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# Lebesgue integration. Detailed proofs to be formalized in Coq

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# Lebesgue integration. Detailed proofs to be formalized in Coq

François Clément, Vincent Martin

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January 2021

Project-Team Serena





# Lebesgue integration. Detailed proofs to be formalized in Coq

François Clément<sup>\*</sup>, Vincent Martin<sup>†</sup>

Project-Team Serena

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**Abstract:** To obtain the highest confidence on the correction of numerical simulation programs implementing the finite element method, one has to formalize the mathematical notions and results that allow to establish the soundness of the method. Sobolev spaces are the correct framework in which most partial derivative equations may be stated and solved. These functional spaces are built on integration and measure theory. Hence, this chapter in functional analysis is a mandatory theoretical cornerstone for the definition of the finite element method. The purpose of this document is to provide the formal proof community with very detailed pen-and-paper proofs of the main results from integration and measure theory.

**Key-words:** measure theory, Lebesgue integration, detailed mathematical proof, formal proof in real analysis

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<sup>\*</sup> Inria & CERMICS, École des Ponts, 77455 Marne-la-Vallée Cedex 2, France. [Francois.Clement@inria.fr](mailto:Francois.Clement@inria.fr).

<sup>†</sup> LMAC, UTC, CS 60319, 60203 Compiègne Cedex, France. [Vincent.Martin@utc.fr](mailto:Vincent.Martin@utc.fr).

**RESEARCH CENTRE  
PARIS**

2 rue Simone Iff - CS 42112  
75589 Paris Cedex 12

# Intégrale de Lebesgue.

## Preuves détaillées en vue d'une formalisation en Coq

**Résumé :** Pour obtenir la plus grande confiance en la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il faut formaliser les notions et résultats mathématiques qui permettent d'établir la justesse de la méthode. Les espaces de Sobolev sont le cadre correct dans lequel la plupart des équations aux dérivées partielles peuvent être énoncées et résolues. La construction de ces espaces fonctionnels repose sur le calcul intégral et la théorie de la mesure. Ce chapitre de l'analyse fonctionnelle est donc un fondement théorique nécessaire à la définition de la méthode des éléments finis. L'objectif de ce document est de fournir à la communauté preuve formelle des preuves papiers très détaillées des principaux résultats du calcul intégral et de la théorie de la mesure.

**Mots-clés :** théorie de la mesure, intégrale de Lebesgue, preuve mathématique détaillée, preuve formelle en analyse réelle

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## 1 Introduction

Formal proof tools are now known to be able to tackle real analysis. For instance, in the field of Ordinary Differential Equations (ODEs) with Isabelle/HOL [36, 35, 37], and Coq [42], or in the field of Partial Differential Equations (PDEs), again with Isabelle/HOL [1], and Coq [5, 6]. In the latter example, the salient aspect is that the round-off error due to the use of IEEE-754 floating-point arithmetic can also be fully taken into account. But the price to pay is that every tiny details of the proofs have to be dealt with, and thus that very detailed pen-and-paper proofs must be available.

Our long term purpose is to formally prove programs using the Finite Element Method (FEM). The FEM is widely used to solve PDEs, mainly because of its well established mathematical foundation, e.g. see [49, 16, 44, 10, 24]. The Lax–Milgram theorem, one of the key ingredients to build the FEM, was already addressed in [17] for a detailed pen-and-paper proof, and in [7] for a formal proof in Coq. The present document is a further contribution toward our ultimate goal.

The Lax–Milgram theorem claims existence and uniqueness of the solution to the weak formulation of a PDE problem and its discrete approximation; it is stated on a Hilbert space, i.e. a complete inner product space. The simplest Hilbert functional spaces relevant for the resolution of PDEs are the spaces  $L^2$  and  $H^1$ . More generally, when stronger results such as the Banach–Nečas–Babuška theorem are involved, one may be interested in the Sobolev spaces  $W^{m,p}$  that are Banach spaces, i.e. complete normed vector spaces.

Let  $d \in \{1, 2, 3\}$  be the spatial dimension. Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . We consider real-valued functions defined almost everywhere over  $\Omega$ . For  $0 < p \leq \infty$ , the Lebesgue space  $L^p(\Omega)$  is the space of measurable functions for which the  $p$ -th power of the absolute value has a finite integral over  $\Omega$ . For any natural number  $m$ , the Sobolev space  $W^{m,p}(\Omega)$  is the vector subspace of  $L^p(\Omega)$  of functions for which all weak derivatives up to order  $m$  also belong to  $L^p(\Omega)$ . The Hilbert spaces correspond to the case  $p = 2$ ,  $H^m(\Omega) \stackrel{\text{def.}}{=} W^{m,2}(\Omega)$ .

There is a huge variety of concepts of integral and integrability in the literature, e.g. see [15, 12], and one may wonder which one to use. Some are overridden by others, some are equivalent, and some have been developed for specific situations, such as vector-valued functions or functions defined on an infinite-dimensional domain. But a few of them have gained popularity, be it for their appropriateness to be taught or their general-purpose nature: namely Riemann, Lebesgue, and Henstock–Kurzweil integration. Lebesgue integral (with Lebesgue measure) is the traditional framework in which the Sobolev functional spaces are defined. Indeed, Riemann integral is disqualified because of its poor results on limit and integral exchange making completeness unreachable, and so is Henstock–Kurzweil integral for its not so obvious extension to the multidimensional case and construction of a complete normed vector space of HK-integrable functions [29, 43].

This document covers all material up to the first properties of the seminormed vector space  $\mathcal{L}^1$  of integrable functions (before switching to the quotient). Among the rich literature on Lebesgue integral theory, it was made using in particular the textbooks [41, 28, 46]. This includes results on the general concepts of measurability, measure and negligibility, and integration of nonnegative measurable functions culminating with the Beppo Levi (monotone convergence) theorem allowing to exchange limit and integral for nondecreasing sequences of measurable functions, and Fatou’s lemma that only gives an upper bound when the sequence is not monotone. These aspects are all formalized in Coq in [8]. In addition, the present document also covers the building of the Lebesgue measure in  $\mathbb{R}$  using Carathéodory’s extension scheme, the integral over a product space and the Tonelli theorem, and the integral for measurable functions with no constant sign and Lebesgue’s dominated convergence theorem.

The paper is organized as follows. The notations are first collected in Section 2. Then, the literature on the subject is briefly reviewed in Section 3. Section 4 gathers some popular proof techniques. The chosen proof paths of the main results are then sketched in Section 5, and fully detailed in Section 6. Section 7 concludes and gives some perspectives. Finally, lists of statements and direct dependencies are gathered in the appendix. The latter are not intended to be printed!



## 2 Notations

In this section (as in most of the document), we use the following conventions:

- capital letters  $X$  and  $Y$  denote “surrounding” sets, typically the domain and codomain of functions,  $A$  and  $B$  are subsets of  $X$ , and  $I$  denotes a set of indices;
- calligraphic letter  $\mathcal{R}$  denotes a binary relation on  $X$  (e.g. equality or inequality);
- the capital letter  $P$  denotes a predicate, i.e. a function taking Boolean values;
- small letters  $x, y$  and  $i$  denote elements of the set using the matching capital letter;
- small letters  $a$  and  $b$  are (extended) real numbers (such that  $a \leq b$ ), and  $n$  and  $p$  are (extended) natural numbers (such that  $n \leq p$ );
- small letter  $f$  denotes a function, e.g. from set  $X$  to set  $Y$ ;
- the capital Greek letter  $\Sigma$  denotes a  $\sigma$ -algebra on  $X$ , and the small Greek letter  $\mu$  denotes a measure on the measurable space  $(X, \Sigma)$ .

The following notations and conventions are used throughout this document.

- Logic:
  - Using a compound (tuple of elements of  $X$ , or subset of  $X$ ) in an expression at a location where only a single element makes sense is a shorthand for the same expression expanded for all elements of the compound; for instance, “ $\forall x, x' \in X$ ” means “ $\forall x \in X, \forall x' \in X$ ”, “ $\forall (x_i)_{i \in I} \in X$ ” means “ $\forall i \in I, x_i \in X$ ”, and “ $x, x' \mathcal{R} x''$ ” means  $x \mathcal{R} x''$  and  $x' \mathcal{R} x''$ ;
  - $x_0 \mathcal{R}_1 x_1 \dots \mathcal{R}_m x_m$  is a shorthand for  $x_0 \mathcal{R}_1 x_1 \wedge x_1 \mathcal{R}_2 x_2 \wedge \dots \wedge x_{m-1} \mathcal{R}_m x_m$ .
- Set theory:
  - $\mathcal{P}(X)$  denotes the power set of  $X$ , i.e. the set of its subsets;
  - $\subsetneq$  denotes proper, or strict, inclusion.
  - $n$ -ary set operations, such as intersection and union, have precedence over binary set operations (no need for big parentheses); for instance,  $\bigcup_{i \in I} A_i \cap B$  means  $(\bigcup_{i \in I} A_i) \cap B$ ;
  - superscript  $^c$  denotes the absolute complement;
  - $\setminus$  denotes the set difference (or relative complement):  $A \setminus B \stackrel{\text{def.}}{=} A \cap B^c$ ;  
when  $B \subset A$ ,  $A \setminus B$  is also called the *local complement of  $B$  (in  $A$ )*;
  - $\uplus$  denotes the disjoint union:  $A \uplus B$  means  $A \cup B$  with the assumption that  $A \cap B = \emptyset$ ;
  - $\bar{\phantom{x}}$  is used to denote a set of traces of subsets, see Definition 213;
  - $\overline{\phantom{x}}$  is used to denote a set of Cartesian products of subsets, see Definition 214;
  - $\text{card}(X)$  is the cardinal of  $X$ , i.e. the number of its elements, with the convention  $\text{card}(X) \stackrel{\text{def.}}{=} \infty$  when  $X$  is infinite;
  - countable means either finite, or infinitely countable:  $X$  is countable iff there exists an injection from  $X$  to  $\mathbb{N}$  iff there exists  $I \subset \mathbb{N}$  and a bijection between  $I$  and  $X$ ;
  - $\mathbb{1}_A$ , or  $\mathbb{1}_A^X$ , denotes the indicator function of the subset  $A$  of  $X$ . It is the function from  $X$  to  $\mathbb{R}$ , or  $\overline{\mathbb{R}}$ , that takes the value 1 for all elements of  $A$ , and the value 0 elsewhere;
  - the set of functions from  $X$  to  $Y$  is either denoted  $Y^X$ , or through the type annotation “ $X \rightarrow Y$ ”. Both compact expressions “let  $f \in Y^X$ ” and “let  $f : X \rightarrow Y$ ” mean “let  $f$  be a function from  $X$  to  $Y$ ”;

- to avoid double parentheses/braces, the expressions involving inverse images are simplified:  $f^{-1}(y)$  is a shorthand for  $f^{-1}(\{y\})$ , and when the codomain  $Y$  is a subset of  $\overline{\mathbb{R}}$ ,  $f^{-1}(a, b)$  is a shorthand for  $f^{-1}((a, b))$  (with the natural variants using (square) brackets when either or both bounds are included);
  - $\{P(f)\}$  is a shorthand for  $\{x \in X \mid P(f(x))\}$ , or  $f^{-1}(\{y \in Y \mid P(y)\})$ ; for instance,  $\{f < a\}$  means either  $f^{-1}[-\infty, a)$  or  $f^{-1}(-\infty, a)$ , see Lemma 555, and Lemma 563;
  - $f^+$  and  $f^-$  denote the nonnegative and nonpositive parts of a numerical function  $f$ , see Definition 373.
- Totally ordered set:
    - when the nonempty set  $X$  is totally ordered, intervals follow the notations used for real numbers; for instance,  $(x_1, x_2]$  represents the subset  $\{x \in X \mid x_1 < x \leq x_2\}$ ;
    - the square-and-round bracket notation is used to not specify whether the bound is included or not:  $(x_1, x_2]$  means either  $(x_1, x_2)$  or  $(x_1, x_2]$ ;
    - the lowest and greatest bounds of  $X$  may be denoted  $-\infty$  and  $+\infty$  (as usual, the plus sign may be omitted); they may belong to  $X$  or not;  $\overline{X}$  denotes  $X \cup \{-\infty, \infty\}$ .
    - rays, or half-lines, are denoted using the square-and-round notation: whether  $\infty$  belongs to  $X$  or not,  $(x, \infty]$  represents  $\{x' \in X \mid x < x'\}$ , i.e. either  $(x, \infty]$  or  $(x, \infty)$ , see Definition 217;
    - $\mathcal{I}_X^{o,p}$  denotes the set of open proper intervals of  $X$ , see Definition 217;
    - $\mathcal{R}_X^o$  denotes the set of open rays of  $X$ , see Definition 217.
    - $\mathcal{I}_X^o$  denotes the set of open intervals of  $X$ , see Definition 217.
  - Numbers:
    - $\lfloor \cdot \rfloor$  denotes the floor function from  $\mathbb{R}$  to  $\mathbb{N}$ :  $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$ ;
    - $\llbracket n..p \rrbracket$  denotes the integer interval  $\llbracket n, p \rrbracket \cap \mathbb{N}$  (with the natural variants using parentheses and/or brackets when exclusion or inclusion of bounds is specified);
    - $\ell(\llbracket a, b \rrbracket)$  denotes the length of interval  $\llbracket a, b \rrbracket$ , see Definition 671.
  - General topology:
    - $\mathcal{T}_X(G)$  denotes the topology generated by  $G \subset \mathcal{P}(X)$ , see Definition 227;
  - Measure theory:
    - $\Pi_X(G)$  denotes the  $\pi$ -system generated by  $G \subset \mathcal{P}(X)$ , see Definition 427;
    - $\mathcal{A}_X(G)$  denotes the set algebra generated by  $G \subset \mathcal{P}(X)$ , see Definition 437;
    - $\mathcal{C}_X(G)$  denotes the monotone class generated by  $G \subset \mathcal{P}(X)$ , see Definition 444;
    - $\Lambda_X(G)$  denotes the  $\lambda$ -system generated by  $G \subset \mathcal{P}(X)$ , see Definition 457;
    - $\Sigma_X(G)$  denotes the  $\sigma$ -algebra generated by  $G \subset \mathcal{P}(X)$ , see Definition 473;
    - $\mathcal{B}(X)$  denotes the Borel  $\sigma$ -algebra generated by the open subsets, see Definition 495;
    - $\bigotimes_{i \in [1..m]} \Sigma_i$  denotes the tensor product of the  $\sigma$ -algebras  $(\Sigma_i)_{i \in [1..m]}$ , see Definition 529;
    - $\mathbf{N}$  denotes the set of negligible subsets, see Definition 614;
    - the annotation “ $\mu$  a.e.” specifies that the proposition is only considered almost everywhere, e.g.  $\stackrel{\mu \text{ a.e.}}{=} , \forall_{\mu \text{ a.e.}}, (X \rightarrow Y)_{\mu \text{ a.e.}}$ , or  $\mathcal{R}_{\mu \text{ a.e.}}$ , see Definition 624;
    - $\delta_Y$  denotes the counting measure associated with  $Y$ ,  $\delta_a$  denotes the Dirac measure at  $a$ , see Lemma 652 and Definition 656;
    - $\mathcal{L}$  denotes the Lebesgue  $\sigma$ -algebra on  $\mathbb{R}$  and  $\lambda$  denotes the (Borel-)Lebesgue measure on  $\mathbb{R}$ , see Definitions 677 and 685, and Theorem 702;

- 
- $\mu_1 \otimes \mu_2$  denotes the tensor product of measures  $\mu_1$  and  $\mu_2$ , see Definition 793, Definition 794, and Lemma 801;
  - $\lambda^{\otimes 2}$  denotes the Lebesgue measure on  $\mathbb{R}^2$ , see Lemma 803;
  - Lebesgue integral:
    - $\mathcal{IF}$  denotes the set of measurable indicator functions, see Definition 709;
    - $\mathcal{SF}$  denotes the vector space of simple functions, see Definition 723;
    - $\mathcal{SF}_+$  denotes the set of nonnegative simple functions, see Definition 732;
    - $\mathcal{M}_+$  denotes the set of nonnegative measurable functions  $X \rightarrow \overline{\mathbb{R}}_+$ , see Definition 578;
    - $\mathcal{M}$  denotes the set of measurable functions  $X \rightarrow \overline{\mathbb{R}}$ , see Definition 560;
    - $\mathcal{M}_{\mathbb{R}}$  denotes the vector space of measurable functions  $X \rightarrow \mathbb{R}$ , see Definition 552;
    - $\int f d\mu$  denotes the (Lebesgue) integral of  $f$  for the measure  $\mu$ , whether the function belongs to  $\mathcal{IF}$ ,  $\mathcal{SF}_+$ ,  $\mathcal{M}_+$ , or  $\mathcal{M}$ , see Section 4.1, Definition 715, Lemmas 737 and 753, and Definition 819;
    - $\int_A f d\mu$  denotes the integral of the restriction of  $f$  to  $A$ , see Lemmas 718, 747, 776, and 825;
    - $\int_a^b f(x) d\mu(x)$  denotes the integral of  $f$  over the interval  $(a, b)$ , see Lemma 828;
    - $\int f(x) dx$  denotes the integral of  $f$  for the Lebesgue measure, see Definition 834;
    - $(\mathcal{L}^1, N_1)$  denotes the seminormed vector space of integrable functions, see Lemma 835, and Definition 845;
    - $\mathcal{I}$  denotes the integral operator for integrable functions, see Lemma 853.

### 3 State of the art

After a quick survey of the wide variety of integrals, and of the means to build Lebesgue integral and Lebesgue measure, we review some works of a few authors, partly from the French school, that provide some details about results in measure theory, Lebesgue integration, and basics of functional analysis.

As usual, proofs provided in the literature are not comprehensive, and we have to cover a series of sources to collect all the details necessary for a formalization in a formal proof tool such as Coq. Usually, Lecture Notes in undergraduate mathematics are very helpful and we selected [31, 32, 33] among many other possible choices.

#### 3.1 A zoology of integrals

The history of integral calculus dates back at least from the Greeks, with the method of exhaustion for the evaluation of areas, or volumes. Since then, mathematicians have constantly endeavored to develop new techniques. The objective is mainly to be able to integrate a wider class of functions. But also to establish more powerful results, such as convergence results, or the Fundamental Theorems of Calculus (expressing that derivation and integration are each other inverse operations). And possibly to fit specific contexts, e.g. driven by applications in physics, or for teaching purposes. For a wide panel, and pros and cons of different approaches, see for instance [18, 15, 12], in which more than an hundred named integrals are listed. But, as pointed out in [47], building a universal integral “will probably be unnecessarily complicated when restricted to a simple setting”.

Considering the case of a numerical function defined on some interval, let us review some of the most popular ones.

**Leibniz–Newton integral (late 17th century).** The Leibniz–Newton (LN-)integral is defined in a descriptive way as the difference of any primitive evaluated at the bounds of the integration interval. All continuous functions are LN-integrable.

This integral possesses interesting properties such as formulas for integration by parts or by substitution, and a uniform convergence theorem. But since the definition is not constructive, there is no regular process to build the primitive of a function, was it continuous. Moreover, other monotone or dominated convergence results must assume that the limit is LN-integrable. And it arises the question of integrability of functions with no primitives.

**Riemann integral (early 19th century).** The Riemann (R-)integral [45] consists in cutting the area “under” the graph of the function into vertical rectangular strips by choosing a subdivision of the integration interval, and in increasing the number of strips by making the step of the subdivisions go down to 0. Equivalently, it might be more convenient to consider the Darboux integral [21], which checks that both upper and lower so-called Darboux sums have the same limit.

This simple approach has similar properties than the LN-integral, with for instance a uniform convergence result, and R-integrable functions are the piecewise continuous function which set of discontinuity points has null measure. But again, monotone convergence and dominated convergence theorems must assume R-integrability of the limit.

**Lebesgue integral (early 20th century).** The key idea of the Lebesgue (L-) approach [39] is to consider subdivisions of the codomain of the function, i.e. cutting the area under the graph into horizontal pieces, which are no longer continuous strips when the function is not concave, and to define a measure for these pieces that generalizes the length of intervals.

This has the great advantage of providing powerful monotone and dominated convergence theorems, and to allow for an abstract setting in which one can handle functions defined on more general spaces than the Euclidean spaces  $\mathbb{R}^n$ . And this paves the road for probability theory. For instance, the Wiener measure used for the study of stochastic processes such as Brownian motion is defined on the infinite-dimensional Banach space of continuous functions from a compact interval

to  $\mathbb{R}^n$ . Moreover, the concept of property satisfied *almost everywhere* opens the way to the  $L^p$  Lebesgue spaces as (complete) normed vector spaces.

The main drawback is the necessity to develop the measure theory to be able to associate a length to the “horizontal pieces”; in general, this is not possible for all subsets of the domain of the function, and leads to the concepts of *measurable subset* and *measurable function*.

Going back to the geometrical interpretation of the area under the graph of the function, it is interesting to note that the opposition Riemann (subdivision of the domain) versus Lebesgue (subdivision of the codomain) reflects in the field of ordinary differential equations where the classical numerical schemes based on a discretization of the time domain are now opposed to the recent quantized state system solvers that are based on the quantization of the state variables, see [14, 25, 26]. Moreover, we may also cite [30] in the field of signal processing where the Lebesgue geometrical scheme is seen as a nonlinear sampling qualified of “noise-free quantization”, that could have impact on electronic design.

**Henstock–Kurzweil integral (mid-20th century).** The Henstock–Kurzweil (HK-)integral, or gauge integral [38, 34, 3] is a generalization of the R-integral for which the subdivision of the integration interval is no longer uniform and driven by the variations of the function to integrate through a so-called *gauge* function.

It is more powerful than the L-integral in the sense that a function is L-integrable if and only if the function and its absolute value are HK-integrable. Unlike with the L-integral, this leads to formulations of the fundamental theorem of calculus without the need for the concept of improper integral. The monotone and dominated convergence theorems are also valid. Moreover, the HK-integral is usually considered more suited for teaching, for it is hardly more complex than the R-integral.

However, unlike with the L-integral, the HK-integral cannot be easily extended to Euclidean spaces  $\mathbb{R}^n$ , and the construction of a Banach space of HK-integrable function is far less obvious than that of the  $L^1$  Lebesgue space [29, 43].

**Daniell integral (early 20th century).** The Daniell (D-) approach [20, 27] is a general scheme that extends an *elementary* integral defined for *elementary* functions to a much wider class of functions by checking that upper and lower elementary integrals of elementary functions share the same limit. In a similar way than the Darboux approach. When applied to the integral of simple functions, this is somehow equivalent to steps 3 and 4 of the Lebesgue scheme described later in Section 4.1. But it is enough to consider the R-integral for the compactly supported continuous functions (a.k.a. the Cauchy integral) to get back the L-integral for the Lebesgue measure on the Euclidean spaces  $\mathbb{R}^n$ .

This approach to the L-integral has the great advantage not to need the concept of measure, which can be reconstructed afterwards by taking the integral of indicator functions of measurable subsets. Moreover, in the mid-20th century, the Daniell approach to L-integral was shown in [4] to be compatible with constructive analysis.

In the present document, we choose the Lebesgue approach.

### 3.2 Lebesgue integral and Lebesgue measure

The Lebesgue integral was originally built above measure theory following the so-called Lebesgue scheme described in Section 4.1. We have seen that it may also be built in a constructive way by applying the Daniell extension scheme to some elementary integral defined for elementary functions, such as the Riemann integral for compactly supported continuous functions. Another alternative using the same basic ingredients consists in the completion of the normed vector space of compactly supported continuous functions and the extension of the Riemann integral which is uniformly continuous [9, 22].

The Lebesgue measure on  $\mathbb{R}$  is a generalization of the length of bounded intervals to a much wider class of subsets. This extension is actually unique and complete. It is defined on the Lebesgue  $\sigma$ -algebra which is generated by the Borel subsets and the negligible subsets. The restriction to the sole Borel subsets (the  $\sigma$ -algebra generated by the open subsets, or the closed subsets) is sometimes called the Borel(-Lebesgue) measure. In the very same way, the Lebesgue measure on the Euclidean spaces  $\mathbb{R}^n$  is a generalization of the  $n$ -volume of rectangular boxes. It is the tensor product of the Lebesgue measure on  $\mathbb{R}$ .

There are three main techniques for the construction of the Lebesgue measure on  $\mathbb{R}$ . The most popular one for teaching is through Carathéodory's extension theorem [13, 23]. The process builds progressively the Lebesgue  $\sigma$ -algebra and the Lebesgue measure with their properties (see Section 4.2). A more abstract approach is based on the Riesz–Markov–Kakutani representation theorem, which associates any positive linear form on the space of compactly supported continuous functions with a unique measure, e.g. see [46]. The third way is not based on measure theory. It follows the Daniell approach to integration, which is also based on the Riemann integration of compactly supported continuous functions, and defines the Lebesgue measure of any measurable subset as the integral of its indicator function.

The construction of non-Lebesgue-measurable subsets requires the use of the axiom of choice, as for instance the Vitali subset [48].

In the present document, we build Lebesgue integral using the Lebesgue scheme (see Section 4.1), and Lebesgue measure through Carathéodory's extension theorem (see Section 4.2).

### 3.3 Our main sources

The main ingredients of measure theory and Lebesgue integration are presented in an elementary manner in the textbook [41], which is very well suited for our purpose. However, some results such as the construction of the Lebesgue measure and the Tonelli theorem are admitted.

Details for the construction of the Lebesgue measure (through the Carathéodory approach), and for the proof of the Tonelli theorem can be found in [28]. This other textbook is quite comprehensive. Many results are presented as exercises with their solution.

We may also cite [46], [2], and [11]. The former for instance for a simpler proof of additivity of the integral for simple functions, and for a construction of the Lebesgue measure through the Riesz–Markov–Kakutani representation theorem. And the latter two for the  $L^p$  Lebesgue spaces and the  $W^{m,p}$  Sobolev spaces.

## 4 Proof techniques

In the framework of integration theory, some proof techniques are used repeatedly. This section describes the most important ones: the “Lebesgue scheme”, the “Carathéodory extension scheme”, and the “Dynkin  $\pi$ - $\lambda$  theorem / monotone class theorem scheme”.

### 4.1 Lebesgue scheme

Consider a set  $X$  equipped with a  $\sigma$ -algebra  $\Sigma$  of measurable subsets, and a measure  $\mu$  that associates a “length” (in  $\overline{\mathbb{R}}_+$ ) to any measurable subset. The set of extended real numbers  $\overline{\mathbb{R}}$  is equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}(\overline{\mathbb{R}})$  that is generated by all open subsets. The purpose is to build, or prove properties of, the integral of “measurable” functions from  $X$  to  $\overline{\mathbb{R}}$ , i.e. for which the inverse image of measurable subsets (of  $\overline{\mathbb{R}}$ ) are measurable subsets (of  $X$ ).

The Lebesgue scheme consists in establishing facts about the integral by working successively inside three embedded functional subsets: the set  $\mathcal{IF}$  of indicator functions of measurable subsets, the vector space  $\mathcal{SF}$  of simple functions (the linear span of  $\mathcal{IF}$ ), and the set  $\mathcal{M}$  of measurable functions.

The four steps of the scheme are the following:

1. establish the fact in the case of indicator functions (in  $\mathcal{IF}$ ) for which the integral is simply the measure of the support;
2. then, generalize the fact to the case of nonnegative simple functions (in  $\mathcal{SF}_+$ ) by nonnegative linear combination of indicator functions;
3. then again, generalize the fact to the case of nonnegative measurable functions (in  $\mathcal{M}_+$ ) by taking the supremum for all lower nonnegative simple functions;
4. finally, generalize the fact to the case of all measurable functions (in  $\mathcal{M}$ ) by taking the difference between the expressions involving nonnegative and nonpositive parts (integrable functions are those for which this difference is well-defined).

Actually, a fifth step can be added to generalize the fact to the case of numeric functions taking their values in  $\mathbb{R}^n$ ,  $\mathbb{C}$ , or  $\mathbb{C}^n$  by considering separately all components.

In the present document, the steps of Lebesgue scheme are mainly used to build the Lebesgue integral, but also to establish some of the properties, such as the Beppo Levi (monotone convergence) theorem and the Tonelli theorem.

### 4.2 Carathéodory’s extension scheme

The Carathéodory extension scheme consists in applying the eponymous theorem to extend a pre-measure (i.e. defined on a ring of subsets, and not on a  $\sigma$ -algebra) into a complete measure on a  $\sigma$ -algebra containing the initial ring of subsets.

Let  $X$  be the ambient set,  $\mu$  be the pre-measure and  $\mathcal{R}$  be the ring of subsets (of  $X$ ). The two steps of the scheme are the following:

1. extend the pre-measure to any subset of  $X$  by taking the infimum of the sum of pre-measures of all coverings with elements of the ring  $\mathcal{R}$  (and the value  $\infty$  when there is no such covering):

$$\mu^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid (A_n)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_n \right\};$$

the theorem ensures that  $\mu^*$  is an outer measure on  $X$ , and that it is an extension of the pre-measure  $\mu$ ;

2. define the set of Carathéodory-measurable subsets as

$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \mid \forall A \subset X, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\};$$

the theorem ensures that  $\Sigma$  is a  $\sigma$ -algebra on  $X$  containing the ring  $\mathcal{R}$ , and that the restriction  $\mu^*|_{\Sigma}$  is a complete measure on  $(X, \Sigma)$ .

The Carathéodory extension scheme is a popular tool to build measures. The prominent example is the Lebesgue measure on  $\mathbb{R}$  that extends the length of open intervals. But for instance, the extension scheme may also be used to build the Lebesgue-Stieltjes measure associated with its cumulative distribution function, and Loeb measures in a nonstandard analysis framework, e.g. see [40, 19].

### 4.3 Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem scheme

A  $\pi$ -system, a set algebra, a monotone class, and a  $\lambda$ -system are all different kinds of subset systems (i.e. subsets of the power set), just like a  $\sigma$ -algebra, but with less basic properties.

Roughly speaking, being a  $\lambda$ -system (resp. a monotone class) is what is missing to a  $\pi$ -system (resp. an algebra of subsets) to be a  $\sigma$ -algebra. To prove that some property  $P$  holds on some  $\sigma$ -algebra  $\Sigma$  generated by some subset system  $G$ , the idea is to proceed in two steps. First, show that the property  $P$  holds on the simpler kind of subset system ( $\pi$ -system, or set algebra) generated by  $G$ , and then establish that the subset of  $\Sigma$  where  $P$  holds is of the less simple kind of subset system ( $\lambda$ -system, or monotone class). Indeed, the Dynkin  $\pi$ - $\lambda$  theorem states that the  $\lambda$ -system generated by a  $\pi$ -system is equal to the  $\sigma$ -algebra generated by the same  $\pi$ -system, and similarly, the monotone class theorem states that the monotone class generated by a set algebra is equal to the  $\sigma$ -algebra generated by the same set algebra.

The five steps of the Dynkin  $\pi$ - $\lambda$  theorem, and monotone class theorem schemes are the following:

1. first define  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ ; obviously, we have  $\mathcal{S} \subset \Sigma$ ;
2. then, show that  $G \subset \mathcal{S}$ ;
3. then, show that the  $\pi$ -system (resp. set algebra)  $\mathcal{U}_X^2(G)$  is a subset of  $\mathcal{S}$ ;
4. then, show that  $\mathcal{S}$  is a  $\lambda$ -system (resp. monotone class);
5. and finally, obtain the other inclusion  $\Sigma \subset \mathcal{S}$ , because from monotonicity of subset system generation, and from the Dynkin  $\pi$ - $\lambda$  theorem (resp. monotone class theorem), i.e. the equality in the middle, we have

$$\Sigma = \Sigma_X(G) \subset \Sigma_X(\mathcal{U}_X^2(G)) = \mathcal{U}_X^1(\mathcal{U}_X^2(G)) \subset \mathcal{U}_X^1(\mathcal{S}) = \mathcal{S}$$

where  $\mathcal{U}_X^1 = \Lambda_X$  (resp.  $\mathcal{C}_X$ ), i.e. the generated  $\lambda$ -system (resp. monotone class), and  $\mathcal{U}_X^2 = \Pi_X$  (resp.  $\mathcal{A}_X$ ), i.e. the generated  $\pi$ -system (resp. set algebra).

In the present document, the fifth step is embodied by Lemma 505 for the Dynkin  $\pi$ - $\lambda$  theorem, and by Lemma 509 for the monotone class theorem. The Dynkin  $\pi$ - $\lambda$  theorem scheme is used once when extending the equality of two measures on a generator  $\pi$ -system to the whole  $\sigma$ -algebra, which is used to establish uniqueness of the Lebesgue measure on  $\mathbb{R}$ . The monotone class theorem scheme is used twice when building the tensor product measure in the context of the product of finite measure spaces: to establish first the measurability of the measure of sections, and then the uniqueness of the tensor product measure.



## 5 Statements and sketches of the proofs

### 5.1 Sketch of the proof of Lebesgue's extended dominated convergence theorem

#### Lebesgue's extended dominated convergence theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$ . Assume that the sequence is  $\mu$ -almost everywhere pointwise convergent towards  $f$ . Assume that  $g$  is  $\mu$ -integrable, and that for all  $n \in \mathbb{N}$ , we have  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, for all  $n \in \mathbb{N}$ ,  $f_n$  is  $\mu$ -integrable,  $f$  is  $\mu$ -integrable, and we have in  $\mathbb{R}$

$$(1) \quad \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

The proof of Lebesgue's extended dominated convergence theorem goes as follows (this proof uses the arithmetic of  $\mathbb{R}$ ):

- $D \stackrel{\text{def.}}{=} \{f = \liminf_{n \rightarrow \infty} f_n\} \cap \{f = \limsup_{n \rightarrow \infty} f_n\} \cap \bigcap_{n \in \mathbb{N}} \{|f_n| \leq g\} \cap g^{-1}(\mathbb{R}_+) \subset X$  is first shown to be measurable with  $\mu(D^c) = 0$ ; thus  $f_n \mathbb{1}_D \stackrel{\mu \text{ a.e.}}{=} f_n$ ,  $f \mathbb{1}_D \stackrel{\mu \text{ a.e.}}{=} f$ , and  $g \mathbb{1}_D \stackrel{\mu \text{ a.e.}}{=} g$ ; moreover,  $g \mathbb{1}_D$  belongs to  $\mathcal{L}^1$ ;
- then, since  $\lim_{n \rightarrow \infty} f_n \mathbb{1}_D = f \mathbb{1}_D$  and  $|f_n \mathbb{1}_D| \leq g \mathbb{1}_D$ , Lebesgue's dominated convergence theorem provides  $f_n \mathbb{1}_D, f \mathbb{1}_D \in \mathcal{L}^1$ , and the equality

$$\int f \mathbb{1}_D d\mu = \lim_{n \rightarrow \infty} \int f_n \mathbb{1}_D d\mu;$$

- finally, the integrability of  $f_n$  and  $f$ , and identity (1) follow from the compatibility of the integral in  $\mathcal{M}$  with almost equality.

For instance, Lebesgue's extended dominated convergence theorem may be used to prove the Leibniz integral rule (differentiation under the integral sign), e.g. for the study of integrals function of their upper bound.

### 5.2 Sketch of the proof of Lebesgue's dominated convergence theorem

#### Lebesgue's dominated convergence theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Assume that the sequence is pointwise convergent towards  $f$ . Let  $g \in \mathcal{L}^1$ . Assume that for all  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then, for all  $n \in \mathbb{N}$ ,  $f_n \in \mathcal{L}^1$ ,  $f \in \mathcal{L}^1$ , the sequence is convergent towards  $f$  in  $\mathcal{L}^1$ , and we have in  $\mathbb{R}$

$$(2) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

The proof of Lebesgue's dominated convergence theorem goes as follows (this proof uses the arithmetic of  $\mathbb{R}$ ):

- monotonicity of integral in  $\mathcal{M}_+$  provides  $f_n, f \in \mathcal{L}^1$ ;
- the sequence  $g_n \stackrel{\text{def.}}{=} |f_n - f|$  is shown to be in  $\mathcal{L}^1 \cap \mathcal{M}_+$  with limit 0;
- the sequence  $2g - g_n$  is also shown to be in  $\mathcal{L}^1 \cap \mathcal{M}_+$ , with limit inferior  $2g$ ;
- then, Fatou's lemma, and linearity of the integral in  $\mathcal{L}^1$  provide the inequality

$$2 \int g d\mu \leq 2 \int g d\mu - \limsup_{n \rightarrow \infty} \int g_n d\mu;$$

thus, a nonnegativeness argument provides the nullity of the limit of the integral of  $g_n$ 's, i.e. the convergence of the  $f_n$ 's towards  $f$  in  $\mathcal{L}^1$ ;

- finally, identity (2) follows from nondecreasingness of the integral in  $\mathcal{L}^1$ , the squeeze theorem, and linearity of the limit.

Lebesgue's dominated convergence theorem is used in the present document to prove Lebesgue's extended dominated convergence theorem.

Lebesgue's dominated convergence theorem admits several variants: with lighter assumptions, or set in  $L^p$  for  $p \in [1, \infty)$ . For instance, it may be used to establish the Fourier inversion formula, or to study the Gibbs phenomenon.

### 5.3 Sketch of the proof of the Tonelli theorem

**Tonelli theorem.** *Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, the function defined for all  $x_i \in X_i$  by  $f_i(x_i) \stackrel{\text{def.}}{=} \int f d\mu_j$  belongs to  $\mathcal{M}_+(X_i, \Sigma_i)$ , and we have in  $\overline{\mathbb{R}}_+$*

$$(3) \quad \int f d(\mu_1 \otimes \mu_2) = \int f_i d\mu_i.$$

The proof of the Tonelli theorem goes as follows (this proof uses the arithmetic of  $\overline{\mathbb{R}}_+$  and the concepts of set algebra and monotone class; it follows the first three steps of Lebesgue scheme, see Section 4.1):

- the result is first established for indicator functions from properties of the measure (in particular the building of the tensor product measure that uses continuity of measures from below and from above);
- then, the result is extended to nonnegative simple functions by taking (positive) linear combination of indicator functions and applying (positive) linearity of the integral in  $\mathcal{M}_+$ ;
- and finally, the result is extended to nonnegative measurable functions by taking the supremum of adapted sequences and applying the Beppo Levi (monotone convergence) theorem.

The Tonelli theorem is used in the present document to establish identities in the cases of restriction, and of tensor product.

### 5.4 Sketch of the proof of Fatou's lemma

**Fatou's lemma.** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then, we have in  $\overline{\mathbb{R}}_+$*

$$(4) \quad \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

The proof of Fatou's lemma goes as follows (this proof uses the arithmetic of  $\overline{\mathbb{R}}_+$ ):

- the sequence  $(\inf_{p \in \mathbb{N}} f_{n+p})_{n \in \mathbb{N}}$  is first shown to be pointwise nondecreasing in  $\mathcal{M}_+$ ;
- then, the Beppo Levi (monotone convergence) theorem provides the equality

$$\int \liminf_{n \in \mathbb{N}} f_n d\mu = \lim_{n \in \mathbb{N}} \int \inf_{p \in \mathbb{N}} f_{n+p} d\mu;$$

- and monotonicity of integral, and infimum and limit inferior definitions provide the inequality

$$\lim_{n \in \mathbb{N}} \int \inf_{p \in \mathbb{N}} f_{n+p} d\mu \leq \liminf_{n \in \mathbb{N}} \int f_n d\mu.$$

Fatou's lemma is used in the present document to prove properties of the integral in  $\mathcal{M}_+$ : identity for pointwise convergent sequences, and Lebesgue's dominated convergence theorem.

The Beppo Levi (monotone convergence) theorem and Fatou's lemma can be established independently of one another, or each one can be obtained as a consequence of the other. Of course, those independent proofs do share a common ingredient to somehow obtain an upper bound for the integral of the limit by the limit of the integrals (see Section 5.5). We chose to establish the Beppo Levi (monotone convergence) theorem first because we needed it earlier to obtain additivity of the integral for nonnegative measurable functions.

## 5.5 Sketch of the proof of the Beppo Levi (monotone convergence) theorem

### Beppo Levi (monotone convergence) theorem.

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Assume that the sequence is pointwise nondecreasing. Then,  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have in  $\overline{\mathbb{R}}_+$

$$(5) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

The proof of the Beppo Levi (monotone convergence) theorem goes as follows (this proof uses the arithmetic of  $\overline{\mathbb{R}}_+$ , it follows the third step of Lebesgue scheme by taking the ‘‘supremum’’ of quantities involving simple functions, see Section 4.1):

- let  $f \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} f_n$ ; monotonicity and completeness arguments provide belonging to  $\mathcal{M}_+$  of  $f = \sup_{n \rightarrow \infty} f_n$ , and the ‘‘easy’’ inequality

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu;$$

- let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ ; using the nondecreasing sequence  $A_n \stackrel{\text{def.}}{=} \{a\varphi \leq f_n\}$  (for  $a \in (0, 1)$  and  $n \in \mathbb{N}$ ), continuity from below of  $\mu$ , linearity, monotonicity and continuity arguments, allows to show the inequality

$$\int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu;$$

- then, taking the supremum over  $\varphi$ 's provides the other inequality

$$\int f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

The Beppo Levi (monotone convergence) theorem is used in the present document to prove properties of the integral in  $\mathcal{M}_+$ : homogeneity at  $\infty$ , additivity,  $\sigma$ -additivity, identity for the integral of restriction, Fatou's lemma, and identity for the integral with the Dirac measure.

## 5.6 Sketch of the proof of Carathéodory's extension theorem

**Carathéodory's extension theorem.** Let  $X$  be a set. Let  $\mathcal{R}$  be a ring of subsets on  $X$  (i.e. closed under complement and union). Let  $\mu$  be a pre-measure defined on  $\mathcal{R}$  (i.e. null on the empty set, and  $\sigma$ -additive). Then,  $\mu$  can be extended into a complete measure on a  $\sigma$ -algebra containing  $\mathcal{R}$ . Moreover, this extension is unique when  $\mu$  is  $\sigma$ -finite.

The proof of Carathéodory's extension theorem goes as follows (this proof uses the arithmetic of  $\overline{\mathbb{R}}_+$  and the concepts of set algebra and monotone class; the existence part may also be known as the Carathéodory extension scheme, see Section 4.2):

- the function  $\mu^*$  defined on  $\mathcal{P}(X)$  by (with the convention  $\inf \emptyset = \infty$ )

$$\forall A \subset X, \quad \mu^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid (A_n)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_n \right\}$$

is first shown to be an outer measure on  $X$  (i.e. null on the empty set, nonnegative, monotone, and  $\sigma$ -subadditive) which restriction to  $\mathcal{R}$  is the pre-measure  $\mu$ ;

- then, the set of subsets

$$\Sigma \stackrel{\text{def.}}{=} \{E \subset X \mid \forall A \subset X, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$$

is shown to be a  $\sigma$ -algebra on  $X$  containing the ring  $\mathcal{R}$ , while the restriction  $\mu^*|_{\Sigma}$  is shown to be a complete measure on  $(X, \Sigma)$  (closedness of  $\Sigma$  under countable union comes from additivity of  $\mu^*$  on  $\Sigma$ );

- moreover, when  $\mu$  is  $\sigma$ -finite (with  $X = \bigsqcup_{n \in \mathbb{N}} X_n$  and  $\mu(X_n) < \infty$ ), two Carathéodory extensions  $\mu_1$  and  $\mu_2$  are first restricted to  $X_n$ , then the set  $\{\mu_1(A \cap X_n) = \mu_2(A \cap X_n)\}$  is shown to be equal to  $\Sigma$ , and finally uniqueness follows from  $\sigma$ -additivity of  $\mu_1$  and  $\mu_2$ .

In the present document, Carathéodory's extension theorem is stated (and proved) in the specific case of the building of the Lebesgue measure on  $\mathbb{R}$ .

## 5.7 Sketch of the proof of the Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem

Both theorems take advantage of the complementarity of being a  $\lambda$ -system and being a  $\pi$ -system (for the Dynkin  $\pi$ - $\lambda$  theorem), or being a monotone class and being a set algebra (for the monotone class theorem). Using the notation of Section 4.3 where  $\mathcal{U}_X^1$  represents the generated  $\lambda$ -system  $\Lambda_X$  (resp. the generated monotone class  $\mathcal{C}_X$ ), and  $\mathcal{U}_X^2$  represents the generated  $\pi$ -system  $\Pi_X$  (resp. the generated set algebra  $\mathcal{A}_X$ ), both theorems share the following abstract form:

### Dynkin $\pi$ - $\lambda$ theorem / monotone class theorem.

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $\mathcal{U}_X^2(G) = G$ . Then,  $\mathcal{U}_X^1(G) = \Sigma_X(G)$ .

Their common proof goes as follows (these proofs use the concepts of  $\pi$ -system and  $\lambda$ -system for the Dynkin  $\pi$ - $\lambda$  theorem, and of set algebra and monotone class for the monotone class theorem):

- first, prove that for all  $G'$ ,  $\mathcal{U}_X^1(G') = G'$  and  $\mathcal{U}_X^2(G') = G'$  implies  $\Sigma_X(G') = G'$ ;
- then, note that  $\mathcal{U}_X^1(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$  (i.e. subset system generation is idempotent);
- then, prove that  $\mathcal{U}_X^2(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$  (i.e. the property of  $G$  is transmitted to  $\mathcal{U}_X^1(G)$ );
- then, apply first result to  $G' \stackrel{\text{def.}}{=} \mathcal{U}_X^1(G)$  and obtain  $\Sigma_X(\mathcal{U}_X^1(G)) = \mathcal{U}_X^1(G)$ ;
- and finally, prove that  $\Sigma_X(\mathcal{U}_X^1(G)) = \Sigma_X(G)$ .

In both cases, the most technical part is the third point.

The Dynkin  $\pi$ - $\lambda$  theorem is used in the present document to extend the equality of two measures on a generator  $\pi$ -system to the whole  $\sigma$ -algebra, which is used to establish uniqueness of the Lebesgue measure on  $\mathbb{R}$ . The monotone class theorem is used in the present document to prove measurability of the measure of sections (in the case of finite measure spaces), and uniqueness of the tensor product measure (also in the finite case). All these three proofs follow the Dynkin  $\pi$ - $\lambda$  theorem / monotone class theorem scheme (see Section 4.3).

## 6 Detailed proofs

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

light gray is for remarks, light green for definitions,  
light blue for lemmas, and light red for theorems.

Definitions and results have a number and a name. Inside the bodies of proofs, pertinent statements are referenced using both their number and name. When appropriate hints are given about the application: either to specify arguments, or to provide justification or consequences; they are underlined. Some useful definitions and results were already stated in [17], which was devoted to the detailed proof of the Lax–Milgram theorem. They are numbered up to 206, and the statements in the present document are numbered starting from 207.

Furthermore, as in [17], the most basic results are supposed to be known and are not detailed further; they are displayed in **bold orange**. These include:

- Logic: tautologies from propositional calculus.
- Set theory:
  - definition and properties of inclusion, intersection, (disjoint) union, complement, set difference, Cartesian product, power set, cardinality, and indicator function (that takes values 0 and 1), such as De Morgan’s laws, monotonicity of intersection, distributivity of intersection over (disjoint) union and set difference, distributivity of the Cartesian product over union, compatibility of intersection with Cartesian product,  $\sigma$ -additivity of the cardinality (with  $\infty$  absorbing element for addition in  $\overline{\mathbb{N}}$ );
  - definition and properties of equivalence and order binary relations;
  - definition and properties of function, composition of functions, inverse image (compatibility with set operations), injective and bijective functions, restriction and extension;
  - countability of finite Cartesian products of countable sets ( $\mathbb{N}^2$ ,  $\mathbb{Q}$ ,  $\mathbb{Q} \times \mathbb{Q}_+^*$ ).
- Algebraic structures: results from group theory.
- Topology: definition and properties of continuous functions.
- Real analysis:
  - properties of the ordered and valued field  $\mathbb{R}$  such as the Archimedean property, density of rational numbers, completeness, sum of the first terms of a geometric series;
  - definition and properties of basic numeric analytic functions such as square root, power function, exponential function, natural logarithm function, and exponentiation;
  - properties of the ordered set  $\overline{\mathbb{R}}$ ;
  - properties of limits in  $\mathbb{R}/\overline{\mathbb{R}}$  (cluster point), compactness of closed and bounded intervals, compatibility of limit with arithmetic operations, the squeeze theorem.

This section is organized as follows. Some technical results in set theory are first collected in Section 6.1. Some results on intervals in totally ordered sets are then gathered in Section 6.2. Then, Section 6.3 is devoted to complements on topological spaces, with the second axiom of countability. Section 6.4 is dedicated to complements on real numbers and their topology, with for instance a fairly comprehensive description of the arithmetic on extended real numbers, and the second-countability of  $\mathbb{R}$  and  $\overline{\mathbb{R}}$ . And Section 6.5 gathers complements on algebraic structures such as seminormed vector space. Then, Section 6.6 is for measurability of subsets and functions, and Section 6.7 for the specific case of sets of numbers. Section 6.8 is for measure spaces, and

Section 6.9 is dedicated to the specific cases involving sets of numbers, including the construction of the Lebesgue measure on  $\mathbb{R}$  through Carathéodory's extension theorem. Finally, the integral of nonnegative functions is addressed in Section 6.10 with for instance the Beppo Levi (monotone convergence) theorem and the Tonelli theorem, and Section 6.11 is for changing sign functions with the seminormed vector space  $\mathcal{L}^1$  and Lebesgue's dominated convergence theorem.

## 6.1 Complements on set theory

**Lemma 207 (compatibility of partition with intersection).** *Let  $X$  be a set. Let  $I \subset \mathbb{N}$ . Let  $A, (X_i)_{i \in I} \subset X$ . Assume that  $X = \bigsqcup_{i \in I} X_i$ . Then, we have  $A = \bigsqcup_{i \in I} (A \cap X_i)$ .*

*Proof.* Direct consequence of **distributivity of intersection over disjoint union**.  $\square$

**Lemma 208 (technical inclusion for countable union).**

*Let  $X$  be a set. Let  $(A_n)_{n \in \mathbb{N}} \subset X$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Then, we have  $\bigcup_{n \in \mathbb{N}} A_{\varphi(n)} \subset \bigcup_{n \in \mathbb{N}} A_n$ .*

*Proof.* Let  $x \in \bigcup_{n \in \mathbb{N}} A_{\varphi(n)}$ . then, from **the definition of union**, there exists  $n \in \mathbb{N}$  such that  $x \in A_{\varphi(n)}$ . Thus, there exists  $m = \varphi(n) \in \mathbb{N}$  such that  $x \in A_m$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{m \in \mathbb{N}} A_m$ .

Therefore, we have  $\bigcup_{n \in \mathbb{N}} A_{\varphi(n)} \subset \bigcup_{n \in \mathbb{N}} A_n$ .  $\square$

**Lemma 209 (order is meaningless in countable union).** *Let  $X$  be a set. Let  $(A_n)_{n \in \mathbb{N}} \subset X$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Assume that  $\varphi$  is bijective. Then, we have  $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A_{\varphi(n)}$ .*

*Proof.* Direct consequence of Lemma 208 (**technical inclusion for countable union**, with  $(A_n)_{n \in \mathbb{N}}$  and  $\varphi$ , then  $(A_{\varphi(n)})_{n \in \mathbb{N}}$  and  $\varphi^{-1}$  which satisfies  $\varphi \circ \varphi^{-1} = \text{Id}_{\mathbb{N}}$ ).  $\square$

**Lemma 210 (definition of double countable union).**

*Let  $X$  be a set. For all  $n, m \in \mathbb{N}$ , let  $A_{n,m} \subset X$ . Let  $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}^2$ . Assume that  $\varphi$  and  $\psi$  are bijective. Then,  $\bigcup_{p \in \mathbb{N}} A_{\varphi(p)} = \bigcup_{p \in \mathbb{N}} A_{\psi(p)}$ . This union is denoted  $\bigcup_{(n,m) \in \mathbb{N}^2} A_{n,m}$ .*

*Proof.* Direct consequence of Lemma 209 (**order is meaningless in countable union**, with  $(A_{\varphi(p)})_{p \in \mathbb{N}}$  and  $\varphi^{-1} \circ \psi$ ).  $\square$

**Lemma 211 (double countable union).** *Let  $X$  be a set. For all  $n, m \in \mathbb{N}$ , let  $A_{n,m} \subset X$ . Then, we have  $\bigcup_{(n,m) \in \mathbb{N}^2} A_{n,m} = \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} A_{n,m} \right)$ .*

*Proof.* From **countability of  $\mathbb{N}^2$** , let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}^2$  be a bijection. Then, from Lemma 210 (**definition of double countable union**), we have  $\bigcup_{(n,m) \in \mathbb{N}^2} A_{n,m} = \bigcup_{p \in \mathbb{N}} A_{\varphi(p)}$  (and this does not depend on the choice for  $\varphi$ ).

Let  $x \in \bigcup_{p \in \mathbb{N}} A_{\varphi(p)}$ . Then, from **the definition of union**, there exists  $p_0 \in \mathbb{N}$  such that  $x \in A_{\varphi(p_0)}$ . Let  $(n_0, m_0) \stackrel{\text{def.}}{=} \varphi(p_0)$ . Thus, we have  $x \in A_{n_0, m_0}$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{m \in \mathbb{N}} A_{n_0, m} \subset \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} A_{n, m} \right)$ .

Conversely, let  $x \in \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} A_{n, m} \right)$ . Then, from **the definition of union**, there exists  $n_0 \in \mathbb{N}$  such that  $x \in \bigcup_{m \in \mathbb{N}} A_{n_0, m}$ , and there exists  $m_0 \in \mathbb{N}$  such that  $x \in A_{n_0, m_0}$ .

Let  $p_0 \stackrel{\text{def.}}{=} \varphi^{-1}(n_0, m_0)$ . Thus, we have  $x \in A_{\varphi(p_0)}$ . Hence, from **the definition of union**, we have  $x \in \bigcup_{p \in \mathbb{N}} A_{\varphi(p)}$ .

Therefore, we have  $\bigcup_{(n,m) \in \mathbb{N}^2} A_{n,m} = \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} A_{n,m} \right)$ .  $\square$

**Lemma 212 (partition of countable union).** *Let  $X$  be a set. Let  $(A_n)_{n \in \mathbb{N}} \subset X$ . Let  $B_0 \stackrel{\text{def.}}{=} A_0$ , and for all  $n \in \mathbb{N}$ , let  $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i \in [0..n]} B_i$ . Then, we have*

$$(6) \quad \forall m, n \in \mathbb{N}, m \neq n \implies B_m \cap B_n = \emptyset,$$

$$(7) \quad \forall n \in \mathbb{N}, \bigcup_{i \in [0..n]} A_i = \bigsqcup_{i \in [0..n]} B_i \quad \text{and} \quad \bigcup_{i \in \mathbb{N}} A_i = \bigsqcup_{i \in \mathbb{N}} B_i.$$

*Proof.* Let  $p \in \mathbb{N}$ . Then, from **the definition of set difference**, and **De Morgan's laws**,

$$B_{p+1} = A_{p+1} \cap \left( \bigcup_{i \in [0..p]} B_i \right)^c = A_{p+1} \cap \bigcap_{i \in [0..p]} B_i^c.$$

Let  $i \in [0..p]$ . Then, from **properties of set operations**,  $B_{p+1}$  and  $B_i$  are disjoint.

Let  $m, n \in \mathbb{N}$ . Assume that  $m \neq n$ , and let  $i \stackrel{\text{def.}}{=} \min(m, n)$  and  $p \stackrel{\text{def.}}{=} \max(m, n) - 1$ . Then, we have  $p \in \mathbb{N}$  and  $i \in [0..p]$ , and thus  $B_m \cap B_n = B_{p+1} \cap B_i = \emptyset$ .

For all  $n \in \mathbb{N}$ , let  $P(n)$  be the property:  $\bigcup_{i \in [0..n]} A_i = \uplus_{i \in [0..n]} B_i$ . **Induction:  $P(0)$ .** Trivial. **Induction:  $P(n)$  implies  $P(n+1)$ .** Let  $n \in \mathbb{N}$ . Assume that  $P(n)$  holds. Let  $B \stackrel{\text{def.}}{=} \bigcup_{i \in [0..n]} B_i$ . Then, from  $P(n)$ , and **properties of set operations (e.g., for all  $X$ , for all  $A, B \subset X$ ,  $A \cup B = A \uplus (B \cap A^c)$ )**, we have

$$\bigcup_{i \in [0..n+1]} A_i = B \cup A_{n+1} = B \uplus (A_{n+1} \cap B^c) = B \uplus B_{n+1} = \uplus_{i \in [0..n+1]} B_i.$$

Hence, for all  $n \in \mathbb{N}$ ,  $P(n)$  holds. Moreover, from **monotonicity of partial union**, and **the definition of the limit of monotone sequence of subsets**, we have  $\bigcup_{i \in \mathbb{N}} A_i = \uplus_{i \in \mathbb{N}} B_i$ .

Therefore, (6) and (7) hold.  $\square$

**Definition 213 (trace of subsets of parties).** Let  $X$  be a set. Let  $Y \subset X$ . Let  $i$  be the canonical injection from  $Y$  to  $X$ . Let  $G \subset \mathcal{P}(X)$ . The notation  $G \bar{\cap} Y$  denotes the set

$$(8) \quad G \bar{\cap} Y \stackrel{\text{def.}}{=} i^{-1}(G) = \{A \cap Y \mid A \in G\}.$$

**Definition 214 (product of subsets of parties).**

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $X_i$  be a set, and  $G_i \subset \mathcal{P}(X_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . The notations  $G_1 \bar{\times} G_2$  and  $\bar{\prod}_{i \in [1..m]} G_i$  respectively denote the sets

$$(9) \quad G_1 \bar{\times} G_2 \stackrel{\text{def.}}{=} \{A_1 \times A_2 \in \mathcal{P}(X_1 \times X_2) \mid A_1 \in G_1 \wedge A_2 \in G_2\}$$

$$(10) \quad \bar{\prod}_{i \in [1..m]} G_i \stackrel{\text{def.}}{=} \left\{ \prod_{i \in [1..m]} A_i \in \mathcal{P}(X) \mid \forall i \in [1..m], A_i \in G_i \right\}.$$

**Lemma 215 (restriction is masking).** Let  $X$  be a set. Let  $A \subset Y \subset X$ . Let  $f : Y \rightarrow \bar{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \bar{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $(\hat{f} \mathbf{1}_A)|_A = f|_A$  and  $(\hat{f} \mathbf{1}_A)|_{A^c} = 0$ .

*Proof.* Direct consequence of **the definition of the indicator function**, and **the definition of restriction of function**.  $\square$

## 6.2 Complements on order theory

*Remark 216.* In the following definition, the strict inequality “ $<$ ” naturally means “ $\leq$  and not equal” (equality is related to the set object).

We recall the notations  $\pm\infty$  to represent the extreme bounds of a totally ordered set; they may belong to the set, or not.



**Definition 217 (interval).** Let  $(X, \leq)$  be a totally ordered nonempty set. Let  $a, b \in X$ .

The subset  $\{x \in X \mid a < x < b\}$  is called the *open proper interval from a to b* (with excluded bounds  $a$  and  $b$ ); it is denoted  $(a, b)$ . The set of open proper intervals for all  $a, b \in X$  is denoted  $\mathcal{I}_X^{o,p}$ .

The subsets  $\{x \in X \mid x < b\}$  and  $\{x \in X \mid a < x\}$  are called *open left ray and open right ray* (with excluded bounds  $a$  and  $b$ ) (or *half-lines*); they are denoted  $(a, \infty]$  and  $[-\infty, b)$ . The set of open rays for all  $a, b \in X$  is denoted  $\mathcal{R}_X^o$ .

The notation  $[-\infty, \infty]$  may be used to represent the whole set  $X$ .

In all cases, square brackets “[, ]” are used to specify that bounds are included, and square-and-round brackets “[, )” are used to avoid specifying inclusion or exclusion of the bounds.

Let  $a, b \in X \cup \{\pm\infty\}$ . The subset  $[a, b]$  is either a proper interval, a ray, or the whole set; it is called *interval*. The set of open intervals for all  $a, b \in X \cup \{\pm\infty\}$  is denoted  $\mathcal{I}_X^o \stackrel{\text{def.}}{=} \mathcal{I}_X^{o,p} \cup \mathcal{R}_X^o \cup \{X\}$ .

*Remark 218.* Note that if  $X$  contains at least two elements,  $\mathcal{I}_X^{o,p}$ ,  $\mathcal{R}_X^o$  and  $\{X\}$  are pairwise disjoint. In particular,  $(a, \infty)$  is an open ray when  $\infty \notin X$ , and an open proper interval when  $\infty \in X$ . More generally, open rays are never open proper intervals. For instance, when  $\infty$  belongs to the set,  $(a, \infty]$  is actually  $(a, \infty)$  (an open subset of  $\overline{\mathbb{R}}$ ), which is distinct from the open proper interval  $(a, \infty)$ .

**Lemma 219 (empty open interval).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Assume that  $X$  is dense-in-itself:

$$(11) \quad \forall x, y \in X, \quad x < y \implies \exists z \in X, \quad x < z < y.$$

Let  $a, b \in X \cup \{\pm\infty\}$ . Then, the open interval  $(a, b)$  is empty iff  $b \leq a$ .

*Proof.* Then, from Definition 217 (*interval*), (11), **transitivity of order (which provides the other implication)**, and **the definition of strict inequality**, we have

$$\begin{aligned} (a, b) = \emptyset &\iff \forall x \in X, \neg(a < x < b) \\ &\iff \neg(\exists x \in X, a < x < b) \iff \neg(a < b) \iff b \leq a. \end{aligned}$$

□

*Remark 220.* Of course, the previous statement is wrong in the presence of isolated points. For instance,  $(0, 1)$  is empty in the discrete sets  $\mathbb{N}$  and  $\mathbb{Z}$ .

*Remark 221.* In the following statement, the left and right square-and-round brackets must remain the same on the sides of each interval. For instance “[1” denotes either “[” or “(”, but it remains identical in the three intervals in (12).

**Lemma 222 (intervals are closed under finite intersection).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Then, the intersection of two proper intervals is a proper interval, the intersection of two rays is either a ray (if they both point towards the same direction), or a proper interval, and the intersection of a proper interval and a ray is a proper interval. In particular, for all  $a, b, c, d \in X \cup \{\pm\infty\}$ , we have (see Remark above)

$$(12) \quad [1a, b]_2 \cap [1c, d]_2 = [1\max(a, c), \min(b, d)]_2.$$

Hence, the closure of  $\mathcal{R}_X^o \cup \{X\}$  under finite intersection is  $\mathcal{I}_X^o$ , and  $\mathcal{I}_X^{o,p}$  and  $\mathcal{I}_X^o$  are closed under finite intersection.

*Proof.* Direct consequence of Definition 217 (*interval*), **totally ordered set properties of X**, and **induction on the number of operands of the finite intersection**. □



**Lemma 223 (empty intersection of open intervals).**

Let  $(X, \leq)$  be a totally ordered nonempty set. Assume that  $X$  is dense-in-itself:

$$(13) \quad \forall x, y \in X, \quad x < y \implies \exists z \in X, \quad x < z < y.$$

Let  $a, b, c, d \in X \cup \{\pm\infty\}$ . Then,  $(a, b) \cap (c, d)$  is empty iff  $b \leq a$ ,  $d \leq c$ ,  $d \leq a$  or  $b \leq c$ .

*Proof.* Direct consequence of Lemma 222 (intervals are closed under finite intersection), Lemma 219 (empty open interval), and **totally ordered set properties of  $X$** .  $\square$

### 6.3 Complements on general topology

*Remark 224.* In [17], we have only covered topology for metric spaces in which subsets are open when they contain a ball centered in each of their points. In the present document, we deal with the general case of topological spaces for which the collection of open subsets is given, e.g. via a topological basis.

**Definition 225 (topological space).** Let  $X$  be a set. A subset  $\mathcal{T}$  of  $\mathcal{P}(X)$  is called *topology* of  $X$  iff  $\emptyset, X \in \mathcal{T}$ , and  $\mathcal{T}$  is closed under (infinite) union and finite intersection.

If so,  $(X, \mathcal{T})$  (or simply  $X$ ) is called *topological space*, elements of  $\mathcal{T}$  are called *open subsets* of  $X$ , and the complement of elements of  $\mathcal{T}$  are called *closed subsets* of  $X$ .

**Lemma 226 (intersection of topologies).**

Let  $X$  and  $I$  be sets. Let  $(\mathcal{T}_i)_{i \in I}$  be topologies on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{T}_i$  is a topology on  $X$ .

*Proof.* Direct consequence of Definition 225 (topological space), and **the definition and properties of intersection and union of subsets**.  $\square$

**Definition 227 (generated topology).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The *topology on  $X$  generated by  $G$*  is the intersection of all topologies on  $X$  containing  $G$ ; it is denoted  $\mathcal{T}_X(G)$ . The generator  $G$  is also called *subbase of the topology*.

**Lemma 228 (generated topology is minimum).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{T}_X(G)$  is the smallest topology on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 227 (generated topology), Lemma 226 (intersection of topologies), and **properties of the intersection**.  $\square$

**Lemma 229 (equivalent definition of generated topology).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Let  $O \subset X$ . Then,  $O \in \mathcal{T}_X(G)$  iff  $O$  is the union of finite intersections of elements of  $G \cup \{X\}$ .

*Proof.* Direct consequence of Definition 225 (topological space), **associativity and commutativity of union, distributivity of intersection and union (both ways)** (thus, the set of unions of finite intersections of elements of  $G$  is a topology of  $X$  containing  $G$ , and contained in all topologies of  $X$  containing  $G$ ), and Lemma 228 (generated topology is minimum).  $\square$

**Definition 230 (topological basis).**

Let  $(X, \mathcal{T})$  be a topological space. Let  $I$  be a set. A set  $\{B_i \in \mathcal{T} \mid i \in I\}$  is called *topological basis* of  $(X, \mathcal{T})$  iff for all  $O \in \mathcal{T}$ , there exists  $J \subset I$  such that  $O = \bigcup_{j \in J} B_j$ .

**Lemma 231 (augmented topological basis).**

Let  $(X, \mathcal{T})$  be a topological space. Let  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Let  $O \in \mathcal{T}$ . Then,  $\mathcal{B} \cup \{O\}$  is a topological basis of  $(X, \mathcal{T})$ .

*Proof.* Direct consequence of Definition 230 (topological basis, with  $O$  open).  $\square$

**Definition 232 (order topology).**

Let  $(X, \leq)$  be a totally ordered set. The topology  $\mathcal{T}_X(\mathcal{R}_X^o)$  is called *order topology on  $X$* .

*Remark 233.* See Definition 217 (*interval*) for the definition of  $\mathcal{R}_X^o$ . Note that the order topology is the standard topology on the totally ordered sets of numbers  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ .

**Lemma 234 (topological basis of order topology).**

Let  $(X, \leq)$  be a totally ordered set. Then,  $\mathcal{I}_X^o$  is a topological basis for the order topology on  $X$ .

*Proof.* Direct consequence of Definition 232 (*order topology*), Lemma 229 (*equivalent definition of generated topology*), Lemma 222 (*intervals are closed under finite intersection*), and Definition 230 (*topological basis*).  $\square$

*Remark 235.* See Definition 217 (*interval*) for the definition of  $\mathcal{I}_X^o$ .

**Lemma 236 (trace topology on subset).**

Let  $(X, \mathcal{T})$  be a topological space, and  $\mathcal{B}$  be a topological basis of  $(X, \mathcal{T})$ . Let  $Y \subset X$ .

Then,  $\mathcal{T}_Y \stackrel{\text{def.}}{=} \mathcal{T} \cap Y$  is a topology of  $Y$ , and  $\mathcal{B} \cap Y$  is a topological basis of  $(Y, \mathcal{T}_Y)$ .

$\mathcal{T}_Y$  is called *trace topology*, and  $(Y, \mathcal{T}_Y)$  is said *topological subspace* of  $(X, \mathcal{T})$ .

*Proof.* Direct consequence of Definition 225 (*topological space*), Definition 230 (*topological basis*), Definition 213 (*trace of subsets of parties*), **distributivity of intersection over union**, and **commutativity of intersection**.  $\square$

**Lemma 237 (box topology on Cartesian product).**

Let  $I$  be a set. For all  $i \in I$ ,

let  $(X_i, \mathcal{T}_i)$  be a topological space and  $\mathcal{B}_i$  be a topological basis of  $(X_i, \mathcal{T}_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in I} X_i$ .

Then,  $\mathcal{T} \stackrel{\text{def.}}{=} \prod_{i \in I} \mathcal{T}_i$  is a topology of  $X$ , and  $\prod_{i \in I} \mathcal{B}_i$  is a topological basis of  $(X, \mathcal{T})$ .

$\mathcal{T}$  is called the *box topology* of  $X$  (induced by the  $\mathcal{T}_i$ 's).

*Proof.* Direct consequence of Definition 225 (*topological space*), Definition 230 (*topological basis*), Definition 214 (*product of subsets of parties*), and **distributivity of the Cartesian product over union**.  $\square$

### 6.3.1 Second axiom of countability

**Definition 238 (second-countability).** A topological space  $(X, \mathcal{T})$  is said *second-countable*, or *completely separable*, iff it admits a countable topological basis,

*Remark 239.* This corresponds to the existence of  $I \subset \mathbb{N}$  in Definition 230 (*topological basis*).

**Lemma 240 (complete countable topological basis).**

Let  $(X, \mathcal{T})$  be a second-countable topological space. Let  $\mathcal{B}$  be a countable topological basis of  $(X, \mathcal{T})$ .

Let  $O \in \mathcal{T}$ . Then,  $\mathcal{B} \cup \{O\}$  is a countable topological basis of  $(X, \mathcal{T})$ .

*Proof.* Direct consequence of Definition 238 (*second-countability*), and Lemma 231 (*augmented topological basis*).  $\square$

**Lemma 241 (compatibility of second-countability with Cartesian product).**

Let  $I$  be a set. For all  $i \in I$ , let  $(X_i, \mathcal{T}_i)$  be a second-countable topological space, and let  $\mathcal{B}_i$  be a countable topological basis of  $(X_i, \mathcal{T}_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in I} X_i$ . Assume that  $X$  is equipped with the box topology  $\mathcal{T} \stackrel{\text{def.}}{=} \prod_{i \in I} \mathcal{T}_i$ . Then,  $\prod_{i \in I} \mathcal{B}_i$  is a countable topological basis of  $(X, \mathcal{T})$ . Hence,  $(X, \mathcal{T})$  is second-countable.

*Proof.* Direct consequence of Lemma 237 (*box topology on Cartesian product*), Definition 214 (*product of subsets of parties*), and **compatibility of finite Cartesian product with countability**.  $\square$

**Lemma 242 (complete countable topological basis of product space).** *Let  $I$  be a set. For all  $i \in I$ , let  $(X_i, \mathcal{T}_i)$  be a second-countable topological space, let  $\mathcal{B}_i$  be a countable topological basis of it, and let  $O_i \in \mathcal{T}_i$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in I} X_i$ . Assume that  $X$  is equipped with the box topology  $\mathcal{T} \stackrel{\text{def.}}{=} \prod_{i \in I} \mathcal{T}_i$ . Then,  $\overline{\prod_{i \in I} \mathcal{B}_i} \cup \{O_i\}$  is a countable topological basis of  $(X, \mathcal{T})$ .*

*Proof.* Direct consequence of Definition 214 (*product of subsets of parties*), Lemma 240 (*complete countable topological basis*), and Lemma 237 (*box topology on Cartesian product*).  $\square$

### 6.3.2 Complements on metric space

**Definition 243 (pseudometric).**

Let  $X$  be a nonempty set. A function  $d : X \times X \rightarrow \mathbb{R}$  is called *pseudometric over  $X$*  iff it is nonnegative, symmetric, it is zero on the diagonal, and it satisfies the triangle inequality:

- (14)  $\forall x, y \in X, \quad d(x, y) \geq 0,$
- (15)  $\forall x, y \in X, \quad d(y, x) = d(x, y),$
- (16)  $\forall x \in X, \quad d(x, x) = 0,$
- (17)  $\forall x, y, z \in X, \quad d(x, z) \leq d(x, y) + d(y, z).$

If so,  $(X, d)$  (or simply  $X$ ) is called *pseudometric space*.

*Remark 244.* A pseudometric becomes a metric when it is also definite. Hence, most results from [17] on metric spaces are still valid on pseudometric spaces.

**Lemma 245 (equivalent definition of convergent sequence).**

Let  $(X, d)$  be a metric space. Let  $(x_n)_{n \in \mathbb{N}}, l \in X$ . Then,  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $l$  iff

$$(18) \quad \forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \quad d(x_n, l) \leq \frac{1}{k+1}.$$

*Proof.* “Left” implies “right”.

Direct consequence of Definition 27 (*convergent sequence*, with  $\varepsilon \stackrel{\text{def.}}{=} \frac{1}{k+1}$ ).

“Right” implies “left”. Assume that (18) holds. Let  $\varepsilon > 0$ . Then, from **the Archimedean property of  $\mathbb{R}$** , and **ordered field properties of  $\mathbb{R}$** , let  $k \in \mathbb{N}$  such that  $k \geq \frac{1}{\varepsilon} - 1$ , i.e.  $\frac{1}{k+1} \leq \varepsilon$ . Thus, from assumption, there exists  $N \in \mathbb{N}$  such that for all  $n \in [N.. \infty)$ , we have  $d(x_n, l) \leq \frac{1}{k+1} \leq \varepsilon$ . Hence, from Definition 27 (*convergent sequence*), the sequence  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $l$ .

Therefore, we have the equivalence.  $\square$

**Lemma 246 (convergent subsequence of Cauchy sequence).**

Let  $(X, d)$  be a metric space. Let  $(x_n)_{n \in \mathbb{N}} \in X$ . Assume that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Let  $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}$ . Assume that  $(n_k)_{k \in \mathbb{N}}$  is nondecreasing and that the subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  is convergent. Then,  $(x_n)_{n \in \mathbb{N}}$  is convergent with the same limit.

*Proof.* From Lemma 29 (*limit is unique*), let  $x \in X$  be the limit of the subsequence. Let  $\varepsilon > 0$ . Then, from Definition 35 (*Cauchy sequence*), let  $N \in \mathbb{N}$  such that for all  $p, q \geq N$ , we have  $d(x_p, x_q) \leq \frac{\varepsilon}{2}$ . Moreover, as  $(n_k)_{k \in \mathbb{N}}$  is increasing, let  $K' \in \mathbb{N}$  such that for all  $k \geq K'$ , we have  $n_k > N$ , and from Definition 27 (*convergent sequence*), let  $K'' \in \mathbb{N}$  such that for all  $k \geq K''$ ,

we have  $d(x_{n_k}, x) \leq \frac{\varepsilon}{2}$ . Let  $K \stackrel{\text{def.}}{=} \max(K', K'')$ . Then, from Definition 17 (*distance*, *triangle inequality*), we have for all  $n \geq N$ ,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from Definition 27 (*convergent sequence*),  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $x$ .  $\square$

**Definition 247 (cluster point).** Let  $(X, d)$  be a metric space. Let  $(x_n)_{n \in \mathbb{N}} \in X$ . A *cluster point* of the sequence is the limit  $x$  of any convergent subsequence of  $(x_n)_{n \in \mathbb{N}}$ :

$$(19) \quad \forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N, \quad d(x_n, x) \leq \varepsilon.$$

## 6.4 Complements on real numbers

### 6.4.1 Real numbers

**Lemma 248 (finite cover of compact interval).**

Let  $a, b, (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{R}$ . Assume that  $a \leq b$  and  $[a, b] \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ . Then, there exists  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b] \subset \bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a$ ,  $b < b_{i_q}$ , and for all  $p \in [0..q-1]$ ,  $a_{i_{p+1}} < b_{i_p}$ .

*Proof.* From **the definition of compactness**, and **compactness of  $[a, b]$** , there exists  $n \in \mathbb{N}$  such that  $[a, b] \subset \bigcup_{j \in [0..n]} (a_j, b_j)$ .

Let  $J \subset [0..n]$  and  $x \in \mathbb{R}$ . Assume that  $[x, b] \subset \bigcup_{j \in J} (a_j, b_j)$  and  $x \leq b$ .

(1). **Next index:**  $\exists i \stackrel{\text{def.}}{=} i_J(x) \in J$ ,  $a_i < x < b_i$ .

Direct consequence of **the definition of union (with  $x \in [x, b]$ )**.

(2). **Next cover:**  $[b_i, b] \subset \bigcup_{j \in J \setminus \{i\}} (a_j, b_j)$ .

**Case  $b < b_i$ .** Trivial. **Case  $b_i \leq b$ .** Then, from (1), we have  $b_i \in (x, b]$ . Let  $y \in [b_i, b] \subset [x, b]$ . Then, from **the definition of union**, there exists  $j \in J$  such that  $a_j < y < b_j$ . Hence, from **transitivity of order in  $\mathbb{R}$** , we have  $b_i < b_j$ , i.e.  $j \neq i$ , and  $[b_i, b] \subset \bigcup_{j \in J \setminus \{i\}} (a_j, b_j)$ .

Let  $i_{-1} \stackrel{\text{def.}}{=} -1$ ,  $J_{-1} \stackrel{\text{def.}}{=} [0..n]$ , and  $b_{i_{-1}} = b_{-1} \stackrel{\text{def.}}{=} a$ . Let  $(i_p)_{p \in [0..q]}$  be the sequence of integers computed by the following algorithm:

```

p  $\stackrel{\text{def.}}{=} -1$ ;
repeat
  |  $i_{p+1} \stackrel{\text{def.}}{=} i_{J_p}(b_{i_p})$ ;
  |  $J_{p+1} \stackrel{\text{def.}}{=} J_p \setminus \{i_{p+1}\}$ ;
  |  $p \stackrel{\text{def.}}{=} p + 1$ ;
until  $b < b_{i_p}$ ;
return  $(i_0, \dots, i_p)$ ;

```

For all  $p \in \{-1\} \cup \mathbb{N}$ , let  $I_p \stackrel{\text{def.}}{=} \{i_0, \dots, i_p\}$  (with  $I_{-1} \stackrel{\text{def.}}{=} \emptyset$ ), and  $P(p)$  be the property:

$$\begin{aligned} \text{card}(I_p) = p + 1 \quad \wedge \quad [0..n] = I_p \uplus J_p \quad \wedge \quad (\forall m \in [0..p], a_{i_m} < b_{i_{m-1}}) \quad \wedge \\ [a, b_{i_p}] \subset \bigcup_{j \in I_p} (a_j, b_j) \quad \wedge \quad [b_{i_p}, b] \subset \bigcup_{j \in J_p} (a_j, b_j). \end{aligned}$$

Let us show that there exists  $q \in [0..n]$  such that  $P(q) \wedge b < b_{i_q}$  holds.

(3). **Initialization:**  $P(-1) \wedge b_{i_{-1}} \leq b$ . Trivial.

(4). **Iterations:**  $\forall p \in \{-1\} \cup \mathbb{N}$ ,  $P(p) \wedge b_{i_p} \leq b$  implies  $P(p+1)$ .

Let  $p \in \{-1\} \cup \mathbb{N}$ . Assume that  $P(p) \wedge b_{i_p} \leq b$  holds.

Let  $J \stackrel{\text{def.}}{=} J_p$  and  $x \stackrel{\text{def.}}{=} b_{i_p}$ . Then, we have  $[x, b] \subset \bigcup_{j \in J} (a_j, b_j)$  and  $x \leq b$ . Thus, from (1), there exists  $i_{p+1} = i_J(x) \in J = J_p$  (i.e.  $i_{p+1} \notin I_p$ ), such that  $a_{i_{p+1}} < b_{i_p} < b_{i_{p+1}}$ . Moreover, from (2), we have  $[b_{i_{p+1}}, b] \subset \bigcup_{j \in J_{p+1}} (a_j, b_j)$ . Then, we have

$$\begin{aligned} \text{card}(I_{p+1}) &= \text{card}(I_p) + 1 = p + 2, \\ [0..n] &= I_p \uplus \{i_{p+1}\} \uplus (J_p \setminus \{i_{p+1}\}) = I_{p+1} \uplus J_{p+1}. \\ a_{i_{p+1}} &< b_{i_p}, \\ [a, b_{i_{p+1}}] &= [a, b_{i_p}] \uplus [b_{i_p}, b_{i_{p+1}}] \subset \bigcup_{j \in I_p} (a_j, b_j) \cup (a_{i_{p+1}}, b_{i_{p+1}}) = \bigcup_{j \in I_{p+1}} (a_j, b_j), \\ [b_{i_{p+1}}, b] &\subset \bigcup_{j \in J_{p+1}} (a_j, b_j). \end{aligned}$$

Hence,  $P(p+1)$  holds.

(5). **Termination:**  $\exists q \in [0..n]$ ,  $P(q) \wedge b < b_{i_q}$ . Assume that  $P(p) \wedge b_{i_p} \leq b$  holds for all  $p$  in  $[0..n]$ . Then, from (4), we have  $P(n+1)$ . Thus, from **additivity of the cardinality**, we have

$$n+1 = \text{card}([0..n]) = \text{card}(I_{n+1}) + \text{card}(J_{p+1}) \geq \text{card}(I_{n+1}) = n+2.$$

Which is impossible. Hence, there exists  $q \in [0..n]$  such that  $P(q) \wedge b < b_{i_q}$  holds.

Moreover, we have  $[a, b] \subset [a, b_{i_q}] \subset \bigcup_{j \in I_q} (a_j, b_j)$ .

Therefore, there exists  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b]$  is included in  $\bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a$ ,  $b < b_{i_q}$ , and for all  $p \in [0..q-1]$ ,  $a_{i_{p+1}} < b_{i_p}$ .  $\square$

**Definition 249 (Hölder conjugates in  $\mathbb{R}$ ).**

Real numbers  $p, q \in (1, \infty)$  are said *Hölder conjugates in  $\mathbb{R}$*  iff  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Lemma 250 (2 is self-Hölder conjugate in  $\mathbb{R}$ ).**

The real number 2 is Hölder conjugate in  $\mathbb{R}$  with itself.

*Proof.* Direct consequence of Definition 249 (*Hölder conjugates in  $\mathbb{R}$* ), and since  $\frac{1}{2} + \frac{1}{2} = 1$ .  $\square$

**Lemma 251 (Young's inequality for products in  $\mathbb{R}$ ).** Let  $p, q \in (1, \infty)$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $a, b \in \mathbb{R}_+$ . Then, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .

*Proof.* **Case  $ab = 0$ .** Then, from **nonnegativeness of exponentiation**, **closedness of multiplicative inverse in  $\mathbb{R}_+^*$** , and of the **multiplication and addition in  $\mathbb{R}_+$** , The right-hand side is nonnegative. hence, the inequality holds.

**Case  $ab > 0$ .** Then, from **the zero-product property in  $\mathbb{R}_+$  (contrapositive)**, we have  $a, b > 0$ . Thus, from **algebraic properties of the natural logarithm function**, Definition 249 (*Hölder conjugates in  $\mathbb{R}$* ), and **concavity of the natural logarithm function (with  $\frac{1}{p} + \frac{1}{q} = 1$ )**, we have

$$\ln(ab) = \ln a + \ln b = \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) \leq \ln \left( \frac{a^p}{p} + \frac{b^q}{q} \right).$$

Hence, from **monotonicity of the exponential function**, and since **the exponential and logarithm functions in  $\mathbb{R}$  are each other inverse**, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .

Therefore, the inequality always holds.  $\square$

**Lemma 252 (Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$ ).**

Let  $a, b \in \mathbb{R}_+$ . Let  $\varepsilon > 0$ . Then, we have  $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ .

*Proof.* Direct consequence of Lemma 250 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ), Lemma 251 (*Young's inequality for products in  $\mathbb{R}$ , with  $p = q \stackrel{\text{def.}}{=} 2$  and  $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}b \in \mathbb{R}_+$* ), and **properties of square root and multiplicative inverse in  $\mathbb{R}_+^*$ , and of multiplication in  $\mathbb{R}_+$** .  $\square$

*Remark 253.* Note that a similar result also holds in the general case of Hölder conjugate numbers  $p, q \in (1, \infty)$ : for all  $a, b \in \mathbb{R}_+$ , for all  $\varepsilon > 0$ , we have  $ab \leq \frac{a^p}{p\varepsilon^{\frac{p}{q}}} + \frac{\varepsilon^{\frac{p}{q}} b^q}{q}$ .

#### 6.4.2 Extended real numbers

**Definition 254** (*extended real numbers,  $\overline{\mathbb{R}}$* ). The set of *extended real numbers* is  $\overline{\mathbb{R}} \stackrel{\text{def.}}{=} \mathbb{R} \uplus \{-\infty, \infty\}$ , and the order in  $\mathbb{R}$  is extended to  $\overline{\mathbb{R}}$  with the following rule:

$$(20) \quad 1. \quad \forall a \in \mathbb{R}, \quad -\infty < a < \infty.$$

**Lemma 255** (*order in  $\overline{\mathbb{R}}$  is total*).  $(\overline{\mathbb{R}}, \leq)$  is a totally ordered set.

*Proof.* Direct consequence of **the definition of total order**, Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ), and **totality of order in  $\mathbb{R}$** .  $\square$

*Remark 256.* The goal of this section is to clarify some properties of basic operations such as addition, multiplication, and exponentiation, when extended to  $\overline{\mathbb{R}}$ .

*Remark 257.* Note that, as with regular real numbers, all results on operations on extended real numbers can be lifted into similar results on functions taking their values in  $\overline{\mathbb{R}}$ .

**Definition 258** (*addition in  $\overline{\mathbb{R}}$* ).

Addition and subtraction in  $\mathbb{R}$  are extended to  $\overline{\mathbb{R}}$  with the following rules:

$$(21) \quad \left\{ \begin{array}{l} 1. \quad \forall a > -\infty, \quad a + \infty = \infty + a \stackrel{\text{def.}}{=} \infty, \\ 2. \quad \forall a < \infty, \quad a + (-\infty) = -\infty + a \stackrel{\text{def.}}{=} -\infty, \\ 3. \quad \infty + (-\infty) \text{ and } -\infty + \infty \text{ are undefined,} \\ 4. \quad -(\pm\infty) = \mp\infty, \\ 5. \quad \forall a, b \in \overline{\mathbb{R}}, \quad a + (-b) \text{ defined} \implies a - b \stackrel{\text{def.}}{=} a + (-b). \end{array} \right.$$

**Lemma 259** (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ).

Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $a + 0 = 0 + a = a$ .

*Proof.* Direct consequence of **abelian group properties of  $(\mathbb{R}, +)$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , new rules 1, and 2 are compatible with the property*).  $\square$

**Lemma 260** (*addition in  $\overline{\mathbb{R}}$  is associative when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a + (b + c)$  and  $(a + b) + c$  are either equal or both undefined.

*Proof.* Direct consequence of **associativity of addition in  $\mathbb{R}$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , new rules 1, 2, and 3 are compatible with associativity*).  $\square$

**Lemma 261** (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*).

Let  $a, b \in \overline{\mathbb{R}}$ . Then,  $a + b$  and  $b + a$  are either equal or both undefined.

*Proof.* Direct consequence of **commutativity of addition in  $\mathbb{R}$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , new rules 1, 2, and 3 are compatible with commutativity*).  $\square$

**Lemma 262** (*infinity-sum property in  $\overline{\mathbb{R}}$* ).

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(22) \quad a + b = \infty \iff (a = \infty \wedge b > -\infty) \vee (a > -\infty \wedge b = \infty)$$

$$(23) \quad a + b = -\infty \iff (a = -\infty \wedge b < \infty) \vee (a < \infty \wedge b = -\infty).$$

Moreover, if we assume that the sum  $a + b$  is well-defined, then we have

$$(24) \quad a + b = \pm\infty \iff a = \pm\infty \vee b = \pm\infty.$$

*Proof.* Direct consequence of Definition 258 (*addition in  $\overline{\mathbb{R}}$* , rules 1, 2 and 3).  $\square$

**Lemma 263** (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*).

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $a \leq b$  iff  $-b \leq -a$ .

*Proof.* Direct consequence of Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ), **monotonicity of additive inverse in  $\mathbb{R}$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$* , rule 4).  $\square$

**Definition 264** (*multiplication in  $\overline{\mathbb{R}}$* ).

Multiplication and division by nonzero in  $\mathbb{R}$  are extended to  $\overline{\mathbb{R}}$  with the following rules:

$$(25) \quad \left\{ \begin{array}{l} 1. \quad \forall a > 0, \quad a \times (\pm\infty) = \pm\infty \times a \stackrel{\text{def.}}{=} \pm\infty, \\ 2. \quad \forall a < 0, \quad a \times (\pm\infty) = \pm\infty \times a \stackrel{\text{def.}}{=} \mp\infty, \\ 3. \quad 0 \times (\pm\infty) \text{ and } \pm\infty \times 0 \text{ are undefined,} \\ 4. \quad \frac{1}{0} \text{ is undefined, and } \frac{1}{\pm\infty} \stackrel{\text{def.}}{=} 0, \\ 5. \quad \forall a, b \in \overline{\mathbb{R}}, \quad \frac{1}{b} \text{ and } a \times \frac{1}{b} \text{ defined} \implies \frac{a}{b} \stackrel{\text{def.}}{=} a \times \frac{1}{b}. \end{array} \right.$$

*Remark 265.* The last rule in Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ) implies that  $\frac{\infty}{\pm\infty}$ ,  $\frac{\mp\infty}{\pm\infty}$ , and  $\frac{a}{0}$  (for all  $a \in \overline{\mathbb{R}}$ ) are undefined. Note that rule 3 is modified in the context of measure theory in Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 266** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a \times (b \times c)$  and  $(a \times b) \times c$  are either equal or both undefined.

*Proof.* Direct consequence of **associativity of multiplication in  $\mathbb{R}$** , and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with associativity).  $\square$

**Lemma 267** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*).

Let  $a, b \in \overline{\mathbb{R}}$ . Then,  $a \times b$  and  $b \times a$  are either equal or both undefined.

*Proof.* Direct consequence of **commutativity of multiplication in  $\mathbb{R}$** , and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* , new rules 1, 2, and 3 are compatible with commutativity).  $\square$

**Lemma 268** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $a \times (b + c)$  and  $(a \times b) + (a \times c)$  are either equal or both undefined.

*Proof.* Direct consequence of **left distributivity of multiplication over addition in  $\mathbb{R}$** , Definition 258 (*addition in  $\overline{\mathbb{R}}$* ), and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ) (new rules are compatible with left distributivity of multiplication over addition).  $\square$

**Lemma 269** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*).

Let  $a, b, c \in \overline{\mathbb{R}}$ . Then,  $(a + b) \times c$  and  $(a \times c) + (b \times c)$  are either equal or both undefined.



*Proof.* Direct consequence of Lemma 267 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined, used twice*), and Lemma 268 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).  $\square$

**Lemma 270 (zero-product property in  $\overline{\mathbb{R}}$ ).** *Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have*

(26) 
$$ab = 0 \iff ab \text{ is defined} \wedge (a = 0 \vee b = 0).$$

*Proof.* Direct consequence of **the zero-product property in  $\mathbb{R}$** , and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rule 3*).  $\square$

**Lemma 271 (infinity-product property in  $\overline{\mathbb{R}}$ ).** *Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have*

(27) 
$$ab = \infty \iff (a = \infty \wedge b > 0) \vee (a = -\infty \wedge b < 0) \vee$$

$(a > 0 \wedge b = \infty) \vee (a < 0 \wedge b = -\infty),$

(28) 
$$ab = -\infty \iff (a = -\infty \wedge b > 0) \vee (a = \infty \wedge b < 0) \vee$$

$(a > 0 \wedge b = -\infty) \vee (a < 0 \wedge b = \infty).$

*Proof.* **“Left” implies “right”.** Assume that  $ab = \pm\infty$ . Then, from Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have  $a = \pm\infty$  or  $b = \pm\infty$ . Assume that the other operand is zero. Then, from Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rule 3*), the product  $ab$  is undefined. Which is impossible.

Assume that  $ab = \infty$ . Then, from Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have either  $a = \infty$  and  $b > 0$ ,  $a = -\infty$  and  $b < 0$ ,  $b = \infty$  and  $a > 0$ , or  $b = -\infty$  and  $a < 0$ .

Similarly, assume now that  $ab = -\infty$ . Then, again from Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*), we have either  $a = -\infty$  and  $b > 0$ ,  $a = \infty$  and  $b < 0$ ,  $b = -\infty$  and  $a > 0$ , or  $b = \infty$  and  $a < 0$ .

**“Right” implies “left”.** Direct consequence of Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , rules 1 and 2*).

Therefore, we have the equivalences.  $\square$

**Lemma 272 (finite-product property in  $\overline{\mathbb{R}}$ ).** *Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have*

(29) 
$$ab \text{ is defined} \wedge ab \in \mathbb{R} \iff a, b \in \mathbb{R}.$$

*Proof.* Direct consequence of Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , new rules are not finite-product rules*).  $\square$

**Definition 273 (absolute value in  $\overline{\mathbb{R}}$ ).**  
The absolute value in  $\mathbb{R}$  is extended to  $\overline{\mathbb{R}}$  with the following rule:

(30) 
$$|\pm\infty| = \infty.$$

**Lemma 274 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ).**  
*Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a| = \max(-a, a)$ .*

*Proof.* Direct consequence of **the definition of absolute value in  $\mathbb{R}$** , and Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 275 (bounded absolute value in  $\overline{\mathbb{R}}$ ).**  
*Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $|a| \leq b$  iff  $-b \leq a \leq b$ .*

*Proof.* Direct consequence of Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ,  $-a \leq b$  and  $a \leq b$* ), and Lemma 263 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone,  $-b \leq a$* ).  $\square$



**Lemma 276 (bounded absolute value in  $\overline{\mathbb{R}}$  (strict)).**

Let  $a, b \in \overline{\mathbb{R}}$ . Then, we have  $|a| < b$  iff  $-b < a < b$ .

*Proof.* Direct consequence of Lemma 274 (equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ,  $-a < b$  and  $a < b$ ), and Lemma 263 (additive inverse in  $\overline{\mathbb{R}}$  is monotone, contrapositive both ways,  $-b < a$ ).  $\square$

**Lemma 277 (finite absolute value in  $\overline{\mathbb{R}}$ ).** Let  $a \in \overline{\mathbb{R}}$ . Then, we have  $|a|$  finite iff  $a$  finite.

*Proof.* Direct consequence of Lemma 276 (bounded absolute value in  $\overline{\mathbb{R}}$  (strict), with  $b \stackrel{\text{def.}}{=} \infty$ ).  $\square$

**Lemma 278 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative).** The absolute value in  $\overline{\mathbb{R}}$  is nonnegative.

*Proof.* Direct consequence of Definition 273 (absolute value in  $\overline{\mathbb{R}}$ ), nonnegativeness of the absolute value in  $\mathbb{R}$ . and nonnegativeness of  $\infty$ .  $\square$

**Lemma 279 (absolute value in  $\overline{\mathbb{R}}$  is even).** The absolute value in  $\overline{\mathbb{R}}$  is even.

*Proof.* Direct consequence of evenness of the absolute value in  $\mathbb{R}$ . and Definition 273 (absolute value in  $\overline{\mathbb{R}}$ , new rule is compatible with evenness).  $\square$

**Lemma 280 (absolute value in  $\overline{\mathbb{R}}$  is definite).** The absolute value in  $\overline{\mathbb{R}}$  is definite.

*Proof.* Direct consequence of definiteness of the absolute value in  $\mathbb{R}$ , and Definition 273 (absolute value in  $\overline{\mathbb{R}}$ , new rule is not a zero-absolute-value rule).  $\square$

**Lemma 281 (absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a + b$  is well-defined. Then, we have  $|a + b| \leq |a| + |b|$ .

*Proof.* Direct consequence of Definition 273 (absolute value in  $\overline{\mathbb{R}}$ ), the triangle inequality for the absolute value in  $\mathbb{R}$ , Definition 258 (addition in  $\overline{\mathbb{R}}$ ,  $\infty$  is absorbing for addition in  $\overline{\mathbb{R}}_+$ ), and Definition 254 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $\infty$  is the maximal element).  $\square$

**Definition 282 (exponential and logarithm in  $\overline{\mathbb{R}}$ ).**

The exponential function in  $\mathbb{R}$  and the natural logarithm function in  $\mathbb{R}_+^*$  are respectively extended to  $\overline{\mathbb{R}}$  and  $\overline{\mathbb{R}}_+$  with the following rules:

$$(31) \quad \exp(-\infty) = 0, \quad \exp \infty = \infty, \quad \ln 0 = -\infty, \quad \text{and} \quad \ln \infty = \infty.$$

**Lemma 283 (exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse).**

The exponential function in  $\overline{\mathbb{R}}$  and the natural logarithm function in  $\overline{\mathbb{R}}_+$  are each other inverse.

*Proof.* Direct consequence of properties of the exponential and natural logarithm functions in  $\mathbb{R}$ , and Definition 282 (exponential and logarithm in  $\overline{\mathbb{R}}$ , new rules are compatible with the property).  $\square$

**Definition 284 (exponentiation in  $\overline{\mathbb{R}}$ ).** Exponentiation in  $\mathbb{R}$  (with either positive base and any exponent, or zero base and positive exponent) is extended to  $\overline{\mathbb{R}}$  with the following rule:

$$(32) \quad \forall a \in \overline{\mathbb{R}}_+, \forall b \in \overline{\mathbb{R}}, \quad b \ln a \text{ defined} \implies a^b \stackrel{\text{def.}}{=} \exp(b \ln a).$$

**Lemma 285 (exponentiation in  $\overline{\mathbb{R}}$ ).**

Exponentiation in  $\overline{\mathbb{R}}$  is the function  $\exp : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_+$  defined by

$$(33) \quad \begin{cases} 1. & \forall a \in \mathbb{R}_+^*, \forall b \in \mathbb{R}, \quad a^b \stackrel{\text{def.}}{=} \exp(b \ln a) \in \mathbb{R}_+^*, \\ 2. & \forall a \in [0, 1), \forall b > 0, \quad a^\infty = 0^b \stackrel{\text{def.}}{=} 0 \quad \wedge \quad a^{-\infty} = \infty^b \stackrel{\text{def.}}{=} \infty, \\ 3. & \forall a \in (1, \infty], \forall b < 0, \quad a^\infty = 0^b \stackrel{\text{def.}}{=} \infty \quad \wedge \quad a^{-\infty} = \infty^b \stackrel{\text{def.}}{=} 0, \\ 4. & 0^0, \infty^0, \text{ and } 1^{\pm\infty} \text{ are undefined.} \end{cases}$$

*Proof.* Direct consequence of **the definition of exponentiation in  $\mathbb{R}$** , Definition 284 (*exponentiation in  $\mathbb{R}$* ), Definition 282 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ), and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 286 (topology of  $\overline{\mathbb{R}}$ ).**

The totally ordered set  $(\overline{\mathbb{R}}, \leq)$  is equipped with the order topology  $\mathcal{T}_{\overline{\mathbb{R}}}(\mathcal{R}_{\overline{\mathbb{R}}}^o)$  generated by the open rays, for which the open intervals  $\mathcal{I}_{\overline{\mathbb{R}}}^o$  constitute a topological basis.

*Proof.* Direct consequence of Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ), Definition 232 (*order topology*), and Lemma 234 (*topological basis of order topology*).  $\square$

*Remark 287.* Note that  $\overline{\mathbb{R}}$  is actually metrizable since it is homeomorphic to a bounded segment, e.g.  $[-\pi/2, \pi/2]$  or  $[-1, 1]$ . Indeed, metrics can be defined on  $\overline{\mathbb{R}}$ , for instance by using the arctangent function, or the hyperbolic tangent function.

**Lemma 288 (trace topology on  $\mathbb{R}$ ).**

Let  $\overline{\mathcal{T}}$  be the order topology on  $\overline{\mathbb{R}}$ . Then, the trace topology  $\overline{\mathcal{T}}_{\mathbb{R}}$  is equal to the order topology on  $\mathbb{R}$ .

*Proof.* Direct consequence of Lemma 236 (*trace topology on subset*), Lemma 234 (*topological basis of order topology*), since  $\mathbb{R} \in \mathcal{I}_{\overline{\mathbb{R}}}^o \subset \mathcal{I}_{\mathbb{R}}^o$ , and Definition 230 (*topological basis*).  $\square$

*Remark 289.* As a consequence, open subsets of  $\mathbb{R}$  are also open subsets of  $\overline{\mathbb{R}}$ .

**Lemma 290 (convergence towards  $-\infty$ ).**

Let  $(x_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ . Then, we have  $\lim_{n \rightarrow \infty} x_n = -\infty$  iff

$$(34) \quad \forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \quad x_n \leq -k.$$

If so, the sequence is said convergent towards  $-\infty$ .

*Proof.* Direct consequence of Lemma 286 (*topology of  $\overline{\mathbb{R}}$* ), **the definition of the limit using neighborhoods**, **the Archimedean property of  $\overline{\mathbb{R}}$** , and **totally ordered set properties of  $\overline{\mathbb{R}}$** .  $\square$

**Lemma 291 (continuity of addition in  $\overline{\mathbb{R}}$ ).** Addition in  $\overline{\mathbb{R}}$  is continuous when defined.

*Proof.* Direct consequence of **continuity of addition in  $\mathbb{R}$** , **unboundedness of addition when an operand tends towards  $\pm\infty$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$* , rules 1 and 2).  $\square$

**Lemma 292 (continuity of multiplication in  $\overline{\mathbb{R}}$ ).**

Multiplication in  $\overline{\mathbb{R}}$  is continuous when defined.

*Proof.* Direct consequence of **continuity of multiplication in  $\mathbb{R}$** , **unboundedness of multiplication when an operand tends towards  $\pm\infty$  and the other is nonzero**, and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* , rules 1 and 2).  $\square$

**Lemma 293 (absolute value in  $\overline{\mathbb{R}}$  is continuous).** The absolute value in  $\overline{\mathbb{R}}$  is continuous.

*Proof.* Direct consequence of **continuity of the absolute value in  $\mathbb{R}$** , and Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* , absolute value is closed in  $\{\pm\infty\}$ ).  $\square$

### 6.4.2.1 Nonnegative extended real numbers

**Lemma 294** (*addition in  $\overline{\mathbb{R}}_+$  is closed*). *Addition in  $\overline{\mathbb{R}}_+$  is closed.*

*Proof.* Direct consequence of **closedness of addition in  $\mathbb{R}_+$** , and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 295** (*addition in  $\overline{\mathbb{R}}_+$  is associative*). *Addition in  $\overline{\mathbb{R}}_+$  is associative.*

*Proof.* Direct consequence of Lemma 260 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*), and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 296** (*addition in  $\overline{\mathbb{R}}_+$  is commutative*). *Addition in  $\overline{\mathbb{R}}_+$  is commutative.*

*Proof.* Direct consequence of Lemma 261 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*), and Definition 258 (*addition in  $\overline{\mathbb{R}}$ , undefined forms of rule 3 cannot occur*).  $\square$

**Lemma 297** (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ). *Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

(35) 
$$a + b = \infty \iff a = \infty \vee b = \infty.$$

*Proof.* Direct consequence of Lemma 262 (*infinity-sum property in  $\overline{\mathbb{R}}$* ).  $\square$

**Lemma 298** (*series are convergent in  $\overline{\mathbb{R}}_+$* ).  
Let  $(a_n)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Then, we have  $\sum_{n \in \mathbb{N}} a_n \in \overline{\mathbb{R}}_+$ .

*Proof.* Direct consequence of **completeness of  $\overline{\mathbb{R}}_+$** .  $\square$

**Lemma 299** (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* ). *Let  $(a_p)_{p \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ .*  
Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Assume that  $\varphi$  is injective. Then, we have  $\sum_{j \in \mathbb{N}} a_{\varphi(j)} \leq \sum_{p \in \mathbb{N}} a_p$ .

*Proof.* Let

$$A = \sum_{j \in \mathbb{N}} a_{\varphi(j)} \stackrel{\text{def.}}{=} \lim_{i \rightarrow \infty} \sum_{j \in [0..i]} a_{\varphi(j)} \quad \text{and} \quad B = \sum_{p \in \mathbb{N}} a_p \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} \sum_{p \in [0..n]} a_p.$$

Then, from Lemma 298 (*series are convergent in  $\overline{\mathbb{R}}_+$* ), we have  $A, B \in \overline{\mathbb{R}}_+$ . Let  $i \in \mathbb{N}$ . Let

$$n \stackrel{\text{def.}}{=} \max\{\varphi(j) \mid j \in [0..i]\}.$$

Then, from **the definition of the maximum**, we have  $\varphi([0..i]) \subset [0..n] \subset \mathbb{N}$ . Thus, from **totally ordered set properties of  $\overline{\mathbb{R}}_+$** , we have

$$\sum_{j \in [0..i]} a_{\varphi(j)} \leq \sum_{p \in [0..n]} a_p \leq B.$$

Hence, from **monotonicity of the limit (when  $n \rightarrow \infty$ )**, we have  $A \leq B$ .

Therefore, we have  $\sum_{j \in \mathbb{N}} a_{\varphi(j)} \leq \sum_{p \in \mathbb{N}} a_p$ .  $\square$

**Lemma 300** (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* ).  
Let  $(a_p)_{p \in \mathbb{N}} \in \overline{\mathbb{R}}_+$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ . Assume that  $\varphi$  is bijective. Then,  $\sum_{p \in \mathbb{N}} a_p = \sum_{j \in \mathbb{N}} a_{\varphi(j)}$ .

*Proof.* Direct consequence of Lemma 299 (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* , first used with the sequence  $(a_p)_{p \in \mathbb{N}}$  and the function  $\varphi$ , and then with the sequence  $(b_j)_{j \in \mathbb{N}} \stackrel{\text{def.}}{=} (a_{\varphi(j)})_{j \in \mathbb{N}}$  and the function  $\varphi^{-1}$  which satisfies  $\varphi \circ \varphi^{-1} = \text{Id}_{\mathbb{N}}$ ).  $\square$

**Lemma 301 (definition of double series in  $\overline{\mathbb{R}}_+$ ).**

Let  $(a_{p,q})_{(p,q) \in \mathbb{N}^2} \in \overline{\mathbb{R}}_+$ . Let  $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}^2$ . Assume that  $\varphi$  and  $\psi$  are bijections. Then, we have  $\sum_{j \in \mathbb{N}} a_{\varphi(j)} = \sum_{j \in \mathbb{N}} a_{\psi(j)}$ . This sum is denoted  $\sum_{(p,q) \in \mathbb{N}^2} a_{p,q}$ .

*Proof.* Direct consequence of Lemma 300 (order is meaningless in series in  $\overline{\mathbb{R}}_+$ , with  $(a_{\varphi(j)})_{j \in \mathbb{N}}$  and  $\varphi^{-1} \circ \psi$ ).  $\square$

**Lemma 302 (double series in  $\overline{\mathbb{R}}_+$ ).**

Let  $(a_{p,q})_{(p,q) \in \mathbb{N}^2} \in \overline{\mathbb{R}}_+$ . Then, we have  $\sum_{(p,q) \in \mathbb{N}^2} a_{p,q} = \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ .

*Proof.* From **countability of  $\mathbb{N}^2$** , let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}^2$  be a bijection. Then, from Lemma 301 (definition of double series in  $\overline{\mathbb{R}}_+$ ), let  $A \stackrel{\text{def.}}{=} \sum_{(p,q) \in \mathbb{N}^2} a_{p,q} = \sum_{j \in \mathbb{N}} a_{\varphi(j)}$  (the sum does not depend on the choice for  $\varphi$ ). Let  $B \stackrel{\text{def.}}{=} \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ . Then, from Lemma 298 (series are convergent in  $\overline{\mathbb{R}}_+$ ), we have  $A, B \in \overline{\mathbb{R}}_+$ .

Let  $i \in \mathbb{N}$ . Let  $n \stackrel{\text{def.}}{=} \max\{\pi_1(\varphi(j)) \mid j \in [0..i]\}$  and  $m \stackrel{\text{def.}}{=} \max\{\pi_2(\varphi(j)) \mid j \in [0..i]\}$  where the functions  $\pi_1$  and  $\pi_2$  are defined by  $\forall (p, q) \in \mathbb{N}^2, \pi_1(p, q) = p$  and  $\pi_2(p, q) = q$ . Then, from **the definition of the maximum**, we have  $\varphi([0..i]) \subset [0..n] \times [0..m] \subset \mathbb{N}^2$ . Thus, from **totally ordered set properties of  $\overline{\mathbb{R}}_+$** , we have

$$\sum_{j \in [0..i]} a_{\varphi(j)} \leq \sum_{p \in [0..n]} \left( \sum_{q \in [0..m]} a_{p,q} \right) \leq B.$$

Hence, from **monotonicity of the limit (when  $i \rightarrow \infty$ )**, we have  $A \leq B$ .

Let  $n, m \in \mathbb{N}$ . Let  $i \stackrel{\text{def.}}{=} \max\{\varphi^{-1}(p, q) \mid (p, q) \in [0..n] \times [0..m]\}$ . Then, from **the definition of the maximum**, we have  $[0..n] \times [0..m] \subset \varphi([0..i]) = \mathbb{N}^2$ . Thus, from **totally ordered set properties of  $\overline{\mathbb{R}}_+$** , we have

$$\sum_{p \in [0..n]} \left( \sum_{q \in [0..m]} a_{p,q} \right) \leq \sum_{j \in [0..i]} a_{\varphi(j)} \leq A.$$

Hence, from **monotonicity of the limit (when  $m \rightarrow \infty$ , then  $n \rightarrow \infty$ )**, we have  $B \leq A$ .

Therefore, we have  $\sum_{(p,q) \in \mathbb{N}^2} a_{p,q} = \sum_{p \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} a_{p,q} \right)$ .  $\square$

**Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ ).**

Multiplication and division in  $\overline{\mathbb{R}}$  are restricted to  $\overline{\mathbb{R}}_+$  by replacing the fourth rule in (25) by

$$(36) \quad 4'. \quad \frac{1}{0} \stackrel{\text{def.}}{=} \infty, \text{ and } \frac{1}{\infty} \stackrel{\text{def.}}{=} 0.$$

*Remark 304.* When restricting to nonnegative extended numbers, making multiplicative inverse a bijection from  $\overline{\mathbb{R}}_+$  onto itself through Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ ) implies that for all  $a > 0$ ,  $\frac{a}{0} \stackrel{\text{def.}}{=} \infty$ . The expressions  $0 \times \infty$ ,  $\frac{\infty}{\infty}$  and  $\frac{0}{0}$  remain undefined.

**Lemma 305 (multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined).**

When they are defined, multiplication and division in  $\overline{\mathbb{R}}_+$  are closed.

*Proof.* Direct consequence of **closedness of multiplication and division by nonzero in  $\overline{\mathbb{R}}_+$** , Definition 264 (multiplication in  $\overline{\mathbb{R}}$ ), and Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ ).  $\square$

**Lemma 306 (zero-product property in  $\overline{\mathbb{R}}_+$ ).** *Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(37) \quad ab = 0 \iff ab \text{ is defined} \wedge (a = 0 \vee b = 0).$$

*Proof.* Direct consequence of Lemma 270 (zero-product property in  $\overline{\mathbb{R}}$ ), and Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not a zero-product rule).  $\square$

**Lemma 307 (infinity-product property in  $\overline{\mathbb{R}}_+$ ).** *Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(38) \quad ab = \infty \iff (a = \infty \wedge b \neq 0) \vee (a \neq 0 \wedge b = \infty).$$

*Proof.* Direct consequence of Lemma 271 (infinity-product property in  $\overline{\mathbb{R}}$ ), and Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not an infinity-product rule).  $\square$

**Lemma 308 (finite-product property in  $\overline{\mathbb{R}}_+$ ).** *Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have*

$$(39) \quad ab \text{ is defined} \wedge ab \in \mathbb{R}_+ \iff a, b \in \mathbb{R}_+.$$

*Proof.* Direct consequence of Lemma 272 (finite-product property in  $\overline{\mathbb{R}}$ ), closedness of multiplication in  $\mathbb{R}_+$ , and Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ , new rule is not a finite-product rule).  $\square$

#### 6.4.2.2 In the context of measure theory

**Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory (and probability), the third rule in (25) is replaced by

$$(40) \quad 3'. \quad 0 \times (\pm\infty) = \pm\infty \times 0 \stackrel{\text{def.}}{=} 0.$$

*Remark 310.* In the context of measure theory, making 0 an absorbing element for multiplication in  $\overline{\mathbb{R}}$  through Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)) implies that  $\frac{\infty}{\pm\infty} = \frac{-\infty}{\pm\infty} \stackrel{\text{def.}}{=} 0$ . The expression  $\frac{a}{0}$  (for all  $a \in \overline{\mathbb{R}}$ ) remains undefined. Note that multiplication in  $\overline{\mathbb{R}}$  is always well-defined in this context.

**Lemma 311 (zero-product property in  $\overline{\mathbb{R}}$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}$ . Then, we have*

$$(41) \quad ab = 0 \iff a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 270 (zero-product property in  $\overline{\mathbb{R}}$ ), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule 3' is compatible with the property).  $\square$

**Lemma 312 (infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)).**

*In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}$ . Then, we have*

$$(42) \quad ab = \infty \iff (a = \infty \wedge b > 0) \vee (a = -\infty \wedge b < 0) \vee (a > 0 \wedge b = \infty) \vee (a < 0 \wedge b = -\infty),$$

$$(43) \quad ab = -\infty \iff (a = -\infty \wedge b > 0) \vee (a = \infty \wedge b < 0) \vee (a > 0 \wedge b = -\infty) \vee (a < 0 \wedge b = \infty).$$

*Proof.* Direct consequence of Lemma 271 (infinity-product property in  $\overline{\mathbb{R}}$ ), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule is not an infinity-product rule).  $\square$

**Lemma 313 (finite-product property in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}$ . Then, we have

$$(44) \quad ab \in \mathbb{R} \iff a, b \in \mathbb{R} \vee a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 312 (infinity-product property in  $\overline{\mathbb{R}}$  (measure theory), contrapositive), **De Morgan's laws**, and **distributivity of logical conjunction over logical disjunction**.  $\square$

**Lemma 314 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)).**

In the context of measure theory, multiplication and division are total functions  $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ .

In particular, we have  $\frac{\infty}{\infty} = \frac{0}{0} \stackrel{\text{def.}}{=} 0$ , and for all  $a > 0$ ,  $\frac{a}{0} \stackrel{\text{def.}}{=} \infty$ .

*Proof.* Direct consequence of Definition 264 (multiplication in  $\overline{\mathbb{R}}$ ), Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ ), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)).  $\square$

*Remark 315.* Of course, in the context of measure theory, multiplication and division are no longer continuous on the whole boundary of  $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$ .

**Lemma 316 (multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)).**

In the context of measure theory, multiplication in  $\overline{\mathbb{R}}_+$  is associative.

*Proof.* Direct consequence of Lemma 266 (multiplication in  $\overline{\mathbb{R}}$  is associative when defined), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 317 (multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)).**

In the context of measure theory, multiplication in  $\overline{\mathbb{R}}_+$  is commutative.

*Proof.* Direct consequence of Lemma 267 (multiplication in  $\overline{\mathbb{R}}$  is commutative when defined), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 318 (multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)).**

In the context of measure theory, multiplication is (left and right) distributive over addition in  $\overline{\mathbb{R}}_+$ .

*Proof.* Direct consequence of Lemma 268 (multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined), Lemma 269 (multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), no longer undefined forms).  $\square$

**Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(45) \quad ab = 0 \iff a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 306 (zero-product property in  $\overline{\mathbb{R}}_+$ ), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule 3' is compatible with the property).  $\square$

**Lemma 320 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(46) \quad ab = \infty \iff (a = \infty \wedge b > 0) \vee (b = \infty \wedge a > 0).$$

*Proof.* Direct consequence of Lemma 307 (infinity-product property in  $\overline{\mathbb{R}}_+$ ), and Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory), new rule is not an infinity-product rule).  $\square$

**Lemma 321 (finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have

$$(47) \quad ab \in \mathbb{R}_+ \iff a, b \in \mathbb{R}_+ \vee a = 0 \vee b = 0.$$

*Proof.* Direct consequence of Lemma 313 (finite-product property in  $\overline{\mathbb{R}}$  (measure theory)), and **closedness of multiplication in  $\mathbb{R}_+$** .  $\square$

**Lemma 322 (exponentiation in  $\overline{\mathbb{R}}$  (measure theory)).**

In the context of measure theory, the fourth rule in (33) is replaced by

$$(48) \quad 4'. \quad 0^0 = \infty^0 = 1^{\pm\infty} \stackrel{\text{def.}}{=} 1.$$

Thus, we have

$$(49) \quad \forall a \in \overline{\mathbb{R}}_+, a^0 = 1 \quad \text{and} \quad \forall b \in \overline{\mathbb{R}}, 1^b = 1.$$

*Proof.* Direct consequence of Definition 284 (exponentiation in  $\overline{\mathbb{R}}$ ), Definition 309 (multiplication in  $\overline{\mathbb{R}}$  (measure theory)), and Lemma 285 (exponentiation in  $\overline{\mathbb{R}}$ , new rule only affects rule 4).  $\square$

**Definition 323 (Hölder conjugates).**

Extended numbers  $p, q \in [1, \infty]$  are said *Hölder conjugates* iff  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Remark 324.* Definition 323 (Hölder conjugates) implies that extended numbers 1 and  $\infty$  are Hölder conjugate since from rule 4' of Definition 303 (multiplication in  $\overline{\mathbb{R}}_+$ ), we have  $\frac{1}{\infty} = 0$ .

**Lemma 325 (Young's inequality for products (measure theory)).**

In the context of measure theory, let  $p, q \in (1, \infty)$ . Assume that  $p$  and  $q$  are Hölder conjugates. Let  $a, b \in \overline{\mathbb{R}}_+$ . Then, we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ .

*Proof.* From Lemma 314 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)), Lemma 285 (exponentiation in  $\overline{\mathbb{R}}$ , with  $p, q \neq 0, \pm\infty$ ), **closedness of multiplicative inverse in  $\mathbb{R}_+^*$** , Lemma 305 (multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined), Definition 264 (multiplication in  $\overline{\mathbb{R}}$ , rule 5 with  $b = p, q \in \mathbb{R}_+^*$ ), and Lemma 294 (addition in  $\overline{\mathbb{R}}_+$  is closed),  $ab, a^p, b^q, \frac{a^p}{p}, \frac{b^q}{q}$ , and  $\frac{a^p}{p} + \frac{b^q}{q}$  are well-defined, and belong to  $\overline{\mathbb{R}}_+$ .

**Case  $ab = \infty$ .** Then, from Lemma 320 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory),  $a = \infty$  or  $b = \infty$ ), Lemma 285 (exponentiation in  $\overline{\mathbb{R}}$ , rule 2 with  $b = p, q > 0$ ), Definition 264 (multiplication in  $\overline{\mathbb{R}}$ , rule 1 with  $a \stackrel{\text{def.}}{=} \frac{1}{p}, \frac{1}{q} > 0$ ), and Definition 258 (addition in  $\overline{\mathbb{R}}$ , rule 1, the other operand is not  $-\infty$ ), the right-hand side equals  $\infty$ . Thus, the (in)equality holds.

**Case  $ab = 0$ .** Then, the inequality holds since the right-hand side is nonnegative.

**Case  $ab \in \mathbb{R}_+^*$ .** Direct consequence of Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory), contrapositive), Lemma 307 (infinity-product property in  $\overline{\mathbb{R}}_+$ , contrapositive), and Lemma 251 (Young's inequality for products in  $\mathbb{R}$ ).

Therefore, the inequality always holds.  $\square$

**Lemma 326 (Young's inequality for products, case  $p = 2$  (measure theory)).**

In the context of measure theory, let  $a, b \in \overline{\mathbb{R}}_+$ . Let  $\varepsilon \in \mathbb{R}_+^*$ . Then, we have  $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ .

*Proof.* Direct consequence of Lemma 250 (2 is self-Hölder conjugate in  $\mathbb{R}$ ), Lemma 325 (Young's inequality for products (measure theory), with  $p = q \stackrel{\text{def.}}{=} 2$  and  $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}b \in \overline{\mathbb{R}}_+$ ), **properties of square root and multiplicative inverse in  $\mathbb{R}_+^*$** , Lemma 314 (multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)), Lemma 316 (multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)), and Lemma 317 (multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)).  $\square$



### 6.4.3 Second countability and real numbers

#### Definition 327 (connected component in $\mathbb{R}$ ).

Let  $A \subset \mathbb{R}$ . Let  $x \in A$ . The *connected component* of  $A$  containing  $x$  is the union of all open intervals  $I$  containing  $x$  and included in  $A$ ; it is denoted  $I_x^A$ .

#### Lemma 328 (connected component of open subset of $\mathbb{R}$ is open interval).

Let  $O \subset \mathbb{R}$  be open. Let  $x \in O$ . Then,  $I_x^O$  is an open interval contained in  $O$ .

*Proof.* Note that from Definition 22 (*open subset*), there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subset O$ . Then, from Definition 327 (*connected component in  $\mathbb{R}$* ),  $(x - \varepsilon, x + \varepsilon) \subset I_x^O$ . Thus,  $I_x^O$  is nonempty, contains  $x$ . Moreover, from Definition 225 (*topological space, closedness under union*),  $I_x^O$  is open.

Let  $a, b, c \in \mathbb{R}$  such that  $a < c < b$  and  $a, b \in I_x^O$ . Then, from Definition 327 (*connected component in  $\mathbb{R}$* ), and **the definition of union**, there exists intervals  $I_a(x)$  and  $I_b(x)$  such that  $x, a \in I_a(x) \subset I_x^O \subset O$  and  $x, b \in I_b(x) \subset I_x^O \subset O$ . **Case  $c = x$ .** Then, we have  $c = x \in I_x^O$ . **Case  $c < x$ .** Then, we have  $a < c < x$ . Thus, since  $a, x \in I_a(x)$ , and  $I_a(x)$  is an interval, we have successively  $c \in (a, x) \subset I_a(x) \subset I_x^O$ . **Case  $x < c$ .** Then, we have  $x < c < b$ . Thus, since  $x, b \in I_b(x)$ , and  $I_b(x)$  is an interval, we have successively  $c \in (x, b) \subset I_b(x) \subset I_x^O$ . Hence, we always have  $c \in I_x^O$ . Thus, from Definition 217 (*interval, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$* ),  $I_x^O$  is an interval.

Therefore,  $I_x^O$  is an open interval contained in  $O$ .  $\square$

#### Lemma 329 (connected component of open subset of $\mathbb{R}$ is maximal).

Let  $O \subset \mathbb{R}$  be open. Let  $x, y \in O$ . Assume that  $y \in I_x^O$ . Then, we have  $I_y^O = I_x^O$ .

*Proof.* From Definition 327 (*connected component in  $\mathbb{R}$ , union*), there exists an open interval  $I$  such that  $y \in I$  and  $x \in I \subset O$ . Thus, we also have  $x \in I_y^O$ . But, from Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*),  $I_x^O$  is an open interval such that  $y \in I_x^O \subset O$ , and  $I_y^O$  is an open interval such that  $x \in I_y^O \subset O$ . Thus, from Definition 327 (*connected component in  $\mathbb{R}$* ), we have  $I_x^O \subset I_y^O$  and  $I_y^O \subset I_x^O$ . Therefore, we have  $I_x^O = I_y^O$ .  $\square$

#### Lemma 330 (connected components of open subset of $\mathbb{R}$ are equal or disjoint).

Let  $O \subset \mathbb{R}$  be open. Let  $x, y \in O$ . Then, we have either  $I_x^O = I_y^O$ , or  $I_x^O \cap I_y^O = \emptyset$ .

*Proof.* Assume that  $I_x^O \cap I_y^O \neq \emptyset$ . Then, there exists  $z \in I_x^O \cap I_y^O$ . Thus, from Lemma 329 (*connected component of open subset of  $\mathbb{R}$  is maximal*), we have  $I_z^O = I_x^O$  and  $I_z^O = I_y^O$ . Therefore,  $I_x^O$  and  $I_y^O$  are equal or disjoint.  $\square$

#### Theorem 331 (countable connected components of open subsets of $\mathbb{R}$ ).

Let  $O \subset \mathbb{R}$  be open. Then,  $O$  is a countable union of disjoint open intervals.

*Proof.* Let  $x \in O$ . Then, from Definition 22 (*open subset*), there exists  $\varepsilon > 0$  such that the open interval  $(x - \varepsilon, x + \varepsilon)$  is included in  $O$ . From **density of rational numbers in  $\mathbb{R}$** , there exists  $q \in \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon)$ . Then, by construction, we have  $x \in (x - \varepsilon, x + \varepsilon) \subset I_q^O$ . Thus, we have the inclusion  $O \subset \bigcup_{q \in \mathbb{Q} \cap O} I_q^O$ .

Conversely, let  $q \in \mathbb{Q} \cap O$ . Then, from Definition 327 (*connected component in  $\mathbb{R}$* ), we have  $I_q^O \subset O$ . Thus, we also have the other inclusion  $\bigcup_{q \in \mathbb{Q} \cap O} I_q^O \subset O$ . Hence, the equality.

Therefore, from **countability of  $\mathbb{Q}$** , Lemma 330 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*), and after eliminating doubles in the union,  $O$  is a countable union of disjoint open intervals.  $\square$

#### Lemma 332 (rational approximation of lower bound of open interval).

Let  $a, b \in \mathbb{R}$ . Assume that  $a < b$ . Then, there exists a sequence  $(a_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$  that is nonincreasing with limit  $a$ .



*Proof.* Note that  $-\infty \leq a < b \leq \infty$ . From **density of rational numbers in  $\mathbb{R}$** , let  $a_0 \in (a, b) \cap \mathbb{Q}$ .

**Case  $a = -\infty$ .** Then, from **density of rational numbers in  $\mathbb{R}$** , for all  $n \in \mathbb{N}$ , let  $a_{n+1}$  be in  $(-\infty, -2|a_n|) \cap \mathbb{Q}$ . Hence, from **ordered field properties of  $\mathbb{R}$** , and **the Archimedean property of  $\mathbb{R}$** , the sequence  $(a_n)_{n \in \mathbb{N}}$  belongs to  $(a, b)$ , and is nonincreasing with limit  $a = -\infty$ .

**Case  $a$  finite.** Then, from **density of rational numbers in  $\mathbb{R}$** , for all  $n \in \mathbb{N}$ , let  $a_{n+1}$  be in  $(a, \frac{a+a_n}{2}) \cap \mathbb{Q} \subset (a, b) \cap \mathbb{Q}$ . Let  $n \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have  $a_{n+1} < a_n$  and  $0 < a_{n+1} - a < \frac{a_n - a}{2}$ . Thus, a trivial finite induction on index  $i$  shows that

$$\forall i \in [0..n+1], \quad 0 < a_{n+1} - a < \frac{a_i - a}{2^{n+1-i}} < \frac{a_0 - a}{2^{n+1}}.$$

Hence, from **the squeeze theorem**, the sequence  $(a_n)_{n \in \mathbb{N}}$  is nonincreasing with limit  $a$ .  $\square$

**Lemma 333 (rational approximation of upper bound of open interval).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ . Then, there exists a sequence  $(b_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$  that is nonincreasing with limit  $b$ .

*Proof.* Direct consequence of Lemma 332 (**rational approximation of lower bound of open interval**, with  $a \stackrel{\text{def.}}{=} -b$  and  $b \stackrel{\text{def.}}{=} -a$ ).  $\square$

**Lemma 334 (open intervals with rational bounds cover open interval).** Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ . Then, there exists  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{Q}$  such that  $(a, b) = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ .

*Proof.* Direct consequence of Lemma 332 (**rational approximation of lower bound of open interval**, there exists  $(a_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$ ), Lemma 333 (**rational approximation of upper bound of open interval**, there exists  $(b_n)_{n \in \mathbb{N}} \in \mathbb{Q} \cap (a, b)$ ), Definition 27 (**convergent sequence**, with  $\varepsilon \stackrel{\text{def.}}{=} x - a$ , then  $\varepsilon \stackrel{\text{def.}}{=} b - x$  for all  $x \in (a, b)$  (hence,  $(a, b) \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ ), and Definition 225 (**topological space**, closedness under union ( $\bigcup_{n \in \mathbb{N}} (a_n, b_n) \subset (a, b)$ )).  $\square$

**Theorem 335 ( $\mathbb{R}$  is second-countable).**

Let  $d$  be the Euclidean distance on  $\mathbb{R}$ .

Then,  $\{(a, b) \mid a, b \in \mathbb{Q} \wedge a < b\}$  is a topological basis of  $(\mathbb{R}, d)$ . Hence,  $(\mathbb{R}, d)$  is second-countable.

*Proof.* Direct consequence of Theorem 331 (**countable connected components of open subsets of  $\mathbb{R}$** ), Lemma 334 (**open intervals with rational bounds cover open interval**), Definition 230 (**topological basis**), **countability of  $\mathbb{Q} \times \mathbb{Q}_+$** , and Definition 238 (**second-countability**).  $\square$

**Lemma 336 ( $\mathbb{R}^n$  is second-countable).**

Let  $n \in [2..\infty)$ .

Let  $d$  be the Euclidean distance on  $\mathbb{R}^n$ . Then,  $\left\{ \prod_{i \in [1..n]} (a_i, b_i) \mid \forall i \in [1..n], a_i, b_i \in \mathbb{Q} \wedge a_i < b_i \right\}$  is a topological basis of  $(\mathbb{R}^n, d)$ . Hence,  $(\mathbb{R}^n, d)$  is second-countable.

*Proof.* Direct consequence of Theorem 335 ( **$\mathbb{R}$  is second-countable**), Lemma 241 (**compatibility of second-countability with Cartesian product**), Lemma 237 (**box topology on Cartesian product**) and Definition 238 (**second-countability**).  $\square$

**Lemma 337 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ ).**

Let  $a, b \in \overline{\mathbb{R}}$ . Assume that  $a < b$ . Then, there exists  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{Q}$  such that

$$(50) \quad (a, b) = \bigcup_{n \in \mathbb{N}} (a_n, b_n), \quad [-\infty, b) = \bigcup_{n \in \mathbb{N}} [-\infty, b_n) \quad \text{and} \quad (a, \infty] = \bigcup_{n \in \mathbb{N}} (a_n, \infty].$$

*Proof.* Direct consequence of Lemma 334 (**open intervals with rational bounds cover open interval**), Lemma 332 (**rational approximation of lower bound of open interval**), and Lemma 333 (**rational approximation of upper bound of open interval**).  $\square$

**Lemma 338** ( $\overline{\mathbb{R}}$  is second-countable).

Let  $\overline{\mathcal{T}} \stackrel{\text{def.}}{=} \overline{\mathcal{T}_{\mathbb{R}}(\mathcal{R}_{\mathbb{R}}^o)}$  be the order topology on  $\overline{\mathbb{R}}$ . Then, the open intervals with rational bounds constitute a topological basis of  $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$ . Hence,  $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$  is second-countable.

*Proof.* Direct consequence of Lemma 286 (topology of  $\overline{\mathbb{R}}$ ), Theorem 331 (countable connected components of open subsets of  $\mathbb{R}$ , similar proof for  $\overline{\mathbb{R}}$ ), Lemma 337 (open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$ ), Definition 230 (topological basis), countability of  $\mathbb{Q}^2$ , compatibility of union with countability, and Definition 238 (second-countability).  $\square$

#### 6.4.4 Infimum, supremum

**Lemma 339** (extrema of constant function). Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $a \in \mathbb{R}$ . Assume that  $f$  is constant of value  $a$ . Then,

$$(51) \quad \sup(f(X)) = \max(f(X)) = \inf(f(X)) = \min(f(X)) = a.$$

*Proof.* Direct consequence of the definition of constant function, Lemma 8 (equivalent definition of maximum), Definition 7 (maximum), Lemma 15 (equivalent definition of minimum), and Definition 14 (minimum).  $\square$

**Lemma 340** (equivalent definition of finite infimum). Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \mathbb{R}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$ .

*Proof.* “Left” implies “right”. Assume first that  $l = \inf(f(X))$ . Then, from Definition 9 (infimum),  $l$  is a (finite) lower bound for  $f(X)$ . Thus, from Lemma 13 (finite infimum discrete), for all  $n \in \mathbb{N}$ , there exists  $x'_n \in X$  such that  $f(x'_n) < l + \frac{1}{n+1}$ . Let  $x_0 \stackrel{\text{def.}}{=} x'_0$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} \stackrel{\text{def.}}{=} \arg \min(f(x_n), f(x'_{n+1}))$ . Then, for all  $n \in \mathbb{N}$ , we have

$$f(x_{n+1}) \leq f(x_n) \leq f(x'_n) < l + \frac{1}{n+1}.$$

Thus, from the definition of monotone sequence, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing. Moreover, let  $k \in \mathbb{N}$ . Let  $N \stackrel{\text{def.}}{=} k$ . Let  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, from ordered field properties of  $\mathbb{R}$ , we have  $f(x_n) \leq f(x_N) = f(x_k) < l + \frac{1}{k+1}$ . Hence, from Lemma 245 (equivalent definition of convergent sequence), the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is convergent with limit  $l$ .

“Right” implies “left”. Conversely, assume now that  $l$  is a (finite) lower bound of  $f(X)$ , and that there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing with limit  $l$ . Let  $n \in \mathbb{N}$ , and let  $\varepsilon_n \stackrel{\text{def.}}{=} \frac{1}{2(n+1)} > 0$ . Then, from Definition 27 (convergent sequence), let  $N \in \mathbb{N}$  such that for all  $p \geq N$ ,  $|f(x_p) - l| < \varepsilon_n$ . Let  $x'_n \stackrel{\text{def.}}{=} x_N$ . Then, from ordered field properties of  $\mathbb{R}$ , we have  $f(x'_n) \leq l + \varepsilon_n < l + \frac{1}{n+1}$ . Hence, from Lemma 13 (finite infimum discrete), we have  $l = \inf(f(X))$ .

Therefore, we have the equivalence.  $\square$

**Lemma 341** (equivalent definition of finite infimum in  $\overline{\mathbb{R}}$ ). Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \mathbb{R}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$ .

*Proof.* Case  $-\infty \in f(X)$ . Then, from the definition of  $-\infty$ , and Definition 9 (infimum, lower bound),  $-\infty$  is the only lower bound of  $f(X)$ . Thus, since  $l \in \mathbb{R}$ , both propositions “ $l = \inf(f(X))$ ” and “ $l$  is a lower bound of  $f(X)$ ” are wrong. Hence, we have the equivalence.

Case  $f(X) \subset \mathbb{R}$ . Direct consequence of Lemma 340 (equivalent definition of finite infimum).

**Case  $-\infty \notin f(X)$ , and  $\infty \in f(X)$ .** Then, from Lemma 339 (*extrema of constant function*, contrapositive with  $\inf(f(X)) \neq \infty$ ), the function cannot be constant of value  $\infty$ . Thus, since  $-\infty \notin f(X)$ , let  $\tilde{x} \in X$  such that  $f(\tilde{x}) \in \mathbb{R}$ . Let  $\pi_f : X \rightarrow X$  be the projection defined by

$$\pi_f(x) \stackrel{\text{def.}}{=} \begin{cases} x & \text{when } f(x) \neq \infty, \\ \tilde{x} & \text{otherwise.} \end{cases}$$

Let  $\tilde{f} \stackrel{\text{def.}}{=} f \circ \pi_f$ . Then, by construction, we have  $\tilde{f}(X) \subset \mathbb{R}$  and  $\tilde{f} \leq f$ .

Let  $l$  be a lower bound of  $\tilde{f}(X)$ . Then,  $l$  is also a lower bound of  $f(X)$ . Conversely, let  $l$  be a lower bound of  $f(X)$ . Let  $x \in X$ . Then, from the definitions of  $\pi_f$  and  $\tilde{f}$ , we have

$$l \leq f(\pi_f(x)) = \tilde{f}(x).$$

Thus,  $l$  is also a lower bound of  $\tilde{f}(X)$ . Hence,  $f(X)$  and  $\tilde{f}(X)$  have the same lower bounds.

Moreover, from Definition 9 (*infimum, lower bound*),  $\inf(f(X))$  is also a lower bound of  $\tilde{f}(X)$ , and  $\inf(\tilde{f}(X))$  is also a lower bound of  $f(X)$ . Thus, from Definition 9 (*infimum, greatest lower bound*), we have  $\inf(f(X)) \leq \inf(\tilde{f}(X))$ , and  $\inf(\tilde{f}(X)) \leq \inf(f(X))$ . Hence, from **totally ordered set properties of  $\overline{\mathbb{R}}$** , we have  $\inf(\tilde{f}(X)) = \inf(f(X))$ .

Furthermore, let  $(\tilde{x}_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Then, from the definition of  $\pi_f$  and  $\tilde{f}$ , the sequence  $(x_n)_{n \in \mathbb{N}} \stackrel{\text{def.}}{=} \pi_f(\tilde{x}_n)_{n \in \mathbb{N}} \in X$  is such that both sequences  $(f(x_n))_{n \in \mathbb{N}}$  and  $(\tilde{f}(\tilde{x}_n))_{n \in \mathbb{N}}$  are identical. Conversely, assume that  $(x_n)_{n \in \mathbb{N}} \in X$  is such that  $(f(x_n))_{n \in \mathbb{N}}$  is in  $\mathbb{R}$ . Then, from the definition of  $\pi_f$  and  $\tilde{f}$ , both sequences  $(f(x_n))_{n \in \mathbb{N}}$  and  $(\tilde{f}(x_n))_{n \in \mathbb{N}}$  are identical.

Therefore, from what precedes, we have the equivalences  $l = \inf(f(X))$  iff  $l = \inf(\tilde{f}(X))$  iff  $l$  is a lower bound of  $\tilde{f}(X)$ , and there exists a sequence  $(\tilde{x}_n)_{n \in \mathbb{N}} \in X$  such that  $(\tilde{f}(\tilde{x}_n))_{n \in \mathbb{N}}$  is nonincreasing with limit  $l$  iff  $l$  is a lower bound of  $f(X)$ , and there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \mathbb{R}$  is nonincreasing with limit  $l$   $\square$

**Lemma 342 (equivalent definition of infimum).** *Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $l \in \overline{\mathbb{R}}$ . Then, we have  $l = \inf(f(X))$  iff  $l$  is a lower bound of  $f(X)$ , and there exists  $(x_n)_{n \in \mathbb{N}} \in X$  such that  $(f(x_n))_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$  is nonincreasing with limit  $l$ .*

*Proof.* **Case  $\inf(f(X)) = -\infty$ .**

**“Left” implies “right”.** Assume first that  $l = \inf(f(X)) = -\infty$ . Then, from **the definition of  $-\infty$** ,  $l$  is a lower bound of  $f(X)$ . Moreover, from Lemma 11 (*finite infimum, contrapositive*), for all  $m \in \mathbb{R}$ , there exists  $x \in X$  such that  $f(x) < m$ . For all  $n \in \mathbb{N}$ , let  $x'_n \in X$  such that  $f(x'_n) < -n$ . Let  $x_0 \stackrel{\text{def.}}{=} x'_0$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} \stackrel{\text{def.}}{=} \arg \min(f(x_n), f(x'_{n+1}))$ . Then, for all  $n \in \mathbb{N}$ , we have  $f(x_{n+1}) \leq f(x_n) \leq f(x'_n) < -n$ . Thus, from **the definition of monotone sequence**, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is nonincreasing. Moreover, let  $k \in \mathbb{N}$ . Let  $N \stackrel{\text{def.}}{=} k$ . Let  $n \in \mathbb{N}$  such that  $n \geq N$ . Then, from **totally ordered set properties of  $\overline{\mathbb{R}}$** , we have

$$f(x_n) \leq f(x_N) = f(x_k) < -k.$$

Hence, from Lemma 290 (*convergence towards  $-\infty$* ), the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is convergent in  $\overline{\mathbb{R}}$  with limit  $-\infty = l$ .

**“Right” implies “left”.** Conversely, assume now that  $l$  is a lower bound of  $f(X)$ . Then, from Definition 9 (*infimum, greatest lower bound*), and **the definition of  $-\infty$** , we have

$$-\infty \leq l \leq \inf(f(X)) = -\infty.$$

Hence, we have  $l = -\infty = \inf(f(X))$ .

**Case  $\inf(f(X)) = \infty$ .**

Then, from Definition 9 (*infimum, lower bound*), and **the definition of  $\infty$** , we have for all  $x \in X$ ,  $\infty = \inf(f(X)) \leq f(x) \leq \infty$ . Which means that  $f$  is a constant function of value  $\infty$ . Thus,

from **the definition of  $\infty$** , every extended number is a lower bound of  $f(X) = \{\infty\}$ . Moreover, for all sequence  $(x_n)_{n \in \mathbb{N}} \in X$ , the sequence  $(f(x_n))_{n \in \mathbb{N}}$  is stationary of value  $\infty$ . Thus, from **the definition of monotone sequence**, and Lemma 34 (*stationary sequence is convergent*),  $(f(x_n))_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$  is nonincreasing with limit  $\infty$ . For instance, the stationary sequence  $(\tilde{x})_{n \in \mathbb{N}}$  for some  $\tilde{x} \in X \neq \emptyset$  is such a sequence. Hence, we have the equivalence  $l = \infty (= \inf(f(X)))$  iff there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in X$  such that the sequence  $(f(x_n))_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$  is nonincreasing with limit  $l$ .

**Case  $\inf(f(X)) \in \mathbb{R}$ .**

Direct consequence of Lemma 341 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ).

Therefore, we always have the equivalence.  $\square$

**Lemma 343 (*infimum is smaller than supremum*).**

Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $\inf(f(X)) \leq \sup(f(X))$ .

*Proof.* Direct consequence of Definition 9 (*infimum, lower bound*), Definition 2 (*supremum, upper bound*), and Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total, transitivity*).  $\square$

**Lemma 344 (*infimum is monotone*).**

Let  $X$  and  $Y$  be nonempty sets.

Assume that  $Y \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $\inf(f(X)) \leq \inf(f(Y))$ .

*Proof.* Direct consequence of Definition 9 (*infimum, lower bound with  $X$ , then greatest lower bound with  $Y$* ), and Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total, transitivity*).  $\square$

**Lemma 345 (*supremum is monotone*).**

Let  $X$  and  $Y$  be nonempty sets.

Assume that  $Y \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $\sup(f(Y)) \leq \sup(f(X))$ .

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 344 (*infimum is monotone*), and **monotonicity of additive inverse**.  $\square$

**Lemma 346 (*compatibility of infimum with absolute value*).**

Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $|\inf(f(X))| \leq \sup(|f(X)|)$ .

*Proof.* From Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ,  $f \leq |f|$* ), Lemma 344 (*infimum is monotone*), Lemma 343 (*infimum is smaller than supremum*), Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*), and Lemma 345 (*supremum is monotone*), we have

$$\inf(f(X)) \leq \inf |f(X)| \leq \sup |f(X)| \quad \text{and} \quad \sup(-f(X)) \leq \sup |f(X)|.$$

**Case  $\inf(f(X)) \geq 0$ .** Then, from Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ), we have  $|\inf(f(X))| = \inf(f(X)) \leq \sup |f(X)|$ . **Case  $\inf(f(X)) < 0$ .** Then, from Lemma 10 (*duality infimum-supremum*), we have  $|\inf(f(X))| = -\inf(f(X)) = \sup(-f(X)) \leq \sup |f(X)|$ .  $\square$

**Lemma 347 (*compatibility of supremum with absolute value*).**

Let  $X$  be a nonempty set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $|\sup(f(X))| \leq \sup(|f(X)|)$ .

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 346 (*compatibility of infimum with absolute value, with  $(-f_n)_{n \in \mathbb{N}}$* ), and Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*).  $\square$

**Lemma 348 (*compatibility of translation with infimum*).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $p \in \mathbb{N}$ , we have  $\inf_{n \in \mathbb{N}} f_n \leq \inf_{n \in \mathbb{N}} f_{n+p}$ .

*Proof.* Direct consequence of Lemma 344 (*infimum is monotone, with  $[p.. \infty) \times X \subset \mathbb{N} \times X$* ).  $\square$

**Lemma 349 (*compatibility of translation with supremum*).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $p \in \mathbb{N}$ , we have  $\sup_{n \in \mathbb{N}} f_{n+p} \leq \sup_{n \in \mathbb{N}} f_n$ .

*Proof.* Direct consequence of Lemma 345 (*supremum is monotone*, with  $[p..∞) \times X \subset \mathbb{N} \times X$ ).  $\square$

**Lemma 350 (infimum of sequence is monotone).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that for all  $n \in \mathbb{N}$ ,  $f_n \leq g_n$ . Then, we have  $\inf_{n \in \mathbb{N}} f_n \leq \inf_{n \in \mathbb{N}} g_n$ .

*Proof.* Direct consequence of Definition 9 (*infimum*, with  $X \stackrel{\text{def.}}{=} \mathbb{N} \times X$ , lower bound for  $\inf_{n \in \mathbb{N}} f_n$  and greatest lower bound for  $\inf_{n \in \mathbb{N}} g_n$ ), and Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*, *transitivity*).  $\square$

**Lemma 351 (supremum of sequence is monotone).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that for all  $n \in \mathbb{N}$ ,  $f_n \leq g_n$ . Then, we have  $\sup_{n \in \mathbb{N}} f_n \leq \sup_{n \in \mathbb{N}} g_n$ .

*Proof.* Direct consequence of Lemma 10 (*duality infimum-supremum*), Lemma 350 (*infimum of sequence is monotone*, with  $-g_n \leq -f_n$ ), and **monotonicity of additive inverse**.  $\square$

#### 6.4.5 Limit inferior, limit superior

**Lemma 352 (limit inferior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $x \in X$ ,  $(\inf_{p \in \mathbb{N}} f_{n+p}(x))_{n \in \mathbb{N}}$  is nondecreasing, and we have

$$(52) \quad \forall x \in X, \quad \lim_{n \rightarrow \infty} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right) = \sup_{n \in \mathbb{N}} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right) \in \overline{\mathbb{R}}.$$

The limit inferior of the sequence is the function  $X \rightarrow \overline{\mathbb{R}}$  defined by

$$(53) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} \left( \inf_{p \in \mathbb{N}} f_{n+p}(x) \right).$$

*Proof.* Direct consequence of Lemma 344 (*infimum is monotone*), and **completeness of  $\overline{\mathbb{R}}$  (a nondecreasing sequence is convergent and its limit is its least upper bound)**.  $\square$

**Lemma 353 (limit inferior is  $\infty$ ).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $x \in X$ . Assume that  $\liminf_{n \rightarrow \infty} f_n(x) = \infty$ . Then, we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, from **the definition of  $\infty$** , we have  $a < \liminf_{n \rightarrow \infty} f_n(x)$ . Thus, from Lemma 352 (*limit inferior*), and **the definition of the limit**, there exists  $N \in \mathbb{N}$  such that  $a < \inf_{p \in \mathbb{N}} f_{N+p}(x)$ . Hence, from Definition 9 (*infimum*, lower bound), we have for all  $p \in \mathbb{N}$ ,  $a < \inf_{p \in \mathbb{N}} f_{N+p}(x) \leq f_{N+p}(x)$ . Therefore, from **the definition of the limit**, we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$ .  $\square$

**Lemma 354 (equivalent definition of the limit inferior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $x \in X$ ,  $\liminf_{n \rightarrow \infty} f_n(x)$  is the smallest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

*Proof.* Let  $x \in X$ . For all  $n \in \mathbb{N}$ , let  $F_n^-(x) \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p}(x)$ .

Let  $\underline{f}(x) \stackrel{\text{def.}}{=} \liminf_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} F_n^-(x) \in \overline{\mathbb{R}}$ .

Let us first show that  $\underline{f}(x)$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x)$  is finite.** Let  $\varepsilon > 0$ . Let  $M \in \mathbb{N}$ . Then, from Lemma 352 (*limit inferior*,  $(F_n^-(x))_{n \in \mathbb{N}}$  is nondecreasing), and Definition 27 (*convergent sequence*, with  $(F_n^-(x))_{n \in \mathbb{N}}$ ), there exists  $N \in \mathbb{N}$  such that we have

$$\forall k \in [N..∞), \quad \underline{f}(x) - \varepsilon \leq F_k^-(x) \leq \underline{f}(x).$$

Let  $K \stackrel{\text{def.}}{=} \max(M, N)$ . Then, from Lemma 11 (*finite infimum*, for  $F_K^-$ ), there exists  $k \geq K \geq M$  such that

$$\underline{f}(x) - \varepsilon \leq F_K^-(x) \leq f_k(x) < F_K^-(x) + \varepsilon \leq \underline{f}(x) + \varepsilon.$$

Hence, from Definition 247 (*cluster point*),  $\underline{f}(x)$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x) = -\infty$ .** From Lemma 352 (*limit inferior*), and Definition 2 (*supremum*, *upper bound*), we have for all  $n \in \mathbb{N}$ ,  $F_n^-(x) = -\infty$ . Let  $n \in \mathbb{N}$ . Let  $a \in \mathbb{R}$ . Then,  $F_n^-(x) < a$ . Thus, from Definition 9 (*infimum*, *greatest lower bound*, *contrapositive*), there exists  $P \in \mathbb{N}$  such that  $f_{n+P}(x) < a$ . Hence, from **the definition of cluster point in  $-\infty$** ,  $\underline{f}(x) = -\infty$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

**Case  $\underline{f}(x) = \infty$ .** Then, from Lemma 353 (*limit inferior is  $\infty$* ), we have  $\lim_{n \rightarrow \infty} f_n(x) = \infty$ , and from **the fact that the limit of a convergent sequence is its only cluster point**,  $\underline{f}(x) = \infty$  is a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

Now, let  $f(x)$  be a cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ . Let us show that  $\underline{f}(x) \leq f(x)$ .

**Case  $f(x) = \infty$ .** Then, from **the definition of  $\infty$** , we have  $\underline{f}(x) \leq f(x)$ .

**Case  $f(x) < \infty$ .** Let  $a \in \mathbb{R}$  such that  $f(x) < a$ . Let  $\varepsilon \stackrel{\text{def.}}{=} \frac{a-f(x)}{2} > 0$ . Let  $n \in \mathbb{N}$ . Then, from Definition 247 (*cluster point*), there exists  $k \geq n$  such that  $f_k(x) > f(x) + \varepsilon < a$ . Thus, from Definition 9 (*infimum*, *lower bound*),  $F_n^-(x) < a$ . Hence, from **monotonicity of the limit**, and Lemma 352 (*limit inferior*), we have  $\underline{f}(x) = \lim_{n \rightarrow \infty} F_n^-(x) \leq a$ . Since this is true for all  $a > f(x)$ , we also have  $\underline{f}(x) \leq f(x)$ .

Therefore, for all  $x \in X$ ,  $\underline{f}(x)$  is the smallest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .  $\square$

**Lemma 355 (*limit inferior is invariant by translation*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \mathbb{R}$ . Then, we have

$$(54) \quad \forall k \in \mathbb{N}, \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_{k+n}(x) = \liminf_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 354 (*equivalent definition of the limit inferior*), Definition 247 (*cluster point*), and **compatibility of translation with limit**.  $\square$

**Lemma 356 (*limit inferior is monotone*).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \mathbb{R}$ . Assume that  $(f_n)_{n \in \mathbb{N}} \leq (g_n)_{n \in \mathbb{N}}$  from some rank:

$$(55) \quad \exists N \in \mathbb{N}, \forall n \in [N, \infty), \forall x \in X, \quad f_n(x) \leq g_n(x).$$

Then, we have

$$(56) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq \liminf_{n \rightarrow \infty} g_n(x).$$

*Proof.* Direct consequence of Lemma 355 (*limit inferior is invariant by translation*, with  $k \stackrel{\text{def.}}{=} N$ ), Lemma 352 (*limit inferior*), Lemma 350 (*infimum of sequence is monotone*), and Lemma 351 (*supremum of sequence is monotone*).  $\square$

**Lemma 357 (*limit superior*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ .

Then, for all  $x \in X$ ,  $(\sup_{p \in \mathbb{N}} f_{n+p}(x))_{n \in \mathbb{N}}$  is nonincreasing, and we have

$$(57) \quad \forall x \in X, \quad \lim_{n \rightarrow \infty} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right) = \inf_{n \in \mathbb{N}} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right) \quad \text{in } \overline{\mathbb{R}}.$$

The limit superior of the sequence is the function  $X \rightarrow \overline{\mathbb{R}}$  defined by

$$(58) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} \lim_{n \in \mathbb{N}} \left( \sup_{p \in \mathbb{N}} f_{n+p}(x) \right).$$



*Proof.* Let  $x \in X$ . For all  $n \in \mathbb{N}$ , let  $F_n(x) \stackrel{\text{def.}}{=} \{f_{n+p}(x) \mid p \in \mathbb{N}\}$ . Then, we have

$$\forall n \in \mathbb{N}, \quad \sup F_n(x) = \sup_{p \in \mathbb{N}} f_{n+p}(x) \quad \wedge \quad F_n(x) \subset F_{n+1}(x) \subset \overline{\mathbb{R}}.$$

Thus, the sequence  $(\sup F_n(x))_{n \in \mathbb{N}}$  is nonincreasing in  $\overline{\mathbb{R}}$ . Therefore, from **completeness of  $\overline{\mathbb{R}}$** , the sequence is convergent and its limit is its greatest lower bound  $\inf_{n \in \mathbb{N}}(\sup F_n(x))$ .  $\square$

**Lemma 358 (duality limit inferior-limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(59) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) = - \liminf_{n \rightarrow \infty} (-f_n(x)).$$

*Proof.* Direct consequence of Lemma 357 (*limit superior*), Lemma 352 (*limit inferior*), **linearity and uniqueness of the limit**, and Lemma 10 (*duality infimum-supremum*, with  $X \stackrel{\text{def.}}{=} \mathbb{N}$ ).  $\square$

**Lemma 359 (equivalent definition of limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, for all  $x \in X$ ,  $\limsup_{n \rightarrow \infty} f_n(x)$  is the largest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .

*Proof.* Let  $x \in X$ . Let  $\bar{f}(x) \stackrel{\text{def.}}{=} \limsup_{n \rightarrow \infty} f_n(x)$ . Then, from Lemma 358 (*duality limit inferior-limit superior*), and Lemma 354 (*equivalent definition of the limit inferior*),  $-\bar{f}(x)$  is the smallest cluster point of the sequence  $(-f_n(x))_{n \in \mathbb{N}}$ . Therefore, from **linearity of the limit**, and **totally ordered set properties of  $\overline{\mathbb{R}}$** ,  $\bar{f}(x)$  is the largest cluster point of the sequence  $(f_n(x))_{n \in \mathbb{N}}$ .  $\square$

**Lemma 360 (limit inferior is smaller than limit superior).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Then, we have

$$(60) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq \limsup_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 354 (*equivalent definition of the limit inferior*), Lemma 359 (*equivalent definition of limit superior*), and Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*) transitivity.  $\square$

**Lemma 361 (limit superior is monotone).**

Let  $X$  be a nonempty set.

Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $(f_n)_{n \in \mathbb{N}} \leq (g_n)_{n \in \mathbb{N}}$  from some rank:

$$(61) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq g_n(x).$$

Then, we have

$$(62) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq \limsup_{n \rightarrow \infty} g_n(x).$$

*Proof.* Direct consequence of **monotonicity of additive inverse in  $\overline{\mathbb{R}}$** , Lemma 356 (*limit inferior is monotone*, with  $-g_n \leq -f_n$ ), and Lemma 358 (*duality limit inferior-limit superior*).  $\square$

**Lemma 362 (compatibility of limit inferior with absolute value).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}_+$ . Then, we have

$$(63) \quad \forall x \in X, \quad \left| \liminf_{n \rightarrow \infty} f_n(x) \right| \leq \limsup_{n \rightarrow \infty} |f_n(x)|.$$

*Proof.* From Lemma 346 (*compatibility of infimum with absolute value*, with  $X \stackrel{\text{def.}}{=} [n.. \infty) \times X$ ), for all  $x \in X$ , for all  $n \in \mathbb{N}$ , we have

$$\left| \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| \leq \sup_{p \in \mathbb{N}} |f_{n+p}(x)|.$$

Therefore, from Lemma 352 (*limit inferior*), **compatibility of the limit with the absolute value, monotonicity of the limit (when  $n \rightarrow \infty$ )**, and Lemma 357 (*limit superior*), we have

$$\left| \liminf_{n \rightarrow \infty} f_n(x) \right| = \left| \lim_{n \rightarrow \infty} \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| = \lim_{n \rightarrow \infty} \left| \inf_{p \in \mathbb{N}} f_{n+p}(x) \right| \leq \lim_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} |f_{n+p}(x)| = \limsup_{n \rightarrow \infty} |f_n(x)|.$$

□

**Lemma 363 (compatibility of limit superior with absolute value).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}_+$ . Then, we have

$$(64) \quad \forall x \in X, \quad \left| \limsup_{n \rightarrow \infty} f_n(x) \right| \leq \limsup_{n \rightarrow \infty} |f_n(x)|.$$

*Proof.* Direct consequence of Lemma 358 (*duality limit inferior-limit superior*), Lemma 362 (*compatibility of limit inferior with absolute value, with  $-f_n$* ), and Lemma 279 (*absolute value in  $\mathbb{R}$  is even*). □

**Definition 364 (pointwise convergence).**

Let  $X$  be a nonempty set. A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions  $X \rightarrow \overline{\mathbb{R}}$  is said *pointwise convergent* iff for all  $x \in X$ ,  $(f_n(x))_{n \in \mathbb{N}}$  is convergent in  $\overline{\mathbb{R}}$ .

**Lemma 365 (limit inferior and limit superior of pointwise convergent).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that the sequence is pointwise convergent. Then, we have

$$(65) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Definition 364 (*pointwise convergence*), Lemma 354 (*equivalent definition of the limit inferior*), Lemma 359 (*equivalent definition of limit superior*), and **the fact that a convergent sequence has its limit as the unique cluster point**. □

**Lemma 366 (limit inferior bounded from below).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $m \in \overline{\mathbb{R}}$ . Assume that the sequence is bounded by  $m$  from below from some rank:

$$(66) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad m \leq f_n(x).$$

Then, we have

$$(67) \quad \forall x \in X, \quad m \leq \liminf_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 356 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence, constant sequence is stationary ( $N = 0$ )*), and Lemma 365 (*limit inferior and limit superior of pointwise convergent, with the constant sequence of value  $m$* ). □

**Lemma 367 (limit inferior bounded from above).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $M \in \overline{\mathbb{R}}$ . Assume that the sequence is bounded by  $M$  from above from some rank:

$$(68) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq M.$$

Then, we have

$$(69) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) \leq M.$$



*Proof.* Direct consequence of Lemma 356 (*limit inferior is monotone*), Lemma 34 (*stationary sequence is convergent*), Definition 33 (*stationary sequence, constant sequence is stationary ( $N = 0$ )*), and Lemma 365 (*limit inferior and limit superior of pointwise convergent, with the constant sequence of value  $M$* ).  $\square$

**Lemma 368 (*limit superior bounded from below*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $m \in \overline{\mathbb{R}}$ . Assume that the sequence is bounded by  $m$  from below from some rank:

$$(70) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad m \leq f_n(x).$$

Then, we have

$$(71) \quad \forall x \in X, \quad m \leq \limsup_{n \rightarrow \infty} f_n(x).$$

*Proof.* Direct consequence of Lemma 367 (*limit inferior bounded from above, with  $f_n \stackrel{\text{def.}}{=} -f_n$  and  $M \stackrel{\text{def.}}{=} -m$* ), and Lemma 358 (*duality limit inferior-limit superior*).  $\square$

**Lemma 369 (*limit superior bounded from above*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Let  $M \in \overline{\mathbb{R}}$ . Assume that the sequence is bounded by  $M$  from above from some rank:

$$(72) \quad \exists N \in \mathbb{N}, \forall n \in [N.. \infty), \forall x \in X, \quad f_n(x) \leq M.$$

Then, we have

$$(73) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq M.$$

*Proof.* Direct consequence of Lemma 366 (*limit inferior bounded from below, with  $f_n \stackrel{\text{def.}}{=} -f_n$  and  $m \stackrel{\text{def.}}{=} -M$* ), and Lemma 358 (*duality limit inferior-limit superior*).  $\square$

**Lemma 370 (*limit inferior, limit superior and pointwise convergence*).**

Let  $X$  be a nonempty set. Let  $(f_n)_{n \in \mathbb{N}} : X \rightarrow \overline{\mathbb{R}}$ . Assume that

$$(74) \quad \forall x \in X, \quad \limsup_{n \rightarrow \infty} f_n(x) \leq \liminf_{n \rightarrow \infty} f_n(x).$$

Then, the sequence is pointwise convergent and

$$(75) \quad \forall x \in X, \quad \liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(x).$$

*Proof.* Let  $x \in X$ . Then, from Lemma 360 (*limit inferior is smaller than limit superior*), we have  $\liminf_{n \rightarrow \infty} f_n(x) = \limsup_{n \rightarrow \infty} f_n(x) \stackrel{\text{def.}}{=} f(x) \in \overline{\mathbb{R}}$ .

**Case  $f(x) = \infty$ .** Then, from Lemma 353 (*limit inferior is  $\infty$* ), we have

$$\lim_{n \rightarrow \infty} f_n(x) = \infty = f(x).$$

**Case  $f(x) = -\infty$ .** Then, from Lemma 358 (*duality limit inferior-limit superior*), and Lemma 353 (*limit inferior is  $\infty$* ), we have  $\lim_{n \rightarrow \infty} f_n(x) = -\infty = f(x)$ .

**Case  $f(x) \in \mathbb{R}$ .** Let  $\varepsilon > 0$ . Then, we have

$$f(x) - \varepsilon < \liminf_{n \rightarrow \infty} f_n(x) = f(x) = \limsup_{n \rightarrow \infty} f_n(x) < f(x) + \varepsilon.$$

Thus, from Lemma 354 (*equivalent definition of the limit inferior*), there exists  $N^- \in \mathbb{N}$  such that for all  $n \geq N^-$ , we have  $f(x) - \varepsilon < f_n(x)$ , and from Lemma 359 (*equivalent definition of limit superior*), there exists  $N^+ \in \mathbb{N}$  such that for all  $n \geq N^+$ , we have  $f_n(x) < f(x) + \varepsilon$ .

Let  $N \stackrel{\text{def.}}{=} \max(N^-, N^+)$ . Then, for all  $n \geq N$ , we have  $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$ . Therefore, from Definition 27 (*convergent sequence*), the sequence  $(f_n(x))_{n \in \mathbb{N}}$  is convergent with limit  $f(x)$ .  $\square$

#### 6.4.6 Truncating a function

##### Definition 371 (*finite part*).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . The *finite part* of  $f$  is the function  $f \mathbb{1}_{f^{-1}(\mathbb{R})}$ .

##### Lemma 372 (*finite part is finite*).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, the finite part of  $f$  is finite.

*Proof.* Direct consequence of Definition 371 (*finite part*), Definition 254 (*extended real numbers*,  $\overline{\mathbb{R}}, \overline{\mathbb{R}} = \{\pm\infty\} \uplus \mathbb{R}$ ), **properties of inverse image** ( $X = f^{-1}(\overline{\mathbb{R}}) = f^{-1}(\pm\infty) \uplus f^{-1}(\mathbb{R})$ ), and **the definition of the indicator function**.  $\square$

##### Definition 373 (*nonnegative and nonpositive parts*).

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . The functions  $f^+ \stackrel{\text{def.}}{=} \max(f, 0)$  and  $f^- \stackrel{\text{def.}}{=} \max(-f, 0)$  are respectively called the *nonnegative part* of  $f$  and the *nonpositive part* of  $f$ .

##### Lemma 374 (*equivalent definition of nonnegative and nonpositive parts*).

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f^+ = f \mathbb{1}_{f^{-1}(\overline{\mathbb{R}}_+)}$  and  $f^- = -f \mathbb{1}_{f^{-1}(\overline{\mathbb{R}}_-)}$ .

*Proof.* Direct consequence of Definition 373 (*nonnegative and nonpositive parts*), **the definition of the maximum**, and **the definition of the indicator function**.  $\square$

##### Lemma 375 (*nonnegative and nonpositive parts are nonnegative*).

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f^+, f^- \geq 0$ .

*Proof.* Direct consequence of Definition 373 (*nonnegative and nonpositive parts*), and **the definition of the maximum**.  $\square$

##### Lemma 376 (*nonnegative and nonpositive parts are orthogonal*).

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Let  $x \in X$ . Then, we have

$$(76) \quad f^+(x) = 0 \text{ (and } f^-(x) = -f(x)) \quad \vee \quad f^-(x) = 0 \text{ (and } f^+(x) = f(x)).$$

*Proof.* Direct consequence of **the definition of the maximum**, **the partition of extended real numbers into negative, zero, and positive extended numbers**.  $\square$

##### Lemma 377 (*decomposition into nonnegative and nonpositive parts*).

Let  $X$  be a set. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ .

*Proof.* Direct consequence of Lemma 376 (*nonnegative and nonpositive parts are orthogonal*,  $f^+$  and  $f^-$  cannot take value  $\infty$  at the same point), Definition 258 (*addition in  $\overline{\mathbb{R}}$* ), and Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).  $\square$

##### Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*).

Let  $X$  be a set. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  such that their sum is well-defined. Then, we have

$$(77) \quad (f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+.$$

*Proof.* Let  $x \in X$ .

**Case  $f(x)$  and  $g(x)$  finite.** From Lemma 377 (*decomposition into nonnegative and nonpositive parts*, with  $f + g$ , then  $f$  and  $g$ ), we have

$$(f + g)^+(x) - (f + g)^-(x) = f(x) + g(x) = (f^+(x) - f^-(x)) + (g^+(x) - g^-(x)).$$

Then, from Definition 373 (*nonnegative and nonpositive parts*),  $(f + g)^+(x)$ ,  $(f + g)^-(x)$ ,  $f^+(x)$ ,  $f^-(x)$ ,  $g^+(x)$  and  $g^-(x)$ , are finite. Hence, from **abelian group properties of  $\mathbb{R}$** , we have

$$(f + g)^+(x) + f^-(x) + g^-(x) = (f + g)^-(x) + f^+(x) + g^+(x).$$

**Case  $f(x)$  or  $g(x)$  is  $\infty$ .** Then, from Definition 258 (*addition in  $\overline{\mathbb{R}}$* , rule 3 cannot occur), and Lemma 376 (*nonnegative and nonpositive parts are orthogonal*), we have

$$(f + g)^+(x) = f(x) + g(x) = \infty \quad \text{and} \quad (f^+(x) = \infty \quad \vee \quad g^+(x) = \infty).$$

Thus, from Lemma 375 (*nonnegative and nonpositive parts are nonnegative*), Lemma 297 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ), Lemma 295 (*addition in  $\overline{\mathbb{R}}_+$  is associative*), and Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*), we have

$$(f + g)^+(x) + f^-(x) + g^-(x) = \infty \quad \text{and} \quad (f + g)^-(x) + f^+(x) + g^+(x) = \infty.$$

Hence, identity (77) is satisfied.

**Case  $f(x)$  or  $g(x)$  is  $-\infty$ .** Same reasoning with the nonpositive parts.

Therefore, we always have the identity. □

**Lemma 379 (compatibility of nonpositive and nonnegative parts with mask).**

Let  $X$  be a set. Let  $A \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $(f \mathbb{1}_A)^\pm = f^\pm \mathbb{1}_A$

*Proof.* Direct consequence of Definition 373 (*nonnegative and nonpositive parts*), and **nonnegativeness of the indicator function**. □

**Lemma 380 (compatibility of nonpositive and nonnegative parts with restriction).**

Let  $X$  be a set. Let  $A \subset X$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $(f|_A)^\pm = f^\pm|_A$ .

*Proof.* Direct consequence of Definition 373 (*nonnegative and nonpositive parts*), and **compatibility of min/max with restriction of function**. □

## 6.5 Complements on algebraic structures

### 6.5.1 Vector space

**Definition 381 (relation compatible with vector operations).** Let  $(E, +_E, \cdot_E)$  be a vector space. An equivalence relation  $\mathcal{R}$  on  $E$  is said *compatible with the vector operations* iff

$$(78) \quad \forall u, u', v, v' \in E, \quad u \mathcal{R} u' \wedge v \mathcal{R} v' \Rightarrow (u +_E v) \mathcal{R} (u' +_E v'),$$

$$(79) \quad \forall \lambda \in \mathbb{K}, \forall u, u' \in E, \quad u \mathcal{R} u' \Rightarrow (\lambda \cdot_E u) \mathcal{R} (\lambda \cdot_E u').$$

**Lemma 382 (quotient vector operations).**

Let  $(E, +_E, \cdot_E)$  be a vector space. Let  $\mathcal{R}$  be an equivalence relation on  $E$ . Assume that  $\mathcal{R}$  is compatible with the vector operations. Then, the mappings  $[+] : E/\mathcal{R} \times E/\mathcal{R} \rightarrow E/\mathcal{R}$  and  $[\cdot] : \mathbb{K} \times E/\mathcal{R} \rightarrow E/\mathcal{R}$  defined by

$$(80) \quad \forall u, v \in E, \quad [u] [+] [v] \stackrel{\text{def.}}{=} [u +_E v],$$

$$(81) \quad \forall \lambda \in \mathbb{K}, \forall u \in E, \quad \lambda [\cdot] [u] \stackrel{\text{def.}}{=} [\lambda \cdot_E u].$$

are well-defined. These mappings are called quotient vector operations induced on  $E/\mathcal{R}$ .

*Proof.* Let  $u, v \in E$ . Let  $\lambda \in \mathbb{K}$ . Let  $u', v' \in E$  such that  $u \mathcal{R} u'$  and  $v \mathcal{R} v'$ , i.e. such that  $[u'] = [u]$  and  $[v'] = [v]$ . Then, from Definition 381 (*relation compatible with vector operations*), we have  $(u +_E v) \mathcal{R} (u' +_E v')$  and  $(\lambda \cdot_E u) \mathcal{R} (\lambda \cdot_E u')$ , i.e.  $[u' +_E v'] = [u +_E v]$  and  $[\lambda \cdot_E u'] = [\lambda \cdot_E u]$ . Therefore, the quotient vector operations  $[+]$  and  $[\cdot]$  defined by (80) and (81) do not depend on the choice of the representative of classes, i.e. they are well-defined.  $\square$

**Lemma 383 (quotient vector space, equivalence relation).**

Let  $(E, +_E, \cdot_E)$  be a vector space. Let  $\mathcal{R}$  be an equivalence relation on  $E$ . Assume that  $\mathcal{R}$  is compatible with the vector operations. Let  $[+]$  and  $[\cdot]$  be the quotient vector operations induced on the quotient set  $E/\mathcal{R}$ . Then,  $(E/\mathcal{R}, [+], [\cdot])$  is a vector space.

*Proof.* From **group theory**,  $(E/\mathcal{R}, [+])$  is an abelian group with identity element  $[0_E]$ . Distributivity of the quotient scalar multiplication over quotient vector addition and field addition, compatibility of the quotient scalar multiplication with field multiplication, and 1 is the identity element for the quotient scalar multiplication are direct consequences of Definition 61 (*vector space*), and Lemma 382 (*quotient vector operations*). Therefore, from Definition 61 (*vector space*),  $(E/\mathcal{R}, [+], [\cdot])$  is a vector space.  $\square$

**Lemma 384 (quotient vector space).** Let  $E$  be a vector space. Let  $F$  be a vector subspace of  $E$ . Let  $\mathcal{R}$  be the relation defined on  $E$  by for all  $u, v \in E$ ,  $u \mathcal{R} v \Leftrightarrow v - u \in F$ .

Then,  $\mathcal{R}$  is an equivalence relation compatible with the vector operations of  $E$ .

The quotient set  $E/\mathcal{R}$  equipped with the quotient vector operations is thus a vector space called quotient vector space of  $E$  by  $F$ ; it is denoted  $E/F$ . For all  $u \in E$ , the class of  $u$  is denoted  $u + F$ . The quotient vector operations are still denoted  $+$  and  $\cdot$  (the latter may be omitted).

*Proof.* Let  $u, u', v, v' \in E$ . Let  $\lambda \in \mathbb{K}$ . Assume that  $u \mathcal{R} u'$  and  $v \mathcal{R} v'$ , i.e.  $u' - u, v' - v \in F$ . Then, from Definition 61 (*vector space, vector addition is associative, and scalar multiplication is distributive over field addition*), and Lemma 81 (*closed under vector operations is subspace*), we have  $(u' + v') - (u + v) = (u' - u) + (v' - v) \in F$  and  $(\lambda u') - (\lambda u) = \lambda(u' - u) \in F$ , i.e.  $(u + v) \mathcal{R} (u' + v')$  and  $(\lambda u) \mathcal{R} (\lambda u')$ . Hence, from Definition 381 (*relation compatible with vector operations*), the equivalence relation  $\mathcal{R}$  is compatible with the vector operations. Therefore, from Lemma 383 (*quotient vector space, equivalence relation*), the quotient set  $E/\mathcal{R}$  equipped with the induced quotient vector operations is a vector space.  $\square$

**Lemma 385 (linear map on quotient vector space).**

Let  $E, F$  be vector spaces.

Let  $G$  be a vector subspace of  $E$ . Let  $f$  be a linear map from  $E$  to  $F$ . Assume that  $G \subset \ker(f)$ .

Then, the function  $[f] \stackrel{\text{def.}}{=} (u + G \mapsto f(u))$  is a linear map from  $E/G$  to  $F$ .

*Proof.* Let  $u \in E$ . Let  $u' \in u + G$ . Then, from hypotheses, we have  $u' - u \in G \subset \ker(f)$ . Thus, from Definition 64 (*linear map*), and Definition 101 (*kernel*), we have  $f(u') = f(u)$ . Hence, the function  $[f]$  does not depend on the choice of the representative of equivalence classes, it is well-defined.

Let  $(u + G), (v + G) \in E/G$ . Let  $\lambda \in \mathbb{K}$ . Then, from Lemma 384 (*quotient vector space*), Lemma 382 (*quotient vector operations*), and Definition 64 (*linear map*), we have

$$\begin{aligned} [f](a(u + G)) &= [f]((au) + G) = f(au) = af(u) = a[f](u + G), \\ [f]((u + G) + (v + G)) &= [f]((u + v) + G) = f(u + v) = f(u) + f(v) = [f](u + G) + [f](v + G). \end{aligned}$$

Therefore, from Definition 64 (*linear map*),  $[f]$  is a linear map from  $E/G$  to  $F$ .  $\square$

### 6.5.2 Algebra over a field

*Remark 386.* The following algebraic structure “algebra over a field” is not to be confused with the concept of “set algebra” defined in Section 6.6.3.

*Remark 387.* Most results on algebras over a field are valid on algebras over a ring, for which the property of vector space over the field is replaced by that of module over the ring.

**Definition 388 (algebra over a field).**

Let  $\mathbb{K}$  be a field. A set  $E$  equipped with three algebra operations (a vector addition  $+$ , a scalar multiplication  $\cdot$ , and a vector multiplication  $\times$ ), is called *algebra (over field  $\mathbb{K}$ )*, or  $\mathbb{K}$ -algebra, iff  $(E, +, \cdot)$  is a  $\mathbb{K}$ -vector space, and vector multiplication is bilinear (or left and right distributive over vector addition, and compatible with scalars):

$$(82) \quad \forall u, v, w \in E, \quad (u + v) \times w = (u \times w) + (v \times w),$$

$$(83) \quad \forall u, v, w \in E, \quad u \times (v + w) = (u \times v) + (u \times w),$$

$$(84) \quad \forall \lambda, \mu \in \mathbb{K}, \forall u, v \in E, \quad (\lambda \cdot u) \times (\mu \cdot v) = (\lambda\mu) \cdot (u \times v).$$

*Remark 389.* The  $\cdot$  and  $\times$  infix signs in the scalar and vector multiplications may be omitted.

**Lemma 390 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra).**

Let  $\mathbb{K}$  be a field. It is an algebra over itself.

*Proof.* Direct consequence of Definition 388 (algebra over a field), and field properties of  $\mathbb{K}$ .  $\square$

**Definition 391 (inherited algebra operations).**

Let  $X$  be a nonempty set. Let  $\mathbb{K}$  be a field. Let  $(E, +_E, \cdot_E, \times_E)$  be an algebra over field  $\mathbb{K}$ . The algebra operations inherited on  $E^X$  are the mappings  $+_{E^X}$  and  $\cdot_{E^X}$  of Definition 91 (inherited vector operations), and the mapping  $\times_{E^X} : E^X \times E^X \rightarrow E^X$  defined by

$$(85) \quad \forall f, g \in E^X, \forall x \in X, \quad (f \times_{E^X} g)(x) \stackrel{\text{def.}}{=} f(x) \times_E g(x).$$

*Remark 392.* Usually, inherited algebra operations are denoted the same way as the algebra operations of the target algebra.

**Lemma 393 (algebra of functions to algebra).**

Let  $\mathbb{K}$  be a field. Let  $X$  be a nonempty set. Let  $(E, +_E, \cdot_E, \times_E)$  be a  $\mathbb{K}$ -algebra. Let  $+_{E^X}, \cdot_{E^X}$  and  $\times_{E^X}$  be the algebra operations inherited on  $E^X$ . Then,  $(E^X, +_{E^X}, \cdot_{E^X}, \times_{E^X})$  is a  $\mathbb{K}$ -algebra.

*Proof.* From Lemma 93 (space of functions to space),  $(E^X, +_{E^X}, \cdot_{E^X})$  is a  $\mathbb{K}$ -vector space. Then, bilinearity of the inherited vector multiplication is a direct consequence of Definition 388 (algebra over a field), and Definition 391 (inherited algebra operations). Therefore, from Definition 388 (algebra over a field),  $E^X$  equipped with  $+_{E^X}, \cdot_{E^X}$  and  $\times_{E^X}$  is a  $\mathbb{K}$ -algebra.  $\square$

**Lemma 394 ( $\mathbb{K}^X$  is algebra).**

Let  $\mathbb{K}$  be a field. Let  $X$  be a nonempty set. Then,  $\mathbb{K}^X$  is a  $\mathbb{K}$ -algebra.

*Proof.* Direct consequence of Lemma 390 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra), and Lemma 393 (algebra of functions to algebra).  $\square$

**Definition 395 (subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $(E, +, \cdot, \times)$  be a  $\mathbb{K}$ -algebra. A subset  $F$  of  $E$  equipped with the restrictions  $+|_F, \cdot|_F$  and  $\times|_F$  of the algebra operations to  $F$  is called ( $\mathbb{K}$ -)subalgebra of  $E$  iff  $(F, +|_F, \cdot|_F, \times|_F)$  is a  $\mathbb{K}$ -algebra.

*Remark 396.* Usually, restrictions  $+|_F, \cdot|_F$ , and  $\times|_F$  are still denoted  $+, \cdot$  and  $\times$ .

**Lemma 397 (vector subspace and closed under multiplication is subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $(E, +, \cdot, \times)$  be a  $\mathbb{K}$ -algebra. Let  $F \subset E$ . Then,  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$  iff  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and  $F$  is closed under vector multiplication:

$$(86) \quad \forall u, v \in F, \quad u \times v \in F.$$

*Proof.* “Left” implies “right”. Assume first that  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$ . Then, from Definition 395 (*subalgebra*,  $F$  is a  $\mathbb{K}$ -algebra), Definition 388 (*algebra over a field*),  $(F, +|_F, \cdot|_F)$  is a  $\mathbb{K}$ -vector space, and  $F$  is closed under the restriction to  $F$  of the three operations. Thus, from Definition 77 (*subspace*),  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and  $F$  is closed under vector multiplication.

“Right” implies “left”. Conversely, assume now that  $F$  is a  $\mathbb{K}$ -vector subspace of  $E$ , and that it is closed under vector multiplication. Then, from Definition 77 (*subspace*),  $(F, +|_F, \cdot|_F)$  is a  $\mathbb{K}$ -vector space. Moreover, from Lemma 81 (*closed under vector operations is subspace*),  $F$  is also closed under vector addition and scalar multiplication. Thus, since  $F$  is a subset of  $E$ , and  $E$  is a  $\mathbb{K}$ -algebra, properties (82) to (84) are trivially satisfied over  $F$  with the restrictions of the three operations. Hence, from Definition 388 (*algebra over a field*), and Definition 395 (*subalgebra*),  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$ .

Therefore, we have the equivalence.  $\square$

**Lemma 398 (closed under algebra operations is subalgebra).**

Let  $\mathbb{K}$  be a field. Let  $E$  be a  $\mathbb{K}$ -algebra. Let  $F \subset E$ . Then,  $F$  is a  $\mathbb{K}$ -subalgebra of  $E$  iff  $0_E \in F$ , and  $F$  is closed under vector addition, scalar and vector multiplications:

$$(87) \quad \forall u, v \in F, \quad u + v \in F,$$

$$(88) \quad \forall \lambda \in \mathbb{K}, \forall u \in F, \quad \lambda u \in F,$$

$$(89) \quad \forall u, v \in F, \quad u \times v \in F.$$

*Proof.* Direct consequence of Lemma 397 (*vector subspace and closed under multiplication is subalgebra*), and Lemma 81 (*closed under vector operations is subspace*).  $\square$

### 6.5.3 Seminormed vector space

**Definition 399 (seminorm).**

Let  $\mathbb{K}$  be a field. Let  $E$  be a  $\mathbb{K}$ -vector space. A function  $\|\cdot\| : E \rightarrow \mathbb{R}$  is called *seminorm over  $E$*  iff it is absolutely homogeneous of degree 1, and it satisfies the triangle inequality:

$$(90) \quad \forall \lambda \in \mathbb{K}, \forall u \in E, \quad \|\lambda u\| = |\lambda| \|u\|,$$

$$(91) \quad \forall u, v \in E, \quad \|u + v\| \leq \|u\| + \|v\|.$$

If so,  $(E, \|\cdot\|)$  (or simply  $E$ ) is called *seminormed ( $\mathbb{K}$ -)vector space*.

*Remark 400.* Most results from [17] on normed vector spaces can be generalized to the case of seminormed vector space, sometimes with slight modifications of the statement. In particular, the associated distance becomes a pseudometric.

**Lemma 401 (definite seminorm is norm).** Let  $(E, \|\cdot\|)$  be a seminormed vector space. Then,  $(E, \|\cdot\|)$  is a normed vector space iff  $\|\cdot\|$  is definite, i.e. for all  $u \in E$ ,  $\|u\| = 0 \Leftrightarrow u = 0$ .

*Proof.* Direct consequence of Definition 399 (*seminorm*), Definition 106 (*normed vector space*), and Definition 105 (*norm*).  $\square$

## 6.6 Measurable space

*Remark 402.* We recall that  $\mathcal{P}(X)$  denotes the power set of  $X$ .

### 6.6.1 Subset systems

*Remark 403.* Subset systems of a set  $X$  are subsets of  $\mathcal{P}(X)$ .

**Lemma 404 (nonempty and with empty or full).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . If  $\mathcal{S}$  contains either the empty set or the full set, then it is nonempty. Assume that  $\mathcal{S}$  is nonempty and closed under complement. Then, it contains the empty set if it is closed under intersection, and it contains the full set if it is closed under union.*

*Proof.* Direct consequence of **the identities  $A \cap A^c = \emptyset$ , and  $A \cup A^c = X$ .** □

**Lemma 405 (with empty and full).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then, it contains the empty set iff it contains the full set.*

*Proof.* Direct consequence of **the identities  $\emptyset^c = X$ , and  $X^c = \emptyset$ .** □

*Remark 406.*

The local complement is the set difference when the first operand contains the second one.

**Lemma 407 (closedness under local complement and complement).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set and is closed under local complement. Then, it is closed under complement.*

*Proof.* Direct consequence of **the identity  $A^c = X \setminus A$ .** □

**Lemma 408 (closedness under disjoint union and local complement).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then,  $\mathcal{S}$  is closed under disjoint union iff it is closed under local complement.*

*Proof.* Direct consequence of **the identity  $A \setminus B = (A^c \cup B)^c$ , monotonicity of complement (then  $A \cap B = \emptyset \Leftrightarrow B \subset A^c$ ), and the identity  $A \cup B = (A^c \setminus B)^c$ .** □

**Lemma 409 (closedness under set difference and local complement).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .*

*If  $\mathcal{S}$  is closed under set difference, then it is closed under local complement.*

*If  $\mathcal{S}$  is closed under intersection and local complement, then it is closed under set difference.*

*Proof.* Direct consequence of **the identity  $A \setminus B = A \setminus (A \cap B)$  with  $A \cap B \subset A$ .** □

**Lemma 410 (closedness under intersection and set difference).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .*

*If  $\mathcal{S}$  is closed under complement and intersection, then it is closed under set difference.*

*If  $\mathcal{S}$  is closed under set difference, then it is closed under intersection.*

*Proof.* Direct consequence of **the identity  $A \cap B = A \setminus (A \setminus B)$ .** □

**Lemma 411 (closedness under union and intersection).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement.*

*Then,  $\mathcal{S}$  is closed under union iff it is closed under intersection.*

*Proof.* Direct consequence of **De Morgan's laws.** □

**Lemma 412 (closedness under union and set difference).**

*Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement.*

*Then,  $\mathcal{S}$  is closed under union iff it is closed under set difference.*



*Proof.* Direct consequence of Lemma 411 (*closedness under union and intersection*), and Lemma 410 (*closedness under intersection and set difference*).  $\square$

**Lemma 413 (*closedness under finite operations*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ .  $\mathcal{S}$  is closed under finite intersection iff it is closed under intersection.  $\mathcal{S}$  is closed under finite union iff it is closed under union.  $\mathcal{S}$  is closed under finite disjoint union iff it is closed under disjoint union.*

*Proof.* Direct consequence of **induction on the number of operands**.  $\square$

**Lemma 414 (*closedness under finite union and intersection*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then,  $\mathcal{S}$  is closed under finite union iff it is closed under finite intersection, and it is closed under finite monotone union iff it is closed under finite monotone intersection.*

*Proof.* Direct consequence of Lemma 413 (*closedness under finite operations*), Lemma 411 (*closedness under union and intersection*), **monotonicity of complement**, and **De Morgan's laws**.  $\square$

*Remark 415.* Note that obviously, closedness under (finite or countable) union implies closedness under (finite or countable) disjoint union. In the same way, closedness under finite or countable intersection (resp. union) implies closedness under finite or countable monotone intersection (resp. union).

*Remark 416.* Note that closedness under a countable subset operation actually means that the subset system is closed under this operation with at most a countable number of operands, i.e. it is also valid for a finite number (e.g. two). Except for countable disjoint union (see next lemma).

**Lemma 417 (*closedness under countable and finite disjoint union*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the empty set and is closed under countable disjoint union. Then, it is closed under finite disjoint union.*

*Proof.* Direct consequence of **extension of finite disjoint union into countable disjoint union using the empty set**.  $\square$

**Lemma 418 (*closedness under countable disjoint union and local complement*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  contains the full set and is closed under complement and countable disjoint union. Then, it is closed under local complement.*

*Proof.* Direct consequence of Lemma 405 (*with empty and full,  $\mathcal{S}$  contains the empty set*), Lemma 417 (*closedness under countable and finite disjoint union,  $\mathcal{S}$  closed under finite disjoint union*), Lemma 413 (*closedness under finite operations,  $\mathcal{S}$  closed under disjoint union*), and Lemma 408 (*closedness under disjoint union and local complement*).  $\square$

**Lemma 419 (*closedness under countable union and intersection*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement. Then,  $\mathcal{S}$  is closed under countable union iff it is closed under countable intersection, and it is closed under countable monotone union iff it is closed under countable monotone intersection.*

*Proof.* Direct consequence of **De Morgan's laws**, and **monotonicity of complement**.  $\square$

**Lemma 420 (*closedness under countable disjoint and monotone union*).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under local complement and countable disjoint union. Then, it is closed under countable monotone union.*



*Proof.* Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$ . Assume that for all  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ .

Let  $B_0 \stackrel{\text{def.}}{=} A_0$ , and for all  $n \in \mathbb{N}$ , let  $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{p \in [0..n]} B_p$ . Then, from Lemma 212 (*partition of countable union*), the sequence  $(B_n)_{n \in \mathbb{N}}$  is pairwise disjoint, for all  $n \in \mathbb{N}$ , we have  $A_n = \bigcup_{i \in [0..n]} A_i = \bigsqcup_{i \in [0..n]} B_i$ , and  $\bigcup_{n \in \mathbb{N}} A_n = \bigsqcup_{n \in \mathbb{N}} B_n$ . Moreover,  $B_0 = A_0 \in \mathcal{S}$ , and for all  $n \in \mathbb{N}$ , we have  $B_{n+1} = A_{n+1} \setminus A_n \in \mathcal{S}$ . Hence,  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable monotone union.  $\square$

**Lemma 421 (closedness under countable monotone and disjoint union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement, local complement and countable monotone union. Then, it is closed under countable disjoint union.

*Proof.* Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$ . Assume that for all  $p, q \in \mathbb{N}$ ,  $p \neq q$  implies  $A_p \cap A_q = \emptyset$ .

For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} \bigsqcup_{p \in [0..n]} A_p$ . Then, from **properties of union**, the sequence  $(B_n)_{n \in \mathbb{N}}$  is nondecreasing, for all  $n \in \mathbb{N}$ , we have  $\bigsqcup_{p \in [0..n]} A_p = \bigcup_{p \in [0..n]} B_p = B_n$ , and  $\bigsqcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$ . Moreover,  $B_0 = A_0 \in \mathcal{S}$ , and from **De Morgan's laws**, and **the definition of set difference**, for all  $n \in \mathbb{N}$ , we have  $B_n \cap A_{n+1} = \bigsqcup_{p \in [0..n]} A_p \cap A_{n+1} = \emptyset$  (i.e.  $B_n \subset A_{n+1}^c$ ), and

$$B_{n+1} = A_{n+1} \sqcup \bigsqcup_{p \in [0..n]} A_p = A_{n+1} \sqcup B_n = (A_{n+1}^c \cap B_n^c)^c = (A_{n+1}^c \setminus B_n)^c.$$

Then, from a trivial induction, for all  $n \in \mathbb{N}$ , we have  $B_n \in \mathcal{S}$ . Hence,  $\bigsqcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable disjoint union.  $\square$

**Lemma 422 (closedness under countable disjoint union and countable union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement, intersection and countable disjoint union. Then, it is closed under countable union.

*Proof.* From Lemma 411 (*closedness under union and intersection*), and Lemma 413 (*closedness under finite operations*),  $\mathcal{S}$  is closed under finite union.

Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$ . Let  $B_0 \stackrel{\text{def.}}{=} A_0$ , and for all  $n \in \mathbb{N}$ , let  $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{p \in [0..n]} B_p$ . Then, from Lemma 212 (*partition of countable union*), the sequence  $(B_n)_{n \in \mathbb{N}}$  is pairwise disjoint, for all  $n \in \mathbb{N}$ , we have  $\bigcup_{p \in [0..n]} A_p = \bigsqcup_{p \in [0..n]} B_p$ , and  $\bigcup_{n \in \mathbb{N}} A_n = \bigsqcup_{n \in \mathbb{N}} B_n$ . Moreover,  $B_0 \in \mathcal{S}$ , and from Lemma 410 (*closedness under intersection and set difference*), for all  $n \in \mathbb{N}$ , we have  $B_{n+1} = A_{n+1} \setminus \bigcup_{p \in [0..n]} A_p \in \mathcal{S}$ . Hence,  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$ .

Therefore,  $\mathcal{S}$  is closed under countable union.  $\square$

**Lemma 423 (closedness under countable monotone union and countable union).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is closed under complement, union and countable monotone union. Then, it is closed under countable union.

*Proof.* Direct consequence of Lemma 411 (*closedness under union and intersection*,  $\mathcal{S}$  is closed under intersection), Lemma 410 (*closedness under intersection and set difference*,  $\mathcal{S}$  is closed under set difference), Lemma 409 (*closedness under set difference and local complement*,  $\mathcal{S}$  is closed under local set difference), Lemma 421 (*closedness under countable monotone and disjoint union*,  $\mathcal{S}$  is closed under disjoint union), and Lemma 422 (*closedness under countable disjoint union and countable union*).  $\square$

### 6.6.2 Pi-system

**Definition 424 ( $\pi$ -system).**

Let  $X$  be a set.

A subset  $\Pi$  of  $\mathcal{P}(X)$  is called  $\pi$ -system on  $X$  iff it is nonempty and closed under finite intersection:

$$(92) \quad \Pi \neq \emptyset,$$

$$(93) \quad \forall n \in \mathbb{N}, \forall (A_p)_{p \in [0..n]} \in \Pi, \bigcap_{p \in [0..n]} A_p \in \Pi.$$

*Remark 425.* Note that adding the empty set to a  $\pi$ -system keeps it a  $\pi$ -system. We may replace in Definition 424 ( $\pi$ -system) the nonempty condition (92) by the “contain-the-empty-set” condition, and still maintain the Dynkin  $\pi$ - $\lambda$  theorem, see Theorem 503.

**Lemma 426 (intersection of  $\pi$ -systems).** Let  $X$  and  $I$  be sets. Let  $(\Pi_i)_{i \in I}$  be  $\pi$ -systems on  $X$ . Then,  $\bigcap_{i \in I} \Pi_i$  is closed under intersection, i.e. it is a  $\pi$ -system iff it is nonempty.

*Proof.* Direct consequence of Definition 424 ( $\pi$ -system), and **the definition of intersection**.  $\square$

**Definition 427 (generated  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . The  $\pi$ -system generated by  $G$  is the intersection of all  $\pi$ -systems on  $X$  containing  $G$ ; it is denoted  $\Pi_X(G)$ .

**Lemma 428 (generated  $\pi$ -system is minimum).** Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then,  $\Pi_X(G)$  is the smallest  $\pi$ -system on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 427 (generated  $\pi$ -system), Lemma 426 (intersection of  $\pi$ -systems), and **properties of the intersection**.  $\square$

**Lemma 429 ( $\pi$ -system generation is monotone).** Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $\emptyset \neq G_1 \subset G_2$ . Then, we have  $\Pi_X(G_1) \subset \Pi_X(G_2)$ .

*Proof.* Direct consequence of **the definition of inclusion ( $G_2 \neq \emptyset$ )**, and Lemma 428 (generated  $\pi$ -system is minimum), with  $G \stackrel{\text{def.}}{=} G_2$ , then  $G \stackrel{\text{def.}}{=} G_1$ .  $\square$

**Lemma 430 ( $\pi$ -system generation is idempotent).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S} \neq \emptyset$ . Then,  $\mathcal{S}$  is a  $\pi$ -system on  $X$  iff  $\Pi_X(\mathcal{S}) = \mathcal{S}$ .

*Proof.* Direct consequence of **reflexivity of inclusion**, and Lemma 428 (generated  $\pi$ -system is minimum),  $\mathcal{S}$  and  $\Pi_X(\mathcal{S})$  are both  $\pi$ -systems containing  $\mathcal{S}$ .  $\square$

### 6.6.3 Set algebra

*Remark 431.* The following concept of “set algebra” is not to be confused with the algebraic structure “algebra over a field” defined in Section 6.5.2.

**Definition 432 (set algebra).** Let  $X$  be a set. A subset  $\mathcal{A}$  of  $\mathcal{P}(X)$  is called *set algebra on  $X$*  iff it contains the empty set, it is closed under complement, and under finite union:

$$(94) \quad \emptyset \in \mathcal{A},$$

$$(95) \quad \forall A \in \mathcal{A}, \quad A^c \in \mathcal{A},$$

$$(96) \quad \forall n \in \mathbb{N}, \forall (A_i)_{i \in [0..n]} \in \mathcal{A}, \quad \bigcup_{i \in [0..n]} A_i \in \mathcal{A}.$$

**Lemma 433 (equivalent definition of set algebra).**

Let  $X$  be a set. Let  $\mathcal{A} \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}$  is a set algebra on  $X$  iff (95) holds and

$$(97) \quad \emptyset \in \mathcal{A} \quad \vee \quad X \in \mathcal{A} \quad \vee \quad \mathcal{A} \neq \emptyset,$$

$$(98) \quad \forall A, B \in \mathcal{A}, \quad A \cup B \in \mathcal{A} \quad \vee \quad \forall A, B \in \mathcal{A}, \quad A \cap B \in \mathcal{A}.$$

*Proof.* Direct consequence of Definition 432 (set algebra), Lemma 404 (nonempty and with empty or full), Lemma 405 (with empty and full), Lemma 413 (closedness under finite operations), and Lemma 414 (closedness under finite union and intersection).  $\square$

**Lemma 434 (other equivalent definition of set algebra).** Let  $X$  be a set. Let  $\mathcal{A} \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}$  is a set algebra on  $X$  iff it contains the full set  $X$ , and it is closed under set difference:

$$(99) \quad X \in \mathcal{A}.$$

$$(100) \quad \forall A, B \in \mathcal{A}, \quad A \setminus B \in \mathcal{A}.$$

*Proof.* Direct consequence of Lemma 433 (equivalent definition of set algebra), Lemma 410 (closedness under intersection and set difference, both ways), Lemma 409 (closedness under set difference and local complement), and Lemma 407 (closedness under local complement and complement).  $\square$

**Lemma 435 (set algebra is closed under local complement).**

Let  $X$  be a set. Let  $\mathcal{A}$  be a set algebra on  $X$ . Then,  $\mathcal{A}$  is closed under local complement.

*Proof.* Direct consequence of Lemma 434 (other equivalent definition of set algebra), and Lemma 409 (closedness under set difference and local complement).  $\square$

**Lemma 436 (intersection of set algebras).**

Let  $X$  and  $I$  be sets. Let  $(\mathcal{A}_i)_{i \in I}$  be set algebras on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{A}_i$  is a set algebra on  $X$ .

*Proof.* Direct consequence of Definition 432 (set algebra), and the definition of intersection.  $\square$

**Definition 437 (generated set algebra).** Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The set algebra generated by  $G$  is the intersection of all set algebras on  $X$  containing  $G$ ; it is denoted  $\mathcal{A}_X(G)$ .

**Lemma 438 (generated set algebra is minimum).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{A}_X(G)$  is the smallest set algebra on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 437 (generated set algebra), Lemma 436 (intersection of set algebras), and properties of the intersection.  $\square$

**Lemma 439 (set algebra generation is monotone).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1 \subset G_2$ . Then, we have  $\mathcal{A}_X(G_1) \subset \mathcal{A}_X(G_2)$ .

*Proof.* Lemma 428 (generated  $\pi$ -system is minimum, with  $G \stackrel{\text{def.}}{=} G_2$ , then  $G \stackrel{\text{def.}}{=} G_1$ ).  $\square$

**Lemma 440 (set algebra generation is idempotent).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Then,  $\mathcal{S}$  is a set algebra on  $X$  iff  $\mathcal{A}_X(\mathcal{S}) = \mathcal{S}$ .

*Proof.* Direct consequence of reflexivity of inclusion, Lemma 438 (generated set algebra is minimum,  $\mathcal{S}$  and  $\mathcal{A}_X(\mathcal{S})$  are both set algebras containing  $\mathcal{S}$ ).  $\square$

**Lemma 441 (explicit set algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G$  contains the full set, is closed under intersection, and satisfies the property:

$$(101) \quad \forall A \in G, \exists B_1, B_2 \in G, \quad B_1 \cap B_2 = \emptyset \quad \wedge \quad A^c = B_1 \uplus B_2.$$

Then, the set algebra generated by  $G$  is the set of finite disjoint unions of elements of  $G$ :

$$(102) \quad \mathcal{A}_X(G) = \left\{ \biguplus_{p \in [0..n]} A_p \mid n \in \mathbb{N} \quad \wedge \quad \forall p \in [0..n], A_p \in G \right. \\ \left. \wedge \quad \forall p, q \in [0..n], p \neq q \Rightarrow A_p \cap A_q = \emptyset \right\}.$$

*Proof.* Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(G)$ . Let  $\mathcal{U}$  be the set of finite disjoint unions of elements of  $G$ .

(0).  $\mathcal{U} \subset \mathcal{A}$ . From Lemma 438 (*generated set algebra is minimum*,  $G \subset \mathcal{A}$ ), and Definition 432 (*set algebra*,  $\mathcal{A}$  is closed under finite union), we have  $\mathcal{U} \subset \mathcal{A}$ .

(1a).  $G \subset \mathcal{U}$ . Direct consequence of the definition of  $\mathcal{U}$  (with  $n \stackrel{\text{def.}}{=} 0$ ).

(1b).  $X \in \mathcal{U}$ . Direct consequence of (1a).

(1c).  $\mathcal{U}$  is closed under complement. Let  $A \in \mathcal{U}$ .

From the definition of  $\mathcal{U}$ , let  $n \in \mathbb{N}$  and  $(A_p)_{p \in [0..n]} \in G$  such that for all  $p, q \in [0..n]$ ,  $p \neq q$  implies  $A_p \cap A_q = \emptyset$ , and  $A = \bigsqcup_{p \in [0..n]} A_p$ . Let  $p \in [0..n]$ . Then, from (101), let  $B_p^0, B_p^1 \in G$  such that  $B_p^0 \cap B_p^1 = \emptyset$  and  $A_p^c = B_p^0 \sqcup B_p^1$ .

Let  $I \stackrel{\text{def.}}{=} \{0, 1\}^{[0..n]}$  (its cardinality is  $2^{n+1}$ ). For all  $\varphi \in I$ , let  $C^\varphi \stackrel{\text{def.}}{=} \bigcap_{p \in [0..n]} B_p^{\varphi(p)}$ . Let  $\varphi \in I$ . Then, from Lemma 413 (*closedness under finite operations*,  $G$  is closed under finite intersection), we have  $C^\varphi \in G$ . Let  $\psi \in I$ . Assume that  $\varphi \neq \psi$ , i.e. there exists  $q \in [0..n]$  such that  $\varphi(q) \neq \psi(q)$ . Then, from **the definition of intersection**, we have  $C^\varphi \subset B_q^{\varphi(q)}$  and  $C^\psi \subset B_q^{\psi(q)}$ . Hence, from **monotonicity of intersection**, we have

$$C^\varphi \cap C^\psi \subset B_q^{\varphi(q)} \cap B_q^{\psi(q)} = \emptyset.$$

Moreover, from **De Morgan's laws**, and **distributivity of intersection over union**, we have

$$A^c = \bigcap_{p \in [0..n]} (B_p^0 \sqcup B_p^1) = \bigcup_{\varphi \in I} \left( \bigcap_{p \in [0..n]} B_p^{\varphi(p)} \right) = \bigsqcup_{\varphi \in I} C^\varphi.$$

Thus,  $A^c \in \mathcal{U}$ . Hence,  $\mathcal{U}$  is closed under complement.

(1d).  $\mathcal{U}$  is closed under intersection. Let  $A^0, A^1 \in \mathcal{U}$ .

Let  $\alpha \in \{0, 1\}$ . From the definition of  $\mathcal{U}$ , let  $n^\alpha \in \mathbb{N}$ , let  $(A_{p^\alpha}^\alpha)_{p^\alpha \in [0..n^\alpha]} \in G$  such that for all  $p^\alpha, q^\alpha \in [0..n^\alpha]$ ,  $p^\alpha \neq q^\alpha$  implies  $A_{p^\alpha}^\alpha \cap A_{q^\alpha}^\alpha = \emptyset$ , and  $A^\alpha = \bigsqcup_{p^\alpha \in [0..n^\alpha]} A_{p^\alpha}^\alpha$ .

For all  $p^0 \in [0..n^0]$ , for all  $p^1 \in [0..n^1]$ , let  $B_{p^0, p^1} \stackrel{\text{def.}}{=} A_{p^0}^0 \cap A_{p^1}^1 \in G$ . Let  $n \stackrel{\text{def.}}{=} n^0 n^1 + n^0 + n^1$ . Let  $\varphi : [0..n] \rightarrow [0..n^0] \times [0..n^1]$  be a bijection (their common cardinality is  $n+1 = (n^0+1)(n^1+1)$ ). Let  $p, q \in [0..n]$ . Assume that  $p \neq q$ . Let  $p^0, q^0 \in [0..n^0]$  and  $p^1, q^1 \in [0..n^1]$  such that

$$(p^0, p^1) = \varphi(p) \quad \text{and} \quad (q^0, q^1) = \varphi(q).$$

Then, from **the definition of bijection and injection**, we have  $p^0 \neq q^0$  or  $p^1 \neq q^1$ , i.e.  $A_{p^0}^0 \cap A_{q^0}^0 = \emptyset$  or  $A_{p^1}^1 \cap A_{q^1}^1 = \emptyset$ . Thus, from **associativity and commutativity of intersection**, and since **the  $\emptyset$  is absorbing for intersection**, we have

$$B_{\varphi(p)} \cap B_{\varphi(q)} = (A_{p^0}^0 \cap A_{p^1}^1) \cap (A_{q^0}^0 \cap A_{q^1}^1) = (A_{p^0}^0 \cap A_{q^0}^0) \cap (A_{p^1}^1 \cap A_{q^1}^1) = \emptyset.$$

Moreover, from **left and right distributivity of intersection over union**, and **associativity and commutativity of union**, we have

$$A^0 \cap A^1 = \left( \bigcup_{p^0 \in [0..n^0]} A_{p^0}^0 \right) \cap \left( \bigcup_{p^1 \in [0..n^1]} A_{p^1}^1 \right) = \bigcup_{(p^0, p^1) \in [0..n^0] \times [0..n^1]} B_{p^0, p^1} = \bigsqcup_{p \in [0..n]} B_{\varphi(p)}.$$

Thus,  $A^0 \cap A^1 \in \mathcal{U}$ . Hence,  $\mathcal{U}$  is closed under intersection.

(2).  $\mathcal{A} \subset \mathcal{U}$ . From (1b), (1c), (1d), Lemma 433 (*equivalent definition of set algebra*,  $\mathcal{U}$  is set algebra), (1a), and Lemma 438 (*generated set algebra is minimum*), we have  $\mathcal{A} \subset \mathcal{U}$ .

Therefore, from (0) and (2), we have  $\mathcal{A} = \mathcal{U}$ . □

#### 6.6.4 Monotone class

**Definition 442 (monotone class).** Let  $X$  be a set. A subset  $\mathcal{C}$  of  $\mathcal{P}(X)$  is called *monotone class on  $X$*  iff it is closed under countable monotone union and intersection:

$$(103) \quad \forall (A_n)_{n \in \mathbb{N}} \in \mathcal{C}, \quad (\forall n \in \mathbb{N}, A_n \subset A_{n+1}) \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{C},$$

$$(104) \quad \forall (A_n)_{n \in \mathbb{N}} \in \mathcal{C}, \quad (\forall n \in \mathbb{N}, A_n \supset A_{n+1}) \implies \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{C}.$$

**Lemma 443 (intersection of monotone classes).** Let  $X$  and  $I$  be sets. Let  $(\mathcal{C}_i)_{i \in I}$  be a family of monotone classes on  $X$ . Then,  $\bigcap_{i \in I} \mathcal{C}_i$  is a monotone class on  $X$ .

*Proof.*

Direct consequence of Definition 442 (monotone class), and **the definition of intersection**.  $\square$

**Definition 444 (generated monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The *monotone class generated by  $G$*  is the intersection of all monotone classes on  $X$  containing  $G$ ; it is denoted by  $\mathcal{C}_X(G)$ .

**Lemma 445 (generated monotone class is minimum).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\mathcal{C}_X(G)$  is the smallest monotone class on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 444 (generated monotone class), Lemma 443 (intersection of monotone classes), and **properties of the intersection**.  $\square$

**Lemma 446 (monotone class generation is monotone).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1 \subset G_2$ . Then, we have  $\mathcal{C}_X(G_1) \subset \mathcal{C}_X(G_2)$ .

*Proof.* Lemma 445 (generated monotone class is minimum, with  $G \stackrel{\text{def.}}{=} G_2$ , then  $G \stackrel{\text{def.}}{=} G_1$ ).  $\square$

**Lemma 447 (monotone class generation is idempotent).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Then,  $\mathcal{S}$  is a monotone class on  $X$  iff  $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$ .

*Proof.* Direct consequence of **reflexivity of inclusion**, Lemma 445 (generated monotone class is minimum,  $\mathcal{S}$  and  $\mathcal{C}_X(\mathcal{S})$  are both monotone classes containing  $\mathcal{S}$ ).  $\square$

**Definition 448 (monotone class and symmetric set difference).**

Let  $X$  be a set. Let  $\mathcal{C}$  be a monotone class on  $X$ . For all  $A \subset X$ , let  $\mathcal{C}_A^\setminus$  be the subset system defined by

$$(105) \quad \mathcal{C}_A^\setminus \stackrel{\text{def.}}{=} \{B \subset X \mid A \setminus B \in \mathcal{C} \wedge B \setminus A \in \mathcal{C}\}.$$

**Lemma 449 ( $\mathcal{C}^\setminus$  is symmetric).**

Let  $X$  be a set. Let  $\mathcal{C}$  be a monotone class on  $X$ . Then, for all  $A, B \subset X$ , we have  $B \in \mathcal{C}_A^\setminus$  iff  $A \in \mathcal{C}_B^\setminus$ .

*Proof.* Direct consequence of Definition 448 (monotone class and symmetric set difference).  $\square$

**Lemma 450 ( $\mathcal{C}^\setminus$  is monotone class).**

Let  $X$  be a set. Let  $\mathcal{C}$  be a monotone class on  $X$ . Then, for all  $A \subset X$ ,  $\mathcal{C}_A^\setminus$  is a monotone class on  $X$ .

*Proof.* Let  $A \subset X$ . Let  $(B_n)_{n \in \mathbb{N}} \in \mathcal{C}_A^\setminus$ . Let  $n \in \mathbb{N}$ . Then, from Definition 448 (*monotone class and symmetric set difference*), we have  $A \setminus B_n \in \mathcal{C}$  and  $B_n \setminus A \in \mathcal{C}$ .

Assume first that, for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ . Let  $n \in \mathbb{N}$ . Then, from **monotonicity of set difference**, we have  $A \setminus B_n \supset A \setminus B_{n+1}$  and  $B_n \setminus A \subset B_{n+1} \setminus A$ . Let  $B \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} B_n$ . Then, from **the definition of set difference**, **De Morgan's laws**, Definition 448 (*monotone class and symmetric set difference*), and Definition 442 (*monotone class*), we have  $A \setminus B = \bigcap_{n \in \mathbb{N}} (A \setminus B_n) \in \mathcal{C}$  and  $B \setminus A = \bigcup_{n \in \mathbb{N}} (B_n \setminus A) \in \mathcal{C}$ . Thus, from Definition 448 (*monotone class and symmetric set difference*), we have  $B \in \mathcal{C}_A^\setminus$ . Hence,  $\mathcal{C}_A^\setminus$  is closed under nondecreasing union.

Assume now that, for all  $n \in \mathbb{N}$ ,  $B_n \supset B_{n+1}$ . Let  $n \in \mathbb{N}$ . Then, from **monotonicity of set difference**, we have  $A \setminus B_n \subset A \setminus B_{n+1}$  and  $B_n \setminus A \supset B_{n+1} \setminus A$ . Let  $B \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} B_n$ . Then, from **the definition of set difference**, **De Morgan's laws**, Definition 448 (*monotone class and symmetric set difference*), and Definition 442 (*monotone class*), we have  $A \setminus B = \bigcup_{n \in \mathbb{N}} (A \setminus B_n) \in \mathcal{C}$  and  $B \setminus A = \bigcap_{n \in \mathbb{N}} (B_n \setminus A) \in \mathcal{C}$ . Thus, from Definition 448 (*monotone class and symmetric set difference*), we have  $B \in \mathcal{C}_A^\setminus$ . Hence,  $\mathcal{C}_A^\setminus$  is closed under nonincreasing intersection.

Therefore, from Definition 442 (*monotone class*),  $\mathcal{C}_A^\setminus$  is a monotone class on  $X$ .  $\square$

**Lemma 451 (monotone class closed under set difference).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is closed under set difference.

Then,  $\mathcal{C}_X(G)$  is closed under set difference.

*Proof.* For all  $A \subset X$ , we use the simplified notation  $\mathcal{C}_A^\setminus \stackrel{\text{def.}}{=} (\mathcal{C}_X(G))_A^\setminus$ .

(1).  $\forall A \in G$ ,  $\mathcal{C}_X(G) \subset \mathcal{C}_A^\setminus$ . Let  $A \in G$ .

Let  $B \in G$ . Then, we have  $A \setminus B$ ,  $B \setminus A \in G$ . Thus, from Lemma 445 (*generated monotone class is minimum*,  $G \subset \mathcal{C}_X(G)$ ), and Definition 448 (*monotone class and symmetric set difference*), we have  $B \in \mathcal{C}_A^\setminus$ , i.e.  $G \subset \mathcal{C}_A^\setminus$ . Hence, from Lemma 450 ( $\mathcal{C}^\setminus$  is monotone class), and Lemma 445 (*generated monotone class is minimum*), we have  $\mathcal{C}_X(G) \subset \mathcal{C}_A^\setminus$ .

(2).  $\forall B \in \mathcal{C}_X(G)$ ,  $\mathcal{C}_X(G) \subset \mathcal{C}_B^\setminus$ . Let  $B \in \mathcal{C}_X(G)$ .

Let  $A \in G$ . Then, from (1), we have  $B \in \mathcal{C}_A^\setminus$ , and from Lemma 449 ( $\mathcal{C}^\setminus$  is symmetric), we have  $A \in \mathcal{C}_B^\setminus$ , i.e.  $G \subset \mathcal{C}_B^\setminus$ . Hence, from Lemma 450 ( $\mathcal{C}^\setminus$  is monotone class), and Lemma 445 (*generated monotone class is minimum*), we have  $\mathcal{C}_X(G) \subset \mathcal{C}_B^\setminus$ .

Let  $A, B \in \mathcal{C}_X(G)$ . Then, from (2), we have  $B \in \mathcal{C}_A^\setminus$ , and from Definition 448 (*monotone class and symmetric set difference*), we have  $A \setminus B \in \mathcal{C}_X(G)$ .

Therefore,  $\mathcal{C}_X(G)$  is closed under set difference.  $\square$

**Lemma 452 (monotone class generated by set algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G$  is a set algebra on  $X$ .

*Proof.* Direct consequence of Lemma 434 (*other equivalent definition of set algebra*,  $G$  contains the full set, and is closed under set difference), Lemma 445 (*generated monotone class is minimum*,  $X \in G \subset \mathcal{C}_X(G)$ ), Lemma 451 (*monotone class closed under set difference*,  $\mathcal{C}_X(G)$  is closed under set difference), and Lemma 434 (*other equivalent definition of set algebra*).  $\square$

### 6.6.5 Lambda-system

#### Definition 453 ( $\lambda$ -system).

Let  $X$  be a set. A subset  $\Lambda$  of  $\mathcal{P}(X)$  is called  $\lambda$ -system on  $X$ , or *Dynkin-system on  $X$*  iff it contains the full set, it is closed under complement, and under countable disjoint union:

$$(106) \quad X \in \Lambda,$$

$$(107) \quad \forall A \in \Lambda, \quad A^c \in \Lambda,$$

$$(108) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Lambda, \quad (\forall p, q \in \mathbb{N}, p \neq q \Rightarrow A_p \cap A_q = \emptyset) \quad \Longrightarrow \quad \biguplus_{n \in \mathbb{N}} A_n \in \Lambda.$$

#### Lemma 454 (equivalent definition of $\lambda$ -system).

Let  $X$  be a set. Let  $\Lambda \subset \mathcal{P}(X)$ . Then,  $\Lambda$  is a  $\lambda$ -system on  $X$  iff (106) holds, and it is closed under local complement, and under countable monotone union:

$$(109) \quad \forall A, B \in \Lambda, \quad B \subset A \Rightarrow A \setminus B \in \Lambda,$$

$$(110) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Lambda, \quad (\forall n \in \mathbb{N}, A_n \subset A_{n+1}) \quad \Longrightarrow \quad \bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

*Proof.* “Left” implies “right”.

Direct consequence of Definition 453 ( $\lambda$ -system), Lemma 418 (closedness under countable disjoint union and local complement,  $\Lambda$  is closed under local complement), and Lemma 420 (closedness under countable disjoint and monotone union,  $\Lambda$  is closed under countable monotone union).

“Right” implies “left”. Direct consequence of Lemma 407 (closedness under local complement and complement,  $\Lambda$  is closed under complement), and Lemma 421 (closedness under countable monotone and disjoint union,  $\Lambda$  is closed under countable disjoint union).  $\square$

#### Lemma 455 (other properties of $\lambda$ -system).

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then,  $\Lambda$  is nonempty, contains the empty set, and is closed under countable monotone intersection.

*Proof.* Direct consequence of Definition 453 ( $\lambda$ -system, closedness under complement), Lemma 404 (nonempty and with empty or full), Lemma 405 (with empty and full), Lemma 454 (equivalent definition of  $\lambda$ -system), and Lemma 419 (closedness under countable union and intersection).  $\square$

#### Lemma 456 (intersection of $\lambda$ -systems).

Let  $X$  and  $I$  be sets. Let  $(\Lambda_i)_{i \in I}$  be  $\lambda$ -systems on  $X$ . Then, if  $\bigcap_{i \in I} \Lambda_i$  is a  $\lambda$ -system on  $X$ .

*Proof.* Direct consequence of Definition 453 ( $\lambda$ -system), and the definition of intersection.  $\square$

#### Definition 457 (generated $\lambda$ -system).

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The  $\lambda$ -system generated by  $G$  is the intersection of all  $\lambda$ -systems on  $X$  containing  $G$ ; it is denoted  $\Lambda_X(G)$ .

#### Lemma 458 (generated $\lambda$ -system is minimum).

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\Lambda_X(G)$  is the smallest  $\lambda$ -system on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 457 (generated  $\lambda$ -system), Lemma 456 (intersection of  $\lambda$ -systems), and properties of the intersection.  $\square$

#### Lemma 459 ( $\lambda$ -system generation is monotone).

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1 \subset G_2$ . Then, we have  $\Lambda_X(G_1) \subset \Lambda_X(G_2)$ .

*Proof.* Lemma 458 (generated  $\lambda$ -system is minimum, with  $G \stackrel{\text{def.}}{=} G_2$ , then  $G \stackrel{\text{def.}}{=} G_1$ ).  $\square$



**Lemma 460 ( $\lambda$ -system generation is idempotent).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Then,  $\mathcal{S}$  is a  $\lambda$ -system on  $X$  iff  $\Lambda_X(\mathcal{S}) = \mathcal{S}$ .

*Proof.* Direct consequence of **reflexivity of inclusion**, Lemma 458 (*generated  $\lambda$ -system is minimum*,  $\mathcal{S}$  and  $\Lambda_X(\mathcal{S})$  are both  $\lambda$ -systems containing  $\mathcal{S}$ ).  $\square$

**Definition 461 ( $\lambda$ -system and intersection).**Let  $X$  be a set.

Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . For all  $A \subset X$ , let  $\Lambda_A^\cap$  be the subset system defined by

$$(111) \quad \Lambda_A^\cap \stackrel{\text{def.}}{=} \{B \in \Lambda \mid A \cap B \in \Lambda\}.$$

**Lemma 462 ( $\Lambda^\cap$  is symmetric).**

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then, for all  $A, B \in \Lambda$ , we have  $B \in \Lambda_A^\cap$  iff  $A \in \Lambda_B^\cap$ .

*Proof.* Direct consequence of Definition 461 ( *$\lambda$ -system and intersection*).  $\square$

**Lemma 463 ( $\Lambda^\cap$  is  $\lambda$ -system).**

Let  $X$  be a set. Let  $\Lambda$  be a  $\lambda$ -system on  $X$ . Then, for all  $A \in \Lambda$ ,  $\Lambda_A^\cap$  is a  $\lambda$ -system on  $X$ .

*Proof.* Let  $A \in \Lambda$ .

From Definition 453 ( *$\lambda$ -system,  $X \in \Lambda$* ), and **the identity  $A \cap X = A \in \Lambda$** , we have  $X \in \Lambda_A^\cap$ .

Let  $B, C \in \Lambda_A^\cap$ , i.e.  $B, C, A \cap B, A \cap C \in \Lambda$ . Assume that  $C \subset B$ .

Then, from Lemma 454 (*equivalent definition of  $\lambda$ -system, closedness under local complement*), we have  $B \setminus C \in \Lambda$ . Moreover, from **distributivity of intersection over set difference**, **monotonicity of intersection**, and Lemma 454 (*equivalent definition of  $\lambda$ -system, closedness under local complement*), we have  $(A \cap C) \subset (A \cap B)$  and  $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) \in \Lambda$ . Hence,  $\Lambda_A^\cap$  is closed under local complement.

Let  $(B_n)_{n \in \mathbb{N}} \in \Lambda_A^\cap$ , i.e. for all  $n \in \mathbb{N}$ ,  $B_n, A \cap B_n \in \Lambda$ . Assume that for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ . Then, from Lemma 454 (*equivalent definition of  $\lambda$ -system, closedness under countable monotone union*), we have  $\bigcup_{n \in \mathbb{N}} B_n \in \Lambda$ . Moreover, from **distributivity of intersection over union**, **monotonicity of intersection**, and Lemma 454 (*equivalent definition of  $\lambda$ -system, closedness under countable monotone union*), we have

$$(A \cap B_n) \subset (A \cap B_{n+1}) \quad \text{and} \quad A \cap \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \Lambda.$$

Hence,  $\Lambda_A^\cap$  is closed under countable monotone union.

Therefore, from Lemma 454 (*equivalent definition of  $\lambda$ -system*),  $\Lambda_A^\cap$  is a  $\lambda$ -system on  $X$ .  $\square$

**Lemma 464 ( $\lambda$ -system with intersection).**Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $G$  is closed under intersection. Then,  $\forall A \in \Lambda_X(G)$ ,  $(\Lambda_X(G))_A^\cap = \Lambda_X(G)$ .

*Proof.* For all  $A \subset X$ , we use the simplified notation  $\Lambda_A^\cap \stackrel{\text{def.}}{=} (\Lambda_X(G))_A^\cap$ .

(0). From Definition 461 ( *$\lambda$ -system and intersection*), we have  $\forall A \in \Lambda_X(G)$ ,  $\Lambda_A^\cap \subset \Lambda_X(G)$ .

(1).  $\forall A \in G$ ,  $\Lambda_X(G) \subset \Lambda_A^\cap$ . Let  $A \in G$ .

Let  $B \in G$ . Then, from Lemma 458 (*generated  $\lambda$ -system is minimum,  $G \subset \Lambda_X(G)$* ), and Definition 461 ( *$\lambda$ -system and intersection*), we have  $B \in \Lambda_A^\cap$ , i.e.  $G \subset \Lambda_A^\cap$ . Hence, from Lemma 463 ( *$\Lambda^\cap$  is  $\lambda$ -system*, with  $A \in G \subset \Lambda_X(G)$ ), and Lemma 458 (*generated  $\lambda$ -system is minimum*), we have  $\Lambda_X(G) \subset \Lambda_A^\cap$ .

(2).  $\forall B \in \Lambda_X(G)$ ,  $\Lambda_X(G) \subset \Lambda_B^\cap$ . Let  $B \in \Lambda_X(G)$ .

Let  $A \in G$ . Then, from (1), we have  $B \in \Lambda_X(G) \subset \Lambda_A^\cap$ . Thus, from Lemma 458 (*generated  $\lambda$ -system is minimum,  $G \subset \Lambda_X(G)$* ), and Lemma 462 ( *$\Lambda^\cap$  is symmetric, with  $A, B \in \Lambda_X(G)$* ), we have  $A \in \Lambda_B^\cap$ , i.e.  $G \subset \Lambda_B^\cap$ . Hence, from Lemma 463 ( *$\Lambda^\cap$  is  $\lambda$ -system*), and Lemma 458 (*generated  $\lambda$ -system is minimum*), we have  $\Lambda_X(G) \subset \Lambda_B^\cap$ .

Therefore, from (0) and (2), we have the equality.  $\square$



**Lemma 465 ( $\lambda$ -system closed under intersection).** *Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is closed under intersection. Then,  $\Lambda_X(G)$  is closed under intersection.*

*Proof.* Let  $A, B \in \Lambda_X(G)$ . Then, from Lemma 464 ( $\lambda$ -system with intersection,  $B$  belongs to  $\Lambda_X(G) = (\Lambda_X(G))_{\Delta}^{\cap}$ ), and Definition 461 ( $\lambda$ -system and intersection), we have  $A \cap B \in \Lambda_X(G)$ .

Therefore,  $\Lambda_X(G)$  is closed under intersection.  $\square$

**Lemma 466 ( $\lambda$ -system generated by  $\pi$ -system).** *Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a  $\pi$ -system on  $X$ . Then,  $\Lambda_X(G)$  is a  $\pi$ -system on  $X$ .*

*Proof.* Direct consequence of Definition 424 ( $\pi$ -system,  $G \neq \emptyset$  and  $G$  is closed under finite intersection), Lemma 413 (closedness under finite operations,  $G$  is closed under intersection), Lemma 458 (generated  $\lambda$ -system is minimum,  $\emptyset \neq G \subset \Lambda_X(G)$ ), Lemma 465 ( $\lambda$ -system closed under intersection,  $\Lambda_X(G)$  is closed under intersection), Lemma 413 (closedness under finite operations,  $\Lambda_X(G)$  is closed under finite intersection), and Definition 424 ( $\pi$ -system).  $\square$

### 6.6.6 Sigma-algebra

**Definition 467 ( $\sigma$ -algebra).** Let  $X$  be a set. A subset  $\Sigma$  of  $\mathcal{P}(X)$  is called  $\sigma$ -algebra on  $X$  iff it contains the empty set, it is closed under complement, and under countable union:

$$(112) \quad \emptyset \in \Sigma,$$

$$(113) \quad \forall A \in \Sigma, \quad A^c \in \Sigma,$$

$$(114) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \quad \bigcup_{i \in I} A_i \in \Sigma.$$

If so,  $(X, \Sigma)$  is called *measurable space*, and elements of  $\Sigma$  are said  $(\Sigma)$ -measurable.

**Lemma 468 (equivalent definition of  $\sigma$ -algebra).**

*Let  $X$  be a set. Let  $\Sigma \subset \mathcal{P}(X)$ . Then,  $\Sigma$  is a  $\sigma$ -algebra on  $X$  iff (113) holds and*

$$(115) \quad \emptyset \in \Sigma \quad \vee \quad X \in \Sigma \quad \vee \quad \Sigma \neq \emptyset,$$

$$(116) \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \bigcup_{i \in I} A_i \in \Sigma \quad \vee \quad \forall I \subset \mathbb{N}, \forall (A_i)_{i \in I} \in \Sigma, \bigcap_{i \in I} A_i \in \Sigma.$$

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra), Lemma 404 (nonempty and with empty or full), Lemma 405 (with empty and full), and Lemma 419 (closedness under countable union and intersection).  $\square$

**Lemma 469 ( $\sigma$ -algebra is set algebra).**

*Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is a set algebra on  $X$ .*

*Proof.* Direct consequence of Definition 432 (set algebra), and Definition 467 ( $\sigma$ -algebra).  $\square$

**Lemma 470 ( $\sigma$ -algebra is closed under set difference).**

*Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is closed under set difference and local complement.*

*Proof.* Direct consequence of Lemma 469 ( $\sigma$ -algebra is set algebra), Lemma 434 (other equivalent definition of set algebra), and Lemma 435 (set algebra is closed under local complement).  $\square$

**Lemma 471 (other properties of  $\sigma$ -algebra).**

*Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is closed under countable monotone intersection and union, and countable disjoint union.*

*Proof.* Direct consequence of Lemma 468 (equivalent definition of  $\sigma$ -algebra).  $\square$

**Lemma 472 (intersection of  $\sigma$ -algebras).**

Let  $X$  and  $I$  be sets. Let  $(\Sigma_i)_{i \in I}$  be  $\sigma$ -algebras on  $X$ . Then,  $\bigcap_{i \in I} \Sigma_i$  is a  $\sigma$ -algebra on  $X$ .

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra), and **the definition of intersection**.  $\square$

**Definition 473 (generated  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . The  $\sigma$ -algebra generated by  $G$  is the intersection of all  $\sigma$ -algebras on  $X$  containing  $G$ ; it is denoted  $\Sigma_X(G)$ .

**Lemma 474 (generated  $\sigma$ -algebra is minimum).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then,  $\Sigma_X(G)$  is the smallest  $\sigma$ -algebra on  $X$  containing  $G$ .

*Proof.* Direct consequence of Definition 473 (generated  $\sigma$ -algebra), Lemma 472 (intersection of  $\sigma$ -algebras), and **properties of the intersection**.  $\square$

**Lemma 475 ( $\sigma$ -algebra generation is monotone).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ . Assume that  $G_1 \subset G_2$ . Then, we have  $\Sigma_X(G_1) \subset \Sigma_X(G_2)$ .

*Proof.* Lemma 474 (generated  $\sigma$ -algebra is minimum, with  $G \stackrel{\text{def.}}{=} G_2$ , then  $G \stackrel{\text{def.}}{=} G_1$ ).  $\square$

**Lemma 476 ( $\sigma$ -algebra generation is idempotent).**

Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Then,  $\mathcal{S}$  is a  $\sigma$ -algebra on  $X$  iff  $\Sigma_X(\mathcal{S}) = \mathcal{S}$ .

*Proof.* Direct consequence of **reflexivity of inclusion**, Lemma 474 (generated  $\sigma$ -algebra is minimum,  $\mathcal{S}$  and  $\Sigma_X(\mathcal{S})$  are both  $\sigma$ -algebras containing  $\mathcal{S}$ ).  $\square$

**Lemma 477 ( $\sigma$ -algebra is  $\pi$ -system).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is a  $\pi$ -system on  $X$ .

*Proof.* Direct consequence of Definition 424 ( $\pi$ -system), and Lemma 468 (equivalent definition of  $\sigma$ -algebra).  $\square$

**Lemma 478 ( $\sigma$ -algebra contains  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Pi_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 474 (generated  $\sigma$ -algebra is minimum,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $G \subset \Sigma_X(G)$ ), Lemma 477 ( $\sigma$ -algebra is  $\pi$ -system), and Lemma 428 (generated  $\pi$ -system is minimum).  $\square$

**Lemma 479 ( $\pi$ -system contains  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ , and that  $\Pi_X(G)$  is closed under complement and countable disjoint union. Then, we have  $\Sigma_X(G) \subset \Pi_X(G)$ .

*Proof.* Direct consequence of Lemma 428 (generated  $\pi$ -system is minimum,  $\Pi_X(G)$  is a  $\pi$ -system, and  $G \subset \Pi_X(G)$ ), Definition 424 ( $\pi$ -system,  $\Pi_X(G)$  is nonempty, and closed under intersection), Lemma 404 (nonempty and with empty or full,  $\emptyset \in \Pi_X(G)$ ), Lemma 422 (closedness under countable disjoint union and countable union,  $\Pi_X(G)$  is closed under countable union), Definition 467 ( $\sigma$ -algebra,  $\Pi_X(G)$  is a  $\sigma$ -algebra), and Lemma 474 (generated  $\sigma$ -algebra is minimum).  $\square$

**Lemma 480 ( $\sigma$ -algebra generated by  $\pi$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $G \neq \emptyset$ . Then, we have  $\Sigma_X(\Pi_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 478 ( $\sigma$ -algebra contains  $\pi$ -system,  $\Pi_X(G) \subset \Sigma_X(G)$ ), Lemma 474 (generated  $\sigma$ -algebra is minimum,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $\Sigma_X(\Pi_X(G)) \subset \Sigma_X(G)$ ), Lemma 428 (generated  $\pi$ -system is minimum,  $G \subset \Pi_X(G)$ ), and Lemma 475 ( $\sigma$ -algebra generation is monotone,  $\Sigma_X(G)$  is included in  $\Sigma_X(\Pi_X(G))$ ).  $\square$

**Lemma 481 ( $\sigma$ -algebra contains set algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{A}_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 474 (*generated  $\sigma$ -algebra is minimum*,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $G \subset \Sigma_X(G)$ ), Lemma 469 ( *$\sigma$ -algebra is set algebra*), and Lemma 438 (*generated set algebra is minimum*).  $\square$

**Lemma 482 (set algebra contains  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\mathcal{A}_X(G)$  is closed under countable monotone union. Then, we have  $\Sigma_X(G) \subset \mathcal{A}_X(G)$ .

*Proof.* Direct consequence of Lemma 438 (*generated set algebra is minimum*,  $\mathcal{A}_X(G)$  is a set algebra, and  $G \subset \mathcal{A}_X(G)$ ), Definition 432 (*set algebra*,  $\mathcal{A}_X(G)$  contains the empty set, is closed under complement and union), Lemma 423 (*closedness under countable monotone union and countable union*,  $\mathcal{A}_X(G)$  is closed under countable union), Definition 467 ( *$\sigma$ -algebra*,  $\mathcal{A}_X(G)$  is a  $\sigma$ -algebra), and Lemma 474 (*generated  $\sigma$ -algebra is minimum*).  $\square$

**Lemma 483 ( $\sigma$ -algebra generated by set algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\mathcal{A}_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 481 ( *$\sigma$ -algebra contains set algebra*,  $\mathcal{A}_X(G) \subset \Sigma_X(G)$ ), Lemma 474 (*generated  $\sigma$ -algebra is minimum*,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $\Sigma_X(\mathcal{A}_X(G)) \subset \Sigma_X(G)$ ), Lemma 438 (*generated set algebra is minimum*,  $G \subset \mathcal{A}_X(G)$ ), and Lemma 475 ( *$\sigma$ -algebra generation is monotone*,  $\Sigma_X(G)$  is included in  $\Sigma_X(\mathcal{A}_X(G))$ ).  $\square$

**Lemma 484 ( $\sigma$ -algebra is monotone class).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is a monotone class on  $X$ .

*Proof.* Direct consequence of Definition 442 (*monotone class*), and Lemma 471 (*other properties of  $\sigma$ -algebra*).  $\square$

**Lemma 485 ( $\sigma$ -algebra contains monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\mathcal{C}_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 474 (*generated  $\sigma$ -algebra is minimum*,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $G \subset \Sigma_X(G)$ ), Lemma 484 ( *$\sigma$ -algebra is monotone class*), and Lemma 445 (*generated monotone class is minimum*).  $\square$

**Lemma 486 (monotone class contains  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Assume that  $\mathcal{C}_X(G)$  contains the empty set, and is closed under complement and union. Then, we have  $\Sigma_X(G) \subset \mathcal{C}_X(G)$ .

*Proof.* Direct consequence of Lemma 445 (*generated monotone class is minimum*,  $\mathcal{C}_X(G)$  is a monotone class, and  $G \subset \mathcal{C}_X(G)$ ), Definition 442 (*monotone class*,  $\mathcal{C}_X(G)$  is closed under countable monotone union), Lemma 423 (*closedness under countable monotone union and countable union*,  $\mathcal{C}_X(G)$  is closed under countable union), Definition 467 ( *$\sigma$ -algebra*,  $\mathcal{C}_X(G)$  is a  $\sigma$ -algebra), and Lemma 474 (*generated  $\sigma$ -algebra is minimum*).  $\square$

**Lemma 487 ( $\sigma$ -algebra generated by monotone class).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\mathcal{C}_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 485 ( *$\sigma$ -algebra contains monotone class*,  $\mathcal{C}_X(G) \subset \Sigma_X(G)$ ), Lemma 474 (*generated  $\sigma$ -algebra is minimum*,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $\Sigma_X(\mathcal{C}_X(G)) \subset \Sigma_X(G)$ ), Lemma 445 (*generated monotone class is minimum*,  $G \subset \mathcal{C}_X(G)$ ), and Lemma 475 ( *$\sigma$ -algebra generation is monotone*,  $\Sigma_X(G) \subset \Sigma_X(\mathcal{C}_X(G))$ ).  $\square$

**Lemma 488 ( $\sigma$ -algebra is  $\lambda$ -system).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\Sigma$  is a  $\lambda$ -system on  $X$ .

*Proof.* Direct consequence of Definition 453 ( $\lambda$ -system), Lemma 468 (equivalent definition of  $\sigma$ -algebra), and Lemma 471 (other properties of  $\sigma$ -algebra).  $\square$

**Lemma 489 ( $\sigma$ -algebra contains  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Lambda_X(G) \subset \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 474 (generated  $\sigma$ -algebra is minimum,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $G \subset \Sigma_X(G)$ ), Lemma 488 ( $\sigma$ -algebra is  $\lambda$ -system), and Lemma 458 (generated  $\lambda$ -system is minimum).  $\square$

**Lemma 490 ( $\lambda$ -system contains  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ .

Assume that  $\Lambda_X(G)$  is closed under intersection. Then, we have  $\Sigma_X(G) \subset \Lambda_X(G)$ .

*Proof.* Direct consequence of Lemma 458 (generated  $\lambda$ -system is minimum,  $\Lambda_X(G)$  is a  $\lambda$ -system, and  $G \subset \Lambda_X(G)$ ), Definition 453 ( $\lambda$ -system,  $\Lambda_X(G)$  contains the full set, is closed under complement and countable disjoint union), Lemma 422 (closedness under countable disjoint union and countable union,  $\Lambda_X(G)$  is closed under countable union), Lemma 468 (equivalent definition of  $\sigma$ -algebra,  $\Lambda_X(G)$  is a  $\sigma$ -algebra), and Lemma 474 (generated  $\sigma$ -algebra is minimum).  $\square$

**Lemma 491 ( $\sigma$ -algebra generated by  $\lambda$ -system).**

Let  $X$  be a set. Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_X(\Lambda_X(G)) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 489 ( $\sigma$ -algebra contains  $\lambda$ -system,  $\Lambda_X(G) \subset \Sigma_X(G)$ ), Lemma 474 (generated  $\sigma$ -algebra is minimum,  $\Sigma_X(G)$  is a  $\sigma$ -algebra, and  $\Sigma_X(\Lambda_X(G)) \subset \Sigma_X(G)$ ), Lemma 458 (generated  $\lambda$ -system is minimum,  $G \subset \Lambda_X(G)$ ), and Lemma 475 ( $\sigma$ -algebra generation is monotone,  $\Sigma_X(G)$  is included in  $\Sigma_X(\Lambda_X(G))$ ).  $\square$

**Lemma 492 (other  $\sigma$ -algebra generator).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ .

Assume that  $G_1 \subset \Sigma_X(G_2)$  and  $G_2 \subset \Sigma_X(G_1)$ . Then, we have  $\Sigma_X(G_1) = \Sigma_X(G_2)$ .

*Proof.* From Lemma 475 ( $\sigma$ -algebra generation is monotone), and Lemma 476 ( $\sigma$ -algebra generation is idempotent), we have

$$\Sigma_X(G_1) \subset \Sigma_X(\Sigma_X(G_2)) = \Sigma_X(G_2) \quad \text{and} \quad \Sigma_X(G_2) \subset \Sigma_X(\Sigma_X(G_1)) = \Sigma_X(G_1).$$

Therefore, we have  $\Sigma_X(G_1) = \Sigma_X(G_2)$ .  $\square$

**Lemma 493 (complete generated  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $G_1, G_2 \subset \mathcal{P}(X)$ .

Assume that  $G_2 \subset \Sigma_X(G_1)$ . Then, we have  $\Sigma_X(G_1 \cup G_2) = \Sigma_X(G_1)$ .

*Proof.* From Lemma 474 (generated  $\sigma$ -algebra is minimum), and Lemma 475 ( $\sigma$ -algebra generation is monotone), we have  $G_1 \subset \Sigma_X(G_1) \subset \Sigma_X(G_1 \cup G_2)$ . From Lemma 474 (generated  $\sigma$ -algebra is minimum), and **monotonicity of union**, we have  $G_1 \cup G_2 \subset \Sigma_X(G_1)$ .

Therefore, from Lemma 492 (other  $\sigma$ -algebra generator), we have  $\Sigma_X(G_1 \cup G_2) = \Sigma_X(G_1)$ .  $\square$

**Lemma 494 (countable  $\sigma$ -algebra generator).**

Let  $X$  be a set. Let  $G_1 \subset G_2 \subset \mathcal{P}(X)$ . Assume that elements of  $G_2$  are countable unions of elements of  $G_1$ . Then, we have  $\Sigma_X(G_1) = \Sigma_X(G_2)$ .

*Proof.* Direct consequence of Lemma 474 (generated  $\sigma$ -algebra is minimum,  $G_1 \subset G_2 \subset \Sigma_X(G_2)$ ), Definition 467 ( $\sigma$ -algebra, closedness under countable union, thus  $G_2 \subset \Sigma_X(G_1)$ ), and Lemma 492 (other  $\sigma$ -algebra generator).  $\square$

**Definition 495 (Borel  $\sigma$ -algebra).**

Let  $(X, \mathcal{T})$  be a topological space. The *Borel  $\sigma$ -algebra of  $X$*  is the  $\sigma$ -algebra generated by the open subsets; it is denoted  $\mathcal{B}(X) \stackrel{\text{def.}}{=} \Sigma_X(\mathcal{T})$ .

$\mathcal{B}(X)$ -measurable subsets are called *Borel subsets of  $X$* .

**Lemma 496 (some Borel subsets).**

Let  $(X, \mathcal{T})$  be a topological space. Then, open and closed subsets are Borel subsets of  $X$ .

Moreover, if the space is separable, then, countable subsets are Borel subsets of  $X$ .

*Proof.* From Definition 495 (Borel  $\sigma$ -algebra), and Definition 467 ( $\sigma$ -algebra), the Borel subsets are the elements of the Borel  $\sigma$ -algebra.

Let  $Y$  be an open subset of  $X$ . Then, from Definition 495 (Borel  $\sigma$ -algebra), and Lemma 474 (*generated  $\sigma$ -algebra is minimum*), we have  $Y \in \mathcal{T} \subset \mathcal{B}(X)$ .

Let  $Y$  be a closed subset of  $X$ . Then, from Definition 225 (*topological space, closed subset*), we have  $Y^c \in \mathcal{T} \subset \mathcal{B}(X)$ . Hence, from Definition 467 ( $\sigma$ -algebra, *closedness under complement*), and since **involutiveness of complement**, we have  $Y = (Y^c)^c \in \mathcal{B}(X)$ .

Let  $Y$  be a countable subset of  $X$ . Then, from **the definition of countability**, there exists  $I \subset \mathbb{N}$ , and  $(x_i)_{i \in I} \in X$  such that  $Y = \bigcup_{i \in I} \{x_i\}$ . Let  $i \in I$ . Then, from **closedness of singletons in a separable space**,  $\{x_i\}$  is closed, thus it belongs to  $\mathcal{B}(X)$ . Hence, from Definition 467 ( $\sigma$ -algebra, *closedness under countable union*), we have  $Y \in \mathcal{B}(X)$ .

Therefore, open, closed, and countable subsets are Borel subsets of  $X$ .  $\square$

**Lemma 497 (countable Borel  $\sigma$ -algebra generator).**

Let  $(X, \mathcal{T})$  be a topological space. Let  $G \subset \mathcal{T}$ . Assume that all open subsets of  $X$  are countable unions of elements of  $G$ . Then, we have  $\mathcal{B}(X) = \Sigma_X(G)$ .

*Proof.* Direct consequence of Lemma 494 (*countable  $\sigma$ -algebra generator, with  $G' \stackrel{\text{def.}}{=} \mathcal{T}$* ), and Definition 495 (Borel  $\sigma$ -algebra,  $\mathcal{B}(X) = \Sigma_X(\mathcal{T})$ ).  $\square$

*Remark 498.* The previous lemma means that if  $G$  contains a countable topological basis of  $(X, \mathcal{T})$  (and possibly other open subsets), then the Borel  $\sigma$ -algebra of  $X$  is generated by  $G$ .

*Remark 499.* Note that metric spaces are separable topological spaces, thus Lemma 496 (*some Borel subsets*), and Lemma 497 (*countable Borel  $\sigma$ -algebra generator*) apply in metric spaces.

*Remark 500.* Note that in the following proof, the point (1a) does not depend on the hypotheses of the Lemma. It could be an independent lemma that characterizes  $\Sigma_1 \overline{\times} \Sigma_2$ .

**Lemma 501 (set algebra generated by product of  $\sigma$ -algebras).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ , and  $\overline{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$ . Then,  $\mathcal{A}_X(\overline{\Sigma})$  is the set of finite disjoint unions of elements of  $\overline{\Sigma}$ .

*Proof.* (1).  $X \in \overline{\Sigma}$ . Direct consequence of Definition 214 (*product of subsets of parties*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, contains full set*).

(2).  $\overline{\Sigma}$  is closed under intersection. Let  $A, B \in \overline{\Sigma}$ .

From Definition 214 (*product of subsets of parties*), let  $A_1, B_1 \in \Sigma_1$  and  $A_2, B_2 \in \Sigma_2$  such that  $A = A_1 \times A_2$  and  $B = B_1 \times B_2$ . Then, from **compatibility of intersection with Cartesian product**, and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under intersection*), we have  $A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2)$  with  $A_1 \cap B_1 \in \Sigma_1$  and  $A_2 \cap B_2 \in \Sigma_2$ . Thus,  $A \cap B \in \overline{\Sigma}$ . Hence,  $\overline{\Sigma}$  is closed under intersection.

(3).  $\forall A \in \overline{\Sigma}, \exists B, C \in \overline{\Sigma}, B \cap C = \emptyset \wedge A^c = B \uplus C$ . Let  $A \in \overline{\Sigma}$ .

From Definition 214 (*product of subsets of parties*), let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$  such that  $A = A_1 \times A_2$ .

Let  $B \stackrel{\text{def.}}{=} X_1 \times A_2^c$  and  $C \stackrel{\text{def.}}{=} A_1^c \times A_2$ . Then, from Definition 214 (*product of subsets of parties*), Lemma 468 (*equivalent definition of  $\sigma$ -algebra*, contains full set, and closedness under complement), and **set operations properties**, we have  $B, C \in \Sigma$ ,  $B \cap C = \emptyset$ , and  $A^c = B \uplus C$ .

Therefore, from (1), (2), (3), and Lemma 441 (*explicit set algebra*, with  $G \stackrel{\text{def.}}{=} \Sigma$ ),  $\mathcal{A}_X(\Sigma)$  is the set of finite disjoint unions of elements of  $\Sigma$ .  $\square$

### 6.6.7 Dynkin $\pi$ - $\lambda$ theorem

**Lemma 502 ( $\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is a  $\pi$ -system on  $X$  and a  $\lambda$ -system on  $X$ . Then,  $\mathcal{S}$  is a  $\sigma$ -algebra on  $X$ .*

*Proof.* Direct consequence of Lemma 460 ( *$\lambda$ -system generation is idempotent*,  $\Lambda_X(\mathcal{S}) = \mathcal{S}$ ), Definition 424 ( *$\pi$ -system*,  $\Lambda_X(\mathcal{S})$  is  $\pi$ -system), Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*,  $\Sigma_X(\mathcal{S})$  is included in  $\Lambda_X(\mathcal{S})$ ), Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*,  $\Lambda_X(\mathcal{S}) \subset \Sigma_X(\mathcal{S})$ ), and Lemma 476 ( *$\sigma$ -algebra generation is idempotent*,  $\Sigma_X(\mathcal{S}) = \Lambda_X(\mathcal{S}) = \mathcal{S}$ ).  $\square$

**Theorem 503 (Dynkin  $\pi$ - $\lambda$  theorem).**

*Let  $X$  be a set. Let  $\Pi$  be a  $\pi$ -system on  $X$ . Then, we have  $\Lambda_X(\Pi) = \Sigma_X(\Pi)$ .*

*Proof.* Direct consequence of Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*,  $\Lambda_X(\Pi)$  is  $\pi$ -system), Lemma 458 (*generated  $\lambda$ -system is minimum*,  $\Lambda_X(\Pi)$  is  $\lambda$ -system), Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*,  $\Lambda_X(\Pi)$  is  $\sigma$ -algebra), Lemma 476 ( *$\sigma$ -algebra generation is idempotent*,  $\Sigma_X(\Lambda_X(\Pi)) = \Lambda_X(\Pi)$ ), and Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*,  $\Sigma_X(\Lambda_X(\Pi)) = \Sigma_X(\Pi)$ ).  $\square$

*Remark 504.* Note that the Dynkin  $\pi$ - $\lambda$  theorem may take the following form: if a  $\lambda$ -system contains a  $\pi$ -system, then it also contains the  $\sigma$ -algebra generated by the  $\pi$ -system.

Similarly, next lemma is an application lemma for Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*). It is used to prove Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*) in Section 6.8.3, itself later used to prove uniqueness of the Lebesgue measure.

**Lemma 505 (usage of Dynkin  $\pi$ - $\lambda$  theorem).**

*Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate over  $\mathcal{P}(X)$ , and let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ . Let  $G \subset \mathcal{P}(X)$  be a nonempty generator of  $\Sigma$ . Assume that  $\Pi_X(G) \subset \mathcal{S}$ , and that  $\mathcal{S}$  is a  $\lambda$ -system. Then, we have  $\mathcal{S} = \Sigma$ , i.e.  $P$  holds for all measurable subsets in  $\Sigma$ .*

*Proof.* From the definition of  $\mathcal{S}$ , we have  $\mathcal{S} \subset \Sigma$ .

Let  $\Pi \stackrel{\text{def.}}{=} \Pi_X(G)$ . Then, from Definition 473 (*generated  $\sigma$ -algebra*,  $\Sigma = \Sigma_X(G)$ ), Lemma 428 (*generated  $\pi$ -system is minimum*,  $G \subset \Pi$ ), Lemma 475 ( *$\sigma$ -algebra generation is monotone*,  $\Sigma_X(G)$  is included in  $\Sigma_X(\Pi)$ ), Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*,  $\Sigma_X(\Pi) = \Lambda_X(\Pi)$ ), Lemma 459 ( *$\lambda$ -system generation is monotone*,  $\Lambda_X(\Pi) \subset \Lambda_X(\mathcal{S})$ ), and Lemma 460 ( *$\lambda$ -system generation is idempotent*,  $\Lambda_X(\mathcal{S}) = \mathcal{S}$ ), we have  $\Sigma \subset \mathcal{S}$ .

Therefore, we have the equality, i.e.  $P$  holds for all subsets in  $\Sigma$ .  $\square$

### 6.6.8 Monotone class theorem

**Lemma 506 (algebra and monotone class is  $\sigma$ -algebra).** *Let  $X$  be a set. Let  $\mathcal{S} \subset \mathcal{P}(X)$ . Assume that  $\mathcal{S}$  is a set algebra on  $X$  and a monotone class on  $X$ . Then,  $\mathcal{S}$  is a  $\sigma$ -algebra on  $X$ .*

*Proof.* Direct consequence of Lemma 447 (*monotone class generation is idempotent*,  $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$ ), Definition 432 (*set algebra*,  $\mathcal{C}_X(\mathcal{S})$  is set algebra), Lemma 486 (*monotone class contains  $\sigma$ -algebra*,  $\Sigma_X(\mathcal{S}) \subset \mathcal{C}_X(\mathcal{S})$ ), Lemma 485 ( *$\sigma$ -algebra contains monotone class*,  $\mathcal{C}_X(\mathcal{S}) \subset \Sigma_X(\mathcal{S})$ ), and Lemma 476 ( *$\sigma$ -algebra generation is idempotent*,  $\Sigma_X(\mathcal{S}) = \mathcal{C}_X(\mathcal{S}) = \mathcal{S}$ ).  $\square$



**Theorem 507 (monotone class).**

Let  $X$  be a set. Let  $\mathcal{A}$  be a set algebra on  $X$ . Then,  $\mathcal{C}_X(\mathcal{A}) = \Sigma_X(\mathcal{A})$ .

*Proof.* Direct consequence of Lemma 452 (monotone class generated by set algebra,  $\mathcal{C}_X(\mathcal{A})$  is set algebra), Lemma 445 (generated monotone class is minimum,  $\mathcal{C}_X(\mathcal{A})$  is monotone class), Lemma 506 (algebra and monotone class is  $\sigma$ -algebra,  $\mathcal{C}_X(\mathcal{A})$  is  $\sigma$ -algebra), Lemma 476 ( $\sigma$ -algebra generation is idempotent,  $\Sigma_X(\mathcal{C}_X(\mathcal{A})) = \mathcal{C}_X(\mathcal{A})$ ), and Lemma 487 ( $\sigma$ -algebra generated by monotone class,  $\Sigma_X(\mathcal{C}_X(\mathcal{A})) = \Sigma_X(\mathcal{A})$ ).  $\square$

*Remark 508.* Note that the monotone class theorem may take the following form: if a monotone class contains a set algebra, then it also contains the  $\sigma$ -algebra generated by the algebra.

Similarly, next lemma is an application lemma for Theorem 507 (monotone class). It is used to prove Lemma 791 (measurability of measure of section (finite)), and Lemma 801 (uniqueness of tensor product measure) in Section 6.10.4 in the context of product spaces.

**Lemma 509 (usage of monotone class theorem).**

Let  $(X, \Sigma)$  be a measurable space. Let  $P$  be a predicate over  $\mathcal{P}(X)$ , and let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid P(A)\}$ . Let  $G \subset \mathcal{P}(X)$  be a generator of  $\Sigma$ . Assume that  $\mathcal{A}_X(G) \subset \mathcal{S}$ , and that  $\mathcal{S}$  is a monotone class. Then, we have  $\mathcal{S} = \Sigma$ , i.e.  $P$  holds for all measurable subsets in  $\Sigma$ .

*Proof.* From the definition of  $\mathcal{S}$ , we have  $\mathcal{S} \subset \Sigma$ .

Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(G)$ . Then, from Definition 473 (generated  $\sigma$ -algebra,  $\Sigma = \Sigma_X(G)$ ), Lemma 438 (generated set algebra is minimum,  $G \subset \mathcal{A}$ ), Lemma 475 ( $\sigma$ -algebra generation is monotone,  $\Sigma_X(G) \subset \Sigma_X(\mathcal{A})$ ), Theorem 507 (monotone class,  $\Sigma_X(\mathcal{A}) = \mathcal{C}_X(\mathcal{A})$ ), Lemma 446 (monotone class generation is monotone,  $\mathcal{C}_X(\mathcal{A}) \subset \mathcal{C}_X(\mathcal{S})$ ), and Lemma 447 (monotone class generation is idempotent,  $\mathcal{C}_X(\mathcal{S}) = \mathcal{S}$ ), we have  $\Sigma \subset \mathcal{S}$ .

Therefore, we have the equality, i.e.  $P$  holds for all subsets in  $\Sigma$ .  $\square$

**6.6.9 Measurable function**

**Definition 510 (measurable function).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. A function  $f : X \rightarrow X'$  is said *measurable* (for  $\Sigma$  and  $\Sigma'$ ) iff

$$(117) \quad f^{-1}(\Sigma') \subset \Sigma \quad \text{i.e. } \forall A' \in \Sigma', f^{-1}(A') \in \Sigma.$$

If  $\Sigma$  and  $\Sigma'$  are the Borel  $\sigma$ -algebras of  $X$  and  $X'$ , then  $f$  is called *Borel function*.

**Lemma 511 (inverse  $\sigma$ -algebra).**

Let  $X$  be a set. Let  $(X', \Sigma')$  be a measurable space. Let  $f : X \rightarrow X'$ . Then,  $\Sigma = f^{-1}(\Sigma')$  is the smallest  $\sigma$ -algebra of  $f^{-1}(X') \subset X$  such that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

It is called the inverse  $\sigma$ -algebra of  $\Sigma'$  by  $f$ .

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra), **homogeneity of inverse image, compatibility of inverse image with complement, union and intersection**, and Definition 510 (measurable function).  $\square$

**Lemma 512 (image  $\sigma$ -algebra).**

Let  $(X, \Sigma)$  be a measurable space. Let  $X'$  be a set. Let  $f : X \rightarrow X'$ . Then,  $\Sigma' = \Sigma_f \stackrel{\text{def.}}{=} \{A' \subset X' \mid f^{-1}(A') \in \Sigma\}$  is the largest  $\sigma$ -algebra of  $f(X)$  such that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . It is called the image  $\sigma$ -algebra of  $\Sigma$  by  $f$ .

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra), **compatibility of inverse image with complement, union and intersection, homogeneity of inverse image ( $f^{-1}(\emptyset) = \emptyset$ )**, and Definition 510 (measurable function).  $\square$

**Lemma 513 (identity function is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Then, the identity function is measurable for  $\Sigma$  (and  $\Sigma$ ).

*Proof.* Direct consequence of **the definition of the identity function**, and Definition 510 (*measurable function*).  $\square$

**Lemma 514 (constant function is measurable).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Assume that  $f$  is constant. Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

*Proof.* Let  $c' \in X'$  be the constant value taken by the function  $f$ . Let  $A' \in \Sigma'$ . **Case  $c' \in A'$ .** Then, we have  $f^{-1}(A') = X$ . **Case  $c' \notin A'$ .** Then, we have  $f^{-1}(A') = \emptyset$ . Thus, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  $f^{-1}(A')$  always belongs to the  $\sigma$ -algebra  $\Sigma$ . Therefore,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .  $\square$

**Lemma 515 (inverse image of generating family).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Then, we have

$$(118) \quad \forall G' \subset \mathcal{P}(X'), \quad \Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G')).$$

*Proof.* **“Left” included in “right”.** Let  $G' \subset \mathcal{P}(X')$ . Then, from Lemma 474 (*generated  $\sigma$ -algebra is minimum*), we have  $G' \subset \Sigma_{X'}(G')$ . Thus, from **monotonicity of inverse image**,  $f^{-1}(G')$  is a subset of  $f^{-1}(\Sigma_{X'}(G'))$ . Hence, from Lemma 475 ( *$\sigma$ -algebra generation is monotone*), Lemma 511 (*inverse  $\sigma$ -algebra,  $f^{-1}(\Sigma_{X'}(G'))$  is  $\sigma$ -algebra*), and Lemma 476 ( *$\sigma$ -algebra generation is idempotent*), we have  $\Sigma_X(f^{-1}(G')) \subset \Sigma_X(f^{-1}(\Sigma_{X'}(G'))) = f^{-1}(\Sigma_{X'}(G'))$ .

**“Right” included in “left”.** Conversely, let  $A' \in G' \subset \mathcal{P}(X')$ . Then, from **monotonicity of inverse image**, and Lemma 474 (*generated  $\sigma$ -algebra is minimum*), we have

$$f^{-1}(A') \in f^{-1}(G') \subset \Sigma_X(f^{-1}(G')).$$

Thus, from Lemma 512 (*image  $\sigma$ -algebra*), we have  $G' \subset (\Sigma_X(f^{-1}(G')))_f$ , and from Lemma 474 (*generated  $\sigma$ -algebra is minimum*), we have  $\Sigma_{X'}(G') \subset (\Sigma_X(f^{-1}(G')))_f$ . Hence, from Lemma 512 (*image  $\sigma$ -algebra*), and **the definition of inverse image**, we have

$$f^{-1}(\Sigma_{X'}(G')) \subset \Sigma_X(f^{-1}(G')).$$

Therefore, we have  $\Sigma_X(f^{-1}(G')) = f^{-1}(\Sigma_{X'}(G'))$ .  $\square$

**Lemma 516 (equivalent definition of measurable function).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $f : X \rightarrow X'$ . Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$  iff

$$(119) \quad \exists G' \subset \mathcal{P}(X'), \quad \Sigma_{X'}(G') = \Sigma' \implies f^{-1}(G') \subset \Sigma.$$

*Proof.* **“Left” implies “right”.** Assume first that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . Let  $G' \stackrel{\text{def.}}{=} \Sigma'$ . Then, from Lemma 476 ( *$\sigma$ -algebra generation is idempotent*), we have  $\Sigma_{X'}(\Sigma') = \Sigma'$ . Hence, from Definition 510 (*measurable function*), we have  $f^{-1}(G') = f^{-1}(\Sigma') \subset \Sigma$ .

**“Right” implies “left”.** Conversely, assume now that there exists  $G' \subset \mathcal{P}(X')$  such that  $\Sigma_{X'}(G') = \Sigma'$  and  $f^{-1}(G') \subset \Sigma$ . Then, from Lemma 515 (*inverse image of generating family*), Lemma 475 ( *$\sigma$ -algebra generation is monotone*), and Lemma 476 ( *$\sigma$ -algebra generation is idempotent*), we have  $f^{-1}(\Sigma') = f^{-1}(\Sigma_{X'}(G')) = \Sigma_X(f^{-1}(G')) \subset \Sigma_X(\Sigma) = \Sigma$ .

Therefore, we have the equivalence.  $\square$

**Lemma 517 (continuous is measurable).**

Let  $(X, \mathcal{T})$  and  $(X', \mathcal{T}')$  be topological spaces. Assume that  $X$  and  $X'$  are equipped with their Borel  $\sigma$ -algebras. Let  $f : X \rightarrow X'$ . Assume that  $f$  is continuous. Then,  $f$  is a Borel function.



*Proof.* Let  $O' \in \mathcal{T}'$ . Then, from **the definition of continuity**,  $f^{-1}(O') \in \mathcal{T}$ . Thus, from Lemma 474 (**generated  $\sigma$ -algebra is minimum**), and Definition 495 (**Borel  $\sigma$ -algebra**), we have  $\mathcal{B}(X') = \Sigma_{X'}(\mathcal{T}')$  and  $f^{-1}(\mathcal{T}') \subset \mathcal{T} \subset \Sigma_X(\mathcal{T}) = \mathcal{B}(X)$ . Therefore, from Lemma 516 (**equivalent definition of measurable function**), and Definition 510 (**measurable function**),  $f$  is a Borel function from  $X$  to  $X'$ .  $\square$

**Lemma 518 (compatibility of measurability with composition).**

Let  $(X, \Sigma)$ ,  $(X', \Sigma')$ , and  $(X'', \Sigma'')$  be measurable spaces. Let  $f : X \rightarrow X'$  and  $g : X' \rightarrow X''$ . Assume that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ , and that  $g$  is measurable for  $\Sigma'$  and  $\Sigma''$ . Then,  $g \circ f$  is measurable for  $\Sigma$  and  $\Sigma''$ .

*Proof.* Direct consequence of **the definition of composition**, Definition 510 (**measurable function**, with  $f$  and  $g$ ), and **monotonicity of inverse image**.  $\square$

### 6.6.10 Measurable subspace

*Remark 519.* We recall that  $\bar{\cap}$  denotes a set of traces of subsets, see Definition 213 (**trace of subsets of parties**).

**Lemma 520 (trace  $\sigma$ -algebra).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then,  $\Sigma \bar{\cap} Y$  is a  $\sigma$ -algebra of  $Y$ , and  $i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma$ .

$\Sigma \bar{\cap} Y$  is called trace  $\sigma$ -algebra of  $\Sigma$  on  $Y$ . The measurable space  $(Y, \Sigma \bar{\cap} Y)$  is said measurable subspace of  $(X, \Sigma)$ .

*Proof.* From Definition 467 ( **$\sigma$ -algebra**,  $\emptyset \in \Sigma$ ), and since  $\emptyset$  is **absorbing for intersection**, we have  $\emptyset = \emptyset \cap Y \in \Sigma \bar{\cap} Y$ .

Let  $A \in \Sigma \bar{\cap} Y$ . Then, from Definition 213 (**trace of subsets of parties**), let  $B \in \Sigma$  such that  $A = B \cap Y$ . Then, from **the definition of set difference**, **De Morgan's laws**, **distributivity of intersection over union**, **the definition of complement**, and **commutativity of intersection**, we have

$$Y \setminus A = Y \setminus (B \cap Y) = Y \cap (B \cap Y)^c = Y \cap (B^c \cup Y^c) = (Y \cap B^c) \cup (Y \cap Y^c) = B^c \cap Y.$$

Hence, from Definition 467 ( **$\sigma$ -algebra**, closedness under complement), we have  $Y \setminus A \in \Sigma \bar{\cap} Y$ .

Let  $I$  be a set. Let  $(A_i)_{i \in I} \in \Sigma \bar{\cap} Y$ . Then, from Definition 213 (**trace of subsets of parties**), for all  $i \in I$ , let  $B_i \in \Sigma$  such that  $A_i = B_i \cap Y$ . Then, from **distributivity of intersection over union**, we have

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} (B_i \cap Y) = \bigcup_{i \in I} B_i \cap Y.$$

Hence, from Definition 467 ( **$\sigma$ -algebra**, closedness under union), we have  $\bigcup_{i \in I} A_i \in \Sigma \bar{\cap} Y$ .

Therefore, from Definition 467 ( **$\sigma$ -algebra**),  $\Sigma \bar{\cap} Y$  is a  $\sigma$ -algebra of  $Y$ . Moreover, from **the definition of the canonical injection**, and Definition 510 (**measurable function**),  $i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma$ .  $\square$

**Lemma 521 (measurability of measurable subspace).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Then, we have  $Y \in \Sigma$  iff  $\Sigma \bar{\cap} Y = \{A \in \Sigma \mid A \subset Y\}$ .

*Proof.* Let  $A \in \Sigma$  such that  $A \subset Y$ . Then, from Definition 213 (**trace of subsets of parties**),  $A = A \cap Y$  belongs to  $\Sigma \bar{\cap} Y$ . Thus, we always have  $\{A \in \Sigma \mid A \subset Y\} \subset \Sigma \bar{\cap} Y$ .

**“Left” implies “right”.** Assume first that  $Y \in \Sigma$ . Let  $A \in \Sigma$ , and let  $A' \stackrel{\text{def}}{=} A \cap Y$ . Then, from Lemma 468 (**equivalent definition of  $\sigma$ -algebra**, closedness under countable intersection), we have  $A' \in \Sigma$  and  $A' \subset Y$ . Thus, from Definition 213 (**trace of subsets of parties**),  $\Sigma \bar{\cap} Y \subset \{A \in \Sigma \mid A \subset Y\}$ . Hence,  $\Sigma \bar{\cap} Y = \{A \in \Sigma \mid A \subset Y\}$ .

**“Right” implies “left”.** Conversely, assume now that  $\Sigma \bar{\cap} Y = \{A \in \Sigma \mid A \subset Y\}$ . Then, from Lemma 520 (*trace  $\sigma$ -algebra*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, contains full set*), we have  $Y \in \Sigma \bar{\cap} Y \subset \Sigma$ .

Therefore, we have the equivalence.  $\square$

**Lemma 522 (generating measurable subspace).**

Let  $X$  be a set. Let  $Y \subset X$ . Let  $G \subset \mathcal{P}(X)$ . Then, we have  $\Sigma_Y(G \bar{\cap} Y) = \Sigma_X(G) \bar{\cap} Y$ .

*Proof.* Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then, from Definition 213 (*trace of subsets of parties*), we have  $i^{-1}(G) = G \bar{\cap} Y$  and  $i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \bar{\cap} Y$ . Therefore, from Lemma 515 (*inverse image of generating family*), we have

$$\Sigma_Y(G \bar{\cap} Y) = \Sigma_Y(i^{-1}(G)) = i^{-1}(\Sigma_X(G)) = \Sigma_X(G) \bar{\cap} Y.$$

$\square$

**Lemma 523 (Borel sub- $\sigma$ -algebra).**

Let  $(X, \mathcal{T})$  be a topological space.

Let  $Y \subset X$ . Then, we have  $\mathcal{B}(Y) = \mathcal{B}(X) \bar{\cap} Y$ , and  $Y \in \mathcal{B}(X)$  iff  $\mathcal{B}(Y) = \{A \in \mathcal{B}(X) \mid A \subset Y\}$ .

*Proof.* Direct consequence of Definition 495 (*Borel  $\sigma$ -algebra*), **the definition of trace topology on  $Y$** , Lemma 522 (*generating measurable subspace*), and Lemma 521 (*measurability of measurable subspace*).  $\square$

**Lemma 524 (characterization of Borel subsets).**

Let  $(X, \mathcal{T})$  be a topology space.

Let  $(Y_n)_{n \in \mathbb{N}} \in \mathcal{B}(X)$ . Assume that  $X = \biguplus_{n \in \mathbb{N}} Y_n$ . Let  $A \subset X$ . Then, we have

$$(120) \quad A \in \mathcal{B}(X) \iff \forall n \in \mathbb{N}, \quad A \cap Y_n \in \mathcal{B}(Y_n).$$

*Proof.* **“Left” implies “right”.** Assume first that  $A \in \mathcal{B}(X)$ . Let  $n \in \mathbb{N}$ . Then,  $A \cap Y_n \subset Y_n$ , and from Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), we have  $A \cap Y_n \in \mathcal{B}(X)$ . Hence, from Lemma 523 (*Borel sub- $\sigma$ -algebra*), we have  $A \cap Y_n \in \mathcal{B}(Y_n)$ .

**“Right” implies “left”.** Conversely, assume now that for all  $n \in \mathbb{N}$ ,  $A \cap Y_n \in \mathcal{B}(Y_n)$ . Let  $n \in \mathbb{N}$ . Then, since  $Y_n \in \mathcal{B}(X)$ , from Lemma 523 (*Borel sub- $\sigma$ -algebra*), we have  $A \cap Y_n \in \mathcal{B}(X)$ . Hence, from Lemma 207 (*compatibility of partition with intersection*), Definition 495 (*Borel  $\sigma$ -algebra*), and Definition 467 ( *$\sigma$ -algebra, closedness under countable union*), we have

$$A = \biguplus_{n \in \mathbb{N}} (A \cap Y_n) \in \mathcal{B}(X).$$

Therefore, we have the equivalence.  $\square$

**Lemma 525 (source restriction of measurable function).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be

measurable spaces. Let  $f : X \rightarrow X'$ . Let  $Y \subset X$ . Let  $f|_Y$  be the restriction of  $f$  to the source  $Y$ . Assume that  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ . Then,  $f|_Y$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma'$ .

*Proof.* Let  $i$  be the canonical injection from  $Y$  to  $X$ . Then, from Definition 213 (*trace of subsets of parties*), and Lemma 520 (*trace  $\sigma$ -algebra*),  $i^{-1}(\Sigma) = \Sigma \bar{\cap} Y$  is a  $\sigma$ -algebra of  $Y$ . Thus, from Definition 510 (*measurable function*),  $i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma$ . Therefore, from Lemma 518 (*compatibility of measurability with composition*),  $f|_Y = f \circ i$  is measurable for  $\Sigma \bar{\cap} Y$  and  $\Sigma'$ .  $\square$

**Lemma 526 (destination restriction of measurable function).**

Let  $(X, \Sigma)$  and  $(X', \Sigma')$

be measurable spaces. Let  $f : X \rightarrow X'$ . Let  $Y' \subset X'$ . Assume that  $f(X) \subset Y'$ . Let  $f|^{Y'}$  be the restriction of  $f$  to the destination  $Y'$ . Then,  $f$  is measurable for  $\Sigma$  and  $\Sigma'$  iff  $f|^{Y'}$  is measurable for  $\Sigma$  and  $\Sigma' \bar{\cap} Y'$ .

*Proof.* Let  $A' \in \Sigma'$ . Then, from **monotonicity of inverse image and complement**, we have

$$f^{-1}(A' \cap Y'^c) \subset f^{-1}(Y'^c) \subset f^{-1}(f(X)^c) = \emptyset.$$

Then, from **compatibility of inverse image with disjoint union**, and **the definition of destination restriction**, we have

$$f^{-1}(A') = f^{-1}((A' \cap Y') \uplus (A' \cap Y'^c)) = f^{-1}(A' \cap Y') = \left(f|_{Y'}\right)^{-1}(A' \cap Y').$$

Hence,  $f^{-1}(\Sigma') = \left(f|_{Y'}\right)^{-1}(\Sigma' \cap Y')$ . Therefore, from Definition 510 (*measurable function*),  $f|_{Y'}$  is measurable for  $\Sigma$  and  $\Sigma' \cap Y'$ .  $\square$

**Lemma 527 (measurability of function defined on a partition).** Let  $(X, \Sigma)$  and  $(X', \Sigma')$  be measurable spaces. Let  $I \subset \mathbb{N}$ . For all  $i \in I$ , let  $X_i \in \Sigma$  and  $f_i : X \rightarrow X'$ . Assume that for all  $i \in I$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'$ , and that  $X = \bigsqcup_{i \in I} X_i$ . Then, the function defined by  $f_i$  on  $X_i$  for all  $i \in I$  is measurable for  $\Sigma$  and  $\Sigma'$ .

*Proof.* Let  $f$  be the function defined by for all  $i \in I$ , for all  $x \in X_i$ ,  $f(x) \stackrel{\text{def.}}{=} f_i(x)$ . Let  $A' \in \Sigma'$ . Let  $i \in I$ . Then, from Definition 510 (*measurable function, with  $f_i$* ), we have  $f_i^{-1}(A') \in \Sigma$ . Thus, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection and union*), we have  $f^{-1}(A') = \bigcup_{i \in I} (X_i \cap f_i^{-1}(A')) \in \Sigma$ . Hence,  $f^{-1}(\Sigma') \subset \Sigma$ .

Therefore, from Definition 510 (*measurable function*),  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .  $\square$

### 6.6.11 Product of measurable spaces

*Remark 528.* We recall that  $[n..p]$  denotes an interval of integers, and that  $\overline{\prod}$  and  $\overline{\times}$  denote a set of Cartesian products of subsets, see Definition 214 (*product of subsets of parties*).

**Definition 529 (tensor product of  $\sigma$ -algebras).**

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space. Let  $X \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} X_i$  and  $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} \Sigma_i$ . The (*tensor*) *product of the  $\sigma$ -algebras*  $(\Sigma_i)_{i \in [1..m]}$  on  $X$  is the  $\sigma$ -algebra generated by  $\overline{\Sigma}$ ; it is denoted  $\otimes_{i \in [1..m]} \Sigma_i \stackrel{\text{def.}}{=} \Sigma_X(\overline{\Sigma})$ .

**Lemma 530 (product of measurable subsets is measurable).**

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and  $A_i \in \Sigma_i$ .

Let  $\overline{\Sigma} \stackrel{\text{def.}}{=} \overline{\prod}_{i \in [1..m]} \Sigma_i$  and  $\Sigma \stackrel{\text{def.}}{=} \otimes_{i \in [1..m]} \Sigma_i$ . Then,  $\overline{\prod}_{i \in [1..m]} A_i \in \overline{\Sigma} \subset \Sigma$ .

*Proof.* Direct consequence of Definition 214 (*product of subsets of parties*), Definition 529 (*tensor product of  $\sigma$ -algebras*), and Lemma 474 (*generated  $\sigma$ -algebra is minimum,  $\overline{\Sigma} \subset \Sigma_X(\overline{\Sigma})$ ).  $\square$*

**Lemma 531 (measurability of function to product space).**

Let  $m \in [2..\infty)$ . Let  $(X, \Sigma)$  be a measurable space. For all  $i \in [1..m]$ , let  $(X'_i, \Sigma'_i)$  be a measurable space, and  $f_i : X \rightarrow X'_i$ . Then,  $(f_i)_{i \in [1..m]}$  is measurable for  $\Sigma$  and  $\otimes_{i \in [1..m]} \Sigma'_i$  iff for all  $i \in [1..m]$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

*Proof.* Let  $\Sigma' \stackrel{\text{def.}}{=} \otimes_{i \in [1..m]} \Sigma'_i$ .

“Left” implies “right”. Assume first that  $(f_i)_{i \in [1..m]}$  is measurable for  $\Sigma$  and  $\Sigma'$ .

Let  $i \in [1..m]$ . Let  $A'_i \in \Sigma'_i$ . Let  $A' \stackrel{\text{def.}}{=} X'_1 \times \dots \times A'_i \times \dots \times X'_m$ . Then, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra,  $\underline{X'_i} \in \Sigma'_i$* ), Definition 529 (*tensor product of  $\sigma$ -algebras*), and Lemma 530

(*product of measurable subsets is measurable*), we have  $A' \in \Sigma'$ . Thus, from Definition 510 (*measurable function, with  $f$* ), we have  $f_i^{-1}(A'_i) = f^{-1}(A') \in \Sigma$ . Hence, from Definition 510 (*measurable function*),  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

“**Right**” implies “**left**”. Assume now that for all  $i \in [1..m]$ ,  $f_i$  is measurable for  $\Sigma$  and  $\Sigma'_i$ .

Let  $\bar{\Sigma}' \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} \Sigma'_i$  and  $f \stackrel{\text{def.}}{=} (f_i)_{i \in [1..p]}$ . Let  $A' \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} A'_i \in \bar{\Sigma}'$ . From Definition 214 (*product of subsets of parties*), for all  $i \in [1..m]$ , we have  $A'_i \in \Sigma'_i$ . Let  $x \in X$ . Then,  $f(x) \in A'$  iff for all  $i \in [1..m]$ ,  $f_i(x) \in A'_i$ . Thus, we have  $f^{-1}(A') = \bigcap_{i \in [1..m]} f_i^{-1}(A'_i)$ . Hence, from Definition 510 (*measurable function, with  $f_i$* ), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), we have  $f^{-1}(\bar{\Sigma}') = \bigcap_{i \in [1..m]} f_i^{-1}(\Sigma'_i) \subset \Sigma$ . Thus, from Lemma 516 (*equivalent definition of measurable function, with  $G' \stackrel{\text{def.}}{=} \bar{\Sigma}'$* ),  $f$  is measurable for  $\Sigma$  and  $\Sigma'$ .

Therefore, we have the equivalence. □

**Lemma 532 (canonical projection is measurable).**

Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and let  $\pi_i$  be the canonical projection from  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$  onto  $X_i$ . Let  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ . Then,  $\Sigma$  is the smallest  $\sigma$ -algebra on  $X$  such that for all  $i \in [1..m]$ ,  $\pi_i$  is measurable for  $\Sigma$  and  $\Sigma_i$ .

*Proof.* Let  $\Sigma'$  be a  $\sigma$ -algebra on  $X$ . Then, from Lemma 531 (*measurability of function to product space, with  $\Sigma \stackrel{\text{def.}}{=} \Sigma'$ ,  $X'_i \stackrel{\text{def.}}{=} X_i$ ,  $\Sigma'_i \stackrel{\text{def.}}{=} \Sigma_i$ , and  $f \stackrel{\text{def.}}{=} \text{Id}_X = (\pi_i)_{i \in [1..m]}$* ), Definition 510 (*measurable function*), and **reflexivity of inclusion** we have

$$\begin{aligned} \forall i \in [1..m], \pi_i \text{ is measurable for } \Sigma' \text{ and } \Sigma_i \\ \iff \text{Id}_X \text{ is measurable for } \Sigma' \text{ and } \Sigma \\ \iff \text{Id}_X^{-1}(\Sigma) = \Sigma \subset \Sigma'. \end{aligned}$$

□

**Lemma 533 (generating product measurable space).** Let  $m \in [2..\infty)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space,  $G_i \subset \mathcal{P}(X_i)$ , and assume that  $X_i \in G_i$  and  $\Sigma_i = \Sigma_{X_i}(G_i)$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$  and  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ . Then, we have  $\Sigma = \Sigma_X \left( \prod_{i \in [1..m]} G_i \right)$ .

*Proof.* Let  $\bar{G} \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} G_i$ .

$\Sigma \subset \Sigma_X(\bar{G})$ . Let  $i \in [1..m]$ . Let  $\pi_i$  be the canonical projection from  $X$  onto  $X_i$ . Then, from Lemma 515 (*inverse image of generating family*), we have

$$\pi_i^{-1}(\Sigma_i) = \pi_i^{-1}(\Sigma_{X_i}(G_i)) = \Sigma_X(\pi_i^{-1}(G_i)).$$

Moreover, since  $X_i \in G_i$ , and from Definition 214 (*product of subsets of parties*), we have

$$\pi_i^{-1}(G_i) = \{X_1 \times \dots \times A_i \times \dots \times X_m \mid A_i \in G_i\} \subset \bar{G}.$$

Then, from Lemma 475 ( *$\sigma$ -algebra generation is monotone*), we have  $\pi_i^{-1}(\Sigma_i) \subset \Sigma_X(\bar{G})$ . Thus, from Definition 510 (*measurable function*),  $\pi_i$  is measurable for  $\Sigma_X(\bar{G})$  and  $\Sigma_i$ . Hence, from Lemma 532 (*canonical projection is measurable, smallest  $\sigma$ -algebra*), we have  $\Sigma \subset \Sigma_X(\bar{G})$ .

$\Sigma_X(\bar{G}) \subset \Sigma$ . Let  $\bar{\Sigma} \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} \Sigma_i$ .

Direct consequence of Lemma 474 (*generated  $\sigma$ -algebra is minimum,  $G_i \subset \Sigma_i$* ), Definition 214 (*product of subsets of parties,  $\bar{G} \subset \bar{\Sigma}$* ), Lemma 475 ( *$\sigma$ -algebra generation is monotone,  $\Sigma_X(\bar{G})$  is included in  $\Sigma_X(\bar{\Sigma})$* ), and Definition 529 (*tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\bar{\Sigma})$* ),

Therefore, we have the equality. □

*Remark 534.* For the sake of simplicity, we only present the remainder of this section in the case of the product of two measure spaces. When  $i \in \{1, 2\}$ , the complement  $\{1, 2\} \setminus \{i\}$  is  $\{3 - i\}$ .

**Definition 535 (section in Cartesian product).** Let  $X_1$  and  $X_2$  be sets. Let  $A \subset X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . The  $i$ -th section of  $A$  in  $x_i$  is the subset

$$(121) \quad s_i(x_i, A) \stackrel{\text{def.}}{=} \{x_j \in X_j \mid (x_1, x_2) \in A\}.$$

**Lemma 536 (section of product).** Let  $X_1$  and  $X_2$  be sets. Let  $A_1 \in X_1$  and  $A_2 \in X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(122) \quad s_i(x_i, A_1 \times A_2) = \begin{cases} A_j & \text{when } x_i \in A_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

*Proof.* Direct consequence of Definition 535 (section in Cartesian product).  $\square$

**Lemma 537 (compatibility of section with set operations).** Let  $X_1$  and  $X_2$  be sets. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $x_i \in X_i$ . Then, the sections are compatible with the empty set, the complement, countable union and intersection, and are monotone:

$$(123) \quad s_i(x_i, \emptyset) = \emptyset,$$

$$(124) \quad \forall A \subset X, \quad s_i(x_i, A^c) = s_i(x_i, A)^c,$$

$$(125) \quad \forall I \subset \mathbb{N}, \forall (A_n)_{n \in I} \subset X, \quad s_i\left(x_i, \bigcup_{n \in I} A_n\right) = \bigcup_{n \in I} s_i(x_i, A_n),$$

$$(126) \quad \forall I \subset \mathbb{N}, \forall (A_n)_{n \in I} \subset X, \quad s_i\left(x_i, \bigcap_{n \in I} A_n\right) = \bigcap_{n \in I} s_i(x_i, A_n),$$

$$(127) \quad \forall A, B \subset X, \quad A \subset B \implies s_i(x_i, A) \subset s_i(x_i, B).$$

*Proof.* Direct consequences of Definition 535 (section in Cartesian product).  $\square$

**Lemma 538 (measurability of section).** Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have  $s_i(x_i, A) \in \Sigma_j$ .

*Proof.* Let  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \subset X \mid s_i(x_i, A) \in \Sigma_j\}$ .

Let  $A_1 \in \Sigma_1$ , and  $A_2 \in \Sigma_2$ . Then, from Lemma 536 (section of product,  $s_i(x_i, A_1 \times A_2)$  belongs to  $\{\emptyset, A_j\}$ ), Definition 467 ( $\sigma$ -algebra,  $\emptyset, A_j \in \Sigma_j$ ), and the definition of  $\mathcal{S}_i$ , we have  $A_1 \times A_2 \in \mathcal{S}_i$ . Hence, from Definition 214 (product of subsets of parties), we have  $\bar{\Sigma} \subset \mathcal{S}_i$ .

From Lemma 537 (compatibility of section with set operations), and Definition 467 ( $\sigma$ -algebra),  $\mathcal{S}_i$  contains the empty set, and is closed under complement and countable union. Thus, from Definition 467 ( $\sigma$ -algebra),  $\mathcal{S}_i$  is a  $\sigma$ -algebra on  $X$ . Hence, from Definition 529 (tensor product of  $\sigma$ -algebras,  $\Sigma$  is generated by  $\bar{\Sigma}$ ), Lemma 474 (generated  $\sigma$ -algebra is minimum), we have  $\Sigma \subset \mathcal{S}_i$ .

Therefore, for all  $A \in \Sigma$  we have  $s_i(x_i, A) \in \Sigma_j$ .  $\square$

**Lemma 539 (indicator of section).** Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $A \subset X_1 \times X_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, we have

$$(128) \quad \forall x_1 \in X_1, \quad \forall x_2 \in X_2, \quad \mathbb{1}_A(x_1, x_2) = \mathbb{1}_{s_i(x_i, A)}(x_j).$$

*Proof.* Direct consequence of Definition 535 (section in Cartesian product), and the definition of the indicator function.  $\square$

**Lemma 540 (measurability of function from product space).**

Let  $(X_1, \Sigma_1)$ ,  $(X_2, \Sigma_2)$  and  $(X', \Sigma')$  be measurable spaces. Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $f : X_1 \times X_2 \rightarrow X'$ . Assume that  $f$  is measurable for  $\Sigma$  and  $\Sigma_1 \otimes \Sigma_2$ . Then, for all  $x_i \in X_i$ , the function  $(x_j \mapsto f(x_1, x_2))$  is measurable for  $\Sigma_j$  and  $\Sigma'$ .

*Proof.* Direct consequence of **properties of inverse image** ( $(f_{x_i}^j)^{-1}(A') = s_i(x_i, f^{-1}(A'))$ ) **where**  $f_{x_i}^j \stackrel{\text{def.}}{=} (x_j \mapsto f(x_1, x_2))$  **and**  $A' \in \Sigma'$ , Lemma 538 (**measurability of section**,  $(f_{x_i}^j)^{-1}(A') \in \Sigma_j$ ), and Definition 510 (**measurable function**).  $\square$

*Remark 541.* Note that the reciprocal of Lemma 540 (**measurability of function from product space**) is false. Indeed, let us build a counter-example.

Let

$$X_1 = X_2 = X' \stackrel{\text{def.}}{=} \mathbb{R}, \quad \Sigma_1 = \Sigma_2 \stackrel{\text{def.}}{=} \Sigma_{\mathbb{R}}(\{\{x\}\}_{x \in \mathbb{R}}), \quad \text{and} \quad \Sigma' \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R}).$$

Let  $D \stackrel{\text{def.}}{=} \{(x, x) \mid x \in \mathbb{R}\} \subset X_1 \times X_2$ . Then, the tensor product  $\sigma$ -algebra  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$  is also generated by all singletons of  $\mathbb{R}^2$ , and  $D$  is not measurable. Let  $f = \mathbf{1}_D$ . Then,  $f$  is not measurable for  $\Sigma$  and  $\Sigma'$ , but for all  $x \in X$ ,  $(y \mapsto f(x, y)) = (y \mapsto f(y, x)) = \mathbf{1}_{\{x\}}$  is obviously measurable for  $\Sigma$  and  $\Sigma'$ .

## 6.7 Measurability and numbers

*Remark 542.* We address in this section the specific case of the measurable spaces of numbers  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , and of measurable functions to them.

### 6.7.1 Borel subset of numbers

#### 6.7.1.1 Borel subset of real numbers

**Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ ).**

$\mathcal{B}(\mathbb{R})$  is generated by any of the following sets of intervals:

$$(129) \quad \{(a, b)\}_{a < b}, \quad \{[a, b)\}_{a < b}, \quad \{[a, b]\}_{a < b}, \quad \{(a, b]\}_{a < b},$$

$$(130) \quad \{(-\infty, b)\}_{b \in \mathbb{R}}, \quad \{(-\infty, b]\}_{b \in \mathbb{R}}, \quad \{(a, \infty)\}_{a \in \mathbb{R}}, \quad \{[a, \infty)\}_{a \in \mathbb{R}}.$$

*Proof.* From Theorem 331 (**countable connected components of open subsets of  $\mathbb{R}$** ), and Lemma 497 (**countable Borel  $\sigma$ -algebra generator**), we have  $\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b})$ .

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following countable unions:

$$\begin{aligned} [a, b] &= \{a\} \cup (a, b), \quad [a, b] = \{a\} \cup (a, b) \cup \{b\}, \quad (a, b] = (a, b) \cup \{b\}, \\ (a, b) &= \bigcup_{n \in \mathbb{N}} \left[ a + \frac{1}{n+1}, b \right) = \bigcup_{n \in \mathbb{N}} \left[ a + \frac{1}{n+1}, b - \frac{1}{n+1} \right] = \bigcup_{n \in \mathbb{N}} \left( a, b - \frac{1}{n+1} \right). \end{aligned}$$

Thus, from Definition 473 (**generated  $\sigma$ -algebra**), Definition 467 ( **$\sigma$ -algebra, closedness under countable union**), and Lemma 496 (**some Borel subsets, singletons are Borel subsets**), we have

$$\begin{aligned} \{(a, b)\}_{a < b}, \{[a, b)\}_{a < b}, \{(a, b]\}_{a < b} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(a, b)\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{[a, b)\}_{a < b}), \Sigma_{\mathbb{R}}(\{(a, b]\}_{a < b}), \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}). \end{aligned}$$

Hence, from Lemma 492 (**other  $\sigma$ -algebra generator**), we have

$$\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{[a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(a, b]\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}).$$

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following (finite) unions and intersections:

$$\begin{aligned} (-\infty, b] &= (-\infty, b) \cup \{b\}, & [a, \infty) &= \{a\} \cup (a, \infty), \\ [a, b) &= (-\infty, a)^c \cap (-\infty, b) = [a, \infty) \cap [b, \infty)^c, \\ (a, b] &= (-\infty, a]^c \cap (-\infty, b] = (a, \infty) \cap (b, \infty)^c. \end{aligned}$$

Thus, since  $(-\infty, b)$  and  $(a, \infty)$  are open, and from Definition 473 (*generated  $\sigma$ -algebra*), Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under complement, countable union and intersection*), and Lemma 496 (*some Borel subsets, singletons are Borel subsets*), we have

$$\begin{aligned} \{(-\infty, b)\}_{b \in \mathbb{R}}, \{[a, \infty)\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(-\infty, b]\}_{b \in \mathbb{R}}, \{(a, \infty)\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}), \\ \{(a, b)\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{(-\infty, b)\}_{b \in \mathbb{R}}), \Sigma_{\mathbb{R}}(\{[a, \infty)\}_{a \in \mathbb{R}}), \\ \{(a, b]\}_{a < b} &\subset \Sigma_{\mathbb{R}}(\{(-\infty, b]\}_{b \in \mathbb{R}}), \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}). \end{aligned}$$

Hence, from Lemma 492 (*other  $\sigma$ -algebra generator*), we have

$$\begin{aligned} \mathcal{B}(\mathbb{R}) &= \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(-\infty, b)\}_{b \in \mathbb{R}}) = \Sigma_{\mathbb{R}}(\{[a, \infty)\}_{a \in \mathbb{R}}), \\ \mathcal{B}(\mathbb{R}) &= \Sigma_{\mathbb{R}}(\{(a, b]\}_{a < b}) = \Sigma_{\mathbb{R}}(\{(-\infty, b]\}_{b \in \mathbb{R}}) = \Sigma_{\mathbb{R}}(\{(a, \infty)\}_{a \in \mathbb{R}}). \end{aligned}$$

Therefore, all eight sets of intervals generate the Borel  $\sigma$ -algebra of  $\mathbb{R}$ .  $\square$

**Lemma 544 (countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$ ).**

In Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ ), finite bounds of intervals,  $a$  and  $b$ , may be taken rational.

*Proof.* Same proof as for Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ ), but with using the countability of the topological basis of  $\mathbb{R}$  equipped with the Euclidean distance through Theorem 335 ( $\mathbb{R}$  is second-countable) instead of Theorem 331 (countable connected components of open subsets of  $\mathbb{R}$ ).  $\square$

### 6.7.1.2 Borel subset of extended real numbers

**Lemma 545 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$ ).**

$\mathcal{B}(\overline{\mathbb{R}})$  is generated by any of the following sets of intervals:

$$(131) \quad \{[-\infty, b)\}_{b \in \mathbb{R}}, \{[-\infty, b]\}_{b \in \mathbb{R}}, \{(a, \infty)\}_{a \in \mathbb{R}}, \{[a, \infty)\}_{a \in \mathbb{R}}.$$

*Proof.* From Theorem 331 (countable connected components of open subsets of  $\mathbb{R}$ ), similar proof in  $\overline{\mathbb{R}}$ , and Lemma 497 (countable Borel  $\sigma$ -algebra generator), we have

$$\mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, b)\}_{a < b} \cup \{[-\infty, b)\}_{b \in \mathbb{R}} \cup \{(a, \infty)\}_{a \in \mathbb{R}}).$$

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following countable unions and (finite) intersections:

$$\begin{aligned} [-\infty, b) &= \bigcup_{n \in \mathbb{N}} \left(b - \frac{1}{n+1}, \infty\right)^c, & (a, \infty) &= \bigcup_{n \in \mathbb{N}} \left[-\infty, a + \frac{1}{n+1}\right)^c, \\ (a, b) &= \bigcup_{n \in \mathbb{N}} \left[-\infty, a + \frac{1}{n+1}\right)^c \cap [-\infty, b) = (a, \infty) \cap \bigcup_{n \in \mathbb{N}} \left(b - \frac{1}{n+1}, \infty\right)^c. \end{aligned}$$

Thus, from Definition 473 (*generated  $\sigma$ -algebra*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under complement, countable union and intersection*), we have

$$\begin{aligned} \{[-\infty, b)\}_{b \in \mathbb{R}}, \{[a, \infty)\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, b)\}_{a < b} \cup \{[-\infty, b)\}_{b \in \mathbb{R}} \cup \{(a, \infty)\}_{a \in \mathbb{R}}), \\ \{(a, b)\}_{a < b}, \{[-\infty, b)\}_{b \in \mathbb{R}}, \{(a, \infty)\}_{a \in \mathbb{R}} &\subset \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}), \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty)\}_{a \in \mathbb{R}}). \end{aligned}$$



Hence, from Lemma 492 (*other  $\sigma$ -algebra generator*), we have

$$\mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, b)\}_{a < b} \cup \{[-\infty, b)\}_{b \in \mathbb{R}} \cup \{(a, \infty]\}_{a \in \mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty]\}_{a \in \mathbb{R}}).$$

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then, we have the following complements:

$$\begin{aligned} [-\infty, b] &= (b, \infty]^c, & [a, \infty] &= [-\infty, a)^c, \\ (a, \infty] &= [-\infty, a]^c, & [-\infty, b) &= [b, \infty)^c. \end{aligned}$$

Thus, from Definition 473 (*generated  $\sigma$ -algebra*), Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under complement, countable union and intersection*), and Lemma 492 (*other  $\sigma$ -algebra generator*), we have

$$\begin{aligned} \{[-\infty, b)\}_{b \in \mathbb{R}} &\subset \mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty]\}_{a \in \mathbb{R}}), \\ \{(a, \infty]\}_{a \in \mathbb{R}} &\subset \mathcal{B}(\overline{\mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}), \\ \{(a, \infty]\}_{a \in \mathbb{R}} &\subset \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}), \\ \{[-\infty, b)\}_{b \in \mathbb{R}} &\subset \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty]\}_{a \in \mathbb{R}}). \end{aligned}$$

Hence, from Lemma 492 (*other  $\sigma$ -algebra generator*), we have

$$\begin{aligned} \mathcal{B}(\overline{\mathbb{R}}) &= \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty]\}_{a \in \mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}), \\ \mathcal{B}(\overline{\mathbb{R}}) &= \Sigma_{\overline{\mathbb{R}}}(\{[-\infty, b)\}_{b \in \mathbb{R}}) = \Sigma_{\overline{\mathbb{R}}}(\{(a, \infty]\}_{a \in \mathbb{R}}). \end{aligned}$$

Therefore, all four sets of intervals generate the Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$ .  $\square$

**Lemma 546 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ).** *Let  $A \subset \overline{\mathbb{R}}$ . Then,  $A \in \mathcal{B}(\overline{\mathbb{R}})$  iff  $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ .*

*Proof.* Direct consequence of Definition 495 (*Borel  $\sigma$ -algebra,  $\mathbb{R}$  (open subset) belongs to  $\mathcal{B}(\overline{\mathbb{R}})$* ), Definition 467 ( *$\sigma$ -algebra,  $\{-\infty, \infty\} = \mathbb{R}^c \in \mathcal{B}(\overline{\mathbb{R}})$* ), Lemma 524 (*characterization of Borel subsets, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$ ,  $n \stackrel{\text{def.}}{=} 2$ ,  $Y_1 \stackrel{\text{def.}}{=} \mathbb{R}$ , and  $Y_2 = \{-\infty, \infty\}$* ), Lemma 523 (*Borel sub- $\sigma$ -algebra, with  $X \stackrel{\text{def.}}{=} \overline{\mathbb{R}}$  and  $Y = \{-\infty, \infty\}$ , i.e.  $\mathcal{B}(\{-\infty, \infty\}) = \mathcal{P}(\{-\infty, \infty\})$ , and thus  $A \cap \{-\infty, \infty\}$  always belongs to  $\mathcal{B}(\{-\infty, \infty\})$* ).  $\square$

*Remark 547.* In other words, Borel subsets of  $\overline{\mathbb{R}}$  are Borel subsets of  $\mathbb{R}$ , or the union of Borel subsets of  $\mathbb{R}$  with  $\{-\infty\}$ ,  $\{\infty\}$ , or  $\{-\infty, \infty\}$ .

### 6.7.1.3 Borel subset of nonnegative numbers

**Lemma 548 (Borel  $\sigma$ -algebra of  $\mathbb{R}_+$ ).** *We have  $\mathcal{B}(\mathbb{R}_+) = \{A \in \mathcal{B}(\mathbb{R}) \mid A \subset \mathbb{R}_+\}$ .*

*Proof.* In  $\mathbb{R}$ , we have  $\mathbb{R}_+ = [0, \infty) = (-\infty, 0)^c$ . Thus, from Definition 467 ( *$\sigma$ -algebra, closedness under complement*), and Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),  $\mathbb{R}_+$  belongs to  $\mathcal{B}(\mathbb{R})$ . Therefore, from Lemma 523 (*Borel sub- $\sigma$ -algebra*), we have  $\mathcal{B}(\mathbb{R}_+) = \{A \in \mathcal{B}(\mathbb{R}) \mid A \subset \mathbb{R}_+\}$ .  $\square$

**Lemma 549 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}_+}$ ).**

$\mathcal{B}(\overline{\mathbb{R}_+})$  is generated by any of the following sets of intervals:

$$(132) \quad \{[0, b)\}_{b \in \mathbb{R}_+}, \quad \{[0, b]\}_{b \in \mathbb{R}_+}, \quad \{(a, \infty]\}_{a \in \mathbb{R}_+}, \quad \{[a, \infty]\}_{a \in \mathbb{R}_+}.$$

*Proof.* Direct consequence of Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ), and Lemma 522 (*generating measurable subspace*).  $\square$

### 6.7.1.4 Product of Borel subsets of numbers

**Lemma 550 (Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ ).**

Let  $n \in [2.. \infty)$ . For all  $i \in [1..n]$ , let  $\mathcal{B}_i \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$ . Then, we have

$$(133) \quad \mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^{n \otimes} \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..n]} \mathcal{B}_i.$$

*Proof.* Let  $G \stackrel{\text{def.}}{=} \{(a, b) \mid a, b \in \mathbb{Q} \wedge a < b\}$ ,  $G' \stackrel{\text{def.}}{=} G \cup \{\mathbb{R}\}$ , and  $\overline{G'} \stackrel{\text{def.}}{=} \prod_{i \in [1..n]} G'$ . Then, from Lemma 544 (*countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$* ), Lemma 468 (*equivalent definition of  $\sigma$ -algebra, contains full set*), and Lemma 493 (*complete generated  $\sigma$ -algebra, with  $G' \stackrel{\text{def.}}{=} \{\mathbb{R}\} \subset \mathcal{B}(\mathbb{R})$ ), we have*

$$\mathcal{B}(\mathbb{R}) = \Sigma_{\mathbb{R}}(G) = \Sigma_{\mathbb{R}}(G').$$

Hence, from Lemma 533 (*generating product measurable space*), we have  $\mathcal{B}(\mathbb{R})^{n \otimes} = \Sigma_{\mathbb{R}^n}(\overline{G'})$ .

Let  $d$  be the Euclidean distance on  $\mathbb{R}^n$ . Then, from Lemma 336 ( *$\mathbb{R}^n$  is second-countable,  $\overline{G}$  countable topological basis of  $(\mathbb{R}^n, d)$* ), and Lemma 242 (*complete countable topological basis of product space, with  $A_i \stackrel{\text{def.}}{=} \mathbb{R}$  open in  $\mathbb{R}$* ),  $\overline{G'}$  is a countable topological basis of  $(\mathbb{R}^n, d)$ . Hence, from Lemma 497 (*countable Borel  $\sigma$ -algebra generator, with  $X \stackrel{\text{def.}}{=} \mathbb{R}^n$  and  $G \stackrel{\text{def.}}{=} \overline{G'}$ ), we have  $\mathcal{B}(\mathbb{R}^n) = \Sigma_{\mathbb{R}^n}(\overline{G'})$ .*

Therefore, we have the equality  $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^{n \otimes}$ .  $\square$

*Remark 551.* Note that Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ) also holds for the product of subsets of  $\mathbb{R}$ . In particular, this more general result could be used to prove that  $\mathcal{B}(\mathbb{R}_+^n) = (\mathcal{B}(\mathbb{R}_+))^{n \otimes}$ .

Note also that identifying  $\mathbb{R}^2$  with  $\mathbb{C}$  via the canonical isometry  $((x, y) \mapsto x + iy)$ , allows to identify open subsets of  $\mathbb{R}^2$  to those of  $\mathbb{C}$ . Hence, a function from some measurable space  $(X, \Sigma)$  to  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$  is measurable iff its real and imaginary parts ( $\text{Re } f$  and  $\text{Im } f$ ) are measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{R})$ .

## 6.7.2 Measurable numeric function

### 6.7.2.1 Measurable numeric function to $\mathbb{R}$

**Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ).**

Let  $(X, \Sigma)$  be a measurable space. The set of functions  $X \rightarrow \mathbb{R}$  that are measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{R})$  is called the *vector space of finite-valued measurable functions (over  $X$ )*; it is denoted  $\mathcal{M}_{\mathbb{R}}(X, \Sigma)$  (or simply  $\mathcal{M}_{\mathbb{R}}$ ).

*Remark 553.* The set  $\mathcal{M}_{\mathbb{R}}$  is shown below to be a vector space; hence its name.

**Lemma 554 (measurability of indicator function).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \subset X$ . Then, we have  $\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}$  iff  $A \in \Sigma$ .

*Proof.*  $\mathbb{1}_A$  takes the values 0 and 1. Let  $A' \in \mathcal{B}(\mathbb{R})$ .

**Case 0, 1  $\in A'$ .** Then,  $\mathbb{1}_A^{-1}(A') = X$ . **Case 0  $\in A'$  and 1  $\notin A'$ .** Then,  $\mathbb{1}_A^{-1}(A') = A^c$ .

**Case 0  $\notin A'$  and 1  $\in A'$ .** Then,  $\mathbb{1}_A^{-1}(A') = A$ . **Case 0, 1  $\notin A'$ .** Then,  $\mathbb{1}_A^{-1}(A') = \emptyset$ .

Thus, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  $\mathbb{1}_A^{-1}(A')$  belongs to the  $\sigma$ -algebra  $\Sigma$  iff  $A \in \Sigma$ . Therefore, from Definition 467 ( *$\sigma$ -algebra*), and Definition 552 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ),  $\mathbb{1}_A \in \mathcal{M}_{\mathbb{R}}$  iff  $A \in \Sigma$ .  $\square$

**Lemma 555 (measurability of numeric function to  $\mathbb{R}$ ).** Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then,  $f \in \mathcal{M}_{\mathbb{R}}$  iff one of the following conditions is satisfied:

$$(134) \quad \forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a) \in \Sigma,$$

$$(135) \quad \forall a \in \mathbb{R}, \quad \{f \leq a\} \stackrel{\text{def.}}{=} f^{-1}(-\infty, a] \in \Sigma,$$

$$(136) \quad \forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty) \in \Sigma,$$

$$(137) \quad \forall a \in \mathbb{R}, \quad \{f \geq a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty) \in \Sigma.$$

*Proof.* “Left” implies “right”. Direct consequence of Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 510 (measurable function), Definition 495 (Borel  $\sigma$ -algebra), Lemma 496 (some Borel subsets), open intervals of  $\mathbb{R}$  are open subsets, and closed intervals of  $\mathbb{R}$ , are closed subsets.

“Right” implies “left”. Direct consequence of Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ ), Lemma 516 (equivalent definition of measurable function), and Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ).

Therefore, we have the equivalence.  $\square$

**Lemma 556 (inverse image is measurable in  $\mathbb{R}$ ).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}_{\mathbb{R}}$ . Then, we have

$$(138) \quad \forall a \in \mathbb{R}, \quad \{f = a\} \stackrel{\text{def.}}{=} f^{-1}(a) \in \Sigma.$$

*Proof.* Direct consequence of Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 510 (measurable function), Lemma 496 (some Borel subsets, singletons are measurable).  $\square$

**Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  is algebra).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\mathcal{M}_{\mathbb{R}}$  is a subalgebra of  $(\mathbb{R}^X, +, \cdot, \times)$ .

*Proof.* Let  $f, g \in \mathcal{M}_{\mathbb{R}}$ . Then, from Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Lemma 531 (measurability of function to product space), and Lemma 550 (Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ , with  $m = 2$ ), the function  $(f, g) : X \rightarrow \mathbb{R}^2$  is measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{R}^2)$ . Moreover, from continuity of addition and multiplication from  $\mathbb{R}^2$  to  $\mathbb{R}$ , Lemma 517 (continuous is measurable), and Lemma 518 (compatibility of measurability with composition),  $f + g$  and  $fg$  are measurable for  $\Sigma$  and  $\mathcal{B}(\mathbb{R})$ .

Let  $f : X \rightarrow \mathbb{R}$ . Let  $a \in \mathbb{R}$ . Then, from Lemma 514 (constant function is measurable), and Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), we have  $a \in \mathcal{M}_{\mathbb{R}}$ . Thus, from the previous result, we have  $af \in \mathcal{M}_{\mathbb{R}}$ .

Therefore, from Lemma 390 ( $\mathbb{K}$  is  $\mathbb{K}$ -algebra, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), Lemma 393 (algebra of functions to algebra, with  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ ), and Lemma 398 (closed under algebra operations is subalgebra),  $\mathcal{M}_{\mathbb{R}}$  is a subalgebra of  $(\mathbb{R}^X, +, \cdot, \times)$ .  $\square$

*Remark 558.* Note that we may also show that  $\mathcal{M}_{\mathbb{C}}$ , the space of measurable functions to  $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ , is a subalgebra of  $(\mathbb{C}, +, \cdot, \times)$ .

**Lemma 559 ( $\mathcal{M}_{\mathbb{R}}$  is vector space).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\mathcal{M}_{\mathbb{R}}$  is a vector space with the zero function as zero.

*Proof.* Direct consequence of Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  is algebra), Definition 395 (subalgebra), Definition 388 (algebra over a field,  $\mathcal{M}_{\mathbb{R}}$  is a vector space), and Lemma 398 (closed under algebra operations is subalgebra,  $0_{\mathcal{M}_{\mathbb{R}}} = 0_{\mathbb{R}^X}$  is the zero function).  $\square$

### 6.7.2.2 Measurable numeric function to $\overline{\mathbb{R}}$

**Definition 560** ( $\mathcal{M}$ , set of measurable numeric functions).

Let  $(X, \Sigma)$  be a measurable space. The set of functions  $X \rightarrow \overline{\mathbb{R}}$  that are measurable for  $\Sigma$  and  $\mathcal{B}(\overline{\mathbb{R}})$  is called the *set of measurable functions (over  $X$ )*; it is denoted  $\mathcal{M}(X, \Sigma)$  (or simply  $\mathcal{M}$ ).

*Remark 561.*

We use the convention of keeping the name of functions  $X \rightarrow \mathbb{R}$  when they are “extended” as functions  $X \rightarrow \overline{\mathbb{R}}$ . This allows us to consider  $\mathcal{M}_{\mathbb{R}}$  as a subset of  $\mathcal{M}$  in the next statement.

**Lemma 562** ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ).

Let  $(X, \Sigma)$  be a measurable space.

Let  $\mathbb{R}^X$  be the set of functions from  $X$  to  $\mathbb{R}$ . Then, we have  $\mathcal{M}_{\mathbb{R}} = \mathcal{M} \cap \mathbb{R}^X$ .

*Proof.*  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M} \cap \mathbb{R}^X$ . Let  $f \in \mathcal{M}_{\mathbb{R}}$ . Let  $A \in \mathcal{B}(\overline{\mathbb{R}})$ . Then, from Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), Definition 254 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $\overline{\mathbb{R}} = \mathbb{R} \uplus \{\pm\infty\}$ ), **compatibility of inverse image with union**, Lemma 546 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ,  $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ ), and Definition 510 (measurable function), we have  $f \in \mathbb{R}^X$ , and

$$f^{-1}(A) = f^{-1}(A \cap \mathbb{R}) \cup f^{-1}(A \cap \{\pm\infty\}) = f^{-1}(A \cap \mathbb{R}) \in \Sigma.$$

Hence, from Definition 510 (measurable function), and Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions),  $f \in \mathcal{M} \cap \mathbb{R}^X$ .

$\mathcal{M} \cap \mathbb{R}^X \subset \mathcal{M}_{\mathbb{R}}$ . Let  $f \in \mathcal{M} \cap \mathbb{R}^X$ . Let  $A \in \mathcal{B}(\mathbb{R})$ . Then, from Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Lemma 546 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ ,  $A \cap \mathbb{R} = A \in \mathcal{B}(\mathbb{R})$ ), and Definition 510 (measurable function), we have  $A \in \mathcal{B}(\overline{\mathbb{R}})$ , and  $f^{-1}(A) \in \Sigma$ . Hence, from Definition 510 (measurable function), and Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$ ), we have  $f \in \mathcal{M}_{\mathbb{R}}$ .

Therefore, we have the equality. □

**Lemma 563** (measurability of numeric function).

Let  $(X, \Sigma)$  be a measurable space.

Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f \in \mathcal{M}$  iff one of the following conditions is satisfied:

$$(139) \quad \forall a \in \mathbb{R}, \quad \{f < a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a) \in \Sigma,$$

$$(140) \quad \forall a \in \mathbb{R}, \quad \{f \leq a\} \stackrel{\text{def.}}{=} f^{-1}[-\infty, a] \in \Sigma,$$

$$(141) \quad \forall a \in \mathbb{R}, \quad \{f > a\} \stackrel{\text{def.}}{=} f^{-1}(a, \infty] \in \Sigma,$$

$$(142) \quad \forall a \in \mathbb{R}, \quad \{f \geq a\} \stackrel{\text{def.}}{=} f^{-1}[a, \infty] \in \Sigma.$$

*Proof.* **“Left” implies “right”.** Direct consequence of Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Definition 510 (measurable function), Definition 495 (Borel  $\sigma$ -algebra), Lemma 496 (some Borel subsets), **open intervals of  $\overline{\mathbb{R}}$  are open subsets**, and **closed intervals of  $\overline{\mathbb{R}}$  are closed subsets**.

**“Right” implies “left”.** Direct consequence of Lemma 545 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$ ), Lemma 516 (equivalent definition of measurable function), and Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions).

Therefore, we have the equivalence. □

**Lemma 564** (inverse image is measurable).

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then, we have

$$(143) \quad \forall a \in \overline{\mathbb{R}}, \quad \{f = a\} \stackrel{\text{def.}}{=} f^{-1}(a) \in \Sigma.$$

*Proof.* Direct consequence of Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Definition 510 (measurable function), Lemma 496 (some Borel subsets, singletons are measurable).  $\square$

**Lemma 565** ( $\mathcal{M}$  is closed under finite part). *Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Let  $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then,  $A \in \Sigma$  and (the finite part of  $f$ )  $f\mathbf{1}_A \in \mathcal{M}_{\mathbb{R}}(\subset \mathcal{M})$ .*

*Proof.* From Definition 254 (extended real numbers,  $\overline{\mathbb{R}}$ ), we have  $\mathbb{R} = \{\pm\infty\}^c$ . Let

$$B \stackrel{\text{def.}}{=} f^{-1}(\{\pm\infty\}) = f^{-1}(-\infty) \uplus f^{-1}(\infty).$$

Then, from **compatibility of inverse image with disjoint union**, Lemma 564 (inverse image is measurable), and Definition 467 ( $\sigma$ -algebra, closedness under union), we have  $B \in \Sigma$ . Hence, from **compatibility of inverse image with complement**, and Definition 467 ( $\sigma$ -algebra, closedness under complement), we have  $A = B^c \in \Sigma$ .

From Definition 371 (finite part), let  $\hat{f} \stackrel{\text{def.}}{=} f\mathbf{1}_A$  be the finite part of  $f$ . Therefore, from Lemma 372 (finite part is finite), **the definition of the indicator function**, Lemma 514 (constant function is measurable,  $0 \in \mathcal{M}$ ), Lemma 527 (measurability of function defined on a partition, with  $(X', \Sigma') \stackrel{\text{def.}}{=} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ , and the partition  $X = A \uplus B$ ), and Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ), we have  $\hat{f} \in \mathcal{M} \cap \mathbb{R}^X = \mathcal{M}_{\mathbb{R}}$ .  $\square$

**Lemma 566** ( $\mathcal{M}$  is closed under addition when defined). *Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$ . Assume that  $f$  and  $g$  never take opposite infinite values at the same point. Then,  $f + g$  is well-defined, and belongs to  $\mathcal{M}$ .*

*Proof.* By hypothesis, the subsets  $\{f = \infty\} \cap \{g = -\infty\}$  and  $\{f = -\infty\} \cap \{g = \infty\}$  are empty. Thus, the whole space reduces to the partition  $X = X_{\pm\infty}^c \uplus X_{\infty} \uplus X_{-\infty}$  (the notation  $\uplus$  is for disjoint union) where

$$\begin{aligned} X_{\pm\infty}^c &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}), \\ X_{\infty} &\stackrel{\text{def.}}{=} \{f = \infty\} \cup \{g = \infty\}, \\ X_{-\infty} &\stackrel{\text{def.}}{=} \{f = -\infty\} \cup \{g = -\infty\}. \end{aligned}$$

Then, from Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Definition 510 (measurable function, for  $f$  and  $g$ ), Definition 495 (Borel  $\sigma$ -algebra,  $\mathbb{R}$  open in  $\overline{\mathbb{R}}$  belongs to  $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection), and Lemma 564 (inverse image is measurable, with  $a \stackrel{\text{def.}}{=} \pm\infty$ ),  $X_{\pm\infty}^c$ ,  $X_{\infty}$  and  $X_{-\infty}$  belong to  $\Sigma$ .

Let  $\tilde{f}$  and  $\tilde{g}$  be the finite parts of  $f$  and  $g$ . Then, from Definition 371 (finite part), and Definition 258 (addition in  $\overline{\mathbb{R}}$ ), the sum  $f + g$  can be defined on the partition by

$$f + g \stackrel{\text{def.}}{=} \begin{cases} \tilde{f} + \tilde{g} & \text{on } X_{\pm\infty}^c, \\ \infty & \text{on } X_{\infty}, \\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 565 ( $\mathcal{M}$  is closed under finite part, with  $f$  and  $g$ ), Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  is algebra, sum), Lemma 514 (constant function is measurable), and Lemma 527 (measurability of function defined on a partition), we have  $f + g \in \mathcal{M}$ .  $\square$

**Lemma 567** ( $\mathcal{M}$  is closed under finite sum). *Let  $(X, \Sigma)$  be a measurable space. Let  $I$  be a finite set. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Assume that the  $f_i$ 's never take opposite infinite values at the same point, i.e. for all  $i, j \in I$ ,  $i \neq j$  implies  $\{f_i = \infty\} \cap \{f_j = -\infty\} = \emptyset$ . Then,  $\sum_{i \in I} f_i$  is well-defined, and belongs to  $\mathcal{M}$ .*

*Proof.* Direct consequence of Lemma 566 ( $\mathcal{M}$  is closed under addition when defined), and induction on the cardinality of  $I$ .  $\square$

**Lemma 568** ( $\mathcal{M}$  is closed under multiplication).

Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$ . Then,  $fg$  is well-defined, and belongs to  $\mathcal{M}$ .

*Proof.* From Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)), and Lemma 307 (infinity-product property in  $\overline{\mathbb{R}}_+$ ), the whole space can be partitioned into  $X = X_{\pm\infty}^c \uplus X_0 \uplus X_\infty \uplus X_{-\infty}$  (the notation  $\uplus$  is for disjoint union) where

$$\begin{aligned} X_{\pm\infty}^c &\stackrel{\text{def.}}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}), \\ X_0 &\stackrel{\text{def.}}{=} (\{|f| = \infty\} \cap \{g = 0\}) \cup (\{f = 0\} \cap \{|g| = \infty\}), \\ X_\infty &\stackrel{\text{def.}}{=} (\{f = \infty\} \cap \{g > 0\}) \cup (\{f = -\infty\} \cap \{g < 0\}) \\ &\quad \cup (\{f > 0\} \cap \{g = \infty\}) \cup (\{f < 0\} \cap \{g = -\infty\}), \\ X_{-\infty} &\stackrel{\text{def.}}{=} (\{f = -\infty\} \cap \{g > 0\}) \cup (\{f = \infty\} \cap \{g < 0\}) \\ &\quad \cup (\{f > 0\} \cap \{g = -\infty\}) \cup (\{f < 0\} \cap \{g = \infty\}). \end{aligned}$$

Then, from Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Definition 510 (measurable function, for  $f$  and  $g$ ), Definition 495 (Borel  $\sigma$ -algebra,  $\mathbb{R}$  open in  $\overline{\mathbb{R}}$  belongs to  $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 564 (inverse image is measurable, with  $a \stackrel{\text{def.}}{=} \pm\infty, 0$ ) Lemma 563 (measurability of numeric function, with  $a \stackrel{\text{def.}}{=} 0$ ), and Lemma 468 (equivalent definition of  $\sigma$ -algebra) closedness under countable union and intersection, we have  $X_{\pm\infty}^c, X_0, X_\infty, X_{-\infty} \in \Sigma$ .

Let  $\tilde{f}$  and  $\tilde{g}$  be the finite parts of  $f$  and  $g$ . Then, from Definition 371 (finite part), Definition 264 (multiplication in  $\mathbb{R}$ ), and Definition 309 (multiplication in  $\mathbb{R}$  (measure theory)), the product  $fg$  can be defined on the partition by

$$fg \stackrel{\text{def.}}{=} \begin{cases} \tilde{f}\tilde{g} & \text{on } X_{\pm\infty}^c, \\ 0 & \text{on } X_0, \\ \infty & \text{on } X_\infty, \\ -\infty & \text{on } X_{-\infty}. \end{cases}$$

Therefore, from Lemma 565 ( $\mathcal{M}$  is closed under finite part, with  $f$  and  $g$ ), Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  is algebra, product), Lemma 514 (constant function is measurable), and Lemma 527 (measurability of function defined on a partition), we have  $fg \in \mathcal{M}$ .  $\square$

**Lemma 569** ( $\mathcal{M}$  is closed under finite product).

Let  $(X, \Sigma)$  be a measurable space. Let  $I$  be a finite set. Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then,  $\prod_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 568 ( $\mathcal{M}$  is closed under multiplication), and induction on the cardinality of  $I$ .  $\square$

**Lemma 570** ( $\mathcal{M}$  is closed under scalar multiplication).

Let  $(X, \Sigma)$  be a measurable space. Let  $a \in \overline{\mathbb{R}}$ . Let  $f \in \mathcal{M}$ . Then, we have  $af \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 568 ( $\mathcal{M}$  is closed under multiplication), and Lemma 514 (constant function is measurable).  $\square$

**Lemma 571** ( $\mathcal{M}$  is closed under infimum).

Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$ . Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then, we have  $\inf_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, from Definition 9 (*infimum*), Lemma 563 (*measurability of numeric function, with  $f_i$* ), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), we have

$$\left\{ \inf_{i \in I} f_i \geq a \right\} = \bigcap_{i \in I} \{f_i \geq a\} \in \Sigma.$$

Therefore, from Lemma 563 (*measurability of numeric function, with  $\inf_{i \in I} f_i$* ),  $\inf_{i \in I} f_i$  belongs to  $\mathcal{M}$ .  $\square$

**Lemma 572 ( $\mathcal{M}$  is closed under supremum).**

Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$ . Let  $(f_i)_{i \in I} \in \mathcal{M}$ . Then, we have  $\sup_{i \in I} f_i \in \mathcal{M}$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, from Definition 2 (*supremum*), Lemma 563 (*measurability of numeric function, with all the  $f_i$ 's*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*),

$$\left\{ \sup_{i \in I} f_i \leq a \right\} = \bigcap_{i \in I} \{f_i \leq a\} \in \Sigma.$$

Therefore, from Lemma 563 (*measurability of numeric function, with  $\sup_{i \in I} f_i$* ),  $\sup_{i \in I} f_i$  belongs to  $\mathcal{M}$ .  $\square$

**Lemma 573 ( $\mathcal{M}$  is closed under limit inferior).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Then, we have  $\liminf_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 571 ( $\mathcal{M}$  is closed under infimum,  $F_n^- \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p}$  belongs to  $\mathcal{M}$ ), Lemma 572 ( $\mathcal{M}$  is closed under supremum,  $f \stackrel{\text{def.}}{=} \sup_{n \in \mathbb{N}} F_n^- \in \mathcal{M}$ ), and Lemma 352 (*limit inferior,  $\liminf_{n \rightarrow \infty} f_n = f$* ).  $\square$

**Lemma 574 ( $\mathcal{M}$  is closed under limit superior).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Then, we have  $\limsup_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 572 ( $\mathcal{M}$  is closed under supremum,  $F_n^+ \stackrel{\text{def.}}{=} \sup_{p \in \mathbb{N}} f_{n+p}$  belongs to  $\mathcal{M}$ ), Lemma 571 ( $\mathcal{M}$  is closed under infimum,  $\bar{f} \stackrel{\text{def.}}{=} \inf_{n \in \mathbb{N}} F_n^+ \in \mathcal{M}$ ), and Lemma 357 (*limit superior,  $\limsup_{n \rightarrow \infty} f_n = \bar{f}$* ).  $\square$

**Lemma 575 ( $\mathcal{M}$  is closed under limit when pointwise convergent).**

Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Assume that  $(f_n(x))_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}$ . Then, we have  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}$ .

*Proof.* Direct consequence of Lemma 370 (*limit inferior, limit superior and pointwise convergence*), Lemma 573 ( $\mathcal{M}$  is closed under limit inferior), or Lemma 574 ( $\mathcal{M}$  is closed under limit superior).  $\square$

**Lemma 576 (measurability and masking).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f \in \mathcal{M}$  iff for all  $A \in \Sigma$ ,  $f \mathbb{1}_A \in \mathcal{M}$ .

*Proof.* “Left” implies “right”. Direct consequence of Lemma 554 (*measurability of indicator function*), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), and Lemma 568 ( $\mathcal{M}$  is closed under multiplication).

“Right” implies “left”. Direct consequence of Definition 467 ( $\sigma$ -algebra,  $X \in \Sigma$ ), the definition of the indicator function ( $\mathbb{1}_X = \mathbf{1}$ ), and Definition 264 (*multiplication in  $\overline{\mathbb{R}}$ , 1 is unity*).

Therefore, we have the equivalence.  $\square$



**Lemma 577 (measurability of restriction).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f}|_A \in \mathcal{M}(X, \Sigma)$ .

*Proof.* Let  $B' \in \mathcal{B}(\overline{\mathbb{R}})$ . **Case  $0 \notin B'$ .** Then, from **the definition of the indicator function**, we have  $(\hat{f}|_A)^{-1}(B') = f^{-1}(B') \cap A$ . **Case  $0 \in B'$ .** Then, from **compatibility of inverse image with disjoint union**, **the definition of the indicator function**, and **associativity of disjoint union**, we have

$$\begin{aligned} (\hat{f}|_A)^{-1}(B') &= (\hat{f}|_A)^{-1}(B' \setminus \{0\}) \uplus (\hat{f}|_A)^{-1}(0) \\ &= (f^{-1}(B' \setminus \{0\}) \cap A) \uplus ((f^{-1}(0) \cap A) \uplus A^c) \\ &= (f^{-1}(B') \cap A) \uplus A^c. \end{aligned}$$

Moreover, from **the definition of restriction of function**, we have  $f|_A^{-1}(B') = f^{-1}(B') \cap A$ . Hence, we have  $(\hat{f}|_A)^{-1}(B') = f|_A^{-1}(B') \uplus C$  with  $C \in \{\emptyset, A^c\}$ .

Let  $B \subset A$ . Assume first that  $B \in \Sigma$ . Then, from Definition 467 ( **$\sigma$ -algebra, closedness under complement and countable union**), we have  $B \uplus A^c \in \Sigma$ . Conversely, assume now that  $B \uplus A^c \in \Sigma$ . Then, from Lemma 468 (**equivalent definition of  $\sigma$ -algebra, closedness under countable intersection**), we have  $(B \uplus A^c) \cap A = B \in \Sigma$ . Hence, we have the equivalence  $B \in \Sigma$  iff  $B \uplus A^c \in \Sigma$ .

Therefore, from Definition 510 (**measurable function**), and Definition 560 ( **$\mathcal{M}$ , set of measurable numeric functions**), we have the equivalence  $f|_A \in \mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f}|_A \in \mathcal{M}(X, \Sigma)$ .  $\square$

**6.7.2.3 Nonnegative measurable numeric function****Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).**

Let  $(X, \Sigma)$  be a measurable space. The subset of nonnegative measurable functions (over  $X$ ) is denoted  $\mathcal{M}_+(X, \Sigma)$  (or simply  $\mathcal{M}_+$ ); it is defined by  $\mathcal{M}_+(X, \Sigma) \stackrel{\text{def.}}{=} \{f \in \mathcal{M} \mid f(X) \subset \overline{\mathbb{R}}_+\}$ .

**Lemma 579 (measurability of nonnegative and nonpositive parts).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then, we have  $f \in \mathcal{M}$  iff  $f^+, f^- \in \mathcal{M}_+$ .

*Proof.* **“Left” implies “right”.** Direct consequence of Definition 373 (**nonnegative and nonpositive parts**), Lemma 570 ( **$\mathcal{M}$  is closed under scalar multiplication,  $-f \in \mathcal{M}$** ), Lemma 514 (**constant function is measurable,  $0 \in \mathcal{M}$** ), Lemma 572 ( **$\mathcal{M}$  is closed under supremum, maximum is supremum**), Lemma 375 (**nonnegative and nonpositive parts are nonnegative**), and Definition 578 ( **$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions**).

**“Right” implies “left”.** Direct consequence of Definition 578 ( **$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,  $\mathcal{M}_+ \subset \mathcal{M}$** ), Lemma 570 ( **$\mathcal{M}$  is closed under scalar multiplication,  $-f^- \in \mathcal{M}$** ), Lemma 377 (**decomposition into nonnegative and nonpositive parts,  $f^+ - f^-$  is well-defined**), and Lemma 566 ( **$\mathcal{M}$  is closed under addition when defined**).

Therefore, we have the equivalence.  $\square$

**Lemma 580 ( $\mathcal{M}_+$  is closed under finite part).**

Let  $(X, \Sigma)$  be a measurable space.

Let  $f \in \mathcal{M}_+$ . Let  $A \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then,  $A \in \Sigma$  and (the finite part of  $f$ )  $f|_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Definition 578 ( **$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions**), Lemma 565 ( **$\mathcal{M}$  is closed under finite part**), **nonnegativeness of the indicator function**, and **closedness of multiplication in  $\mathbb{R}_+$** .  $\square$

**Lemma 581 ( $\mathcal{M}$  is closed under absolute value).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then, we have  $|f| \in \mathcal{M}_+ \subset \mathcal{M}$ .

Moreover, if  $f \in \mathcal{M}_{\mathbb{R}}$ , then we have  $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 293 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*), Lemma 517 (*continuous is measurable*), Lemma 518 (*compatibility of measurability with composition*), Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions,  $|f| \in \mathcal{M}$* ), Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is non-negative*), Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions,  $|f| \in \mathcal{M}_+$* ), **closedness of absolute value in  $\mathbb{R}$** , and Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$ ,  $|f| \in \mathcal{M}_{\mathbb{R}}$* ).  $\square$

**Lemma 582 ( $\mathcal{M}_+$  is closed under addition).** *Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}_+$ . Then,  $f + g$  is well-defined, and belongs to  $\mathcal{M}_+$ .*

*Proof.* Direct consequence of Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), Lemma 566 ( $\mathcal{M}$  *is closed under addition when defined,  $f$  and  $g$  cannot take opposite infinite values*), and Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*).  $\square$

**Lemma 583 ( $\mathcal{M}_+$  is closed under multiplication).** *Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}_+$ . Then, we have  $fg \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), Lemma 568 ( $\mathcal{M}$  *is closed under multiplication*), and Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*).  $\square$

**Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication).** *Let  $(X, \Sigma)$  be a measurable space. Let  $a \in \overline{\mathbb{R}}_+$ . Let  $f \in \mathcal{M}_+$ . Then, we have  $af \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), Lemma 570 ( $\mathcal{M}$  *is closed under scalar multiplication*), and Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*).  $\square$

**Lemma 585 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent).** *Let  $(X, \Sigma)$  be a measurable space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Assume that  $(f_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}_+$ . Then, we have  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ .*

*Proof.* Direct consequence of Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), Lemma 575 ( $\mathcal{M}$  *is closed under limit when pointwise convergent*), and **completeness of  $\overline{\mathbb{R}}_+$** .  $\square$

**Lemma 586 ( $\mathcal{M}_+$  is closed under countable sum).** *Let  $(X, \Sigma)$  be a measurable space. Let  $I \subset \mathbb{N}$ . Let  $(f_i)_{i \in I} \in \mathcal{M}_+$ . Then,  $\sum_{i \in I} f_i$  is well-defined, and belongs to  $\mathcal{M}_+$ .*

*Proof.* Direct consequence of Lemma 298 (*series are convergent in  $\overline{\mathbb{R}}_+$* ), Lemma 582 ( $\mathcal{M}_+$  *is closed under addition*), induction on  $n \in \mathbb{N}$  (with  $g_n \stackrel{\text{def.}}{=} \sum_{i \in I \cap [0..n]} f_i \in \mathcal{M}_+$ , with the convention that a sum indexed by the empty set is zero), and Lemma 585 ( $\mathcal{M}_+$  *is closed under limit when pointwise convergent, with  $f_n \stackrel{\text{def.}}{=} g_n$* ).  $\square$

#### 6.7.2.4 Tensor product of measurable numeric functions

**Definition 587 (tensor product of numeric functions).** Let  $m \in [2.. \infty)$ . For all  $i \in [1..m]$ , let  $X_i$  be a set, and  $f_i : X_i \rightarrow \overline{\mathbb{R}}$ . Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$ . The *tensor product of  $(f_i)_{i \in [1..m]}$*  is the function  $\bigotimes_{i \in [1..m]} f_i : X \rightarrow \overline{\mathbb{R}}$  defined by

$$(144) \quad \forall x \stackrel{\text{def.}}{=} (x_i)_{i \in [1..m]} \in X, \quad \left( \bigotimes_{i \in [1..m]} f_i \right) (x) \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} f_i(x_i).$$

**Lemma 588 (measurability of tensor product of numeric functions).**

Let  $m \in [2..∞)$ . For all  $i \in [1..m]$ , let  $(X_i, \Sigma_i)$  be a measurable space, and  $f_i \in \mathcal{M}(X_i, \Sigma_i)$ .

Then, we have  $\bigotimes_{i \in [1..m]} f_i \in \mathcal{M}\left(\prod_{i \in [1..m]} X_i, \bigotimes_{i \in [1..m]} \Sigma_i\right)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} \prod_{i \in [1..m]} X_i$  and  $\Sigma \stackrel{\text{def.}}{=} \bigotimes_{i \in [1..m]} \Sigma_i$ . Let  $i \in [1..m]$ . Let  $\pi_i$  be the canonical projection from  $X$  onto  $X_i$ . Then, from Definition 560 ( $\mathcal{M}$ , set of measurable numeric functions), Lemma 532 (canonical projection is measurable), and Lemma 518 (compatibility of measurability with composition),  $f_i \circ \pi_i$  belongs to  $\mathcal{M}(X, \Sigma)$ . Therefore, from Definition 587 (tensor product of numeric functions), and Lemma 569 ( $\mathcal{M}$  is closed under finite product, with  $I \stackrel{\text{def.}}{=} [1..m]$ ), we have

$$\bigotimes_{i \in [1..m]} f_i = \prod_{i \in [1..m]} (f_i \circ \pi_i) \in \mathcal{M}(X, \Sigma).$$

□

## 6.8 Measure space

### 6.8.1 Measure

*Remark 589.* We recall that  $\uplus$  denotes disjoint union.

**Definition 590 (additivity over measurable space).** Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  is said *additive* iff for all  $n \in \mathbb{N}$ , for all  $(A_i)_{i \in [0..n]} \in \Sigma$ ,

$$(145) \quad (\forall p, q \in [0..n], p \neq q \Rightarrow A_p \cap A_q = \emptyset) \implies \mu \left( \biguplus_{i \in [0..n]} A_i \right) = \sum_{i \in [0..n]} \mu(A_i).$$

**Definition 591 ( $\sigma$ -additivity over measurable space).** Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  is said  *$\sigma$ -additive* iff for all  $I \subset \mathbb{N}$ , for all  $(A_i)_{i \in I} \in \Sigma$ ,

$$(146) \quad (\forall p, q \in I, p \neq q \Rightarrow A_p \cap A_q = \emptyset) \implies \mu \left( \biguplus_{i \in I} A_i \right) = \sum_{i \in I} \mu(A_i).$$

*Remark 592.* Note that from Definition 467 ( $\sigma$ -algebra, closed under countable union), both Definition 590 (additivity over measurable space) and Definition 591 ( $\sigma$ -additivity over measurable space) are well-defined.

**Lemma 593 ( $\sigma$ -additivity implies additivity).** Let  $(X, \Sigma)$  be a measurable space. Let  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is  $\sigma$ -additive. Then,  $\mu$  is additive.

*Proof.* Direct consequence of Definition 591 ( $\sigma$ -additivity over measurable space, with  $I \stackrel{\text{def.}}{=} [0..n]$ ), and Definition 590 (additivity over measurable space). □

**Definition 594 (measure).**

Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  is called *measure on  $(X, \Sigma)$*  iff it is nonnegative,  $\mu(\emptyset) = 0$ , and it is  $\sigma$ -additive. If so,  $(X, \Sigma, \mu)$  is called *measure space*.

*Remark 595.* The above statement is actually the definition of a *nonnegative* measure, but as we do not consider “signed” measures in this document, we omit the qualifier “nonnegative”.

**Lemma 596 (measure over countable partition).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (B_i)_{i \in I} \in \Sigma$ . Assume that  $X = \bigsqcup_{i \in I} B_i$ . Then, for all  $i \in I$ ,  $A \cap B_i \in \Sigma$ , and we have

$$(147) \quad \mu(A) = \sum_{i \in I} \mu(A \cap B_i).$$

*Proof.* Direct consequence of Lemma 207 (compatibility of partition with intersection), Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection), Definition 594 (measure,  $\sigma$ -additivity), and Definition 591 ( $\sigma$ -additivity over measurable space).  $\square$

**Lemma 597 (measure is monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is nondecreasing over  $\Sigma$ :

$$(148) \quad \forall A, B \in \Sigma, \quad A \subset B \implies \mu(A) \leq \mu(B).$$

Moreover, if  $\mu(A)$  is finite, then we have  $\mu(B \setminus A) = \mu(B) - \mu(A)$ .

*Proof.* Let  $A, B \in \Sigma$ . Assume that  $A \subset B$ . Then, from the definition of set difference, and Lemma 470 ( $\sigma$ -algebra is closed under set difference), we have  $A \cap (B \setminus A) = \emptyset$ ,  $B = A \sqcup (B \setminus A)$  and  $B \setminus A \in \Sigma$ . Therefore, from Definition 594 (measure, nonnegativeness), and Definition 591 ( $\sigma$ -additivity over measurable space) we have  $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$ .

Assume now that  $\mu(A) < \infty$ . Then, from Definition 258 (addition in  $\mathbb{R}$ , rule 5 applies), we have  $\mu(B \setminus A) = \mu(B) - \mu(A)$ .  $\square$

**Lemma 598 (measure satisfies the finite Boole inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  satisfies the finite Boole inequality:

$$(149) \quad \forall n \in \mathbb{N}, \forall (A_i)_{i \in [0..n]} \in \Sigma, \quad \mu \left( \bigcup_{i \in [0..n]} A_i \right) \leq \sum_{i \in [0..n]} \mu(A_i).$$

*Proof.* For  $n \in \mathbb{N}$ , let  $P(n)$  be the property  $\forall (A_i)_{i \in [0..n]} \in \Sigma, \mu \left( \bigcup_{i \in [0..n]} A_i \right) \leq \sum_{i \in [0..n]} \mu(A_i)$ .

**Induction:  $P(0)$ .** Trivial.

**Induction:  $P(n)$  implies  $P(n+1)$ .** Let  $n \in \mathbb{N}$ . Assume that  $P(n)$  holds.

For all  $i \in [0..(n+1)]$ , let  $A_i \in \Sigma$ . From Definition 467 ( $\sigma$ -algebra, closedness under countable union), let  $B \stackrel{\text{def.}}{=} \bigcup_{i \in [0..n]} A_i \in \Sigma$ . Then, from the definition of set difference, associativity of union, and Lemma 470 ( $\sigma$ -algebra is closed under set difference), we have

$$B \cap (A_{n+1} \setminus B) = \emptyset, \quad \bigcup_{i \in [0..(n+1)]} A_i = B \sqcup (A_{n+1} \setminus B), \quad \text{and} \quad A_{n+1} \setminus B \in \Sigma.$$

Thus, from Definition 594 (measure,  $\sigma$ -additivity), Definition 591 ( $\sigma$ -additivity over measurable space, with  $\text{card}(I) = 2$ ),  $P(n)$ , Lemma 597 (measure is monotone, with  $A_{n+1} \setminus B \subset A_{n+1}$ ), and monotonicity of addition, we have

$$\mu \left( \bigcup_{i \in [0..(n+1)]} A_i \right) = \mu(B \sqcup (A_{n+1} \setminus B)) = \mu(B) + \mu(A_{n+1} \setminus B) \leq \sum_{i \in [0..(n+1)]} \mu(A_i).$$

Hence,  $P(n+1)$  holds.

Therefore, by induction, we have  $P(n)$  for all  $n \in \mathbb{N}$ .  $\square$

**Definition 599 (continuity from below).**

Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  is said *continuous from below* iff

$$(150) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, A_n \subset A_{n+1}) \implies \mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

**Lemma 600 (measure is continuous from below).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is continuous from below. Moreover, we have

$$(151) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, A_n \subset A_{n+1}) \implies \mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \sup_{n \in \mathbb{N}} \mu(A_n).$$

*Proof.* Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma$ . Assume that the sequence  $(A_n)_{n \in \mathbb{N}}$  is nondecreasing.

Let  $B_0 \stackrel{\text{def.}}{=} A_0 \in \Sigma$ . for all  $n \in \mathbb{N}$ , let  $B_{n+1} \stackrel{\text{def.}}{=} A_{n+1} \setminus \bigcup_{i \in [0..n]} B_i$ . Then, from Lemma 212 (*partition of countable union*), the sequence  $(B_n)_{n \in \mathbb{N}}$  is pairwise disjoint, for all  $n \in \mathbb{N}$ , we have  $B_{n+1} = A_{n+1} \setminus \bigcup_{i \in [0..n]} A_i = A_{n+1} \setminus A_n$ , and  $\bigcup_{n \in \mathbb{N}} A_n = \biguplus_{n \in \mathbb{N}} B_n$ . Hence, from Lemma 470 ( *$\sigma$ -algebra is closed under set difference*), and Definition 467 ( *$\sigma$ -algebra, closedness under countable union*), we have for all  $n \in \mathbb{N}$ ,  $B_n, \biguplus_{i \in [0..n]} B_i, \bigcup_{n \in \mathbb{N}} A_n, \biguplus_{n \in \mathbb{N}} B_n \in \Sigma$ .

Let  $p, q \in \mathbb{N}$  such that  $p < q$ . Then,  $B_p \subset A_p \subset A_{q-1}$  and  $B_q \cap A_{q-1} = \emptyset$ . Thus, from **properties of intersection**, we have  $B_p \cap B_q = \emptyset$ . Hence, from Definition 594 (*measure*), Definition 591 ( *$\sigma$ -additivity over measurable space*), and **the definition of the sum of a sequence of positive numbers**, we have

$$\begin{aligned} \mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) &= \mu \left( \biguplus_{n \in \mathbb{N}} B_n \right) = \sum_{n \in \mathbb{N}} \mu(B_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i \in [0..n]} \mu(B_i) = \lim_{n \rightarrow \infty} \mu \left( \biguplus_{i \in [0..n]} B_n \right) = \lim_{n \rightarrow \infty} \mu(A_n). \end{aligned}$$

Therefore, from Definition 599 (*continuity from below*),  $\mu$  is continuous from below.

Moreover, Equation (151) is a direct consequence of Lemma 597 (*measure is monotone*), and **properties of nondecreasing sequences in  $\overline{\mathbb{R}}_+$** .  $\square$

**Definition 601 (continuity from above).**

Let  $(X, \Sigma)$  be a measurable space. A function  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$  is said *continuous from above* iff

$$(152) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, A_n \supset A_{n+1} \quad \wedge \quad \exists n_0 \in \mathbb{N}, \mu(A_{n_0}) < \infty)$$

$$(153) \quad \implies \mu \left( \bigcap_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n) < \infty.$$

**Lemma 602 (measure is continuous from above).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is continuous from above. Moreover, we have

$$(154) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, A_n \supset A_{n+1} \quad \wedge \quad \exists n_0 \in \mathbb{N}, \mu(A_{n_0}) < \infty)$$

$$(155) \quad \implies \mu \left( \bigcap_{n \in \mathbb{N}} A_n \right) = \inf_{n \in \mathbb{N}} \mu(A_n) < \infty.$$

*Proof.* Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma$ . Assume that the sequence  $(A_n)_{n \in \mathbb{N}}$  is nonincreasing. Then, from **properties of the intersection and of the limit**, we have

$$\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \geq n_0} A_n \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_{n_0+n}).$$

For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} A_{n_0} \setminus A_{n_0+n}$  (thus  $B_0 = \emptyset$ ). Then, from **the definition and properties of set difference**, Lemma 470 ( *$\sigma$ -algebra is closed under set difference*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*),  $(B_n)_{n \in \mathbb{N}}$  is a non-decreasing sequence of measurable subsets such that  $\bigcup_{n \in \mathbb{N}} B_n = A_{n_0} \setminus \bigcap_{n \geq n_0} A_n \in \Sigma$ . Thus, from Lemma 600 (*measure is continuous from below*), Definition 599 (*continuity from below*), and Lemma 597 (*measure is monotone, with  $\mu(A_{n_0}) < \infty$* ), we have

$$\mu(A_{n_0}) - \mu\left(\bigcap_{n \geq n_0} A_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} (\mu(A_{n_0}) - \mu(A_{n_0+n})).$$

Hence, from **linearity of the limit, additive group properties of  $\mathbb{R}$** , and Lemma 597 (*measure is monotone*), we have

$$\mu\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \mu\left(\bigcap_{n \geq n_0} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_{n_0+n}) = \lim_{n \rightarrow \infty} \mu(A_n) \leq \mu(A_{n_0}) < \infty.$$

Therefore, from Definition 601 (*continuity from above*),  $\mu$  is continuous from above.

Moreover, Equation (154) is a direct consequence of Lemma 597 (*measure is monotone*), and **properties of nonincreasing sequences in  $\overline{\mathbb{R}}_+$** .  $\square$

**Lemma 603 (*measure satisfies the Boole inequality*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  satisfies the Boole inequality:

$$(156) \quad \forall (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n).$$

*Proof.* Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma$ . For all  $n \in \mathbb{N}$ , from Definition 467 ( *$\sigma$ -algebra, closedness under countable union*), let  $B_n \stackrel{\text{def.}}{=} \bigcup_{i \in [0..n]} A_i \in \Sigma$ . Then, from **properties of union**, and Definition 467 ( *$\sigma$ -algebra, closedness under countable union*), the sequence  $(B_n)_{n \in \mathbb{N}}$  is nondecreasing and

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \in \Sigma.$$

Thus, from Lemma 600 (*measure is continuous from below*), we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sup_{n \in \mathbb{N}} \mu(B_n).$$

Let  $n \in \mathbb{N}$ . Then, from Lemma 598 (*measure satisfies the finite Boole inequality*), and Definition 594 (*measure, nonnegativeness*), we have

$$\mu(B_n) = \mu\left(\bigcup_{i \in [0..n]} A_i\right) \leq \sum_{i \in [0..n]} \mu(A_i) \leq \sum_{n \in \mathbb{N}} \mu(A_n).$$

Therefore, from Definition 2 (*supremum, least upper bound*), we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n).$$

$\square$

**Lemma 604 (equivalent definition of measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\mu : \Sigma \rightarrow \overline{\mathbb{R}}$ . Assume that  $\mu$  is nonnegative, and that  $\mu(\emptyset) = 0$ . Then,  $\mu$  is a measure on  $(X, \Sigma)$  iff it is additive and continuous from below.

*Proof.* “Left” implies “right”. Direct consequence of Definition 594 (measure), Lemma 593 ( $\sigma$ -additivity implies additivity), and Lemma 600 (measure is continuous from below).

“Right” implies “left”. Assume that  $\mu$  is additive and continuous from below.

Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \in \Sigma$ . Assume that for all  $p, q \in I$ ,  $p \neq q \Rightarrow A_p \cap A_q = \emptyset$ . For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} \biguplus_{i \in I, i \in [0..n]} A_i$ . Then, from Definition 467 ( $\sigma$ -algebra, closedness under countable union),  $(B_n)_{n \in \mathbb{N}}$  is a nondecreasing sequence of measurable subsets such that  $\bigcup_{n \in \mathbb{N}} B_n = \biguplus_{i \in I} A_i \in \Sigma$ . Thus, from Definition 599 (continuity from below), and Definition 590 (additivity over measurable space), we have

$$\begin{aligned} \mu \left( \biguplus_{i \in I} A_i \right) &= \mu \left( \bigcup_{n \in \mathbb{N}} B_n \right) = \lim_{n \rightarrow \infty} \mu(B_n) \\ &= \lim_{n \rightarrow \infty} \mu \left( \biguplus_{i \in I, i \in [0..n]} A_i \right) = \lim_{n \rightarrow \infty} \sum_{i \in I, i \in [0..n]} \mu(A_i) = \sum_{i \in I} \mu(A_i). \end{aligned}$$

Hence, from Definition 591 ( $\sigma$ -additivity over measurable space), and Definition 594 (measure),  $\mu$  is a measure on  $(X, \Sigma)$ .

Therefore, we have the equivalence. □

**Definition 605 (finite measure).**

Let  $(X, \Sigma, \mu)$  be a measure space.

The measure  $\mu$  is said *finite* iff  $\mu(X) < \infty$ . If so, the measure space  $(X, \Sigma, \mu)$  is also said *finite*.

**Lemma 606 (finite measure is bounded).**

Let  $(X, \Sigma, \mu)$  be a finite measure space. Then,  $\mu$  is bounded. □

*Proof.* Direct consequence of Lemma 468 (equivalent definition of  $\sigma$ -algebra, contains full set), and Lemma 597 (measure is monotone,  $\mu(A) \leq \mu(X) < \infty$ ). □

**Definition 607 ( $\sigma$ -finite measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. The measure  $\mu$  is said  $\sigma$ -finite iff

$$(157) \quad \exists (A_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, \quad \mu(A_n) < \infty) \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} A_n.$$

If so, the measure space  $(X, \Sigma, \mu)$  is also said  $\sigma$ -finite.

**Lemma 608 (equivalent definition of  $\sigma$ -finite measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then, the measure  $\mu$  is  $\sigma$ -finite iff

$$(158) \quad \exists (B_n)_{n \in \mathbb{N}} \in \Sigma, \quad (\forall n \in \mathbb{N}, \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) < \infty) \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} B_n.$$

*Proof.* “Left” implies “right”. From Definition 607 ( $\sigma$ -finite measure), there exists  $(A_n)_{n \in \mathbb{N}} \in \Sigma$  such that, for all  $n \in \mathbb{N}$ ,  $\mu(A_n) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} A_n$ . For all  $n \in \mathbb{N}$ , from Definition 467 ( $\sigma$ -algebra, closedness under countable union), let  $B_n \stackrel{\text{def.}}{=} \bigcup_{p \in [0..n]} A_p \in \Sigma$ .

Let  $n \in \mathbb{N}$ . Then, from **associativity of union**, we have  $B_{n+1} = B_n \cup A_{n+1}$ . Thus, we have  $B_n \subset B_{n+1}$ , and  $\bigcup_{p \in [0..n]} B_p = B_n = \bigcup_{p \in [0..n]} A_p$ . Hence, from Lemma 598 (measure satisfies



the finite Boole inequality), closedness of addition in  $\mathbb{R}_+$ , and the definition of countable union, we have, for all  $n \in \mathbb{N}$ ,  $\mu(B_n) \leq \sum_{p \in [0..n]} \mu(A_p) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} B_n$ .

“Right” implies “left”. Trivial.

Therefore, we have the equivalence.  $\square$

**Definition 609 (diffuse measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Assume that  $\Sigma$  contains all singletons of  $X$ . The measure  $\mu$  is said *diffuse* iff for all  $x \in X$ ,  $\mu(\{x\}) = 0$ . If so, the measure space  $(X, \Sigma, \mu)$  is also said *diffuse*.

**Lemma 610 (finite measure is  $\sigma$ -finite).**

Let  $(X, \Sigma, \mu)$  be a finite measure space. Then,  $\mu$  is  $\sigma$ -finite.

*Proof.* Direct consequence of Definition 607 ( $\sigma$ -finite measure, with  $A_n = X$ ), and Definition 605 (finite measure).  $\square$

**Lemma 611 (trace measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y \in \Sigma$ . Then,  $\mu|_{\Sigma \cap Y}$  is a measure on  $(Y, \Sigma \cap Y)$ .

The measure  $\mu_Y \stackrel{\text{def.}}{=} \mu|_{\Sigma \cap Y}$  is called trace measure on  $Y$ . The measure space  $(Y, \Sigma \cap Y, \mu_Y)$  is called trace measure space on  $Y$ .

*Proof.* Direct consequence of Lemma 520 (trace  $\sigma$ -algebra), Lemma 521 (measurability of measurable subspace,  $\Sigma \cap Y \subset \Sigma$ ), and Definition 594 (measure).  $\square$

**Lemma 612 (restricted measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y \in \Sigma$ . Then, the function  $\mu'_Y$  defined on  $\Sigma$  by

$$(159) \quad \forall A \in \Sigma, \quad \mu'_Y(A) \stackrel{\text{def.}}{=} \mu(A \cap Y)$$

is a measure on  $(X, \Sigma)$ .

*Proof.* Direct consequence of Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $\text{card}(I) = 2$ )), properties of intersection, and Definition 594 (measure).  $\square$

*Remark 613.* Note that measures  $\mu_Y$ , from Lemma 611 (trace measure), and  $\mu'_Y$ , from Lemma 612 (restricted measure), are distinct since they are not defined on the same  $\sigma$ -algebra. But they coincide on the trace  $\sigma$ -algebra  $\Sigma \cap Y$ .

## 6.8.2 Negligible subset

**Definition 614 (negligible subset).**

Let  $(X, \Sigma, \mu)$  be a measure space. A subset  $A$  of  $X$  is said  $(\mu)$ -negligible iff there exists  $B \in \Sigma$  such that  $A \subset B$  and  $\mu(B) = 0$ .

The set of  $\mu$ -negligible subsets is denoted  $\mathbf{N}(X, \Sigma, \mu)$  (or simply  $\mathbf{N}$ ).

**Definition 615 (complete measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. The measure  $\mu$  is said *complete* iff  $\mathbf{N}(X, \Sigma, \mu) \subset \Sigma$ . If so, the measure space  $(X, \Sigma, \mu)$  is also said *complete*.

**Definition 616 (considerable subset).**

Let  $(X, \Sigma, \mu)$  be a measure space. A subset  $A$  of  $X$  is said  $(\mu)$ -considerable iff  $A \notin \mathbf{N}(X, \Sigma, \mu)$ .

**Lemma 617 (equivalent definition of considerable subset).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \subset X$ . Then,  $A$  is  $\mu$ -considerable iff for all  $B \in \Sigma$ ,  $A \subset B$  implies  $\mu(B) > 0$ .

*Proof.* Direct consequence of Definition 616 (*considerable subset*), Definition 614 (*negligible subset*), Definition 594 (*measure, nonnegativeness*), and **the tautology**  $\neg(P \wedge Q) \Leftrightarrow (P \Rightarrow \neg Q)$ .  $\square$

*Remark 618.* In the previous lemma, “considerable” naturally means “non-negligible”.

**Lemma 619 (*negligibility of measurable subset*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $A \in \mathbf{N}$  iff  $\mu(A) = 0$ .

*Proof.* Direct consequence of Definition 614 (*negligible subset, with  $B = A$* ).  $\square$

**Lemma 620 (*empty set is negligible*).** Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\emptyset \in \mathbf{N}$ .

*Proof.* Direct consequence of Lemma 619 (*negligibility of measurable subset*), Definition 467 ( *$\sigma$ -algebra,  $\emptyset \in \Sigma$* ), and Definition 594 (*measure,  $\mu(\emptyset) = 0$* ).  $\square$

**Lemma 621 (*compatibility of null measure with countable union*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbf{N}$ . For all  $i \in I$ , let  $A_i \in \Sigma$  such that  $\mu(A_i) = 0$ . Then, we have  $\mu(\bigcup_{i \in I} A_i) = 0$ .

*Proof.* Direct consequence of Definition 467 ( *$\sigma$ -algebra, closedness under countable union*), Definition 594 (*measure, nonnegativeness*), Lemma 598 (*measure satisfies the finite Boole inequality*), Lemma 603 (*measure satisfies the Boole inequality*), **additive group properties of  $\mathbb{R}$** , and Lemma 34 (*stationary sequence is convergent, countable sum of zero terms is zero*).  $\square$

**Lemma 622 ( *$\mathbf{N}$  is closed under countable union*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbf{N}$ . Let  $(A_i)_{i \in I} \in \mathbf{N}$ . Then, we have  $\bigcup_{i \in I} A_i \in \mathbf{N}$ .

*Proof.* Direct consequence of Definition 614 (*negligible subset*), **monotonicity of union**, and Lemma 621 (*compatibility of null measure with countable union*).  $\square$

**Lemma 623 (*subset of negligible is negligible*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \mathbf{N}$ . Then, we have  $\mathcal{P}(A) \subset \mathbf{N}$ .

*Proof.* Direct consequence of Definition 614 (*negligible subset*), **the definition of the power set**, and **transitivity of the inclusion**.  $\square$

**Definition 624 (*property almost satisfied*).**

Let  $(X, \Sigma, \mu)$  be a measure space. A predicate  $P$  defined on  $X$  is said *satisfied ( $\mu$ -)almost everywhere* iff  $\{\neg P\} \stackrel{\text{def.}}{=} \{x \in X \mid \neg P(x)\} \in \mathbf{N}$ ; this is denoted either “ $P \mu$  a.e.”, “ $\forall x \in X, P(x) \mu$  a.e.”, “ $\forall_{\mu \text{ a.e. } x} \in X, P(x)$ ”, or “for  $\mu$ -almost all  $x \in X, P(x)$ ” (or simply without the mention of the measure  $\mu$ ).

*Remark 625.* When a single binary relation (using infix notation) is involved in  $P$ , the annotation “ $\mu$  a.e.” may be put above the infix operator, as in “ $\stackrel{\mu \text{ a.e.}}{=}$ ” (or simply as in “ $\stackrel{\text{a.e.}}{=}$ ”).

**Lemma 626 (*everywhere implies almost everywhere*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  be a predicate on  $X$ . Then,  $P \Rightarrow P \mu$  a.e.

*Proof.* Direct consequence of Definition 624 (*property almost satisfied*), and Lemma 620 (*empty set is negligible*).  $\square$

**Lemma 627 (*extended almost modus ponens*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  and  $Q$  be predicates on  $X$ . Then,

$$(160) \quad (P \stackrel{\mu \text{ a.e.}}{\implies} Q) \wedge P \mu \text{ a.e.} \implies Q \mu \text{ a.e.}$$

*Proof.* From Definition 624 (*property almost satisfied*), and **modus ponens**, we have

$$\{P \Rightarrow Q\}^c, \{P\}^c \in \mathbf{N} \quad \text{and} \quad B \stackrel{\text{def.}}{=} \{P \Rightarrow Q\} \cap \{P\} \subset \{Q\}.$$

Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 622 ( **$\mathbf{N}$  is closed under countable union**, with  $\text{card}(I) = 2$ ), we have

$$\{Q\}^c \subset B^c \quad \text{and} \quad B^c = \{P \Rightarrow Q\}^c \cup \{P\}^c \in \mathbf{N}.$$

Therefore, from Lemma 623 (*subset of negligible is negligible*), and Definition 624 (*property almost satisfied*), we have for  $\mu$ -almost all  $x \in X$ ,  $Q(x)$ .  $\square$

**Lemma 628 (almost modus ponens).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $P$  and  $Q$  be predicates on  $X$ . Then,

$$(161) \quad (P \implies Q) \wedge P \mu \text{ a.e.} \implies Q \mu \text{ a.e.}$$

*Proof.* Direct consequence of Lemma 626 (*everywhere implies almost everywhere*, with predicate  $P \Rightarrow Q$ ), and Lemma 627 (*extended almost modus ponens*).  $\square$

*Remark 629.*

Lemma 628 (*almost modus ponens*), and Lemma 627 (*extended almost modus ponens*) allow to still use modus ponens in reasoning when predicates are only valid almost everywhere.

**Definition 630 (almost definition).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. A function  $f : X \rightarrow Y$  is said *defined ( $\mu$ -)almost everywhere in  $X$*  iff the property “ $f(x)$  is defined” is satisfied  $\mu$ -almost everywhere.

The set of functions  $X \rightarrow Y$  defined ( $\mu$ -)almost everywhere is denoted  $(Y^X)_{\mu \text{ a.e.}}$ , or through the type annotation  $(X \rightarrow Y)_{\mu \text{ a.e.}}$  (or simply without the mention of the measure  $\mu$ ).

**Definition 631 (almost binary relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . The functions  $f, g : (X \rightarrow Y)_{\mu \text{ a.e.}}$  are said *in relation ( $\mu$ -)almost everywhere through  $\mathcal{R}$*  iff the property “ $f(x) \mathcal{R} g(x)$ ” is satisfied  $\mu$ -almost everywhere; this is denoted  $f \mathcal{R}_{\mu \text{ a.e.}} g$  (or simply  $f \mathcal{R}_{\text{a.e.}} g$ ).

**Lemma 632 (compatibility of almost binary relation with reflexivity).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is reflexive. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also reflexive.

*Proof.* Direct consequence of Definition 631 (*almost binary relation*), Definition 630 (*almost definition*), and Definition 624 (*property almost satisfied*), **the definition of reflexivity ( $\{f \mathcal{R} f\}^c$  is empty)**, Definition 467 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), Definition 594 (*measure*,  $\mu(\emptyset) = 0$ ), and Lemma 619 (*negligibility of measurable subset*,  $\emptyset \in \mathbf{N}$ ).  $\square$

**Lemma 633 (compatibility of almost binary relation with symmetry).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is symmetric. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also symmetric.

*Proof.* Direct consequence of Definition 631 (*almost binary relation*), Definition 630 (*almost definition*), Definition 624 (*property almost satisfied*), Definition 614 (*negligible subset*), and **the definition of symmetry (contrapositive)**.  $\square$

**Lemma 634 (compatibility of almost binary relation with antisymmetry).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is antisymmetric. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also antisymmetric.

*Proof.* Let  $f$  and  $g$  be functions defined almost everywhere in  $X$ . Let  $A_f$  and  $A_g$  be the respective domains of definition of  $f$  and  $g$ . Assume that  $f \mathcal{R}_{\mu \text{ a.e.}} g$  and  $g \mathcal{R}_{\mu \text{ a.e.}} f$ . Then, from Definition 630 (*almost definition*), Definition 631 (*almost binary relation*), and Definition 624 (*property almost satisfied*), we have

$$A_f^c, A_g^c, \{f \mathcal{R} g\}^c, \{g \mathcal{R} f\}^c \in \mathbf{N}.$$

Moreover, from **the definition of antisymmetry**, and **the definition of intersection**, we have

$$B \stackrel{\text{def.}}{=} A_f \cap A_g \cap \{f \mathcal{R} g\} \cap \{g \mathcal{R} f\} \subset A_f \cap A_g \cap \{f = g\} \subset \{f = g\}.$$

Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 622 ( **$\mathbf{N}$  is closed under countable union**), we have

$$\{f = g\}^c \subset B^c \quad \text{and} \quad B^c = A_f^c \cup A_g^c \cup \{f \mathcal{R} g\}^c \cup \{g \mathcal{R} f\}^c \in \mathbf{N}.$$

Therefore, from Lemma 623 (*subset of negligible is negligible*), Definition 624 (*property almost satisfied*), Definition 631 (*almost binary relation*), and Definition 630 (*almost definition*), we have  $f \stackrel{\mu \text{ a.e.}}{=} g$ .  $\square$

**Lemma 635 (compatibility of almost binary relation with transitivity).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is transitive. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also transitive.*

*Proof.* Let  $f, g$  and  $h$  be functions defined almost everywhere in  $X$ . Let  $A_f, A_g$  and  $A_h$  be the respective domains of definition of  $f, g$  and  $h$ . Assume that  $f \mathcal{R}_{\mu \text{ a.e.}} g$  and  $g \mathcal{R}_{\mu \text{ a.e.}} h$ . Then, from Definition 630 (*almost definition*), Definition 631 (*almost binary relation*), Definition 630 (*almost definition*), and Definition 624 (*property almost satisfied*), we have

$$A_f^c, A_g^c, A_h^c, \{f \mathcal{R} g\}^c, \{g \mathcal{R} h\}^c \in \mathbf{N}.$$

Moreover, from **the definition of transitivity**, and **the definition of intersection**, we have

$$B \stackrel{\text{def.}}{=} A_f \cap A_g \cap A_h \cap \{f \mathcal{R} g\} \cap \{g \mathcal{R} h\} \subset A_f \cap A_g \cap A_h \cap \{f \mathcal{R} h\} \subset \{f \mathcal{R} h\}.$$

Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 622 ( **$\mathbf{N}$  is closed under countable union**), we have

$$\{f \mathcal{R} h\}^c \subset B^c \quad \text{and} \quad B^c = A_f^c \cup A_g^c \cup A_h^c \cup \{f \mathcal{R} g\}^c \cup \{g \mathcal{R} h\}^c \in \mathbf{N}.$$

Thus, from Lemma 623 (*subset of negligible is negligible*), Definition 624 (*property almost satisfied*), Definition 631 (*almost binary relation*), and Definition 630 (*almost definition*), we have  $f \mathcal{R}_{\mu \text{ a.e.}} h$ .  $\square$

*Remark 636.* From the similarity of their formal expressions, antisymmetry and transitivity can be abstracted under the form

$$\forall f, g, h : X \rightarrow Y, \quad f \mathcal{R} g \quad \wedge \quad g \mathcal{R} F(f, h) \quad \implies \quad f \mathcal{R}' F(g, h)$$

where the binary relation  $\mathcal{R}'$  is the equality for antisymmetry and  $\mathcal{R}$  for transitivity, and where the function  $F : Y^X \times Y^X \rightarrow Y^X$  is the first projection for antisymmetry and the second projection for transitivity. Moreover, the proofs of Lemma 634 (*compatibility of almost binary relation with antisymmetry*) and Lemma 635 (*compatibility of almost binary relation with transitivity*) only differ from the expression of the two instances of  $\mathcal{R}'$  and  $F$ . Thus, both statements and proofs of these lemmas can be abstracted under more general forms.

**Lemma 637 (almost equivalence is equivalence relation).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is an equivalence relation. Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also an equivalence relation.*

*Proof.* Direct consequence of **the definition of equivalence relation**, Lemma 632 (*compatibility of almost binary relation with reflexivity*), Lemma 633 (*compatibility of almost binary relation with symmetry*), and Lemma 635 (*compatibility of almost binary relation with transitivity*).  $\square$

**Lemma 638 (almost equality is equivalence relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Then,  $\stackrel{\mu \text{ a.e.}}{=}$  is an equivalence relation over  $Y^X$ .

*Proof.* Direct consequence of Lemma 637 (*almost equivalence is equivalence relation, with equality over  $Y$* ).  $\square$

**Lemma 639 (almost order is order relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $\mathcal{R}$  be a binary relation over  $Y$ , lifted into a binary relation over  $Y^X$ . Assume that  $\mathcal{R}$  is an order relation.

Then,  $\mathcal{R}_{\mu \text{ a.e.}}$  is also an order relation.

*Proof.* Direct consequence of **the definition of order relation**, Lemma 632 (*compatibility of almost binary relation with reflexivity*), Lemma 634 (*compatibility of almost binary relation with antisymmetry*), and Lemma 635 (*compatibility of almost binary relation with transitivity*).  $\square$

**Lemma 640 (compatibility of almost binary relation with operator).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a nonempty set. Let  $y_0 \in Y$ . Let  $\mathcal{R}$  and  $\mathcal{R}'$  be binary relations over  $Y$ , lifted into binary relations over  $Y^X$ . Assume that  $y_0 \mathcal{R} y_0$ . Let  $I$  be a nonempty subset of  $\mathbb{N}$ . Let  $\diamond : Y^I \rightarrow Y$ , lifted into an operator  $(Y^X)^I \rightarrow Y^X$ . Assume that for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : X \rightarrow Y$ , we have

$$(162) \quad (\forall i \in I, f_i \mathcal{R} g_i) \implies \diamond(f_i)_{i \in I} \mathcal{R}' \diamond(g_i)_{i \in I}.$$

Then, for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ , we have

$$(163) \quad (\forall i \in I, f_i \mathcal{R}_{\mu \text{ a.e.}} g_i) \implies \diamond(f_i)_{i \in I} \mathcal{R}'_{\mu \text{ a.e.}} \diamond(g_i)_{i \in I}.$$

*Proof.* For all  $i \in I$ , let  $f_i, g_i : (X \rightarrow Y)_{\mu \text{ a.e.}}$ . Assume that for all  $i \in I$ ,  $f_i \mathcal{R}_{\mu \text{ a.e.}} g_i$  holds. For all  $i \in I$ , let  $B_i \stackrel{\text{def.}}{=} \{f_i \mathcal{R} g_i\}$ . For all  $i \in I$ , let  $\tilde{f}_i, \tilde{g}_i : X \rightarrow Y$  defined by

$$\left\{ \begin{array}{l} \tilde{f}_i(x) \stackrel{\text{def.}}{=} f_i(x) \\ \tilde{g}_i(x) \stackrel{\text{def.}}{=} g_i(x) \end{array} \right\} \quad \text{when } x \in \bigcap_{i \in I} B_i,$$

$$\tilde{f}_i(x) = \tilde{g}_i(x) \stackrel{\text{def.}}{=} y_0 \quad \text{otherwise.}$$

Let  $i \in I$ . Then, from Definition 631 (*almost binary relation*), Definition 630 (*almost definition*), and Definition 624 (*property almost satisfied*), we have  $B_i^c \in \mathbf{N}$ . Let  $x \in X$ . Then, by construction, and from reflexivity of  $\mathcal{R}$ , we have  $\tilde{f}_i(x) \mathcal{R} \tilde{g}_i(x)$ . Thus, from assumption,  $\diamond(\tilde{f}_i)_{i \in I}(x) \mathcal{R}' \diamond(\tilde{g}_i)_{i \in I}(x)$  also holds. Moreover, by construction, we have

$$\forall x \in \bigcap_{i \in I} B_i, \quad \diamond(\tilde{f}_i)_{i \in I}(x) = \diamond(f_i)_{i \in I}(x) \quad \text{and} \quad \diamond(\tilde{g}_i)_{i \in I}(x) = \diamond(g_i)_{i \in I}(x).$$

Thus, we have  $\bigcap_{i \in I} B_i \subset \{\diamond(f_i)_{i \in I} \mathcal{R}' \diamond(g_i)_{i \in I}\}$ . Hence, from **monotonicity of complement**, **De Morgan's laws**, and Lemma 622 ( **$\mathbf{N}$  is closed under countable union**), we have

$$\{\diamond(f_i)_{i \in I} \mathcal{R}' \diamond(g_i)_{i \in I}\}^c \subset \bigcup_{i \in I} \{f_i \mathcal{R} g_i\}^c \in \mathbf{N}.$$

Therefore, from Definition 624 (*property almost satisfied*), Definition 631 (*almost binary relation*), and Definition 630 (*almost definition*), we have  $\diamond(f_i)_{i \in I} \mathcal{R}'_{\mu \text{ a.e.}} \diamond(g_i)_{i \in I}$ .  $\square$

**Lemma 641 (compatibility of almost equality with operator).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $I \subset \mathbb{N}$ . Let  $\diamond : Y^I \rightarrow Y$ , lifted into an operator  $(Y^X)^I \rightarrow Y^X$ . Let  $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ . Assume that for all  $i \in I$ ,  $f_i \stackrel{\mu \text{ a.e.}}{=} g_i$ . Then, we have  $\diamond(f_i)_{i \in I} \stackrel{\mu \text{ a.e.}}{=} \diamond(g_i)_{i \in I}$ .

*Proof.* Direct consequence of Lemma 640 (compatibility of almost binary relation with operator, with  $\mathcal{R} = \mathcal{R}' \stackrel{\text{def.}}{=} \text{equality}$ ), **reflexivity of equality**, and **the definition of function ((for all  $i \in I$ ,  $f_i = g_i$ ) implies  $\diamond(f_i)_{i \in I} = \diamond(g_i)_{i \in I}$ )**.  $\square$

**Lemma 642 (compatibility of almost inequality with operator).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be an ordered set. Let  $\mathcal{R}'$  be a binary relation on  $Y$ , lifted into a binary relation on  $Y^X$ . Let  $I \subset \mathbb{N}$ . Let  $\diamond : Y^I \rightarrow Y$ , lifted into an operator  $(Y^X)^I \rightarrow Y^X$ . Assume that for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : X \rightarrow Y$ , we have

$$(164) \quad (\forall i \in I, f_i \leq g_i) \implies \diamond(f_i)_{i \in I} \mathcal{R}' \diamond(g_i)_{i \in I}.$$

Then, for all  $(f_i)_{i \in I}, (g_i)_{i \in I} : (X \rightarrow Y)_{\mu \text{ a.e.}}$ , we have

$$(165) \quad (\forall i \in I, f_i \stackrel{\mu \text{ a.e.}}{\leq} g_i) \implies \diamond(f_i)_{i \in I} \mathcal{R}'_{\mu \text{ a.e.}} \diamond(g_i)_{i \in I}.$$

*Proof.* Direct consequence of Lemma 640 (compatibility of almost binary relation with operator, with  $\mathcal{R} \stackrel{\text{def.}}{=} \leq$ ), and **reflexivity of inequality**.  $\square$

*Remark 643.* The previous generic results apply to unary operators  $\diamond$  such as the scalar multiplication, or the absolute value (for which  $\text{card}(I) = 1$ ), as well as to binary (or  $n$ -ary) operators  $\diamond$  such as the addition, the multiplication, the maximum, or the minimum (for which  $\text{card}(I) \geq 2$  is finite). But it also applies to operators  $\diamond$  taking a countable number of arguments such as the infimum, the supremum, or the limit of a pointwise convergent sequence.

*Remark 644.* Note that the previous result may be useful with  $\mathcal{R}'$  distinct from  $\leq$ . For instance, when the operator  $\diamond$  is the scalar multiplication by a negative number, we need  $\mathcal{R}' = \geq$ .

**Lemma 645 (local injectivity implies almost local injectivity).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $Y$  be a set. Let  $a \in Y$ , lifted into a constant function in  $Y^X$ . Let  $\diamond : Y \rightarrow Y$ , lifted into an operator  $Y^X \rightarrow Y^X$ . Assume that  $\diamond$  is “injective in  $a$ ”:

$$(166) \quad \forall f \in Y^X, \quad \diamond(f) = \diamond(a) \implies f = a.$$

Then,  $\diamond$  is “almost injective in  $a$ ”:

$$(167) \quad \forall f \in (Y^X)_{\mu \text{ a.e.}}, \quad \diamond(f) \stackrel{\mu \text{ a.e.}}{=} \diamond(a) \implies f \stackrel{\mu \text{ a.e.}}{=} a.$$

*Proof.* Assume that  $\diamond$  is “injective in  $a$ ”, i.e.  $\{\diamond(f) = \diamond(a)\} \subset \{f = a\}$ . Then, since **complement is a function**, we also have  $\{\diamond(f) = \diamond(a)\}^c \supset \{f = a\}^c$ . Thus, from Definition 631 (almost binary relation, with equality), Definition 630 (almost definition), Definition 624 (property almost satisfied), and Definition 614 (negligible subset), we have

$$\begin{aligned} \diamond(f) \stackrel{\mu \text{ a.e.}}{=} \diamond(a) &\implies \exists B \in \Sigma, \{\diamond(f) = \diamond(a)\}^c \subset B \wedge \mu(B) = 0 \\ &\implies \exists B \in \Sigma, \{f = a\}^c \subset B \wedge \mu(B) = 0 \\ &\implies f \stackrel{\mu \text{ a.e.}}{=} a. \end{aligned}$$

Therefore,  $\diamond$  is “almost injective in  $a$ ” (for which equality is replaced by equality  $\mu$ -almost everywhere).  $\square$

*Remark 646.* Note that Lemma 641 (compatibility of almost equality with operator) provides the implication in the other direction.



### 6.8.3 Uniqueness condition

*Remark 647.* Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*) will be useful to establish in Section 6.9.2 uniqueness of the Lebesgue measure that generalizes the length of bounded intervals.

*Remark 648.* The following proof follows the Dynkin  $\pi$ - $\lambda$  theorem scheme (see Section 4.3).

#### Lemma 649 (*uniqueness of measures extended from a $\pi$ -system*).

Let  $(X, \Sigma)$  be a measurable space. Let  $\mu_1$  and  $\mu_2$  be two measures on  $(X, \Sigma)$ . Let  $G \subset \mathcal{P}(X)$ . Assume that  $G$  is a  $\pi$ -system, a generator of  $\Sigma$ , contains a sequence  $(X_n)_{n \in \mathbb{N}}$  of subsets of finite measure  $\mu_1$  covering  $X$ , and that both measures coincide on  $G$ . Then, we have  $\mu_1 = \mu_2$ .

*Proof.* Let  $A \in G$ . Assume that  $\mu_1(A) < \infty$  (e.g.  $A \stackrel{\text{def.}}{=} X_0$ ).

Let  $B \in \Sigma$ . Then, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection with  $\text{card}(I) = 2$ ), we have  $B \cap A \in \Sigma$ . Let  $\mathcal{S}_A \stackrel{\text{def.}}{=} \{B \in \Sigma \mid \mu_1(B \cap A) = \mu_2(B \cap A)\}$ .

(1).  $G \subset \mathcal{S}_A$ . Let  $B \in G$ . Then, from Definition 424 ( *$\pi$ -system*, closedness under intersection), we have  $B \cap A \in G$ . Thus, we have  $B \in \mathcal{S}_A$ . Hence,  $G \subset \mathcal{S}_A$ .

(2).  $\mathcal{S}_A$  is  $\lambda$ -system.

From **the identity**  $X \cap A = A$ , we have  $X \in \mathcal{S}_A$ .

Let  $B_1, B_2 \in \mathcal{S}_A$ . Assume that  $B_1 \subset B_2$ . Then, from **distributivity of intersection over local complement**, and Lemma 597 (*measure is monotone*, with  $\mu_1$  and  $\mu_2$ ), we have

$$\mu_2(B_1 \cap A) = \mu_1(B_1 \cap A) \leq \mu_1(A) < \infty,$$

and

$$\begin{aligned} \mu_1((B_2 \setminus B_1) \cap A) &= \mu_1(B_2 \cap A \setminus B_1 \cap A) = \mu_1(B_2 \cap A) - \mu_1(B_1 \cap A) \\ &= \mu_2(B_2 \cap A) - \mu_2(B_1 \cap A) = \mu_2(B_2 \cap A \setminus B_1 \cap A) = \mu_2((B_2 \setminus B_1) \cap A). \end{aligned}$$

Then,  $B_2 \setminus B_1 \in \mathcal{S}_A$ . Thus,  $\mathcal{S}_A$  is closed under local complement.

Let  $(B_n)_{n \in \mathbb{N}} \in \mathcal{S}_A$ . Assume that for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ . Then, from **distributivity of intersection over union**, Lemma 600 (*measure is continuous from below*, with  $\mu_1$  and  $\mu_2$ ), and Definition 599 (*continuity from below*), we have

$$\begin{aligned} \mu_1\left(\bigcup_{n \in \mathbb{N}} B_n \cap A\right) &= \mu_1\left(\bigcup_{n \in \mathbb{N}} (B_n \cap A)\right) = \lim_{n \rightarrow \infty} \mu_1(B_n \cap A) \\ &= \lim_{n \rightarrow \infty} \mu_2(B_n \cap A) = \mu_2\left(\bigcup_{n \in \mathbb{N}} (B_n \cap A)\right) = \mu_2\left(\bigcup_{n \in \mathbb{N}} B_n \cap A\right). \end{aligned}$$

Then,  $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{S}_A$ . Thus,  $\mathcal{S}_A$  is closed under countable monotone union.

Hence, from Lemma 454 (*equivalent definition of  $\lambda$ -system*),  $\mathcal{S}_A$  is a  $\lambda$ -system on  $X$ .

(3).  $\mathcal{S}_A = \Sigma$ . Direct consequence of Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*, with  $P$  being the function  $B \mapsto \mu_1(B \cap A) = \mu_2(B \cap A)$ ).

(4).  $\mu_1 = \mu_2$ . Let  $A \in \Sigma$ .

Let  $Y_0 = X_0$ , and for all  $n \in \mathbb{N}$ , let  $Y_{n+1} \stackrel{\text{def.}}{=} X_{n+1} \setminus \bigcup_{p \in [0..n]} Y_p$ , and  $A_n \stackrel{\text{def.}}{=} A \cap Y_n$ . Then, from Lemma 212 (*partition of countable union*), **associativity of intersection**, and **distributivity of intersection over union**,  $(A_n)_{n \in \mathbb{N}}$  is pairwise disjoint, and we have  $\bigsqcup_{n \in \mathbb{N}} A_n = A$ .

Let  $n \in \mathbb{N}$ . Then, from **the definition of set difference** ( $Y_n \subset X_n$ ), Lemma 597 (*measure is monotone*,  $\mu_1(Y_n) \leq \mu_1(X_n) < \infty$ ), and (3), we have  $A \in \mathcal{S}_{Y_n}$  (i.e.  $\mu_1(A_n) = \mu_2(A_n)$ ). Thus, from Definition 594 (*measure,  $\mu_1$  and  $\mu_2$  are  $\sigma$ -additive*), and Definition 591 ( *$\sigma$ -additivity over measurable space*), we have  $\mu_1(A) = \sum_{n \in \mathbb{N}} \mu_1(A_n) = \sum_{n \in \mathbb{N}} \mu_2(A_n) = \mu_2(A)$ .

Therefore, we have the equality  $\mu_1 = \mu_2$ .  $\square$



#### 6.8.4 Some measures

**Lemma 650 (trivial measure).**

Let  $(X, \Sigma)$  be a measurable space. Then, the zero function on  $\Sigma$  is a measure on  $(X, \Sigma)$ .

It is called the trivial measure on  $(X, \Sigma)$ , and  $(X, \Sigma, 0)$  is called the trivial measure space.

*Proof.* Direct consequence of Definition 594 (measure), and ordered group properties of  $\mathbb{R}$ .  $\square$

**Lemma 651 (equivalent definition of trivial measure).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $\mu$  is the trivial measure iff  $\mu(X) = 0$ .

*Proof.* Direct consequence of Lemma 650 (trivial measure), Definition 594 (measure,  $\mu$  is non-negative), and Lemma 597 (measure is monotone,  $\mu$  is nonpositive).  $\square$

**Lemma 652 (counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $\delta_Y$  be the function defined by

$$(168) \quad \forall A \in \Sigma, \quad \delta_Y(A) \stackrel{\text{def.}}{=} \text{card}(A \cap Y)$$

with the convention that the cardinality of an infinite set is  $\infty$ . Then,  $\delta_Y$  is a measure on  $(X, \Sigma)$ .

It is called the counting measure (associated with  $Y$ ).

*Proof.* From the definition of the cardinality, and the definition of  $\infty$ , the function  $\delta_Y$  is nonnegative. Moreover, from Definition 467 ( $\sigma$ -algebra,  $\emptyset \in \Sigma$ ), and the definition of the empty set, we have  $\delta_Y(\emptyset) = \text{card}(\emptyset) = 0$ .

Let  $I \subset \mathbb{N}$ . Let  $(A_i)_{i \in I} \in \Sigma$ . Assume that for all  $p, q \in I$ ,  $p \neq q$  implies  $A_p \cap A_q = \emptyset$ . Then, from distributivity of intersection over disjoint union, and  $\sigma$ -additivity of the cardinality (with the convention that  $\infty$  is absorbing element for addition in  $\overline{\mathbb{N}}$ ), we have

$$\delta_Y \left( \biguplus_{i \in I} A_i \right) = \text{card} \left( \biguplus_{i \in I} A_i \cap Y \right) = \text{card} \left( \biguplus_{i \in I} (A_i \cap Y) \right) = \sum_{i \in I} \text{card}(A_i \cap Y) = \sum_{i \in I} \delta_Y(A_i).$$

Therefore, from Definition 591 ( $\sigma$ -additivity over measurable space), and Definition 594 (measure),  $\delta_Y$  is a measure on  $(X, \Sigma)$ .  $\square$

**Lemma 653 (finiteness of counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Then,  $\delta_Y$  is finite iff  $Y$  is finite.

*Proof.* Direct consequence of Lemma 468 (equivalent definition of  $\sigma$ -algebra, contains full set), Lemma 652 (counting measure,  $\delta_Y(X) = \text{card}(Y)$ ), and Definition 605 (finite measure).  $\square$

**Lemma 654 ( $\sigma$ -finite counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Assume that  $\Sigma$  contains all singletons of  $X$ .

Then,  $\delta_Y$  is  $\sigma$ -finite implies  $Y$  is countable, and  $X$  is countable implies  $\delta_Y$  is  $\sigma$ -finite.

*Proof.* From Definition 467 ( $\sigma$ -algebra, closedness under countable union), The  $\sigma$ -algebra  $\Sigma$  contains all countable subsets of  $X$ . Hence, if  $X$  is countable, we have  $\Sigma = \mathcal{P}(X)$ .

**First implication.** Assume that  $\delta_Y$  is  $\sigma$ -finite.

Then, from Lemma 608 (equivalent definition of  $\sigma$ -finite measure), let  $(B_n)_{n \in \mathbb{N}} \in \Sigma$  such that for all  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$ ,  $\delta_Y(B_n) = \text{card}(B_n \cap Y) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} B_n$ . For all  $n \in \mathbb{N}$ , let  $A_n \stackrel{\text{def.}}{=} B_n \cap Y$ . Let  $n \in \mathbb{N}$ . Then, from De Morgan's laws, and Lemma 652 (counting measure), we have  $Y = \bigcup_{n \in \mathbb{N}} A_n$ , and  $\text{card}(A_n) = \delta_Y(B_n) < \infty$ . Hence, from countability of countable union of finite subsets,  $Y$  is countable.

**Second implication.** Assume that  $X$  is countable. Then, we have  $\Sigma = \mathcal{P}(X)$ .

From **the definition of countability**, let  $\varphi$  be a bijection from  $I \subset \mathbb{N}$  onto  $X$ . For all  $n \in \mathbb{N}$ , let  $A_n \stackrel{\text{def.}}{=} \varphi([0..n] \cap I) \in \Sigma$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 652 (*counting measure*), **monotonicity of the cardinality**, **preservation of the cardinality by bijection**, **compatibility of direct image with union**, and **De Morgan's laws**, we have

$$\delta_Y(A_n) = \text{card}(A_n \cap Y) \leq \text{card}(A_n) = \text{card}([0..n] \cap I) \leq n + 1 < \infty,$$

$$\bigcup_{n \in \mathbb{N}} A_n = \varphi \left( \bigcup_{n \in \mathbb{N}} ([0..n] \cap I) \right) = \varphi \left( \bigcup_{n \in \mathbb{N}} [0..n] \cap I \right) = \varphi(I) = X.$$

Hence, from Definition 607 ( *$\sigma$ -finite measure*), the counting measure  $\delta_Y$  is  $\sigma$ -finite.

Therefore, we have both implications.  $\square$

*Remark 655.* Counting measures are usually considered in the case where  $Y = X$  is countable, and  $\Sigma$  is the discrete  $\sigma$ -algebra  $\mathcal{P}(X)$ ; typically the  $\sigma$ -finite measure space  $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \delta_{\mathbb{N}})$ .

**Definition 656 (Dirac measure).** Let  $(X, \Sigma)$  be a measurable space. The counting measure associated with  $\{a\}$  is called *Dirac measure (at  $a$ )*; it is also denoted  $\delta_a \stackrel{\text{def.}}{=} \delta_{\{a\}}$ .

**Lemma 657 (equivalent definition of Dirac measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \in \Sigma$ . Then, we have  $\delta_a(A) = \mathbb{1}_A(a)$ .

*Proof.* Direct consequence of Definition 656 (*Dirac measure*), and Lemma 652 (*counting measure*,  $\text{card}(A \cap \{a\}) = 1$  when  $a \in A$  and 0 otherwise).  $\square$

**Lemma 658 (Dirac measure is finite).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\delta_a$  is finite and  $\delta_a(X) = 1$ .

*Proof.* Direct consequence of Definition 656 (*Dirac measure*), Lemma 653 (*finiteness of counting measure*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra*,  $\text{card}(X \cap \{a\}) = 1$ ).  $\square$

## 6.9 Measure and numbers

### 6.9.1 Negligibility and numbers

**Definition 659 (summability domain).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g : X \rightarrow \overline{\mathbb{R}}$ . The *summability domain* of  $f$  and  $g$  is denoted  $\mathcal{D}^+(f, g)$ ; it is defined by

$$(169) \quad \mathcal{D}^+(f, g) \stackrel{\text{def.}}{=} [(\{f = \infty\} \cap \{g = -\infty\}) \cup (\{f = -\infty\} \cap \{g = \infty\})]^c.$$

**Lemma 660 (summability on summability domain).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Then,  $f(x) + g(x)$  is well-defined iff  $x \in \mathcal{D}^+(f, g)$ .

*Proof.* Direct consequence of Definition 258 (*addition in  $\overline{\mathbb{R}}$* ), and Definition 659 (*summability domain*).  $\square$

**Lemma 661 (measurability of summability domain).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Then,  $\mathcal{D}^+(f, g) \in \Sigma$ .

*Proof.* Direct consequence of Lemma 564 (*inverse image is measurable*), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra*) closedness under countable intersection, union and complement.  $\square$

**Lemma 662 (negligibility of summability domain).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Then,  $f + g$  is well-defined almost everywhere iff  $[\mathcal{D}^+(f, g)]^c \in \mathbf{N}$ .*

*Proof.* Direct consequence of Lemma 660 (summability on summability domain), and Definition 624 (property almost satisfied),  $\square$

**Lemma 663 (almost sum).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Let  $A \stackrel{\text{def.}}{=} \mathcal{D}^+(f, g)$ . Let  $\tilde{f} \stackrel{\text{def.}}{=} f\mathbf{1}_A$  and  $\tilde{g} \stackrel{\text{def.}}{=} g\mathbf{1}_A$ . Assume that  $f + g$  is well-defined almost everywhere. Then, we have  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$ ,  $\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g$ ,  $\mathcal{D}^+(\tilde{f}, \tilde{g}) = X$ , and  $\tilde{f}, \tilde{g}, \tilde{f} + \tilde{g} \in \mathcal{M}$ .  
The sum  $\tilde{f} + \tilde{g}$  is called the almost sum of  $f$  and  $g$ ; it is denoted  $f \stackrel{\mu \text{ a.e.}}{+} g$ .*

*Proof.* From Lemma 662 (negligibility of summability domain), and Definition 624 (property almost satisfied, with  $\{\tilde{f} \neq f\} = \{\tilde{g} \neq g\} = A^c$ ), we have  $\tilde{f} \stackrel{\mu \text{ a.e.}}{=} f$  and  $\tilde{g} \stackrel{\mu \text{ a.e.}}{=} g$ . From Lemma 661 (measurability of summability domain), Lemma 554 (measurability of indicator function), and Lemma 568 ( $\mathcal{M}$  is closed under multiplication), we have  $\tilde{f}, \tilde{g} \in \mathcal{M}$ . Hence, from Definition 258 (addition in  $\mathbb{R}$ ), and Lemma 566 ( $\mathcal{M}$  is closed under addition when defined),  $\tilde{f} + \tilde{g}$  is well-defined, and we have  $\tilde{f} + \tilde{g} \in \mathcal{M}$ .  $\square$

**Lemma 664 (compatibility of almost sum with almost equality).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f + g$  is well-defined almost everywhere. Let  $f', g' \in \mathcal{M}$ . Assume that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ .  
Then,  $f' + g' \in \mathcal{M}$ , and we have  $f' + g' \stackrel{\mu \text{ a.e.}}{=} \tilde{f} + \tilde{g}$ .*

*Proof.* From Lemma 620 (empty set is negligible), Lemma 662 (negligibility of summability domain), and Lemma 566 ( $\mathcal{M}$  is closed under addition when defined),  $f' + g'$  is well-defined, and we have  $f' + g' \in \mathcal{M}$ . let  $A \stackrel{\text{def.}}{=} \mathcal{D}^+(f, g)$ ,  $\tilde{f} \stackrel{\text{def.}}{=} f\mathbf{1}_A$  and  $\tilde{g} \stackrel{\text{def.}}{=} g\mathbf{1}_A$ . Therefore, from Lemma 638 (almost equality is equivalence relation, transitivity), Lemma 641 (compatibility of almost equality with operator, with the binary operator addition), and Lemma 663 (almost sum), we have  $f' \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} \tilde{g}$ , and  $f' + g' \stackrel{\mu \text{ a.e.}}{=} \tilde{f} + \tilde{g} \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ .  $\square$

*Remark 665.*

Note that from Lemma 663 (almost sum), such functions  $f'$  and  $g'$  actually exist in  $\mathcal{M}$ .

**Lemma 666 (almost sum is sum).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $\mathcal{D}^+(f, g) = X$ . Then,  $f + g \in \mathcal{M}$ , and we have  $f \stackrel{\mu \text{ a.e.}}{+} g = f + g$ .*

*Proof.* Direct consequence of Definition 659 (summability domain,  $f + g$  is well-defined everywhere), and Lemma 663 (almost sum, with  $\tilde{f} = f$  and  $\tilde{g} = g$ ).  $\square$

**Lemma 667 (absolute value is almost definite).** *Let  $(X, \Sigma, \mu)$  be a measure space. Then, the absolute value, lifted into an operator  $\mathbb{R}^X \rightarrow \mathbb{R}^X$ , is almost definite:*

$$(170) \quad \forall f \in (\mathbb{R}^X)_{\mu \text{ a.e.}}, \quad |f| \stackrel{\mu \text{ a.e.}}{=} 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Direct consequence of Lemma 645 (local injectivity implies almost local injectivity, with  $Y \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $a \stackrel{\text{def.}}{=} 0$ , and the unary operator  $|\cdot|$ ), Lemma 641 (compatibility of almost equality with operator, with  $Y \stackrel{\text{def.}}{=} \mathbb{R}$ , and the unary operator  $|\cdot|$ ), and Lemma 280 (absolute value in  $\overline{\mathbb{R}}$  is definite).  $\square$

**Lemma 668 (masking almost nowhere).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $A \in \Sigma$ . Assume that  $\mu(A^c) = 0$ . Then, we have  $f \stackrel{\mu \text{ a.e.}}{=} f\mathbf{1}_A$ .*

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra, closedness under complement), the definition of the indicator function ( $A \subset \{f = f\mathbf{1}_A\}$ ), monotonicity of complement ( $\{f = f\mathbf{1}_A\}^c \subset A^c$ ), Definition 614 (negligible subset). and Definition 624 (property almost satisfied).  $\square$

**Lemma 669 (finite nonnegative part).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $A \in \Sigma$ . Let  $A_f \stackrel{\text{def.}}{=} A \cap f^{-1}(\mathbb{R}_+)$ . Let  $\tilde{f}_A \stackrel{\text{def.}}{=} f\mathbf{1}_{A_f}$ . Then,  $A_f \in \Sigma$  and  $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .  
Moreover, assume that  $f$  is  $\mu$ -almost everywhere finite and nonnegative, and that  $\mu(A^c) = 0$ . Then, we have  $\mu(A_f^c) = 0$ , i.e.  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_A$ .

*Proof.* Let  $B \stackrel{\text{def.}}{=} f^{-1}(\overline{\mathbb{R}_+})$  and  $C \stackrel{\text{def.}}{=} f^{-1}(\mathbb{R})$ . Then, from compatibility of inverse image with intersection, the definition of  $A_f$ , and properties of the indicator function, we have

$$f^{-1}(\mathbb{R}_+) = B \cap C, \quad A_f = A \cap B \cap C, \quad \text{and} \quad \mathbf{1}_{A_f} = \mathbf{1}_A \mathbf{1}_B \mathbf{1}_C.$$

Hence, from Lemma 374 (equivalent definition of nonnegative and nonpositive parts), and Definition 371 (finite part),  $\tilde{f}_A$  is the finite part of  $(f\mathbf{1}_A)^+$ . Therefore, from Lemma 576 (measurability and masking,  $f\mathbf{1}_A \in \mathcal{M}$ ), Lemma 579 (measurability of nonnegative and nonpositive parts, with  $f\mathbf{1}_A$ ), and Lemma 580 ( $\mathcal{M}_+$  is closed under finite part, with  $(f\mathbf{1}_A)^+$ ), we have  $\tilde{f}_A \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ .

Assume now that  $f$  is  $\mu$ -almost everywhere finite and nonnegative, and that  $\mu(A^c) = 0$ . Then, from Definition 467 ( $\sigma$ -algebra, closedness under complement), Definition 624 (property almost satisfied), and Lemma 619 (negligibility of measurable subset), we have  $B^c, C^c \in \Sigma$ , and  $\mu(B^c) = \mu(C^c) = 0$ . Hence, from Definition 467 ( $\sigma$ -algebra, closedness under complement), De Morgan's laws, and Lemma 598 (measure satisfies the finite Boole inequality), we have

$$A_f^c \in \Sigma \quad \text{and} \quad \mu(A_f^c) \leq \mu(A^c) + \mu(B^c) + \mu(C^c) = 0.$$

Therefore, from Definition 594 (measure, nonnegativeness), and Lemma 668 (masking almost nowhere), we have  $\mu(A_f^c) = 0$  and  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_A$ .  $\square$

## 6.9.2 The Lebesgue measure

*Remark 670.* We recall the denotation  $[, ]$  to not specify open or closed bounds for intervals.

**Definition 671 (length of interval).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . The length of intervals from  $a$  to  $b$  is  $\ell([a, b]) \stackrel{\text{def.}}{=} b - a$ .

**Lemma 672 (length is nonnegative).** Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then,  $\ell((a, b)) \geq 0$ .

*Proof.* Direct consequence of Definition 671 (length of interval).  $\square$

**Lemma 673 (length is homogeneous).**

We have  $\ell(\emptyset) = 0$ .

*Proof.* Let  $a \in \mathbb{R}$ . Then, for instance, we have  $\emptyset = (a, a)$ . Therefore, from Definition 671 (length of interval), we have  $\ell(\emptyset) = 0$ .  $\square$

**Lemma 674 (length of partition).**

Let  $a, b, c \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have

$$(171) \quad \ell((a, b) \cap (c, \infty)) + \ell((a, b) \cap (c, \infty)^c) = \ell((a, b)).$$

*Proof.* Let  $I \stackrel{\text{def.}}{=} (a, b)$ , and  $E \stackrel{\text{def.}}{=} (c, \infty)$ . We have  $E^c = (-\infty, c]$ .

**Case  $c \leq a$ .** Then, we have  $I \cap E = I$  and  $I \cap E^c = \emptyset$ . Hence, from Lemma 673 (length is homogeneous), we have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .

**Case  $a < c < b$ .** Then, we have  $I \cap E = (a, c)$  and  $I \cap E^c = [c, b)$ . Hence, from Definition 671 (*length of interval*), and **additive abelian group properties of  $\mathbb{R}$** , we have

$$\ell(I \cap E) + \ell(I \cap E^c) = \ell((a, c)) + \ell([c, b)) = (c - a) + (b - c) = b - a = \ell(I).$$

**Case  $b \leq c$ .** Then, we have  $I \cap E = \emptyset$  and  $I \cap E^c = I$ . Hence, from Lemma 673 (*length is homogeneous*), we have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .

Therefore, we always have  $\ell(I \cap E) + \ell(I \cap E^c) = \ell(I)$ .  $\square$

**Definition 675 (set of countable cover with bounded open intervals).**

Let  $A \subset \mathbb{R}$ . The set of countable covers of  $A$  with bounded open intervals is

$$(172) \quad C_A \stackrel{\text{def.}}{=} \left\{ ((a_n, b_n))_{n \in \mathbb{N}} \mid (\forall n \in \mathbb{N}, a_n, b_n \in \mathbb{R} \wedge a_n \leq b_n) \quad \wedge \quad A \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n) \right\}.$$

**Lemma 676 (set of countable cover with bounded open intervals is nonempty).**

Let  $A \subset \mathbb{R}$ . Then, we have  $C_A \neq \emptyset$ .

*Proof.* For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} (-n, n)$ . Then, from **the Archimedean property of  $\mathbb{R}$** , we have  $\mathbb{R} \subset \bigcup_{n \in \mathbb{N}} I_n$ . Therefore, from Definition 675 (*set of countable cover with bounded open intervals*), and **properties of inclusion**, we have  $(I_n)_{n \in \mathbb{N}} \in C_A$ .  $\square$

**Definition 677 ( $\lambda^*$ , Lebesgue measure candidate).**

The Lebesgue measure candidate is the function  $\lambda^* : \mathcal{P}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}_+$  defined by

$$(173) \quad \forall A \subset \mathbb{R}, \quad \lambda^*(A) \stackrel{\text{def.}}{=} \inf \left\{ \sum_{n \in \mathbb{N}} \ell(I_n) \mid (I_n)_{n \in \mathbb{N}} \in C_A \right\}.$$

*Remark 678.*

Note that  $\lambda^*$  defined in Definition 677 ( *$\lambda^*$ , Lebesgue measure candidate*) is not  $\sigma$ -additive on  $\mathcal{P}(\mathbb{R})$ , e.g. see [28, Ex. 2.28 pp. 86–87]. Thus, it is not a measure on  $\mathcal{P}(\mathbb{R})$ . In fact, there is no measure on  $\mathcal{P}(\mathbb{R})$  that generalizes the length of intervals, e.g. see [28, Ex. 2.29 pp. 87–88].

**Lemma 679 ( $\lambda^*$  is nonnegative).**

Let  $A \subset \mathbb{R}$ . Then, we have  $\lambda^*(A) \geq 0$ .

*Proof.* Direct consequence of Lemma 672 (*length is nonnegative*), and **monotonicity of infimum**.  $\square$

**Lemma 680 ( $\lambda^*$  is homogeneous).**

We have  $\lambda^*(\emptyset) = 0$ .

*Proof.* From Lemma 679 ( *$\lambda^*$  is nonnegative*), we have  $\lambda(\emptyset) \geq 0$ .

For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} (0, 0) = \emptyset$ . Let  $n \in \mathbb{N}$ . Then, from Definition 671 (*length of interval*), and **field properties of  $\mathbb{R}$** , we have  $\ell(I_n) = 0 - 0 = 0$ . Thus, we have  $\sum_{n \in \mathbb{N}} \ell(I_n) = 0$ . Moreover, we have  $\emptyset \subset \bigcup_{n \in \mathbb{N}} I_n$ . Thus, we have  $(I_n)_{n \in \mathbb{N}} \in C_\emptyset$ . Hence, from Definition 677 ( *$\lambda^*$ , Lebesgue measure candidate*), and Definition 9 (*infimum, lower bound*), we have  $\lambda^*(\emptyset) \leq 0$ .

Therefore  $\lambda^*(\emptyset) = 0$ .  $\square$

**Lemma 681 ( $\lambda^*$  is monotone).** Let  $A, B \subset \mathbb{R}$ . Assume that  $A \subset B$ . Then,  $\lambda^*(A) \leq \lambda^*(B)$ .

*Proof.* Let  $(I_n)_{n \in \mathbb{N}} \in C_B$ . Then, from Definition 675 (*set of countable cover with bounded open intervals*), and **transitivity of inclusion**, we have  $A \subset B \subset \bigcup_{n \in \mathbb{N}} I_n$ . Thus, from Definition 675 (*set of countable cover with bounded open intervals*), we have  $(I_n)_{n \in \mathbb{N}} \in C_A$ . Hence, from Definition 677 ( *$\lambda^*$ , Lebesgue measure candidate*), and Definition 9 (*infimum, lower bound, for  $A$* ), we have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \ell(I_n)$ .

Therefore, from Definition 9 (*infimum, greatest lower bound, for  $B$* ), and Definition 677 ( *$\lambda^*$ , Lebesgue measure candidate*), we have  $\lambda^*(A) \leq \lambda^*(B)$ .  $\square$

**Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive).** Let  $(A_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ . Then,  $\lambda^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .

*Proof.* Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ .

**Case  $\exists n_0 \in \mathbb{N}$  such that  $\lambda^*(A_{n_0}) = \infty$ .** Then, from **totally ordered set properties of  $\overline{\mathbb{R}}_+$** , we have  $\lambda^*(\bigcup_{n \in \mathbb{N}} A_n) \leq \infty = \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .

**Case  $\forall n \in \mathbb{N}$ ,  $\lambda^*(A_n) < \infty$ .** Let  $\varepsilon > 0$ . Let  $n \in \mathbb{N}$ . Then, from Definition 677 ( $\lambda^*$ , *Lebesgue measure candidate*), and Lemma 11 (*finite infimum*), there exists  $(I_{n,m})_{m \in \mathbb{N}} \in C_{A_n}$  such that  $\sum_{m \in \mathbb{N}} \ell(I_{n,m}) < \lambda^*(A_n) + \frac{\varepsilon}{2^{n+1}}$ . Let  $\varphi : \mathbb{N} \rightarrow \mathbb{N}^2$  be a bijection. Then, from Definition 675 (*set of countable cover with bounded open intervals*), Lemma 211 (*double countable union*), Lemma 210 (*definition of double countable union*), and **countability of  $\mathbb{N}^2$** , we have

$$A = \bigcup_{n \in \mathbb{N}} A_n \subset \bigcup_{n \in \mathbb{N}} \left( \bigcup_{m \in \mathbb{N}} I_{n,m} \right) = \bigcup_{(n,m) \in \mathbb{N}^2} I_{n,m} = \bigcup_{p \in \mathbb{N}} I_{\varphi(p)}.$$

Then, from Definition 675 (*set of countable cover with bounded open intervals*),  $(I_{\varphi(n)})_{n \in \mathbb{N}}$  belongs to  $C_A$ , and thus, from Definition 677 ( $\lambda^*$ , *Lebesgue measure candidate*), Definition 9 (*infimum, lower bound*), Lemma 301 (*definition of double series in  $\mathbb{R}_+$* ), Lemma 302 (*double series in  $\mathbb{R}_+$* ), and **additive properties of  $\mathbb{R}_+$** , we have

$$\lambda^*(A) \leq \sum_{p \in \mathbb{N}} \ell(I_{\varphi(p)}) = \sum_{(n,m) \in \mathbb{N}^2} \ell(I_{n,m}) = \sum_{n \in \mathbb{N}} \left( \sum_{m \in \mathbb{N}} \ell(I_{n,m}) \right) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n) + \varepsilon.$$

Hence, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .

Therefore, we always have  $\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \lambda^*(A_n)$ .  $\square$

**Lemma 683 ( $\lambda^*$  generalizes length of intervals).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have  $\lambda^*([a, b]) = b - a$ .

*Proof.* Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ .

**(1). Closed interval.** Let  $\varepsilon > 0$ . Then, we have  $[a, b] \subset (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$ . Thus, from Definition 675 (*set of countable cover with bounded open intervals*), Definition 677 ( $\lambda^*$ , *Lebesgue measure candidate*), Definition 9 (*infimum, lower bound*), Lemma 680 ( $\lambda^*$  is homogeneous), Definition 671 (*length of interval*), and **ordered field properties of  $\mathbb{R}$** , the sequence  $((a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}), \emptyset, \dots)$  belongs to  $C_{[a,b]}$ , and  $\lambda^*([a, b]) \leq \ell(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) = b - a + \varepsilon$ . Hence, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\lambda^*([a, b]) \leq b - a$ .

Let  $((a_n, b_n))_{n \in \mathbb{N}} \in C_{[a,b]}$  (with  $\forall n \in \mathbb{N}$ ,  $a_n \leq b_n$ ). Then, from Definition 675 (*set of countable cover with bounded open intervals*), and Lemma 248 (*finite cover of compact interval*), there exists  $q \in \mathbb{N}$  and  $(i_p)_{p \in [0..q]} \in \mathbb{N}$  pairwise distinct such that  $[a, b] \subset \bigcup_{p \in [0..q]} (a_{i_p}, b_{i_p})$  with  $a_{i_0} < a$ ,  $b < b_{i_q}$ , and for all  $p \in [0..q-1]$ ,  $a_{i_{p+1}} < b_{i_p}$ . Then, from **ordered field properties of  $\mathbb{R}$** , Definition 671 (*length of interval*), and **totally ordered set properties of  $\overline{\mathbb{R}}_+$** , we have

$$\begin{aligned} b - a < b_{i_q} - a_{i_0} &\leq b_{i_q} + \sum_{p \in [0..q-1]} (-a_{i_{p+1}} + b_{i_p}) - a_{i_0} = \sum_{p \in [0..q]} (b_{i_p} - a_{i_p}) \\ &= \sum_{p \in [0..q]} \ell((a_{i_p}, b_{i_p})) \leq \sum_{n \in \mathbb{N}} \ell((a_n, b_n)). \end{aligned}$$

Hence, from Definition 677 ( $\lambda^*$ , *Lebesgue measure candidate*), and Definition 9 (*infimum, greatest lower bound*), we have  $b - a \leq \lambda^*([a, b])$ .

Therefore, we have  $\lambda^*([a, b]) = b - a$ .

**(2). Open interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$\left[ a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2} \right] \subset (a, b) \subset [a, b].$$



Hence, from (1), and Lemma 681 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*((a, b)) \leq b - a$ . Therefore, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\lambda^*((a, b)) = b - a$ .

**(3). Left-open right-closed interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$[a + \varepsilon, b] \subset (a, b] \subset [a, b].$$

Hence, from (1), and Lemma 681 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*((a, b]) \leq b - a$ . Therefore, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\lambda^*((a, b]) = b - a$ .

**(4). Left-closed right-open interval.** Let  $\varepsilon > 0$ . Assume that  $\varepsilon < b - a$ . Then, we have

$$(a, b - \varepsilon] \subset [a, b) \subset [a, b].$$

Hence, from (1), and Lemma 681 ( $\lambda^*$  is monotone), we have  $b - a - \varepsilon \leq \lambda^*([a, b)) \leq b - a$ . Therefore, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\lambda^*([a, b)) = b - a$ .  $\square$

*Remark 684.*

Note that similar proofs using monotonicity provide for all  $a \in \mathbb{R}$ ,  $\lambda^*((-\infty, a]) = \lambda^*([a, \infty)) = \infty$ .

**Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).**

The set  $\mathcal{L}$  of subsets of  $\mathbb{R}$  defined by

$$(174) \quad \mathcal{L} \stackrel{\text{def.}}{=} \{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\}$$

is called the *Lebesgue  $\sigma$ -algebra*.

*Remark 686.* The set  $\mathcal{L}$  is shown below to be a  $\sigma$ -algebra; hence its name.

**Lemma 687 (equivalent definition of  $\mathcal{L}$ ).**

We have

$$(175) \quad \mathcal{L} = \{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \leq \lambda^*(A)\}.$$

*Proof.* Direct consequence of Lemma 207 (*compatibility of partition with intersection, with card(I) equals 2*), Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive), Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total, antisymmetry*), and Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).  $\square$

**Lemma 688 ( $\mathcal{L}$  is closed under complement).**

$\mathcal{L}$  is closed under complement.

*Proof.* Let  $E \in \mathcal{L}$ . Let  $A \subset \mathbb{R}$ . Then, from **the double complement law**, Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*), and Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra), we have

$$\lambda^*(A \cap E^c) + \lambda^*(A \cap (E^c)^c) = \lambda^*(A \cap E^c) + \lambda^*(A \cap E) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) = \lambda^*(A).$$

Therefore,  $E^c \in \mathcal{L}$ .  $\square$

**Lemma 689 ( $\mathcal{L}$  is closed under finite union).**

$\mathcal{L}$  is closed under finite union.

*Proof.* For all  $n \in [2.. \infty)$ , let  $P(n)$  be the property:  $\forall (E_i)_{i \in [1..n]} \in \mathcal{L}, \bigcup_{i \in [1..n]} E_i \in \mathcal{L}$ .

**Induction:  $P(2)$ .** Let  $E_1, E_2 \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} E_1 \cup E_2$ .

Let  $A \subset \mathbb{R}$ . Then, from **properties of intersection, union and complement**, we have

$$A \cap E = A \cap (E_1 \uplus (E_1^c \cap E_2)) = (A \cap E_1) \cup (A \cap (E_1^c \cap E_2)).$$

Thus, from Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive), **associativity of intersection**, and Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra,  $E_2$ , then  $E_1$  belong to  $\mathcal{L}$ ), we have

$$\begin{aligned} \lambda^*(A \cap E) + \lambda^*(A \cap E^c) &= \lambda^*((A \cap E_1) \cup (A \cap (E_1^c \cap E_2))) + \lambda^*(A \cap (E_1^c \cap E_2^c)) \\ &\leq \lambda^*(A \cap E_1) + \lambda^*((A \cap E_1^c) \cap E_2) + \lambda^*((A \cap E_1^c) \cap E_2^c) \\ &= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^c) \\ &= \lambda^*(A). \end{aligned}$$



Hence, from Lemma 687 (*equivalent definition of  $\mathcal{L}$* ), we have  $E \in \mathcal{L}$ .

**Induction;  $P(n)$  implies  $P(n+1)$ .** Let  $n \in [2..∞)$ . Assume that  $P(n)$  holds.

Let  $(E_i)_{i \in [1..n+1]} \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} \bigcup_{i \in [1..n]} E_i$ . Then, from  $P(n)$ , we have  $E \in \mathcal{L}$ , and from **associativity of union**, and  $P(2)$ , we have  $\bigcup_{i \in [1..n+1]} E_i = E \cup E_{n+1} \in \mathcal{L}$ .

Therefore,  $P(n)$  holds for all  $n \in [2..∞)$ .  $\square$

**Lemma 690 ( $\mathcal{L}$  is closed under finite intersection).**  $\mathcal{L}$  is closed under finite intersection.

*Proof.* Direct consequence of **De Morgan's laws**, Lemma 688 ( $\mathcal{L}$  is closed under complement), and Lemma 689 ( $\mathcal{L}$  is closed under finite union).  $\square$

**Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ ).** *Let  $n \in [2..∞)$ . Let  $(E_i)_{i \in [1..n]} \in \mathcal{L}$ . Assume that for all  $i, j \in [1..n]$ ,  $i \neq j \Rightarrow E_i \cap E_j = \emptyset$ . Then,  $\lambda^*$  is additive on  $\mathcal{L}$ :*

$$(176) \quad \forall A \subset \mathbb{R}, \quad \lambda^* \left( A \cap \biguplus_{i \in [1..n]} E_i \right) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i).$$

*Proof.* For all  $n \in [2..∞)$ , let  $P(n)$  be the property: for all  $(E_i)_{i \in [1..n]} \in \mathcal{L}$ ,

$$(\forall i, j \in [1..n], i \neq j \Rightarrow E_i \cap E_j = \emptyset) \implies \lambda^* \left( A \cap \biguplus_{i \in [1..n]} E_i \right) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i).$$

**Induction:  $P(2)$ .** Let  $E_1, E_2 \in \mathcal{L}$ . Assume that  $E_1 \cap E_2 = \emptyset$ . Let  $A \subset \mathbb{R}$ . Then, from **properties of intersection, union, and complement**, Lemma 207 (*compatibility of partition with intersection*, with  $\text{card}(I) = 2$ ), and Definition 685 ( $\mathcal{L}$ , *Lebesgue  $\sigma$ -algebra*,  $E_1 \in \mathcal{L}$ ), we have

$$\begin{aligned} \lambda^*(A \cap (E_1 \uplus E_2)) &= \lambda^*((A \cap E_1) \uplus (A \cap E_2)) \\ &= \lambda^*([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1) + \lambda^*([(A \cap E_1) \uplus (A \cap E_2)] \cap E_1^c) \\ &= \lambda^*(A \cap E_1) + \lambda^*(A \cap E_2). \end{aligned}$$

**Induction:  $P(n)$  implies  $P(n+1)$ .** Let  $n \in [2..∞)$ . Assume that  $P(n)$  holds.

Let  $(E_i)_{i \in [1..n+1]} \in \mathcal{L}$ . Assume that for all  $i, j \in [1..n+1]$ ,  $i \neq j$  implies  $E_i \cap E_j = \emptyset$ . Let  $E$  be the disjoint union  $\biguplus_{i \in [1..n]} E_i$ . Then, from **properties of intersection, union, and complement**,  $P(2)$ , and  $P(n)$ , we have  $E \cap E_{n+1} = \emptyset$ , and

$$\lambda^*(A \cap (E \uplus E_{n+1})) = \lambda^*(A \cap E) + \lambda^*(A \cap E_{n+1}) = \sum_{i \in [1..n]} \lambda^*(A \cap E_i) + \lambda^*(A \cap E_{n+1}).$$

Therefore,  $P(n)$  holds for all  $n \in [2..∞)$ .  $\square$

*Remark 692.* The additive property of  $\lambda^*$  per se actually corresponds to the case  $A = \mathbb{R}$ .

**Lemma 693 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ).**  $\lambda^*$  is  $\sigma$ -additive on  $(\mathbb{R}, \mathcal{L})$ .

*Proof.* Let  $(E_n)_{n \in \mathbb{N}} \in \mathcal{L}$ . Assume that for all  $i, j \in \mathbb{N}$ ,  $i \neq j$  implies  $E_i \cap E_j = \emptyset$ .

Let  $E \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} E_n$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ , with  $A \stackrel{\text{def.}}{=} \mathbb{R}$ ), and Lemma 681 ( $\lambda^*$  is monotone), we have

$$\sum_{i \in [0..n]} \lambda^*(E_i) = \lambda^* \left( \bigcup_{i \in [0..n]} E_i \right) \leq \lambda^*(E).$$

Thus, from **monotonicity of the limit**, we have

$$\sum_{n \in \mathbb{N}} \lambda^*(E_n) \leq \lambda^*(E).$$

Hence, from Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive), we have

$$\lambda^*(E) = \sum_{n \in \mathbb{N}} \lambda^*(E_n).$$

Therefore, from Definition 591 ( $\sigma$ -additivity over measurable space),  $\lambda^*$  is  $\sigma$ -additive over  $(\mathbb{R}, \mathcal{L})$ .  $\square$

**Lemma 694 (partition of countable union in  $\mathcal{L}$ ).**

Let  $(E_n)_{n \in \mathbb{N}} \in \mathcal{L}$ . Let  $F_0 \stackrel{\text{def.}}{=} E_0$ , and for all  $n \in \mathbb{N}$ , let  $F_{n+1} \stackrel{\text{def.}}{=} E_{n+1} \setminus \bigcup_{i \in [0..n]} F_i$ . Then,

$$(177) \quad \forall n \in \mathbb{N}, \quad F_n \in \mathcal{L},$$

$$(178) \quad \forall m, n \in \mathbb{N}, \quad m \neq n \implies F_m \cap F_n = \emptyset,$$

$$(179) \quad \forall n \in \mathbb{N}, \quad \bigcup_{i \in [0..n]} E_i = \bigsqcup_{i \in [0..n]} F_i.$$

*Proof.* For all  $n \in \mathbb{N}$ , let  $P(n)$  be the property:  $\forall i \in \mathbb{N}, i \leq n \implies F_i \in \mathcal{L}$ . **Induction:  $P(0)$ .** Trivial. **Induction:  $P(n)$  implies  $P(n+1)$ .** Direct consequence of Lemma 689 ( $\mathcal{L}$  is closed under finite union), Lemma 688 ( $\mathcal{L}$  is closed under complement), and Lemma 690 ( $\mathcal{L}$  is closed under finite intersection). Therefore, for all  $n \in \mathbb{N}$ ,  $P(n)$  holds.

Equations (178) and (179) are direct consequence of Lemma 212 (*partition of countable union*).  $\square$

**Lemma 695 ( $\mathcal{L}$  is closed under countable union).**  $\mathcal{L}$  is closed under countable union.

*Proof.* Let  $I \subset \mathbb{N}$ . Let  $(E_i)_{i \in I} \in \mathcal{L}$ . Let  $E \stackrel{\text{def.}}{=} \bigcup_{i \in I} E_i$ .

**Case  $\text{card}(I) < \infty$ .** Then, from Lemma 689 ( $\mathcal{L}$  is closed under finite union), we have  $E \in \mathcal{L}$ .

**Case  $\text{card}(I) = \infty$ .** Let  $\varphi : \mathbb{N} \rightarrow I$  be a bijection. Then, from **associativity of union**, we have  $E = \bigcup_{n \in \mathbb{N}} E_{\varphi(n)}$ . Let  $F_0 \stackrel{\text{def.}}{=} E_{\varphi(0)}$ , and for all  $n \in \mathbb{N}$ , let  $F_{n+1} \stackrel{\text{def.}}{=} E_{\varphi(n+1)} \setminus \bigcup_{i \in [0..n]} F_i$ .

Let  $n \in \mathbb{N}$ . Then, from Lemma 694 (*partition of countable union in  $\mathcal{L}$* ,  $\bigsqcup_{i \in [0..n]} F_i$  is equal to  $\bigcup_{i \in [0..n]} E_{\varphi(i)}$ ), **properties of union**, and **monotonicity of complement**, we have

$$E^c \subset F^c \quad \text{where } F \stackrel{\text{def.}}{=} \bigsqcup_{i \in [0..n]} F_i.$$

Let  $A \subset \mathbb{R}$ . Then, from Lemma 681 ( $\lambda^*$  is monotone, with  $A \cap E^c \subset A \cap F^c$ ), Lemma 694 (*partition of countable union in  $\mathcal{L}$* , the  $F_i$ 's are disjoint in  $\mathcal{L}$ ), Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ ), Lemma 689 ( $\mathcal{L}$  is closed under finite union,  $F \in \mathcal{L}$ ), and Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra, with  $F$ ), we have

$$\begin{aligned} \sum_{i \in [0..n]} \lambda^*(A \cap F_i) + \lambda^*(A \cap E^c) &\leq \sum_{i \in [0..n]} \lambda^*(A \cap F_i) + \lambda^*(A \cap F^c) \\ &= \lambda^*(A \cap F) + \lambda^*(A \cap F^c) = \lambda^*(A). \end{aligned}$$

Thus, from Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive), **distributivity of intersection over union**, and **monotonicity of the limit**, we have

$$\lambda^* \left( A \cap \bigcup_{n \in \mathbb{N}} F_n \right) + \lambda^*(A \cap E^c) \leq \sum_{n \in \mathbb{N}} \lambda^*(A \cap F_n) + \lambda^*(A \cap E^c) \leq \lambda^*(A).$$

Hence, since **the limit is a function**, we have  $\bigcup_{n \in \mathbb{N}} F_n = \bigcup_{n \in \mathbb{N}} E_{\varphi(n)} = E$ , and

$$\lambda^*(A \cap E) + \lambda^*(A \cap E^c) \leq \lambda^*(A).$$

Therefore, from Lemma 687 (*equivalent definition of  $\mathcal{L}$* ), we have  $E \in \mathcal{L}$ .  $\square$

**Lemma 696 (rays are Lebesgue-measurable).**

Let  $a \in \mathbb{R}$ . Then, we have

$$(180) \quad (a, \infty), (-\infty, a], (-\infty, a), [a, \infty) \in \mathcal{L}.$$

*Proof.* (1).  $(a, \infty) \in \mathcal{L}$ .

Let  $A \subset \mathbb{R}$ . Let  $\alpha \stackrel{\text{def.}}{=} \lambda^*(A \cap (a, \infty)) + \lambda^*(A \cap (-\infty, a])$ . Let us show that  $\alpha \leq \lambda^*(A)$ .

**Case  $\lambda^*(A) = \infty$ .** Trivial. **Case  $\lambda^*(A)$  finite.** Let  $\varepsilon > 0$ . From Definition 677 ( $\lambda^*$ , *Lebesgue measure candidate*), and Lemma 11 (*finite infimum*), let  $(I_n)_{n \in \mathbb{N}} \in C_A$  such that

$$\lambda^*(A) \leq \sum_{n \in \mathbb{N}} \ell(I_n) \leq \lambda^*(A) + \varepsilon.$$

For all  $n \in \mathbb{N}$ , let  $I'_n \stackrel{\text{def.}}{=} I_n \cap (a, \infty)$  and  $I''_n \stackrel{\text{def.}}{=} I_n \cap (-\infty, a]$ . Let  $n \in \mathbb{N}$ . From Lemma 222 (*intervals are closed under finite intersection*),  $I'_n$  and  $I''_n$  are intervals (possibly empty). Then, from Lemma 683 ( $\lambda^*$  *generalizes length of intervals*), and since  $I'_n$  and  $I''_n$  are contiguous, we have  $\lambda^*(I'_n) + \lambda^*(I''_n) = \ell(I'_n) + \ell(I''_n) = \ell(I_n)$ . Hence, from Lemma 681 ( $\lambda^*$  *is monotone*), and Lemma 682 ( $\lambda^*$  *is  $\sigma$ -subadditive*), we have

$$\alpha \leq \lambda^*\left(\bigcup_{n \in \mathbb{N}} I'_n\right) + \lambda^*\left(\bigcup_{n \in \mathbb{N}} I''_n\right) \leq \sum_{n \in \mathbb{N}} (\lambda^*(I'_n) + \lambda^*(I''_n)) = \sum_{n \in \mathbb{N}} \ell(I_n) \leq \lambda^*(A) + \varepsilon.$$

Therefore, from **monotonicity of the limit (when  $\varepsilon \rightarrow 0^+$ )**, we have  $\alpha \leq \lambda^*(A)$ , and from Lemma 687 (*equivalent definition of  $\mathcal{L}$* ), we have  $(a, \infty) \in \mathcal{L}$ .

(2).  $(-\infty, a] \in \mathcal{L}$ .

Direct consequence of Lemma 688 ( $\mathcal{L}$  *is closed under complement*), and (1).

(3).  $(-\infty, a) \in \mathcal{L}$ .

Direct consequence of Lemma 695 ( $\mathcal{L}$  *is closed under countable union*), and (2) with

$$(-\infty, a) = \bigcup_{n \in \mathbb{N}} \left(-\infty, a - \frac{1}{n+1}\right].$$

(4).  $[a, \infty) \in \mathcal{L}$ .

Direct consequence of Lemma 688 ( $\mathcal{L}$  *is closed under complement*), and (3).  $\square$

**Lemma 697 (intervals are Lebesgue-measurable).**

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have

$$(181) \quad (a, b), [a, b], [a, b), (a, b] \in \mathcal{L}.$$

*Proof.* Direct consequence of Lemma 690 ( $\mathcal{L}$  *is closed under finite intersection*), and Lemma 696 (*rays are Lebesgue-measurable*) with

$$\begin{aligned} (a, b) &= (-\infty, b) \cap (a, \infty), & [a, b] &= (-\infty, b] \cap [a, \infty), \\ [a, b) &= (-\infty, b) \cap [a, \infty), & (a, b] &= (-\infty, b] \cap (a, \infty). \end{aligned}$$

$\square$

**Lemma 698 ( $\mathcal{L}$  is  $\sigma$ -algebra).**

$(\mathbb{R}, \mathcal{L})$  is a measurable space.

*Proof.* Let  $A \subset \mathbb{R}$ . Then, from **properties of the empty set and of intersection**, and Lemma 680 ( $\lambda^*$  is homogeneous), we have  $\lambda^*(A \cap \emptyset) + \lambda^*(A \cap \emptyset^c) = \lambda^*(\emptyset) + \lambda^*(A \cap \mathbb{R}) = \lambda^*(A)$ . Hence,  $\emptyset \in \mathcal{L}$ .

Therefore, from Lemma 688 ( $\mathcal{L}$  is closed under complement), Lemma 695 ( $\mathcal{L}$  is closed under countable union), and Definition 467 ( $\sigma$ -algebra, measurable space),  $(\mathbb{R}, \mathcal{L})$  is a measurable space.  $\square$

**Lemma 699** ( $\lambda^*$  is measure on  $\mathcal{L}$ ).  $(\mathbb{R}, \mathcal{L}, \lambda^*_{|\mathcal{L}})$  is a measure space.

*Proof.* Direct consequence of Lemma 698 ( $\mathcal{L}$  is  $\sigma$ -algebra), Lemma 679 ( $\lambda^*$  is nonnegative), Lemma 680 ( $\lambda^*$  is homogeneous), Lemma 693 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ), and Definition 594 (measure).  $\square$

**Lemma 700** ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ). We have  $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$ .

*Proof.* Direct consequence of Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ ), Lemma 697 (intervals are Lebesgue-measurable), and Lemma 475 ( $\sigma$ -algebra generation is monotone).  $\square$

**Lemma 701** ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ).  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^*_{|\mathcal{B}(\mathbb{R})})$  is a measure space.

*Proof.* Direct consequence of Lemma 699 ( $\lambda^*$  is measure on  $\mathcal{L}$ ), and Lemma 700 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ).  $\square$

**Theorem 702** (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).

There exists a unique measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  that generalizes the length of bounded open intervals.

This measure is denoted  $\lambda \stackrel{\text{def.}}{=} \lambda^*_{|\mathcal{B}(\mathbb{R})}$ ; it is called the (Borel-)Lebesgue measure on Borel subsets (of  $\mathbb{R}$ ).

*Proof. Existence.* Direct consequence of Lemma 701 ( $\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$ ), and Lemma 683 ( $\lambda^*$  generalizes length of intervals, for open intervals).

**Uniqueness.** Let  $\mu$  be a measure on  $\mathcal{B}(\mathbb{R})$  such that, for all  $a, b \in \mathbb{R}$ ,  $a < b$  implies  $\mu((a, b)) = b - a$ .

Let  $G \stackrel{\text{def.}}{=} \{(a, b)\}_{a < b}$ .

From Definition 217 (interval,  $(a, a) = \emptyset$ ), Lemma 222 (intervals are closed under finite intersection), and Definition 424 ( $\pi$ -system),  $G = \{(a, b)\}_{a < b} \cup \{\emptyset\}$  is a  $\pi$ -system.

From Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ , with  $(a, b)$ ), Definition 467 ( $\sigma$ -algebra,  $\emptyset \in \mathcal{B}(\mathbb{R})$ ), and Lemma 493 (complete generated  $\sigma$ -algebra, with  $G \stackrel{\text{def.}}{=} \{(a, b)\}_{a < b}$  and  $G' \stackrel{\text{def.}}{=} \{\emptyset\}$ ), we have

$$\Sigma_{\mathbb{R}}(G) = \Sigma_{\mathbb{R}}(\{(a, b)\}_{a < b}) = \mathcal{B}(\mathbb{R}).$$

For all  $n \in \mathbb{N}$ , let  $I_n \stackrel{\text{def.}}{=} (-n, n)$ . Then, from **the Archimedean property of  $\mathbb{R}$** , we have  $\mathbb{R} = \bigcup_{n \in \mathbb{N}} I_n$ . Let  $n \in \mathbb{N}$ . Then, we have  $I_n \in G$ , and  $\mu(I_n) = 2n < \infty$ .

From Definition 594 (measure,  $\mu(\emptyset) = 0 = \lambda^*_{|\mathcal{B}(\mathbb{R})}(\emptyset)$ ),  $\mu$  and  $\lambda^*_{|\mathcal{B}(\mathbb{R})}$  coincide on  $G$ .

Therefore, from Lemma 649 (uniqueness of measures extended from a  $\pi$ -system, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$ ,  $\Sigma \stackrel{\text{def.}}{=} \mathcal{B}(\mathbb{R})$ , and  $X_n \stackrel{\text{def.}}{=} I_n$ ), we have, for all  $A \in \mathcal{B}(\mathbb{R})$ ,  $\mu(A) = \lambda^*_{|\mathcal{B}(\mathbb{R})}(A)$ .  $\square$

*Remark 703.* Note that  $\mathbf{N}(\mathbb{R}, \mathcal{L}, \lambda^*) = \mathbf{N}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \subset \mathcal{L}$ , e.g. see [28, Ex. 2.33 pp. 92–94].

**Lemma 704** (Lebesgue measure generalizes length of intervals).

Let  $a, b \in \mathbb{R}$ . Assume that  $a \leq b$ . Then, we have  $\lambda([a, b]) = b - a$ .

*Proof.* Direct consequence of Theorem 702 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), and Lemma 683 ( $\lambda^*$  generalizes length of intervals).  $\square$

*Remark 705.* Note that similarly to  $\lambda^*$ , we can prove for all  $a \in \mathbb{R}$ ,  $\lambda((-\infty, a]) = \lambda([a, \infty)) = \infty$ .

**Lemma 706 (Lebesgue measure is  $\sigma$ -finite).** The measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is  $\sigma$ -finite.

*Proof.* Direct consequence of Theorem 702 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), Lemma 683 ( $\lambda^*$  generalizes length of intervals, with  $b = -a \in \mathbb{N}$ ), the Archimedean property of  $\mathbb{R}$ , and Definition 607 ( $\sigma$ -finite measure, with  $A_n \stackrel{\text{def.}}{=} (-n, n)$ ).  $\square$

**Lemma 707 (Lebesgue measure is diffuse).** The measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is diffuse.

*Proof.* Direct consequence of Theorem 702 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ), Lemma 683 ( $\lambda^*$  generalizes length of intervals, with  $b = a$ ), and Definition 609 (diffuse measure,  $\{a\} = [a, a]$ ).  $\square$

## 6.10 Integration of nonnegative functions

### 6.10.1 Integration of indicator functions

*Remark 708.* In this section, functions take their values in  $\{0, 1\}$ , and the expressions involving integrals are taken in  $\overline{\mathbb{R}}_+$ .

**Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions).**

Let  $(X, \Sigma)$  be a measurable space. The set of measurable indicator functions is denoted  $\mathcal{IF}(X, \Sigma)$  (or simply  $\mathcal{IF}$ ); it is defined by  $\mathcal{IF}(X, \Sigma) \stackrel{\text{def.}}{=} \{\mathbb{1}_A \mid A \in \Sigma\}$ .

**Lemma 710 (indicator and support are each other inverse).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \in \Sigma$ , let  $f \in \mathcal{IF}$ . Then, we have

$$(182) \quad \mathbb{1}_A \in \mathcal{IF}, \quad \{f \neq 0\} \in \Sigma, \quad \{\mathbb{1}_A \neq 0\} = A \quad \text{and} \quad \mathbb{1}_{\{f \neq 0\}} = f.$$

*Proof.* Direct consequence of Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions), the definition of the indicator function, and the definition of inverse image.  $\square$

**Lemma 711 ( $\mathcal{IF}$  is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathcal{IF} \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions), Lemma 554 (measurability of indicator function), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), non-negativeness of indicator function, and Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).  $\square$

*Remark 712.*

We recall the notation  $\mathbb{1}_U^V$  to denote the indicator function of  $U$  defined on  $V$  (when  $U \subset V$ ).

**Lemma 713 ( $\mathcal{IF}$  is closed under extension by zero).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$  and  $f|_A \in \mathcal{IF}(A, \Sigma \cap A)$ . Then, we have  $\hat{f}\mathbb{1}_A \in \mathcal{IF}(X, \Sigma)$ .

*Proof.* Direct consequence of Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions,  $f|_A = \mathbb{1}_B^A$  with  $B \in \Sigma$  such that  $B \subset A$ ), and Lemma 215 (restriction is masking,  $\hat{f}\mathbb{1}_A = \mathbb{1}_B^X$ ).  $\square$

**Lemma 714 ( $\mathcal{IF}$  is closed under restriction).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{IF}(X, \Sigma)$ . Let  $A \in \Sigma$ . Then, we have  $f|_A \in \mathcal{IF}(A, \Sigma \cap A)$ .

*Proof.* Direct consequence of Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions,  $f = \mathbb{1}_B^X$  with  $B \in \Sigma$ ), the rule for the restriction of an indicator function ( $f|_A = \mathbb{1}_{B \cap A}^A$ ), and Lemma 521 (measurability of measurable subspace,  $B \cap A \in \Sigma \cap A$ ).  $\square$

**Definition 715 (integral in  $\mathcal{IF}$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{IF}$ . The integral of  $f$  (for the measure  $\mu$ ) is denoted  $\int f d\mu$ ; it is defined by

$$(183) \quad \int f d\mu \stackrel{\text{def.}}{=} \mu(\{f \neq 0\}) \in \overline{\mathbb{R}}.$$

**Lemma 716 (equivalent definition of integral in  $\mathcal{IF}$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $\int \mathbb{1}_A d\mu = \mu(A)$ .

*Proof.* Direct consequence of Definition 715 (integral in  $\mathcal{IF}$ ), and Lemma 710 (indicator and support are each other inverse).  $\square$

**Lemma 717 (additivity of integral in  $\mathcal{IF}$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $n \in \mathbb{N}$ . Let  $(A_i)_{i \in [0..n]} \in \Sigma$ . Assume that for all  $p, q \in [0..n]$ ,  $p \neq q$  implies  $A_p \cap A_q = \emptyset$ . Then, we have

$$(184) \quad \int \mathbb{1}_{\bigsqcup_{i \in [0..n]} A_i} d\mu = \sum_{i \in [0..n]} \int \mathbb{1}_{A_i} d\mu.$$

*Proof.* Direct consequence of Lemma 716 (equivalent definition of integral in  $\mathcal{IF}$ ), Definition 594 (measure,  $\sigma$ -additivity), Lemma 593 ( $\sigma$ -additivity implies additivity), and Definition 590 (additivity over measurable space).  $\square$

**Lemma 718 (integral in  $\mathcal{IF}$  over subset).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{IF}(A, \Sigma \cap A)$  iff  $\hat{f}\mathbb{1}_A \in \mathcal{IF}(X, \Sigma)$ .

If so, there exists  $B \in \Sigma$  such that  $B \subset A$ ,  $f|_A = \mathbb{1}_B^A$ ,  $\hat{f}\mathbb{1}_A = \mathbb{1}_B^X$ , and we have

$$(185) \quad \int f|_A d\mu_A = \int \hat{f}\mathbb{1}_A d\mu.$$

This integral is denoted  $\int_A f d\mu$ ; it is called integral of  $f$  over  $A$ .

*Proof. Equivalence.* Direct consequence of Lemma 713 ( $\mathcal{IF}$  is closed under extension by zero), and Lemma 714 ( $\mathcal{IF}$  is closed under restriction).

**Identity.** Direct consequence of Lemma 215 (restriction is masking), Lemma 710 (indicator and support are each other inverse,  $f|_A = \mathbb{1}_B^A$  and  $\hat{f}\mathbb{1}_A = \mathbb{1}_B^X$  with  $B \in \Sigma$  such that  $B \subset A$ ), Lemma 716 (equivalent definition of integral in  $\mathcal{IF}$ , with  $\mu \stackrel{\text{def.}}{=} \mu_A$ , then  $A \stackrel{\text{def.}}{=} B$ ), and Lemma 611 (trace measure).  $\square$

**Lemma 719 (integral in  $\mathcal{IF}$  over subset is additive).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $n \in \mathbb{N}$ . Let  $A, (A_i)_{i \in [0..n]} \in \Sigma$ . Assume that  $(A_i)_{i \in [0..n]}$  is a partition of  $A$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f}\mathbb{1}_A \in \mathcal{IF}$  iff for all  $i \in [0..n]$ ,  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{IF}$ . If so, we have

$$(186) \quad \int_A f d\mu = \sum_{i \in [0..n]} \int_{A_i} f d\mu.$$

*Proof.* “Left” implies “right”. Direct consequence of Lemma 710 (*indicator and support are each other inverse*,  $\hat{f}\mathbb{1}_A = \mathbb{1}_B$  with  $B \in \Sigma$  such that  $B \subset A$ ), Lemma 468 (*equivalent definition of  $\sigma$ -algebra*, closedness under countable intersection ( $B_i \stackrel{\text{def.}}{=} B \cap A_i \in \Sigma$ )), and **the formula for the indicator of intersection (product)**, thus  $\hat{f}\mathbb{1}_{A_i} = \mathbb{1}_{B_i}$ .

“Right” implies “left”. Direct consequence of Lemma 710 (*indicator and support are each other inverse*,  $\hat{f}\mathbb{1}_{A_i} = \mathbb{1}_{B_i}$  with  $B_i \in \Sigma$  such that  $B_i \subset A_i$ ), **monotonicity of intersection**, Definition 467 ( *$\sigma$ -algebra*,  $B \stackrel{\text{def.}}{=} \bigcap_{i \in [0..n]} B_i \in \Sigma$ ), **the formula for the indicator of disjoint union (sum)**, thus  $\hat{f}\mathbb{1}_A = \mathbb{1}_B$ , and **left distributivity of multiplication over addition in  $\mathbb{R}$**  ( $\hat{f}\mathbb{1}_A = \mathbb{1}_B$ ).

Therefore, we have the equivalence.

**Identity.** Direct consequence of Lemma 718 (*integral in  $\mathcal{IF}$  over subset*, with  $A$  and the  $A_i$ 's), **the formula for the indicator of disjoint union (sum)**, **left distributivity of multiplication over addition in  $\mathbb{R}$** , and Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ).  $\square$

*Remark 720.*

In the following statement, when  $Y$  is uncountable, the sum (of nonnegative values) is understood as the supremum for all finite subsets (which is also correct of course in the countable case):

$$(187) \quad \sum_{y \in Y} f(y) \stackrel{\text{def.}}{=} \sup_{\substack{Z \subset Y \\ \text{card}(Z) < \infty}} \sum_{z \in Z} f(z).$$

**Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{IF}$ . Then, we have

$$(188) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 710 (*indicator and support are each other inverse*), Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* , with  $A = \{f \neq 0\} \in \Sigma$ ), Lemma 652 (*counting measure*), **the definition of the cardinality**, and **the definition of the indicator function**.  $\square$

### 6.10.2 Integration of nonnegative simple functions

*Remark 722.* In this section, functions take their values in  $\mathbb{R}$ .

**Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).**

Let  $(X, \Sigma)$  be a measurable space. The vector subspace of finite linear combinations of indicator functions of measurable subsets is called *vector space of simple functions*; it is denoted  $\mathcal{SF}(X, \Sigma)$  (or simply  $\mathcal{SF}$ ); it is defined by  $\mathcal{SF}(X, \Sigma) \stackrel{\text{def.}}{=} \text{span}(\mathcal{IF}(X, \Sigma))$ .

**Lemma 724 ( *$\mathcal{SF}$  is vector space*).**

Let  $(X, \Sigma)$  be a measurable space. Then,  $\mathcal{SF}$  is a vector subspace of  $\mathbb{R}^X$ .

*Proof.* Direct consequence of Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*), and **the definition of the (linear) span**.  $\square$

**Lemma 725 ( *$\mathcal{SF}$  simple representation*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \in \mathcal{SF}$  iff there exists  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}$  and  $(A_i)_{i \in [0..n]} \in \Sigma$  such that  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ .

If so, for all  $x \in X \setminus \bigcup_{i \in [0..n]} A_i$ , we have  $f(x) = 0$ .



*Proof. Equivalence.* Direct consequence of Definition 723 ( $\mathcal{SF}$ , vector space of simple functions), Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions), and Lemma 710 (indicator and support are each other inverse).

**Property.**

Direct consequence of **the definition of the indicator function**, and **field properties of  $\mathbb{R}$  (0 is absorbing element for multiplication and identity element for addition)**.  $\square$

*Remark 726.* We recall that  $f^{-1}(y)$  denotes the subset  $f^{-1}(\{y\})$ .

**Lemma 727 ( $\mathcal{SF}$  canonical representation).**

Let  $(X, \Sigma)$  be a measurable space.

Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \in \mathcal{SF}$  iff  $f(X)$  is nonempty, finite, and for all  $y$  in  $f(X)$ ,  $f^{-1}(y)$  belongs to  $\Sigma$ . If so, we have the following canonical representation:

$$(189) \quad f = \sum_{y \in f(X)} y \mathbb{1}_{f^{-1}(y)},$$

i.e. there exists  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}$ , and  $(A_i)_{i \in [0..n]} \in \Sigma$ , such that

$$(190) \quad \forall p, q \in [0..n], \quad p \neq q \implies a_p \neq a_q,$$

$$(191) \quad \forall i \in [0..n], \quad A_i = f^{-1}(a_i) \neq \emptyset,$$

$$(192) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(193) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}.$$

*Proof. “Left” implies “right”.* Assume first that  $f \in \mathcal{SF}$ . Then, from Lemma 725 ( $\mathcal{SF}$  simple representation), there exists  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}$ , and  $(A_i)_{i \in [0..n]} \in \Sigma$ , such that  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ . Thus, from **the definition of the indicator function**,  $f$  can only take the values  $\sum_{i \in [0..n]} a_i \delta_i$  where  $(\delta_i)_{i \in [0..n]} \in \{0, 1\}$ . Hence, the cardinality of  $f(X)$  is at most  $2^{n+1}$ .

Let  $(\delta_i)_{i \in [0..n]} \in \{0, 1\}$ . Let  $y \stackrel{\text{def.}}{=} \sum_{i \in [0..n]} a_i \delta_i \in f(X)$ . Let  $I_y \subset [0..n]$  such that  $i \in I_y$  iff  $\delta_i = 1$ . Then, we have<sup>1</sup>  $y = \sum_{i \in I_y} a_i$ . Unfortunately, several partial sums of  $a_i$ 's may lead to the same value  $y$ . Thus, from Lemma 468 (**equivalent definition of  $\sigma$ -algebra**, closedness under countable intersection and union), we have

$$f^{-1}(y) = \bigcup_{\substack{I \subset \mathcal{P}([0..n]) \\ \sum_{i \in I} a_i = \sum_{i \in I_y} a_i = y}} \left( \bigcap_{i \in I_y} A_i \right) \in \Sigma.$$

From **properties of inverse image**, the collection  $(f^{-1}(y))_{y \in f(X)}$  constitutes a finite partition of the whole set  $X$ . Let  $x \in f^{-1}(y)$ . Then, from **field properties of  $\mathbb{R}$** , we have

$$f(x) = y = y \times 1 = y \mathbb{1}_{f^{-1}(y)}(x).$$

Hence, we have  $f = \sum_{y \in f(X)} y \mathbb{1}_{f^{-1}(y)}$ .

**“Right” implies “left”.** Let  $n \stackrel{\text{def.}}{=} \text{card}(f(X)) - 1 \in \mathbb{N}$ . Let  $(a_i)_{i \in [0..n]} \in \mathbb{R}$  be the  $n + 1$  distinct values taken by the function  $f$ , i.e. such that  $f(X) = \{a_i \mid i \in [0..n]\}$ . For all  $i \in [0..n]$ , let

$$A_i \stackrel{\text{def.}}{=} f^{-1}(a_i) \in \Sigma.$$

<sup>1</sup>For instance, the case  $I_y = \emptyset$  may correspond to  $y = 0$  when  $0 \notin \{a_i \mid i \in [0..n]\}$ .

Then, from **the definition and properties of image and inverse image**, and **the definition of partition**, the  $n$  subsets  $(A_i)_{i \in [0..n]}$  are nonempty, and constitutes a partition of  $X$ . Thus, from **the definition of the indicator function**, we have  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ . Hence, from Lemma 725 ( **$\mathcal{SF}$  simple representation**), we have  $f \in \mathcal{SF}$ .

Therefore, we have the equivalence, and both representations.  $\square$

**Lemma 728 ( $\mathcal{SF}$  is measurable).**

Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathcal{IF} \subset \mathcal{SF} \subset \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 723 ( **$\mathcal{SF}$ , vector space of simple functions**), Lemma 554 (**measurability of indicator function**), Lemma 557 ( **$\mathcal{M}_{\mathbb{R}}$  is algebra**), and Lemma 562 ( **$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$** ).  $\square$

**Lemma 729 ( $\mathcal{SF}$  is closed under extension by zero).**

Let  $(X, \Sigma)$  be a measurable space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$  and  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$  and  $f|_A \in \mathcal{SF}(A, \Sigma \cap A)$ . Then, we have  $\hat{f} \mathbb{1}_A \in \mathcal{SF}(X, \Sigma)$ .

*Proof.* Direct consequence of Definition 723 ( **$\mathcal{SF}$ , vector space of simple functions**,  $f|_A$  is a linear combination of  $f_i$ 's in  $\mathcal{IF}(A, \Sigma \cap A)$ ), and Lemma 713 ( **$\mathcal{IF}$  is closed under extension by zero**, with  $\hat{f}_i$  function from  $X$  to  $\overline{\mathbb{R}}$  such that  $(\hat{f}_i)|_Y = f_i$ ).  $\square$

**Lemma 730 ( $\mathcal{SF}$  is closed under restriction).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{SF}(X, \Sigma)$ . Let  $A \in \Sigma$ . Then, we have  $f|_A \in \mathcal{SF}(A, \Sigma \cap A)$ .

*Proof.* Direct consequence of Definition 723 ( **$\mathcal{SF}$ , vector space of simple functions**), and Lemma 714 ( **$\mathcal{IF}$  is closed under restriction**).  $\square$

*Remark 731.* From now on, functions take their values in  $\mathbb{R}_+$ , and the expressions involving integrals are taken in  $\mathbb{R}_+$ .

**Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).**

Let  $(X, \Sigma)$  be a measurable space. The subset of nonnegative simple functions is denoted  $\mathcal{SF}_+(X, \Sigma)$  (or simply  $\mathcal{SF}_+$ ); it is defined by  $\mathcal{SF}_+(X, \Sigma) \stackrel{\text{def.}}{=} \{f \in \mathcal{SF}(X, \Sigma) \mid f(X) \subset \mathbb{R}_+\}$ .

**Lemma 733 ( $\mathcal{SF}_+$  canonical representation).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then,  $f \in \mathcal{SF}_+$  iff there exists  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$  and  $(A_i)_{i \in [0..n]} \in \Sigma$  such that

$$(194) \quad \forall p, q \in [0..n], \quad p \neq q \implies a_p \neq a_q,$$

$$(195) \quad \forall i \in [0..n], \quad A_i = f^{-1}(a_i) \neq \emptyset,$$

$$(196) \quad X = \bigsqcup_{i \in [0..n]} A_i,$$

$$(197) \quad f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$$

which can also be written (with  $f^{-1}(y) \in \Sigma$  for all  $y \in f(X)$ )

$$(198) \quad f = \sum_{y \in f(X)} y \mathbb{1}_{f^{-1}(y)}.$$

*Proof.* “Left” implies “right”.

Direct consequence of Lemma 727 ( $\mathcal{SF}$  canonical representation, since  $\mathcal{SF}_+ \subset \mathcal{SF}$ ), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions,  $a_i \in f(X) \subset \mathbb{R}_+$ ).

“Right” implies “left”.

Direct consequence of Lemma 727 ( $\mathcal{SF}$  canonical representation,  $f \in \mathcal{SF}$ ), nonnegative additivity of  $\mathbb{R}_+$  ( $f \geq 0$ ), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).  $\square$

**Lemma 734** ( $\mathcal{SF}_+$  simple representation).

Let  $(X, \Sigma)$  be a measurable space. Let  $f : X \rightarrow \mathbb{R}$ . Then, we have  $f \in \mathcal{SF}_+$  iff there exists  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$  and  $(A_i)_{i \in [0..n]} \in \Sigma$  such that  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ .

If so, for all  $x \in X \setminus \bigcup_{i \in [0..n]} A_i$ , we have  $f(x) = 0$ .

*Proof.* “Left” implies “right”.

Direct consequence of Lemma 733 ( $\mathcal{SF}_+$  canonical representation).

“Right” implies “left”.

Direct consequence of Lemma 725 ( $\mathcal{SF}$  simple representation,  $f \in \mathcal{SF}$ ), nonnegative additivity of  $\mathbb{R}_+$  ( $f \geq 0$ ), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions).

Therefore, we have the equivalence.

**Property.** Direct consequence of Lemma 725 ( $\mathcal{SF}$  simple representation).  $\square$

**Lemma 735** ( $\mathcal{SF}_+$  is closed under positive linear operations).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Let  $a \in \mathbb{R}_+$ . Then, we have  $f + g, af \in \mathcal{SF}_+$ .

*Proof.* Direct consequence of Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 724 ( $\mathcal{SF}$  is vector space), Definition 61 (vector space), and closedness of addition and multiplication by a nonnegative number in  $\mathbb{R}_+$ .  $\square$

**Lemma 736** ( $\mathcal{SF}_+$  is measurable).

Let  $(X, \Sigma)$  be a measurable space. Then, we have  $\mathcal{IF} \subset \mathcal{SF}_+ \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+ \subset \mathcal{M}$ .

*Proof.* Direct consequence of Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions), nonnegativeness of the indicator function, Lemma 728 ( $\mathcal{SF}$  is measurable), Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions).  $\square$

**Lemma 737** (integral in  $\mathcal{SF}_+$ ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, we have  $f = \sum_{y \in f(X)} y \mathbb{1}_{f^{-1}(y)}$ , and the sum  $\sum_{y \in f(X)} y \mu(f^{-1}(y))$  is well-defined.

The integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by

$$(199) \quad \int f d\mu \stackrel{\text{def.}}{=} \sum_{y \in f(X)} y \mu(f^{-1}(y)) \in \overline{\mathbb{R}}_+.$$

*Proof.* Direct consequence of Lemma 733 ( $\mathcal{SF}_+$  canonical representation), Lemma 716 (equivalent definition of integral in  $\mathcal{IF}$ ), Definition 594 (measure, nonnegativeness), Lemma 314 (multiplication in  $\mathbb{R}_+$  is closed (measure theory),  $y \in \mathbb{R}_+ \subset \overline{\mathbb{R}}_+$ ), and Lemma 294 (addition in  $\mathbb{R}_+$  is closed).  $\square$

**Lemma 738** (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{IF}$ . Then the values of  $\int f d\mu$  provided by Definition 715 (integral in  $\mathcal{IF}$ ), and Lemma 737 (integral in  $\mathcal{SF}_+$ ) coincide.

*Proof.* Direct consequence of Definition 715 (*integral in  $\mathcal{IF}$* ), and Lemma 737 (*integral in  $\mathcal{SF}_+$* ).  $\square$

**Lemma 739 (*integral in  $\mathcal{SF}_+$  of indicator function*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $\mathbb{1}_A \in \mathcal{SF}_+$  and  $\int \mathbb{1}_A d\mu = \mu(A)$ .

*Proof.* Direct consequence of Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ), and Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ).  $\square$

**Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Let  $y \in f(X)$ . Then, we have

$$(200) \quad \mu(f^{-1}(y)) = \sum_{z \in g(X)} \mu(f^{-1}(y) \cap g^{-1}(z)).$$

*Proof.* Direct consequence of Lemma 736 ( *$\mathcal{SF}_+$  is measurable*), Lemma 496 (*some Borel subsets, singletons are measurable*), Lemma 556 (*inverse image is measurable in  $\mathbb{R}$ ,  $f^{-1}(y), g^{-1}(z) \in \Sigma$* ), Lemma 727 ( *$\mathcal{SF}$  canonical representation,  $X = \bigsqcup_{z \in g(X)} g^{-1}(z)$* ), and Lemma 596 (*measure over countable partition, with  $A \stackrel{\text{def.}}{=} f^{-1}(y)$ ,  $B_i \stackrel{\text{def.}}{=} g^{-1}(z)$  and  $\text{card}(I) = \text{card}(g(X))$ ).  $\square$*

**Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Let  $y \in f(X)$ . Then, we have

$$(201) \quad \sum_{z \in g(X)} (y + z) \mu(f^{-1}(y) \cap g^{-1}(z)) = \sum_{t \in (f+g)(X)} t \mu(f^{-1}(y) \cap (f+g)^{-1}(t)).$$

*Proof.* For all  $z, t \in \mathbb{R}$ , let  $A(z) \stackrel{\text{def.}}{=} f^{-1}(y) \cap g^{-1}(z)$  and  $B(t) \stackrel{\text{def.}}{=} f^{-1}(y) \cap (f+g)^{-1}(t)$ . Then, from Lemma 735 ( *$\mathcal{SF}_+$  is closed under positive linear operations*), and Lemma 736 ( *$\mathcal{SF}_+$  is measurable*), we have  $f, g, f+g \in \mathcal{SF}_+ \subset \mathcal{M}_+ \subset \mathcal{M}$ .

Let  $z, t \in \mathbb{R}$ . Then, from Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*), Lemma 556 (*inverse image is measurable in  $\mathbb{R}$ ,  $f^{-1}(y), g^{-1}(z), (f+g)^{-1}(t) \in \Sigma$* ), and Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection*), we have  $A(z), B(t) \in \Sigma$ .

Let  $\mathcal{A} \stackrel{\text{def.}}{=} \{z \in g(X) \mid \mu(A(z)) > 0\}$  and  $\mathcal{B} \stackrel{\text{def.}}{=} \{t \in (f+g)(X) \mid \mu(B(t)) > 0\}$ .

(1).  $\forall z \in g(X), A(z) = B(y+z)$ .

Direct consequence of the definition of  $A(z)$  and  $B(y+z)$ , **the definition of the addition of functions to  $\mathbb{R}$** , and **additive abelian group properties of  $\mathbb{R}$  (with  $y \neq \infty$ )**.

(2).  $\forall t \in (f+g)(X), B(t) = A(-y+t)$ .

Direct consequence of the definition of  $B(t)$  and  $A(-y+t)$ , **the definition of the addition of functions to  $\mathbb{R}$** , and **additive abelian group properties of  $\mathbb{R}$  (with  $y \neq \infty$ )**.

(3).  $\tau_y = (z \mapsto y+z) : \mathcal{A} \rightarrow \mathcal{B}$  is a bijection.

From **additive abelian group properties of  $\mathbb{R}$  (with  $y \neq \infty$ )**, the translation  $\tau_y$  is a bijection from  $\mathbb{R}$  onto itself.

Let  $z \in \mathcal{A}$ . Then, from the definition of  $\mathcal{A}$ , we have  $z \in g(X)$  and  $\mu(A(z)) > 0$ . Thus, from Definition 594 (*measure,  $\mu(\emptyset) = 0$  (contrapositive)*), we have  $A(z) \neq \emptyset$ . Let  $x \in A(z)$ . Then, from the definition of  $A(z)$ , we have  $f(x) = y$  and  $g(x) = z$ . Thus, from the definition of the addition of functions to  $\mathbb{R}$ , we have  $(f+g)(x) = y+z$ , i.e.  $y+z \in (f+g)(X)$ . Moreover, from (1), we have  $\mu(B(y+z)) = \mu(A(z)) > 0$ . Thus, from the definition of  $\mathcal{B}$ , we have  $\tau_y(z) = y+z \in \mathcal{B}$ . Hence, we have  $\tau_y(\mathcal{A}) \subset \mathcal{B}$ .

Conversely, let  $t \in \mathcal{B}$ . Then, from the definition of  $\mathcal{B}$ , we have  $t \in (f+g)(X)$  and  $\mu(B(t)) > 0$ . Thus, from Definition 594 (*measure,  $\mu(\emptyset) = 0$  (contrapositive)*), we have  $B(t) \neq \emptyset$ . Let  $x \in B(t)$ . Then, from the definition of  $B(t)$ , we have  $f(x) = y$  and  $(f+g)(x) = t$ . Thus, from the definition of the addition of functions to  $\mathbb{R}$ , and **additive abelian group properties of  $\mathbb{R}$  (with  $y \neq \infty$ )**, we

have  $g(x) = -y + t$ , i.e.  $-y + t \in g(X)$ . Moreover, from (2), we have  $\mu(A(-y + t)) = \mu(B(t)) > 0$ . Thus, from the definition of  $\mathcal{A}$ , we have  $\tau_y^{-1}(t) = -y + t \in \mathcal{A}$ . Then, from **properties of inverse functions**, we have  $t = \tau_y(\tau_y^{-1}(t)) \in \tau_y(\mathcal{A})$ . Hence, we have  $\mathcal{B} \subset \tau_y(\mathcal{A})$ .

Finally,  $\tau_y$  is a bijection from  $\mathcal{A}$  onto  $\tau_y(\mathcal{A}) = \mathcal{B}$ .

Therefore, from the definitions of  $\mathcal{A}$  and  $\tau_y$ , (1), (3), and the definition of  $\mathcal{B}$ , we have

$$\begin{aligned} \sum_{z \in g(X)} (y + z) \mu(A(z)) &= \sum_{z \in \mathcal{A}} (y + z) \mu(A(z)) = \sum_{z \in \mathcal{A}} \tau_y(z) \mu(B(\tau_y(z))) \\ &= \sum_{t \in \tau_y(\mathcal{A})} t \mu(B(t)) = \sum_{t \in \mathcal{B}} t \mu(B(t)) = \sum_{t \in (f+g)(X)} t \mu(B(t)). \end{aligned}$$

□

*Remark 742.* Note that the previous lemma is still valid when no assumption is made on the sign of functions  $f$  and  $g$ , and for any  $y \in \mathbb{R}$ . Moreover, the equalities  $A(z) = B(y + z)$  and  $B(t) = A(-y + t)$  are still valid for any  $z, t \in \mathbb{R}$ , in which case these sets may be empty.

**Lemma 743 (integral in  $\mathcal{SF}_+$  is positive linear).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Let  $a \in \mathbb{R}_+$ . Then,  $f + g, af \in \mathcal{SF}_+$ , and we have*

$$(202) \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu \quad \text{and} \quad \int af d\mu = a \int f d\mu.$$

*Proof.* From Lemma 735 ( **$\mathcal{SF}_+$  is closed under positive linear operations**), and Lemma 736 ( **$\mathcal{SF}_+$  is measurable**), we have  $f + g, f \in \mathcal{SF}_+ \subset \mathcal{M}_+ \subset \mathcal{M}$ .

Let  $y, z, t \in \mathbb{R}$ . Then, from Definition 560 ( **$\mathcal{M}$ , set of measurable numeric functions**), Lemma 556 (**inverse image is measurable in  $\mathbb{R}$ ,  $f^{-1}(y), g^{-1}(z), (f + g)^{-1}(t) \in \Sigma$** ), and Lemma 468 (**equivalent definition of  $\sigma$ -algebra, closedness under countable intersection**), we have  $f^{-1}(y) \cap g^{-1}(z), f^{-1}(y) \cap (f + g)^{-1}(t) \in \Sigma$ .

From Lemma 737 (**integral in  $\mathcal{SF}_+$** ), Lemma 740 (**decomposition of measure in  $\mathcal{SF}_+$** ), and Lemma 318 (**multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)**), we have

$$\int f d\mu = \sum_{y \in f(X)} y \mu(f^{-1}(y)) = \sum_{y \in f(X)} \sum_{z \in g(X)} y \mu(f^{-1}(y) \cap g^{-1}(z)).$$

In the very same way, we have

$$\int g d\mu = \sum_{z \in g(X)} z \mu(g^{-1}(z)) = \sum_{z \in g(X)} \sum_{y \in f(X)} z \mu(g^{-1}(z) \cap f^{-1}(y)).$$

Hence, from Lemma 295 (**addition in  $\overline{\mathbb{R}}_+$  is associative**), Lemma 296 (**addition in  $\overline{\mathbb{R}}_+$  is commutative**), **symmetry of intersection**, Lemma 318 (**multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)**), Lemma 741 (**change of variable in sum in  $\mathcal{SF}_+$** ), Lemma 740 (**decomposition of measure in  $\mathcal{SF}_+$ , with  $(f + g), f$  and  $t$** ), and Lemma 737 (**integral in  $\mathcal{SF}_+$** ), we have

$$\begin{aligned} \int f d\mu + \int g d\mu &= \sum_{y \in f(X)} \sum_{z \in g(X)} (y + z) \mu(f^{-1}(y) \cap g^{-1}(z)) \\ &= \sum_{y \in f(X)} \sum_{t \in (f+g)(X)} t \mu(f^{-1}(y) \cap (f + g)^{-1}(t)) \\ &= \sum_{t \in (f+g)(X)} t \sum_{y \in f(X)} \mu((f + g)^{-1}(t) \cap f^{-1}(y)) \\ &= \sum_{t \in (f+g)(X)} t \mu((f + g)^{-1}(t)) \\ &= \int (f + g) d\mu. \end{aligned}$$

**Case  $a = 0$ .** Then, from Lemma 724 ( $\mathcal{SF}$  is vector space), Definition 61 (vector space,  $0 \in \mathcal{SF}$ ), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), we have  $0f = 0 \in \mathcal{SF}_+$ . Hence, from Lemma 737 (integral in  $\mathcal{SF}_+$ ), and Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)), we have

$$\int 0f d\mu = \int 0 d\mu = 0 \times \mu(X) = 0 = 0 \times \int f d\mu.$$

**Case  $a > 0$ .** Let  $y \in f(X)$ . Let  $z = ay \in (af)(X)$ . Then, from Definition 264 (multiplication in  $\overline{\mathbb{R}}$ , for all  $a \in \mathbb{R}_+^*$ ,  $\frac{a}{a} = 1$ ), we have  $(af)^{-1}(z) = f^{-1}(y)$ . Thus, from Lemma 737 (integral in  $\mathcal{SF}_+$ ), we have

$$\int af d\mu = \sum_{z \in (af)(X)} z \mu((af)^{-1}(z)) = \sum_{y \in f(X)} ay \mu(f^{-1}(y)) = a \int f d\mu.$$

Therefore, for all  $a \in \mathbb{R}_+$ , we have  $\int af d\mu = a \int f d\mu$ .  $\square$

**Lemma 744 (equivalent definition of the integral in  $\mathcal{SF}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, for all  $n \in \mathbb{N}$ ,  $(a_i)_{i \in [0..n]} \in \mathbb{R}_+$ , and  $(A_i)_{i \in [0..n]} \in \Sigma$  such that  $f = \sum_{i \in [0..n]} a_i \mathbb{1}_{A_i}$ , we have

$$(203) \quad \int f d\mu = \sum_{i \in [0..n]} a_i \mu(A_i).$$

*Proof.* Direct consequence of Lemma 734 ( $\mathcal{SF}_+$  simple representation, thus such representation exists), Lemma 743 (integral in  $\mathcal{SF}_+$  is positive linear), and Lemma 739 (integral in  $\mathcal{SF}_+$  of indicator function).  $\square$

**Lemma 745 (integral in  $\mathcal{SF}_+$  is monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{SF}_+$ . Then, we have

$$(204) \quad f \leq g \implies \int f d\mu \leq \int g d\mu.$$

*Proof.* Assume that  $f \leq g$ . Then, from Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Lemma 724 ( $\mathcal{SF}$  is vector space), and Definition 61 (vector space,  $(\mathcal{SF}, +)$  is an abelian group), we have  $g = f + (g - f)$  with  $f, g - f \in \mathcal{SF}_+$ . Therefore, from Lemma 743 (integral in  $\mathcal{SF}_+$  is positive linear), and Lemma 737 (integral in  $\mathcal{SF}_+$ , nonnegativeness with  $g - f \in \mathcal{SF}_+$ ), we have  $\int g d\mu = \int f d\mu + \int (g - f) d\mu \geq \int f d\mu$ .  $\square$

**Lemma 746 (integral in  $\mathcal{SF}_+$  is continuous).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(205) \quad \int f d\mu = \sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu.$$

*Proof.* From Definition 2 (supremum, upper bound), and since  $f \in \mathcal{SF}_+$ , we have

$$\int f d\mu \leq \sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu.$$

Conversely, let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ . Then, from Lemma 745 (integral in  $\mathcal{SF}_+$  is monotone), and Definition 2 (supremum, least upper bound), we have

$$\sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu \leq \int f d\mu.$$

Therefore, we have the equality.  $\square$

**Lemma 747 (integral in  $\mathcal{SF}_+$  over subset).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{SF}_+(A, \Sigma \cap A)$  iff  $\hat{f}\mathbb{1}_A \in \mathcal{SF}_+(X, \Sigma)$ . If so, we have*

$$(206) \quad \int f|_A d\mu_A = \int \hat{f}\mathbb{1}_A d\mu.$$

*This integral is denoted  $\int_A f d\mu$ ; it is called integral of  $f$  over  $A$ .*

*Proof. Equivalence.* Direct consequence of Lemma 729 ( $\mathcal{SF}$  is closed under extension by zero), and Lemma 730 ( $\mathcal{SF}$  is closed under restriction).

**Identity.** Direct consequence of Definition 723 ( $\mathcal{SF}$ , vector space of simple functions,  $f|_A$  is a linear combination of  $f_i$ 's in  $\mathcal{IF}(A, \Sigma \cap A)$ ) Lemma 743 (integral in  $\mathcal{SF}_+$  is positive linear), Lemma 738 (integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$ ), and Lemma 718 (integral in  $\mathcal{IF}$  over subset, with  $\hat{f}_i : X \rightarrow \overline{\mathbb{R}}$  such that  $(\hat{f}_i)|_Y = f_i$ ).  $\square$

**Lemma 748 (integral in  $\mathcal{SF}_+$  over subset is additive).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $n \in \mathbb{N}$ . Let  $A, (A_i)_{i \in [0..n]} \in \Sigma$ . Assume that  $(A_i)_{i \in [0..n]}$  is a partition of  $A$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f}\mathbb{1}_A \in \mathcal{SF}_+$  iff for all  $i \in [0..n]$ ,  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{SF}_+$ . If so, we have*

$$(207) \quad \int_A f d\mu = \sum_{i \in [0..n]} \int_{A_i} f d\mu.$$

*Proof. (1).* Let  $W \in \Sigma$ . Assume that  $W \subset Y$ . Then, from Definition 723 ( $\mathcal{SF}$ , vector space of simple functions), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), we have  $\hat{f}\mathbb{1}_W \in \mathcal{SF}_+$  iff there exist  $p \in \mathbb{N}$ ,  $(a_j)_{j \in [0..p]} \in \mathbb{R}_+$ , and  $(f_j)_{j \in [0..p]} \in \mathcal{IF}$  such that

$$\hat{f}\mathbb{1}_W = \sum_{j \in [0..p]} a_j f_j.$$

**(2).** Let  $g \in \mathcal{IF}$ . Let  $i \in [0..n]$ . Then, from Definition 709 ( $\mathcal{IF}$ , set of measurable indicator functions,  $g = \mathbb{1}_B$  with  $B \in \Sigma$ ), **the formula for the indicator of intersection (product)**, and Lemma 468 (**equivalent definition of  $\sigma$ -algebra**, closedness under countable intersection), we have  $g\mathbb{1}_{A_i} = \mathbb{1}_{B \cap A_i} \in \mathcal{IF}$ .

**“Left” implies “right”.** Assume that  $\hat{f}\mathbb{1}_A \in \mathcal{SF}_+$ .

Then, from (1) (with  $W \stackrel{\text{def.}}{=} A$ ), let  $p \in \mathbb{N}$ ,  $(a_j)_{j \in [0..p]} \in \mathbb{R}_+$ , and  $(f_j)_{j \in [0..p]} \in \mathcal{IF}$  such that

$$\hat{f}\mathbb{1}_A = \sum_{j \in [0..p]} a_j f_j.$$

Let  $i \in [0..n]$ . Let  $j \in [0..p]$ . Let  $g_j^i \stackrel{\text{def.}}{=} f_j \mathbb{1}_{A_i}$ . Then, from (2) (with  $g \stackrel{\text{def.}}{=} f_j$ ), we have  $g_j^i \in \mathcal{IF}$ . Moreover, since  $A_i \subset A$  (i.e.  $A_i = A \cap A_i$ ), from **the formula for the indicator of intersection (product)**, and **field properties of  $\mathbb{R}$** , we have

$$\hat{f}\mathbb{1}_{A_i} = \hat{f}\mathbb{1}_{A \cap A_i} = \left( \hat{f}\mathbb{1}_A \right) \mathbb{1}_{A_i} = \sum_{j \in [0..p]} a_j f_j \mathbb{1}_{A_i} = \sum_{j \in [0..p]} a_j g_j^i.$$

Hence, from (1) (with  $W \stackrel{\text{def.}}{=} A_i$ ), we have  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{SF}_+$ .



**“Right” implies “left”.** Assume that for all  $i \in [0..n]$ ,  $\hat{f}\mathbf{1}_{A_i} \in \mathcal{SF}_+$ .

Then, from (1) (with  $W \stackrel{\text{def.}}{=} A_i$ ), for all  $i \in [0..n]$ , let  $p^i \in \mathbb{N}$ ,  $(a_{j^i}^i)_{j^i \in [0..p^i]} \in \mathbb{R}_+$ , and  $(f_{j^i}^i)_{j^i \in [0..p^i]}$  in  $\mathcal{IF}$  such that  $\hat{f}\mathbf{1}_{A_i} = \sum_{j^i \in [0..p^i]} a_{j^i}^i f_{j^i}^i$ . Let  $p \stackrel{\text{def.}}{=} -1 + \sum_{i \in [0..n]} (1 + p^i)$ ,

$$(a_j)_{j \in [0..p]} \stackrel{\text{def.}}{=} \left( (a_{j^i}^i)_{j^i \in [0..p^i]} \right)_{i \in [0..n]} \in \mathbb{R}_+, \quad \text{and} \quad (f_j)_{j \in [0..p]} \stackrel{\text{def.}}{=} \left( (f_{j^i}^i \mathbf{1}_{A_i})_{j^i \in [0..p^i]} \right)_{i \in [0..n]} \in \mathcal{IF}.$$

Then, from **the formula for the indicator of disjoint union (sum)**, **field properties of  $\mathbb{R}$** , and (2) (with  $g \stackrel{\text{def.}}{=} f_{j^i}^i$ ), we have

$$\hat{f}\mathbf{1}_A = \sum_{i \in [0..n]} \hat{f}\mathbf{1}_{A_i} = \sum_{i \in [0..n]} \sum_{j^i \in [0..p^i]} a_{j^i}^i (f_{j^i}^i \mathbf{1}_{A_i}) = \sum_{j \in [0..p]} a_j f_j.$$

Hence, from (1) (with  $W \stackrel{\text{def.}}{=} A$ ), we have  $\hat{f}\mathbf{1}_A \in \mathcal{SF}_+$ .

Therefore, we have the equivalence.

**Identity.** Direct consequence of Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*, with  $A$  and the  $A_i$ 's), and **the formula for the indicator of disjoint union (sum)**.  $\square$

**Lemma 749 (integral in  $\mathcal{SF}_+$  for counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(208) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 734 ( *$\mathcal{SF}_+$  simple representation*), Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ), Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*), and **associativity and commutativity of (possibly uncountable) addition in  $\mathbb{R}_+$** .  $\square$

**Lemma 750 (integral in  $\mathcal{SF}_+$  for counting measure on  $\mathbb{N}$ ).**

Let  $f \in \mathcal{SF}_+(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ . Then, we have

$$(209) \quad \int f d\delta_{\mathbb{N}} = \sum_{n \in \mathbb{N}} f(n).$$

*Proof.* Direct consequence of Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*, with  $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$  and  $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$ ).  $\square$

**Lemma 751 (integral in  $\mathcal{SF}_+$  for Dirac measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\{a\} \in \Sigma$ . Let  $f \in \mathcal{SF}_+$ . Then, we have

$$(210) \quad \int f d\delta_a = f(a).$$

*Proof.* Direct consequence of Definition 656 (*Dirac measure*), and Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*, with  $Y \stackrel{\text{def.}}{=} \{a\}$ ).  $\square$

### 6.10.3 Integration of nonnegative measurable functions

*Remark 752.* In this section, functions take their values in  $\overline{\mathbb{R}}_+$ , and the expressions involving integrals are also taken in  $\overline{\mathbb{R}}_+$ .

**Lemma 753 (integral in  $\mathcal{M}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\{\int \varphi d\mu \mid \varphi \in \mathcal{SF}_+ \wedge \varphi \leq f\}$  admits a supremum.

The integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by

$$(211) \quad \int f d\mu \stackrel{\text{def.}}{=} \sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu \in \overline{\mathbb{R}}_+.$$

A function  $f : X \rightarrow \overline{\mathbb{R}}_+$  is said  $\mu$ -integrable (in  $\mathcal{M}_+$ ) iff  $f \in \mathcal{M}_+$  and  $\int f d\mu < \infty$ .

*Proof.* Direct consequence of Lemma 737 (integral in  $\mathcal{SF}_+$ ), and Definition 2 (supremum, with the Lebesgue integral in  $\mathcal{SF}_+$  over  $\{\varphi \in \mathcal{SF}_+ \mid \varphi \leq f\}$ ),  $\square$

**Lemma 754 (integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{SF}_+$ . Then, the values of  $\int f d\mu$  provided by Lemma 737 (integral in  $\mathcal{SF}_+$ ), and Lemma 753 (integral in  $\mathcal{M}_+$ ) coincide.

*Proof.* Direct consequence of Lemma 737 (integral in  $\mathcal{SF}_+$ ), Lemma 746 (integral in  $\mathcal{SF}_+$  is continuous), and Lemma 753 (integral in  $\mathcal{M}_+$ ).  $\square$

**Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Then, we have  $\mathbb{1}_A \in \mathcal{M}_+$  and  $\int \mathbb{1}_A d\mu = \mu(A)$ .

*Proof.* Direct consequence of Lemma 736 ( $\mathcal{SF}_+$  is measurable), Lemma 754 (integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$ ), and Lemma 739 (integral in  $\mathcal{SF}_+$  of indicator function).  $\square$

**Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $a \in \mathbb{R}_+$ . Then,  $af \in \mathcal{M}_+$ , and we have

$$(212) \quad \int af d\mu = a \int f d\mu.$$

*Proof.* From Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  $af \in \mathcal{M}_+$ .

**Case  $a = 0$ .** Then, from Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), Definition 723 ( $\mathcal{SF}$ , vector space of simple functions), and Definition 61 (vector space,  $0 \in \mathcal{SF}$ ), we have

$$0f = 0 \in \mathcal{SF}_+ \subset \mathcal{M}_+.$$

Hence, from Lemma 753 (integral in  $\mathcal{M}_+$ ), Lemma 744 (equivalent definition of the integral in  $\mathcal{SF}_+$ ), Lemma 737 (integral in  $\mathcal{SF}_+$ ), and Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)), we have

$$\int 0f d\mu = \int 0 d\mu = 0 = 0 \times \int f d\mu.$$

**Case  $a > 0$ .** Then, from Lemma 316 (multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)), and Definition 264 (multiplication in  $\overline{\mathbb{R}}$ , for all  $a \in \mathbb{R}_+^*$ ,  $\frac{a}{a} = 1$ ), we have  $(\star) \forall b \in \overline{\mathbb{R}}_+, \frac{1}{a}(ab) = b$ .

Let  $\varphi \in \mathcal{SF}_+$  such that  $\varphi \leq f$ . Then, from Lemma 735 ( $\mathcal{SF}_+$  is closed under positive linear operations), we have  $a\varphi \in \mathcal{SF}_+$ . Moreover, from compatibility of multiplication by a positive number with order in  $\overline{\mathbb{R}}$ , property  $(\star)$ , Lemma 743 (integral in  $\mathcal{SF}_+$  is positive linear, positive homogeneity), Lemma 753 (integral in  $\mathcal{M}_+$ ), and Definition 2 (supremum, upper bound), we have  $a\varphi \leq af$  and

$$\int \varphi d\mu = \frac{1}{a} \int a\varphi d\mu \leq \frac{1}{a} \int af d\mu.$$

Thus, from Lemma 753 (*integral in  $\mathcal{M}_+$* ), Definition 2 (*supremum, least upper bound*), and property  $(\star)$ , we have

$$a \int f d\mu = a \sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu \leq a \frac{1}{a} \int af d\mu = \int af d\mu.$$

Moreover, the same result also holds for function  $af \in \mathcal{M}_+$  and number  $\frac{1}{a} > 0$ , and with property  $(\star)$ , we have  $\int af d\mu \leq a \int \frac{1}{a}(af) d\mu = a \int f d\mu$ . Hence, we have the equality.

Therefore, we always have the equality.  $\square$

**Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $0 \in \mathcal{M}_+$ , and we have  $\int 0 d\mu = 0$ .

*Proof.* Direct consequence of Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $a \stackrel{\text{def.}}{=} 0$* ), Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ), and Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*).  $\square$

**Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have

$$(213) \quad f \leq g \implies \int f d\mu \leq \int g d\mu.$$

*Proof.* Assume that  $f \leq g$ . Then,  $\{\varphi \in \mathcal{SF}_+ \mid \varphi \leq f\} \subset \{\varphi \in \mathcal{SF}_+ \mid \varphi \leq g\}$ . Therefore, from Lemma 753 (*integral in  $\mathcal{M}_+$* ), and **monotonicity of supremum**, we have

$$\int f d\mu \leq \int g d\mu.$$

$\square$

**Theorem 759 (*Beppo Levi, monotone convergence*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Assume that the sequence is pointwise nondecreasing. Then,  $\lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+$ , and we have

$$(214) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* From **convergence of monotone sequences in  $\overline{\mathbb{R}}$** , and **completeness of  $\overline{\mathbb{R}}$** , the existence of the pointwise limit  $f$  is guaranteed in  $\overline{\mathbb{R}}$ , and we have  $f = \lim_{n \rightarrow \infty} f_n = \sup_{n \in \mathbb{N}} f_n$ .

From Lemma 585 ( **$\mathcal{M}_+$  is closed under limit when pointwise convergent**), we have  $f \in \mathcal{M}_+$ . Then, from Lemma 753 (*integral in  $\mathcal{M}_+$* ), the integral of  $f$  is well-defined. Moreover, from Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), and **completeness of the extended real numbers**, the sequence  $(\int f_n d\mu)_{n \in \mathbb{N}}$  is nondecreasing, hence convergent in  $\overline{\mathbb{R}}$  towards

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

Thus, from Definition 2 (*supremum, upper bound*), and Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), for all  $n \in \mathbb{N}$ , we have

$$\int f_n d\mu \leq \int \sup_{n \in \mathbb{N}} f_n d\mu = \int f d\mu.$$

Hence, from Definition 2 (*supremum, least upper bound*), we have

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu \leq \int f d\mu.$$

Let  $a \in (0, 1)$ . Let  $\varphi \in \mathcal{SF}_+$ . Assume that  $\varphi \leq f$ . Let  $n \in \mathbb{N}$ . Let  $A_n \stackrel{\text{def.}}{=} \{a\varphi \leq f_n\} \subset X$ . Then, from Definition 723 ( $\mathcal{SF}$ , *vector space of simple functions*,  $\underline{-a\varphi \in \mathcal{SF}}$ ), Lemma 728 ( $\mathcal{SF}$  is *measurable*,  $\underline{-a\varphi \in \mathcal{M}}$ ), and Lemma 566 ( $\mathcal{M}$  is *closed under addition when defined*,  $\underline{-a\varphi}$  takes *finite values*), we have  $f_n - a\varphi \in \mathcal{M}$ . Thus, from Lemma 563 (*measurability of numeric function*, with  $\underline{f_n - a\varphi}$ ), and Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*), we have  $A_n \in \Sigma$ , and  $\mathbb{1}_{A_n} \in \mathcal{M}_+$ . Moreover, from **monotonicity of addition in  $\mathbb{R}_+$** , and **transitivity of order**, the sequence  $(A_n)_{n \in \mathbb{N}}$  is nondecreasing.

Let  $x \in X$ . **Case  $\varphi(x) = 0$** . Then, for all  $n \in \mathbb{N}$ ,  $x \in A_n$ . **Case  $\varphi(x) > 0$** . Then, since  $0 < a < 1$ , we have  $a\varphi(x) < \varphi(x) \leq f(x)$ . Thus, from **monotonicity of the limit**, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a\varphi(x) \leq f_n(x)$ , i.e.  $x \in A_n$ . Hence, we have  $X = \bigcup_{n \in \mathbb{N}} A_n$ .

Let  $B \in \Sigma$ . Then, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*, *closedness under countable intersection* (with  $\text{card}(I) = 2$ )), **monotonicity of intersection**, and **distributivity of intersection over union**, the sequence  $(B \cap A_n)_{n \in \mathbb{N}}$  belongs to  $\Sigma$ , is nondecreasing, and  $\bigcup_{n \in \mathbb{N}} (B \cap A_n) = B$ . Thus, from Lemma 600 (*measure is continuous from below*), and Definition 599 (*continuity from below*), we have  $\mu(B) = \mu(\bigcup_{n \in \mathbb{N}} (B \cap A_n)) = \lim_{n \rightarrow \infty} \mu(B \cap A_n)$ .

Then, from Lemma 744 (*equivalent definition of the integral in  $\mathcal{SF}_+$* ), there exists  $k \in \mathbb{N}$ ,  $(a_i)_{i \in [0..k]} \in \mathbb{R}_+$ , and  $(B_i)_{i \in [0..k]} \in \Sigma$  such that  $\varphi = \sum_{i \in [0..k]} a_i \mathbb{1}_{B_i}$  and  $\int \varphi d\mu = \sum_{i \in [0..k]} a_i \mu(B_i)$ . Thus, from Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*), we have

$$\forall n \in \mathbb{N}, \int \varphi \mathbb{1}_{A_n} d\mu = \int \sum_{i \in [0..k]} a_i \mathbb{1}_{B_i \cap A_n} d\mu = \sum_{i \in [0..k]} a_i \mu(B_i \cap A_n).$$

Hence, from the previous result on the measure  $\mu$ , **linearity of the limit**, and Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*, with  $a > 0$ ), we have

$$\int \varphi d\mu = \sum_{i \in [0..k]} a_i \lim_{n \rightarrow \infty} \mu(B_i \cap A_n) = \lim_{n \rightarrow \infty} \int \varphi \mathbb{1}_{A_n} d\mu = \frac{1}{a} \lim_{n \rightarrow \infty} \int a\varphi \mathbb{1}_{A_n} d\mu.$$

Let  $n \in \mathbb{N}$ . Then, from the definition of the  $A_n$ 's, and **the definition of the indicator function**, we have  $a\varphi \mathbb{1}_{A_n} \leq f_n \mathbb{1}_{A_n} \leq f_n$ . Thus, from **compatibility of the multiplication by a positive number with order**, Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), and **monotonicity and linearity of the limit**, we have

$$\int \varphi d\mu \leq \frac{1}{a} \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Then, from **left continuity of the division at  $a = 1$** , and **monotonicity of the limit**,

$$\forall \varphi \in \mathcal{SF}_+, \varphi \leq f \Rightarrow \int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Hence, from Lemma 753 (*integral in  $\mathcal{M}_+$* ), and Definition 2 (*supremum*, *least upper bound*),

$$\int f d\mu = \sup_{\varphi \in \mathcal{SF}_+, \varphi \leq f} \int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Therefore, we have  $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$ . □

**Lemma 760** (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Then,  $\infty f \in \mathcal{M}_+$ , and we have

$$(215) \quad \int \infty f d\mu = \infty \int f d\mu.$$

*Proof.* From Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*,  $\underline{f \geq 0}$ ), **ordered set properties of  $\mathbb{R}_+$** , Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*), and

Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  $(nf)_{n \in \mathbb{N}}$  is a nondecreasing sequence in  $\mathcal{M}_+$  such that  $\lim_{n \rightarrow \infty} nf = \infty f \in \mathcal{M}_+$ . Therefore, from Theorem 759 (Beppo Levi, monotone convergence), Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous), and **positive homogeneity of the limit in  $\overline{\mathbb{R}}_+$** , we have

$$\begin{aligned} \int \infty f \, d\mu &= \int \left( \lim_{n \rightarrow \infty} n \right) f \, d\mu = \int \left( \lim_{n \rightarrow \infty} nf \right) \, d\mu \\ &= \lim_{n \rightarrow \infty} \left( \int nf \, d\mu \right) = \lim_{n \rightarrow \infty} \left( n \int f \, d\mu \right) = \left( \lim_{n \rightarrow \infty} n \right) \int f \, d\mu = \infty \int f \, d\mu. \end{aligned}$$

□

**Definition 761 (adapted sequence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$ . The sequence  $(f_n)_{n \in \mathbb{N}}$  is called *adapted sequence for  $f$*  iff it is nondecreasing and  $f = \lim_{n \in \mathbb{N}} f_n = \sup_{n \in \mathbb{N}} f_n$ .

**Lemma 762 (adapted sequence in  $\mathcal{M}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . For all  $n \in \mathbb{N}$ , let  $f_n : X \rightarrow \overline{\mathbb{R}}_+$  defined by

$$(216) \quad \forall x \in X, \quad f_n(x) \stackrel{\text{def.}}{=} \begin{cases} \frac{\lfloor 2^n f(x) \rfloor}{2^n} & \text{when } f(x) < n, \\ n & \text{otherwise.} \end{cases}$$

Then,  $(f_n)_{n \in \mathbb{N}}$  is an adapted sequence for  $f$ .

*Proof.* Let  $n \in \mathbb{N}$ . Let  $x \in X$ . Assume that  $f(x) < n$ . Let  $i \stackrel{\text{def.}}{=} \lfloor 2^n f(x) \rfloor \in \mathbb{N}$ . Then, from **the definition of the floor function**, we have  $0 \leq i \leq 2^n f(x) < i + 1 \leq n2^n$ . Thus, for all  $i \in \mathbb{N}$  such that  $0 \leq i < n2^n$ , for all  $x \in X$  such that  $\frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}$ , we have  $f_n(x) = \frac{i}{2^n}$ . Hence,

$$f_n = \sum_{0 \leq i < n2^n} \frac{i}{2^n} \mathbf{1}_{\{\frac{i}{2^n} \leq f < \frac{i+1}{2^n}\}} + n \mathbf{1}_{\{f \geq n\}}.$$

Moreover, from Lemma 563 (measurability of numeric function), and Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $\text{card}(I) = 2$ )), we have

$$\left\{ \frac{i}{2^n} \leq f < \frac{i+1}{2^n} \right\} = \left\{ \frac{i}{2^n} \leq f \right\} \cap \left\{ f < \frac{i+1}{2^n} \right\} \in \Sigma \quad \text{and} \quad \{f \geq n\} \in \Sigma.$$

Hence, from Definition 723 ( $\mathcal{SF}$ , vector space of simple functions), and Definition 732 ( $\mathcal{SF}_+$ , subset of nonnegative simple functions), we have  $f_n \in \mathcal{SF}_+$ .

Let  $n \in \mathbb{N}$ . Let  $x \in X$ . **Case  $n + 1 \leq f(x)$ .** Then, we have

$$f_n(x) = n < n + 1 = f_{n+1}(x).$$

**Case  $f(x) < n + 1$ .** Let  $i \stackrel{\text{def.}}{=} \lfloor 2^{n+1} f(x) \rfloor$ . Then, we have

$$f_{n+1}(x) = \frac{i}{2^{n+1}} \quad \text{with } 0 \leq \frac{i}{2^{n+1}} \leq f(x) < \frac{i+1}{2^{n+1}}.$$

**Case  $\frac{i}{2^{n+1}} \geq n$ .** Then,  $f(x) \geq n$  and we have

$$f_n(x) = n \leq \frac{i}{2^{n+1}} = f_{n+1}(x).$$

**Case  $\frac{i}{2^{n+1}} < n$  and  $i$  even.** Let  $j \stackrel{\text{def.}}{=} \frac{i}{2} \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have

$$\frac{j}{2^n} \leq f(x) < \frac{j}{2^n} + \frac{1}{2^{n+1}} < \frac{j+1}{2^n}.$$

Thus, from **the definition of the floor function**, we have  $j = \lfloor 2^n f(x) \rfloor$  and

$$f_n(x) = \frac{j}{2^n} = \frac{i}{2^{n+1}} = f_{n+1}(x).$$

**Case  $\frac{i}{2^{n+1}} < n$  and  $i$  odd.** Let  $j \stackrel{\text{def.}}{=} \frac{i-1}{2} \in \mathbb{N}$ . Then, from **ordered field properties of  $\mathbb{R}$** , we have

$$\frac{j}{2^n} \leq \frac{j}{2^n} + \frac{1}{2^{n+1}} \leq f(x) < \frac{j+1}{2^n}.$$

Thus, from **the definition of the floor function**, we have  $j = \lfloor 2^n f(x) \rfloor$  and

$$f_n(x) = \frac{j}{2^n} = \frac{i-1}{2^{n+1}} < \frac{i}{2^{n+1}} = f_{n+1}(x).$$

Thus, we always have  $f_n(x) \leq f_{n+1}(x)$ . Hence, the sequence  $(f_n)_{n \in \mathbb{N}}$  is nondecreasing.

Let  $x \in X$ . We have  $f(x) \in \overline{\mathbb{R}}_+$ . **Case  $f(x) = \infty$ .** Then, for all  $n \in \mathbb{N}$ , we have  $f_n(x) = n$ . Hence,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ . **Case  $f(x) \in \mathbb{R}_+$ .** Then, from **the Archimedean property of  $\mathbb{R}$** , there exists  $N \in \mathbb{N}$  such that  $f(x) < N$ . Let  $n \in \mathbb{N}$  such that  $N \leq n$ . Then, from **the definition of the floor function**, we have

$$2^n f_n(x) = \lfloor 2^n f(x) \rfloor \leq 2^n f(x) < \lfloor 2^n f(x) \rfloor + 1 = 2^n f_n(x) + 1.$$

Thus, from **ordered field properties of  $\mathbb{R}$  (with  $2^n > 0$ )**, we have  $f(x) - \frac{1}{2^n} < f_n(x) \leq f(x)$ . Hence, from **the squeeze theorem**, we have  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

Therefore, from Definition 761 (**adapted sequence**),  $(f_n)_{n \in \mathbb{N}}$  is an adapted sequence for  $f$ .  $\square$

**Lemma 763 (usage of adapted sequences).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}_+$ . Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  be an adapted sequence of  $f$ . Then, we have*

$$(217) \quad \int f \, d\mu = \int \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

*Proof.* Direct consequence of Definition 761 (**adapted sequence**), Lemma 762 (**adapted sequence in  $\mathcal{M}_+$** ), Theorem 759 (**Beppo Levi, monotone convergence**), and Lemma 754 (**integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$** ).  $\square$

**Lemma 764 (integral in  $\mathcal{M}_+$  is additive).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then,  $f + g \in \mathcal{M}_+$ , and we have*

$$(218) \quad \int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

*Proof.* From Lemma 582 ( **$\mathcal{M}_+$  is closed under addition**), we have  $f + g \in \mathcal{M}_+$ .

From Lemma 762 (**adapted sequence in  $\mathcal{M}_+$** ), let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  be adapted sequences for  $f$  and  $g$ . Then, from Lemma 743 (**integral in  $\mathcal{SF}_+$  is positive linear, additivity**), **monotonicity of addition**, and **additivity of the limit**,  $(f_n + g_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+$  is an adapted sequence for  $f + g$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 743 (**integral in  $\mathcal{SF}_+$  is positive linear, additivity**), we have

$$\int (f_n + g_n) \, d\mu = \int f_n \, d\mu + \int g_n \, d\mu.$$

Therefore, from **linearity of the limit when  $n$  goes to infinity**, and Lemma 763 (**usage of adapted sequences**), we have

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

$\square$

**Lemma 765 (integral in  $\mathcal{M}_+$  is positive linear).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Let  $a \in \overline{\mathbb{R}}_+$ . Then,  $f + g, af \in \mathcal{M}_+$ , and we have

$$(219) \quad \int (f + g) d\mu = \int f d\mu + \int g d\mu \quad \text{and} \quad \int af d\mu = a \int f d\mu.$$

*Proof.* Direct consequence of Lemma 764 (integral in  $\mathcal{M}_+$  is additive), Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous), and Lemma 760 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ).  $\square$

**Lemma 766 (integral in  $\mathcal{M}_+$  is  $\sigma$ -additive).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then,  $\sum_{n \in \mathbb{N}} f_n \in \mathcal{M}_+$ , and we have

$$(220) \quad \int \left( \sum_{n \in \mathbb{N}} f_n \right) d\mu = \sum_{n \in \mathbb{N}} \left( \int f_n d\mu \right).$$

*Proof.* Direct consequence of Lemma 586 ( $\mathcal{M}_+$  is closed under countable sum), and Theorem 759 (Beppo Levi, monotone convergence, with nondecreasing sequence  $(\sum_{i \in [0..n]} f_i)_{n \in \mathbb{N}}$ ).  $\square$

**Lemma 767 (integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts).** Let  $(X, \Sigma)$  be a measurable space. Let  $f \in \mathcal{M}$ . Then, we have

$$(221) \quad \int |f| d\mu = \int f^+ d\mu + \int f^- d\mu.$$

*Proof.* Direct consequence of Lemma 581 ( $\mathcal{M}$  is closed under absolute value,  $|f|$  belongs to  $\mathcal{M}_+$ ), Lemma 579 (measurability of nonnegative and nonpositive parts,  $f^+, f^- \in \mathcal{M}_+$ ), Lemma 377 (decomposition into nonnegative and nonpositive parts,  $|f| = f^+ + f^-$ ), and Lemma 764 (integral in  $\mathcal{M}_+$  is additive).  $\square$

**Lemma 768 (compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts).**

Let  $(X, \Sigma)$  be a measurable space. Let  $f, g \in \mathcal{M}$  such that  $f + g \in \mathcal{M}$ . Then, we have

$$(222) \quad \int (f + g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f + g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu.$$

*Proof.* Direct consequence of Lemma 378 (compatibility of nonpositive and nonnegative parts with addition), and Lemma 764 (integral in  $\mathcal{M}_+$  is additive).  $\square$

**Lemma 769 (integral in  $\mathcal{M}_+$  is almost definite).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  in  $\mathcal{M}_+$ . Then, we have

$$(223) \quad \int f d\mu = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Let  $A \stackrel{\text{def.}}{=} \{f > 0\} = f^{-1}(0, \infty]$ . Then, from Lemma 563 (measurability of numeric function,  $A \in \Sigma$ ), Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function,  $\mathbb{1}_A \in \mathcal{M}_+$ ), Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication,  $\infty f, \infty \mathbb{1}_A \in \mathcal{M}_+$ ), Lemma 760 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ), Lemma 320 (infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory),  $\infty f = \infty \mathbb{1}_A$ ), and Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function) we have

$$\infty \int f d\mu = \int \infty f d\mu = \int \infty \mathbb{1}_A d\mu = \infty \int \mathbb{1}_A d\mu = \infty \mu(A).$$

Therefore, from Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory), multiplication by  $\infty$  is definite in  $\overline{\mathbb{R}}_+$ ), Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions,



$\{f = 0\}^c = A$ ), Lemma 619 (*negligibility of measurable subset*), and Definition 624 (*property almost satisfied*), we have

$$\begin{aligned} \int f d\mu = 0 &\iff \infty \int f d\mu = 0 \iff \infty \mu(A) = 0 \\ &\iff \mu(A) = 0 \iff \mu(\{f = 0\}^c) = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0. \end{aligned}$$

□

**Lemma 770 (compatibility of integral in  $\mathcal{M}_+$  with almost binary relation).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $\mathcal{R}$  be a binary relation on  $\mathcal{M}_+$ . Let  $\mathcal{R}'$  be a binary relation on  $\overline{\mathbb{R}}_+$ . Assume that we have the properties  $0 \mathcal{R} 0$  and

$$(224) \quad \forall f, g \in \mathcal{M}_+, \quad f \mathcal{R} g \implies \int f d\mu \mathcal{R}' \int g d\mu.$$

Then, we have the property

$$(225) \quad \forall f, g \in \mathcal{M}_+, \quad f \mathcal{R}_{\mu \text{ a.e.}} g \implies \int f d\mu \mathcal{R}' \int g d\mu.$$

*Proof.* Let  $f, g \in \mathcal{M}_+$ . Assume that  $f \mathcal{R}_{\mu \text{ a.e.}} g$  holds. Then, from Definition 631 (*almost binary relation*), Definition 630 (*almost definition*), Definition 624 (*property almost satisfied*), Definition 614 (*negligible subset*), and **properties of complement**, let  $A \in \Sigma$  such that  $\mu(A) = 0$  and  $A^c \subset \{f \mathcal{R} g\}$ , i.e. such that  $f|_{A^c} \mathcal{R} g|_{A^c}$  holds. Thus, from **the definition of the indicator function, and since  $0 \mathcal{R} 0$ , we have  $f\mathbb{1}_{A^c} \mathcal{R} g\mathbb{1}_{A^c}$** . Then, from Definition 467 ( $\sigma$ -algebra,  $A^c \in \Sigma$ ), Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*,  $\mathbb{1}_A, \mathbb{1}_{A^c} \in \mathcal{M}_+$ ), and Lemma 583 ( $\mathcal{M}_+$  is closed under multiplication), we have  $f\mathbb{1}_A, f\mathbb{1}_{A^c}, g\mathbb{1}_A, g\mathbb{1}_{A^c} \in \mathcal{M}_+$ . Thus, from assumption, the property  $\int f\mathbb{1}_{A^c} d\mu \mathcal{R}' \int g\mathbb{1}_{A^c} d\mu$  holds.

Let  $h \in \{f, g\}$ . Then, from **the definition of the indicator function ( $\{\mathbb{1}_A = 0\}^c$  is  $A$ )**, Lemma 619 (*negligibility of measurable subset*,  $\{\mathbb{1}_A = 0\}^c \in \mathbf{N}$ ), Definition 624 (*property almost satisfied*,  $\mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} 0$ ), Lemma 641 (*compatibility of almost equality with operator, with the unary operator left multiplication by  $h$* ), and Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*), we have  $h\mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} h0 = 0$ . Thus, from Lemma 626 (*everywhere implies almost everywhere*), and Lemma 638 (*almost equality is equivalence relation, transitivity*), we have  $h\mathbb{1}_A \stackrel{\mu \text{ a.e.}}{=} 0$ . Hence, from **properties of the indicator function**, Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*), Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*), Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*), and Lemma 259 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ), we have for all  $h \in \{f, g\}$ ,

$$\int h d\mu = \int h(\mathbb{1}_A + \mathbb{1}_{A^c}) d\mu = \int h\mathbb{1}_A d\mu + \int h\mathbb{1}_{A^c} d\mu = \int h\mathbb{1}_{A^c} d\mu.$$

Therefore,  $\int f d\mu \mathcal{R}' \int g d\mu$  also holds. □

**Lemma 771 (compatibility of integral in  $\mathcal{M}_+$  with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have

$$(226) \quad f \stackrel{\mu \text{ a.e.}}{=} g \implies \int f d\mu = \int g d\mu.$$

*Proof.* Direct consequence of Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation, with  $\mathcal{R} = \mathcal{R}' \stackrel{\text{def.}}{=} \text{equality}$* ), and Lemma 753 (*integral in  $\mathcal{M}_+$ , integral is a function*). □

**Lemma 772 (integral in  $\mathcal{M}_+$  is almost monotone).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}_+$ . Then, we have

$$(227) \quad f \stackrel{\mu \text{ a.e.}}{\leq} g \implies \int f \, d\mu \leq \int g \, d\mu.$$

*Proof.* Direct consequence of Lemma 770 (compatibility of integral in  $\mathcal{M}_+$  with almost binary relation, with  $\mathcal{R} = \mathcal{R}' \stackrel{\text{def.}}{=} \text{inequality}$ ), and Lemma 758 (integral in  $\mathcal{M}_+$  is monotone).  $\square$

**Lemma 773 (Bienaymé–Chebyshev inequality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Let  $a \in \overline{\mathbb{R}}_+^*$ . Then, we have

$$(228) \quad a\mu(\{|f| \geq a\}) \leq \int |f| \, d\mu.$$

*Proof.* Let  $A \stackrel{\text{def.}}{=} \{|f| \geq a\}$ . Then, from Lemma 581 ( $\mathcal{M}$  is closed under absolute value,  $|f|$  belongs to  $\mathcal{M}_+ \subset \mathcal{M}$ ), and Lemma 563 (measurability of numeric function), we have  $A \in \Sigma$ . Moreover, from the definition of the indicator function, and Lemma 278 (absolute value in  $\overline{\mathbb{R}}$  is non-negative), we have  $|f| \geq a\mathbb{1}_A$ . Therefore, from Lemma 554 (measurability of indicator function), Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication), Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous), Lemma 760 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ), Lemma 758 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function), we have  $a\mathbb{1}_A \in \mathcal{M}_+$  and

$$\int |f| \, d\mu \geq \int a\mathbb{1}_A \, d\mu = a \int \mathbb{1}_A \, d\mu = a\mu(A).$$

$\square$

**Lemma 774 (integrable in  $\mathcal{M}_+$  is almost finite).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f : X \rightarrow \overline{\mathbb{R}}_+$  be  $\mu$ -integrable in  $\mathcal{M}_+$ . Then, we have  $\mu(f^{-1}(\infty)) = 0$ , i.e.  $f \stackrel{\mu \text{ a.e.}}{<} \infty$ .

*Proof.* Direct consequence of Lemma 753 (integral in  $\mathcal{M}_+$ ,  $f \in \mathcal{M}_+$ ), Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), Lemma 773 (Bienaymé–Chebyshev inequality, with  $a \stackrel{\text{def.}}{=} \infty$ ), Lemma 278 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative,  $|f| = f$ ), Definition 254 (extended real numbers,  $\overline{\mathbb{R}}$ ,  $A \stackrel{\text{def.}}{=} \{|f| \geq \infty\} = f^{-1}(\infty)$ ), Lemma 753 (integral in  $\mathcal{M}_+$ ,  $\int |f| \, d\mu < \infty$ ), compatibility of multiplication with order in  $\overline{\mathbb{R}}$  ( $\mu(A) \leq 0$ ), Definition 594 (measure, nonnegativeness, i.e.  $\mu(A) = 0$ ), Lemma 619 (negligibility of measurable subset,  $A \in \mathbf{N}$ ), and Definition 624 (property almost satisfied,  $A^c = \{f < \infty\}$ ).  $\square$

**Lemma 775 (bounded by integrable in  $\mathcal{M}_+$  is integrable).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$  iff there exists  $g : X \rightarrow \overline{\mathbb{R}}_+$  such that  $g$  is  $\mu$ -integrable in  $\mathcal{M}_+$  and  $f \leq g$ .

*Proof.* “Left” implies “right”. Direct consequence of Lemma 255 (order in  $\overline{\mathbb{R}}$  is total, reflexivity, with  $g \stackrel{\text{def.}}{=} f$ ).

“Right” implies “left”. Direct consequence of Lemma 753 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability), Lemma 758 (integral in  $\mathcal{M}_+$  is monotone), and Lemma 255 (order in  $\overline{\mathbb{R}}$  is total, transitivity).

Therefore, we have the equivalence.  $\square$

**Lemma 776 (integral in  $\mathcal{M}_+$  over subset).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then, we have  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$  iff  $\hat{f}\mathbb{1}_A \in \mathcal{M}_+(X, \Sigma)$ . If so, we have*

$$(229) \quad \int f|_A d\mu_A = \int \hat{f}\mathbb{1}_A d\mu.$$

*This integral is still denoted  $\int_A f d\mu$ ; it is still called integral of  $f$  over  $A$ .*

*Proof. Equivalence.* Direct consequence of Lemma 577 (*measurability of restriction*), Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), and **nonnegativeness of the indicator function**.

**Identity.** Direct consequence of Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* , let  $(\hat{f}_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$  be an adapted sequence for  $\hat{f}\mathbb{1}_A$ ), Lemma 730 ( *$\mathcal{SF}$  is closed under restriction*,  $(\hat{f}_n)|_A$  belongs to  $\mathcal{SF}_+(A, \Sigma \cap A)$ ), **compatibility of restriction of function with monotonicity and limit**, Definition 761 (*adapted sequence*,  $((\hat{f}_n)|_A)_{n \in \mathbb{N}}$  is an adapted sequences for  $f|_A$ ), Lemma 763 (*usage of adapted sequences*), and Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*).  $\square$

**Lemma 777 (integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $(A_i)_{i \in I}$  is a partition of  $A$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f}\mathbb{1}_A = \sum_{i \in I} \hat{f}\mathbb{1}_{A_i}$ , and  $\hat{f}\mathbb{1}_A \in \mathcal{M}_+$  iff for all  $i \in I$ ,  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{M}_+$ . If so, we have*

$$(230) \quad \int_A f d\mu = \sum_{i \in I} \int_{A_i} f d\mu.$$

*Proof. (1).*  $\hat{f}\mathbb{1}_A = \sum_{i \in I} \hat{f}\mathbb{1}_{A_i}$ . Direct consequence of **the formula for the indicator of countable disjoint union (sum)**, and **left distributivity of multiplication over countable addition in  $\mathbb{R}_+$** .

**“Left” implies “right”.** Direct consequence of **the formula for the indicator of intersection ( $\mathbb{1}_{A_i} = \mathbb{1}_{A \cap A_i} = \mathbb{1}_A \mathbb{1}_{A_i}$ )**, **associativity of multiplication in  $\mathbb{R}$** , Lemma 554 (*measurability of indicator function*), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), **nonnegativeness of the indicator function**, Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*), and Lemma 583 ( $\mathcal{M}_+$  *is closed under multiplication*).

**“Right” implies “left”.** Direct consequence of (1), and Lemma 586 ( $\mathcal{M}_+$  *is closed under countable sum*).

Therefore, we have the equivalence.

**Identity.** Direct consequence of Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*, with  $A$ , then  $A_i$ ), (1), and Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*).  $\square$

**Lemma 778 (integral in  $\mathcal{M}_+$  over singleton).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $a \in X$ . Assume that  $\{a\} \in \Sigma$ . Let  $f : X \rightarrow \overline{\mathbb{R}}_+$ . Then, we have  $f\mathbb{1}_{\{a\}} = f(a)\mathbb{1}_{\{a\}} \in \mathcal{M}_+$ , and*

$$(231) \quad \int_{\{a\}} f d\mu = f(a)\mu(\{a\}).$$

*Proof.* From **the definition of the indicator function**, Lemma 554 (*measurability of indicator function*), and Lemma 584 ( $\mathcal{M}_+$  *is closed under nonnegative scalar multiplication*), we have  $f\mathbb{1}_{\{a\}} = f(a)\mathbb{1}_{\{a\}} \in \mathcal{M}_+$ . Then, from Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*, with  $A \stackrel{\text{def.}}{=} \{a\}$ ),

$Y \stackrel{\text{def.}}{=} X$ ), Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear, homogeneity in  $\overline{\mathbb{R}}_+$* ), and Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*), we have

$$\int_{\{a\}} f d\mu = \int f \mathbb{1}_{\{a\}} d\mu = f(a) \int \mathbb{1}_{\{a\}} d\mu = f(a) \mu(\{a\}).$$

□

**Theorem 779 (Fatou's lemma).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Then, we have

$$(232) \quad \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* Let  $n \in \mathbb{N}$ . Let  $g_n \stackrel{\text{def.}}{=} \inf_{p \in \mathbb{N}} f_{n+p}$ . Then, from **monotonicity of infimum**, Lemma 366 (*limit inferior bounded from below, with  $m = 0$* ), Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),  $(g_n)_{n \in \mathbb{N}}$  is a nondecreasing sequence in  $\mathcal{M}_+$ . Thus, from Definition 364 (*pointwise convergence*), **properties of nondecreasing sequences in the ordered set  $\overline{\mathbb{R}}$** , Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),  $(g_n)_{n \in \mathbb{N}}$  is pointwise convergent in  $\overline{\mathbb{R}}_+$  towards a measurable function  $g = \lim_{n \in \mathbb{N}} g_n$ . Moreover, from **the nondecreasing property of the sequence in the ordered set  $\overline{\mathbb{R}}$** , and Lemma 352 (*limit inferior*), we have  $g = \sup_{n \in \mathbb{N}} g_n = \liminf_{n \rightarrow \infty} f_n$ . Hence, from Theorem 759 (*Beppo Levi, monotone convergence, with  $(g_n)_{n \in \mathbb{N}}$* ),  $g$  is measurable and nonnegative, and we have

$$\int \liminf_{n \rightarrow \infty} f_n d\mu = \int g d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu.$$

Let  $n, p \in \mathbb{N}$ . Let  $x \in X$ . Then, from Definition 9 (*infimum, lower bound*), we have

$$g_n(x) \leq f_{n+p}(x).$$

Then, from Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), we have

$$\int g_n d\mu \leq \int f_{n+p} d\mu.$$

Hence, from Definition 9 (*infimum, greatest lower bound*), we have

$$\int g_n d\mu \leq \inf_{p \in \mathbb{N}} \int f_{n+p} d\mu.$$

Therefore, from **monotonicity of the limit in the ordered set  $\overline{\mathbb{R}}$** , and Lemma 352 (*limit inferior*), we have

$$\int \liminf_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

□

**Lemma 780 (integral in  $\mathcal{M}_+$  of pointwise convergent sequence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ . Assume that the sequence is pointwise convergent towards  $f$  such that for all  $n \in \mathbb{N}$ ,  $f_n \leq f$ . Then,  $f \in \mathcal{M}_+$ , and we have

$$(233) \quad \int \lim_{n \rightarrow \infty} f_n d\mu = \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* From Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*), we have

$$f = \lim_{n \rightarrow \infty} f_n \in \mathcal{M}_+.$$

Moreover, from Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), we have

$$\forall n \in \mathbb{N}, \quad \int f_n d\mu \leq \int f d\mu.$$

Thus, from Definition 2 (*supremum, least upper bound*), Lemma 365 (*limit inferior and limit superior of pointwise convergent*), and Theorem 779 (*Fatou's lemma*), we have

$$\limsup_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu = \int \lim_{n \rightarrow \infty} f_n d\mu = \int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$$

Therefore, from Lemma 370 (*limit inferior, limit superior and pointwise convergence*), we have

$$\liminf_{n \rightarrow \infty} \int f_n d\mu = \limsup_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

□

**Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{M}_+$ . Then, we have

$$(234) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* Direct consequence of Lemma 763 (*usage of adapted sequences*), Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*), and **compatibility of (possibly uncountable) addition in  $\overline{\mathbb{R}}_+$  with limit**. □

**Lemma 782 (*integral in  $\mathcal{M}_+$  for counting measure on  $\mathbb{N}$* ).**

Let  $f : \mathbb{N} \rightarrow \overline{\mathbb{R}}_+$  be a nonnegative sequence. Then, we have

$$(235) \quad \int f d\delta_{\mathbb{N}} = \sum_{n \in \mathbb{N}} f(n).$$

*Proof.* Direct consequence of Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*, with  $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$  and  $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$ ). □

*Remark 783.* Note that the previous result makes nonnegative series be Lebesgue integrals for the counting measure on natural numbers. Thus, the theory of nonnegative series can be derived from Lebesgue integration for nonnegative measurable functions.

**Lemma 784 (*integral in  $\mathcal{M}_+$  for Dirac measure*).**

Let  $(X, \Sigma)$  be a measurable space. Let  $\{a\} \in \Sigma$ . Let  $f \in \mathcal{M}_+$ . Then, we have

$$(236) \quad \int f d\delta_a = f(a).$$

*Proof.* Direct consequence of Definition 656 (*Dirac measure*), and Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*, with  $Y \stackrel{\text{def.}}{=} \{a\}$ ). □

#### 6.10.4 Measure and integration over product space

*Remark 785.* For the sake of simplicity, we only present the remainder of this section in the case of the product of two measure spaces. When  $i \in \{1, 2\}$ , the complement  $\{1, 2\} \setminus \{i\}$  is  $\{3 - i\}$ . The definition of sections and some of their properties were presented in Section 6.6.11.

**Lemma 786 (countable union of sections is measurable).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma_1 \otimes \Sigma_2$ . Let  $i$  be in  $\{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(237) \quad s_i \left( x_i, \bigcup_{n \in \mathbb{N}} A_n \right) = \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j.$$

*Proof.* Direct consequence of Lemma 537 (compatibility of section with set operations, with countable union), Definition 467 ( $\sigma$ -algebra, closedness under countable union), and Lemma 538 (measurability of section).  $\square$

**Lemma 787 (countable intersection of sections is measurable).**

Let  $(X_1, \Sigma_1)$  and  $(X_2, \Sigma_2)$  be measurable spaces. Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma_1 \otimes \Sigma_2$ . Let  $i$  be in  $\{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Let  $x_i \in X_i$ . Then, we have

$$(238) \quad s_i \left( x_i, \bigcap_{n \in \mathbb{N}} A_n \right) = \bigcap_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j.$$

*Proof.* Direct consequence of Lemma 537 (compatibility of section with set operations, with countable intersection), Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection), and Lemma 538 (measurability of section).  $\square$

**Lemma 788 (measure of section).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, the function  $F_i^A$  defined by

$$(239) \quad \forall x_i \in X_i, \quad F_i^A(x_i) \stackrel{\text{def.}}{=} \mu_j(s_i(x_i, A))$$

is well-defined on  $X_i$ , and takes its values in  $\overline{\mathbb{R}}_+$ . It is called measure of  $i$ -th section of  $A$ .

*Proof.* Direct consequence of Lemma 538 (measurability of section), and Definition 594 (measure, nonnegativeness).  $\square$

**Lemma 789 (measure of section of product).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then, we have

$$(240) \quad F_i^{A_1 \times A_2} = \mu_j(A_j) \mathbb{1}_{A_i}.$$

*Proof.* Direct consequence of Lemma 788 (measure of section), Lemma 536 (section of product), Definition 594 (measure,  $\mu_j(\emptyset) = 0$ ), Lemma 319 (zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)), and the definition of the indicator function.  $\square$

*Remark 790.* The following proof follows the monotone class theorem scheme (see Section 4.3).

**Lemma 791 (measurability of measure of section (finite)).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be measure spaces. Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Assume that  $\mu_j$  is finite. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Then, we have  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\overline{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \overline{\times} \Sigma_2$ , and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\overline{\Sigma})$ .

Let  $\mathcal{S}_i \stackrel{\text{def.}}{=} \{A \in \Sigma \mid F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)\}$ .

(1).  $\overline{\Sigma} \subset \mathcal{S}_i$ . Let  $A \in \overline{\Sigma}$ . Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$  such that  $A = A_1 \times A_2$ .

From Lemma 789 (measure of section of product,  $F_i^A = \mu_j(A_j) \mathbb{1}_{A_i}$ ), Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function,  $\mathbb{1}_{A_i} \in \mathcal{M}_+(X_i, \Sigma_i)$ ), Definition 594 (measure,  $\mu_j(A_j) \in \overline{\mathbb{R}}_+$ ), and

Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication, with  $a \stackrel{\text{def.}}{=} \mu_j(A_j)$ ),  $F_i^A$  belongs to  $\mathcal{M}_+(X_i, \Sigma_i)$ . Thus, from Lemma 530 (product of measurable subsets is measurable,  $A \in \Sigma$ ), and the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ . Hence, we have  $\bar{\Sigma} \subset \mathcal{S}_i$ .

(2).  $\mathcal{A} \subset \mathcal{S}_i$ . Let  $A \in \mathcal{A}$ .

Then, from Lemma 438 (generated set algebra is minimum), Definition 529 (tensor product of  $\sigma$ -algebras,  $\bar{\Sigma} = \Sigma_X(\bar{\Sigma})$ ), Lemma 481 ( $\sigma$ -algebra contains set algebra,  $\mathcal{A} \subset \Sigma$ ), Lemma 501 (set algebra generated by product of  $\sigma$ -algebras), and Lemma 530 (product of measurable subsets is measurable,  $\bar{\Sigma} \subset \Sigma$ ), we have  $A \in \Sigma$ , and there exists  $n \in \mathbb{N}$ , and  $(A_p)_{p \in [0..n]}$  in  $\bar{\Sigma} \subset \Sigma$  such that, for all  $p, q \in [0..n]$ ,  $p \neq q \Rightarrow A_p \cap A_q = \emptyset$ , and  $A = \bigsqcup_{p \in [0..n]} A_p$ .

Let  $x_i \in X_i$ . Then, from Lemma 788 (measure of section), Lemma 786 (countable union of sections is measurable), Lemma 537 (compatibility of section with set operations, the  $s_i(x_i, A_p)$ 's are pairwise disjoint), Definition 594 (measure,  $\mu_j$  is  $\sigma$ -additive), and Definition 591 ( $\sigma$ -additivity over measurable space, with  $I \stackrel{\text{def.}}{=} [0..n]$  and  $\mu \stackrel{\text{def.}}{=} \mu_j$ ), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j \left( \bigsqcup_{p \in [0..n]} s_i(x_i, A_p) \right) = \sum_{p \in [0..n]} \mu_j(s_i(x_i, A_p)) = \sum_{p \in [0..n]} F_i^{A_p}(x_i).$$

Thus, from (1) ( $A_p \in \mathcal{S}_i$ ), the definition of  $\mathcal{S}_i$ , and Lemma 582 ( $\mathcal{M}_+$  is closed under addition), we have  $A \in \mathcal{S}_i$ . Hence, we have  $\mathcal{A} \subset \mathcal{S}_i$ .

(3).  $\mathcal{S}_i$  is monotone class. Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}_i \subset \Sigma$ .

Let  $x_i \in X_i$ . Let  $n \in \mathbb{N}$ . Then, from Lemma 538 (measurability of section), Lemma 786 (countable union of sections is measurable), and Lemma 787 (countable intersection of sections is measurable), we have

$$s_i(x_i, A_n), \quad s_i \left( x_i, \bigcup_{n \in \mathbb{N}} A_n \right) = \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n), \quad s_i \left( x_i, \bigcap_{n \in \mathbb{N}} A_n \right) = \bigcap_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j.$$

Assume first that, for all  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ .

Let  $x_i \in X_i$ . Then, from Lemma 788 (measure of section), the definition of  $A$ , Lemma 537 (compatibility of section with set operations,  $(s_i(x_i, A_n))_{n \in \mathbb{N}}$  is nondecreasing), and Lemma 600 (measure is continuous from below, with  $\mu_j$ ), we have

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j \left( s_i \left( x_i, \bigcup_{n \in \mathbb{N}} A_n \right) \right) \\ &= \mu_j \left( \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n) \right) = \sup_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sup_{n \in \mathbb{N}} F_i^{A_n}(x_i). \end{aligned}$$

Thus, from the definition of  $\mathcal{S}_i$ , and Lemma 572 ( $\mathcal{M}$  is closed under supremum),  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ . Hence, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ , i.e.  $\mathcal{S}_i$  is closed under nondecreasing union.

Assume now that, for all  $n \in \mathbb{N}$ ,  $A_n \supset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n$ .

Let  $x_i \in X_i$ . Then, from Lemma 788 (measure of section), the definition of  $A$ , Lemma 537 (compatibility of section with set operations,  $(s_i(x_i, A_n))_{n \in \mathbb{N}}$  is nonincreasing), Definition 605 (finite measure, with  $\mu_j$ ), Lemma 606 (finite measure is bounded,  $\mu_j(s_i(x_i, A_0))$  is finite), and Lemma 602 (measure is continuous from above, with  $\mu_j$  and  $n_0 \stackrel{\text{def.}}{=} 0$ ), we have

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j \left( s_i \left( x_i, \bigcap_{n \in \mathbb{N}} A_n \right) \right) \\ &= \mu_j \left( \bigcap_{n \in \mathbb{N}} s_i(x_i, A_n) \right) = \inf_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \inf_{n \in \mathbb{N}} F_i^{A_n}(x_i). \end{aligned}$$



Thus, from the definition of  $\mathcal{S}_i$ , and Lemma 571 ( $\mathcal{M}$  is closed under infimum),  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ . Then, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}_i$ , i.e.  $\mathcal{S}_i$  is closed under nonincreasing intersection.

Hence, from Definition 442 (monotone class),  $\mathcal{S}_i$  is a monotone class.

Therefore, from Definition 529 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\overline{\Sigma})$ ), (2), (3), and Lemma 509 (usage of monotone class theorem, with  $G \stackrel{\text{def.}}{=} \overline{\Sigma}$ ), we have for all  $A \in \Sigma$ ,  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .  $\square$

**Lemma 792 (measurability of measure of section).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Then, we have  $F_i^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

*Proof.* Let  $j \stackrel{\text{def.}}{=} 3 - i$ . From Lemma 608 (equivalent definition of  $\sigma$ -finite measure, with  $\mu_j$ ), there exists  $(B_{j,n})_{n \in \mathbb{N}} \in \Sigma_j$  such that, for all  $n \in \mathbb{N}$ ,  $B_{j,n} \subset B_{j,n+1}$ ,  $\mu_j(B_{j,n}) < \infty$ , and  $X_j = \bigcup_{n \in \mathbb{N}} B_{j,n}$ .

For all  $n \in \mathbb{N}$ , for all  $A \in \Sigma_1 \otimes \Sigma_2$ , from Lemma 612 (restricted measure, with  $Y \stackrel{\text{def.}}{=} B_{j,n}$ ), and Lemma 788 (measure of section, with  $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$ ), let  $\mu_{j,n}$  be the restricted measure, and  $F_{i,n}^A$  be the measure of  $i$ -th section of  $A$  defined by

$$\mu_{j,n} \stackrel{\text{def.}}{=} (\mu_j)'_{B_{j,n}} = (A_j \in \Sigma_j \mapsto \mu_j(A_j \cap B_{j,n})) \quad \text{and} \quad F_{i,n}^A \stackrel{\text{def.}}{=} (x_i \in X_i \mapsto \mu_{j,n}(s_i(x_i, A))).$$

Let  $n \in \mathbb{N}$ . Then, from Definition 605 (finite measure,  $\mu_{j,n}(X) = \mu_j(B_{j,n}) < \infty$ ),  $\mu_{j,n}$  is a finite measure on  $(X_j, \Sigma_j)$ . Moreover, from Lemma 791 (measurability of measure of section (finite), with  $\mu_j \stackrel{\text{def.}}{=} \mu_{j,n}$  finite), we have  $F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i)$ .

Let  $x_i \in X_i$ . Then, from Lemma 538 (measurability of section), the definition of  $B_{j,n}$ , and **distributivity of intersection over union**, we have

$$\Sigma_j \ni s_i(x_i, A) = s_i(x_i, A) \cap \bigcup_{n \in \mathbb{N}} B_{j,n} = \bigcup_{n \in \mathbb{N}} (s_i(x_i, A) \cap B_{j,n}).$$

Thus, from Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $\text{card}(I) = 2$ )), we have  $s_i(x_i, A) \cap B_{j,n} \in \Sigma_j$ . Hence, from Lemma 788 (measure of section), **monotonicity of intersection** ( $(s_i(x_i, A) \cap B_{j,n})_{n \in \mathbb{N}}$  is nondecreasing), Lemma 600 (measure is continuous from below, with  $\mu_j$  and  $A_n \stackrel{\text{def.}}{=} s_i(x_i, A) \cap B_{j,n}$ ), and the definition of  $\mu_{j,n}$ , we have

$$\begin{aligned} F_i^A(x_i) &= \mu_j(s_i(x_i, A)) = \mu_j\left(s_i(x_i, A) \cap \bigcup_{n \in \mathbb{N}} B_{j,n}\right) = \mu_j\left(\bigcup_{n \in \mathbb{N}} (s_i(x_i, A) \cap B_{j,n})\right) \\ &= \sup_{n \in \mathbb{N}} \mu_j(s_i(x_i, A) \cap B_{j,n}) = \sup_{n \in \mathbb{N}} \mu_{j,n}(s_i(x_i, A)) = \sup_{n \in \mathbb{N}} F_{i,n}^A(x_i). \end{aligned}$$

Therefore, from Lemma 572 ( $\mathcal{M}$  is closed under supremum), we have

$$F_i^A = \sup_{n \in \mathbb{N}} F_{i,n}^A \in \mathcal{M}_+(X_i, \Sigma_i).$$

$\square$

**Definition 793 (tensor product measure).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . A measure  $\mu$  on the product measurable space  $(X, \Sigma)$  is called *tensor product measure* (on  $(X, \Sigma)$ ) iff

$$(241) \quad \forall A_1 \in \Sigma_1, \forall A_2 \in \Sigma_2, \quad \mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2).$$

**Definition 794 (candidate tensor product measure).** Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $i \in \{1, 2\}$ . The function defined by

$$(242) \quad \forall A \in \Sigma_1 \otimes \Sigma_2, \quad (\mu_1 \otimes \mu_2)^i(A) \stackrel{\text{def.}}{=} \int F_i^A d\mu_i$$

is called *candidate tensor product measure*.

**Lemma 795 (candidate tensor product measure is tensor product measure).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $i \in \{1, 2\}$ . Then, the candidate tensor product measure  $(\mu_1 \otimes \mu_2)^i$  is a tensor product measure on  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ .

**Measure.** Let  $j \stackrel{\text{def.}}{=} 3 - i$ .

Let  $A \in \Sigma$ . Then, from Definition 794 (candidate tensor product measure), Lemma 792 (measurability of measure of section), and Lemma 753 (integral in  $\mathcal{M}_+$ , nonnegativeness), the value  $(\mu_1 \otimes \mu_2)^i(A) = \int F_i^A d\mu_i$  is well-defined in  $\overline{\mathbb{R}}_+$ , and from Lemma 788 (measure of section), it is equal to  $\int \mu_j(s_i(x_i, A)) d\mu_i$ .

From Lemma 537 (compatibility of section with set operations, empty set), Definition 594 (measure,  $\mu_j(\emptyset) = 0$ ), and Lemma 757 (integral in  $\mathcal{M}_+$  of zero is zero), we have

$$(\mu_1 \otimes \mu_2)^i(\emptyset) = \int \mu_j(s_i(x_i, \emptyset)) d\mu_i = \int \mu_j(\emptyset) d\mu_i = \int 0 d\mu_i = 0.$$

Let  $(A_n)_{n \in \mathbb{N}} \in \Sigma$ . Assume that, for all  $p, q \in \mathbb{N}$ ,  $p \neq q$  implies  $A_p \cap A_q = \emptyset$ . Let  $A \stackrel{\text{def.}}{=} \bigsqcup_{n \in \mathbb{N}} A_n$ . Let  $x_i \in X_i$ . Let  $n \in \mathbb{N}$ . Let  $p, q \in \mathbb{N}$ . Assume that  $p \neq q$ . Then, from Lemma 538 (measurability of section), the definition of the  $A_n$ 's, Lemma 786 (countable union of sections is measurable), and Lemma 537 (compatibility of section with set operations, with intersection and empty set), we have  $s_i(x_i, A_n) \in \Sigma_j$ , and

$$\begin{aligned} s_i(x_i, A) &= s_i\left(x_i, \bigcup_{n \in \mathbb{N}} A_n\right) = \bigcup_{n \in \mathbb{N}} s_i(x_i, A_n) \in \Sigma_j, \\ s_i(x_i, A_p) \cap s_i(x_i, A_q) &= s_i(x_i, A_p \cap A_q) = s_i(x_i, \emptyset) = \emptyset. \end{aligned}$$

Thus, from Lemma 788 (measure of section), Definition 594 (measure,  $\mu_j$  is  $\sigma$ -additive), Definition 591 ( $\sigma$ -additivity over measurable space), we have

$$F_i^A(x_i) = \mu_j(s_i(x_i, A)) = \mu_j\left(\bigsqcup_{n \in \mathbb{N}} s_i(x_i, A_n)\right) = \sum_{n \in \mathbb{N}} \mu_j(s_i(x_i, A_n)) = \sum_{n \in \mathbb{N}} F_i^{A_n}(x_i).$$

Then, from Lemma 766 (integral in  $\mathcal{M}_+$  is  $\sigma$ -additive), and Definition 794 (candidate tensor product measure), we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \sum_{n \in \mathbb{N}} F_i^{A_n} d\mu_i = \sum_{n \in \mathbb{N}} \int F_i^{A_n} d\mu_i = \sum_{n \in \mathbb{N}} (\mu_1 \otimes \mu_2)^i(A_n).$$

Hence, from Definition 591 ( $\sigma$ -additivity over measurable space),  $(\mu_1 \otimes \mu_2)^i$  is  $\sigma$ -additive.

Therefore, from Definition 594 (measure),  $(\mu_1 \otimes \mu_2)^i$  is a measure on  $(X, \Sigma)$ .

**Identity.** Let  $A_1 \in \Sigma_1$ . Let  $A_2 \in \Sigma_2$ . Let  $A \stackrel{\text{def.}}{=} A_1 \times A_2$ .

Then, from Lemma 530 (product of measurable subsets is measurable), we have  $A \in \Sigma$ . Hence, from Definition 794 (candidate tensor product measure), Lemma 789 (measure of section of product), Definition 594 (measure,  $\mu_j$  is nonnegative), Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous), Lemma 755 (integral in  $\mathcal{M}_+$  of indicator function, with  $\mu \stackrel{\text{def.}}{=} \mu_i$ ), Lemma 317 (multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)), and since  $\{1, 2\} = \{i, j\}$ , we have

$$(\mu_1 \otimes \mu_2)^i(A) = \int \mu_j(A_j) \mathbb{1}_{A_i} d\mu_i = \mu_j(A_j) \mu_i(A_i) = \mu_1(A_1) \mu_2(A_2),$$

Therefore, from Definition 793 (tensor product measure),  $(\mu_1 \otimes \mu_2)^i$  is a tensor product measure.  $\square$

**Lemma 796 (tensor product of finite measures).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be finite measure spaces. Let  $\mu$  be a tensor product measure on  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ . Then,  $\mu$  is finite.

*Proof.* Direct consequence of Definition 594 (measure), Lemma 468 (equivalent definition of  $\sigma$ -algebra,  $X_1 \in \Sigma_1$ ,  $X_2 \in \Sigma_2$ , and  $X_1 \times X_2 \in \Sigma_1 \otimes \Sigma_2$ ), Definition 793 (tensor product measure), Definition 605 (finite measure), and **closedness of multiplication in  $\mathbb{R}_+$** .  $\square$

**Lemma 797 (tensor product of  $\sigma$ -finite measures).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $\mu$  be a tensor product measure on  $(X, \Sigma)$ . Then,  $\mu$  is  $\sigma$ -finite.

In particular, let  $i \in \{1, 2\}$ . Let  $(B_{i,n})_{n \in \mathbb{N}} \in \Sigma_i$  such that

$$(243) \quad (\forall n \in \mathbb{N}, \quad B_{i,n} \subset B_{i,n+1} \quad \wedge \quad \mu_i(B_{i,n}) < \infty) \quad \wedge \quad X_i = \bigcup_{n \in \mathbb{N}} B_{i,n}.$$

For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$ . Then, we have

$$(244) \quad (\forall n \in \mathbb{N}, \quad B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) < \infty) \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} B_n.$$

*Proof.* Existence of the  $B_{i,n}$ 's comes from Lemma 608 (equivalent definition of  $\sigma$ -finite measure).

Let  $n \in \mathbb{N}$ . Then, from the definition of the  $B_n$ 's, Lemma 530 (product of measurable subsets is measurable), **monotonicity of Cartesian product**, Definition 793 (tensor product measure), and **closedness of multiplication in  $\mathbb{R}_+$** , we have

$$B_n \in \Sigma \quad \wedge \quad B_n \subset B_{n+1} \quad \wedge \quad \mu(B_n) = \mu_1(B_{1,n}) \mu_2(B_{2,n}) < \infty.$$

Moreover, from **monotonicity of union**, we have  $\bigcup_{n \in \mathbb{N}} B_n \subset X$ . Conversely, let  $(x_1, x_2) \in X$ . Let  $i \in \{1, 2\}$ . Then, from Definition 607 ( $\sigma$ -finite measure, with  $\mu_i$ ), there exists  $n_i \in \mathbb{N}$  such that  $x_i \in B_{i,n_i}$ . Thus, from monotonicity of the  $B_{i,n}$ 's, we have  $x_i \in B_{i, \max(n_1, n_2)}$ . Then, from the definition of the  $B_n$ 's, we have  $(x_1, x_2) \in B_{\max(n_1, n_2)}$ , i.e.  $X \subset \bigcup_{n \in \mathbb{N}} B_n$ . Hence, we have  $X = \bigcup_{n \in \mathbb{N}} B_n$ .

Therefore, from Definition 607 ( $\sigma$ -finite measure),  $\mu$  is  $\sigma$ -finite.  $\square$

*Remark 798.* The following proof follows the monotone class theorem scheme (see Section 4.3).

**Lemma 799 (uniqueness of tensor product measure (finite)).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be finite measure spaces. Then, there exists a unique tensor product measure on the product measurable space  $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ .

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\bar{\Sigma} \stackrel{\text{def.}}{=} \Sigma_1 \bar{\times} \Sigma_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $m$  and  $m'$  be two tensor product measures on  $(X, \Sigma)$ . Let  $\mathcal{S} \stackrel{\text{def.}}{=} \{A \in \Sigma \mid m(A) = m'(A)\}$ . Let  $\mathcal{A} \stackrel{\text{def.}}{=} \mathcal{A}_X(\bar{\Sigma})$ .

(1).  $\bar{\Sigma} \subset \mathcal{S}$ . Direct consequence of Lemma 530 (product of measurable subsets is measurable), Definition 793 (tensor product measure, with  $m$  and  $m'$ ), and the definition of  $\mathcal{S}$ .

(2).  $\mathcal{A} \subset \mathcal{S}$ . Let  $A \in \mathcal{A}$ .

Then, from Definition 529 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\bar{\Sigma})$ ), Lemma 481 ( $\sigma$ -algebra contains set algebra,  $\mathcal{A} \subset \Sigma$ ), Lemma 501 (set algebra generated by product of  $\sigma$ -algebras), and Lemma 530 (product of measurable subsets is measurable,  $\bar{\Sigma} \subset \Sigma$ ), we have  $A \in \Sigma$ , and there exists  $n \in \mathbb{N}$ , and  $(A_p)_{p \in [0..n]} \in \bar{\Sigma} \subset \Sigma$  such that, for all  $p, q \in [0..n]$ ,  $p \neq q \Rightarrow A_p \cap A_q = \emptyset$ , and  $A = \biguplus_{p \in [0..n]} A_p$ . Thus, from Definition 594 (measure,  $\sigma$ -additivity for  $m$  and  $m'$ ), Lemma 593

( $\sigma$ -additivity implies additivity), Definition 590 (additivity over measurable space), (1) ( $A_p \in \mathcal{S}$ ), the definition of  $\mathcal{S}$ , and since **addition in  $\overline{\mathbb{R}}_+$  is a function**, we have

$$m(A) = m\left(\bigsqcup_{p \in [0..n]} A_p\right) = \sum_{p \in [0..n]} m(A_p) = \sum_{p \in [0..n]} m'(A_p) = m'\left(\bigsqcup_{p \in [0..n]} A_p\right) = m'(A).$$

Then, from the definition of  $\mathcal{S}$ , we have  $A \in \mathcal{S}$ . Hence, we have  $\mathcal{A} \subset \mathcal{S}$ .

**(3).  $\mathcal{S}$  is monotone class.** Let  $(A_n)_{n \in \mathbb{N}} \in \mathcal{S} \subset \Sigma$ .

Then, from Lemma 468 (equivalent definition of  $\sigma$ -algebra, closedness under countable union and intersection), we have  $\bigcup_{n \in \mathbb{N}} A_n, \bigcap_{n \in \mathbb{N}} A_n \in \Sigma$ .

Assume first that, for all  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcup_{n \in \mathbb{N}} A_n$ .

Then, from Lemma 600 (measure is continuous from below, with  $m$  and  $m'$ ), the definition of  $\mathcal{S}$ , and since **supremum is a function**, we have

$$m(A) = m\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sup_{n \in \mathbb{N}} m(A_n) = \sup_{n \in \mathbb{N}} m'(A_n) = m'\left(\bigcup_{n \in \mathbb{N}} A_n\right) = m'(A).$$

Hence, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}$ , i.e.  $\mathcal{S}$  is closed under nondecreasing union.

Assume now that, for all  $n \in \mathbb{N}$ ,  $A_n \supset A_{n+1}$ . Let  $A \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} A_n$ .

Then, from Lemma 796 (tensor product of finite measures, with  $m$  and  $m'$ ), Lemma 606 (finite measure is bounded,  $m(A_0), m'(A_0) < \infty$ ), Lemma 602 (measure is continuous from above, with  $\mu \stackrel{\text{def.}}{=} m, m'$  and  $n_0 \stackrel{\text{def.}}{=} 0$ ), the definition of  $\mathcal{S}$ , and since **infimum is a function**, we have

$$m(A) = m\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} m(A_n) = \inf_{n \in \mathbb{N}} m'(A_n) = m'\left(\bigcap_{n \in \mathbb{N}} A_n\right) = m'(A).$$

Thus, from the definition of  $\mathcal{S}_i$ , we have  $A \in \mathcal{S}$ , i.e.  $\mathcal{S}$  is closed under nonincreasing intersection.

Hence, from Definition 442 (monotone class),  $\mathcal{S}$  is a monotone class.

Therefore, from Definition 529 (tensor product of  $\sigma$ -algebras,  $\Sigma = \Sigma_X(\overline{\Sigma})$ ), (2), (3), and Lemma 509 (usage of monotone class theorem, with  $G \stackrel{\text{def.}}{=} \overline{\Sigma}$ ), we have  $m = m'$ .  $\square$

*Remark 800.* The uniqueness part of the following proof reproduces the schemes of the proof of Lemma 792 (measurability of measure of section).

**Lemma 801 (uniqueness of tensor product measure).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$  and  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Then, there exists a unique tensor product measure on  $(X, \Sigma)$ .

This measure is denoted  $\mu_1 \otimes \mu_2$ , and for all  $i \in \{1, 2\}$  with  $j \stackrel{\text{def.}}{=} 3 - i$ , we have

$$(245) \quad \forall A \in \Sigma, \quad (\mu_1 \otimes \mu_2)(A) = \int \mu_j(s_i(x_i, A)) d\mu_i.$$

It is called the (tensor) product measure of  $\mu_1$  and  $\mu_2$ , and  $(X, \Sigma, \mu_1 \otimes \mu_2)$  is called (tensor) product measure space.

*Proof. Uniqueness.* Let  $m$  and  $m'$  be two tensor product measures on  $(X, \Sigma)$ .

From Lemma 608 (equivalent definition of  $\sigma$ -finite measure), for all  $i \in \{1, 2\}$ , there exists  $(B_{i,n})_{n \in \mathbb{N}} \in \Sigma_i$  such that, for all  $n \in \mathbb{N}$ ,  $B_{i,n} \subset B_{i,n+1}$ ,  $\mu_i(B_{i,n}) < \infty$ , and  $X_i = \bigcup_{n \in \mathbb{N}} B_{i,n}$ . For all  $n \in \mathbb{N}$ , let  $B_n \stackrel{\text{def.}}{=} B_{1,n} \times B_{2,n}$ . Then, from Lemma 797 (tensor product of  $\sigma$ -finite measures, with  $\mu \stackrel{\text{def.}}{=} m, m'$ ), we have for all  $n \in \mathbb{N}$ ,  $B_n \in \Sigma$ ,  $B_n \subset B_{n+1}$ ,  $m(B_n), m'(B_n) < \infty$ , and  $X = \bigcup_{n \in \mathbb{N}} B_n$ .

For all  $i \in \{1, 2\}$ , for all  $n \in \mathbb{N}$ , from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*,  $(A \in \Sigma$  implies  $A \cap B_n \in \Sigma)$ ), and Lemma 612 (*restricted measure*, with  $Y \stackrel{\text{def.}}{=} B_{i,n}$  and  $\mu \stackrel{\text{def.}}{=} \mu_i$ , then  $Y \stackrel{\text{def.}}{=} B_n$  and  $\mu \stackrel{\text{def.}}{=} m, m'$ ), let  $\mu_{i,n}$ ,  $m_n$  and  $m'_n$  be the restricted measures defined by

$$\begin{aligned} \mu_{i,n} &\stackrel{\text{def.}}{=} (\mu_i)'_{B_{i,n}} = (A_i \in \Sigma_i \mapsto \mu_i(A_i \cap B_{i,n})), \\ m_n &\stackrel{\text{def.}}{=} m'_{B_n} = (A \in \Sigma \mapsto m(A \cap B_n)) \quad \text{and} \quad m'_n \stackrel{\text{def.}}{=} (m')'_{B_n} = (A \in \Sigma \mapsto m'(A \cap B_n)). \end{aligned}$$

Let  $i \in \{1, 2\}$ . Let  $n \in \mathbb{N}$ . Then, from Definition 605 (*finite measure*, with  $\mu_{i,n}(X_i) = \mu_i(B_{i,n})$ ,  $m_n(X) = m(B_n)$  and  $m'_n(X) = m'(B_n)$  finite),  $\mu_{i,n}$ ,  $m_n$  and  $m'_n$  are finite measures.

Let  $n \in \mathbb{N}$ . Let  $A_1 \in \Sigma_1$  and  $A_2 \in \Sigma_2$ . Then, from Lemma 530 (*product of measurable subsets is measurable*,  $A_1 \times A_2 \in \Sigma$ ), the definitions of  $m_n$ ,  $B_n$ ,  $\mu_{1,n}$ ,  $\mu_{2,n}$  and  $m'_n$ , **compatibility of intersection with Cartesian product**, and Definition 793 (*tensor product measure*, with  $\mu \stackrel{\text{def.}}{=} m, m'$ ), we have

$$\begin{aligned} m_n(A_1 \times A_2) &= m((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = m((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) \\ &= \mu_1(A_1 \cap B_{1,n}) \mu_2(A_2 \cap B_{2,n}) = \mu_{1,n}(A_1) \mu_{2,n}(A_2) \\ &= m'((A_1 \cap B_{1,n}) \times (A_2 \cap B_{2,n})) = m'((A_1 \times A_2) \cap (B_{1,n} \times B_{2,n})) = m'_n(A_1 \times A_2). \end{aligned}$$

Thus, from Definition 793 (*tensor product measure*, with  $\mu_1 \stackrel{\text{def.}}{=} \mu_{1,n}$ ,  $\mu_2 \stackrel{\text{def.}}{=} \mu_{2,n}$  and  $\mu \stackrel{\text{def.}}{=} m_n, m'_n$ ),  $m_n$  and  $m'_n$  are tensor product measures on  $(X, \Sigma)$ . Hence, from Lemma 799 (*uniqueness of tensor product measure (finite)*), we have  $m_n = m'_n$ .

Let  $A \in \Sigma$ . Then, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra*,  $A \cap B_n \in \Sigma$ , then  $\bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \Sigma$ ), **distributivity of intersection over union**, **monotonicity of intersection** ( $(A \cap B_n)_{n \in \mathbb{N}}$  is nondecreasing), Lemma 600 (*measure is continuous from below*, with  $m$ ,  $m'$  and  $A_n \stackrel{\text{def.}}{=} A \cap B_n$ ), the definition of  $m_n$  and  $m'_n$ , and since **supremum is a function**, we have

$$\begin{aligned} m(A) &= m\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = m\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) \\ &= \sup_{n \in \mathbb{N}} m(A \cap B_n) = \sup_{n \in \mathbb{N}} m_n(A) = \sup_{n \in \mathbb{N}} m'_n(A) = \sup_{n \in \mathbb{N}} m'(A \cap B_n) \\ &= m'\left(\bigcup_{n \in \mathbb{N}} (A \cap B_n)\right) = m'\left(A \cap \bigcup_{n \in \mathbb{N}} B_n\right) = m'(A). \end{aligned}$$

Therefore, we have  $m = m'$ .

**Identity.** Direct consequence of the uniqueness result above, Lemma 795 (*candidate tensor product measure is tensor product measure*), Definition 794 (*candidate tensor product measure*), and Lemma 788 (*measure of section*).  $\square$

**Lemma 802 (negligibility of measurable section).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Let  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ . Let  $A \in \Sigma_1 \otimes \Sigma_2$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ . Then,  $A$  is  $\mu$ -negligible iff  $\mu(A) = 0$  iff for almost all  $x_i \in X_i$ ,  $\mu_j(s_i(x_i, A)) = 0$  iff for almost all  $x_i \in X_i$ ,  $s_i(x_i, A)$  is  $\mu_j$ -negligible.

*Proof.* Direct consequence of Lemma 792 (*measurability of measure of section*,  $s_i(x_i, A) \in \Sigma_j$ ) Lemma 619 (*negligibility of measurable subset*, with  $A$ , then  $s_i(x_i, A)$ ), Lemma 801 (*uniqueness of tensor product measure*), Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*, with  $\mu_i$  and  $(x_i \mapsto \mu_j(s_i(x_i, A)))$ ).  $\square$

**Lemma 803 (Lebesgue measure on  $\mathbb{R}^2$ ).**

There is a unique measure on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$  that generalizes the area of bounded open boxes.

It is denoted  $\lambda^{\otimes 2} \stackrel{\text{def.}}{=} \lambda \otimes \lambda$ , and it is called the Lebesgue measure on (Borel subsets of)  $\mathbb{R}^2$ .

*Proof.* Direct consequence of Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ), Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*), and Lemma 801 (*uniqueness of tensor product measure*).  $\square$

**Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).**

Let  $a_1, b_1, a_2, b_2 \in \mathbb{R}$ . Assume that  $a_1 \leq b_1$  and  $a_2 \leq b_2$ . Then, we have

$$(246) \quad \lambda^{\otimes 2}([a_1, b_1] \times [a_2, b_2]) = (b_1 - a_1)(b_2 - a_2).$$

*Proof.* Direct consequence of Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ), Definition 793 (*tensor product measure*), Lemma 704 (*Lebesgue measure generalizes length of intervals*), and **the definition of the area of boxes of  $\mathbb{R}^2$** .  $\square$

**Lemma 805 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*).**

Let  $A \subset \mathbb{R}^2$  be a box (i.e. the Cartesian product of two intervals). Then, we have  $\lambda^{\otimes 2}(A) = 0$  iff  $A$  is a line (i.e. at least one of the two intervals is a singleton).

*Proof.* Direct consequence of Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes, with  $a_1 = b_1$  or  $a_2 = b_2$* ), and Lemma 319 (*zero-product property in  $\mathbb{R}_+$  (measure theory)*).  $\square$

**Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).**

The measure space  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$  is  $\sigma$ -finite.

*Proof.* Direct consequence of Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ), Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*), and Lemma 797 (*tensor product of  $\sigma$ -finite measures*).  $\square$

**Lemma 807 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*).**

The measure space  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda^{\otimes 2})$  is diffuse.

*Proof.* Direct consequence of Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes, with  $a_1 = b_1$  and  $a_2 = b_2$* ), and Definition 609 (*diffuse measure,  $\{(a_1, a_2)\} = [a_1, a_1] \times [a_2, a_2]$* ).  $\square$

**Theorem 808 (*Tonelli*).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$  be  $\sigma$ -finite measure spaces.

Let  $f \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ . Let  $i \in \{1, 2\}$ . Let  $j \stackrel{\text{def.}}{=} 3 - i$ .

Then,  $f_i \stackrel{\text{def.}}{=} (x_i \in X_i \mapsto \int f d\mu_j) \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have

$$(247) \quad \int f d(\mu_1 \otimes \mu_2) = \int f_i d\mu_i.$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ , and  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ .

(1). For  $f \in \mathcal{IF}(X, \Sigma)$ .

Let  $A \stackrel{\text{def.}}{=} \{f \neq 0\}$ . Then, from Lemma 710 (*indicator and support are each other inverse*), we have  $A \in \Sigma$  and  $f = \mathbf{1}_A$ . Let  $x_i \in X_i$ . Then, from Lemma 538 (*measurability of section*), Lemma 539 (*indicator of section*), and Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*), we have

$$s_i(x_i, A) \in \Sigma_j \quad \text{and} \quad \int \mathbf{1}_A d\mu_j = \int \mathbf{1}_{s_i(x_i, A)} d\mu_j = \mu_j(s_i(x_i, A)).$$

Hence, from Lemma 788 (*measure of section*), and Lemma 792 (*measurability of measure of section*), we have

$$f_i = F_i^A \in \mathcal{M}_+(X, \Sigma).$$

Therefore, from Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*), and Lemma 801 (*uniqueness of tensor product measure*), we have

$$\int f d\mu = \mu(A) = \int F_i^A d\mu_i = \int f_i d\mu_i.$$

(2). For  $f \in \mathcal{SF}_+(X, \Sigma)$ .

From Lemma 734 ( $\mathcal{SF}_+$  simple representation), let  $n \in \mathbb{N}$ ,  $(a_k)_{k \in [0..n]} \in \mathbb{R}_+$  and  $(A_k)_{k \in [0..n]} \in \Sigma$  such that  $f = \sum_{k \in [0..n]} a_k \mathbb{1}_{A_k}$ . Let  $k \in [0..n]$ . Then, from (1), we have

$$F_i^{A_k} \stackrel{\text{def.}}{=} \int \mathbb{1}_{A_k} d\mu_j \in \mathcal{M}_+(X, \Sigma) \quad \text{and} \quad \int \mathbb{1}_{A_k} d\mu = \int F_i^{A_k} d\mu_j.$$

Hence, from Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear, with  $\mu_j$* ), we have

$$f_i = \sum_{k \in [0..n]} a_k F_i^{A_k} \in \mathcal{M}_+(X_i, \Sigma_i).$$

Therefore, from Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear, with  $\mu$ , then  $\mu_j$* ), we have

$$\begin{aligned} \int f d\mu &= \sum_{k \in [0..n]} a_k \left( \int \mathbb{1}_{A_k} d\mu \right) = \sum_{k \in [0..n]} a_k \left( \int F_i^{A_k} d\mu_i \right) \\ &= \int \left( \sum_{k \in [0..n]} a_k F_i^{A_k} \right) d\mu_i = \int f_i d\mu_i. \end{aligned}$$

(3). For  $f \in \mathcal{M}_+(X, \Sigma)$ .

From Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ), let  $(\varphi_n)_{n \in \mathbb{N}} \in \mathcal{SF}_+(X, \Sigma)$  be an adapted sequence for  $f$ . Let  $n \in \mathbb{N}$ . Then, from (2), we have

$$\varphi_n^i \stackrel{\text{def.}}{=} \int \varphi_n d\mu_j \in \mathcal{M}_+(X_i, \Sigma_i) \quad \text{and} \quad \int \varphi_n d\mu = \int \varphi_n^i d\mu_i.$$

Hence, from Lemma 763 (*usage of adapted sequences, with  $\mu_j$* ), and Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*), we have

$$f_i = \lim_{n \rightarrow \infty} \varphi_n^i \in \mathcal{M}_+(X_i, \Sigma_i).$$

Therefore, from Lemma 763 (*usage of adapted sequences, with  $\mu$ , then  $\mu_j$* ), we have

$$\int f d\mu = \lim_{n \rightarrow \infty} \int \varphi_n d\mu = \lim_{n \rightarrow \infty} \int \varphi_n^i d\mu_i = \int \lim_{n \rightarrow \infty} \varphi_n^i d\mu_i = \int f_i d\mu_i.$$

□

**Lemma 809 (Tonelli on subset).**

Let  $(X_1, \Sigma_1, \mu_1)$  and  $(X_2, \Sigma_2, \mu_2)$

be  $\sigma$ -finite measure spaces. Let  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ . Let  $A \in \Sigma$ . Let  $Y \subset X_1 \times X_2$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}_+$ . Assume that  $f|_A \in \mathcal{M}_+(A, \Sigma \cap A)$ . Let  $i \in \{1, 2\}$  with  $j \stackrel{\text{def.}}{=} 3 - i$ .

Then,  $f_i^A \stackrel{\text{def.}}{=} (x_i \mapsto \int_{s_i(x_i, A)} f d\mu_j) \in \mathcal{M}_+(X_i, \Sigma_i)$ , and we have

$$(248) \quad \int_A f d(\mu_1 \otimes \mu_2) = \int f_i^A d\mu_i.$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ . Let  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}_+$  such that  $\hat{f}|_Y = f$ . Then, from Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*), we have

$$\hat{f} \mathbb{1}_A \in \mathcal{M}_+(X, \Sigma) \quad \text{and} \quad \int_A f d\mu = \int \hat{f} \mathbb{1}_A d\mu.$$

Hence, from Theorem 808 (*Tonelli, with  $f \stackrel{\text{def.}}{=} \hat{f} \mathbb{1}_A$* ), we have

$$\hat{f}_i \stackrel{\text{def.}}{=} \left( x_i \mapsto \int \hat{f} \mathbb{1}_A d\mu_j \right) \in \mathcal{M}_+(X_i, \Sigma_i) \quad \text{and} \quad \int \hat{f} \mathbb{1}_A d\mu = \int \hat{f}_i d\mu_i.$$



Therefore, from Lemma 539 (*indicator of section*), and Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*), we have  $\hat{f}_i(x_i) = \int \hat{f} \mathbf{1}_{s_i(x_i, A)} d\mu_j = \int_{s_i(x_i, A)} f d\mu_j$ , i.e.

$$f_i^A = \hat{f}_i \in \mathcal{M}_+(X_i, \Sigma_i) \quad \text{and} \quad \int_A f d\mu = \int f_i^A d\mu_j.$$

□

**Lemma 810 (*Tonelli for tensor product*).**

For all  $i \in \{1, 2\}$ , let  $(X_i, \Sigma_i, \mu_i)$  be a  $\sigma$ -finite measure space, and  $f_i \in \mathcal{M}_+(X_i, \Sigma_i)$ . Then,  $f_1 \otimes f_2 \in \mathcal{M}_+(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ , and we have

$$(249) \quad \int (f_1 \otimes f_2) d(\mu_1 \otimes \mu_2) = \left( \int f_1 d\mu_1 \right) \left( \int f_2 d\mu_2 \right).$$

*Proof.* Let  $X \stackrel{\text{def.}}{=} X_1 \times X_2$ ,  $\Sigma \stackrel{\text{def.}}{=} \Sigma_1 \otimes \Sigma_2$ ,  $\mu \stackrel{\text{def.}}{=} \mu_1 \otimes \mu_2$ , and  $f \stackrel{\text{def.}}{=} f_1 \otimes f_2$ .

Then, from Definition 587 (*tensor product of numeric functions*), Lemma 588 (*measurability of tensor product of numeric functions*), Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*), and Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*),  $f$  belongs to  $\mathcal{M}_+(X, \Sigma)$ .

Then, from Theorem 808 (*Tonelli, with  $i \stackrel{\text{def.}}{=} 1$* ), Definition 587 (*tensor product of numeric functions*), Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $f \stackrel{\text{def.}}{=} f_2$  and  $a \stackrel{\text{def.}}{=} f_1(x_1)$* ), and Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*), we have

$$\int f d\mu = \int (f)_1 d\mu_1 \quad \text{where } \forall x_1 \in X_1, (f)_1(x_1) \stackrel{\text{def.}}{=} \int f_1(x_1) f_2 d\mu_2 = \left( \int f_2 d\mu_2 \right) f_1(x_1).$$

Therefore, from Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $a \stackrel{\text{def.}}{=} \int f_2 d\mu_2$  and  $f \stackrel{\text{def.}}{=} f_1$* ), Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*), we have

$$\int f d\mu = \int \left( \int f_2 d\mu_2 \right) f_1 d\mu_1 = \left( \int f_1 d\mu_1 \right) \left( \int f_2 d\mu_2 \right).$$

□

## 6.11 Integration of real functions

*Remark 811.* From now on, the expressions involving integrals are taken in  $\mathbb{R}$ , thus functions also take almost all their values in  $\mathbb{R}$ .

### 6.11.1 Definition of the integral

**Definition 812 (*integrability*).** Let  $(X, \Sigma, \mu)$  be a measure space. A function  $f : X \rightarrow \overline{\mathbb{R}}$  is said  $\mu$ -integrable (in  $\mathcal{M}$ ) iff  $f^+$  and  $f^-$  are  $\mu$ -integrable in  $\mathcal{M}_+$ .

**Lemma 813 (*integrable is measurable*).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f$  is  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have  $f \in \mathcal{M}$ .

*Proof.* Direct consequence of Definition 812 (*integrability*), Lemma 753 (*integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability with  $f^+$  and  $f^-$* ), and Lemma 579 (*measurability of nonnegative and nonpositive parts*). □

**Lemma 814 (*equivalent definition of integrability*).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f \in \mathcal{M}$  and  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$ .

*Proof.* “Left” implies “right”. Direct consequence of Definition 812 (*integrability*), Lemma 813 (*integrable is measurable*), Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*), Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*), Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*), and Lemma 753 (*integral in  $\mathcal{M}_+$ ,  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$* ).

“Right” implies “left”. Direct consequence of Lemma 579 (*measurability of nonnegative and nonpositive parts*), Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*), Lemma 297 (*infinity-sum property in  $\overline{\mathbb{R}}_+$ , contrapositive*), Lemma 753 (*integral in  $\mathcal{M}_+$ ,  $f^+$  and  $f^-$  are  $\mu$ -integrable in  $\mathcal{M}_+$* ), and Definition 812 (*integrability*).

Therefore, we have the equivalence.  $\square$

**Lemma 815 (compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $f$  is nonnegative. Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ .

*Proof.* Direct consequence of Lemma 814 (*equivalent definition of integrability*), and Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$ ,  $|f| = f$* ).  $\square$

**Lemma 816 (integrable is almost finite).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f : X \rightarrow \overline{\mathbb{R}}$  be  $\mu$ -integrable in  $\mathcal{M}$ . Then, we have  $\mu(f^{-1}(\pm\infty)) = 0$ , i.e.  $|f| \stackrel{\mu \text{ a.e.}}{<} \infty$ .

*Proof.* Direct consequence of Lemma 814 (*equivalent definition of integrability,  $|f|$  is integrable in  $\mathcal{M}_+$* ), Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even,  $f^{-1}(\pm\infty) = |f|^{-1}(\infty)$* ), and Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite, with  $|f|$* ).  $\square$

**Lemma 817 (almost bounded by integrable is integrable).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff there exists  $g : X \rightarrow \overline{\mathbb{R}}_+$  such that  $g$  is  $\mu$ -integrable in  $\mathcal{M}_+$  and  $|f| \stackrel{\mu \text{ a.e.}}{\leq} g$ .

*Proof.* “Left” implies “right”. Direct consequence of Lemma 814 (*equivalent definition of integrability, with  $g \stackrel{\text{def.}}{=} |f|$* ), Lemma 639 (*almost order is order relation, reflexivity*).

“Right” implies “left”. Direct consequence of Lemma 581 ( *$\mathcal{M}$  is closed under absolute value,  $|f| \in \mathcal{M}_+$* ), Lemma 772 (*integral in  $\mathcal{M}_+$  is almost monotone, with  $|f|$  and  $g$* ), Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total, transitivity*), Lemma 753 (*integral in  $\mathcal{M}_+$ ,  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$* ), and Lemma 814 (*equivalent definition of integrability*).

Therefore, we have the equivalence.  $\square$

**Lemma 818 (bounded by integrable is integrable).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ .

Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff there exists  $g : X \rightarrow \overline{\mathbb{R}}_+$   $\mu$ -integrable in  $\mathcal{M}_+$  such that  $|f| \leq g$ .

*Proof.* “Left” implies “right”. Direct consequence of Lemma 814 (*equivalent definition of integrability, with  $g \stackrel{\text{def.}}{=} |f|$* ), and Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total, reflexivity*).

“Right” implies “left”. Direct consequence of Lemma 626 (*everywhere implies almost everywhere*), and Lemma 817 (*almost bounded by integrable is integrable*).

Therefore, we have the equivalence.  $\square$

**Definition 819 (integral).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . The Lebesgue integral of  $f$  (for the measure  $\mu$ ) is still denoted  $\int f d\mu$ ; it is defined by

$$(250) \quad \int f d\mu \stackrel{\text{def.}}{=} \int f^+ d\mu - \int f^- d\mu \in \mathbb{R}.$$

**Lemma 820 (compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Assume that  $f$  is nonnegative. Then, both Lemma 753 (integral in  $\mathcal{M}_+$ ), and Definition 819 (integral) provide the same value for the integral of  $f$ .

*Proof.* Direct consequence of Definition 373 (nonnegative and nonpositive parts,  $f^+ = f$  and  $f^- = 0$ ), Definition 812 (integrability), Lemma 753 (integral in  $\mathcal{M}_+$ ,  $f$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ), Definition 819 (integral), and Lemma 757 (integral in  $\mathcal{M}_+$  of zero is zero).  $\square$

**Lemma 821 (integral of zero is zero).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then, 0 is  $\mu$ -integrable in  $\mathcal{M}$ , and  $\int 0 d\mu = 0$ .

*Proof.* Direct consequence of Lemma 820 (compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ), and Lemma 757 (integral in  $\mathcal{M}_+$  of zero is zero).  $\square$

**Definition 822 (merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . The integral of  $f$  (for the measure  $\mu$ ) exists iff  $f$  is nonnegative (and  $\int f d\mu \in \overline{\mathbb{R}}_+$ ), or  $f$  is  $\mu$ -integrable (and  $\int f d\mu \in \mathbb{R}$ ).

**Lemma 823 (compatibility of integral with almost equality).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $g \in \mathcal{M}$ . Assume that  $g \stackrel{\mu \text{ a.e.}}{=} f$ . Then,  $g$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(251) \quad \int g d\mu = \int f d\mu.$$

*Proof.* Direct consequence of Lemma 641 (compatibility of almost equality with operator, with the binary operator max and the unary operator additive inverse,  $g^+ \stackrel{\mu \text{ a.e.}}{=} f^+$  and  $g^- \stackrel{\mu \text{ a.e.}}{=} f^-$ ), Lemma 771 (compatibility of integral in  $\mathcal{M}_+$  with almost equality), Definition 812 (integrability), and Definition 819 (integral).  $\square$

*Remark 824.* From Lemma 816 (integrable is almost finite), integrable functions can only take infinite values on a negligible subset, and from Lemma 823 (compatibility of integral with almost equality), the integral keeps the same value when the function is modified on a negligible subset. Therefore, we can restrict the study of integrable functions to the sole case of real-valued functions.

### 6.11.2 Notations for specific cases

**Lemma 825 (integral over subset).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $A \in \Sigma$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $f|_A$  is  $\mu_A$ -integrable in  $\mathcal{M}(A, \Sigma \cap A)$  iff  $\hat{f}\mathbb{1}_A$  is  $\mu$ -integrable in  $\mathcal{M}(X, \Sigma)$ . If so, the function  $f$  is said  $\mu$ -integrable over  $A$ , and we have

$$(252) \quad \int f|_A d\mu_A = \int \hat{f}\mathbb{1}_A d\mu.$$

This integral is still denoted  $\int_A f d\mu$ ; it is still called integral of  $f$  over  $A$ .

*Proof. Equivalence.* Direct consequence of Lemma 380 (compatibility of nonpositive and nonnegative parts with restriction), Lemma 379 (compatibility of nonpositive and nonnegative parts with mask,  $(\hat{f})^\pm$  is an extension of  $f^\pm$  to  $X$ ), Lemma 776 (integral in  $\mathcal{M}_+$  over subset,  $\int f|_A d\mu_A$  equals  $\int (\hat{f})^\pm \mathbb{1}_A d\mu$  and is finite), Definition 812 (integrability), Lemma 753 (integral in  $\mathcal{M}_+$ , definition

of integrability in  $\mathcal{M}_+$ ), Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*, with  $f^\pm$  and  $\hat{f} \stackrel{\text{def.}}{=} (f)^\pm$ ), and **the tautology**

$$\left\{ \begin{array}{l} P \Leftrightarrow Q \wedge R \\ \hat{P} \Leftrightarrow \hat{Q} \wedge \hat{R} \\ Q \Leftrightarrow \hat{Q} \\ Q \vee \hat{Q} \Rightarrow (R \Leftrightarrow \hat{R}) \end{array} \right. \implies (P \Leftrightarrow \hat{P}).$$

**Identity.** Direct consequence of Definition 819 (*integral*, with  $f|_A$ , then  $\hat{f}\mathbb{1}_A$ ), and Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*, with  $f \stackrel{\text{def.}}{=} f^\pm$  and  $\hat{f} \stackrel{\text{def.}}{=} (f)^\pm$ ).  $\square$

**Lemma 826 (*integral over subset is  $\sigma$ -additive*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $I \subset \mathbb{N}$ . Let  $A, (A_i)_{i \in I} \in \Sigma$ . Assume that  $A = \bigsqcup_{i \in I} A_i$ . Let  $Y \subset X$  such that  $A \subset Y$ . Let  $f : Y \rightarrow \overline{\mathbb{R}}$ . Let  $\hat{f} : X \rightarrow \overline{\mathbb{R}}$ . Assume that  $\hat{f}|_Y = f$ . Then,  $\hat{f}\mathbb{1}_A \in \mathcal{M}$  iff for all  $i \in I$ ,  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{M}$ .

Moreover, if  $\sum_{i \in I} |\hat{f}\mathbb{1}_{A_i}|$  is  $\mu$ -integrable, then  $f$  is  $\mu$ -integrable over  $A$ , and we have

$$(253) \quad \int_A f d\mu = \sum_{i \in I} \int_{A_i} f d\mu.$$

*Proof. Equivalence.* Direct consequence of Lemma 814 (*equivalent definition of integrability*), Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*, equivalence, same proof for  $\hat{f}\mathbb{1}_A \in \mathcal{M}$  iff for all  $i \in I$ ,  $\hat{f}\mathbb{1}_{A_i} \in \mathcal{M}$  since the countable sum is well-defined).

**$f$  is  $\mu$ -integrable over  $A$ .** Direct consequence of **the countable triangle inequality for the absolute value in  $\mathbb{R}$** , Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), and Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*,  $\int |\hat{f}\mathbb{1}_A| d\mu \leq \sum_{i \in I} \int |\hat{f}\mathbb{1}_{A_i}| d\mu$ ).

**Identity.** Direct consequence of Definition 819 (*integral*, with  $f \stackrel{\text{def.}}{=} \hat{f}\mathbb{1}_A$ ), Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*, with  $f^+$  and  $f^-$ ), **associativity of addition in  $\mathbb{R}$  (both countable sums are finite)**.  $\square$

**Lemma 827 (*integral over singleton*).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $a \in X$ . Assume that  $\{a\} \in \Sigma$ . Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable over  $\{a\}$  iff

$$(254) \quad (f(a) \text{ and } \mu(\{a\}) \text{ are finite}) \vee f(a) = 0 \vee \mu(\{a\}) = 0.$$

If so, we have

$$(255) \quad \int_{\{a\}} f d\mu = f(a)\mu(\{a\}).$$

*Proof.* From Lemma 825 (*integral over subset*,  $\mu$ -integrability over subset), Lemma 814 (*equivalent definition of integrability*), Lemma 753 (*integral in  $\mathcal{M}_+$* ,  $\mu$ -integrability in  $\mathcal{M}_+$ ), **non-negativeness of the indicator function**, Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*, with  $|f|$ ,  $|f\mathbb{1}_{\{a\}}| = |f(a)|\mathbb{1}_{\{a\}} \in \mathcal{M}_+$ ), Lemma 321 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*), **closedness of absolute value in  $\mathbb{R}$** , Definition 273 (*absolute value in  $\mathbb{R}$* , absolute value is closed

in  $\{\pm\infty\}$ ), and Lemma 280 (*absolute value in  $\overline{\mathbb{R}}$  is definite*), we have

$$\begin{aligned}
f \text{ is } \mu\text{-integrable over } \{a\} &\iff f\mathbb{1}_{\{a\}} \text{ is } \mu\text{-integrable in } \mathcal{M} \\
&\iff |f\mathbb{1}_{\{a\}}| \text{ is } \mu\text{-integrable in } \mathcal{M}_+ \\
&\iff |f\mathbb{1}_{\{a\}}| \in \mathcal{M}_+ \wedge \int |f\mathbb{1}_{\{a\}}| d\mu < \infty \\
&\iff |f(a)|\mu(\{a\}) < \infty \\
&\iff (f(a) \in \mathbb{R} \wedge \mu(\{a\}) \in \mathbb{R}_+) \\
&\quad \vee f(a) = 0 \quad \vee \mu(\{a\}) = 0.
\end{aligned}$$

Assume that  $f$  is  $\mu$ -integrable over  $\{a\}$ . Then, from Lemma 825 (*integral over subset*,  $f\mathbb{1}_{\{a\}}$  is  $\mu$ -integrable in  $\mathcal{M}$ ), Definition 812 (*integrability*), Definition 373 (*nonnegative and nonpositive parts*), and **nonnegativeness of the indicator function**,  $(f\mathbb{1}_{\{a\}})^+ = f^+\mathbb{1}_{\{a\}}$  and  $(f\mathbb{1}_{\{a\}})^- = f^-\mathbb{1}_{\{a\}}$  are  $\mu$ -integrable in  $\mathcal{M}_+$ . Moreover, from Lemma 825 (*integral over subset*), Definition 819 (*integral*), Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*), and Lemma 377 (*decomposition into nonnegative and nonpositive parts*,  $f(a) = f^+(a) - f^-(a)$ ), we have

$$\begin{aligned}
\int_{\{a\}} f d\mu &= \int f\mathbb{1}_{\{a\}} d\mu = \int (f\mathbb{1}_{\{a\}})^+ d\mu - \int (f\mathbb{1}_{\{a\}})^- d\mu \\
&= f^+(a)\mu(\{a\}) - f^-(a)\mu(\{a\}) = f(a)\mu(\{a\}).
\end{aligned}$$

□

**Lemma 828 (*integral over interval*).**

Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a measure space. Assume that  $\mu$  is diffuse. Let  $a, b \in \overline{\mathbb{R}}$  such that  $a \leq b$ . Let  $f$  be a  $\mu$ -integrable function over  $(a, b)$ . Then, the integral over the interval remains the same when the interval is closed at one or both of its finite extremities.

The integral of  $f$  over an interval with extremities  $a$  and  $b$  is called integral of  $f$  from  $a$  to  $b$ ; it is denoted  $\int_a^b f(x) d\mu(x)$ . When  $a < b$ , we assume the convention  $\int_b^a f(x) d\mu(x) \stackrel{\text{def.}}{=} -\int_a^b f(x) d\mu(x)$ .

*Proof.* Direct consequence of Definition 609 (*diffuse measure*), Lemma 826 (*integral over subset is  $\sigma$ -additive*, on  $[a, b] = \{a\} \uplus (a, b) \uplus \{b\}$ ), Lemma 496 (*some Borel subsets*, singletons are measurable), Lemma 827 (*integral over singleton*), and Definition 609 (*diffuse measure*). □

**Lemma 829 (*Chasles relation, integral over split intervals*).**

Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a measure space. Assume that  $\mu$  is diffuse. Let  $a, b, c \in \overline{\mathbb{R}}$ . Let  $f$  be a  $\mu$ -integrable function over  $(\min(a, b, c), \max(a, b, c))$ . Then, we have

$$(256) \quad \int_a^b f(x) d\mu(x) = \int_a^c f(x) d\mu(x) + \int_c^b f(x) d\mu(x).$$

*Proof.* Direct consequence of Lemma 826 (*integral over subset is  $\sigma$ -additive*). Lemma 828 (*integral over interval*, convention for reverse bounds), and **field properties of  $\mathbb{R}$** .

It is trivial when  $a \leq c \leq b$ . For instance, when  $c < b < a$ , we have

$$\int_c^a f(x) d\mu(x) = \int_c^b f(x) d\mu(x) + \int_b^a f(x) d\mu(x).$$

Thus,

$$\begin{aligned}
\int_a^b f(x) d\mu(x) &= -\int_b^a f(x) d\mu(x) \\
&= -\int_c^a f(x) d\mu(x) + \int_c^b f(x) d\mu(x) \\
&= \int_a^c f(x) d\mu(x) + \int_c^b f(x) d\mu(x).
\end{aligned}$$

□

**Lemma 830 (integral for counting measure).**

Let  $(X, \Sigma)$  be a measurable space. Let  $Y \subset X$ . Let  $f \in \mathcal{M}$ . Assume that  $\sum_{y \in Y} |f(y)|$  is finite. Then,  $f$  is  $\delta_Y$ -integrable in  $\mathcal{M}$ , and we have

$$(257) \quad \int f d\delta_Y = \sum_{y \in Y} f(y).$$

*Proof.* From Lemma 781 (integral in  $\mathcal{M}_+$  for counting measure,  $\int |f| d\delta_Y$  is finite), Lemma 753 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability in  $\mathcal{M}_+$ ), and Lemma 814 (equivalent definition of integrability),  $f$  is  $\delta_Y$ -integrable in  $\mathcal{M}$ . Then, from Definition 819 (integral), Lemma 781 (integral in  $\mathcal{M}_+$  for counting measure), Lemma 377 (decomposition into nonnegative and nonpositive parts), and **associativity and commutativity of (possibly uncountable) addition for absolutely convergent sums**, we have

$$\int f d\delta_Y = \int f^+ d\delta_Y - \int f^- d\delta_Y = \sum_{y \in Y} f^+(y) - \sum_{y \in Y} f^-(y) = \sum_{y \in Y} f(y).$$

□

**Lemma 831 (integral for counting measure on  $\mathbb{N}$ ).**Let  $f : \mathbb{N} \rightarrow \overline{\mathbb{R}}$  be a sequence.

Assume that  $\sum_{n \in \mathbb{N}} |f(n)|$  is finite. Then,  $f$  is  $\delta_Y$ -integrable in  $\mathcal{M}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ , and we have

$$(258) \quad \int f d\delta_{\mathbb{N}} = \sum_{n \in \mathbb{N}} f(n).$$

*Proof.* Direct consequence of Lemma 830 (integral for counting measure, with  $Y = X \stackrel{\text{def.}}{=} \mathbb{N}$  and  $\Sigma \stackrel{\text{def.}}{=} \mathcal{P}(\mathbb{N})$ ). □

*Remark 832.* Note that the previous result makes absolutely convergent series be Lebesgue integrals for the counting measure on natural numbers. Thus, the theory of absolutely converging series can be derived from the theory of Lebesgue integration.

**Lemma 833 (integral for Dirac measure).** Let  $(X, \Sigma)$  be a measurable space. Let  $\{a\} \in \Sigma$ . Let  $f \in \mathcal{M}$ . Assume that  $f(a)$  is finite. Then,  $f$  is  $\delta_a$ -integrable in  $\mathcal{M}$ , and we have

$$(259) \quad \int f d\delta_a = f(a).$$

*Proof.* Direct consequence of Definition 656 (Dirac measure), and Lemma 830 (integral for counting measure, with  $Y \stackrel{\text{def.}}{=} \{a\}$ ). □

**Definition 834 (integral for Lebesgue measure on  $\mathbb{R}$ ).**

Let  $a, b \in \overline{\mathbb{R}}$ . Let  $Y$  be an interval with extremities  $\min(a, b)$  and  $\max(a, b)$ . Let  $f : Y \rightarrow \mathbb{R}$ . Assume that  $f$  is  $\lambda$ -integrable over  $Y$  (for the Lebesgue measure  $\lambda$ ). The integral of  $f$  from  $a$  to  $b$  is denoted  $\int_a^b f(x) dx$ .

### 6.11.3 The seminormed vector space $\mathcal{L}^1$

**Lemma 835 (seminorm  $\mathcal{L}^1$ ).**Let  $(X, \Sigma, \mu)$  be a measure space. The function

$$(260) \quad N_1 \stackrel{\text{def.}}{=} \left( f \in \mathcal{M} \mapsto \int |f| d\mu \in \overline{\mathbb{R}}_+ \right)$$

is well-defined; it is called seminorm  $\mathcal{L}^1$ .

*Proof.* Direct consequence of Lemma 581 ( $\mathcal{M}$  is closed under absolute value,  $|f| \in \mathcal{M}_+$ ), Definition 822 (merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ), and Lemma 753 (integral in  $\mathcal{M}_+$ , nonnegativeness).  $\square$

*Remark 836.*

The function  $N_1$  is shown below to be a seminorm on the vector space  $\mathcal{L}^1$ , hence its name.

**Lemma 837 (integrable is finite seminorm  $\mathcal{L}^1$ ).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in X \rightarrow \overline{\mathbb{R}}$ . Then,  $f$  is  $\mu$ -integrable in  $\mathcal{M}$  iff  $f \in \mathcal{M}$  and  $N_1(f) < \infty$ .*

*Proof.* “Left” implies “right”. Direct consequence of Lemma 814 (equivalent definition of integrability), Lemma 753 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability), and Lemma 835 (seminorm  $\mathcal{L}^1$ ).

“Right” implies “left”. Direct consequence of Lemma 581 ( $\mathcal{M}$  is closed under absolute value,  $|f| \in \mathcal{M}_+$ ), Lemma 835 (seminorm  $\mathcal{L}^1$ ), Lemma 753 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability), and Lemma 814 (equivalent definition of integrability).  $\square$

**Lemma 838 (compatibility of  $N_1$  with almost equality).** *Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{M}$ . Assume that  $f \stackrel{\mu \text{ a.e.}}{=} g$ . Then, we have  $N_1(f) = N_1(g)$ .*

*Proof.* Direct consequence of Lemma 835 (seminorm  $\mathcal{L}^1$ ), Lemma 641 (compatibility of almost equality with operator, with the unary operator absolute value), and Lemma 823 (compatibility of integral with almost equality).  $\square$

**Lemma 839 ( $N_1$  is almost definite).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $N_1$  is almost definite:*

$$(261) \quad \forall f \in \mathcal{M}, \quad N_1(f) = 0 \iff f \stackrel{\mu \text{ a.e.}}{=} 0.$$

*Proof.* Direct consequence of Lemma 835 (seminorm  $\mathcal{L}^1$ ), Lemma 769 (integral in  $\mathcal{M}_+$  is almost definite), and Lemma 667 (absolute value is almost definite).  $\square$

**Lemma 840 ( $N_1$  is absolutely homogeneous).**

*Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $N_1$  is absolutely homogeneous of degree 1:*

$$(262) \quad \forall \lambda \in \overline{\mathbb{R}}, \forall f \in \mathcal{M}, \quad N_1(\lambda f) = |\lambda| N_1(f).$$

*Proof.* Direct consequence of Lemma 835 (seminorm  $\mathcal{L}^1$ ), **multiplicativity of the absolute value**, Lemma 756 (integral in  $\mathcal{M}_+$  is positive homogeneous), and Lemma 760 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ).  $\square$

**Lemma 841 (integral is homogeneous).**

*Let  $(X, \Sigma, \mu)$  be a measure space.*

*Let  $f$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $a \in \mathbb{R}$ . Then,  $af$  is  $\mu$ -integrable in  $\mathcal{M}$ , and we have*

$$(263) \quad \int af \, d\mu = a \int f \, d\mu.$$

*Proof.* From Lemma 813 (integrable is measurable,  $f \in \mathcal{M}$ ), Lemma 570 ( $\mathcal{M}$  is closed under scalar multiplication,  $af \in \mathcal{M}$ ), Lemma 840 ( $N_1$  is absolutely homogeneous,  $N_1(af) = |a|N_1(f) < \infty$ ), Lemma 835 (seminorm  $\mathcal{L}^1$ ), Lemma 753 (integral in  $\mathcal{M}_+$ , definition of  $\mu$ -integrability in  $\mathcal{M}_+$ ), and Lemma 814 (equivalent definition of integrability),  $af$  is  $\mu$ -integrable in  $\mathcal{M}$ . Hence, from Lemma 579 (measurability of nonnegative and nonpositive parts), Definition 373 (nonnegative and nonpositive parts), and **ordered set properties of  $\overline{\mathbb{R}}$** , we have  $f^\pm \in \mathcal{M}_+$ ,

$$(af)^\pm = af^\pm \in \mathcal{M}_+ \text{ when } a \geq 0, \quad \text{and} \quad (af)^\pm = -af^\mp \in \mathcal{M}_+ \text{ when } a < 0.$$



Therefore, from Definition 819 (*integral, with  $af$* ), Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous, with  $af^\pm$  or  $-af^\mp$* ), **field properties of  $\mathbb{R}$  (all integrals below are finite)**, and Definition 819 (*integral, with  $f$* ), we have in both cases

$$\int af \, d\mu = \int (af)^+ \, d\mu - \int (af)^- \, d\mu = a \int f^+ \, d\mu - a \int f^- \, d\mu = a \int f \, d\mu.$$

□

*Remark 842.*

In both following statements, the summability domain  $\mathcal{D}^+(f, g)$  and the almost sum  $f \stackrel{\mu \text{ a.e.}}{+} g$  are respectively defined in Definition 659 (*summability domain*), and Lemma 663 (*almost sum*).

**Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f, g \in \mathcal{M}$ . Assume that  $f + g$  is well-defined almost everywhere. Let  $f', g' \in \mathcal{M}$ . Assume that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and we have

$$(264) \quad N_1(f' + g') = N_1(f \stackrel{\mu \text{ a.e.}}{+} g) \leq N_1(f) + N_1(g).$$

*Proof.* From Lemma 663 (*almost sum*), and Lemma 664 (*compatibility of almost sum with almost equality*), we have  $f \stackrel{\mu \text{ a.e.}}{+} g, f' + g' \in \mathcal{M}$  and  $f' + g' \stackrel{\mu \text{ a.e.}}{=} f \stackrel{\mu \text{ a.e.}}{+} g$ . Then, from Lemma 581 ( $\mathcal{M}$  is closed under absolute value), we have  $|f|, |g|, |f'|, |g'|, |f' + g'| \in \mathcal{M}_+$ . Therefore, from Lemma 838 (*compatibility of  $N_1$  with almost equality*), **symmetry of equality**, Lemma 835 (*seminorm  $\mathcal{L}^1$ , with  $f' + g'$* ), Lemma 281 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*), Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*), Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*), Lemma 835 (*seminorm  $\mathcal{L}^1$ , with  $f'$  and  $g'$* ), and Lemma 838 (*compatibility of  $N_1$  with almost equality*), we have

$$\begin{aligned} N_1(f \stackrel{\mu \text{ a.e.}}{+} g) &= N_1(f' + g') = \int |f' + g'| \, d\mu \\ &\leq \int (|f'| + |g'|) \, d\mu = \int |f'| \, d\mu + \int |g'| \, d\mu = N_1(f') + N_1(g') = N_1(f) + N_1(g). \end{aligned}$$

□

**Lemma 844 (*integral is additive*).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g$  be  $\mu$ -integrable in  $\mathcal{M}$ . Let  $f', g' \in \mathcal{M}$ . Assume that  $f' \stackrel{\mu \text{ a.e.}}{=} f$ ,  $g' \stackrel{\mu \text{ a.e.}}{=} g$ , and  $\mathcal{D}^+(f', g') = X$ . Then,  $f + g$  is well-defined almost everywhere,  $f' + g'$  and  $f \stackrel{\mu \text{ a.e.}}{+} g$  are  $\mu$ -integrable in  $\mathcal{M}$ , and we have

$$(265) \quad \int (f' + g') \, d\mu = \int (f \stackrel{\mu \text{ a.e.}}{+} g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

*Proof.* From Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),  $f, g \in \mathcal{M}$ , and  $N_1(f), N_1(g) < \infty$ . From Lemma 816 (*integrable is almost finite*), and Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  $f + g$  is well-defined almost everywhere. Then, from Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ), and **closedness of addition in  $\mathbb{R}_+$** , we have  $f' + g', f \stackrel{\mu \text{ a.e.}}{+} g \in \mathcal{M}$ , and  $N_1(f' + g') = N_1(f \stackrel{\mu \text{ a.e.}}{+} g) < \infty$ . Hence, from Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),  $f' + g'$  and  $f \stackrel{\mu \text{ a.e.}}{+} g$  are  $\mu$ -integrable in  $\mathcal{M}$ . Moreover, from Lemma 823 (*compatibility of integral with almost equality*),  $f'$  and  $g'$  are also  $\mu$ -integrable in  $\mathcal{M}$ .

From Lemma 579 (*measurability of nonnegative and nonpositive parts*), we have

$$f'^{\pm}, g'^{\pm}, (f' + g')^{\pm} \in \mathcal{M}_+.$$

Therefore, from Lemma 823 (*compatibility of integral with almost equality*), **symmetry of equality**, Definition 819 (*integral, with  $f' + g'$* ), Lemma 768 (*compatibility of integral in  $\mathcal{M}_+$  with*

nonpositive and nonnegative parts, with  $f'$  and  $g'$ ), **field properties of  $\mathbb{R}$  (all integrals below are finite)**, Definition 819 (*integral, with  $f'$  and  $g'$* ), and Lemma 823 (*compatibility of integral with almost equality, with  $f$  and  $g$* ), we have

$$\begin{aligned} \int (f \overset{\mu \text{ a.e.}}{+} g) d\mu &= \int (f' + g') d\mu = \int (f' + g')^+ d\mu - \int (f' + g')^- d\mu \\ &= \int f'^+ d\mu + \int g'^+ d\mu - \left( \int f'^- d\mu + \int g'^- d\mu \right) \\ &= \left( \int f'^+ d\mu - \int f'^- d\mu \right) + \left( \int g'^+ d\mu - \int g'^- d\mu \right) \\ &= \int f' d\mu + \int g' d\mu = \int f d\mu + \int g d\mu. \end{aligned}$$

□

**Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions).**

Let  $(X, \Sigma, \mu)$  be a measure space. The *vector space of integrable functions* is denoted  $\mathcal{L}^1(X, \Sigma, \mu)$  (or simply  $\mathcal{L}^1$ ); it is defined by

$$(266) \quad \mathcal{L}^1(X, \Sigma, \mu) \stackrel{\text{def.}}{=} \{f \in \mathcal{M}_{\mathbb{R}} \mid N_1(f) < \infty\}.$$

*Remark 846.* The set  $\mathcal{L}^1$  is shown below to be a (seminormed) vector space, hence its name.

**Lemma 847 (equivalent definition of  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space.

Let  $f : X \rightarrow \overline{\mathbb{R}}$ . Then,  $f \in \mathcal{L}^1$  iff  $f$  is finite and  $\mu$ -integrable in  $\mathcal{M}$ .

*Proof.* “**Left**” implies “**right**”. Direct consequence of Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $f \in \mathcal{M} \cap \mathbb{R}^X$ ), Lemma 835 (seminorm  $\mathcal{L}^1$ ), Lemma 753 (integral in  $\mathcal{M}_+$ ,  $|f|$  is  $\mu$ -integrable in  $\mathcal{M}_+$ ), and Lemma 814 (equivalent definition of integrability).

“**Right**” implies “**left**”. Direct consequence of Lemma 814 (equivalent definition of integrability), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $f \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 753 (integral in  $\mathcal{M}_+$ ), Lemma 835 (seminorm  $\mathcal{L}^1$ ), and Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions).

Therefore, we have the equivalence. □

**Lemma 848 (Minkowski inequality in  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f, g \in \mathcal{L}^1$ . Then, we have  $N_1(f + g) \leq N_1(f) + N_1(g)$ .

*Proof.* Direct consequence of Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions,  $f$  and  $g$  belong to  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), Definition 659 (summability domain,  $\mathcal{D}^+(f, g) = X$ ), Lemma 666 (almost sum is sum), Lemma 626 (everywhere implies almost everywhere), and Lemma 843 (Minkowski inequality in  $\mathcal{M}$ ). □

**Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space).**

Let  $(X, \Sigma, \mu)$  be a measure space. Then,  $(\mathcal{L}^1, N_1)$  is a seminormed vector space.

*Proof.* From Lemma 559 ( $\mathcal{M}_{\mathbb{R}}$  is vector space, with  $0 : X \rightarrow \mathbb{R}$  as zero), Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), Lemma 626 (everywhere implies almost everywhere), and Lemma 839 ( $N_1$  is almost definite,  $N_1(0) = 0$ ),  $\mathcal{L}^1$  is a subset of the vector space  $\mathcal{M}_{\mathbb{R}}$  containing the zero, and we have  $N_1(\mathcal{L}^1) \subset \mathbb{R}$ .

Let  $a \in \mathbb{R}$ . Let  $f, g \in \mathcal{L}^1 \subset \mathcal{M}_{\mathbb{R}}$ . Then, from Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), we have  $N_1(f), N_1(g) < \infty$ . Moreover, from Lemma 559 ( $\mathcal{M}_{\mathbb{R}}$  is vector space), we have

$af, f + g \in \mathcal{M}_{\mathbb{R}}$ . Thus, from Lemma 840 ( $N_1$  is absolutely homogeneous), Lemma 848 (Minkowski inequality in  $\mathcal{L}^1$ ), and **closedness of multiplication and addition in  $\mathbb{R}$** , we have

$$N_1(af) = |a|N_1(f) < \infty \quad \text{and} \quad N_1(f + g) \leq N_1(f) + N_1(g) < \infty$$

Hence, from Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), we have  $af, f + g \in \mathcal{L}^1$ ,  $N_1$  is absolutely homogeneous of degree 1, and  $N_1$  satisfies the triangle inequality.

Therefore, from Lemma 81 (closed under vector operations is subspace,  $\mathcal{L}^1$  is a vector subspace of  $\mathcal{M}_{\mathbb{R}}$ ), Definition 77 (subspace,  $\mathcal{L}^1$  is a vector space), and Definition 399 (seminorm,  $N_1$  is a seminorm over  $\mathcal{L}^1$ ),  $(\mathcal{L}^1, N_1)$  is a seminormed vector space.  $\square$

**Definition 850 (convergence in  $\mathcal{L}^1$ ).** Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{L}^1$ . The sequence  $(f_n)_{n \in \mathbb{N}}$  is said *convergent towards  $f$  in  $\mathcal{L}^1$*  iff  $\lim_{n \rightarrow \infty} N_1(f_n - f) = 0$ .

**Lemma 851 ( $\mathcal{L}^1$  is closed under absolute value).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{L}^1$ . Then, we have  $|f| \in \mathcal{L}^1$ .

*Proof.* Direct consequence of Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), Lemma 581 ( $\mathcal{M}$  is closed under absolute value,  $|f| \in \mathcal{M}_{\mathbb{R}}$ ), **idempotence of the absolute value**, and Lemma 835 (seminorm  $\mathcal{L}^1$ ,  $N_1(|f|) = N_1(f)$ ).  $\square$

**Lemma 852 (bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $f \in \mathcal{M}$ . Then, we have the equivalence

$$(267) \quad f \in \mathcal{L}^1 \iff \exists g \in \mathcal{L}^1, \quad |f| \leq g.$$

*Proof.* “Left” implies “right”. Direct consequence of Lemma 851 ( $\mathcal{L}^1$  is closed under absolute value,  $g \stackrel{\text{def.}}{=} |f| \in \mathcal{L}^1$ ), and **reflexivity of order in  $\mathbb{R}$** .

“Right” implies “left”. From Lemma 847 (equivalent definition of  $\mathcal{L}^1$ , with  $g$ ), Lemma 278 (absolute value in  $\mathbb{R}$  is nonnegative), Lemma 255 (order in  $\mathbb{R}$  is total, transitivity, thus  $|f|$  is finite and  $g$  is nonnegative), Lemma 277 (finite absolute value in  $\mathbb{R}$ ), and Lemma 815 (compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$ , with  $g$ ), we have  $f$  finite and  $g$   $\mu$ -integrable in  $\mathcal{M}_+$ . Hence, from Lemma 818 (bounded by integrable is integrable,  $f$   $\mu$ -integrable in  $\mathcal{M}$ ), and Lemma 847 (equivalent definition of  $\mathcal{L}^1$ , with  $f$ ), we have  $f \in \mathcal{L}^1$ .

Therefore, we have the equivalence.  $\square$

**Lemma 853 (integral is nonnegative linear form on  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $\mathcal{I} : \mathcal{L}^1 \rightarrow \mathbb{R}$  be the function defined by for all  $f \in \mathcal{L}^1$ ,  $\mathcal{I}(f) \stackrel{\text{def.}}{=} \int f d\mu$ . Then,  $\mathcal{I}$  is a nonnegative linear form on  $\mathcal{L}^1$ ; hence it is nondecreasing.

Moreover, for all  $f \in \mathcal{L}^1$ , we have  $|\mathcal{I}(f)| \leq \mathcal{I}(|f|) = N_1(f)$ .

*Proof.* (1). **Linearity.** Direct consequence of Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space), Lemma 841 (integral is homogeneous), Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions,  $\mathcal{L}^1 \subset \mathcal{M}_{\mathbb{R}}$ ), Definition 258 (addition in  $\overline{\mathbb{R}}$ , addition is well-defined in  $\mathcal{L}^1$ ), and Lemma 844 (integral is additive, with  $h \stackrel{\text{def.}}{=} f + g$ ).

(2). **Nonnegativeness.** Let  $f \in \mathcal{L}^1$ . Assume that  $f \geq 0$ . Then, from Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions,  $f \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ,  $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ ), and Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), we have  $f \in \mathcal{M}_+$ . Hence, from the definition of  $\mathcal{I}$ , and Lemma 753 (integral in  $\mathcal{M}_+$ ), we have  $\mathcal{I}(f) = \int f d\mu \geq 0$ .

Therefore, from Definition 64 (linear map), and Definition 66 (linear form),  $\mathcal{I}$  is a nonnegative linear form on  $\mathcal{L}^1$ .

**(3). Monotonicity.** Let  $f, g \in \mathcal{L}^1$ . Assume that  $f \leq g$ . Then, from Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space), Definition 61 (vector space, additive abelian group properties), and ordered field properties of  $\mathbb{R}$ , we have  $g = f + (g - f)$  with  $g - f \geq 0$ . Thus, from the definition of  $\mathcal{I}$  (with  $f$ ), (2), (1), and the definition of  $\mathcal{I}$  (with  $g$ ), we have

$$\mathcal{I}(f) = \int f \, d\mu \leq \int f \, d\mu + \int (g - f) \, d\mu = \int g \, d\mu = \mathcal{I}(g).$$

Hence,  $\mathcal{I}$  is nondecreasing.

**(4). Inequality.** Let  $f \in \mathcal{L}^1$ . Then, from Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions), Lemma 835 (seminorm  $\mathcal{L}^1$ ), Definition 812 (integrability), and Lemma 814 (equivalent definition of integrability),  $f^+$ ,  $f^-$  and  $|f|$  are  $\mu$ -integrable in  $\mathcal{M}_+$ , i.e. all integrals below are finite. Hence, from the definition of  $\mathcal{I}$  (with  $f$ ), Definition 819 (integral), the triangle inequality for the absolute value in  $\mathbb{R}$ , Lemma 375 (nonnegative and nonpositive parts are nonnegative), (2), Lemma 767 (integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts), the definition of  $\mathcal{I}$  (with  $|f|$ ), and Lemma 835 (seminorm  $\mathcal{L}^1$ ), we have

$$|\mathcal{I}(f)| = \left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leq \int f^+ \, d\mu + \int f^- \, d\mu = \int |f| \, d\mu = \mathcal{I}(|f|) = N_1(f).$$

□

**Lemma 854 (constant function is  $\mathcal{L}^1$ ).**

Let  $(X, \Sigma, \mu)$  be a finite measure space. Let  $a \in \mathbb{R}$ . Then,  $f \stackrel{\text{def.}}{=} (x \mapsto a) \in \mathcal{L}^1$ , and we have

$$(268) \quad \int f \, d\mu = a\mu(X).$$

*Proof.* Direct consequence of Definition 467 ( $\sigma$ -algebra,  $X \in \Sigma$ ), Definition 723 ( $\mathcal{SF}$ , vector space of simple functions,  $|f| = |a|\mathbb{1}_X \in \mathcal{SF}_+$ ), Lemma 737 (integral in  $\mathcal{SF}_+$ ,  $\int |f| \, d\mu = |a|\mu(X) < \infty$ ), Lemma 754 (integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$ ), Lemma 820 (compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ), and Definition 819 (integral,  $\int f \, d\mu = \text{sgn}(a) \int |f| \, d\mu = a\mu(X)$ ). □

**Lemma 855 (first mean value theorem).**

Let  $(X, \Sigma, \mu)$  be a finite measure space. Assume that  $\mu$  is nonzero. Let  $f \in \mathcal{M}$ . Assume that  $f$  is bounded. Then,  $f \in \mathcal{L}^1$ , and we have

$$(269) \quad \inf(f(X)) \leq \frac{1}{\mu(X)} \int f \, d\mu \leq \sup(f(X)).$$

Moreover, inequalities are strict iff  $f$  is not equal to one of its bounds  $\mu$ -almost everywhere.

*Proof.* Let  $m \stackrel{\text{def.}}{=} \inf(f(X))$  and  $M = \sup(f(X))$ .

**Case  $0 \leq m \leq M$ .** Then,  $-m \leq M$  and  $|f| \leq M$ . **Case  $m < 0 \leq M$ .** Then,  $|f| \leq \max(-m, M)$ . **Case  $m \leq M < 0$ .** Then,  $M \leq -m$  and  $|f| \leq -m$ . Thus, in all cases, we have  $|f| \leq g$  where  $g = \max(-m, M)$ . Hence, from Lemma 854 (constant function is  $\mathcal{L}^1$ ,  $g \in \mathcal{L}^1$ ), and Lemma 852 (bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$ ), we have  $f \in \mathcal{L}^1$ . Therefore, from Lemma 853 (integral is nonnegative linear form on  $\mathcal{L}^1$ ,  $\mathcal{I}$  is nondecreasing), and ordered field properties of  $\mathbb{R}$  with  $0 < \mu(X) < \infty$ , we have

$$m \leq \frac{1}{\mu(X)} \int f \, d\mu \leq M.$$

Moreover, from Lemma 854 (constant function is  $\mathcal{L}^1$ ,  $(x \mapsto m) \in \mathcal{L}^1$ ), Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space,  $f - m \in \mathcal{L}^1$ ), Lemma 853 (integral is nonnegative linear form on  $\mathcal{L}^1$ ,  $\int (f - m) \, d\mu = 0$ ), Lemma 820 (compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$ ), Lemma 769 (integral

in  $\mathcal{M}_+$  is almost definite, with  $f - m \geq 0$ ), and Lemma 641 (compatibility of almost equality with operator, with the unary operator translation by  $m$ ), we have

$$\int f d\mu = m\mu(X) \iff \int (f - m) d\mu = 0 \iff f - m \stackrel{\mu \text{ a.e.}}{=} 0 \iff f \stackrel{\mu \text{ a.e.}}{=} m.$$

Similarly, with  $M - f \in \mathcal{L}^1 \cap \mathcal{M}_+$ , we have  $\int f d\mu = M\mu(X) \iff f \stackrel{\mu \text{ a.e.}}{=} M$ . Hence, since  $(P \iff Q) \iff (\neg P \iff \neg Q)$ , we have strict inequalities in (269) iff  $f$  is not equal to  $m$  or  $M$   $\mu$ -almost everywhere.  $\square$

**Lemma 856 (variant of first mean value theorem).**

Let  $(X, \Sigma, \mu)$  be a finite measure space. Assume that  $\mu$  is nonzero. Let  $f \in \mathcal{M}$ ,  $m \stackrel{\text{def.}}{=} \inf(f(X))$  and  $M \stackrel{\text{def.}}{=} \sup(f(X))$ . Assume that  $f$  is bounded, not equal to  $m$  or  $M$   $\mu$ -almost everywhere, and  $(m, M) \subset f(X)$ . Then,  $f \in \mathcal{L}^1$ , and we have

$$(270) \quad \exists x \in X, \quad \int f d\mu = f(x)\mu(X).$$

*Proof.* Direct consequence of Lemma 855 (first mean value theorem, with strict inequalities), **ordered field properties of  $\mathbb{R}$  with  $0 < \mu(X) < \infty$** , and Definition 217 (interval, with  $X \stackrel{\text{def.}}{=} \mathbb{R}$ ), there exists  $y \in (m, M)$  such that  $\int f d\mu = y\mu(X)$ .  $\square$

**Theorem 857 (Lebesgue, dominated convergence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}} \in \mathcal{M}$ . Assume that the sequence is pointwise convergent towards  $f$ . Let  $g \in \mathcal{L}^1$ . Assume that for all  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then, for all  $n \in \mathbb{N}$ ,  $f_n \in \mathcal{L}^1$ ,  $f \in \mathcal{L}^1$ , the sequence is convergent towards  $f$  in  $\mathcal{L}^1$ , and we have

$$(271) \quad \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* From Lemma 575 ( $\mathcal{M}$  is closed under limit when pointwise convergent), we have  $f \in \mathcal{M}$ . Moreover, from Lemma 293 (absolute value in  $\overline{\mathbb{R}}$  is continuous), and **monotonicity of the limit in  $\overline{\mathbb{R}}$** , we have  $|f| \leq g$ .

Let  $n \in \mathbb{N}$ . Then, from Lemma 852 (bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$ ), we have  $f_n, f \in \mathcal{L}^1$ . Thus, from Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space), Definition 399 (seminorm), Definition 61 (vector space,  $(\mathcal{L}^1, +)$  is an abelian group), Lemma 851 ( $\mathcal{L}^1$  is closed under absolute value), **linearity and compatibility of the limit with the absolute value**, Lemma 278 (absolute value in  $\overline{\mathbb{R}}$  is nonnegative), and Lemma 280 (absolute value in  $\overline{\mathbb{R}}$  is definite), we have

$$g_n \stackrel{\text{def.}}{=} |f_n - f| \in \mathcal{L}^1 \cap \mathcal{M}_+ \quad \wedge \quad \lim_{n \rightarrow \infty} g_n = 0.$$

Moreover, from **the triangle inequality for the absolute value**, and **ordered field properties of  $\mathbb{R}$** , we have  $g_n \leq |f_n| + |-f| \leq 2g$ . Thus, from Lemma 849 ( $\mathcal{L}^1$  is seminormed vector space), Definition 399 (seminorm), Definition 61 (vector space,  $(\mathcal{L}^1, +)$  is an abelian group), Definition 845 ( $\mathcal{L}^1$ , vector space of integrable functions,  $\mathcal{L}^1 \subset \mathcal{M}$ ), and Definition 578 ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions), we have  $2g - g_n \in \mathcal{L}^1 \cap \mathcal{M}_+$ , and from Lemma 365 (limit inferior and limit superior of pointwise convergent), we have  $\liminf_{n \rightarrow \infty} (2g - g_n) = \lim_{n \rightarrow \infty} (2g - g_n) = 2g$ .

From Theorem 779 (Fatou's lemma, with  $f_n \stackrel{\text{def.}}{=} 2g - g_n$ ), Lemma 853 (integral is nonnegative linear form on  $\mathcal{L}^1$ , linearity), and Lemma 358 (duality limit inferior-limit superior), we have

$$\begin{aligned} 2 \int g d\mu &= \int \liminf_{n \rightarrow \infty} (2g - g_n) d\mu \leq \liminf_{n \rightarrow \infty} \int (2g - g_n) d\mu \\ &= \liminf_{n \rightarrow \infty} \left( 2 \int g d\mu - \int g_n d\mu \right) = 2 \int g d\mu - \limsup_{n \rightarrow \infty} \int g_n d\mu. \end{aligned}$$

Thus, from **ordered field properties of  $\mathbb{R}$  (with  $\int g d\mu$  finite)**,  $\limsup_{n \rightarrow \infty} \int g_n d\mu \leq 0$ . Hence, from Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$ ,  $\int g_n d\mu \geq 0$* ), Lemma 366 (*limit inferior bounded from below,  $\liminf_{n \rightarrow \infty} \int g_n d\mu \geq 0$* ), Lemma 370 (*limit inferior, limit superior and pointwise convergence, with  $\limsup \leq 0 \leq \liminf$* ), the definition of the  $g_n$ 's, and Lemma 835 (*seminorm  $\mathcal{L}^1$* ), we have

$$0 = \liminf_{n \rightarrow \infty} \int g_n d\mu = \limsup_{n \rightarrow \infty} \int g_n d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu = \lim_{n \rightarrow \infty} N_1(f_n - f).$$

Therefore, from Definition 850 (*convergence in  $\mathcal{L}^1$* ),  $(f_n)_{n \in \mathbb{N}}$  is convergent towards  $f$  in  $\mathcal{L}^1$ .

Moreover, from Lemma 278 (*absolute value in  $\mathbb{R}$  is nonnegative*), Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ), and the definition of the  $g_n$ 's, we have

$$0 \leq \left| \int f_n d\mu - \int f d\mu \right| = \left| \int (f_n - f) d\mu \right| \leq \int |f_n - f| d\mu = \int g_n d\mu.$$

Thus, from **the squeeze theorem**, and **linearity of the limit**, we have

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu = \int \lim_{n \rightarrow \infty} f_n d\mu.$$

□

**Theorem 858 (Lebesgue, extended dominated convergence).**

Let  $(X, \Sigma, \mu)$  be a measure space. Let  $(f_n)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$ . Assume that the sequence is  $\mu$ -almost everywhere pointwise convergent towards  $f$ , that  $g$  is  $\mu$ -integrable, and that for all  $n \in \mathbb{N}$ , we have  $|f_n| \stackrel{\mu \text{ a.e.}}{\leq} g$ . Then, for all  $n \in \mathbb{N}$ ,  $f_n$  is  $\mu$ -integrable,  $f$  is  $\mu$ -integrable, the sequence  $(N_1(f_n - f))_{n \in \mathbb{N}}$  converges towards 0, and we have

$$(272) \quad \int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* For all  $n \in \mathbb{N}$ , let

$$\tilde{B} \stackrel{\text{def.}}{=} \{f = \liminf_{n \rightarrow \infty} f_n\} \cap \{f = \limsup_{n \rightarrow \infty} f_n\} \quad \text{and} \quad \tilde{C}_n \stackrel{\text{def.}}{=} \{|f_n| \leq g\}.$$

Let  $n \in \mathbb{N}$ . Then, from Definition 624 (*property almost satisfied*), Definition 614 (*negligible subset*), Definition 467 ( *$\sigma$ -algebra, closedness under complement*), **monotonicity of complement**, and since **involutiveness of complement**, let  $B, C_n \in \Sigma$  such that

$$B \subset \tilde{B}, \quad C_n \subset \tilde{C}_n \quad \text{and} \quad \mu(B^c) = \mu(C_n^c) = 0.$$

Let  $C \stackrel{\text{def.}}{=} \bigcap_{n \in \mathbb{N}} C_n$  and  $D \stackrel{\text{def.}}{=} B \cap C$ . Then, from Lemma 468 (*equivalent definition of  $\sigma$ -algebra, closedness under countable intersection (with  $I = \mathbb{N}$ , then  $\text{card}(I) = 2$ )*), **De Morgan's laws**, and Lemma 621 (*compatibility of null measure with countable union, with  $I = \mathbb{N}$ , then  $\text{card}(I) = 2$ )*), we have  $C, D \in \Sigma$ , and  $\mu(C^c) = \mu(D^c) = 0$ . For all  $n \in \mathbb{N}$ , let

$$A \stackrel{\text{def.}}{=} D \cap g^{-1}(\mathbb{R}_+), \quad \tilde{f}_n \stackrel{\text{def.}}{=} f_n \mathbf{1}_A, \quad \tilde{f} \stackrel{\text{def.}}{=} f \mathbf{1}_A, \quad \tilde{g} \stackrel{\text{def.}}{=} g \mathbf{1}_A.$$

Let  $n \in \mathbb{N}$ . Then, from Lemma 669 (*finite nonnegative part, with  $g$* ), we have  $A \in \Sigma$ ,  $\tilde{g} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_+$ ,  $\mu(A^c) = 0$ , and  $g \stackrel{\mu \text{ a.e.}}{=} \tilde{g}$ . Hence, from Lemma 576 (*measurability and masking, with  $f_n$  and  $f$* ), and Lemma 668 (*masking almost nowhere, with  $f_n$  and  $f$* ), we have  $\tilde{f}_n, \tilde{f} \in \mathcal{M}$ ,  $f_n \stackrel{\mu \text{ a.e.}}{=} \tilde{f}_n$  and  $f \stackrel{\mu \text{ a.e.}}{=} \tilde{f}$ .

From Lemma 641 (*compatibility of almost equality with operator, with the unary operator absolute value*), we have  $|\tilde{g}| \stackrel{\mu \text{ a.e.}}{=} |g|$ . Thus, from Lemma 823 (*compatibility of integral with*

*almost equality, with  $|g|$  and  $|\tilde{g}|$* ), Lemma 814 (*equivalent definition of integrability, with  $g$* ), and Lemma 835 (*seminorm  $\mathcal{L}^1$* ), we have  $N_1(\tilde{g}) = N_1(g) < \infty$ . Hence, from Definition 845 ( $\mathcal{L}^1$ , *vector space of integrable functions*), we have  $\tilde{g} \in \mathcal{L}^1$ .

Let  $x \in X$ . **Case  $x \in A$ .** Then, from **the definition of the indicator function**, and since  $A \subset B \subset \tilde{B}$ , we have  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x) = \tilde{f}(x)$ . Moreover, for all  $n \in \mathbb{N}$ , since  $A \subset C_n \subset \tilde{C}_n$ , we have  $|\tilde{f}_n(x)| = |f_n(x)| \leq g(x) = \tilde{g}(x)$ . **Case  $x \notin A$ .** Then, from **the definition of the indicator function**, we have for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n(x) = \tilde{f}(x) = 0 = \tilde{g}(x)$ . Thus, we have  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = 0 = \tilde{f}(x)$ , and  $|\tilde{f}_n(x)| = 0 \leq 0 = \tilde{g}(x)$ . Hence, in all cases, from Theorem 857 (*Lebesgue, dominated convergence*), we have for all  $n \in \mathbb{N}$ ,  $\tilde{f}_n, \tilde{f} \in \mathcal{L}^1$ , and

$$\int \tilde{f} d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n d\mu.$$

Therefore, from Lemma 641 (*compatibility of almost equality with operator, with the unary operator absolute value,  $|\tilde{f}_n| \stackrel{\mu \text{ a.e.}}{=} |f_n|$  and  $|\tilde{f}| \stackrel{\mu \text{ a.e.}}{=} |f|$* ), Lemma 823 (*compatibility of integral with almost equality, with  $f_n$  and  $\tilde{f}_n$ , then  $f$  and  $\tilde{f}$* ), Lemma 814 (*equivalent definition of integrability, with  $f_n$ , then  $f$* ), Lemma 835 (*seminorm  $\mathcal{L}^1$* ), and Definition 845 ( $\mathcal{L}^1$ , *vector space of integrable functions*), for all  $n \in \mathbb{N}$ ,  $f_n$  and  $f$  are  $\mu$ -integrable in  $\mathcal{M}$ , and

$$\int f d\mu = \int \tilde{f} d\mu = \lim_{n \rightarrow \infty} \int \tilde{f}_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

□



## 7 Conclusions, perspectives

We have presented very detailed proofs of the main basic results in measure theory and Lebesgue integration such as the Beppo Levi (monotone convergence) theorem, Fatou's lemma, the Tonelli theorem, and Lebesgue's dominated convergence theorem.

The short-term purpose of this work was to help the formalization in a formal proof assistant such as Coq of the basic concepts in measure theory and Lebesgue integration and of the proofs of their main properties. A first milestone towards this is dedicated to the integral of nonnegative measurable functions [8] where special attention was paid to the formalization of  $\sigma$ -algebras and of simple functions.

Our mean-term purpose is now to continue up to the formalization of  $L^p$  Lebesgue spaces and of  $W^{m,p}$  Sobolev spaces as Banach spaces, and in particular  $L^2$  and  $H^m \stackrel{\text{def.}}{=} W^{m,2}$  as Hilbert spaces. This will include parts of the distribution theory.

The long-term purpose of these studies is the formal proof of programs implementing the Finite Element Method. As a consequence, after having addressed the formalization of the Lax–Milgram theorem [17, 7], we will also have to write very detailed pen-and-paper proofs for the concepts and results of the interpolation and approximation theory to define the Finite Element Method itself.

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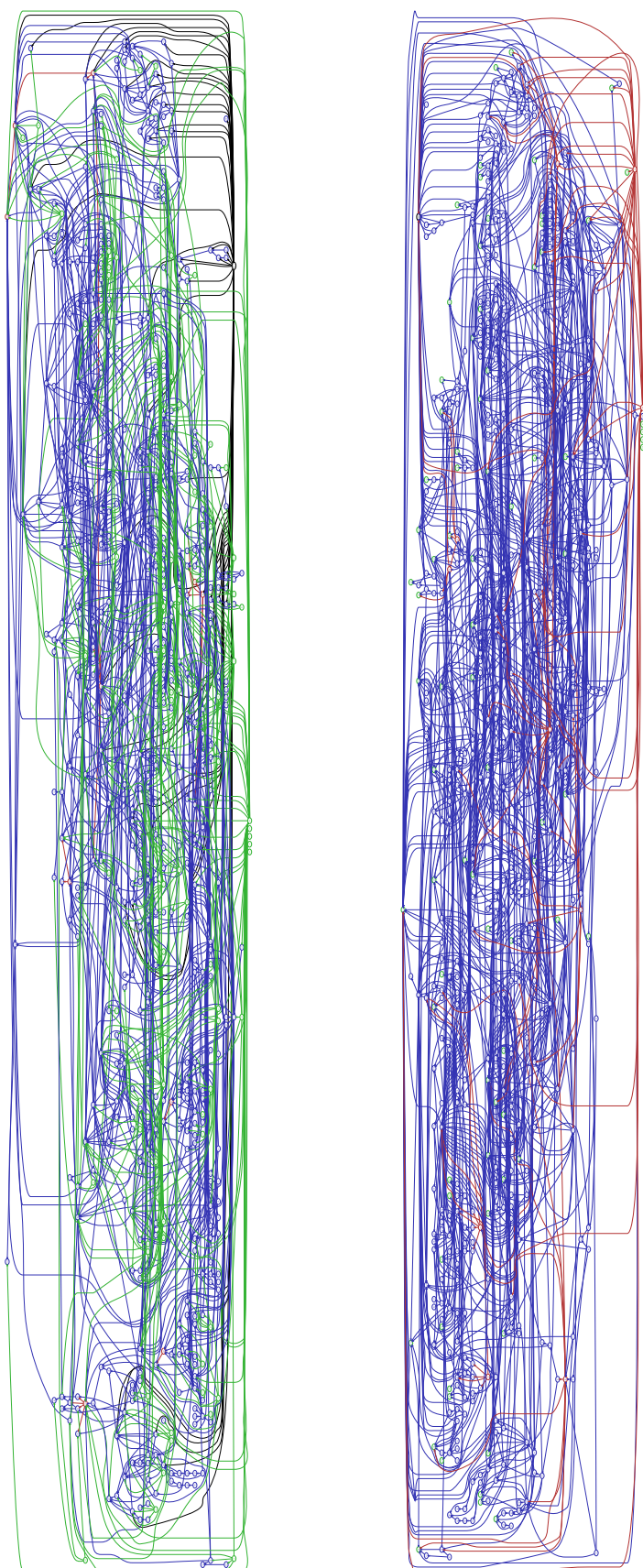


Figure 1: Dependency graphs. Top (or left): depends directly from... Bottom (or right): is a direct dependency of... Arrows are colored according to the nature of their foot: green is for definitions, blue for lemmas, and red for theorems. All dependencies are detailed in Appendices B and C.

## B Depends directly from...

**Lemma 207** (*compatibility of partition with intersection*)  
has no direct dependency.

**Lemma 208** (*technical inclusion for countable union*)  
has no direct dependency.

**Lemma 209** (*order is meaningless in countable union*)  
depends directly from:  
Lemma 208 (*technical inclusion for countable union*).

**Lemma 210** (*definition of double countable union*)  
depends directly from:  
Lemma 209 (*order is meaningless in countable union*).

**Lemma 211** (*double countable union*)  
depends directly from:  
Lemma 210 (*definition of double countable union*).

**Lemma 212** (*partition of countable union*)  
has no direct dependency.

**Lemma 215** (*restriction is masking*)  
has no direct dependency.

**Lemma 219** (*empty open interval*)  
depends directly from:  
Definition 217 (*interval*).

**Lemma 222** (*intervals are closed under finite intersection*)  
depends directly from:  
Definition 217 (*interval*).

**Lemma 223** (*empty intersection of open intervals*)  
depends directly from:  
Lemma 219 (*empty open interval*),  
Lemma 222 (*intervals are closed under finite intersection*).

**Lemma 226** (*intersection of topologies*)  
depends directly from:  
Definition 225 (*topological space*).

**Lemma 228** (*generated topology is minimum*)  
depends directly from:  
Lemma 226 (*intersection of topologies*),  
Definition 227 (*generated topology*).

**Lemma 229** (*equivalent definition of generated topology*)  
depends directly from:  
Definition 225 (*topological space*),  
Lemma 228 (*generated topology is minimum*).

**Lemma 231** (*augmented topological basis*)  
depends directly from:  
Definition 230 (*topological basis*).

**Lemma 234** (*topological basis of order topology*)

depends directly from:

- Lemma 222 (*intervals are closed under finite intersection*),
- Lemma 229 (*equivalent definition of generated topology*),
- Definition 230 (*topological basis*),
- Definition 232 (*order topology*).

**Lemma 236** (*trace topology on subset*)

depends directly from:

- Definition 213 (*trace of subsets of parties*),
- Definition 225 (*topological space*),
- Definition 230 (*topological basis*).

**Lemma 237** (*box topology on Cartesian product*)

depends directly from:

- Definition 214 (*product of subsets of parties*),
- Definition 225 (*topological space*),
- Definition 230 (*topological basis*).

**Lemma 240** (*complete countable topological basis*)

depends directly from:

- Lemma 231 (*augmented topological basis*),
- Definition 238 (*second-countability*).

**Lemma 241** (*compatibility of second-countability with Cartesian product*)

depends directly from:

- Definition 214 (*product of subsets of parties*),
- Lemma 237 (*box topology on Cartesian product*).

**Lemma 242** (*complete countable topological basis of product space*)

depends directly from:

- Definition 214 (*product of subsets of parties*),
- Lemma 237 (*box topology on Cartesian product*),
- Lemma 240 (*complete countable topological basis*).

**Lemma 245** (*equivalent definition of convergent sequence*)

depends directly from:

- Statement(s) from [17].

**Lemma 246** (*convergent subsequence of Cauchy sequence*)

depends directly from:

- Statement(s) from [17].

**Lemma 248** (*finite cover of compact interval*)

has no direct dependency.

**Lemma 250** (*2 is self-Hölder conjugate in  $\mathbb{R}$* )

depends directly from:

- Definition 249 (*Hölder conjugates in  $\mathbb{R}$* ).

**Lemma 251** (*Young's inequality for products in  $\mathbb{R}$* )

depends directly from:

- Definition 249 (*Hölder conjugates in  $\mathbb{R}$* ).

**Lemma 252** (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* )

depends directly from:

- Lemma 250 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),
- Lemma 251 (*Young's inequality for products in  $\mathbb{R}$* ).



**Lemma 255** (*order in  $\overline{\mathbb{R}}$  is total*)

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ).**Lemma 259** (*zero is identity element for addition in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).**Lemma 260** (*addition in  $\overline{\mathbb{R}}$  is associative when defined*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).**Lemma 261** (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).**Lemma 262** (*infinity-sum property in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).**Lemma 263** (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*)

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).**Lemma 266** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*)

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 267** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*)

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 268** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 269** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*)

depends directly from:

Lemma 267 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),Lemma 268 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*).**Lemma 270** (*zero-product property in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 271** (*infinity-product property in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 272** (*finite-product property in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).**Lemma 274** (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* )

depends directly from:

Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).

- Lemma 275** (*bounded absolute value in  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Lemma 263 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 276** (*bounded absolute value in  $\overline{\mathbb{R}}$  (strict)*)  
 depends directly from:  
 Lemma 263 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),  
 Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 277** (*finite absolute value in  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Lemma 276 (*bounded absolute value in  $\overline{\mathbb{R}}$  (strict)*).
- Lemma 278** (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*)  
 depends directly from:  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 279** (*absolute value in  $\overline{\mathbb{R}}$  is even*)  
 depends directly from:  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 280** (*absolute value in  $\overline{\mathbb{R}}$  is definite*)  
 depends directly from:  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 281** (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*)  
 depends directly from:  
 Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 283** (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*)  
 depends directly from:  
 Definition 282 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ).
- Lemma 285** (*exponentiation in  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 282 (*exponential and logarithm in  $\overline{\mathbb{R}}$* ),  
 Definition 284 (*exponentiation in  $\overline{\mathbb{R}}$* ).
- Lemma 286** (*topology of  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Definition 232 (*order topology*),  
 Lemma 234 (*topological basis of order topology*),  
 Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ).
- Lemma 288** (*trace topology on  $\mathbb{R}$* )  
 depends directly from:  
 Definition 230 (*topological basis*),  
 Lemma 234 (*topological basis of order topology*),  
 Lemma 236 (*trace topology on subset*).
- Lemma 290** (*convergence towards  $-\infty$* )  
 depends directly from:  
 Lemma 286 (*topology of  $\overline{\mathbb{R}}$* ).

- Lemma 291** (*continuity of addition in  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).
- Lemma 292** (*continuity of multiplication in  $\overline{\mathbb{R}}$* )  
 depends directly from:  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ).
- Lemma 293** (*absolute value in  $\overline{\mathbb{R}}$  is continuous*)  
 depends directly from:  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ).
- Lemma 294** (*addition in  $\overline{\mathbb{R}}_+$  is closed*)  
 depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ).
- Lemma 295** (*addition in  $\overline{\mathbb{R}}_+$  is associative*)  
 depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 260 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*).
- Lemma 296** (*addition in  $\overline{\mathbb{R}}_+$  is commutative*)  
 depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 261 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*).
- Lemma 297** (*infinity-sum property in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 262 (*infinity-sum property in  $\overline{\mathbb{R}}$* ).
- Lemma 298** (*series are convergent in  $\overline{\mathbb{R}}_+$* )  
 has no direct dependency.
- Lemma 299** (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 298 (*series are convergent in  $\overline{\mathbb{R}}_+$* ).
- Lemma 300** (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 299 (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* ).
- Lemma 301** (*definition of double series in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 300 (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* ).
- Lemma 302** (*double series in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 298 (*series are convergent in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 301 (*definition of double series in  $\overline{\mathbb{R}}_+$* ).
- Lemma 305** (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*)  
 depends directly from:  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 303 (*multiplication in  $\overline{\mathbb{R}}_+$* ).
- Lemma 306** (*zero-product property in  $\overline{\mathbb{R}}_+$* )  
 depends directly from:  
 Lemma 270 (*zero-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 303 (*multiplication in  $\overline{\mathbb{R}}_+$* ).

**Lemma 307** (*infinity-product property in  $\overline{\mathbb{R}}_+$* )

depends directly from:

Lemma 271 (*infinity-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 303 (*multiplication in  $\overline{\mathbb{R}}_+$* ).

**Lemma 308** (*finite-product property in  $\overline{\mathbb{R}}_+$* )

depends directly from:

Lemma 272 (*finite-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 303 (*multiplication in  $\overline{\mathbb{R}}_+$* ).

**Lemma 311** (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

depends directly from:

Lemma 270 (*zero-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 312** (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

depends directly from:

Lemma 271 (*infinity-product property in  $\overline{\mathbb{R}}$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 313** (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*)

depends directly from:

Lemma 312 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 314** (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*)

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 303 (*multiplication in  $\overline{\mathbb{R}}_+$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 316** (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*)

depends directly from:

Lemma 266 (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 317** (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*)

depends directly from:

Lemma 267 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 318** (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*)

depends directly from:

Lemma 268 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),  
 Lemma 269 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 319** (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

depends directly from:

Lemma 306 (*zero-product property in  $\overline{\mathbb{R}}_+$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 320** (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

depends directly from:

Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 321** (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

depends directly from:

Lemma 313 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*).**Lemma 322** (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*)

depends directly from:

Definition 284 (*exponentiation in  $\overline{\mathbb{R}}$* ),Lemma 285 (*exponentiation in  $\overline{\mathbb{R}}$* ),Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*).**Lemma 325** (*Young's inequality for products (measure theory)*)

depends directly from:

Lemma 251 (*Young's inequality for products in  $\mathbb{R}$* ),Definition 258 (*addition in  $\mathbb{R}$* ),Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),Lemma 285 (*exponentiation in  $\overline{\mathbb{R}}$* ),Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),Lemma 305 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*),Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),Lemma 320 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*).**Lemma 326** (*Young's inequality for products, case  $p = 2$  (measure theory)*)

depends directly from:

Lemma 250 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),Lemma 316 (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*),Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*),Lemma 325 (*Young's inequality for products (measure theory)*).**Lemma 328** (*connected component of open subset of  $\mathbb{R}$  is open interval*)

depends directly from:

Statement(s) from [17],

Definition 217 (*interval*),Definition 225 (*topological space*),Definition 327 (*connected component in  $\mathbb{R}$* ).**Lemma 329** (*connected component of open subset of  $\mathbb{R}$  is maximal*)

depends directly from:

Definition 327 (*connected component in  $\mathbb{R}$* ),Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*).**Lemma 330** (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*)

depends directly from:

Lemma 329 (*connected component of open subset of  $\mathbb{R}$  is maximal*).**Theorem 331** (*countable connected components of open subsets of  $\mathbb{R}$* )

depends directly from:

Statement(s) from [17],

Definition 327 (*connected component in  $\mathbb{R}$* ),Lemma 330 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*).**Lemma 332** (*rational approximation of lower bound of open interval*)

has no direct dependency.

**Lemma 333** (*rational approximation of upper bound of open interval*)

depends directly from:

Lemma 332 (*rational approximation of lower bound of open interval*).

**Lemma 334** (*open intervals with rational bounds cover open interval*)

depends directly from:

Statement(s) from [17],

Definition 225 (*topological space*),

Lemma 332 (*rational approximation of lower bound of open interval*),

Lemma 333 (*rational approximation of upper bound of open interval*).

**Theorem 335** ( *$\mathbb{R}$  is second-countable*)

depends directly from:

Definition 230 (*topological basis*),

Definition 238 (*second-countability*),

Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),

Lemma 334 (*open intervals with rational bounds cover open interval*).

**Lemma 336** ( *$\mathbb{R}^n$  is second-countable*)

depends directly from:

Lemma 237 (*box topology on Cartesian product*),

Definition 238 (*second-countability*),

Lemma 241 (*compatibility of second-countability with Cartesian product*),

Theorem 335 ( *$\mathbb{R}$  is second-countable*).

**Lemma 337** (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* )

depends directly from:

Lemma 332 (*rational approximation of lower bound of open interval*),

Lemma 333 (*rational approximation of upper bound of open interval*),

Lemma 334 (*open intervals with rational bounds cover open interval*).

**Lemma 338** ( *$\overline{\mathbb{R}}$  is second-countable*)

depends directly from:

Definition 230 (*topological basis*),

Definition 238 (*second-countability*),

Lemma 286 (*topology of  $\overline{\mathbb{R}}$* ),

Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),

Lemma 337 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Lemma 339** (*extrema of constant function*)

depends directly from:

Statement(s) from [17].

**Lemma 340** (*equivalent definition of finite infimum*)

depends directly from:

Statement(s) from [17],

Lemma 245 (*equivalent definition of convergent sequence*).

**Lemma 341** (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* )

depends directly from:

Statement(s) from [17],

Lemma 339 (*extrema of constant function*),

Lemma 340 (*equivalent definition of finite infimum*).

**Lemma 342** (*equivalent definition of infimum*)

depends directly from:

Statement(s) from [17],

Lemma 290 (*convergence towards  $-\infty$* ),  
Lemma 341 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ).

**Lemma 343 (*infimum is smaller than supremum*)**

depends directly from:  
Statement(s) from [17],  
Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*).

**Lemma 344 (*infimum is monotone*)**

depends directly from:  
Statement(s) from [17],  
Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*).

**Lemma 345 (*supremum is monotone*)**

depends directly from:  
Statement(s) from [17],  
Lemma 344 (*infimum is monotone*).

**Lemma 346 (*compatibility of infimum with absolute value*)**

depends directly from:  
Statement(s) from [17],  
Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ),  
Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
Lemma 343 (*infimum is smaller than supremum*),  
Lemma 344 (*infimum is monotone*),  
Lemma 345 (*supremum is monotone*).

**Lemma 347 (*compatibility of supremum with absolute value*)**

depends directly from:  
Statement(s) from [17],  
Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
Lemma 346 (*compatibility of infimum with absolute value*).

**Lemma 348 (*compatibility of translation with infimum*)**

depends directly from:  
Lemma 344 (*infimum is monotone*).

**Lemma 349 (*compatibility of translation with supremum*)**

depends directly from:  
Lemma 345 (*supremum is monotone*).

**Lemma 350 (*infimum of sequence is monotone*)**

depends directly from:  
Statement(s) from [17],  
Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*).

**Lemma 351 (*supremum of sequence is monotone*)**

depends directly from:  
Statement(s) from [17],  
Lemma 350 (*infimum of sequence is monotone*).

**Lemma 352 (*limit inferior*)**

depends directly from:  
Lemma 344 (*infimum is monotone*).

**Lemma 353 (*limit inferior is  $\infty$* )**

depends directly from:  
Statement(s) from [17],  
Lemma 352 (*limit inferior*).



**Lemma 354** (*equivalent definition of the limit inferior*)

depends directly from:  
 Statement(s) from [17],  
 Definition 247 (*cluster point*),  
 Lemma 352 (*limit inferior*),  
 Lemma 353 (*limit inferior is  $\infty$* ).

**Lemma 355** (*limit inferior is invariant by translation*)

depends directly from:  
 Definition 247 (*cluster point*),  
 Lemma 354 (*equivalent definition of the limit inferior*).

**Lemma 356** (*limit inferior is monotone*)

depends directly from:  
 Lemma 350 (*infimum of sequence is monotone*),  
 Lemma 351 (*supremum of sequence is monotone*),  
 Lemma 352 (*limit inferior*),  
 Lemma 355 (*limit inferior is invariant by translation*).

**Lemma 357** (*limit superior*)

has no direct dependency.

**Lemma 358** (*duality limit inferior-limit superior*)

depends directly from:  
 Statement(s) from [17],  
 Lemma 352 (*limit inferior*),  
 Lemma 357 (*limit superior*).

**Lemma 359** (*equivalent definition of limit superior*)

depends directly from:  
 Lemma 354 (*equivalent definition of the limit inferior*),  
 Lemma 358 (*duality limit inferior-limit superior*).

**Lemma 360** (*limit inferior is smaller than limit superior*)

depends directly from:  
 Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 354 (*equivalent definition of the limit inferior*),  
 Lemma 359 (*equivalent definition of limit superior*).

**Lemma 361** (*limit superior is monotone*)

depends directly from:  
 Lemma 356 (*limit inferior is monotone*),  
 Lemma 358 (*duality limit inferior-limit superior*).

**Lemma 362** (*compatibility of limit inferior with absolute value*)

depends directly from:  
 Lemma 346 (*compatibility of infimum with absolute value*),  
 Lemma 352 (*limit inferior*),  
 Lemma 357 (*limit superior*).

**Lemma 363** (*compatibility of limit superior with absolute value*)

depends directly from:  
 Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*),  
 Lemma 358 (*duality limit inferior-limit superior*),  
 Lemma 362 (*compatibility of limit inferior with absolute value*).

**Lemma 365** (*limit inferior and limit superior of pointwise convergent*)

depends directly from:

Lemma 354 (*equivalent definition of the limit inferior*),

Lemma 359 (*equivalent definition of limit superior*),

Definition 364 (*pointwise convergence*).

**Lemma 366** (*limit inferior bounded from below*)

depends directly from:

Statement(s) from [17],

Lemma 356 (*limit inferior is monotone*),

Lemma 365 (*limit inferior and limit superior of pointwise convergent*).

**Lemma 367** (*limit inferior bounded from above*)

depends directly from:

Statement(s) from [17],

Lemma 356 (*limit inferior is monotone*),

Lemma 365 (*limit inferior and limit superior of pointwise convergent*).

**Lemma 368** (*limit superior bounded from below*)

depends directly from:

Lemma 358 (*duality limit inferior-limit superior*),

Lemma 367 (*limit inferior bounded from above*).

**Lemma 369** (*limit superior bounded from above*)

depends directly from:

Lemma 358 (*duality limit inferior-limit superior*),

Lemma 366 (*limit inferior bounded from below*).

**Lemma 370** (*limit inferior, limit superior and pointwise convergence*)

depends directly from:

Statement(s) from [17],

Lemma 353 (*limit inferior is  $\infty$* ),

Lemma 354 (*equivalent definition of the limit inferior*),

Lemma 358 (*duality limit inferior-limit superior*),

Lemma 359 (*equivalent definition of limit superior*),

Lemma 360 (*limit inferior is smaller than limit superior*).

**Lemma 372** (*finite part is finite*)

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 371 (*finite part*).

**Lemma 374** (*equivalent definition of nonnegative and nonpositive parts*)

depends directly from:

Definition 373 (*nonnegative and nonpositive parts*).

**Lemma 375** (*nonnegative and nonpositive parts are nonnegative*)

depends directly from:

Definition 373 (*nonnegative and nonpositive parts*).

**Lemma 376** (*nonnegative and nonpositive parts are orthogonal*)

has no direct dependency.

**Lemma 377** (*decomposition into nonnegative and nonpositive parts*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),

Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 376 (*nonnegative and nonpositive parts are orthogonal*).

**Lemma 378** (*compatibility of nonpositive and nonnegative parts with addition*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),

Lemma 295 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),

Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),

Lemma 297 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ),

Definition 373 (*nonnegative and nonpositive parts*),

Lemma 375 (*nonnegative and nonpositive parts are nonnegative*),

Lemma 376 (*nonnegative and nonpositive parts are orthogonal*),

Lemma 377 (*decomposition into nonnegative and nonpositive parts*).

**Lemma 379** (*compatibility of nonpositive and nonnegative parts with mask*)

depends directly from:

Definition 373 (*nonnegative and nonpositive parts*).

**Lemma 380** (*compatibility of nonpositive and nonnegative parts with restriction*)

depends directly from:

Definition 373 (*nonnegative and nonpositive parts*).

**Lemma 382** (*quotient vector operations*)

depends directly from:

Definition 381 (*relation compatible with vector operations*).

**Lemma 383** (*quotient vector space, equivalence relation*)

depends directly from:

Statement(s) from [17],

Lemma 382 (*quotient vector operations*).

**Lemma 384** (*quotient vector space*)

depends directly from:

Statement(s) from [17],

Definition 381 (*relation compatible with vector operations*),

Lemma 383 (*quotient vector space, equivalence relation*).

**Lemma 385** (*linear map on quotient vector space*)

depends directly from:

Statement(s) from [17],

Lemma 382 (*quotient vector operations*),

Lemma 384 (*quotient vector space*).

**Lemma 390** ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*)

depends directly from:

Definition 388 (*algebra over a field*).

**Lemma 393** (*algebra of functions to algebra*)

depends directly from:

Statement(s) from [17],

Definition 388 (*algebra over a field*),

Definition 391 (*inherited algebra operations*).

**Lemma 394** ( *$\mathbb{K}^X$  is algebra*)

depends directly from:

Lemma 390 ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*),

Lemma 393 (*algebra of functions to algebra*).

**Lemma 397** (*vector subspace and closed under multiplication is subalgebra*)

depends directly from:

Statement(s) from [17],  
 Definition 388 (*algebra over a field*),  
 Definition 395 (*subalgebra*).

**Lemma 398** (*closed under algebra operations is subalgebra*)

depends directly from:  
 Statement(s) from [17],  
 Lemma 397 (*vector subspace and closed under multiplication is subalgebra*).

**Lemma 401** (*definite seminorm is norm*)

depends directly from:  
 Statement(s) from [17],  
 Definition 399 (*seminorm*).

**Lemma 404** (*nonempty and with empty or full*)

has no direct dependency.

**Lemma 405** (*with empty and full*)

has no direct dependency.

**Lemma 407** (*closedness under local complement and complement*)

has no direct dependency.

**Lemma 408** (*closedness under disjoint union and local complement*)

has no direct dependency.

**Lemma 409** (*closedness under set difference and local complement*)

has no direct dependency.

**Lemma 410** (*closedness under intersection and set difference*)

has no direct dependency.

**Lemma 411** (*closedness under union and intersection*)

has no direct dependency.

**Lemma 412** (*closedness under union and set difference*)

depends directly from:  
 Lemma 410 (*closedness under intersection and set difference*),  
 Lemma 411 (*closedness under union and intersection*).

**Lemma 413** (*closedness under finite operations*)

has no direct dependency.

**Lemma 414** (*closedness under finite union and intersection*)

depends directly from:  
 Lemma 411 (*closedness under union and intersection*),  
 Lemma 413 (*closedness under finite operations*).

**Lemma 417** (*closedness under countable and finite disjoint union*)

has no direct dependency.

**Lemma 418** (*closedness under countable disjoint union and local complement*)

depends directly from:  
 Lemma 405 (*with empty and full*),  
 Lemma 408 (*closedness under disjoint union and local complement*),  
 Lemma 413 (*closedness under finite operations*),  
 Lemma 417 (*closedness under countable and finite disjoint union*).

**Lemma 419** (*closedness under countable union and intersection*)

has no direct dependency.

**Lemma 420** (*closedness under countable disjoint and monotone union*)

depends directly from:

Lemma 212 (*partition of countable union*).

**Lemma 421** (*closedness under countable monotone and disjoint union*)

has no direct dependency.

**Lemma 422** (*closedness under countable disjoint union and countable union*)

depends directly from:

Lemma 212 (*partition of countable union*),

Lemma 410 (*closedness under intersection and set difference*),

Lemma 411 (*closedness under union and intersection*),

Lemma 413 (*closedness under finite operations*).

**Lemma 423** (*closedness under countable monotone union and countable union*)

depends directly from:

Lemma 409 (*closedness under set difference and local complement*),

Lemma 410 (*closedness under intersection and set difference*),

Lemma 411 (*closedness under union and intersection*),

Lemma 421 (*closedness under countable monotone and disjoint union*),

Lemma 422 (*closedness under countable disjoint union and countable union*).

**Lemma 426** (*intersection of  $\pi$ -systems*)

depends directly from:

Definition 424 ( *$\pi$ -system*).

**Lemma 428** (*generated  $\pi$ -system is minimum*)

depends directly from:

Lemma 426 (*intersection of  $\pi$ -systems*),

Definition 427 (*generated  $\pi$ -system*).

**Lemma 429** ( *$\pi$ -system generation is monotone*)

depends directly from:

Lemma 428 (*generated  $\pi$ -system is minimum*).

**Lemma 430** ( *$\pi$ -system generation is idempotent*)

depends directly from:

Lemma 428 (*generated  $\pi$ -system is minimum*).

**Lemma 433** (*equivalent definition of set algebra*)

depends directly from:

Lemma 404 (*nonempty and with empty or full*),

Lemma 405 (*with empty and full*),

Lemma 413 (*closedness under finite operations*),

Lemma 414 (*closedness under finite union and intersection*),

Definition 432 (*set algebra*).

**Lemma 434** (*other equivalent definition of set algebra*)

depends directly from:

Lemma 407 (*closedness under local complement and complement*),

Lemma 409 (*closedness under set difference and local complement*),

Lemma 410 (*closedness under intersection and set difference*),

Lemma 433 (*equivalent definition of set algebra*).

**Lemma 435** (*set algebra is closed under local complement*)

depends directly from:

Lemma 409 (*closedness under set difference and local complement*),Lemma 434 (*other equivalent definition of set algebra*).**Lemma 436** (*intersection of set algebras*)

depends directly from:

Definition 432 (*set algebra*).**Lemma 438** (*generated set algebra is minimum*)

depends directly from:

Lemma 436 (*intersection of set algebras*),Definition 437 (*generated set algebra*).**Lemma 439** (*set algebra generation is monotone*)

depends directly from:

Lemma 428 (*generated  $\pi$ -system is minimum*).**Lemma 440** (*set algebra generation is idempotent*)

depends directly from:

Lemma 438 (*generated set algebra is minimum*).**Lemma 441** (*explicit set algebra*)

depends directly from:

Lemma 413 (*closedness under finite operations*),Definition 432 (*set algebra*),Lemma 433 (*equivalent definition of set algebra*),Lemma 438 (*generated set algebra is minimum*).**Lemma 443** (*intersection of monotone classes*)

depends directly from:

Definition 442 (*monotone class*).**Lemma 445** (*generated monotone class is minimum*)

depends directly from:

Lemma 443 (*intersection of monotone classes*),Definition 444 (*generated monotone class*).**Lemma 446** (*monotone class generation is monotone*)

depends directly from:

Lemma 445 (*generated monotone class is minimum*).**Lemma 447** (*monotone class generation is idempotent*)

depends directly from:

Lemma 445 (*generated monotone class is minimum*).**Lemma 449** ( *$\mathcal{C}^{\setminus}$  is symmetric*)

depends directly from:

Definition 448 (*monotone class and symmetric set difference*).**Lemma 450** ( *$\mathcal{C}^{\setminus}$  is monotone class*)

depends directly from:

Definition 442 (*monotone class*),Definition 448 (*monotone class and symmetric set difference*).**Lemma 451** (*monotone class closed under set difference*)

depends directly from:

Lemma 445 (*generated monotone class is minimum*),

Definition 448 (*monotone class and symmetric set difference*),

Lemma 449 ( *$\mathcal{C}^{\setminus}$  is symmetric*),

Lemma 450 ( *$\mathcal{C}^{\setminus}$  is monotone class*).

**Lemma 452 (*monotone class generated by set algebra*)**

depends directly from:

Lemma 434 (*other equivalent definition of set algebra*),

Lemma 445 (*generated monotone class is minimum*),

Lemma 451 (*monotone class closed under set difference*).

**Lemma 454 (*equivalent definition of  $\lambda$ -system*)**

depends directly from:

Lemma 407 (*closedness under local complement and complement*),

Lemma 418 (*closedness under countable disjoint union and local complement*),

Lemma 420 (*closedness under countable disjoint and monotone union*),

Lemma 421 (*closedness under countable monotone and disjoint union*),

Definition 453 ( *$\lambda$ -system*).

**Lemma 455 (*other properties of  $\lambda$ -system*)**

depends directly from:

Lemma 404 (*nonempty and with empty or full*),

Lemma 405 (*with empty and full*),

Lemma 419 (*closedness under countable union and intersection*),

Definition 453 ( *$\lambda$ -system*),

Lemma 454 (*equivalent definition of  $\lambda$ -system*).

**Lemma 456 (*intersection of  $\lambda$ -systems*)**

depends directly from:

Definition 453 ( *$\lambda$ -system*).

**Lemma 458 (*generated  $\lambda$ -system is minimum*)**

depends directly from:

Lemma 456 (*intersection of  $\lambda$ -systems*),

Definition 457 (*generated  $\lambda$ -system*).

**Lemma 459 ( *$\lambda$ -system generation is monotone*)**

depends directly from:

Lemma 458 (*generated  $\lambda$ -system is minimum*).

**Lemma 460 ( *$\lambda$ -system generation is idempotent*)**

depends directly from:

Lemma 458 (*generated  $\lambda$ -system is minimum*).

**Lemma 462 ( *$\Lambda^{\cap}$  is symmetric*)**

depends directly from:

Definition 461 ( *$\lambda$ -system and intersection*).

**Lemma 463 ( *$\Lambda^{\cap}$  is  $\lambda$ -system*)**

depends directly from:

Definition 453 ( *$\lambda$ -system*),

Lemma 454 (*equivalent definition of  $\lambda$ -system*).

**Lemma 464 ( *$\lambda$ -system with intersection*)**

depends directly from:

Lemma 458 (*generated  $\lambda$ -system is minimum*),

Definition 461 ( *$\lambda$ -system and intersection*),

Lemma 462 ( *$\Lambda^{\cap}$  is symmetric*),

Lemma 463 ( *$\Lambda^{\cap}$  is  $\lambda$ -system*).



**Lemma 465** ( *$\lambda$ -system closed under intersection*)

depends directly from:

Definition 461 ( *$\lambda$ -system and intersection*),Lemma 464 ( *$\lambda$ -system with intersection*).**Lemma 466** ( *$\lambda$ -system generated by  $\pi$ -system*)

depends directly from:

Lemma 413 (*closedness under finite operations*),Definition 424 ( *$\pi$ -system*),Lemma 458 (*generated  $\lambda$ -system is minimum*),Lemma 465 ( *$\lambda$ -system closed under intersection*).**Lemma 468** (*equivalent definition of  $\sigma$ -algebra*)

depends directly from:

Lemma 404 (*nonempty and with empty or full*),Lemma 405 (*with empty and full*),Lemma 419 (*closedness under countable union and intersection*),Definition 467 ( *$\sigma$ -algebra*).**Lemma 469** ( *$\sigma$ -algebra is set algebra*)

depends directly from:

Definition 432 (*set algebra*),Definition 467 ( *$\sigma$ -algebra*).**Lemma 470** ( *$\sigma$ -algebra is closed under set difference*)

depends directly from:

Lemma 434 (*other equivalent definition of set algebra*),Lemma 435 (*set algebra is closed under local complement*),Lemma 469 ( *$\sigma$ -algebra is set algebra*).**Lemma 471** (*other properties of  $\sigma$ -algebra*)

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).**Lemma 472** (*intersection of  $\sigma$ -algebras*)

depends directly from:

Definition 467 ( *$\sigma$ -algebra*).**Lemma 474** (*generated  $\sigma$ -algebra is minimum*)

depends directly from:

Lemma 472 (*intersection of  $\sigma$ -algebras*),Definition 473 (*generated  $\sigma$ -algebra*).**Lemma 475** ( *$\sigma$ -algebra generation is monotone*)

depends directly from:

Lemma 474 (*generated  $\sigma$ -algebra is minimum*).**Lemma 476** ( *$\sigma$ -algebra generation is idempotent*)

depends directly from:

Lemma 474 (*generated  $\sigma$ -algebra is minimum*).**Lemma 477** ( *$\sigma$ -algebra is  $\pi$ -system*)

depends directly from:

Definition 424 ( *$\pi$ -system*),Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 478** ( *$\sigma$ -algebra contains  $\pi$ -system*)

depends directly from:

- Lemma 428 (*generated  $\pi$ -system is minimum*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Lemma 477 ( *$\sigma$ -algebra is  $\pi$ -system*).

**Lemma 479** ( *$\pi$ -system contains  $\sigma$ -algebra*)

depends directly from:

- Lemma 404 (*nonempty and with empty or full*),
- Lemma 422 (*closedness under countable disjoint union and countable union*),
- Definition 424 ( *$\pi$ -system*),
- Lemma 428 (*generated  $\pi$ -system is minimum*),
- Definition 467 ( *$\sigma$ -algebra*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Lemma 480** ( *$\sigma$ -algebra generated by  $\pi$ -system*)

depends directly from:

- Lemma 428 (*generated  $\pi$ -system is minimum*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Lemma 475 ( *$\sigma$ -algebra generation is monotone*),
- Lemma 478 ( *$\sigma$ -algebra contains  $\pi$ -system*).

**Lemma 481** ( *$\sigma$ -algebra contains set algebra*)

depends directly from:

- Lemma 438 (*generated set algebra is minimum*),
- Lemma 469 ( *$\sigma$ -algebra is set algebra*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Lemma 482** (*set algebra contains  $\sigma$ -algebra*)

depends directly from:

- Lemma 423 (*closedness under countable monotone union and countable union*),
- Definition 432 (*set algebra*),
- Lemma 438 (*generated set algebra is minimum*),
- Definition 467 ( *$\sigma$ -algebra*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Lemma 483** ( *$\sigma$ -algebra generated by set algebra*)

depends directly from:

- Lemma 438 (*generated set algebra is minimum*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Lemma 475 ( *$\sigma$ -algebra generation is monotone*),
- Lemma 481 ( *$\sigma$ -algebra contains set algebra*).

**Lemma 484** ( *$\sigma$ -algebra is monotone class*)

depends directly from:

- Definition 442 (*monotone class*),
- Lemma 471 (*other properties of  $\sigma$ -algebra*).

**Lemma 485** ( *$\sigma$ -algebra contains monotone class*)

depends directly from:

- Lemma 445 (*generated monotone class is minimum*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Lemma 484 ( *$\sigma$ -algebra is monotone class*).

**Lemma 486** (*monotone class contains  $\sigma$ -algebra*)

depends directly from:

- Lemma 423 (*closedness under countable monotone union and countable union*),

Definition 442 (*monotone class*),  
 Lemma 445 (*generated monotone class is minimum*),  
 Definition 467 ( $\sigma$ -algebra),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Lemma 487 ( $\sigma$ -algebra generated by monotone class)**

depends directly from:  
 Lemma 445 (*generated monotone class is minimum*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 485 ( *$\sigma$ -algebra contains monotone class*).

**Lemma 488 ( $\sigma$ -algebra is  $\lambda$ -system)**

depends directly from:  
 Definition 453 ( *$\lambda$ -system*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 471 (*other properties of  $\sigma$ -algebra*).

**Lemma 489 ( $\sigma$ -algebra contains  $\lambda$ -system)**

depends directly from:  
 Lemma 458 (*generated  $\lambda$ -system is minimum*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 488 ( *$\sigma$ -algebra is  $\lambda$ -system*).

**Lemma 490 ( $\lambda$ -system contains  $\sigma$ -algebra)**

depends directly from:  
 Lemma 422 (*closedness under countable disjoint union and countable union*),  
 Definition 453 ( *$\lambda$ -system*),  
 Lemma 458 (*generated  $\lambda$ -system is minimum*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Lemma 491 ( $\sigma$ -algebra generated by  $\lambda$ -system)**

depends directly from:  
 Lemma 458 (*generated  $\lambda$ -system is minimum*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*).

**Lemma 492 (*other  $\sigma$ -algebra generator*)**

depends directly from:  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 476 ( *$\sigma$ -algebra generation is idempotent*).

**Lemma 493 (*complete generated  $\sigma$ -algebra*)**

depends directly from:  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 492 (*other  $\sigma$ -algebra generator*).

**Lemma 494 (*countable  $\sigma$ -algebra generator*)**

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 492 (*other  $\sigma$ -algebra generator*).

**Lemma 496** (*some Borel subsets*)

depends directly from:

Definition 225 (*topological space*),

Definition 467 ( *$\sigma$ -algebra*),

Lemma 474 (*generated  $\sigma$ -algebra is minimum*),

Definition 495 (*Borel  $\sigma$ -algebra*).

**Lemma 497** (*countable Borel  $\sigma$ -algebra generator*)

depends directly from:

Lemma 494 (*countable  $\sigma$ -algebra generator*),

Definition 495 (*Borel  $\sigma$ -algebra*).

**Lemma 501** (*set algebra generated by product of  $\sigma$ -algebras*)

depends directly from:

Definition 214 (*product of subsets of parties*),

Lemma 441 (*explicit set algebra*),

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 502** ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*)

depends directly from:

Definition 424 ( *$\pi$ -system*),

Lemma 460 ( *$\lambda$ -system generation is idempotent*),

Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),

Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*),

Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*).

**Theorem 503** (*Dynkin  $\pi$ - $\lambda$  theorem*)

depends directly from:

Lemma 458 (*generated  $\lambda$ -system is minimum*),

Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*),

Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),

Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),

Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*).

**Lemma 505** (*usage of Dynkin  $\pi$ - $\lambda$  theorem*)

depends directly from:

Lemma 428 (*generated  $\pi$ -system is minimum*),

Lemma 459 ( *$\lambda$ -system generation is monotone*),

Lemma 460 ( *$\lambda$ -system generation is idempotent*),

Definition 473 (*generated  $\sigma$ -algebra*),

Lemma 475 ( *$\sigma$ -algebra generation is monotone*),

Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 506** (*algebra and monotone class is  $\sigma$ -algebra*)

depends directly from:

Definition 432 (*set algebra*),

Lemma 447 (*monotone class generation is idempotent*),

Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),

Lemma 485 ( *$\sigma$ -algebra contains monotone class*),

Lemma 486 (*monotone class contains  $\sigma$ -algebra*).

**Theorem 507** (*monotone class*)

depends directly from:

Lemma 445 (*generated monotone class is minimum*),

Lemma 452 (*monotone class generated by set algebra*),

Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),

Lemma 487 ( *$\sigma$ -algebra generated by monotone class*),  
 Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*).

**Lemma 509 (*usage of monotone class theorem*)**

depends directly from:

Lemma 438 (*generated set algebra is minimum*),  
 Lemma 446 (*monotone class generation is monotone*),  
 Lemma 447 (*monotone class generation is idempotent*),  
 Definition 473 (*generated  $\sigma$ -algebra*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Theorem 507 (*monotone class*).

**Lemma 511 (*inverse  $\sigma$ -algebra*)**

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Definition 510 (*measurable function*).

**Lemma 512 (*image  $\sigma$ -algebra*)**

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Definition 510 (*measurable function*).

**Lemma 513 (*identity function is measurable*)**

depends directly from:

Definition 510 (*measurable function*).

**Lemma 514 (*constant function is measurable*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 515 (*inverse image of generating family*)**

depends directly from:

Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),  
 Lemma 511 (*inverse  $\sigma$ -algebra*),  
 Lemma 512 (*image  $\sigma$ -algebra*).

**Lemma 516 (*equivalent definition of measurable function*)**

depends directly from:

Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),  
 Definition 510 (*measurable function*),  
 Lemma 515 (*inverse image of generating family*).

**Lemma 517 (*continuous is measurable*)**

depends directly from:

Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Definition 495 (*Borel  $\sigma$ -algebra*),  
 Definition 510 (*measurable function*),  
 Lemma 516 (*equivalent definition of measurable function*).

**Lemma 518 (*compatibility of measurability with composition*)**

depends directly from:

Definition 510 (*measurable function*).

**Lemma 520** (*trace  $\sigma$ -algebra*)

depends directly from:

- Definition 213 (*trace of subsets of parties*),
- Definition 467 ( *$\sigma$ -algebra*),
- Definition 510 (*measurable function*).

**Lemma 521** (*measurability of measurable subspace*)

depends directly from:

- Definition 213 (*trace of subsets of parties*),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Lemma 520 (*trace  $\sigma$ -algebra*).

**Lemma 522** (*generating measurable subspace*)

depends directly from:

- Definition 213 (*trace of subsets of parties*),
- Lemma 515 (*inverse image of generating family*).

**Lemma 523** (*Borel sub- $\sigma$ -algebra*)

depends directly from:

- Definition 495 (*Borel  $\sigma$ -algebra*),
- Lemma 521 (*measurability of measurable subspace*),
- Lemma 522 (*generating measurable subspace*).

**Lemma 524** (*characterization of Borel subsets*)

depends directly from:

- Lemma 207 (*compatibility of partition with intersection*),
- Definition 467 ( *$\sigma$ -algebra*),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Definition 495 (*Borel  $\sigma$ -algebra*),
- Lemma 523 (*Borel sub- $\sigma$ -algebra*).

**Lemma 525** (*source restriction of measurable function*)

depends directly from:

- Definition 213 (*trace of subsets of parties*),
- Definition 510 (*measurable function*),
- Lemma 518 (*compatibility of measurability with composition*),
- Lemma 520 (*trace  $\sigma$ -algebra*).

**Lemma 526** (*destination restriction of measurable function*)

depends directly from:

- Definition 510 (*measurable function*).

**Lemma 527** (*measurability of function defined on a partition*)

depends directly from:

- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Definition 510 (*measurable function*).

**Lemma 530** (*product of measurable subsets is measurable*)

depends directly from:

- Definition 214 (*product of subsets of parties*),
- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Definition 529 (*tensor product of  $\sigma$ -algebras*).

**Lemma 531** (*measurability of function to product space*)

depends directly from:

- Definition 214 (*product of subsets of parties*),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Definition 510 (*measurable function*),  
 Lemma 516 (*equivalent definition of measurable function*),  
 Definition 529 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 530 (*product of measurable subsets is measurable*).

**Lemma 532 (*canonical projection is measurable*)**

depends directly from:

Definition 510 (*measurable function*),  
 Lemma 531 (*measurability of function to product space*).

**Lemma 533 (*generating product measurable space*)**

depends directly from:

Definition 214 (*product of subsets of parties*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Lemma 475 ( *$\sigma$ -algebra generation is monotone*),  
 Definition 510 (*measurable function*),  
 Lemma 515 (*inverse image of generating family*),  
 Definition 529 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 532 (*canonical projection is measurable*).

**Lemma 536 (*section of product*)**

depends directly from:

Definition 535 (*section in Cartesian product*).

**Lemma 537 (*compatibility of section with set operations*)**

depends directly from:

Definition 535 (*section in Cartesian product*).

**Lemma 538 (*measurability of section*)**

depends directly from:

Definition 214 (*product of subsets of parties*),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 474 (*generated  $\sigma$ -algebra is minimum*),  
 Definition 529 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 536 (*section of product*),  
 Lemma 537 (*compatibility of section with set operations*).

**Lemma 539 (*indicator of section*)**

depends directly from:

Definition 535 (*section in Cartesian product*).

**Lemma 540 (*measurability of function from product space*)**

depends directly from:

Definition 510 (*measurable function*),  
 Lemma 538 (*measurability of section*).

**Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* )**

depends directly from:

Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 473 (*generated  $\sigma$ -algebra*),  
 Lemma 492 (*other  $\sigma$ -algebra generator*),  
 Lemma 496 (*some Borel subsets*),  
 Lemma 497 (*countable Borel  $\sigma$ -algebra generator*).



**Lemma 544** (*countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$* )

depends directly from:

- Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Theorem 335 ( *$\mathbb{R}$  is second-countable*),
- Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ).

**Lemma 545** (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* )

depends directly from:

- Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Definition 473 (*generated  $\sigma$ -algebra*),
- Lemma 492 (*other  $\sigma$ -algebra generator*),
- Lemma 497 (*countable Borel  $\sigma$ -algebra generator*).

**Lemma 546** (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* )

depends directly from:

- Definition 467 ( *$\sigma$ -algebra*),
- Definition 495 (*Borel  $\sigma$ -algebra*),
- Lemma 523 (*Borel sub- $\sigma$ -algebra*),
- Lemma 524 (*characterization of Borel subsets*).

**Lemma 548** (*Borel  $\sigma$ -algebra of  $\mathbb{R}_+$* )

depends directly from:

- Definition 467 ( *$\sigma$ -algebra*),
- Lemma 523 (*Borel sub- $\sigma$ -algebra*),
- Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ).

**Lemma 549** (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}_+}$* )

depends directly from:

- Lemma 522 (*generating measurable subspace*),
- Lemma 545 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ).

**Lemma 550** (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* )

depends directly from:

- Lemma 242 (*complete countable topological basis of product space*),
- Lemma 336 ( *$\mathbb{R}^n$  is second-countable*),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Lemma 493 (*complete generated  $\sigma$ -algebra*),
- Lemma 497 (*countable Borel  $\sigma$ -algebra generator*),
- Lemma 533 (*generating product measurable space*),
- Lemma 544 (*countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$* ).

**Lemma 554** (*measurability of indicator function*)

depends directly from:

- Definition 467 ( *$\sigma$ -algebra*),
- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Definition 552 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ).

**Lemma 555** (*measurability of numeric function to  $\mathbb{R}$* )

depends directly from:

- Definition 495 (*Borel  $\sigma$ -algebra*),
- Lemma 496 (*some Borel subsets*),
- Definition 510 (*measurable function*),
- Lemma 516 (*equivalent definition of measurable function*),
- Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),
- Definition 552 ( *$\mathcal{M}_{\mathbb{R}}$ , vector space of measurable numeric functions to  $\mathbb{R}$* ).

**Lemma 556** (*inverse image is measurable in  $\mathbb{R}$* )

depends directly from:

Lemma 496 (*some Borel subsets*),

Definition 510 (*measurable function*),

Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ).

**Lemma 557** ( $\mathcal{M}_{\mathbb{R}}$  *is algebra*)

depends directly from:

Lemma 390 ( $\mathbb{K}$  *is  $\mathbb{K}$ -algebra*),

Lemma 393 (*algebra of functions to algebra*),

Lemma 398 (*closed under algebra operations is subalgebra*),

Lemma 514 (*constant function is measurable*),

Lemma 517 (*continuous is measurable*),

Lemma 518 (*compatibility of measurability with composition*),

Lemma 531 (*measurability of function to product space*),

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ),

Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ).

**Lemma 559** ( $\mathcal{M}_{\mathbb{R}}$  *is vector space*)

depends directly from:

Definition 388 (*algebra over a field*),

Definition 395 (*subalgebra*),

Lemma 398 (*closed under algebra operations is subalgebra*),

Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  *is algebra*).

**Lemma 562** ( $\mathcal{M}$  *and finite is  $\mathcal{M}_{\mathbb{R}}$* )

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 510 (*measurable function*),

Lemma 546 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),

Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ),

Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions*).

**Lemma 563** (*measurability of numeric function*)

depends directly from:

Definition 495 (*Borel  $\sigma$ -algebra*),

Lemma 496 (*some Borel subsets*),

Definition 510 (*measurable function*),

Lemma 516 (*equivalent definition of measurable function*),

Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ),

Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions*).

**Lemma 564** (*inverse image is measurable*)

depends directly from:

Lemma 496 (*some Borel subsets*),

Definition 510 (*measurable function*),

Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions*).

**Lemma 565** ( $\mathcal{M}$  *is closed under finite part*)

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Definition 371 (*finite part*),

Lemma 372 (*finite part is finite*),

Definition 467 ( $\sigma$ -*algebra*),

Lemma 514 (*constant function is measurable*),

Lemma 527 (*measurability of function defined on a partition*),

Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 564 (*inverse image is measurable*).

**Lemma 566** ( *$\mathcal{M}$  is closed under addition when defined*)

depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 371 (*finite part*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 495 (*Borel  $\sigma$ -algebra*),  
 Definition 510 (*measurable function*),  
 Lemma 514 (*constant function is measurable*),  
 Lemma 527 (*measurability of function defined on a partition*),  
 Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),  
 Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*),  
 Lemma 564 (*inverse image is measurable*),  
 Lemma 565 ( *$\mathcal{M}$  is closed under finite part*).

**Lemma 567** ( *$\mathcal{M}$  is closed under finite sum*)

depends directly from:  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*).

**Lemma 568** ( *$\mathcal{M}$  is closed under multiplication*)

depends directly from:  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Definition 371 (*finite part*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 495 (*Borel  $\sigma$ -algebra*),  
 Definition 510 (*measurable function*),  
 Lemma 514 (*constant function is measurable*),  
 Lemma 527 (*measurability of function defined on a partition*),  
 Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),  
 Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*),  
 Lemma 563 (*measurability of numeric function*),  
 Lemma 564 (*inverse image is measurable*),  
 Lemma 565 ( *$\mathcal{M}$  is closed under finite part*).

**Lemma 569** ( *$\mathcal{M}$  is closed under finite product*)

depends directly from:  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 570** ( *$\mathcal{M}$  is closed under scalar multiplication*)

depends directly from:  
 Lemma 514 (*constant function is measurable*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 571** ( *$\mathcal{M}$  is closed under infimum*)

depends directly from:  
 Statement(s) from [17],  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 563 (*measurability of numeric function*).

**Lemma 572** ( *$\mathcal{M}$  is closed under supremum*)

depends directly from:

Statement(s) from [17],  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 563 (*measurability of numeric function*).

**Lemma 573** ( *$\mathcal{M}$  is closed under limit inferior*)

depends directly from:  
 Lemma 352 (*limit inferior*),  
 Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 572 ( *$\mathcal{M}$  is closed under supremum*).

**Lemma 574** ( *$\mathcal{M}$  is closed under limit superior*)

depends directly from:  
 Lemma 357 (*limit superior*),  
 Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 572 ( *$\mathcal{M}$  is closed under supremum*).

**Lemma 575** ( *$\mathcal{M}$  is closed under limit when pointwise convergent*)

depends directly from:  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*),  
 Lemma 573 ( *$\mathcal{M}$  is closed under limit inferior*),  
 Lemma 574 ( *$\mathcal{M}$  is closed under limit superior*).

**Lemma 576** (*measurability and masking*)

depends directly from:  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 554 (*measurability of indicator function*),  
 Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 577** (*measurability of restriction*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 510 (*measurable function*),  
 Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*).

**Lemma 579** (*measurability of nonnegative and nonpositive parts*)

depends directly from:  
 Definition 373 (*nonnegative and nonpositive parts*),  
 Lemma 375 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 377 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 514 (*constant function is measurable*),  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 570 ( *$\mathcal{M}$  is closed under scalar multiplication*),  
 Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),  
 Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**Lemma 580** ( *$\mathcal{M}_+$  is closed under finite part*)

depends directly from:  
 Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),  
 Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*).

**Lemma 581** ( *$\mathcal{M}$  is closed under absolute value*)

depends directly from:  
 Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 293 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),

Lemma 517 (*continuous is measurable*),  
 Lemma 518 (*compatibility of measurability with composition*),  
 Definition 552 ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* ),  
 Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions*),  
 Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**Lemma 582** ( $\mathcal{M}_+$  *is closed under addition*)

depends directly from:  
 Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 566 ( $\mathcal{M}$  *is closed under addition when defined*),  
 Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**Lemma 583** ( $\mathcal{M}_+$  *is closed under multiplication*)

depends directly from:  
 Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 568 ( $\mathcal{M}$  *is closed under multiplication*),  
 Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**Lemma 584** ( $\mathcal{M}_+$  *is closed under nonnegative scalar multiplication*)

depends directly from:  
 Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 570 ( $\mathcal{M}$  *is closed under scalar multiplication*),  
 Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**Lemma 585** ( $\mathcal{M}_+$  *is closed under limit when pointwise convergent*)

depends directly from:  
 Lemma 575 ( $\mathcal{M}$  *is closed under limit when pointwise convergent*),  
 Definition 578 ( $\mathcal{M}_+$ , *subset of nonnegative measurable numeric functions*).

**Lemma 586** ( $\mathcal{M}_+$  *is closed under countable sum*)

depends directly from:  
 Lemma 298 (*series are convergent in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 582 ( $\mathcal{M}_+$  *is closed under addition*),  
 Lemma 585 ( $\mathcal{M}_+$  *is closed under limit when pointwise convergent*).

**Lemma 588** (*measurability of tensor product of numeric functions*)

depends directly from:  
 Lemma 518 (*compatibility of measurability with composition*),  
 Lemma 532 (*canonical projection is measurable*),  
 Definition 560 ( $\mathcal{M}$ , *set of measurable numeric functions*),  
 Lemma 569 ( $\mathcal{M}$  *is closed under finite product*),  
 Definition 587 (*tensor product of numeric functions*).

**Lemma 593** ( $\sigma$ -*additivity implies additivity*)

depends directly from:  
 Definition 590 (*additivity over measurable space*),  
 Definition 591 ( $\sigma$ -*additivity over measurable space*).

**Lemma 596** (*measure over countable partition*)

depends directly from:  
 Lemma 207 (*compatibility of partition with intersection*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 591 ( $\sigma$ -*additivity over measurable space*),  
 Definition 594 (*measure*).

**Lemma 597** (*measure is monotone*)

depends directly from:

Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 470 ( *$\sigma$ -algebra is closed under set difference*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*).

**Lemma 598** (*measure satisfies the finite Boole inequality*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 470 ( *$\sigma$ -algebra is closed under set difference*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*),  
 Lemma 597 (*measure is monotone*).

**Lemma 600** (*measure is continuous from below*)

depends directly from:  
 Lemma 212 (*partition of countable union*),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 470 ( *$\sigma$ -algebra is closed under set difference*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*),  
 Lemma 597 (*measure is monotone*),  
 Definition 599 (*continuity from below*).

**Lemma 602** (*measure is continuous from above*)

depends directly from:  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 470 ( *$\sigma$ -algebra is closed under set difference*),  
 Lemma 597 (*measure is monotone*),  
 Definition 599 (*continuity from below*),  
 Lemma 600 (*measure is continuous from below*),  
 Definition 601 (*continuity from above*).

**Lemma 603** (*measure satisfies the Boole inequality*)

depends directly from:  
 Statement(s) from [17],  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 594 (*measure*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 600 (*measure is continuous from below*).

**Lemma 604** (*equivalent definition of measure*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 590 (*additivity over measurable space*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Lemma 593 ( *$\sigma$ -additivity implies additivity*),  
 Definition 594 (*measure*),  
 Definition 599 (*continuity from below*),  
 Lemma 600 (*measure is continuous from below*).

**Lemma 606** (*finite measure is bounded*)

depends directly from:  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 597 (*measure is monotone*).

**Lemma 608** (*equivalent definition of  $\sigma$ -finite measure*)

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Definition 607 ( *$\sigma$ -finite measure*).

**Lemma 610** (*finite measure is  $\sigma$ -finite*)

depends directly from:  
 Definition 605 (*finite measure*),  
 Definition 607 ( *$\sigma$ -finite measure*).

**Lemma 611** (*trace measure*)

depends directly from:  
 Lemma 520 (*trace  $\sigma$ -algebra*),  
 Lemma 521 (*measurability of measurable subspace*),  
 Definition 594 (*measure*).

**Lemma 612** (*restricted measure*)

depends directly from:  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 594 (*measure*).

**Lemma 617** (*equivalent definition of considerable subset*)

depends directly from:  
 Definition 594 (*measure*),  
 Definition 614 (*negligible subset*),  
 Definition 616 (*considerable subset*).

**Lemma 619** (*negligibility of measurable subset*)

depends directly from:  
 Definition 614 (*negligible subset*).

**Lemma 620** (*empty set is negligible*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 594 (*measure*),  
 Lemma 619 (*negligibility of measurable subset*).

**Lemma 621** (*compatibility of null measure with countable union*)

depends directly from:  
 Statement(s) from [17],  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 594 (*measure*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 603 (*measure satisfies the Boole inequality*).

**Lemma 622** ( *$\mathbb{N}$  is closed under countable union*)

depends directly from:  
 Definition 614 (*negligible subset*),  
 Lemma 621 (*compatibility of null measure with countable union*).

**Lemma 623** (*subset of negligible is negligible*)

depends directly from:  
 Definition 614 (*negligible subset*).

**Lemma 626** (*everywhere implies almost everywhere*)

depends directly from:  
 Lemma 620 (*empty set is negligible*),  
 Definition 624 (*property almost satisfied*).



**Lemma 627** (*extended almost modus ponens*)

depends directly from:

Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),

Lemma 623 (*subset of negligible is negligible*),

Definition 624 (*property almost satisfied*).

**Lemma 628** (*almost modus ponens*)

depends directly from:

Lemma 626 (*everywhere implies almost everywhere*),

Lemma 627 (*extended almost modus ponens*).

**Lemma 632** (*compatibility of almost binary relation with reflexivity*)

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),

Definition 594 (*measure*),

Lemma 619 (*negligibility of measurable subset*),

Definition 624 (*property almost satisfied*),

Definition 630 (*almost definition*),

Definition 631 (*almost binary relation*).

**Lemma 633** (*compatibility of almost binary relation with symmetry*)

depends directly from:

Definition 614 (*negligible subset*),

Definition 624 (*property almost satisfied*),

Definition 630 (*almost definition*),

Definition 631 (*almost binary relation*).

**Lemma 634** (*compatibility of almost binary relation with antisymmetry*)

depends directly from:

Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),

Lemma 623 (*subset of negligible is negligible*),

Definition 624 (*property almost satisfied*),

Definition 630 (*almost definition*),

Definition 631 (*almost binary relation*).

**Lemma 635** (*compatibility of almost binary relation with transitivity*)

depends directly from:

Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),

Lemma 623 (*subset of negligible is negligible*),

Definition 624 (*property almost satisfied*),

Definition 630 (*almost definition*),

Definition 631 (*almost binary relation*).

**Lemma 637** (*almost equivalence is equivalence relation*)

depends directly from:

Lemma 632 (*compatibility of almost binary relation with reflexivity*),

Lemma 633 (*compatibility of almost binary relation with symmetry*),

Lemma 635 (*compatibility of almost binary relation with transitivity*).

**Lemma 638** (*almost equality is equivalence relation*)

depends directly from:

Lemma 637 (*almost equivalence is equivalence relation*).

**Lemma 639** (*almost order is order relation*)

depends directly from:

Lemma 632 (*compatibility of almost binary relation with reflexivity*),

Lemma 634 (*compatibility of almost binary relation with antisymmetry*),  
 Lemma 635 (*compatibility of almost binary relation with transitivity*).

**Lemma 640 (*compatibility of almost binary relation with operator*)**

depends directly from:

Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),  
 Definition 624 (*property almost satisfied*),  
 Definition 630 (*almost definition*),  
 Definition 631 (*almost binary relation*).

**Lemma 641 (*compatibility of almost equality with operator*)**

depends directly from:

Lemma 640 (*compatibility of almost binary relation with operator*).

**Lemma 642 (*compatibility of almost inequality with operator*)**

depends directly from:

Lemma 640 (*compatibility of almost binary relation with operator*).

**Lemma 645 (*local injectivity implies almost local injectivity*)**

depends directly from:

Definition 614 (*negligible subset*),  
 Definition 624 (*property almost satisfied*),  
 Definition 630 (*almost definition*),  
 Definition 631 (*almost binary relation*).

**Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*)**

depends directly from:

Lemma 212 (*partition of countable union*),  
 Definition 424 ( *$\pi$ -system*),  
 Lemma 454 (*equivalent definition of  $\lambda$ -system*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*),  
 Lemma 597 (*measure is monotone*),  
 Definition 599 (*continuity from below*),  
 Lemma 600 (*measure is continuous from below*).

**Lemma 650 (*trivial measure*)**

depends directly from:

Definition 594 (*measure*).

**Lemma 651 (*equivalent definition of trivial measure*)**

depends directly from:

Definition 594 (*measure*),  
 Lemma 597 (*measure is monotone*),  
 Lemma 650 (*trivial measure*).

**Lemma 652 (*counting measure*)**

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*).

**Lemma 653 (*finiteness of counting measure*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Definition 605 (*finite measure*),  
 Lemma 652 (*counting measure*).

**Lemma 654** ( *$\sigma$ -finite counting measure*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 607 ( *$\sigma$ -finite measure*),  
 Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),  
 Lemma 652 (*counting measure*).

**Lemma 657** (*equivalent definition of Dirac measure*)

depends directly from:  
 Lemma 652 (*counting measure*),  
 Definition 656 (*Dirac measure*).

**Lemma 658** (*Dirac measure is finite*)

depends directly from:  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 653 (*finiteness of counting measure*),  
 Definition 656 (*Dirac measure*).

**Lemma 660** (*summability on summability domain*)

depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Definition 659 (*summability domain*).

**Lemma 661** (*measurability of summability domain*)

depends directly from:  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 564 (*inverse image is measurable*).

**Lemma 662** (*negligibility of summability domain*)

depends directly from:  
 Definition 624 (*property almost satisfied*),  
 Lemma 660 (*summability on summability domain*).

**Lemma 663** (*almost sum*)

depends directly from:  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 554 (*measurability of indicator function*),  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),  
 Definition 624 (*property almost satisfied*),  
 Lemma 661 (*measurability of summability domain*),  
 Lemma 662 (*negligibility of summability domain*).

**Lemma 664** (*compatibility of almost sum with almost equality*)

depends directly from:  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 620 (*empty set is negligible*),  
 Lemma 638 (*almost equality is equivalence relation*),  
 Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 662 (*negligibility of summability domain*),  
 Lemma 663 (*almost sum*).

**Lemma 666** (*almost sum is sum*)

depends directly from:

Definition 659 (*summability domain*),  
 Lemma 663 (*almost sum*).

**Lemma 667** (*absolute value is almost definite*)

depends directly from:  
 Lemma 280 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 645 (*local injectivity implies almost local injectivity*).

**Lemma 668** (*masking almost nowhere*)

depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Definition 614 (*negligible subset*),  
 Definition 624 (*property almost satisfied*).

**Lemma 669** (*finite nonnegative part*)

depends directly from:  
 Definition 371 (*finite part*),  
 Lemma 374 (*equivalent definition of nonnegative and nonpositive parts*),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 576 (*measurability and masking*),  
 Lemma 579 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 580 ( *$\mathcal{M}_+$  is closed under finite part*),  
 Definition 594 (*measure*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 619 (*negligibility of measurable subset*),  
 Definition 624 (*property almost satisfied*),  
 Lemma 668 (*masking almost nowhere*).

**Lemma 672** (*length is nonnegative*)

depends directly from:  
 Definition 671 (*length of interval*).

**Lemma 673** (*length is homogeneous*)

depends directly from:  
 Definition 671 (*length of interval*).

**Lemma 674** (*length of partition*)

depends directly from:  
 Definition 671 (*length of interval*),  
 Lemma 673 (*length is homogeneous*).

**Lemma 676** (*set of countable cover with bounded open intervals is nonempty*)

depends directly from:  
 Definition 675 (*set of countable cover with bounded open intervals*).

**Lemma 679** ( *$\lambda^*$  is nonnegative*)

depends directly from:  
 Lemma 672 (*length is nonnegative*).

**Lemma 680** ( *$\lambda^*$  is homogeneous*)

depends directly from:  
 Statement(s) from [17],  
 Definition 671 (*length of interval*),  
 Definition 677 ( *$\lambda^*$ , Lebesgue measure candidate*),  
 Lemma 679 ( *$\lambda^*$  is nonnegative*).

**Lemma 681** ( $\lambda^*$  is monotone)

depends directly from:

Statement(s) from [17],

Definition 675 (set of countable cover with bounded open intervals),

Definition 677 ( $\lambda^*$ , Lebesgue measure candidate).**Lemma 682** ( $\lambda^*$  is  $\sigma$ -subadditive)

depends directly from:

Statement(s) from [17],

Lemma 210 (definition of double countable union),

Lemma 211 (double countable union),

Lemma 301 (definition of double series in  $\overline{\mathbb{R}}_+$ ),Lemma 302 (double series in  $\overline{\mathbb{R}}_+$ ),

Definition 675 (set of countable cover with bounded open intervals),

Definition 677 ( $\lambda^*$ , Lebesgue measure candidate).**Lemma 683** ( $\lambda^*$  generalizes length of intervals)

depends directly from:

Statement(s) from [17],

Lemma 248 (finite cover of compact interval),

Definition 671 (length of interval),

Definition 675 (set of countable cover with bounded open intervals),

Definition 677 ( $\lambda^*$ , Lebesgue measure candidate),Lemma 680 ( $\lambda^*$  is homogeneous),Lemma 681 ( $\lambda^*$  is monotone).**Lemma 687** (equivalent definition of  $\mathcal{L}$ )

depends directly from:

Lemma 207 (compatibility of partition with intersection),

Lemma 255 (order in  $\overline{\mathbb{R}}$  is total),Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive),Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).**Lemma 688** ( $\mathcal{L}$  is closed under complement)

depends directly from:

Lemma 296 (addition in  $\overline{\mathbb{R}}_+$  is commutative),Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).**Lemma 689** ( $\mathcal{L}$  is closed under finite union)

depends directly from:

Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive),Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra),Lemma 687 (equivalent definition of  $\mathcal{L}$ ).**Lemma 690** ( $\mathcal{L}$  is closed under finite intersection)

depends directly from:

Lemma 688 ( $\mathcal{L}$  is closed under complement),Lemma 689 ( $\mathcal{L}$  is closed under finite union).**Lemma 691** ( $\lambda^*$  is additive on  $\mathcal{L}$ )

depends directly from:

Lemma 207 (compatibility of partition with intersection),

Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra).**Lemma 693** ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ )

depends directly from:

Definition 591 ( $\sigma$ -additivity over measurable space),

Lemma 681 ( $\lambda^*$  is monotone),  
 Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive),  
 Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ ).

**Lemma 694** (*partition of countable union in  $\mathcal{L}$* )

depends directly from:

Lemma 212 (*partition of countable union*),  
 Lemma 688 ( $\mathcal{L}$  is closed under complement),  
 Lemma 689 ( $\mathcal{L}$  is closed under finite union),  
 Lemma 690 ( $\mathcal{L}$  is closed under finite intersection).

**Lemma 695** ( *$\mathcal{L}$  is closed under countable union*)

depends directly from:

Lemma 681 ( $\lambda^*$  is monotone),  
 Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive),  
 Definition 685 ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra),  
 Lemma 687 (*equivalent definition of  $\mathcal{L}$* ),  
 Lemma 689 ( $\mathcal{L}$  is closed under finite union),  
 Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ ),  
 Lemma 694 (*partition of countable union in  $\mathcal{L}$* ).

**Lemma 696** (*rays are Lebesgue-measurable*)

depends directly from:

Statement(s) from [17],  
 Lemma 222 (*intervals are closed under finite intersection*),  
 Definition 677 ( $\lambda^*$ , Lebesgue measure candidate),  
 Lemma 681 ( $\lambda^*$  is monotone),  
 Lemma 682 ( $\lambda^*$  is  $\sigma$ -subadditive),  
 Lemma 683 ( $\lambda^*$  generalizes length of intervals),  
 Lemma 687 (*equivalent definition of  $\mathcal{L}$* ),  
 Lemma 688 ( $\mathcal{L}$  is closed under complement),  
 Lemma 695 ( $\mathcal{L}$  is closed under countable union).

**Lemma 697** (*intervals are Lebesgue-measurable*)

depends directly from:

Lemma 690 ( $\mathcal{L}$  is closed under finite intersection),  
 Lemma 696 (*rays are Lebesgue-measurable*).

**Lemma 698** ( *$\mathcal{L}$  is  $\sigma$ -algebra*)

depends directly from:

Definition 467 ( $\sigma$ -algebra),  
 Lemma 680 ( $\lambda^*$  is homogeneous),  
 Lemma 688 ( $\mathcal{L}$  is closed under complement),  
 Lemma 695 ( $\mathcal{L}$  is closed under countable union).

**Lemma 699** ( $\lambda^*$  is measure on  $\mathcal{L}$ )

depends directly from:

Definition 594 (*measure*),  
 Lemma 679 ( $\lambda^*$  is nonnegative),  
 Lemma 680 ( $\lambda^*$  is homogeneous),  
 Lemma 693 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ),  
 Lemma 698 ( $\mathcal{L}$  is  $\sigma$ -algebra).

**Lemma 700** ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ )

depends directly from:

Lemma 475 ( $\sigma$ -algebra generation is monotone),

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),  
 Lemma 697 (*intervals are Lebesgue-measurable*).

**Lemma 701** ( *$\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$* )

depends directly from:  
 Lemma 699 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ),  
 Lemma 700 ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* ).

**Theorem 702** (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* )

depends directly from:  
 Definition 217 (*interval*),  
 Lemma 222 (*intervals are closed under finite intersection*),  
 Definition 424 ( *$\pi$ -system*),  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 493 (*complete generated  $\sigma$ -algebra*),  
 Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),  
 Definition 594 (*measure*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),  
 Lemma 701 ( *$\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$* ).

**Lemma 704** (*Lebesgue measure generalizes length of intervals*)

depends directly from:  
 Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),  
 Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 706** (*Lebesgue measure is  $\sigma$ -finite*)

depends directly from:  
 Definition 607 ( *$\sigma$ -finite measure*),  
 Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),  
 Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 707** (*Lebesgue measure is diffuse*)

depends directly from:  
 Definition 609 (*diffuse measure*),  
 Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),  
 Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 710** (*indicator and support are each other inverse*)

depends directly from:  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*).

**Lemma 711** ( *$\mathcal{IF}$  is measurable*)

depends directly from:  
 Lemma 554 (*measurability of indicator function*),  
 Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*).

**Lemma 713** ( *$\mathcal{IF}$  is closed under extension by zero*)

depends directly from:  
 Lemma 215 (*restriction is masking*),  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*).

**Lemma 714** ( *$\mathcal{IF}$  is closed under restriction*)

depends directly from:  
 Lemma 521 (*measurability of measurable subspace*),  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*).



**Lemma 716** (*equivalent definition of integral in  $\mathcal{IF}$* )

depends directly from:

Lemma 710 (*indicator and support are each other inverse*),  
 Definition 715 (*integral in  $\mathcal{IF}$* ).

**Lemma 717** (*additivity of integral in  $\mathcal{IF}$* )

depends directly from:

Definition 590 (*additivity over measurable space*),  
 Lemma 593 ( *$\sigma$ -additivity implies additivity*),  
 Definition 594 (*measure*),  
 Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ).

**Lemma 718** (*integral in  $\mathcal{IF}$  over subset*)

depends directly from:

Lemma 215 (*restriction is masking*),  
 Lemma 611 (*trace measure*),  
 Lemma 710 (*indicator and support are each other inverse*),  
 Lemma 713 ( *$\mathcal{IF}$  is closed under extension by zero*),  
 Lemma 714 ( *$\mathcal{IF}$  is closed under restriction*),  
 Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ).

**Lemma 719** (*integral in  $\mathcal{IF}$  over subset is additive*)

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 710 (*indicator and support are each other inverse*),  
 Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ),  
 Lemma 718 (*integral in  $\mathcal{IF}$  over subset*).

**Lemma 721** (*integral in  $\mathcal{IF}$  for counting measure*)

depends directly from:

Lemma 652 (*counting measure*),  
 Lemma 710 (*indicator and support are each other inverse*),  
 Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ).

**Lemma 724** ( *$\mathcal{SF}$  is vector space*)

depends directly from:

Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).

**Lemma 725** ( *$\mathcal{SF}$  simple representation*)

depends directly from:

Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*),  
 Lemma 710 (*indicator and support are each other inverse*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).

**Lemma 727** ( *$\mathcal{SF}$  canonical representation*)

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 725 ( *$\mathcal{SF}$  simple representation*).

**Lemma 728** ( *$\mathcal{SF}$  is measurable*)

depends directly from:

Lemma 554 (*measurability of indicator function*),  
 Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),  
 Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).

**Lemma 729** ( *$\mathcal{SF}$  is closed under extension by zero*)

depends directly from:

Lemma 713 ( *$\mathcal{IF}$  is closed under extension by zero*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).

**Lemma 730** ( *$\mathcal{SF}$  is closed under restriction*)

depends directly from:

Lemma 714 ( *$\mathcal{IF}$  is closed under restriction*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*).

**Lemma 733** ( *$\mathcal{SF}_+$  canonical representation*)

depends directly from:

Lemma 727 ( *$\mathcal{SF}$  canonical representation*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*).

**Lemma 734** ( *$\mathcal{SF}_+$  simple representation*)

depends directly from:

Lemma 725 ( *$\mathcal{SF}$  simple representation*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*),  
 Lemma 733 ( *$\mathcal{SF}_+$  canonical representation*).

**Lemma 735** ( *$\mathcal{SF}_+$  is closed under positive linear operations*)

depends directly from:

Statement(s) from [17],  
 Lemma 724 ( *$\mathcal{SF}$  is vector space*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*).

**Lemma 736** ( *$\mathcal{SF}_+$  is measurable*)

depends directly from:

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*),  
 Lemma 728 ( *$\mathcal{SF}$  is measurable*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*).

**Lemma 737** (*integral in  $\mathcal{SF}_+$* )

depends directly from:

Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),  
 Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Definition 594 (*measure*),  
 Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ),  
 Lemma 733 ( *$\mathcal{SF}_+$  canonical representation*).

**Lemma 738** (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* )

depends directly from:

Definition 715 (*integral in  $\mathcal{IF}$* ),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ).

**Lemma 739** (*integral in  $\mathcal{SF}_+$  of indicator function*)

depends directly from:

Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ),  
 Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**Lemma 740** (*decomposition of measure in  $\mathcal{SF}_+$* )

depends directly from:

Lemma 496 (*some Borel subsets*),  
 Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),  
 Lemma 596 (*measure over countable partition*),

Lemma 727 (*SF canonical representation*),  
 Lemma 736 (*SF<sub>+</sub> is measurable*).

**Lemma 741 (*change of variable in sum in SF<sub>+</sub>*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),  
 Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*),  
 Definition 594 (*measure*),  
 Lemma 735 (*SF<sub>+</sub> is closed under positive linear operations*),  
 Lemma 736 (*SF<sub>+</sub> is measurable*).

**Lemma 743 (*integral in SF<sub>+</sub> is positive linear*)**

depends directly from:

Statement(s) from [17],  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 295 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),  
 Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),  
 Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*),  
 Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),  
 Definition 560 ( *$\mathcal{M}$ , set of measurable numeric functions*),  
 Lemma 724 (*SF is vector space*),  
 Definition 732 (*SF<sub>+</sub>, subset of nonnegative simple functions*),  
 Lemma 735 (*SF<sub>+</sub> is closed under positive linear operations*),  
 Lemma 736 (*SF<sub>+</sub> is measurable*),  
 Lemma 737 (*integral in SF<sub>+</sub>*),  
 Lemma 740 (*decomposition of measure in SF<sub>+</sub>*),  
 Lemma 741 (*change of variable in sum in SF<sub>+</sub>*).

**Lemma 744 (*equivalent definition of the integral in SF<sub>+</sub>*)**

depends directly from:

Lemma 734 (*SF<sub>+</sub> simple representation*),  
 Lemma 739 (*integral in SF<sub>+</sub> of indicator function*),  
 Lemma 743 (*integral in SF<sub>+</sub> is positive linear*).

**Lemma 745 (*integral in SF<sub>+</sub> is monotone*)**

depends directly from:

Statement(s) from [17],  
 Lemma 724 (*SF is vector space*),  
 Definition 732 (*SF<sub>+</sub>, subset of nonnegative simple functions*),  
 Lemma 737 (*integral in SF<sub>+</sub>*),  
 Lemma 743 (*integral in SF<sub>+</sub> is positive linear*).

**Lemma 746 (*integral in SF<sub>+</sub> is continuous*)**

depends directly from:

Statement(s) from [17],  
 Lemma 745 (*integral in SF<sub>+</sub> is monotone*).

**Lemma 747 (*integral in SF<sub>+</sub> over subset*)**

depends directly from:

Lemma 718 (*integral in IF over subset*),  
 Definition 723 (*SF, vector space of simple functions*),  
 Lemma 729 (*SF is closed under extension by zero*),  
 Lemma 730 (*SF is closed under restriction*),

Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 748 (*integral in  $\mathcal{SF}_+$  over subset is additive*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 709 ( *$\mathcal{IF}$ , set of measurable indicator functions*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*),  
 Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*).

**Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*)**

depends directly from:

Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*),  
 Lemma 734 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**Lemma 750 (*integral in  $\mathcal{SF}_+$  for counting measure on  $\mathbb{N}$* )**

depends directly from:

Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*).

**Lemma 751 (*integral in  $\mathcal{SF}_+$  for Dirac measure*)**

depends directly from:

Definition 656 (*Dirac measure*),  
 Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*).

**Lemma 753 (*integral in  $\mathcal{M}_+$* )**

depends directly from:

Statement(s) from [17],  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ).

**Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* )**

depends directly from:

Lemma 737 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 746 (*integral in  $\mathcal{SF}_+$  is continuous*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ).

**Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*)**

depends directly from:

Lemma 736 ( *$\mathcal{SF}_+$  is measurable*),  
 Lemma 739 (*integral in  $\mathcal{SF}_+$  of indicator function*),  
 Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ).

**Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*)**

depends directly from:

Statement(s) from [17],  
 Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),  
 Lemma 316 (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*),  
 Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*),  
 Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*),  
 Lemma 735 ( *$\mathcal{SF}_+$  is closed under positive linear operations*),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 744 (*equivalent definition of the integral in  $\mathcal{SF}_+$* ),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ).

**Lemma 757** (*integral in  $\mathcal{M}_+$  of zero is zero*)

depends directly from:

Definition 264 (*multiplication in  $\overline{\mathbb{R}}$* ),

Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 758** (*integral in  $\mathcal{M}_+$  is monotone*)

depends directly from:

Lemma 753 (*integral in  $\mathcal{M}_+$* ).

**Theorem 759** (*Beppo Levi, monotone convergence*)

depends directly from:

Statement(s) from [17],

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 563 (*measurability of numeric function*),

Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),

Definition 599 (*continuity from below*),

Lemma 600 (*measure is continuous from below*),

Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*),

Lemma 728 ( *$\mathcal{SF}$  is measurable*),

Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),

Lemma 744 (*equivalent definition of the integral in  $\mathcal{SF}_+$* ),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),

Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*).

**Lemma 760** (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* )

depends directly from:

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 762** (*adapted sequence in  $\mathcal{M}_+$* )

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 563 (*measurability of numeric function*),

Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*),

Definition 732 ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*),

Definition 761 (*adapted sequence*).

**Lemma 763** (*usage of adapted sequences*)

depends directly from:

Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),

Theorem 759 (*Beppo Levi, monotone convergence*),

Definition 761 (*adapted sequence*),

Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ).

**Lemma 764** (*integral in  $\mathcal{M}_+$  is additive*)

depends directly from:

Lemma 582 ( *$\mathcal{M}_+$  is closed under addition*),

Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),

Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 763 (*usage of adapted sequences*).

**Lemma 765** (*integral in  $\mathcal{M}_+$  is positive linear*)

depends directly from:

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*).**Lemma 766** (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*)

depends directly from:

Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*),Theorem 759 (*Beppo Levi, monotone convergence*).**Lemma 767** (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*)

depends directly from:

Lemma 377 (*decomposition into nonnegative and nonpositive parts*),Lemma 579 (*measurability of nonnegative and nonpositive parts*),Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*).**Lemma 768** (*compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts*)

depends directly from:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*).**Lemma 769** (*integral in  $\mathcal{M}_+$  is almost definite*)

depends directly from:

Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),Lemma 320 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),Lemma 563 (*measurability of numeric function*),Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),Lemma 619 (*negligibility of measurable subset*),Definition 624 (*property almost satisfied*),Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).**Lemma 770** (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*)

depends directly from:

Lemma 259 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*),Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),Definition 467 ( *$\sigma$ -algebra*),Lemma 583 ( *$\mathcal{M}_+$  is closed under multiplication*),Definition 614 (*negligible subset*),Lemma 619 (*negligibility of measurable subset*),Definition 624 (*property almost satisfied*),Lemma 626 (*everywhere implies almost everywhere*),Definition 630 (*almost definition*),Definition 631 (*almost binary relation*),Lemma 638 (*almost equality is equivalence relation*),Lemma 641 (*compatibility of almost equality with operator*),Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*),Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*).

**Lemma 771** (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*)

depends directly from:

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 772** (*integral in  $\mathcal{M}_+$  is almost monotone*)

depends directly from:

Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 773** (*Bienaymé–Chebyshev inequality*)

depends directly from:

Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),

Lemma 554 (*measurability of indicator function*),

Lemma 563 (*measurability of numeric function*),

Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),

Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),

Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),

Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ).

**Lemma 774** (*integrable in  $\mathcal{M}_+$  is almost finite*)

depends directly from:

Definition 254 (*extended real numbers,  $\overline{\mathbb{R}}$* ),

Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Definition 594 (*measure*),

Lemma 619 (*negligibility of measurable subset*),

Definition 624 (*property almost satisfied*),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 773 (*Bienaymé–Chebyshev inequality*).

**Lemma 775** (*bounded by integrable in  $\mathcal{M}_+$  is integrable*)

depends directly from:

Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*).

**Lemma 776** (*integral in  $\mathcal{M}_+$  over subset*)

depends directly from:

Lemma 577 (*measurability of restriction*),

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Lemma 730 ( *$\mathcal{SF}$  is closed under restriction*),

Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*),

Definition 761 (*adapted sequence*),

Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 763 (*usage of adapted sequences*).

**Lemma 777** (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*)

depends directly from:

Lemma 554 (*measurability of indicator function*),

Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Lemma 583 ( *$\mathcal{M}_+$  is closed under multiplication*),

Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*),



Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*).

**Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*)**

depends directly from:

Lemma 554 (*measurability of indicator function*),  
 Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*).

**Theorem 779 (*Fatou's lemma*)**

depends directly from:

Statement(s) from [17],  
 Lemma 352 (*limit inferior*),  
 Definition 364 (*pointwise convergence*),  
 Lemma 366 (*limit inferior bounded from below*),  
 Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),  
 Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*)**

depends directly from:

Statement(s) from [17],  
 Lemma 365 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*),  
 Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),  
 Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Theorem 779 (*Fatou's lemma*).

**Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*)**

depends directly from:

Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*),  
 Lemma 763 (*usage of adapted sequences*).

**Lemma 782 (*integral in  $\mathcal{M}_+$  for counting measure on  $\mathbb{N}$* )**

depends directly from:

Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*).

**Lemma 784 (*integral in  $\mathcal{M}_+$  for Dirac measure*)**

depends directly from:

Definition 656 (*Dirac measure*),  
 Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*).

**Lemma 786 (*countable union of sections is measurable*)**

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 537 (*compatibility of section with set operations*),  
 Lemma 538 (*measurability of section*).

**Lemma 787 (*countable intersection of sections is measurable*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 537 (*compatibility of section with set operations*),  
 Lemma 538 (*measurability of section*).

**Lemma 788** (*measure of section*)

depends directly from:

- Lemma 538 (*measurability of section*),
- Definition 594 (*measure*).

**Lemma 789** (*measure of section of product*)

depends directly from:

- Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),
- Lemma 536 (*section of product*),
- Definition 594 (*measure*),
- Lemma 788 (*measure of section*).

**Lemma 791** (*measurability of measure of section (finite)*)

depends directly from:

- Lemma 438 (*generated set algebra is minimum*),
- Definition 442 (*monotone class*),
- Lemma 481 ( *$\sigma$ -algebra contains set algebra*),
- Lemma 501 (*set algebra generated by product of  $\sigma$ -algebras*),
- Lemma 509 (*usage of monotone class theorem*),
- Definition 529 (*tensor product of  $\sigma$ -algebras*),
- Lemma 530 (*product of measurable subsets is measurable*),
- Lemma 537 (*compatibility of section with set operations*),
- Lemma 538 (*measurability of section*),
- Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),
- Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),
- Lemma 582 ( *$\mathcal{M}_+$  is closed under addition*),
- Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),
- Definition 591 ( *$\sigma$ -additivity over measurable space*),
- Definition 594 (*measure*),
- Lemma 600 (*measure is continuous from below*),
- Lemma 602 (*measure is continuous from above*),
- Definition 605 (*finite measure*),
- Lemma 606 (*finite measure is bounded*),
- Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),
- Lemma 786 (*countable union of sections is measurable*),
- Lemma 787 (*countable intersection of sections is measurable*),
- Lemma 788 (*measure of section*),
- Lemma 789 (*measure of section of product*).

**Lemma 792** (*measurability of measure of section*)

depends directly from:

- Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),
- Lemma 538 (*measurability of section*),
- Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),
- Lemma 600 (*measure is continuous from below*),
- Definition 605 (*finite measure*),
- Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),
- Lemma 612 (*restricted measure*),
- Lemma 788 (*measure of section*),
- Lemma 791 (*measurability of measure of section (finite)*).

**Lemma 795** (*candidate tensor product measure is tensor product measure*)

depends directly from:

- Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*),
- Lemma 530 (*product of measurable subsets is measurable*),

Lemma 537 (*compatibility of section with set operations*),  
 Lemma 538 (*measurability of section*),  
 Definition 591 ( *$\sigma$ -additivity over measurable space*),  
 Definition 594 (*measure*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Lemma 786 (*countable union of sections is measurable*),  
 Lemma 788 (*measure of section*),  
 Lemma 789 (*measure of section of product*),  
 Lemma 792 (*measurability of measure of section*),  
 Definition 793 (*tensor product measure*),  
 Definition 794 (*candidate tensor product measure*).

**Lemma 796 (*tensor product of finite measures*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Definition 594 (*measure*),  
 Definition 605 (*finite measure*),  
 Definition 793 (*tensor product measure*).

**Lemma 797 (*tensor product of  $\sigma$ -finite measures*)**

depends directly from:

Lemma 530 (*product of measurable subsets is measurable*),  
 Definition 607 ( *$\sigma$ -finite measure*),  
 Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),  
 Definition 793 (*tensor product measure*).

**Lemma 799 (*uniqueness of tensor product measure (finite)*)**

depends directly from:

Definition 442 (*monotone class*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 481 ( *$\sigma$ -algebra contains set algebra*),  
 Lemma 501 (*set algebra generated by product of  $\sigma$ -algebras*),  
 Lemma 509 (*usage of monotone class theorem*),  
 Definition 529 (*tensor product of  $\sigma$ -algebras*),  
 Lemma 530 (*product of measurable subsets is measurable*),  
 Definition 590 (*additivity over measurable space*),  
 Lemma 593 ( *$\sigma$ -additivity implies additivity*),  
 Definition 594 (*measure*),  
 Lemma 600 (*measure is continuous from below*),  
 Lemma 602 (*measure is continuous from above*),  
 Lemma 606 (*finite measure is bounded*),  
 Definition 793 (*tensor product measure*),  
 Lemma 796 (*tensor product of finite measures*).

**Lemma 801 (*uniqueness of tensor product measure*)**

depends directly from:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 530 (*product of measurable subsets is measurable*),  
 Lemma 600 (*measure is continuous from below*),  
 Definition 605 (*finite measure*),  
 Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),

Lemma 612 (*restricted measure*),  
 Lemma 788 (*measure of section*),  
 Definition 793 (*tensor product measure*),  
 Definition 794 (*candidate tensor product measure*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 797 (*tensor product of  $\sigma$ -finite measures*),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 802 (*negligibility of measurable section*)**

depends directly from:

Lemma 619 (*negligibility of measurable subset*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 792 (*measurability of measure of section*),  
 Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* )**

depends directly from:

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),  
 Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*),  
 Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*)**

depends directly from:

Lemma 704 (*Lebesgue measure generalizes length of intervals*),  
 Definition 793 (*tensor product measure*),  
 Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**Lemma 805 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*)**

depends directly from:

Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*)**

depends directly from:

Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*),  
 Lemma 797 (*tensor product of  $\sigma$ -finite measures*),  
 Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**Lemma 807 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*)**

depends directly from:

Definition 609 (*diffuse measure*),  
 Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Theorem 808 (*Tonelli*)**

depends directly from:

Lemma 538 (*measurability of section*),  
 Lemma 539 (*indicator of section*),  
 Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*),  
 Lemma 710 (*indicator and support are each other inverse*),  
 Lemma 734 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Lemma 763 (*usage of adapted sequences*),  
 Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 788 (*measure of section*),  
 Lemma 792 (*measurability of measure of section*),  
 Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 809** (*Tonelli on subset*)

depends directly from:

Lemma 539 (*indicator of section*),

Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*),

Theorem 808 (*Tonelli*).

**Lemma 810** (*Tonelli for tensor product*)

depends directly from:

Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),

Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*),

Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),

Definition 587 (*tensor product of numeric functions*),

Lemma 588 (*measurability of tensor product of numeric functions*),

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Theorem 808 (*Tonelli*).

**Lemma 813** (*integrable is measurable*)

depends directly from:

Lemma 579 (*measurability of nonnegative and nonpositive parts*),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Definition 812 (*integrability*).

**Lemma 814** (*equivalent definition of integrability*)

depends directly from:

Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),

Lemma 297 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ),

Lemma 579 (*measurability of nonnegative and nonpositive parts*),

Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),

Definition 812 (*integrability*),

Lemma 813 (*integrable is measurable*).

**Lemma 815** (*compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$* )

depends directly from:

Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ),

Lemma 814 (*equivalent definition of integrability*).

**Lemma 816** (*integrable is almost finite*)

depends directly from:

Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*),

Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),

Lemma 814 (*equivalent definition of integrability*).

**Lemma 817** (*almost bounded by integrable is integrable*)

depends directly from:

Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),

Lemma 639 (*almost order is order relation*),

Lemma 753 (*integral in  $\mathcal{M}_+$* ),

Lemma 772 (*integral in  $\mathcal{M}_+$  is almost monotone*),

Lemma 814 (*equivalent definition of integrability*).

**Lemma 818** (*bounded by integrable is integrable*)

depends directly from:

Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 626 (*everywhere implies almost everywhere*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Lemma 817 (*almost bounded by integrable is integrable*).

**Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* )**

depends directly from:  
 Definition 373 (*nonnegative and nonpositive parts*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Definition 812 (*integrability*),  
 Definition 819 (*integral*).

**Lemma 821 (*integral of zero is zero*)**

depends directly from:  
 Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ).

**Lemma 823 (*compatibility of integral with almost equality*)**

depends directly from:  
 Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 771 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),  
 Definition 812 (*integrability*),  
 Definition 819 (*integral*).

**Lemma 825 (*integral over subset*)**

depends directly from:  
 Lemma 379 (*compatibility of nonpositive and nonnegative parts with mask*),  
 Lemma 380 (*compatibility of nonpositive and nonnegative parts with restriction*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*),  
 Definition 812 (*integrability*),  
 Definition 819 (*integral*).

**Lemma 826 (*integral over subset is  $\sigma$ -additive*)**

depends directly from:  
 Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),  
 Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Definition 819 (*integral*).

**Lemma 827 (*integral over singleton*)**

depends directly from:  
 Definition 273 (*absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 280 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 321 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Definition 373 (*nonnegative and nonpositive parts*),  
 Lemma 377 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Definition 812 (*integrability*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Definition 819 (*integral*),  
 Lemma 825 (*integral over subset*).

**Lemma 828 (*integral over interval*)**

depends directly from:

Lemma 496 (*some Borel subsets*),  
 Definition 609 (*diffuse measure*),  
 Lemma 826 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 827 (*integral over singleton*).

**Lemma 829** (*Chasles relation, integral over split intervals*)

depends directly from:  
 Lemma 826 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 828 (*integral over interval*).

**Lemma 830** (*integral for counting measure*)

depends directly from:  
 Lemma 377 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Definition 819 (*integral*).

**Lemma 831** (*integral for counting measure on  $\mathbb{N}$* )

depends directly from:  
 Lemma 830 (*integral for counting measure*).

**Lemma 833** (*integral for Dirac measure*)

depends directly from:  
 Definition 656 (*Dirac measure*),  
 Lemma 830 (*integral for counting measure*).

**Lemma 835** (*seminorm  $\mathcal{L}^1$* )

depends directly from:  
 Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Definition 822 (*merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ).

**Lemma 837** (*integrable is finite seminorm  $\mathcal{L}^1$* )

depends directly from:  
 Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 814 (*equivalent definition of integrability*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ).

**Lemma 838** (*compatibility of  $N_1$  with almost equality*)

depends directly from:  
 Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 823 (*compatibility of integral with almost equality*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ).

**Lemma 839** ( *$N_1$  is almost definite*)

depends directly from:  
 Lemma 667 (*absolute value is almost definite*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ).

**Lemma 840** ( *$N_1$  is absolutely homogeneous*)

depends directly from:  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ).



**Lemma 841** (*integral is homogeneous*)

depends directly from:

- Definition 373 (*nonnegative and nonpositive parts*),
- Lemma 570 ( *$\mathcal{M}$  is closed under scalar multiplication*),
- Lemma 579 (*measurability of nonnegative and nonpositive parts*),
- Lemma 753 (*integral in  $\mathcal{M}_+$* ),
- Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 813 (*integrable is measurable*),
- Lemma 814 (*equivalent definition of integrability*),
- Definition 819 (*integral*),
- Lemma 835 (*seminorm  $\mathcal{L}^1$* ),
- Lemma 840 ( *$N_1$  is absolutely homogeneous*).

**Lemma 843** (*Minkowski inequality in  $\mathcal{M}$* )

depends directly from:

- Lemma 281 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),
- Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),
- Lemma 663 (*almost sum*),
- Lemma 664 (*compatibility of almost sum with almost equality*),
- Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*),
- Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*),
- Lemma 835 (*seminorm  $\mathcal{L}^1$* ),
- Lemma 838 (*compatibility of  $N_1$  with almost equality*).

**Lemma 844** (*integral is additive*)

depends directly from:

- Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),
- Lemma 579 (*measurability of nonnegative and nonpositive parts*),
- Lemma 768 (*compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts*),
- Lemma 816 (*integrable is almost finite*),
- Definition 819 (*integral*),
- Lemma 823 (*compatibility of integral with almost equality*),
- Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),
- Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 847** (*equivalent definition of  $\mathcal{L}^1$* )

depends directly from:

- Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 753 (*integral in  $\mathcal{M}_+$* ),
- Lemma 814 (*equivalent definition of integrability*),
- Lemma 835 (*seminorm  $\mathcal{L}^1$* ),
- Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*).

**Lemma 848** (*Minkowski inequality in  $\mathcal{L}^1$* )

depends directly from:

- Lemma 626 (*everywhere implies almost everywhere*),
- Definition 659 (*summability domain*),
- Lemma 666 (*almost sum is sum*),
- Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ),
- Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*).

**Lemma 849** ( *$\mathcal{L}^1$  is seminormed vector space*)

depends directly from:

- Statement(s) from [17],
- Definition 399 (*seminorm*),
- Lemma 559 ( *$\mathcal{M}_{\mathbb{R}}$  is vector space*),

Lemma 626 (*everywhere implies almost everywhere*),  
 Lemma 839 ( *$N_1$  is almost definite*),  
 Lemma 840 ( *$N_1$  is absolutely homogeneous*),  
 Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*),  
 Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ).

**Lemma 851** ( *$\mathcal{L}^1$  is closed under absolute value*)

depends directly from:

Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
 Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*).

**Lemma 852** (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* )

depends directly from:

Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),  
 Lemma 277 (*finite absolute value in  $\overline{\mathbb{R}}$* ),  
 Lemma 278 (*absolute value in  $\mathbb{R}$  is nonnegative*),  
 Lemma 815 (*compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),  
 Lemma 818 (*bounded by integrable is integrable*),  
 Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),  
 Lemma 851 ( *$\mathcal{L}^1$  is closed under absolute value*).

**Lemma 853** (*integral is nonnegative linear form on  $\mathcal{L}^1$* )

depends directly from:

Statement(s) from [17],  
 Definition 258 (*addition in  $\overline{\mathbb{R}}$* ),  
 Lemma 375 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),  
 Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),  
 Definition 812 (*integrability*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Definition 819 (*integral*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
 Lemma 841 (*integral is homogeneous*),  
 Lemma 844 (*integral is additive*),  
 Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*).

**Lemma 854** (*constant function is  $\mathcal{L}^1$* )

depends directly from:

Definition 467 ( *$\sigma$ -algebra*),  
 Definition 723 ( *$\mathcal{SF}$ , vector space of simple functions*),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),  
 Definition 819 (*integral*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ).

**Lemma 855** (*first mean value theorem*)

depends directly from:

Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*),  
 Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ),

Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),  
 Lemma 854 (*constant function is  $\mathcal{L}^1$* ).

**Lemma 856** (*variant of first mean value theorem*)  
 depends directly from:  
 Definition 217 (*interval*),  
 Lemma 855 (*first mean value theorem*).

**Theorem 857** (*Lebesgue, dominated convergence*)  
 depends directly from:  
 Statement(s) from [17],  
 Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),  
 Lemma 280 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),  
 Lemma 293 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),  
 Lemma 358 (*duality limit inferior-limit superior*),  
 Lemma 365 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 366 (*limit inferior bounded from below*),  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*),  
 Definition 399 (*seminorm*),  
 Lemma 575 ( *$\mathcal{M}$  is closed under limit when pointwise convergent*),  
 Definition 578 ( *$\mathcal{M}_+$ , subset of nonnegative measurable numeric functions*),  
 Theorem 779 (*Fatou's lemma*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
 Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*),  
 Definition 850 (*convergence in  $\mathcal{L}^1$* ),  
 Lemma 851 ( *$\mathcal{L}^1$  is closed under absolute value*),  
 Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Theorem 858** (*Lebesgue, extended dominated convergence*)  
 depends directly from:  
 Definition 467 ( *$\sigma$ -algebra*),  
 Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),  
 Lemma 576 (*measurability and masking*),  
 Definition 614 (*negligible subset*),  
 Lemma 621 (*compatibility of null measure with countable union*),  
 Definition 624 (*property almost satisfied*),  
 Lemma 641 (*compatibility of almost equality with operator*),  
 Lemma 668 (*masking almost nowhere*),  
 Lemma 669 (*finite nonnegative part*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Lemma 823 (*compatibility of integral with almost equality*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
 Definition 845 ( *$\mathcal{L}^1$ , vector space of integrable functions*),  
 Theorem 857 (*Lebesgue, dominated convergence*).

## C Is a direct dependency of...

### Statement(s) from [17]

are direct dependencies of:

- Lemma 245 (*equivalent definition of convergent sequence*),
- Lemma 246 (*convergent subsequence of Cauchy sequence*),
- Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*),
- Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ),
- Lemma 334 (*open intervals with rational bounds cover open interval*),
- Lemma 339 (*extrema of constant function*),
- Lemma 340 (*equivalent definition of finite infimum*),
- Lemma 341 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ),
- Lemma 342 (*equivalent definition of infimum*),
- Lemma 343 (*infimum is smaller than supremum*),
- Lemma 344 (*infimum is monotone*),
- Lemma 345 (*supremum is monotone*),
- Lemma 346 (*compatibility of infimum with absolute value*),
- Lemma 347 (*compatibility of supremum with absolute value*),
- Lemma 350 (*infimum of sequence is monotone*),
- Lemma 351 (*supremum of sequence is monotone*),
- Lemma 353 (*limit inferior is  $\infty$* ),
- Lemma 354 (*equivalent definition of the limit inferior*),
- Lemma 358 (*duality limit inferior-limit superior*),
- Lemma 366 (*limit inferior bounded from below*),
- Lemma 367 (*limit inferior bounded from above*),
- Lemma 370 (*limit inferior, limit superior and pointwise convergence*),
- Lemma 383 (*quotient vector space, equivalence relation*),
- Lemma 384 (*quotient vector space*),
- Lemma 385 (*linear map on quotient vector space*),
- Lemma 393 (*algebra of functions to algebra*),
- Lemma 397 (*vector subspace and closed under multiplication is subalgebra*),
- Lemma 398 (*closed under algebra operations is subalgebra*),
- Lemma 401 (*definite seminorm is norm*),
- Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),
- Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),
- Lemma 603 (*measure satisfies the Boole inequality*),
- Lemma 621 (*compatibility of null measure with countable union*),
- Lemma 680 ( *$\lambda^*$  is homogeneous*),
- Lemma 681 ( *$\lambda^*$  is monotone*),
- Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*),
- Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),
- Lemma 696 (*rays are Lebesgue-measurable*),
- Lemma 735 ( *$\mathcal{SF}_+$  is closed under positive linear operations*),
- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),
- Lemma 745 (*integral in  $\mathcal{SF}_+$  is monotone*),
- Lemma 746 (*integral in  $\mathcal{SF}_+$  is continuous*),
- Lemma 753 (*integral in  $\mathcal{M}_+$* ),
- Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Theorem 759 (*Beppo Levi, monotone convergence*),
- Theorem 779 (*Fatou's lemma*),
- Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),
- Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*),
- Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 207** (*compatibility of partition with intersection*)

is a direct dependency of:

Lemma 524 (*characterization of Borel subsets*),

Lemma 596 (*measure over countable partition*),

Lemma 687 (*equivalent definition of  $\mathcal{L}$* ),

Lemma 691 ( *$\lambda^*$  is additive on  $\mathcal{L}$* ).

**Lemma 208** (*technical inclusion for countable union*)

is a direct dependency of:

Lemma 209 (*order is meaningless in countable union*).

**Lemma 209** (*order is meaningless in countable union*)

is a direct dependency of:

Lemma 210 (*definition of double countable union*).

**Lemma 210** (*definition of double countable union*)

is a direct dependency of:

Lemma 211 (*double countable union*),

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Lemma 211** (*double countable union*)

is a direct dependency of:

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Lemma 212** (*partition of countable union*)

is a direct dependency of:

Lemma 420 (*closedness under countable disjoint and monotone union*),

Lemma 422 (*closedness under countable disjoint union and countable union*),

Lemma 600 (*measure is continuous from below*),

Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),

Lemma 694 (*partition of countable union in  $\mathcal{L}$* ).

**Definition 213** (*trace of subsets of parties*)

is a direct dependency of:

Lemma 236 (*trace topology on subset*),

Lemma 520 (*trace  $\sigma$ -algebra*),

Lemma 521 (*measurability of measurable subspace*),

Lemma 522 (*generating measurable subspace*),

Lemma 525 (*source restriction of measurable function*).

**Definition 214** (*product of subsets of parties*)

is a direct dependency of:

Lemma 237 (*box topology on Cartesian product*),

Lemma 241 (*compatibility of second-countability with Cartesian product*),

Lemma 242 (*complete countable topological basis of product space*),

Lemma 501 (*set algebra generated by product of  $\sigma$ -algebras*),

Lemma 530 (*product of measurable subsets is measurable*),

Lemma 531 (*measurability of function to product space*),

Lemma 533 (*generating product measurable space*),

Lemma 538 (*measurability of section*).

**Lemma 215** (*restriction is masking*)

is a direct dependency of:

Lemma 713 ( *$\mathcal{IF}$  is closed under extension by zero*),

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*).

**Definition 217** (*interval*)

is a direct dependency of:

Lemma 219 (*empty open interval*),

Lemma 222 (*intervals are closed under finite intersection*),

Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*),

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),

Lemma 856 (*variant of first mean value theorem*).

**Lemma 219** (*empty open interval*)

is a direct dependency of:

Lemma 223 (*empty intersection of open intervals*).

**Lemma 222** (*intervals are closed under finite intersection*)

is a direct dependency of:

Lemma 223 (*empty intersection of open intervals*),

Lemma 234 (*topological basis of order topology*),

Lemma 696 (*rays are Lebesgue-measurable*),

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 223** (*empty intersection of open intervals*)

is not yet used.

**Definition 225** (*topological space*)

is a direct dependency of:

Lemma 226 (*intersection of topologies*),

Lemma 229 (*equivalent definition of generated topology*),

Lemma 236 (*trace topology on subset*),

Lemma 237 (*box topology on Cartesian product*),

Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*),

Lemma 334 (*open intervals with rational bounds cover open interval*),

Lemma 496 (*some Borel subsets*).

**Lemma 226** (*intersection of topologies*)

is a direct dependency of:

Lemma 228 (*generated topology is minimum*).

**Definition 227** (*generated topology*)

is a direct dependency of:

Lemma 228 (*generated topology is minimum*).

**Lemma 228** (*generated topology is minimum*)

is a direct dependency of:

Lemma 229 (*equivalent definition of generated topology*).

**Lemma 229** (*equivalent definition of generated topology*)

is a direct dependency of:

Lemma 234 (*topological basis of order topology*).

**Definition 230** (*topological basis*)

is a direct dependency of:

Lemma 231 (*augmented topological basis*),

Lemma 234 (*topological basis of order topology*),

Lemma 236 (*trace topology on subset*),

Lemma 237 (*box topology on Cartesian product*),

Lemma 288 (*trace topology on  $\mathbb{R}$* ),

Theorem 335 ( *$\mathbb{R}$  is second-countable*),

Lemma 338 ( *$\mathbb{R}$  is second-countable*).

**Lemma 231** (*augmented topological basis*)

is a direct dependency of:

Lemma 240 (*complete countable topological basis*).

**Definition 232** (*order topology*)

is a direct dependency of:

Lemma 234 (*topological basis of order topology*),

Lemma 286 (*topology of  $\mathbb{R}$* ).

**Lemma 234** (*topological basis of order topology*)

is a direct dependency of:

Lemma 286 (*topology of  $\mathbb{R}$* ),

Lemma 288 (*trace topology on  $\mathbb{R}$* ).

**Lemma 236** (*trace topology on subset*)

is a direct dependency of:

Lemma 288 (*trace topology on  $\mathbb{R}$* ).

**Lemma 237** (*box topology on Cartesian product*)

is a direct dependency of:

Lemma 241 (*compatibility of second-countability with Cartesian product*),

Lemma 242 (*complete countable topological basis of product space*),

Lemma 336 ( *$\mathbb{R}^n$  is second-countable*).

**Definition 238** (*second-countability*)

is a direct dependency of:

Lemma 240 (*complete countable topological basis*),

Theorem 335 ( *$\mathbb{R}$  is second-countable*),

Lemma 336 ( *$\mathbb{R}^n$  is second-countable*),

Lemma 338 ( *$\mathbb{R}$  is second-countable*).

**Lemma 240** (*complete countable topological basis*)

is a direct dependency of:

Lemma 242 (*complete countable topological basis of product space*).

**Lemma 241** (*compatibility of second-countability with Cartesian product*)

is a direct dependency of:

Lemma 336 ( *$\mathbb{R}^n$  is second-countable*).

**Lemma 242** (*complete countable topological basis of product space*)

is a direct dependency of:

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ).

**Definition 243** (*pseudometric*)

is not yet used.

**Lemma 245** (*equivalent definition of convergent sequence*)

is a direct dependency of:

Lemma 340 (*equivalent definition of finite infimum*).

**Lemma 246** (*convergent subsequence of Cauchy sequence*)

is not yet used.

**Definition 247** (*cluster point*)

is a direct dependency of:

Lemma 354 (*equivalent definition of the limit inferior*),

Lemma 355 (*limit inferior is invariant by translation*).



**Lemma 248** (*finite cover of compact interval*)

is a direct dependency of:

Lemma 683 ( *$\lambda^*$  generalizes length of intervals*).

**Definition 249** (*Hölder conjugates in  $\mathbb{R}$* )

is a direct dependency of:

Lemma 250 (*2 is self-Hölder conjugate in  $\mathbb{R}$* ),

Lemma 251 (*Young's inequality for products in  $\mathbb{R}$* ).

**Lemma 250** (*2 is self-Hölder conjugate in  $\mathbb{R}$* )

is a direct dependency of:

Lemma 252 (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* ),

Lemma 326 (*Young's inequality for products, case  $p = 2$  (measure theory)*).

**Lemma 251** (*Young's inequality for products in  $\mathbb{R}$* )

is a direct dependency of:

Lemma 252 (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* ),

Lemma 325 (*Young's inequality for products (measure theory)*).

**Lemma 252** (*Young's inequality for products in  $\mathbb{R}$ , case  $p = 2$* )

is not yet used.

**Definition 254** (*extended real numbers,  $\overline{\mathbb{R}}$* )

is a direct dependency of:

Lemma 255 (*order in  $\overline{\mathbb{R}}$  is total*),

Lemma 263 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),

Lemma 281 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),

Lemma 286 (*topology of  $\overline{\mathbb{R}}$* ),

Lemma 372 (*finite part is finite*),

Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),

Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),

Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*).

**Lemma 255** (*order in  $\overline{\mathbb{R}}$  is total*)

is a direct dependency of:

Lemma 343 (*infimum is smaller than supremum*),

Lemma 344 (*infimum is monotone*),

Lemma 350 (*infimum of sequence is monotone*),

Lemma 360 (*limit inferior is smaller than limit superior*),

Lemma 687 (*equivalent definition of  $\mathcal{L}$* ),

Lemma 775 (*bounded by integrable in  $\mathcal{M}_+$  is integrable*),

Lemma 817 (*almost bounded by integrable is integrable*),

Lemma 818 (*bounded by integrable is integrable*),

Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ).

**Definition 258** (*addition in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

Lemma 259 (*zero is identity element for addition in  $\overline{\mathbb{R}}$* ),

Lemma 260 (*addition in  $\overline{\mathbb{R}}$  is associative when defined*),

Lemma 261 (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*),

Lemma 262 (*infinity-sum property in  $\overline{\mathbb{R}}$* ),

Lemma 263 (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*),

Lemma 268 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),

Lemma 281 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),

Lemma 291 (*continuity of addition in  $\overline{\mathbb{R}}$* ),

Lemma 294 (*addition in  $\overline{\mathbb{R}}_+$  is closed*),

- Lemma 295 (*addition in  $\overline{\mathbb{R}}_+$  is associative*),
- Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*),
- Lemma 325 (*Young's inequality for products (measure theory)*),
- Lemma 377 (*decomposition into nonnegative and nonpositive parts*),
- Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),
- Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 597 (*measure is monotone*),
- Lemma 660 (*summability on summability domain*),
- Lemma 663 (*almost sum*),
- Lemma 844 (*integral is additive*),
- Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 259** (*zero is identity element for addition in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 260** (*addition in  $\overline{\mathbb{R}}$  is associative when defined*)

is a direct dependency of:

- Lemma 295 (*addition in  $\overline{\mathbb{R}}_+$  is associative*).

**Lemma 261** (*addition in  $\overline{\mathbb{R}}$  is commutative when defined*)

is a direct dependency of:

- Lemma 296 (*addition in  $\overline{\mathbb{R}}_+$  is commutative*).

**Lemma 262** (*infinity-sum property in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 297 (*infinity-sum property in  $\overline{\mathbb{R}}_+$* ).

**Lemma 263** (*additive inverse in  $\overline{\mathbb{R}}$  is monotone*)

is a direct dependency of:

- Lemma 275 (*bounded absolute value in  $\overline{\mathbb{R}}$* ),
- Lemma 276 (*bounded absolute value in  $\overline{\mathbb{R}}$  (strict)*).

**Definition 264** (*multiplication in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 266 (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*),
- Lemma 267 (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*),
- Lemma 268 (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*),
- Lemma 270 (*zero-product property in  $\overline{\mathbb{R}}$* ),
- Lemma 271 (*infinity-product property in  $\overline{\mathbb{R}}$* ),
- Lemma 272 (*finite-product property in  $\overline{\mathbb{R}}$* ),
- Lemma 285 (*exponentiation in  $\overline{\mathbb{R}}$* ),
- Lemma 292 (*continuity of multiplication in  $\overline{\mathbb{R}}$* ),
- Lemma 305 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*),
- Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),
- Lemma 325 (*Young's inequality for products (measure theory)*),
- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 576 (*measurability and masking*),
- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),
- Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*).

**Lemma 266** (*multiplication in  $\overline{\mathbb{R}}$  is associative when defined*)

is a direct dependency of:

- Lemma 316 (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*).

**Lemma 267** (*multiplication in  $\overline{\mathbb{R}}$  is commutative when defined*)

is a direct dependency of:

- Lemma 269 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),
- Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*).

**Lemma 268** (*multiplication in  $\overline{\mathbb{R}}$  is left distributive over addition when defined*)

is a direct dependency of:

- Lemma 269 (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*),
- Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*).

**Lemma 269** (*multiplication in  $\overline{\mathbb{R}}$  is right distributive over addition when defined*)

is a direct dependency of:

- Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*).

**Lemma 270** (*zero-product property in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 306 (*zero-product property in  $\overline{\mathbb{R}}_+$* ),
- Lemma 311 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 271** (*infinity-product property in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),
- Lemma 312 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 272** (*finite-product property in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 308 (*finite-product property in  $\overline{\mathbb{R}}_+$* ).

**Definition 273** (*absolute value in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 274 (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* ),
- Lemma 278 (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*),
- Lemma 279 (*absolute value in  $\overline{\mathbb{R}}$  is even*),
- Lemma 280 (*absolute value in  $\overline{\mathbb{R}}$  is definite*),
- Lemma 281 (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*),
- Lemma 293 (*absolute value in  $\overline{\mathbb{R}}$  is continuous*),
- Lemma 377 (*decomposition into nonnegative and nonpositive parts*),
- Lemma 827 (*integral over singleton*).

**Lemma 274** (*equivalent definition of absolute value in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 275 (*bounded absolute value in  $\overline{\mathbb{R}}$* ),
- Lemma 276 (*bounded absolute value in  $\overline{\mathbb{R}}$  (strict)*),
- Lemma 346 (*compatibility of infimum with absolute value*),
- Lemma 815 (*compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$* ).

**Lemma 275** (*bounded absolute value in  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 276** (*bounded absolute value in  $\overline{\mathbb{R}}$  (strict)*)

is a direct dependency of:

- Lemma 277 (*finite absolute value in  $\overline{\mathbb{R}}$* ).

**Lemma 277** (*finite absolute value in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ).

**Lemma 278** (*absolute value in  $\overline{\mathbb{R}}$  is nonnegative*)

is a direct dependency of:

- Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),
- Lemma 773 (*Bienaymé–Chebyshev inequality*),
- Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),
- Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ),
- Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 279** (*absolute value in  $\overline{\mathbb{R}}$  is even*)

is a direct dependency of:

- Lemma 346 (*compatibility of infimum with absolute value*),
- Lemma 347 (*compatibility of supremum with absolute value*),
- Lemma 363 (*compatibility of limit superior with absolute value*),
- Lemma 816 (*integrable is almost finite*).

**Lemma 280** (*absolute value in  $\overline{\mathbb{R}}$  is definite*)

is a direct dependency of:

- Lemma 667 (*absolute value is almost definite*),
- Lemma 827 (*integral over singleton*),
- Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 281** (*absolute value in  $\overline{\mathbb{R}}$  satisfies triangle inequality*)

is a direct dependency of:

- Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Definition 282** (*exponential and logarithm in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 283 (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*),
- Lemma 285 (*exponentiation in  $\overline{\mathbb{R}}$* ).

**Lemma 283** (*exponential and logarithm in  $\overline{\mathbb{R}}$  are inverse*)

is not yet used.

**Definition 284** (*exponentiation in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 285 (*exponentiation in  $\overline{\mathbb{R}}$* ),
- Lemma 322 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 285** (*exponentiation in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 322 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),
- Lemma 325 (*Young's inequality for products (measure theory)*).

**Lemma 286** (*topology of  $\overline{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 290 (*convergence towards  $-\infty$* ),
- Lemma 338 ( *$\overline{\mathbb{R}}$  is second-countable*).

**Lemma 288** (*trace topology on  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 290** (*convergence towards  $-\infty$* )

is a direct dependency of:

- Lemma 342 (*equivalent definition of infimum*).

**Lemma 291** (*continuity of addition in  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 292** (*continuity of multiplication in  $\overline{\mathbb{R}}$* )

is not yet used.

**Lemma 293** (*absolute value in  $\overline{\mathbb{R}}$  is continuous*)

is a direct dependency of:

Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 294** (*addition in  $\overline{\mathbb{R}}_+$  is closed*)

is a direct dependency of:

Lemma 325 (*Young's inequality for products (measure theory)*),

Lemma 582 ( *$\mathcal{M}_+$  is closed under addition*),

Lemma 737 (*integral in  $\mathcal{SF}_+$* ),

Lemma 814 (*equivalent definition of integrability*).

**Lemma 295** (*addition in  $\overline{\mathbb{R}}_+$  is associative*)

is a direct dependency of:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 296** (*addition in  $\overline{\mathbb{R}}_+$  is commutative*)

is a direct dependency of:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 688 ( *$\mathcal{L}$  is closed under complement*),

Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 297** (*infinity-sum property in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 814 (*equivalent definition of integrability*).

**Lemma 298** (*series are convergent in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 299 (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* ),

Lemma 302 (*double series in  $\overline{\mathbb{R}}_+$* ),

Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*).

**Lemma 299** (*technical upper bound in series in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 300 (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* ).

**Lemma 300** (*order is meaningless in series in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 301 (*definition of double series in  $\overline{\mathbb{R}}_+$* ).

**Lemma 301** (*definition of double series in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 302 (*double series in  $\overline{\mathbb{R}}_+$* ),

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Lemma 302** (*double series in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*).

**Definition 303** (*multiplication in  $\overline{\mathbb{R}}_+$* )

is a direct dependency of:

Lemma 305 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*),

Lemma 306 (*zero-product property in  $\overline{\mathbb{R}}_+$* ),

Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 308 (*finite-product property in  $\overline{\mathbb{R}}_+$* ),  
 Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*).

**Lemma 305 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed when defined*)**  
 is a direct dependency of:  
 Lemma 325 (*Young's inequality for products (measure theory)*).

**Lemma 306 (*zero-product property in  $\overline{\mathbb{R}}_+$* )**  
 is a direct dependency of:  
 Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**Lemma 307 (*infinity-product property in  $\overline{\mathbb{R}}_+$* )**  
 is a direct dependency of:  
 Lemma 320 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 325 (*Young's inequality for products (measure theory)*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 308 (*finite-product property in  $\overline{\mathbb{R}}_+$* )**  
 is not yet used.

**Definition 309 (*multiplication in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 is a direct dependency of:  
 Lemma 311 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 312 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*),  
 Lemma 316 (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*),  
 Lemma 317 (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*),  
 Lemma 318 (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*),  
 Lemma 319 (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 320 (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*),  
 Lemma 322 (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 311 (*zero-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 is not yet used.

**Lemma 312 (*infinity-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 is a direct dependency of:  
 Lemma 313 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*).

**Lemma 313 (*finite-product property in  $\overline{\mathbb{R}}$  (measure theory)*)**  
 is a direct dependency of:  
 Lemma 321 (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*).

**Lemma 314 (*multiplication in  $\overline{\mathbb{R}}_+$  is closed (measure theory)*)**  
 is a direct dependency of:  
 Lemma 325 (*Young's inequality for products (measure theory)*),  
 Lemma 326 (*Young's inequality for products, case  $p = 2$  (measure theory)*),  
 Lemma 583 ( *$\mathcal{M}_+$  is closed under multiplication*),  
 Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 810 (*Tonelli for tensor product*).

**Lemma 316 (*multiplication in  $\overline{\mathbb{R}}_+$  is associative (measure theory)*)**  
 is a direct dependency of:  
 Lemma 326 (*Young's inequality for products, case  $p = 2$  (measure theory)*),  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 317** (*multiplication in  $\overline{\mathbb{R}}_+$  is commutative (measure theory)*)

is a direct dependency of:

- Lemma 326 (*Young's inequality for products, case  $p = 2$  (measure theory)*),
- Lemma 795 (*candidate tensor product measure is tensor product measure*),
- Lemma 810 (*Tonelli for tensor product*).

**Lemma 318** (*multiplication in  $\overline{\mathbb{R}}_+$  is distributive over addition (measure theory)*)

is a direct dependency of:

- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),
- Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 319** (*zero-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

is a direct dependency of:

- Lemma 325 (*Young's inequality for products (measure theory)*),
- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),
- Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),
- Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*),
- Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),
- Lemma 789 (*measure of section of product*),
- Lemma 805 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*).

**Lemma 320** (*infinity-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

is a direct dependency of:

- Lemma 325 (*Young's inequality for products (measure theory)*),
- Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*).

**Lemma 321** (*finite-product property in  $\overline{\mathbb{R}}_+$  (measure theory)*)

is a direct dependency of:

- Lemma 827 (*integral over singleton*).

**Lemma 322** (*exponentiation in  $\overline{\mathbb{R}}$  (measure theory)*)

is not yet used.

**Definition 323** (*Hölder conjugates*)

is not yet used.

**Lemma 325** (*Young's inequality for products (measure theory)*)

is a direct dependency of:

- Lemma 326 (*Young's inequality for products, case  $p = 2$  (measure theory)*).

**Lemma 326** (*Young's inequality for products, case  $p = 2$  (measure theory)*)

is not yet used.

**Definition 327** (*connected component in  $\mathbb{R}$* )

is a direct dependency of:

- Lemma 328 (*connected component of open subset of  $\mathbb{R}$  is open interval*),
- Lemma 329 (*connected component of open subset of  $\mathbb{R}$  is maximal*),
- Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ).

**Lemma 328** (*connected component of open subset of  $\mathbb{R}$  is open interval*)

is a direct dependency of:

- Lemma 329 (*connected component of open subset of  $\mathbb{R}$  is maximal*).

**Lemma 329** (*connected component of open subset of  $\mathbb{R}$  is maximal*)

is a direct dependency of:

- Lemma 330 (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*).



**Lemma 330** (*connected components of open subset of  $\mathbb{R}$  are equal or disjoint*)

is a direct dependency of:

Theorem 331 (*countable connected components of open subsets of  $\mathbb{R}$* ).

**Theorem 331** (*countable connected components of open subsets of  $\mathbb{R}$* )

is a direct dependency of:

Theorem 335 ( *$\mathbb{R}$  is second-countable*),

Lemma 338 ( *$\mathbb{R}$  is second-countable*),

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 544 (*countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ).

**Lemma 332** (*rational approximation of lower bound of open interval*)

is a direct dependency of:

Lemma 333 (*rational approximation of upper bound of open interval*),

Lemma 334 (*open intervals with rational bounds cover open interval*),

Lemma 337 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Lemma 333** (*rational approximation of upper bound of open interval*)

is a direct dependency of:

Lemma 334 (*open intervals with rational bounds cover open interval*),

Lemma 337 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Lemma 334** (*open intervals with rational bounds cover open interval*)

is a direct dependency of:

Theorem 335 ( *$\mathbb{R}$  is second-countable*),

Lemma 337 (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* ).

**Theorem 335** ( *$\mathbb{R}$  is second-countable*)

is a direct dependency of:

Lemma 336 ( *$\mathbb{R}^n$  is second-countable*),

Lemma 544 (*countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$* ).

**Lemma 336** ( *$\mathbb{R}^n$  is second-countable*)

is a direct dependency of:

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ).

**Lemma 337** (*open intervals with rational bounds cover open interval of  $\overline{\mathbb{R}}$* )

is a direct dependency of:

Lemma 338 ( *$\overline{\mathbb{R}}$  is second-countable*).

**Lemma 338** ( *$\overline{\mathbb{R}}$  is second-countable*)

is not yet used.

**Lemma 339** (*extrema of constant function*)

is a direct dependency of:

Lemma 341 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ).

**Lemma 340** (*equivalent definition of finite infimum*)

is a direct dependency of:

Lemma 341 (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* ).

**Lemma 341** (*equivalent definition of finite infimum in  $\overline{\mathbb{R}}$* )

is a direct dependency of:

Lemma 342 (*equivalent definition of infimum*).

**Lemma 342** (*equivalent definition of infimum*)

is not yet used.

**Lemma 343** (*infimum is smaller than supremum*)

is a direct dependency of:

Lemma 346 (*compatibility of infimum with absolute value*).

**Lemma 344** (*infimum is monotone*)

is a direct dependency of:

Lemma 345 (*supremum is monotone*),

Lemma 346 (*compatibility of infimum with absolute value*),

Lemma 348 (*compatibility of translation with infimum*),

Lemma 352 (*limit inferior*).

**Lemma 345** (*supremum is monotone*)

is a direct dependency of:

Lemma 346 (*compatibility of infimum with absolute value*),

Lemma 349 (*compatibility of translation with supremum*).

**Lemma 346** (*compatibility of infimum with absolute value*)

is a direct dependency of:

Lemma 347 (*compatibility of supremum with absolute value*),

Lemma 362 (*compatibility of limit inferior with absolute value*).

**Lemma 347** (*compatibility of supremum with absolute value*)

is not yet used.

**Lemma 348** (*compatibility of translation with infimum*)

is not yet used.

**Lemma 349** (*compatibility of translation with supremum*)

is not yet used.

**Lemma 350** (*infimum of sequence is monotone*)

is a direct dependency of:

Lemma 351 (*supremum of sequence is monotone*),

Lemma 356 (*limit inferior is monotone*).

**Lemma 351** (*supremum of sequence is monotone*)

is a direct dependency of:

Lemma 356 (*limit inferior is monotone*).

**Lemma 352** (*limit inferior*)

is a direct dependency of:

Lemma 353 (*limit inferior is  $\infty$* ),

Lemma 354 (*equivalent definition of the limit inferior*),

Lemma 356 (*limit inferior is monotone*),

Lemma 358 (*duality limit inferior-limit superior*),

Lemma 362 (*compatibility of limit inferior with absolute value*),

Lemma 573 ( *$\mathcal{M}$  is closed under limit inferior*),

Theorem 779 (*Fatou's lemma*).

**Lemma 353** (*limit inferior is  $\infty$* )

is a direct dependency of:

Lemma 354 (*equivalent definition of the limit inferior*),

Lemma 370 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 354** (*equivalent definition of the limit inferior*)

is a direct dependency of:

Lemma 355 (*limit inferior is invariant by translation*),

Lemma 359 (*equivalent definition of limit superior*),

Lemma 360 (*limit inferior is smaller than limit superior*),  
 Lemma 365 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 355** (*limit inferior is invariant by translation*)

is a direct dependency of:  
 Lemma 356 (*limit inferior is monotone*).

**Lemma 356** (*limit inferior is monotone*)

is a direct dependency of:  
 Lemma 361 (*limit superior is monotone*),  
 Lemma 366 (*limit inferior bounded from below*),  
 Lemma 367 (*limit inferior bounded from above*).

**Lemma 357** (*limit superior*)

is a direct dependency of:  
 Lemma 358 (*duality limit inferior-limit superior*),  
 Lemma 362 (*compatibility of limit inferior with absolute value*),  
 Lemma 574 ( *$\mathcal{M}$  is closed under limit superior*).

**Lemma 358** (*duality limit inferior-limit superior*)

is a direct dependency of:  
 Lemma 359 (*equivalent definition of limit superior*),  
 Lemma 361 (*limit superior is monotone*),  
 Lemma 363 (*compatibility of limit superior with absolute value*),  
 Lemma 368 (*limit superior bounded from below*),  
 Lemma 369 (*limit superior bounded from above*),  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*),  
 Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 359** (*equivalent definition of limit superior*)

is a direct dependency of:  
 Lemma 360 (*limit inferior is smaller than limit superior*),  
 Lemma 365 (*limit inferior and limit superior of pointwise convergent*),  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 360** (*limit inferior is smaller than limit superior*)

is a direct dependency of:  
 Lemma 370 (*limit inferior, limit superior and pointwise convergence*).

**Lemma 361** (*limit superior is monotone*)

is not yet used.

**Lemma 362** (*compatibility of limit inferior with absolute value*)

is a direct dependency of:  
 Lemma 363 (*compatibility of limit superior with absolute value*).

**Lemma 363** (*compatibility of limit superior with absolute value*)

is not yet used.

**Definition 364** (*pointwise convergence*)

is a direct dependency of:  
 Lemma 365 (*limit inferior and limit superior of pointwise convergent*),  
 Theorem 779 (*Fatou's lemma*).

**Lemma 365** (*limit inferior and limit superior of pointwise convergent*)

is a direct dependency of:  
 Lemma 366 (*limit inferior bounded from below*),

Lemma 367 (*limit inferior bounded from above*),  
 Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
 Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 366** (*limit inferior bounded from below*)

is a direct dependency of:

Lemma 369 (*limit superior bounded from above*),  
 Theorem 779 (*Fatou's lemma*),  
 Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 367** (*limit inferior bounded from above*)

is a direct dependency of:

Lemma 368 (*limit superior bounded from below*).

**Lemma 368** (*limit superior bounded from below*)

is not yet used.

**Lemma 369** (*limit superior bounded from above*)

is not yet used.

**Lemma 370** (*limit inferior, limit superior and pointwise convergence*)

is a direct dependency of:

Lemma 575 ( *$\mathcal{M}$  is closed under limit when pointwise convergent*),  
 Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
 Theorem 857 (*Lebesgue, dominated convergence*).

**Definition 371** (*finite part*)

is a direct dependency of:

Lemma 372 (*finite part is finite*),  
 Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 669 (*finite nonnegative part*).

**Lemma 372** (*finite part is finite*)

is a direct dependency of:

Lemma 565 ( *$\mathcal{M}$  is closed under finite part*).

**Definition 373** (*nonnegative and nonpositive parts*)

is a direct dependency of:

Lemma 374 (*equivalent definition of nonnegative and nonpositive parts*),  
 Lemma 375 (*nonnegative and nonpositive parts are nonnegative*),  
 Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),  
 Lemma 379 (*compatibility of nonpositive and nonnegative parts with mask*),  
 Lemma 380 (*compatibility of nonpositive and nonnegative parts with restriction*),  
 Lemma 579 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),  
 Lemma 827 (*integral over singleton*),  
 Lemma 841 (*integral is homogeneous*).

**Lemma 374** (*equivalent definition of nonnegative and nonpositive parts*)

is a direct dependency of:

Lemma 669 (*finite nonnegative part*).

**Lemma 375** (*nonnegative and nonpositive parts are nonnegative*)

is a direct dependency of:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),

Lemma 579 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 376** (*nonnegative and nonpositive parts are orthogonal*)

is a direct dependency of:

Lemma 377 (*decomposition into nonnegative and nonpositive parts*),  
 Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*).

**Lemma 377** (*decomposition into nonnegative and nonpositive parts*)

is a direct dependency of:

Lemma 378 (*compatibility of nonpositive and nonnegative parts with addition*),  
 Lemma 579 (*measurability of nonnegative and nonpositive parts*),  
 Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),  
 Lemma 827 (*integral over singleton*),  
 Lemma 830 (*integral for counting measure*).

**Lemma 378** (*compatibility of nonpositive and nonnegative parts with addition*)

is a direct dependency of:

Lemma 768 (*compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts*).

**Lemma 379** (*compatibility of nonpositive and nonnegative parts with mask*)

is a direct dependency of:

Lemma 825 (*integral over subset*).

**Lemma 380** (*compatibility of nonpositive and nonnegative parts with restriction*)

is a direct dependency of:

Lemma 825 (*integral over subset*).

**Definition 381** (*relation compatible with vector operations*)

is a direct dependency of:

Lemma 382 (*quotient vector operations*),  
 Lemma 384 (*quotient vector space*).

**Lemma 382** (*quotient vector operations*)

is a direct dependency of:

Lemma 383 (*quotient vector space, equivalence relation*),  
 Lemma 385 (*linear map on quotient vector space*).

**Lemma 383** (*quotient vector space, equivalence relation*)

is a direct dependency of:

Lemma 384 (*quotient vector space*).

**Lemma 384** (*quotient vector space*)

is a direct dependency of:

Lemma 385 (*linear map on quotient vector space*).

**Lemma 385** (*linear map on quotient vector space*)

is not yet used.

**Definition 388** (*algebra over a field*)

is a direct dependency of:

Lemma 390 ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*),  
 Lemma 393 (*algebra of functions to algebra*),  
 Lemma 397 (*vector subspace and closed under multiplication is subalgebra*),  
 Lemma 559 ( *$\mathcal{M}_{\mathbb{R}}$  is vector space*).

**Lemma 390** ( *$\mathbb{K}$  is  $\mathbb{K}$ -algebra*)

is a direct dependency of:

Lemma 394 ( *$\mathbb{K}^X$  is algebra*),

Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*).

**Definition 391** (*inherited algebra operations*)

is a direct dependency of:

Lemma 393 (*algebra of functions to algebra*).

**Lemma 393** (*algebra of functions to algebra*)

is a direct dependency of:

Lemma 394 ( *$\mathbb{K}^X$  is algebra*),

Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*).

**Lemma 394** ( *$\mathbb{K}^X$  is algebra*)

is not yet used.

**Definition 395** (*subalgebra*)

is a direct dependency of:

Lemma 397 (*vector subspace and closed under multiplication is subalgebra*),

Lemma 559 ( *$\mathcal{M}_{\mathbb{R}}$  is vector space*).

**Lemma 397** (*vector subspace and closed under multiplication is subalgebra*)

is a direct dependency of:

Lemma 398 (*closed under algebra operations is subalgebra*).

**Lemma 398** (*closed under algebra operations is subalgebra*)

is a direct dependency of:

Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),

Lemma 559 ( *$\mathcal{M}_{\mathbb{R}}$  is vector space*).

**Definition 399** (*seminorm*)

is a direct dependency of:

Lemma 401 (*definite seminorm is norm*),

Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 401** (*definite seminorm is norm*)

is not yet used.

**Lemma 404** (*nonempty and with empty or full*)

is a direct dependency of:

Lemma 433 (*equivalent definition of set algebra*),

Lemma 455 (*other properties of  $\lambda$ -system*),

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*).

**Lemma 405** (*with empty and full*)

is a direct dependency of:

Lemma 418 (*closedness under countable disjoint union and local complement*),

Lemma 433 (*equivalent definition of set algebra*),

Lemma 455 (*other properties of  $\lambda$ -system*),

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 407** (*closedness under local complement and complement*)

is a direct dependency of:

Lemma 434 (*other equivalent definition of set algebra*),

Lemma 454 (*equivalent definition of  $\lambda$ -system*).

**Lemma 408** (*closedness under disjoint union and local complement*)

is a direct dependency of:

Lemma 418 (*closedness under countable disjoint union and local complement*).

**Lemma 409** (*closedness under set difference and local complement*)

is a direct dependency of:

Lemma 423 (*closedness under countable monotone union and countable union*),

Lemma 434 (*other equivalent definition of set algebra*),

Lemma 435 (*set algebra is closed under local complement*).

**Lemma 410** (*closedness under intersection and set difference*)

is a direct dependency of:

Lemma 412 (*closedness under union and set difference*),

Lemma 422 (*closedness under countable disjoint union and countable union*),

Lemma 423 (*closedness under countable monotone union and countable union*),

Lemma 434 (*other equivalent definition of set algebra*).

**Lemma 411** (*closedness under union and intersection*)

is a direct dependency of:

Lemma 412 (*closedness under union and set difference*),

Lemma 414 (*closedness under finite union and intersection*),

Lemma 422 (*closedness under countable disjoint union and countable union*),

Lemma 423 (*closedness under countable monotone union and countable union*).

**Lemma 412** (*closedness under union and set difference*)

is not yet used.

**Lemma 413** (*closedness under finite operations*)

is a direct dependency of:

Lemma 414 (*closedness under finite union and intersection*),

Lemma 418 (*closedness under countable disjoint union and local complement*),

Lemma 422 (*closedness under countable disjoint union and countable union*),

Lemma 433 (*equivalent definition of set algebra*),

Lemma 441 (*explicit set algebra*),

Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*).

**Lemma 414** (*closedness under finite union and intersection*)

is a direct dependency of:

Lemma 433 (*equivalent definition of set algebra*).

**Lemma 417** (*closedness under countable and finite disjoint union*)

is a direct dependency of:

Lemma 418 (*closedness under countable disjoint union and local complement*).

**Lemma 418** (*closedness under countable disjoint union and local complement*)

is a direct dependency of:

Lemma 454 (*equivalent definition of  $\lambda$ -system*).

**Lemma 419** (*closedness under countable union and intersection*)

is a direct dependency of:

Lemma 455 (*other properties of  $\lambda$ -system*),

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*).

**Lemma 420** (*closedness under countable disjoint and monotone union*)

is a direct dependency of:

Lemma 454 (*equivalent definition of  $\lambda$ -system*).



**Lemma 421** (*closedness under countable monotone and disjoint union*)

is a direct dependency of:

- Lemma 423 (*closedness under countable monotone union and countable union*),
- Lemma 454 (*equivalent definition of  $\lambda$ -system*).

**Lemma 422** (*closedness under countable disjoint union and countable union*)

is a direct dependency of:

- Lemma 423 (*closedness under countable monotone union and countable union*),
- Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*),
- Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*).

**Lemma 423** (*closedness under countable monotone union and countable union*)

is a direct dependency of:

- Lemma 482 (*set algebra contains  $\sigma$ -algebra*),
- Lemma 486 (*monotone class contains  $\sigma$ -algebra*).

**Definition 424** ( *$\pi$ -system*)

is a direct dependency of:

- Lemma 426 (*intersection of  $\pi$ -systems*),
- Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*),
- Lemma 477 ( *$\sigma$ -algebra is  $\pi$ -system*),
- Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*),
- Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*),
- Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),
- Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 426** (*intersection of  $\pi$ -systems*)

is a direct dependency of:

- Lemma 428 (*generated  $\pi$ -system is minimum*).

**Definition 427** (*generated  $\pi$ -system*)

is a direct dependency of:

- Lemma 428 (*generated  $\pi$ -system is minimum*).

**Lemma 428** (*generated  $\pi$ -system is minimum*)

is a direct dependency of:

- Lemma 429 ( *$\pi$ -system generation is monotone*),
- Lemma 430 ( *$\pi$ -system generation is idempotent*),
- Lemma 439 (*set algebra generation is monotone*),
- Lemma 478 ( *$\sigma$ -algebra contains  $\pi$ -system*),
- Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*),
- Lemma 480 ( *$\sigma$ -algebra generated by  $\pi$ -system*),
- Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 429** ( *$\pi$ -system generation is monotone*)

is not yet used.

**Lemma 430** ( *$\pi$ -system generation is idempotent*)

is not yet used.

**Definition 432** (*set algebra*)

is a direct dependency of:

- Lemma 433 (*equivalent definition of set algebra*),
- Lemma 436 (*intersection of set algebras*),
- Lemma 441 (*explicit set algebra*),
- Lemma 469 ( *$\sigma$ -algebra is set algebra*),
- Lemma 482 (*set algebra contains  $\sigma$ -algebra*),
- Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*).

**Lemma 433** (*equivalent definition of set algebra*)

is a direct dependency of:

- Lemma 434 (*other equivalent definition of set algebra*),
- Lemma 441 (*explicit set algebra*).

**Lemma 434** (*other equivalent definition of set algebra*)

is a direct dependency of:

- Lemma 435 (*set algebra is closed under local complement*),
- Lemma 452 (*monotone class generated by set algebra*),
- Lemma 470 ( *$\sigma$ -algebra is closed under set difference*).

**Lemma 435** (*set algebra is closed under local complement*)

is a direct dependency of:

- Lemma 470 ( *$\sigma$ -algebra is closed under set difference*).

**Lemma 436** (*intersection of set algebras*)

is a direct dependency of:

- Lemma 438 (*generated set algebra is minimum*).

**Definition 437** (*generated set algebra*)

is a direct dependency of:

- Lemma 438 (*generated set algebra is minimum*).

**Lemma 438** (*generated set algebra is minimum*)

is a direct dependency of:

- Lemma 440 (*set algebra generation is idempotent*),
- Lemma 441 (*explicit set algebra*),
- Lemma 481 ( *$\sigma$ -algebra contains set algebra*),
- Lemma 482 (*set algebra contains  $\sigma$ -algebra*),
- Lemma 483 ( *$\sigma$ -algebra generated by set algebra*),
- Lemma 509 (*usage of monotone class theorem*),
- Lemma 791 (*measurability of measure of section (finite)*).

**Lemma 439** (*set algebra generation is monotone*)

is not yet used.

**Lemma 440** (*set algebra generation is idempotent*)

is not yet used.

**Lemma 441** (*explicit set algebra*)

is a direct dependency of:

- Lemma 501 (*set algebra generated by product of  $\sigma$ -algebras*).

**Definition 442** (*monotone class*)

is a direct dependency of:

- Lemma 443 (*intersection of monotone classes*),
- Lemma 450 ( *$\mathcal{C}^{\setminus}$  is monotone class*),
- Lemma 484 ( *$\sigma$ -algebra is monotone class*),
- Lemma 486 (*monotone class contains  $\sigma$ -algebra*),
- Lemma 791 (*measurability of measure of section (finite)*),
- Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 443** (*intersection of monotone classes*)

is a direct dependency of:

- Lemma 445 (*generated monotone class is minimum*).

**Definition 444** (*generated monotone class*)

is a direct dependency of:

Lemma 445 (*generated monotone class is minimum*).

**Lemma 445** (*generated monotone class is minimum*)

is a direct dependency of:

Lemma 446 (*monotone class generation is monotone*),

Lemma 447 (*monotone class generation is idempotent*),

Lemma 451 (*monotone class closed under set difference*),

Lemma 452 (*monotone class generated by set algebra*),

Lemma 485 ( *$\sigma$ -algebra contains monotone class*),

Lemma 486 (*monotone class contains  $\sigma$ -algebra*),

Lemma 487 ( *$\sigma$ -algebra generated by monotone class*),

Theorem 507 (*monotone class*).

**Lemma 446** (*monotone class generation is monotone*)

is a direct dependency of:

Lemma 509 (*usage of monotone class theorem*).

**Lemma 447** (*monotone class generation is idempotent*)

is a direct dependency of:

Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*),

Lemma 509 (*usage of monotone class theorem*).

**Definition 448** (*monotone class and symmetric set difference*)

is a direct dependency of:

Lemma 449 ( *$\mathcal{C}^{\setminus}$  is symmetric*),

Lemma 450 ( *$\mathcal{C}^{\setminus}$  is monotone class*),

Lemma 451 (*monotone class closed under set difference*).

**Lemma 449** ( *$\mathcal{C}^{\setminus}$  is symmetric*)

is a direct dependency of:

Lemma 451 (*monotone class closed under set difference*).

**Lemma 450** ( *$\mathcal{C}^{\setminus}$  is monotone class*)

is a direct dependency of:

Lemma 451 (*monotone class closed under set difference*).

**Lemma 451** (*monotone class closed under set difference*)

is a direct dependency of:

Lemma 452 (*monotone class generated by set algebra*).

**Lemma 452** (*monotone class generated by set algebra*)

is a direct dependency of:

Theorem 507 (*monotone class*).

**Definition 453** ( *$\lambda$ -system*)

is a direct dependency of:

Lemma 454 (*equivalent definition of  $\lambda$ -system*),

Lemma 455 (*other properties of  $\lambda$ -system*),

Lemma 456 (*intersection of  $\lambda$ -systems*),

Lemma 463 ( *$\Lambda^{\cap}$  is  $\lambda$ -system*),

Lemma 488 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*).

**Lemma 454** (*equivalent definition of  $\lambda$ -system*)

is a direct dependency of:

- Lemma 455 (*other properties of  $\lambda$ -system*),  
 Lemma 463 ( *$\Lambda^\cap$  is  $\lambda$ -system*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*).

**Lemma 455** (*other properties of  $\lambda$ -system*)  
 is not yet used.

**Lemma 456** (*intersection of  $\lambda$ -systems*)  
 is a direct dependency of:  
 Lemma 458 (*generated  $\lambda$ -system is minimum*).

**Definition 457** (*generated  $\lambda$ -system*)  
 is a direct dependency of:  
 Lemma 458 (*generated  $\lambda$ -system is minimum*).

**Lemma 458** (*generated  $\lambda$ -system is minimum*)  
 is a direct dependency of:  
 Lemma 459 ( *$\lambda$ -system generation is monotone*),  
 Lemma 460 ( *$\lambda$ -system generation is idempotent*),  
 Lemma 464 ( *$\lambda$ -system with intersection*),  
 Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*),  
 Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*),  
 Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*),  
 Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),  
 Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 459** ( *$\lambda$ -system generation is monotone*)  
 is a direct dependency of:  
 Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 460** ( *$\lambda$ -system generation is idempotent*)  
 is a direct dependency of:  
 Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*),  
 Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Definition 461** ( *$\lambda$ -system and intersection*)  
 is a direct dependency of:  
 Lemma 462 ( *$\Lambda^\cap$  is symmetric*),  
 Lemma 464 ( *$\lambda$ -system with intersection*),  
 Lemma 465 ( *$\lambda$ -system closed under intersection*).

**Lemma 462** ( *$\Lambda^\cap$  is symmetric*)  
 is a direct dependency of:  
 Lemma 464 ( *$\lambda$ -system with intersection*).

**Lemma 463** ( *$\Lambda^\cap$  is  $\lambda$ -system*)  
 is a direct dependency of:  
 Lemma 464 ( *$\lambda$ -system with intersection*).

**Lemma 464** ( *$\lambda$ -system with intersection*)  
 is a direct dependency of:  
 Lemma 465 ( *$\lambda$ -system closed under intersection*).

**Lemma 465** ( *$\lambda$ -system closed under intersection*)  
 is a direct dependency of:  
 Lemma 466 ( *$\lambda$ -system generated by  $\pi$ -system*).

**Lemma 466** ( *$\lambda$ -system generated by  $\pi$ -system*)

is a direct dependency of:

Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Definition 467** ( *$\sigma$ -algebra*)

is a direct dependency of:

Lemma 468 (*equivalent definition of  $\sigma$ -algebra*),

Lemma 469 ( *$\sigma$ -algebra is set algebra*),

Lemma 472 (*intersection of  $\sigma$ -algebras*),

Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*),

Lemma 482 (*set algebra contains  $\sigma$ -algebra*),

Lemma 486 (*monotone class contains  $\sigma$ -algebra*),

Lemma 494 (*countable  $\sigma$ -algebra generator*),

Lemma 496 (*some Borel subsets*),

Lemma 511 (*inverse  $\sigma$ -algebra*),

Lemma 512 (*image  $\sigma$ -algebra*),

Lemma 520 (*trace  $\sigma$ -algebra*),

Lemma 524 (*characterization of Borel subsets*),

Lemma 538 (*measurability of section*),

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 546 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),

Lemma 548 (*Borel  $\sigma$ -algebra of  $\mathbb{R}_+$* ),

Lemma 554 (*measurability of indicator function*),

Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),

Lemma 576 (*measurability and masking*),

Lemma 577 (*measurability of restriction*),

Lemma 598 (*measure satisfies the finite Boole inequality*),

Lemma 600 (*measure is continuous from below*),

Lemma 603 (*measure satisfies the Boole inequality*),

Lemma 604 (*equivalent definition of measure*),

Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),

Lemma 620 (*empty set is negligible*),

Lemma 621 (*compatibility of null measure with countable union*),

Lemma 632 (*compatibility of almost binary relation with reflexivity*),

Lemma 652 (*counting measure*),

Lemma 654 ( *$\sigma$ -finite counting measure*),

Lemma 668 (*masking almost nowhere*),

Lemma 669 (*finite nonnegative part*),

Lemma 698 ( *$\mathcal{L}$  is  $\sigma$ -algebra*),

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),

Lemma 719 (*integral in  $\mathcal{IF}$  over subset is additive*),

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),

Lemma 786 (*countable union of sections is measurable*),

Lemma 854 (*constant function is  $\mathcal{L}^1$* ),

Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 468** (*equivalent definition of  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 471 (*other properties of  $\sigma$ -algebra*),

Lemma 477 ( *$\sigma$ -algebra is  $\pi$ -system*),

Lemma 488 ( *$\sigma$ -algebra is  $\lambda$ -system*),

Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*),

Lemma 501 (*set algebra generated by product of  $\sigma$ -algebras*),

Lemma 514 (*constant function is measurable*),

Lemma 521 (*measurability of measurable subspace*),

Lemma 524 (*characterization of Borel subsets*),  
 Lemma 527 (*measurability of function defined on a partition*),  
 Lemma 531 (*measurability of function to product space*),  
 Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),  
 Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ),  
 Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ),  
 Lemma 554 (*measurability of indicator function*),  
 Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),  
 Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),  
 Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),  
 Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),  
 Lemma 577 (*measurability of restriction*),  
 Lemma 596 (*measure over countable partition*),  
 Lemma 602 (*measure is continuous from above*),  
 Lemma 606 (*finite measure is bounded*),  
 Lemma 612 (*restricted measure*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 653 (*finiteness of counting measure*),  
 Lemma 658 (*Dirac measure is finite*),  
 Lemma 661 (*measurability of summability domain*),  
 Lemma 719 (*integral in  $\mathcal{IF}$  over subset is additive*),  
 Lemma 727 ( *$\mathcal{SF}$  canonical representation*),  
 Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 748 (*integral in  $\mathcal{SF}_+$  over subset is additive*),  
 Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Lemma 787 (*countable intersection of sections is measurable*),  
 Lemma 792 (*measurability of measure of section*),  
 Lemma 796 (*tensor product of finite measures*),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*),  
 Lemma 801 (*uniqueness of tensor product measure*),  
 Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 469 ( $\sigma$ -algebra is set algebra)**

is a direct dependency of:

Lemma 470 ( *$\sigma$ -algebra is closed under set difference*),  
 Lemma 481 ( *$\sigma$ -algebra contains set algebra*).

**Lemma 470 ( $\sigma$ -algebra is closed under set difference)**

is a direct dependency of:

Lemma 597 (*measure is monotone*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 600 (*measure is continuous from below*),  
 Lemma 602 (*measure is continuous from above*).

**Lemma 471 (other properties of  $\sigma$ -algebra)**

is a direct dependency of:

Lemma 484 ( *$\sigma$ -algebra is monotone class*),  
 Lemma 488 ( *$\sigma$ -algebra is  $\lambda$ -system*).

**Lemma 472 (intersection of  $\sigma$ -algebras)**

is a direct dependency of:

Lemma 474 (*generated  $\sigma$ -algebra is minimum*).

**Definition 473 (generated  $\sigma$ -algebra)**

is a direct dependency of:

- Lemma 474 (*generated  $\sigma$ -algebra is minimum*),
- Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),
- Lemma 509 (*usage of monotone class theorem*),
- Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),
- Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ).

**Lemma 474 (generated  $\sigma$ -algebra is minimum)**

is a direct dependency of:

- Lemma 475 ( *$\sigma$ -algebra generation is monotone*),
- Lemma 476 ( *$\sigma$ -algebra generation is idempotent*),
- Lemma 478 ( *$\sigma$ -algebra contains  $\pi$ -system*),
- Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*),
- Lemma 480 ( *$\sigma$ -algebra generated by  $\pi$ -system*),
- Lemma 481 ( *$\sigma$ -algebra contains set algebra*),
- Lemma 482 (*set algebra contains  $\sigma$ -algebra*),
- Lemma 483 ( *$\sigma$ -algebra generated by set algebra*),
- Lemma 485 ( *$\sigma$ -algebra contains monotone class*),
- Lemma 486 (*monotone class contains  $\sigma$ -algebra*),
- Lemma 487 ( *$\sigma$ -algebra generated by monotone class*),
- Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*),
- Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*),
- Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),
- Lemma 493 (*complete generated  $\sigma$ -algebra*),
- Lemma 494 (*countable  $\sigma$ -algebra generator*),
- Lemma 496 (*some Borel subsets*),
- Lemma 515 (*inverse image of generating family*),
- Lemma 517 (*continuous is measurable*),
- Lemma 530 (*product of measurable subsets is measurable*),
- Lemma 533 (*generating product measurable space*),
- Lemma 538 (*measurability of section*).

**Lemma 475 ( $\sigma$ -algebra generation is monotone)**

is a direct dependency of:

- Lemma 480 ( *$\sigma$ -algebra generated by  $\pi$ -system*),
- Lemma 483 ( *$\sigma$ -algebra generated by set algebra*),
- Lemma 487 ( *$\sigma$ -algebra generated by monotone class*),
- Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),
- Lemma 492 (*other  $\sigma$ -algebra generator*),
- Lemma 493 (*complete generated  $\sigma$ -algebra*),
- Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*),
- Lemma 509 (*usage of monotone class theorem*),
- Lemma 515 (*inverse image of generating family*),
- Lemma 516 (*equivalent definition of measurable function*),
- Lemma 533 (*generating product measurable space*),
- Lemma 700 ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* ).

**Lemma 476 ( $\sigma$ -algebra generation is idempotent)**

is a direct dependency of:

- Lemma 492 (*other  $\sigma$ -algebra generator*),
- Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*),
- Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*),
- Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*),
- Theorem 507 (*monotone class*),



Lemma 515 (*inverse image of generating family*),  
 Lemma 516 (*equivalent definition of measurable function*).

**Lemma 477 ( *$\sigma$ -algebra is  $\pi$ -system*)**

is a direct dependency of:

Lemma 478 ( *$\sigma$ -algebra contains  $\pi$ -system*).

**Lemma 478 ( *$\sigma$ -algebra contains  $\pi$ -system*)**

is a direct dependency of:

Lemma 480 ( *$\sigma$ -algebra generated by  $\pi$ -system*).

**Lemma 479 ( *$\pi$ -system contains  $\sigma$ -algebra*)**

is not yet used.

**Lemma 480 ( *$\sigma$ -algebra generated by  $\pi$ -system*)**

is not yet used.

**Lemma 481 ( *$\sigma$ -algebra contains set algebra*)**

is a direct dependency of:

Lemma 483 ( *$\sigma$ -algebra generated by set algebra*),

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 482 (*set algebra contains  $\sigma$ -algebra*)**

is not yet used.

**Lemma 483 ( *$\sigma$ -algebra generated by set algebra*)**

is not yet used.

**Lemma 484 ( *$\sigma$ -algebra is monotone class*)**

is a direct dependency of:

Lemma 485 ( *$\sigma$ -algebra contains monotone class*).

**Lemma 485 ( *$\sigma$ -algebra contains monotone class*)**

is a direct dependency of:

Lemma 487 ( *$\sigma$ -algebra generated by monotone class*),

Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*).

**Lemma 486 (*monotone class contains  $\sigma$ -algebra*)**

is a direct dependency of:

Lemma 506 (*algebra and monotone class is  $\sigma$ -algebra*).

**Lemma 487 ( *$\sigma$ -algebra generated by monotone class*)**

is a direct dependency of:

Theorem 507 (*monotone class*).

**Lemma 488 ( *$\sigma$ -algebra is  $\lambda$ -system*)**

is a direct dependency of:

Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*).

**Lemma 489 ( *$\sigma$ -algebra contains  $\lambda$ -system*)**

is a direct dependency of:

Lemma 491 ( *$\sigma$ -algebra generated by  $\lambda$ -system*),

Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*).

**Lemma 490 ( *$\lambda$ -system contains  $\sigma$ -algebra*)**

is a direct dependency of:

Lemma 502 ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*).

**Lemma 491** ( *$\sigma$ -algebra generated by  $\lambda$ -system*)

is a direct dependency of:

Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 492** (*other  $\sigma$ -algebra generator*)

is a direct dependency of:

Lemma 493 (*complete generated  $\sigma$ -algebra*),

Lemma 494 (*countable  $\sigma$ -algebra generator*),

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ).

**Lemma 493** (*complete generated  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ),

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 494** (*countable  $\sigma$ -algebra generator*)

is a direct dependency of:

Lemma 497 (*countable Borel  $\sigma$ -algebra generator*).

**Definition 495** (*Borel  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 496 (*some Borel subsets*),

Lemma 497 (*countable Borel  $\sigma$ -algebra generator*),

Lemma 517 (*continuous is measurable*),

Lemma 523 (*Borel sub- $\sigma$ -algebra*),

Lemma 524 (*characterization of Borel subsets*),

Lemma 546 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ),

Lemma 555 (*measurability of numeric function to  $\mathbb{R}$* ),

Lemma 563 (*measurability of numeric function*),

Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Lemma 496** (*some Borel subsets*)

is a direct dependency of:

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 555 (*measurability of numeric function to  $\mathbb{R}$* ),

Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),

Lemma 563 (*measurability of numeric function*),

Lemma 564 (*inverse image is measurable*),

Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ),

Lemma 828 (*integral over interval*).

**Lemma 497** (*countable Borel  $\sigma$ -algebra generator*)

is a direct dependency of:

Lemma 543 (*Borel  $\sigma$ -algebra of  $\mathbb{R}$* ),

Lemma 545 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$* ),

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ).

**Lemma 501** (*set algebra generated by product of  $\sigma$ -algebras*)

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 502** ( *$\pi$ -system and  $\lambda$ -system is  $\sigma$ -algebra*)

is a direct dependency of:

Theorem 503 (*Dynkin  $\pi$ - $\lambda$  theorem*).

**Theorem 503** (*Dynkin  $\pi$ - $\lambda$  theorem*)

is a direct dependency of:

Lemma 505 (*usage of Dynkin  $\pi$ - $\lambda$  theorem*).

**Lemma 505** (*usage of Dynkin  $\pi$ - $\lambda$  theorem*)

is a direct dependency of:

Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*).

**Lemma 506** (*algebra and monotone class is  $\sigma$ -algebra*)

is a direct dependency of:

Theorem 507 (*monotone class*).

**Theorem 507** (*monotone class*)

is a direct dependency of:

Lemma 509 (*usage of monotone class theorem*).

**Lemma 509** (*usage of monotone class theorem*)

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Definition 510** (*measurable function*)

is a direct dependency of:

Lemma 511 (*inverse  $\sigma$ -algebra*),

Lemma 512 (*image  $\sigma$ -algebra*),

Lemma 513 (*identity function is measurable*),

Lemma 516 (*equivalent definition of measurable function*),

Lemma 517 (*continuous is measurable*),

Lemma 518 (*compatibility of measurability with composition*),

Lemma 520 (*trace  $\sigma$ -algebra*),

Lemma 525 (*source restriction of measurable function*),

Lemma 526 (*destination restriction of measurable function*),

Lemma 527 (*measurability of function defined on a partition*),

Lemma 531 (*measurability of function to product space*),

Lemma 532 (*canonical projection is measurable*),

Lemma 533 (*generating product measurable space*),

Lemma 540 (*measurability of function from product space*),

Lemma 555 (*measurability of numeric function to  $\mathbb{R}$* ),

Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),

Lemma 562 ( *$\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$* ),

Lemma 563 (*measurability of numeric function*),

Lemma 564 (*inverse image is measurable*),

Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),

Lemma 577 (*measurability of restriction*).

**Lemma 511** (*inverse  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 515 (*inverse image of generating family*).

**Lemma 512** (*image  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 515 (*inverse image of generating family*).

**Lemma 513** (*identity function is measurable*)

is not yet used.

**Lemma 514** (*constant function is measurable*)

is a direct dependency of:

- Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),
- Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 570 ( *$\mathcal{M}$  is closed under scalar multiplication*),
- Lemma 579 (*measurability of nonnegative and nonpositive parts*).

**Lemma 515** (*inverse image of generating family*)

is a direct dependency of:

- Lemma 516 (*equivalent definition of measurable function*),
- Lemma 522 (*generating measurable subspace*),
- Lemma 533 (*generating product measurable space*).

**Lemma 516** (*equivalent definition of measurable function*)

is a direct dependency of:

- Lemma 517 (*continuous is measurable*),
- Lemma 531 (*measurability of function to product space*),
- Lemma 555 (*measurability of numeric function to  $\mathbb{R}$* ),
- Lemma 563 (*measurability of numeric function*).

**Lemma 517** (*continuous is measurable*)

is a direct dependency of:

- Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),
- Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*).

**Lemma 518** (*compatibility of measurability with composition*)

is a direct dependency of:

- Lemma 525 (*source restriction of measurable function*),
- Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*),
- Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*),
- Lemma 588 (*measurability of tensor product of numeric functions*).

**Lemma 520** (*trace  $\sigma$ -algebra*)

is a direct dependency of:

- Lemma 521 (*measurability of measurable subspace*),
- Lemma 525 (*source restriction of measurable function*),
- Lemma 611 (*trace measure*).

**Lemma 521** (*measurability of measurable subspace*)

is a direct dependency of:

- Lemma 523 (*Borel sub- $\sigma$ -algebra*),
- Lemma 611 (*trace measure*),
- Lemma 714 ( *$\mathcal{IF}$  is closed under restriction*).

**Lemma 522** (*generating measurable subspace*)

is a direct dependency of:

- Lemma 523 (*Borel sub- $\sigma$ -algebra*),
- Lemma 549 (*Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}_+$* ).

**Lemma 523** (*Borel sub- $\sigma$ -algebra*)

is a direct dependency of:

- Lemma 524 (*characterization of Borel subsets*),
- Lemma 546 (*Borel subsets of  $\mathbb{R}$  and  $\mathbb{R}$* ),
- Lemma 548 (*Borel  $\sigma$ -algebra of  $\mathbb{R}_+$* ).

**Lemma 524** (*characterization of Borel subsets*)

is a direct dependency of:

Lemma 546 (*Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$* ).

**Lemma 525** (*source restriction of measurable function*)

is not yet used.

**Lemma 526** (*destination restriction of measurable function*)

is not yet used.

**Lemma 527** (*measurability of function defined on a partition*)

is a direct dependency of:

Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),

Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),

Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*).

**Definition 529** (*tensor product of  $\sigma$ -algebras*)

is a direct dependency of:

Lemma 530 (*product of measurable subsets is measurable*),

Lemma 531 (*measurability of function to product space*),

Lemma 533 (*generating product measurable space*),

Lemma 538 (*measurability of section*),

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 530** (*product of measurable subsets is measurable*)

is a direct dependency of:

Lemma 531 (*measurability of function to product space*),

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 795 (*candidate tensor product measure is tensor product measure*),

Lemma 797 (*tensor product of  $\sigma$ -finite measures*),

Lemma 799 (*uniqueness of tensor product measure (finite)*),

Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 531** (*measurability of function to product space*)

is a direct dependency of:

Lemma 532 (*canonical projection is measurable*),

Lemma 557 ( *$\mathcal{M}_{\mathbb{R}}$  is algebra*).

**Lemma 532** (*canonical projection is measurable*)

is a direct dependency of:

Lemma 533 (*generating product measurable space*),

Lemma 588 (*measurability of tensor product of numeric functions*).

**Lemma 533** (*generating product measurable space*)

is a direct dependency of:

Lemma 550 (*Borel  $\sigma$ -algebra of  $\mathbb{R}^n$* ).

**Definition 535** (*section in Cartesian product*)

is a direct dependency of:

Lemma 536 (*section of product*),

Lemma 537 (*compatibility of section with set operations*),

Lemma 539 (*indicator of section*).

**Lemma 536** (*section of product*)

is a direct dependency of:

Lemma 538 (*measurability of section*),

Lemma 789 (*measure of section of product*).

**Lemma 537 (compatibility of section with set operations)**

is a direct dependency of:

Lemma 538 (measurability of section),

Lemma 786 (countable union of sections is measurable),

Lemma 787 (countable intersection of sections is measurable),

Lemma 791 (measurability of measure of section (finite)),

Lemma 795 (candidate tensor product measure is tensor product measure).

**Lemma 538 (measurability of section)**

is a direct dependency of:

Lemma 540 (measurability of function from product space),

Lemma 786 (countable union of sections is measurable),

Lemma 787 (countable intersection of sections is measurable),

Lemma 788 (measure of section),

Lemma 791 (measurability of measure of section (finite)),

Lemma 792 (measurability of measure of section),

Lemma 795 (candidate tensor product measure is tensor product measure),

Theorem 808 (Tonelli).

**Lemma 539 (indicator of section)**

is a direct dependency of:

Theorem 808 (Tonelli),

Lemma 809 (Tonelli on subset).

**Lemma 540 (measurability of function from product space)**

is not yet used.

**Lemma 543 (Borel  $\sigma$ -algebra of  $\mathbb{R}$ )**

is a direct dependency of:

Lemma 544 (countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$ ),

Lemma 548 (Borel  $\sigma$ -algebra of  $\mathbb{R}_+$ ),

Lemma 555 (measurability of numeric function to  $\mathbb{R}$ ),

Lemma 700 ( $\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$ ),

Theorem 702 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ).

**Lemma 544 (countable generator of Borel  $\sigma$ -algebra of  $\mathbb{R}$ )**

is a direct dependency of:

Lemma 550 (Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ ).

**Lemma 545 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}$ )**

is a direct dependency of:

Lemma 549 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}_+$ ),

Lemma 563 (measurability of numeric function).

**Lemma 546 (Borel subsets of  $\overline{\mathbb{R}}$  and  $\mathbb{R}$ )**

is a direct dependency of:

Lemma 562 ( $\mathcal{M}$  and finite is  $\mathcal{M}_{\mathbb{R}}$ ).

**Lemma 548 (Borel  $\sigma$ -algebra of  $\mathbb{R}_+$ )**

is not yet used.

**Lemma 549 (Borel  $\sigma$ -algebra of  $\overline{\mathbb{R}}_+$ )**

is not yet used.

**Lemma 550 (Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ )**

is a direct dependency of:

Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  is algebra).

**Definition 552** ( $\mathcal{M}_{\mathbb{R}}$ , *vector space of measurable numeric functions to  $\mathbb{R}$* )

is a direct dependency of:

- Lemma 554 (*measurability of indicator function*),
- Lemma 555 (*measurability of numeric function to  $\mathbb{R}$* ),
- Lemma 556 (*inverse image is measurable in  $\mathbb{R}$* ),
- Lemma 557 ( $\mathcal{M}_{\mathbb{R}}$  *is algebra*),
- Lemma 562 ( $\mathcal{M}$  *and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 581 ( $\mathcal{M}$  *is closed under absolute value*).

**Lemma 554** (*measurability of indicator function*)

is a direct dependency of:

- Lemma 576 (*measurability and masking*),
- Lemma 663 (*almost sum*),
- Lemma 711 ( $\mathcal{IF}$  *is measurable*),
- Lemma 728 ( $\mathcal{SF}$  *is measurable*),
- Lemma 773 (*Bienaymé–Chebyshev inequality*),
- Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),
- Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*).

**Lemma 555** (*measurability of numeric function to  $\mathbb{R}$* )

is not yet used.

**Lemma 556** (*inverse image is measurable in  $\mathbb{R}$* )

is a direct dependency of:

- Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ),
- Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),
- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 557** ( $\mathcal{M}_{\mathbb{R}}$  *is algebra*)

is a direct dependency of:

- Lemma 559 ( $\mathcal{M}_{\mathbb{R}}$  *is vector space*),
- Lemma 566 ( $\mathcal{M}$  *is closed under addition when defined*),
- Lemma 568 ( $\mathcal{M}$  *is closed under multiplication*),
- Lemma 728 ( $\mathcal{SF}$  *is measurable*).

**Lemma 559** ( $\mathcal{M}_{\mathbb{R}}$  *is vector space*)

is a direct dependency of:

- Lemma 849 ( $\mathcal{L}^1$  *is seminormed vector space*).

**Definition 560** ( $\mathcal{M}$ , *set of measurable numeric functions*)

is a direct dependency of:

- Lemma 562 ( $\mathcal{M}$  *and finite is  $\mathcal{M}_{\mathbb{R}}$* ),
- Lemma 563 (*measurability of numeric function*),
- Lemma 564 (*inverse image is measurable*),
- Lemma 566 ( $\mathcal{M}$  *is closed under addition when defined*),
- Lemma 568 ( $\mathcal{M}$  *is closed under multiplication*),
- Lemma 577 (*measurability of restriction*),
- Lemma 581 ( $\mathcal{M}$  *is closed under absolute value*),
- Lemma 588 (*measurability of tensor product of numeric functions*),
- Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),
- Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*).

**Lemma 562** ( $\mathcal{M}$  *and finite is  $\mathcal{M}_{\mathbb{R}}$* )

is a direct dependency of:

- Lemma 565 ( $\mathcal{M}$  *is closed under finite part*),
- Lemma 576 (*measurability and masking*),



- Lemma 711 ( *$\mathcal{LF}$  is measurable*),
- Lemma 728 ( *$\mathcal{SF}$  is measurable*),
- Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),
- Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),
- Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 563 (*measurability of numeric function*)**

is a direct dependency of:

- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 571 ( *$\mathcal{M}$  is closed under infimum*),
- Lemma 572 ( *$\mathcal{M}$  is closed under supremum*),
- Theorem 759 (*Beppo Levi, monotone convergence*),
- Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),
- Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 773 (*Bienaymé–Chebyshev inequality*).

**Lemma 564 (*inverse image is measurable*)**

is a direct dependency of:

- Lemma 565 ( *$\mathcal{M}$  is closed under finite part*),
- Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 661 (*measurability of summability domain*).

**Lemma 565 ( *$\mathcal{M}$  is closed under finite part*)**

is a direct dependency of:

- Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*),
- Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*),
- Lemma 580 ( *$\mathcal{M}_+$  is closed under finite part*).

**Lemma 566 ( *$\mathcal{M}$  is closed under addition when defined*)**

is a direct dependency of:

- Lemma 567 ( *$\mathcal{M}$  is closed under finite sum*),
- Lemma 579 (*measurability of nonnegative and nonpositive parts*),
- Lemma 582 ( *$\mathcal{M}_+$  is closed under addition*),
- Lemma 663 (*almost sum*),
- Lemma 664 (*compatibility of almost sum with almost equality*),
- Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 567 ( *$\mathcal{M}$  is closed under finite sum*)**

is not yet used.

**Lemma 568 ( *$\mathcal{M}$  is closed under multiplication*)**

is a direct dependency of:

- Lemma 569 ( *$\mathcal{M}$  is closed under finite product*),
- Lemma 570 ( *$\mathcal{M}$  is closed under scalar multiplication*),
- Lemma 576 (*measurability and masking*),
- Lemma 583 ( *$\mathcal{M}_+$  is closed under multiplication*),
- Lemma 663 (*almost sum*).

**Lemma 569 ( *$\mathcal{M}$  is closed under finite product*)**

is a direct dependency of:

- Lemma 588 (*measurability of tensor product of numeric functions*).

**Lemma 570 ( *$\mathcal{M}$  is closed under scalar multiplication*)**

is a direct dependency of:

- Lemma 579 (*measurability of nonnegative and nonpositive parts*),

Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  
 Lemma 841 (integral is homogeneous).

**Lemma 571** ( $\mathcal{M}$  is closed under infimum)

is a direct dependency of:

Lemma 573 ( $\mathcal{M}$  is closed under limit inferior),  
 Lemma 574 ( $\mathcal{M}$  is closed under limit superior),  
 Theorem 779 (Fatou's lemma),  
 Lemma 791 (measurability of measure of section (finite)).

**Lemma 572** ( $\mathcal{M}$  is closed under supremum)

is a direct dependency of:

Lemma 573 ( $\mathcal{M}$  is closed under limit inferior),  
 Lemma 574 ( $\mathcal{M}$  is closed under limit superior),  
 Lemma 579 (measurability of nonnegative and nonpositive parts),  
 Lemma 791 (measurability of measure of section (finite)),  
 Lemma 792 (measurability of measure of section).

**Lemma 573** ( $\mathcal{M}$  is closed under limit inferior)

is a direct dependency of:

Lemma 575 ( $\mathcal{M}$  is closed under limit when pointwise convergent).

**Lemma 574** ( $\mathcal{M}$  is closed under limit superior)

is a direct dependency of:

Lemma 575 ( $\mathcal{M}$  is closed under limit when pointwise convergent).

**Lemma 575** ( $\mathcal{M}$  is closed under limit when pointwise convergent)

is a direct dependency of:

Lemma 585 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent),  
 Theorem 857 (Lebesgue, dominated convergence).

**Lemma 576** (measurability and masking)

is a direct dependency of:

Lemma 669 (finite nonnegative part),  
 Theorem 858 (Lebesgue, extended dominated convergence).

**Lemma 577** (measurability of restriction)

is a direct dependency of:

Lemma 776 (integral in  $\mathcal{M}_+$  over subset).

**Definition 578** ( $\mathcal{M}_+$ , subset of nonnegative measurable numeric functions)

is a direct dependency of:

Lemma 579 (measurability of nonnegative and nonpositive parts),  
 Lemma 580 ( $\mathcal{M}_+$  is closed under finite part),  
 Lemma 581 ( $\mathcal{M}$  is closed under absolute value),  
 Lemma 582 ( $\mathcal{M}_+$  is closed under addition),  
 Lemma 583 ( $\mathcal{M}_+$  is closed under multiplication),  
 Lemma 584 ( $\mathcal{M}_+$  is closed under nonnegative scalar multiplication),  
 Lemma 585 ( $\mathcal{M}_+$  is closed under limit when pointwise convergent),  
 Lemma 711 ( $\mathcal{IF}$  is measurable),  
 Lemma 736 ( $\mathcal{SF}_+$  is measurable),  
 Lemma 760 (integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$ ),  
 Lemma 769 (integral in  $\mathcal{M}_+$  is almost definite),  
 Lemma 774 (integrable in  $\mathcal{M}_+$  is almost finite),  
 Lemma 776 (integral in  $\mathcal{M}_+$  over subset),  
 Lemma 777 (integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive),  
 Lemma 810 (Tonelli for tensor product),

Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),  
Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 579 (*measurability of nonnegative and nonpositive parts*)**

is a direct dependency of:

Lemma 669 (*finite nonnegative part*),  
Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),  
Lemma 813 (*integrable is measurable*),  
Lemma 814 (*equivalent definition of integrability*),  
Lemma 841 (*integral is homogeneous*),  
Lemma 844 (*integral is additive*).

**Lemma 580 ( *$\mathcal{M}_+$  is closed under finite part*)**

is a direct dependency of:

Lemma 669 (*finite nonnegative part*).

**Lemma 581 ( *$\mathcal{M}$  is closed under absolute value*)**

is a direct dependency of:

Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),  
Lemma 773 (*Bienaymé–Chebyshev inequality*),  
Lemma 814 (*equivalent definition of integrability*),  
Lemma 817 (*almost bounded by integrable is integrable*),  
Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),  
Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ),  
Lemma 851 ( *$\mathcal{L}^1$  is closed under absolute value*).

**Lemma 582 ( *$\mathcal{M}_+$  is closed under addition*)**

is a direct dependency of:

Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*),  
Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*),  
Lemma 791 (*measurability of measure of section (finite)*).

**Lemma 583 ( *$\mathcal{M}_+$  is closed under multiplication*)**

is a direct dependency of:

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*).

**Lemma 584 ( *$\mathcal{M}_+$  is closed under nonnegative scalar multiplication*)**

is a direct dependency of:

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
Lemma 773 (*Bienaymé–Chebyshev inequality*),  
Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*),  
Lemma 791 (*measurability of measure of section (finite)*).

**Lemma 585 ( *$\mathcal{M}_+$  is closed under limit when pointwise convergent*)**

is a direct dependency of:

Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*),  
Theorem 759 (*Beppo Levi, monotone convergence*),  
Theorem 779 (*Fatou's lemma*),  
Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
Theorem 808 (*Tonelli*).

**Lemma 586 ( *$\mathcal{M}_+$  is closed under countable sum*)**

is a direct dependency of:

Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*).

**Definition 587 (tensor product of numeric functions)**

is a direct dependency of:

Lemma 588 (*measurability of tensor product of numeric functions*),  
 Lemma 810 (*Tonelli for tensor product*).

**Lemma 588 (measurability of tensor product of numeric functions)**

is a direct dependency of:

Lemma 810 (*Tonelli for tensor product*).

**Definition 590 (additivity over measurable space)**

is a direct dependency of:

Lemma 593 ( *$\sigma$ -additivity implies additivity*),  
 Lemma 604 (*equivalent definition of measure*),  
 Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Definition 591 ( $\sigma$ -additivity over measurable space)**

is a direct dependency of:

Lemma 593 ( *$\sigma$ -additivity implies additivity*),  
 Lemma 596 (*measure over countable partition*),  
 Lemma 597 (*measure is monotone*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 600 (*measure is continuous from below*),  
 Lemma 604 (*equivalent definition of measure*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 652 (*counting measure*),  
 Lemma 693 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),  
 Lemma 791 (*measurability of measure of section (finite)*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*).

**Lemma 593 ( $\sigma$ -additivity implies additivity)**

is a direct dependency of:

Lemma 604 (*equivalent definition of measure*),  
 Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Definition 594 (measure)**

is a direct dependency of:

Lemma 596 (*measure over countable partition*),  
 Lemma 597 (*measure is monotone*),  
 Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 600 (*measure is continuous from below*),  
 Lemma 603 (*measure satisfies the Boole inequality*),  
 Lemma 604 (*equivalent definition of measure*),  
 Lemma 611 (*trace measure*),  
 Lemma 612 (*restricted measure*),  
 Lemma 617 (*equivalent definition of considerable subset*),  
 Lemma 620 (*empty set is negligible*),  
 Lemma 621 (*compatibility of null measure with countable union*),  
 Lemma 632 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 650 (*trivial measure*),  
 Lemma 651 (*equivalent definition of trivial measure*),

Lemma 652 (*counting measure*),  
 Lemma 669 (*finite nonnegative part*),  
 Lemma 699 ( $\lambda^*$  is measure on  $\mathcal{L}$ ),  
 Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ),  
 Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ),  
 Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),  
 Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),  
 Lemma 788 (*measure of section*),  
 Lemma 789 (*measure of section of product*),  
 Lemma 791 (*measurability of measure of section (finite)*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 796 (*tensor product of finite measures*),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 596** (*measure over countable partition*)

is a direct dependency of:

Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ).

**Lemma 597** (*measure is monotone*)

is a direct dependency of:

Lemma 598 (*measure satisfies the finite Boole inequality*),  
 Lemma 600 (*measure is continuous from below*),  
 Lemma 602 (*measure is continuous from above*),  
 Lemma 606 (*finite measure is bounded*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Lemma 651 (*equivalent definition of trivial measure*).

**Lemma 598** (*measure satisfies the finite Boole inequality*)

is a direct dependency of:

Lemma 603 (*measure satisfies the Boole inequality*),  
 Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),  
 Lemma 621 (*compatibility of null measure with countable union*),  
 Lemma 669 (*finite nonnegative part*).

**Definition 599** (*continuity from below*)

is a direct dependency of:

Lemma 600 (*measure is continuous from below*),  
 Lemma 602 (*measure is continuous from above*),  
 Lemma 604 (*equivalent definition of measure*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 600** (*measure is continuous from below*)

is a direct dependency of:

Lemma 602 (*measure is continuous from above*),  
 Lemma 603 (*measure satisfies the Boole inequality*),  
 Lemma 604 (*equivalent definition of measure*),  
 Lemma 649 (*uniqueness of measures extended from a  $\pi$ -system*),  
 Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 791 (*measurability of measure of section (finite)*),  
 Lemma 792 (*measurability of measure of section*),  
 Lemma 799 (*uniqueness of tensor product measure (finite)*),  
 Lemma 801 (*uniqueness of tensor product measure*).

**Definition 601** (*continuity from above*)

is a direct dependency of:

Lemma 602 (*measure is continuous from above*).

**Lemma 602** (*measure is continuous from above*)

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 603** (*measure satisfies the Boole inequality*)

is a direct dependency of:

Lemma 621 (*compatibility of null measure with countable union*).

**Lemma 604** (*equivalent definition of measure*)

is not yet used.

**Definition 605** (*finite measure*)

is a direct dependency of:

Lemma 610 (*finite measure is  $\sigma$ -finite*),

Lemma 653 (*finiteness of counting measure*),

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 792 (*measurability of measure of section*),

Lemma 796 (*tensor product of finite measures*),

Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 606** (*finite measure is bounded*)

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Definition 607** ( *$\sigma$ -finite measure*)

is a direct dependency of:

Lemma 608 (*equivalent definition of  $\sigma$ -finite measure*),

Lemma 610 (*finite measure is  $\sigma$ -finite*),

Lemma 654 ( *$\sigma$ -finite counting measure*),

Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*),

Lemma 797 (*tensor product of  $\sigma$ -finite measures*).

**Lemma 608** (*equivalent definition of  $\sigma$ -finite measure*)

is a direct dependency of:

Lemma 654 ( *$\sigma$ -finite counting measure*),

Lemma 792 (*measurability of measure of section*),

Lemma 797 (*tensor product of  $\sigma$ -finite measures*),

Lemma 801 (*uniqueness of tensor product measure*).

**Definition 609** (*diffuse measure*)

is a direct dependency of:

Lemma 707 (*Lebesgue measure is diffuse*),

Lemma 807 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*),

Lemma 828 (*integral over interval*).

**Lemma 610** (*finite measure is  $\sigma$ -finite*)

is not yet used.

**Lemma 611** (*trace measure*)

is a direct dependency of:

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*).

**Lemma 612** (*restricted measure*)

is a direct dependency of:

- Lemma 792 (*measurability of measure of section*),
- Lemma 801 (*uniqueness of tensor product measure*).

**Definition 614** (*negligible subset*)

is a direct dependency of:

- Lemma 617 (*equivalent definition of considerable subset*),
- Lemma 619 (*negligibility of measurable subset*),
- Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),
- Lemma 623 (*subset of negligible is negligible*),
- Lemma 633 (*compatibility of almost binary relation with symmetry*),
- Lemma 645 (*local injectivity implies almost local injectivity*),
- Lemma 668 (*masking almost nowhere*),
- Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),
- Theorem 858 (*Lebesgue, extended dominated convergence*).

**Definition 615** (*complete measure*)

is not yet used.

**Definition 616** (*considerable subset*)

is a direct dependency of:

- Lemma 617 (*equivalent definition of considerable subset*).

**Lemma 617** (*equivalent definition of considerable subset*)

is not yet used.

**Lemma 619** (*negligibility of measurable subset*)

is a direct dependency of:

- Lemma 620 (*empty set is negligible*),
- Lemma 632 (*compatibility of almost binary relation with reflexivity*),
- Lemma 669 (*finite nonnegative part*),
- Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),
- Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),
- Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),
- Lemma 802 (*negligibility of measurable section*).

**Lemma 620** (*empty set is negligible*)

is a direct dependency of:

- Lemma 626 (*everywhere implies almost everywhere*),
- Lemma 664 (*compatibility of almost sum with almost equality*).

**Lemma 621** (*compatibility of null measure with countable union*)

is a direct dependency of:

- Lemma 622 ( *$\mathbf{N}$  is closed under countable union*),
- Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 622** ( *$\mathbf{N}$  is closed under countable union*)

is a direct dependency of:

- Lemma 627 (*extended almost modus ponens*),
- Lemma 634 (*compatibility of almost binary relation with antisymmetry*),
- Lemma 635 (*compatibility of almost binary relation with transitivity*),
- Lemma 640 (*compatibility of almost binary relation with operator*).

**Lemma 623** (*subset of negligible is negligible*)

is a direct dependency of:

- Lemma 627 (*extended almost modus ponens*),



Lemma 634 (*compatibility of almost binary relation with antisymmetry*),  
 Lemma 635 (*compatibility of almost binary relation with transitivity*).

**Definition 624 (*property almost satisfied*)**

is a direct dependency of:

Lemma 626 (*everywhere implies almost everywhere*),  
 Lemma 627 (*extended almost modus ponens*),  
 Lemma 632 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 633 (*compatibility of almost binary relation with symmetry*),  
 Lemma 634 (*compatibility of almost binary relation with antisymmetry*),  
 Lemma 635 (*compatibility of almost binary relation with transitivity*),  
 Lemma 640 (*compatibility of almost binary relation with operator*),  
 Lemma 645 (*local injectivity implies almost local injectivity*),  
 Lemma 662 (*negligibility of summability domain*),  
 Lemma 663 (*almost sum*),  
 Lemma 668 (*masking almost nowhere*),  
 Lemma 669 (*finite nonnegative part*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
 Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),  
 Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 626 (*everywhere implies almost everywhere*)**

is a direct dependency of:

Lemma 628 (*almost modus ponens*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
 Lemma 818 (*bounded by integrable is integrable*),  
 Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*).

**Lemma 627 (*extended almost modus ponens*)**

is a direct dependency of:

Lemma 628 (*almost modus ponens*).

**Lemma 628 (*almost modus ponens*)**

is not yet used.

**Definition 630 (*almost definition*)**

is a direct dependency of:

Lemma 632 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 633 (*compatibility of almost binary relation with symmetry*),  
 Lemma 634 (*compatibility of almost binary relation with antisymmetry*),  
 Lemma 635 (*compatibility of almost binary relation with transitivity*),  
 Lemma 640 (*compatibility of almost binary relation with operator*),  
 Lemma 645 (*local injectivity implies almost local injectivity*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Definition 631 (*almost binary relation*)**

is a direct dependency of:

Lemma 632 (*compatibility of almost binary relation with reflexivity*),  
 Lemma 633 (*compatibility of almost binary relation with symmetry*),  
 Lemma 634 (*compatibility of almost binary relation with antisymmetry*),  
 Lemma 635 (*compatibility of almost binary relation with transitivity*),  
 Lemma 640 (*compatibility of almost binary relation with operator*),  
 Lemma 645 (*local injectivity implies almost local injectivity*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 632** (*compatibility of almost binary relation with reflexivity*)

is a direct dependency of:

Lemma 637 (*almost equivalence is equivalence relation*),

Lemma 639 (*almost order is order relation*).

**Lemma 633** (*compatibility of almost binary relation with symmetry*)

is a direct dependency of:

Lemma 637 (*almost equivalence is equivalence relation*).

**Lemma 634** (*compatibility of almost binary relation with antisymmetry*)

is a direct dependency of:

Lemma 639 (*almost order is order relation*).

**Lemma 635** (*compatibility of almost binary relation with transitivity*)

is a direct dependency of:

Lemma 637 (*almost equivalence is equivalence relation*),

Lemma 639 (*almost order is order relation*).

**Lemma 637** (*almost equivalence is equivalence relation*)

is a direct dependency of:

Lemma 638 (*almost equality is equivalence relation*).

**Lemma 638** (*almost equality is equivalence relation*)

is a direct dependency of:

Lemma 664 (*compatibility of almost sum with almost equality*),

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*).

**Lemma 639** (*almost order is order relation*)

is a direct dependency of:

Lemma 817 (*almost bounded by integrable is integrable*).

**Lemma 640** (*compatibility of almost binary relation with operator*)

is a direct dependency of:

Lemma 641 (*compatibility of almost equality with operator*),

Lemma 642 (*compatibility of almost inequality with operator*).

**Lemma 641** (*compatibility of almost equality with operator*)

is a direct dependency of:

Lemma 664 (*compatibility of almost sum with almost equality*),

Lemma 667 (*absolute value is almost definite*),

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),

Lemma 823 (*compatibility of integral with almost equality*),

Lemma 838 (*compatibility of  $N_1$  with almost equality*),

Lemma 855 (*first mean value theorem*),

Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 642** (*compatibility of almost inequality with operator*)

is not yet used.

**Lemma 645** (*local injectivity implies almost local injectivity*)

is a direct dependency of:

Lemma 667 (*absolute value is almost definite*).

**Lemma 649** (*uniqueness of measures extended from a  $\pi$ -system*)

is a direct dependency of:

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Lemma 650** (*trivial measure*)

is a direct dependency of:

Lemma 651 (*equivalent definition of trivial measure*).

**Lemma 651** (*equivalent definition of trivial measure*)

is not yet used.

**Lemma 652** (*counting measure*)

is a direct dependency of:

Lemma 653 (*finiteness of counting measure*),

Lemma 654 ( *$\sigma$ -finite counting measure*),

Lemma 657 (*equivalent definition of Dirac measure*),

Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*).

**Lemma 653** (*finiteness of counting measure*)

is a direct dependency of:

Lemma 658 (*Dirac measure is finite*).

**Lemma 654** ( *$\sigma$ -finite counting measure*)

is not yet used.

**Definition 656** (*Dirac measure*)

is a direct dependency of:

Lemma 657 (*equivalent definition of Dirac measure*),

Lemma 658 (*Dirac measure is finite*),

Lemma 751 (*integral in  $\mathcal{SF}_+$  for Dirac measure*),

Lemma 784 (*integral in  $\mathcal{M}_+$  for Dirac measure*),

Lemma 833 (*integral for Dirac measure*).

**Lemma 657** (*equivalent definition of Dirac measure*)

is not yet used.

**Lemma 658** (*Dirac measure is finite*)

is not yet used.

**Definition 659** (*summability domain*)

is a direct dependency of:

Lemma 660 (*summability on summability domain*),

Lemma 666 (*almost sum is sum*),

Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ).

**Lemma 660** (*summability on summability domain*)

is a direct dependency of:

Lemma 662 (*negligibility of summability domain*).

**Lemma 661** (*measurability of summability domain*)

is a direct dependency of:

Lemma 663 (*almost sum*).

**Lemma 662** (*negligibility of summability domain*)

is a direct dependency of:

Lemma 663 (*almost sum*),

Lemma 664 (*compatibility of almost sum with almost equality*).

**Lemma 663** (*almost sum*)

is a direct dependency of:

Lemma 664 (*compatibility of almost sum with almost equality*),

Lemma 666 (*almost sum is sum*),

Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 664** (*compatibility of almost sum with almost equality*)

is a direct dependency of:

Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 666** (*almost sum is sum*)

is a direct dependency of:

Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ).

**Lemma 667** (*absolute value is almost definite*)

is a direct dependency of:

Lemma 839 ( *$N_1$  is almost definite*).

**Lemma 668** (*masking almost nowhere*)

is a direct dependency of:

Lemma 669 (*finite nonnegative part*),

Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 669** (*finite nonnegative part*)

is a direct dependency of:

Theorem 858 (*Lebesgue, extended dominated convergence*).

**Definition 671** (*length of interval*)

is a direct dependency of:

Lemma 672 (*length is nonnegative*),

Lemma 673 (*length is homogeneous*),

Lemma 674 (*length of partition*),

Lemma 680 ( *$\lambda^*$  is homogeneous*),

Lemma 683 ( *$\lambda^*$  generalizes length of intervals*).

**Lemma 672** (*length is nonnegative*)

is a direct dependency of:

Lemma 679 ( *$\lambda^*$  is nonnegative*).

**Lemma 673** (*length is homogeneous*)

is a direct dependency of:

Lemma 674 (*length of partition*).

**Lemma 674** (*length of partition*)

is not yet used.

**Definition 675** (*set of countable cover with bounded open intervals*)

is a direct dependency of:

Lemma 676 (*set of countable cover with bounded open intervals is nonempty*),

Lemma 681 ( *$\lambda^*$  is monotone*),

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*),

Lemma 683 ( *$\lambda^*$  generalizes length of intervals*).

**Lemma 676** (*set of countable cover with bounded open intervals is nonempty*)

is not yet used.

**Definition 677** ( *$\lambda^*$ , Lebesgue measure candidate*)

is a direct dependency of:

Lemma 680 ( *$\lambda^*$  is homogeneous*),

Lemma 681 ( *$\lambda^*$  is monotone*),

Lemma 682 ( *$\lambda^*$  is  $\sigma$ -subadditive*),

Lemma 683 ( *$\lambda^*$  generalizes length of intervals*),

Lemma 696 (*rays are Lebesgue-measurable*).

**Lemma 679** ( $\lambda^*$  is nonnegative)

is a direct dependency of:

Lemma 680 ( $\lambda^*$  is homogeneous),

Lemma 699 ( $\lambda^*$  is measure on  $\mathcal{L}$ ).

**Lemma 680** ( $\lambda^*$  is homogeneous)

is a direct dependency of:

Lemma 683 ( $\lambda^*$  generalizes length of intervals),

Lemma 698 ( $\mathcal{L}$  is  $\sigma$ -algebra),

Lemma 699 ( $\lambda^*$  is measure on  $\mathcal{L}$ ).

**Lemma 681** ( $\lambda^*$  is monotone)

is a direct dependency of:

Lemma 683 ( $\lambda^*$  generalizes length of intervals),

Lemma 693 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ),

Lemma 695 ( $\mathcal{L}$  is closed under countable union),

Lemma 696 (rays are Lebesgue-measurable).

**Lemma 682** ( $\lambda^*$  is  $\sigma$ -subadditive)

is a direct dependency of:

Lemma 687 (equivalent definition of  $\mathcal{L}$ ),

Lemma 689 ( $\mathcal{L}$  is closed under finite union),

Lemma 693 ( $\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$ ),

Lemma 695 ( $\mathcal{L}$  is closed under countable union),

Lemma 696 (rays are Lebesgue-measurable).

**Lemma 683** ( $\lambda^*$  generalizes length of intervals)

is a direct dependency of:

Lemma 696 (rays are Lebesgue-measurable),

Theorem 702 (Carathéodory, Lebesgue measure on  $\mathbb{R}$ ),

Lemma 704 (Lebesgue measure generalizes length of intervals),

Lemma 706 (Lebesgue measure is  $\sigma$ -finite),

Lemma 707 (Lebesgue measure is diffuse).

**Definition 685** ( $\mathcal{L}$ , Lebesgue  $\sigma$ -algebra)

is a direct dependency of:

Lemma 687 (equivalent definition of  $\mathcal{L}$ ),

Lemma 688 ( $\mathcal{L}$  is closed under complement),

Lemma 689 ( $\mathcal{L}$  is closed under finite union),

Lemma 691 ( $\lambda^*$  is additive on  $\mathcal{L}$ ),

Lemma 695 ( $\mathcal{L}$  is closed under countable union).

**Lemma 687** (equivalent definition of  $\mathcal{L}$ )

is a direct dependency of:

Lemma 689 ( $\mathcal{L}$  is closed under finite union),

Lemma 695 ( $\mathcal{L}$  is closed under countable union),

Lemma 696 (rays are Lebesgue-measurable).

**Lemma 688** ( $\mathcal{L}$  is closed under complement)

is a direct dependency of:

Lemma 690 ( $\mathcal{L}$  is closed under finite intersection),

Lemma 694 (partition of countable union in  $\mathcal{L}$ ),

Lemma 696 (rays are Lebesgue-measurable),

Lemma 698 ( $\mathcal{L}$  is  $\sigma$ -algebra).

**Lemma 689** ( $\mathcal{L}$  is closed under finite union)

is a direct dependency of:

Lemma 690 ( *$\mathcal{L}$  is closed under finite intersection*),  
 Lemma 694 (*partition of countable union in  $\mathcal{L}$* ),  
 Lemma 695 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 690** ( *$\mathcal{L}$  is closed under finite intersection*)

is a direct dependency of:

Lemma 694 (*partition of countable union in  $\mathcal{L}$* ),  
 Lemma 697 (*intervals are Lebesgue-measurable*).

**Lemma 691** ( *$\lambda^*$  is additive on  $\mathcal{L}$* )

is a direct dependency of:

Lemma 693 ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* ),  
 Lemma 695 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 693** ( *$\lambda^*$  is  $\sigma$ -additive on  $\mathcal{L}$* )

is a direct dependency of:

Lemma 699 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ).

**Lemma 694** (*partition of countable union in  $\mathcal{L}$* )

is a direct dependency of:

Lemma 695 ( *$\mathcal{L}$  is closed under countable union*).

**Lemma 695** ( *$\mathcal{L}$  is closed under countable union*)

is a direct dependency of:

Lemma 696 (*rays are Lebesgue-measurable*),  
 Lemma 698 ( *$\mathcal{L}$  is  $\sigma$ -algebra*).

**Lemma 696** (*rays are Lebesgue-measurable*)

is a direct dependency of:

Lemma 697 (*intervals are Lebesgue-measurable*).

**Lemma 697** (*intervals are Lebesgue-measurable*)

is a direct dependency of:

Lemma 700 ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* ).

**Lemma 698** ( *$\mathcal{L}$  is  $\sigma$ -algebra*)

is a direct dependency of:

Lemma 699 ( *$\lambda^*$  is measure on  $\mathcal{L}$* ).

**Lemma 699** ( *$\lambda^*$  is measure on  $\mathcal{L}$* )

is a direct dependency of:

Lemma 701 ( *$\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$* ).

**Lemma 700** ( *$\mathcal{B}(\mathbb{R})$  is sub- $\sigma$ -algebra of  $\mathcal{L}$* )

is a direct dependency of:

Lemma 701 ( *$\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$* ).

**Lemma 701** ( *$\lambda^*$  is measure on  $\mathcal{B}(\mathbb{R})$* )

is a direct dependency of:

Theorem 702 (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* ).

**Theorem 702** (*Carathéodory, Lebesgue measure on  $\mathbb{R}$* )

is a direct dependency of:

Lemma 704 (*Lebesgue measure generalizes length of intervals*),  
 Lemma 706 (*Lebesgue measure is  $\sigma$ -finite*),  
 Lemma 707 (*Lebesgue measure is diffuse*),  
 Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ).

**Lemma 704** (*Lebesgue measure generalizes length of intervals*)

is a direct dependency of:

Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Lemma 706** (*Lebesgue measure is  $\sigma$ -finite*)

is a direct dependency of:

Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ),

Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 707** (*Lebesgue measure is diffuse*)

is not yet used.

**Definition 709** ( *$\mathcal{IF}$ , set of measurable indicator functions*)

is a direct dependency of:

Lemma 710 (*indicator and support are each other inverse*),

Lemma 711 ( *$\mathcal{IF}$  is measurable*),

Lemma 713 ( *$\mathcal{IF}$  is closed under extension by zero*),

Lemma 714 ( *$\mathcal{IF}$  is closed under restriction*),

Lemma 725 ( *$\mathcal{SF}$  simple representation*),

Lemma 736 ( *$\mathcal{SF}_+$  is measurable*),

Lemma 748 (*integral in  $\mathcal{SF}_+$  over subset is additive*).

**Lemma 710** (*indicator and support are each other inverse*)

is a direct dependency of:

Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ),

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*),

Lemma 719 (*integral in  $\mathcal{IF}$  over subset is additive*),

Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*),

Lemma 725 ( *$\mathcal{SF}$  simple representation*),

Theorem 808 (*Tonelli*).

**Lemma 711** ( *$\mathcal{IF}$  is measurable*)

is not yet used.

**Lemma 713** ( *$\mathcal{IF}$  is closed under extension by zero*)

is a direct dependency of:

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*),

Lemma 729 ( *$\mathcal{SF}$  is closed under extension by zero*).

**Lemma 714** ( *$\mathcal{IF}$  is closed under restriction*)

is a direct dependency of:

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*),

Lemma 730 ( *$\mathcal{SF}$  is closed under restriction*).

**Definition 715** (*integral in  $\mathcal{IF}$* )

is a direct dependency of:

Lemma 716 (*equivalent definition of integral in  $\mathcal{IF}$* ),

Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ).

**Lemma 716** (*equivalent definition of integral in  $\mathcal{IF}$* )

is a direct dependency of:

Lemma 717 (*additivity of integral in  $\mathcal{IF}$* ),

Lemma 718 (*integral in  $\mathcal{IF}$  over subset*),

Lemma 721 (*integral in  $\mathcal{IF}$  for counting measure*),

Lemma 737 (*integral in  $\mathcal{SF}_+$* ),

Lemma 739 (*integral in  $\mathcal{SF}_+$  of indicator function*).



**Lemma 717** (*additivity of integral in  $\mathcal{IF}$* )

is a direct dependency of:

Lemma 719 (*integral in  $\mathcal{IF}$  over subset is additive*).

**Lemma 718** (*integral in  $\mathcal{IF}$  over subset*)

is a direct dependency of:

Lemma 719 (*integral in  $\mathcal{IF}$  over subset is additive*),

Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*).

**Lemma 719** (*integral in  $\mathcal{IF}$  over subset is additive*)

is not yet used.

**Lemma 721** (*integral in  $\mathcal{IF}$  for counting measure*)

is a direct dependency of:

Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*).

**Definition 723** ( *$\mathcal{SF}$ , vector space of simple functions*)

is a direct dependency of:

Lemma 724 ( *$\mathcal{SF}$  is vector space*),

Lemma 725 ( *$\mathcal{SF}$  simple representation*),

Lemma 728 ( *$\mathcal{SF}$  is measurable*),

Lemma 729 ( *$\mathcal{SF}$  is closed under extension by zero*),

Lemma 730 ( *$\mathcal{SF}$  is closed under restriction*),

Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*),

Lemma 748 (*integral in  $\mathcal{SF}_+$  over subset is additive*),

Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),

Theorem 759 (*Beppo Levi, monotone convergence*),

Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),

Lemma 854 (*constant function is  $\mathcal{L}^1$* ).

**Lemma 724** ( *$\mathcal{SF}$  is vector space*)

is a direct dependency of:

Lemma 735 ( *$\mathcal{SF}_+$  is closed under positive linear operations*),

Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),

Lemma 745 (*integral in  $\mathcal{SF}_+$  is monotone*).

**Lemma 725** ( *$\mathcal{SF}$  simple representation*)

is a direct dependency of:

Lemma 727 ( *$\mathcal{SF}$  canonical representation*),

Lemma 734 ( *$\mathcal{SF}_+$  simple representation*).

**Lemma 727** ( *$\mathcal{SF}$  canonical representation*)

is a direct dependency of:

Lemma 733 ( *$\mathcal{SF}_+$  canonical representation*),

Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ).

**Lemma 728** ( *$\mathcal{SF}$  is measurable*)

is a direct dependency of:

Lemma 736 ( *$\mathcal{SF}_+$  is measurable*),

Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 729** ( *$\mathcal{SF}$  is closed under extension by zero*)

is a direct dependency of:

Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*).

**Lemma 730** ( *$\mathcal{SF}$  is closed under restriction*)

is a direct dependency of:

Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*).

**Definition 732** ( *$\mathcal{SF}_+$ , subset of nonnegative simple functions*)

is a direct dependency of:

Lemma 733 ( *$\mathcal{SF}_+$  canonical representation*),  
 Lemma 734 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 735 ( *$\mathcal{SF}_+$  is closed under positive linear operations*),  
 Lemma 736 ( *$\mathcal{SF}_+$  is measurable*),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 745 (*integral in  $\mathcal{SF}_+$  is monotone*),  
 Lemma 748 (*integral in  $\mathcal{SF}_+$  over subset is additive*),  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ).

**Lemma 733** ( *$\mathcal{SF}_+$  canonical representation*)

is a direct dependency of:

Lemma 734 ( *$\mathcal{SF}_+$  simple representation*),  
 Lemma 737 (*integral in  $\mathcal{SF}_+$* ).

**Lemma 734** ( *$\mathcal{SF}_+$  simple representation*)

is a direct dependency of:

Lemma 744 (*equivalent definition of the integral in  $\mathcal{SF}_+$* ),  
 Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*),  
 Theorem 808 (*Tonelli*).

**Lemma 735** ( *$\mathcal{SF}_+$  is closed under positive linear operations*)

is a direct dependency of:

Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*).

**Lemma 736** ( *$\mathcal{SF}_+$  is measurable*)

is a direct dependency of:

Lemma 740 (*decomposition of measure in  $\mathcal{SF}_+$* ),  
 Lemma 741 (*change of variable in sum in  $\mathcal{SF}_+$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*).

**Lemma 737** (*integral in  $\mathcal{SF}_+$* )

is a direct dependency of:

Lemma 738 (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* ),  
 Lemma 743 (*integral in  $\mathcal{SF}_+$  is positive linear*),  
 Lemma 745 (*integral in  $\mathcal{SF}_+$  is monotone*),  
 Lemma 753 (*integral in  $\mathcal{M}_+$* ),  
 Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* ),  
 Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*),  
 Lemma 854 (*constant function is  $\mathcal{L}^1$* ).

**Lemma 738** (*integral in  $\mathcal{SF}_+$  generalizes integral in  $\mathcal{IF}$* )

is a direct dependency of:

Lemma 739 (*integral in  $\mathcal{SF}_+$  of indicator function*),  
 Lemma 747 (*integral in  $\mathcal{SF}_+$  over subset*),  
 Lemma 749 (*integral in  $\mathcal{SF}_+$  for counting measure*).

**Lemma 739** (*integral in  $\mathcal{SF}_+$  of indicator function*)

is a direct dependency of:

Lemma 744 (*equivalent definition of the integral in  $SF_+$* ),  
 Lemma 755 (*integral in  $M_+$  of indicator function*).

**Lemma 740** (*decomposition of measure in  $SF_+$* )  
 is a direct dependency of:  
 Lemma 743 (*integral in  $SF_+$  is positive linear*).

**Lemma 741** (*change of variable in sum in  $SF_+$* )  
 is a direct dependency of:  
 Lemma 743 (*integral in  $SF_+$  is positive linear*).

**Lemma 743** (*integral in  $SF_+$  is positive linear*)  
 is a direct dependency of:  
 Lemma 744 (*equivalent definition of the integral in  $SF_+$* ),  
 Lemma 745 (*integral in  $SF_+$  is monotone*),  
 Lemma 747 (*integral in  $SF_+$  over subset*),  
 Lemma 756 (*integral in  $M_+$  is positive homogeneous*),  
 Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 764 (*integral in  $M_+$  is additive*).

**Lemma 744** (*equivalent definition of the integral in  $SF_+$* )  
 is a direct dependency of:  
 Lemma 756 (*integral in  $M_+$  is positive homogeneous*),  
 Theorem 759 (*Beppo Levi, monotone convergence*).

**Lemma 745** (*integral in  $SF_+$  is monotone*)  
 is a direct dependency of:  
 Lemma 746 (*integral in  $SF_+$  is continuous*).

**Lemma 746** (*integral in  $SF_+$  is continuous*)  
 is a direct dependency of:  
 Lemma 754 (*integral in  $M_+$  generalizes integral in  $SF_+$* ).

**Lemma 747** (*integral in  $SF_+$  over subset*)  
 is a direct dependency of:  
 Lemma 748 (*integral in  $SF_+$  over subset is additive*),  
 Lemma 776 (*integral in  $M_+$  over subset*).

**Lemma 748** (*integral in  $SF_+$  over subset is additive*)  
 is not yet used.

**Lemma 749** (*integral in  $SF_+$  for counting measure*)  
 is a direct dependency of:  
 Lemma 750 (*integral in  $SF_+$  for counting measure on  $\mathbb{N}$* ),  
 Lemma 751 (*integral in  $SF_+$  for Dirac measure*),  
 Lemma 781 (*integral in  $M_+$  for counting measure*).

**Lemma 750** (*integral in  $SF_+$  for counting measure on  $\mathbb{N}$* )  
 is not yet used.

**Lemma 751** (*integral in  $SF_+$  for Dirac measure*)  
 is not yet used.

**Lemma 753** (*integral in  $M_+$* )  
 is a direct dependency of:  
 Lemma 754 (*integral in  $M_+$  generalizes integral in  $SF_+$* ),  
 Lemma 756 (*integral in  $M_+$  is positive homogeneous*),  
 Lemma 758 (*integral in  $M_+$  is monotone*),

Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 771 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),  
 Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*),  
 Lemma 775 (*bounded by integrable in  $\mathcal{M}_+$  is integrable*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 813 (*integrable is measurable*),  
 Lemma 814 (*equivalent definition of integrability*),  
 Lemma 817 (*almost bounded by integrable is integrable*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),  
 Lemma 825 (*integral over subset*),  
 Lemma 827 (*integral over singleton*),  
 Lemma 830 (*integral for counting measure*),  
 Lemma 835 (*seminorm  $\mathcal{L}^1$* ),  
 Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),  
 Lemma 841 (*integral is homogeneous*),  
 Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 754 (*integral in  $\mathcal{M}_+$  generalizes integral in  $\mathcal{SF}_+$* )**

is a direct dependency of:

Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*),  
 Lemma 763 (*usage of adapted sequences*),  
 Lemma 854 (*constant function is  $\mathcal{L}^1$* ).

**Lemma 755 (*integral in  $\mathcal{M}_+$  of indicator function*)**

is a direct dependency of:

Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
 Lemma 773 (*Bienaymé–Chebyshev inequality*),  
 Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Lemma 791 (*measurability of measure of section (finite)*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Theorem 808 (*Tonelli*).

**Lemma 756 (*integral in  $\mathcal{M}_+$  is positive homogeneous*)**

is a direct dependency of:

Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*),  
 Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 773 (*Bienaymé–Chebyshev inequality*),  
 Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 810 (*Tonelli for tensor product*),  
 Lemma 840 ( *$N_1$  is absolutely homogeneous*),  
 Lemma 841 (*integral is homogeneous*).

**Lemma 757 (*integral in  $\mathcal{M}_+$  of zero is zero*)**

is a direct dependency of:

Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),  
 Lemma 821 (*integral of zero is zero*).

**Lemma 758 (*integral in  $\mathcal{M}_+$  is monotone*)**

is a direct dependency of:

Theorem 759 (*Beppo Levi, monotone convergence*),  
 Lemma 772 (*integral in  $\mathcal{M}_+$  is almost monotone*),

Lemma 773 (*Bienaymé–Chebyshev inequality*),  
 Lemma 775 (*bounded by integrable in  $\mathcal{M}_+$  is integrable*),  
 Theorem 779 (*Fatou’s lemma*),  
 Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),  
 Lemma 826 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Theorem 759 (*Beppo Levi, monotone convergence*)**

is a direct dependency of:

Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* ),  
 Lemma 763 (*usage of adapted sequences*),  
 Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*),  
 Theorem 779 (*Fatou’s lemma*).

**Lemma 760 (*integral in  $\mathcal{M}_+$  is homogeneous at  $\infty$* )**

is a direct dependency of:

Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 769 (*integral in  $\mathcal{M}_+$  is almost definite*),  
 Lemma 773 (*Bienaymé–Chebyshev inequality*),  
 Lemma 840 ( *$N_1$  is absolutely homogeneous*).

**Definition 761 (*adapted sequence*)**

is a direct dependency of:

Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* ),  
 Lemma 763 (*usage of adapted sequences*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*).

**Lemma 762 (*adapted sequence in  $\mathcal{M}_+$* )**

is a direct dependency of:

Lemma 763 (*usage of adapted sequences*),  
 Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*),  
 Theorem 808 (*Tonelli*).

**Lemma 763 (*usage of adapted sequences*)**

is a direct dependency of:

Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*),  
 Lemma 776 (*integral in  $\mathcal{M}_+$  over subset*),  
 Lemma 781 (*integral in  $\mathcal{M}_+$  for counting measure*),  
 Theorem 808 (*Tonelli*).

**Lemma 764 (*integral in  $\mathcal{M}_+$  is additive*)**

is a direct dependency of:

Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*),  
 Lemma 767 (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*),  
 Lemma 768 (*compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts*),  
 Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
 Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 765 (*integral in  $\mathcal{M}_+$  is positive linear*)**

is a direct dependency of:

Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Theorem 808 (*Tonelli*).

**Lemma 766 (*integral in  $\mathcal{M}_+$  is  $\sigma$ -additive*)**

is a direct dependency of:

Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),

Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 826 (*integral over subset is  $\sigma$ -additive*).

**Lemma 767** (*integral in  $\mathcal{M}_+$  of decomposition into nonpositive and nonnegative parts*)

is a direct dependency of:

Lemma 814 (*equivalent definition of integrability*),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 768** (*compatibility of integral in  $\mathcal{M}_+$  with nonpositive and nonnegative parts*)

is a direct dependency of:

Lemma 844 (*integral is additive*).

**Lemma 769** (*integral in  $\mathcal{M}_+$  is almost definite*)

is a direct dependency of:

Lemma 770 (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*),  
 Lemma 802 (*negligibility of measurable section*),  
 Lemma 839 ( *$N_1$  is almost definite*),  
 Lemma 855 (*first mean value theorem*).

**Lemma 770** (*compatibility of integral in  $\mathcal{M}_+$  with almost binary relation*)

is a direct dependency of:

Lemma 771 (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*),  
 Lemma 772 (*integral in  $\mathcal{M}_+$  is almost monotone*).

**Lemma 771** (*compatibility of integral in  $\mathcal{M}_+$  with almost equality*)

is a direct dependency of:

Lemma 823 (*compatibility of integral with almost equality*).

**Lemma 772** (*integral in  $\mathcal{M}_+$  is almost monotone*)

is a direct dependency of:

Lemma 817 (*almost bounded by integrable is integrable*).

**Lemma 773** (*Bienaymé–Chebyshev inequality*)

is a direct dependency of:

Lemma 774 (*integrable in  $\mathcal{M}_+$  is almost finite*).

**Lemma 774** (*integrable in  $\mathcal{M}_+$  is almost finite*)

is a direct dependency of:

Lemma 816 (*integrable is almost finite*).

**Lemma 775** (*bounded by integrable in  $\mathcal{M}_+$  is integrable*)

is not yet used.

**Lemma 776** (*integral in  $\mathcal{M}_+$  over subset*)

is a direct dependency of:

Lemma 777 (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*),  
 Lemma 778 (*integral in  $\mathcal{M}_+$  over singleton*),  
 Lemma 809 (*Tonelli on subset*),  
 Lemma 825 (*integral over subset*).

**Lemma 777** (*integral in  $\mathcal{M}_+$  over subset is  $\sigma$ -additive*)

is a direct dependency of:

Lemma 826 (*integral over subset is  $\sigma$ -additive*).

**Lemma 778** (*integral in  $\mathcal{M}_+$  over singleton*)

is a direct dependency of:

Lemma 827 (*integral over singleton*).

**Theorem 779 (Fatou's lemma)**

is a direct dependency of:

Lemma 780 (*integral in  $\mathcal{M}_+$  of pointwise convergent sequence*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 780 (integral in  $\mathcal{M}_+$  of pointwise convergent sequence)**

is not yet used.

**Lemma 781 (integral in  $\mathcal{M}_+$  for counting measure)**

is a direct dependency of:

Lemma 782 (*integral in  $\mathcal{M}_+$  for counting measure on  $\mathbb{N}$* ),

Lemma 784 (*integral in  $\mathcal{M}_+$  for Dirac measure*),

Lemma 830 (*integral for counting measure*).

**Lemma 782 (integral in  $\mathcal{M}_+$  for counting measure on  $\mathbb{N}$ )**

is not yet used.

**Lemma 784 (integral in  $\mathcal{M}_+$  for Dirac measure)**

is not yet used.

**Lemma 786 (countable union of sections is measurable)**

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 795 (*candidate tensor product measure is tensor product measure*).

**Lemma 787 (countable intersection of sections is measurable)**

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*).

**Lemma 788 (measure of section)**

is a direct dependency of:

Lemma 789 (*measure of section of product*),

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 792 (*measurability of measure of section*),

Lemma 795 (*candidate tensor product measure is tensor product measure*),

Lemma 801 (*uniqueness of tensor product measure*),

Theorem 808 (*Tonelli*).

**Lemma 789 (measure of section of product)**

is a direct dependency of:

Lemma 791 (*measurability of measure of section (finite)*),

Lemma 795 (*candidate tensor product measure is tensor product measure*).

**Lemma 791 (measurability of measure of section (finite))**

is a direct dependency of:

Lemma 792 (*measurability of measure of section*).

**Lemma 792 (measurability of measure of section)**

is a direct dependency of:

Lemma 795 (*candidate tensor product measure is tensor product measure*),

Lemma 802 (*negligibility of measurable section*),

Theorem 808 (*Tonelli*).

**Definition 793 (tensor product measure)**

is a direct dependency of:

Lemma 795 (*candidate tensor product measure is tensor product measure*),

Lemma 796 (*tensor product of finite measures*),

Lemma 797 (*tensor product of  $\sigma$ -finite measures*),



Lemma 799 (*uniqueness of tensor product measure (finite)*),  
 Lemma 801 (*uniqueness of tensor product measure*),  
 Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*).

**Definition 794 (*candidate tensor product measure*)**

is a direct dependency of:

Lemma 795 (*candidate tensor product measure is tensor product measure*),  
 Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 795 (*candidate tensor product measure is tensor product measure*)**

is a direct dependency of:

Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 796 (*tensor product of finite measures*)**

is a direct dependency of:

Lemma 799 (*uniqueness of tensor product measure (finite)*).

**Lemma 797 (*tensor product of  $\sigma$ -finite measures*)**

is a direct dependency of:

Lemma 801 (*uniqueness of tensor product measure*),  
 Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 799 (*uniqueness of tensor product measure (finite)*)**

is a direct dependency of:

Lemma 801 (*uniqueness of tensor product measure*).

**Lemma 801 (*uniqueness of tensor product measure*)**

is a direct dependency of:

Lemma 802 (*negligibility of measurable section*),  
 Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* ),  
 Theorem 808 (*Tonelli*).

**Lemma 802 (*negligibility of measurable section*)**

is not yet used.

**Lemma 803 (*Lebesgue measure on  $\mathbb{R}^2$* )**

is a direct dependency of:

Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*),  
 Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*).

**Lemma 804 (*Lebesgue measure on  $\mathbb{R}^2$  generalizes area of boxes*)**

is a direct dependency of:

Lemma 805 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*),  
 Lemma 807 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*).

**Lemma 805 (*Lebesgue measure on  $\mathbb{R}^2$  is zero on lines*)**

is not yet used.

**Lemma 806 (*Lebesgue measure on  $\mathbb{R}^2$  is  $\sigma$ -finite*)**

is not yet used.

**Lemma 807 (*Lebesgue measure on  $\mathbb{R}^2$  is diffuse*)**

is not yet used.

**Theorem 808 (*Tonelli*)**

is a direct dependency of:

Lemma 809 (*Tonelli on subset*),  
 Lemma 810 (*Tonelli for tensor product*).

**Lemma 809** (*Tonelli on subset*)

is not yet used.

**Lemma 810** (*Tonelli for tensor product*)

is not yet used.

**Definition 812** (*integrability*)

is a direct dependency of:

- Lemma 813 (*integrable is measurable*),
- Lemma 814 (*equivalent definition of integrability*),
- Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),
- Lemma 823 (*compatibility of integral with almost equality*),
- Lemma 825 (*integral over subset*),
- Lemma 827 (*integral over singleton*),
- Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 813** (*integrable is measurable*)

is a direct dependency of:

- Lemma 814 (*equivalent definition of integrability*),
- Lemma 841 (*integral is homogeneous*).

**Lemma 814** (*equivalent definition of integrability*)

is a direct dependency of:

- Lemma 815 (*compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),
- Lemma 816 (*integrable is almost finite*),
- Lemma 817 (*almost bounded by integrable is integrable*),
- Lemma 818 (*bounded by integrable is integrable*),
- Lemma 826 (*integral over subset is  $\sigma$ -additive*),
- Lemma 827 (*integral over singleton*),
- Lemma 830 (*integral for counting measure*),
- Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),
- Lemma 841 (*integral is homogeneous*),
- Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),
- Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),
- Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 815** (*compatibility of integrability in  $\mathcal{M}$  and  $\mathcal{M}_+$* )

is a direct dependency of:

- Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ).

**Lemma 816** (*integrable is almost finite*)

is a direct dependency of:

- Lemma 844 (*integral is additive*).

**Lemma 817** (*almost bounded by integrable is integrable*)

is a direct dependency of:

- Lemma 818 (*bounded by integrable is integrable*).

**Lemma 818** (*bounded by integrable is integrable*)

is a direct dependency of:

- Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ).

**Definition 819** (*integral*)

is a direct dependency of:

- Lemma 820 (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* ),
- Lemma 823 (*compatibility of integral with almost equality*),
- Lemma 825 (*integral over subset*),

Lemma 826 (*integral over subset is  $\sigma$ -additive*),  
 Lemma 827 (*integral over singleton*),  
 Lemma 830 (*integral for counting measure*),  
 Lemma 841 (*integral is homogeneous*),  
 Lemma 844 (*integral is additive*),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),  
 Lemma 854 (*constant function is  $\mathcal{L}^1$* ).

**Lemma 820** (*compatibility of integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* )

is a direct dependency of:

Lemma 821 (*integral of zero is zero*),  
 Lemma 854 (*constant function is  $\mathcal{L}^1$* ),  
 Lemma 855 (*first mean value theorem*).

**Lemma 821** (*integral of zero is zero*)

is not yet used.

**Definition 822** (*merge integral in  $\mathcal{M}$  and  $\mathcal{M}_+$* )

is a direct dependency of:

Lemma 835 (*seminorm  $\mathcal{L}^1$* ).

**Lemma 823** (*compatibility of integral with almost equality*)

is a direct dependency of:

Lemma 838 (*compatibility of  $N_1$  with almost equality*),  
 Lemma 844 (*integral is additive*),  
 Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 825** (*integral over subset*)

is a direct dependency of:

Lemma 827 (*integral over singleton*).

**Lemma 826** (*integral over subset is  $\sigma$ -additive*)

is a direct dependency of:

Lemma 828 (*integral over interval*),  
 Lemma 829 (*Chasles relation, integral over split intervals*).

**Lemma 827** (*integral over singleton*)

is a direct dependency of:

Lemma 828 (*integral over interval*).

**Lemma 828** (*integral over interval*)

is a direct dependency of:

Lemma 829 (*Chasles relation, integral over split intervals*).

**Lemma 829** (*Chasles relation, integral over split intervals*)

is not yet used.

**Lemma 830** (*integral for counting measure*)

is a direct dependency of:

Lemma 831 (*integral for counting measure on  $\mathbb{N}$* ),  
 Lemma 833 (*integral for Dirac measure*).

**Lemma 831** (*integral for counting measure on  $\mathbb{N}$* )

is not yet used.

**Lemma 833** (*integral for Dirac measure*)

is not yet used.

**Definition 834** (*integral for Lebesgue measure on  $\mathbb{R}$* )

is not yet used.

**Lemma 835** (*seminorm  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 837 (*integrable is finite seminorm  $\mathcal{L}^1$* ),  
 Lemma 838 (*compatibility of  $N_1$  with almost equality*),  
 Lemma 839 ( *$N_1$  is almost definite*),  
 Lemma 840 ( *$N_1$  is absolutely homogeneous*),  
 Lemma 841 (*integral is homogeneous*),  
 Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ),  
 Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),  
 Lemma 851 ( *$\mathcal{L}^1$  is closed under absolute value*),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),  
 Theorem 857 (*Lebesgue, dominated convergence*),  
 Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 837** (*integrable is finite seminorm  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 844 (*integral is additive*).

**Lemma 838** (*compatibility of  $N_1$  with almost equality*)

is a direct dependency of:

Lemma 843 (*Minkowski inequality in  $\mathcal{M}$* ).

**Lemma 839** ( *$N_1$  is almost definite*)

is a direct dependency of:

Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*).

**Lemma 840** ( *$N_1$  is absolutely homogeneous*)

is a direct dependency of:

Lemma 841 (*integral is homogeneous*),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*).

**Lemma 841** (*integral is homogeneous*)

is a direct dependency of:

Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Lemma 843** (*Minkowski inequality in  $\mathcal{M}$* )

is a direct dependency of:

Lemma 844 (*integral is additive*),  
 Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ).

**Lemma 844** (*integral is additive*)

is a direct dependency of:

Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ).

**Definition 845** ( *$\mathcal{L}^1$ , vector space of integrable functions*)

is a direct dependency of:

Lemma 847 (*equivalent definition of  $\mathcal{L}^1$* ),  
 Lemma 848 (*Minkowski inequality in  $\mathcal{L}^1$* ),  
 Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*),  
 Lemma 851 ( *$\mathcal{L}^1$  is closed under absolute value*),  
 Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),  
 Theorem 857 (*Lebesgue, dominated convergence*),  
 Theorem 858 (*Lebesgue, extended dominated convergence*).

**Lemma 847** (*equivalent definition of  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ).

**Lemma 848** (*Minkowski inequality in  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 849 ( *$\mathcal{L}^1$  is seminormed vector space*).

**Lemma 849** ( *$\mathcal{L}^1$  is seminormed vector space*)

is a direct dependency of:

Lemma 853 (*integral is nonnegative linear form on  $\mathcal{L}^1$* ),

Lemma 855 (*first mean value theorem*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Definition 850** (*convergence in  $\mathcal{L}^1$* )

is a direct dependency of:

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 851** ( *$\mathcal{L}^1$  is closed under absolute value*)

is a direct dependency of:

Lemma 852 (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* ),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 852** (*bounded by  $\mathcal{L}^1$  is  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 855 (*first mean value theorem*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 853** (*integral is nonnegative linear form on  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 855 (*first mean value theorem*),

Theorem 857 (*Lebesgue, dominated convergence*).

**Lemma 854** (*constant function is  $\mathcal{L}^1$* )

is a direct dependency of:

Lemma 855 (*first mean value theorem*).

**Lemma 855** (*first mean value theorem*)

is a direct dependency of:

Lemma 856 (*variant of first mean value theorem*).

**Lemma 856** (*variant of first mean value theorem*)

is not yet used.

**Theorem 857** (*Lebesgue, dominated convergence*)

is a direct dependency of:

Theorem 858 (*Lebesgue, extended dominated convergence*).

**Theorem 858** (*Lebesgue, extended dominated convergence*)



**RESEARCH CENTRE  
PARIS**

2 rue Simone Iff - CS 42112  
75589 Paris Cedex 12

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