölder inequality*),

Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*),

Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),

Lemma 1310 (*Minkowski equality (nonnegative functions)*),

Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),

Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ),

Lemma 1339 (*convergence in  $\mathcal{L}^p$* ),

Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1347 (*Lebesgue, dominated convergence*),

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1288** ( *$N_p$  with trivial measure*)

is explicitly cited in the proof of:

Lemma 1317 ( *$\mathcal{L}^p$  with trivial measure*),

Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 1289** ( *$N_p$  of indicator function*)

is explicitly cited in the proof of:

Lemma 1301 ( *$N_p$  of constant function*).

**Lemma 1290** ( *$N_p$  is nonnegative*)

is explicitly cited in the proof of:

Lemma 1304 (*Hölder inequality*),

Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ).

**Lemma 1291** (*link between  $N_1$  and  $N_p$* )

is explicitly cited in the proof of:

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),  
 Lemma 1305 (*generalized Hölder inequality*),  
 Lemma 1306 (*compatibility of  $N_p$  with Hölder conjugates*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ).

**Lemma 1292** (*integrable is finite  $N_1$* )

is explicitly cited in the proof of:

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*).

**Lemma 1293** ( *$N_\infty$  bounded is almost bounded*)

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1294** (*compatibility of  $N_p$  with power function*)

is explicitly cited in the proof of:

Lemma 1295 (*compatibility of  $N_p$  with absolute value*),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 1295** (*compatibility of  $N_p$  with absolute value*)

is explicitly cited in the proof of:

Lemma 1298 ( *$N_p$  is monotone for nonnegative functions*),  
 Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1296** ( *$N_p$  is almost definite*)

is explicitly cited in the proof of:

Lemma 1304 (*Hölder inequality*),  
 Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*).

**Lemma 1297** (*compatibility of  $N_p$  with almost equality*)

is explicitly cited in the proof of:

Lemma 1299 ( *$N_p$  is almost monotone for almost nonnegative functions*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 1310 (*Minkowski equality (nonnegative functions)*),  
 Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*),  
 Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Lemma 1298** ( *$N_p$  is monotone for nonnegative functions*)

is explicitly cited in the proof of:

Lemma 1299 ( *$N_p$  is almost monotone for almost nonnegative functions*).

**Lemma 1299** ( *$N_p$  is almost monotone for almost nonnegative functions*)

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 1300** ( *$N_p$  is absolutely homogeneous*)

is explicitly cited in the proof of:

Lemma 1301 ( *$N_p$  of constant function*),

Lemma 1304 (*Hölder inequality*),  
 Lemma 1311 (*Minkowski equality (linearly dependent functions)*),  
 Lemma 1327 ( $\mathcal{L}^p$  is closed under scalar multiplication),  
 Lemma 1337 ( $\mathcal{L}^p$  is seminormed vector space).

**Lemma 1301** ( *$N_p$  of constant function*)

is explicitly cited in the proof of:  
 Lemma 1328 (*constant function is  $\mathcal{L}^p$* ).

**Lemma 1302** (*generalized Hölder inequality (with  $\infty$ )*)

is explicitly cited in the proof of:  
 Lemma 1304 (*Hölder inequality*),  
 Lemma 1305 (*generalized Hölder inequality*).

**Lemma 1304** (*Hölder inequality*)

is explicitly cited in the proof of:  
 Lemma 1305 (*generalized Hölder inequality*),  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1305** (*generalized Hölder inequality*)

is explicitly cited in the proof of:  
 Lemma 1334 (*generalized Hölder inequality in  $\mathcal{L}^p$* ).

**Lemma 1306** (*compatibility of  $N_p$  with Hölder conjugates*)

is explicitly cited in the proof of:  
 Lemma 1309 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 1309** (*Minkowski inequality in  $\mathcal{M}$* )

is explicitly cited in the proof of:  
 Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ).

**Lemma 1310** (*Minkowski equality (nonnegative functions)*)

is not yet used.

**Lemma 1311** (*Minkowski equality (linearly dependent functions)*)

is explicitly cited in the proof of:  
 Lemma 1312 (*Minkowski equality (almost linearly dependent functions)*).

**Lemma 1312** (*Minkowski equality (almost linearly dependent functions)*)

is not yet used.

**Definition 1314** ( *$\mathcal{L}^p$ , vector space of  $p$ -powered integrable functions*)

is explicitly cited in the proof of:  
 Lemma 1316 (*equivalent definition of  $\mathcal{L}^p$* ),  
 Lemma 1317 ( *$\mathcal{L}^p$  with trivial measure*),  
 Lemma 1318 ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* ),  
 Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ),  
 Lemma 1322 ( *$\mathcal{L}^p$  is closed under absolute value*),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1324 (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* ),  
 Lemma 1326 (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* ),  
 Lemma 1327 ( *$\mathcal{L}^p$  is closed under scalar multiplication*),  
 Lemma 1328 (*constant function is  $\mathcal{L}^p$* ),  
 Lemma 1334 (*generalized Hölder inequality in  $\mathcal{L}^p$* ),  
 Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ),  
 Lemma 1336 (*Minkowski inequality in  $\mathcal{L}^p$* ),  
 Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),

Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1316** (*equivalent definition of  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1320 (*integral in  $\mathcal{L}^1$  is almost monotone*),  
 Lemma 1321 (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* ),  
 Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Lemma 1339 (*convergence in  $\mathcal{L}^p$* ),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1345 (*Fatou’s lemma in  $\mathcal{L}^1$* ),  
 Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1349 (*Lebesgue, extended dominated convergence*),  
 Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),  
 Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 1317** ( *$\mathcal{L}^p$  with trivial measure*)

is explicitly cited in the proof of:

Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 1318** ( *$\mathcal{L}^p$  and nonnegative is  $\mathcal{M}_+$* )

is explicitly cited in the proof of:

Lemma 1319 (*integral is positive linear form on  $\mathcal{L}^1$* ),  
 Lemma 1329 (*first mean value inequality*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1319** (*integral is positive linear form on  $\mathcal{L}^1$* )

is explicitly cited in the proof of:

Lemma 1329 (*first mean value inequality*),  
 Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),  
 Theorem 1347 (*Lebesgue, dominated convergence*).

**Lemma 1320** (*integral in  $\mathcal{L}^1$  is almost monotone*)

is not yet used.

**Lemma 1321** (*link between  $\mathcal{L}^1$  and  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Lemma 1322** ( *$\mathcal{L}^p$  is closed under absolute value*)

is explicitly cited in the proof of:

Lemma 1323 (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* ),  
 Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1323** (*bounded by  $\mathcal{L}^p$  is  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1329 (*first mean value inequality*),  
 Theorem 1347 (*Lebesgue, dominated convergence*),  
 Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1324** (*finite  $N_p$  has finite representative in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Lemma 1326** (*integrable and nonnegative has finite representative in  $\mathcal{L}^1$* )

is explicitly cited in the proof of:

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1345 (*Fatou's lemma in  $\mathcal{L}^1$* ).

**Lemma 1327** ( *$\mathcal{L}^p$  is closed under scalar multiplication*)

is explicitly cited in the proof of:

Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*).

**Lemma 1328** (*constant function is  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1329 (*first mean value inequality*).

**Lemma 1329** (*first mean value inequality*)

is explicitly cited in the proof of:

Lemma 1331 (*first mean value theorem*).

**Lemma 1331** (*first mean value theorem*)

is not yet used.

**Lemma 1334** (*generalized Hölder inequality in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1335 (*interpolation inequality in  $\mathcal{L}^p$* ).

**Lemma 1335** (*interpolation inequality in  $\mathcal{L}^p$* )

is not yet used.

**Lemma 1336** (*Minkowski inequality in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1337 ( *$\mathcal{L}^p$  is seminormed vector space*),

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1337** ( *$\mathcal{L}^p$  is seminormed vector space*)

is explicitly cited in the proof of:

Lemma 1339 (*convergence in  $\mathcal{L}^p$* ),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1347 (*Lebesgue, dominated convergence*),

Theorem 1349 (*Lebesgue, extended dominated convergence*),

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1339** (*convergence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1340 (*equivalent definition of convergence in  $\mathcal{L}^p$* ),

Lemma 1341 (*limits in  $\mathcal{L}^p$  are almost equal*),

Theorem 1343 (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* ),

Theorem 1347 (*Lebesgue, dominated convergence*),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1340** (*equivalent definition of convergence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ),

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Lemma 1341** (*limits in  $\mathcal{L}^p$  are almost equal*)

is explicitly cited in the proof of:

Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).

**Theorem 1343** (*Beppo Levi, monotone convergence in  $\mathcal{L}^1$* )

is explicitly cited in the proof of:

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ).

**Theorem 1345** (*Fatou’s lemma in  $\mathcal{L}^1$* )

**Theorem 1347** (*Lebesgue, dominated convergence*)

is explicitly cited in the proof of:

Theorem 1349 (*Lebesgue, extended dominated convergence*).

**Theorem 1349** (*Lebesgue, extended dominated convergence*)

is explicitly cited in the proof of:

Theorem 1351 (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* ).

**Theorem 1351** (*Lebesgue, extended dominated convergence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Theorem 1352 (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* ),

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Theorem 1352** (*Beppo Levi, monotone convergence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Theorem 1354 (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* ).

**Theorem 1354** (*Riesz–Fischer, convergence of Cauchy sequence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1355 (*subsequence of convergent sequence in  $\mathcal{L}^p$* ).

**Lemma 1355** (*subsequence of convergent sequence in  $\mathcal{L}^p$* )

is explicitly cited in the proof of:

Lemma 1356 (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* ).

**Lemma 1356** (*almost pointwise convergence and convergence in  $\mathcal{L}^p$* )

is not yet used.



**RESEARCH CENTRE  
PARIS**

2 rue Simone Iff - CS 42112  
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