# Lebesgue integration. Detailed proofs to be formalized in Coq 

François Clément, Vincent Martin

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## Lebesgue integration. Detailed proofs to be formalized in Coq

François Clément, Vincent Martin

## RESEARCH <br> REPORT <br> $\mathbf{N}^{\circ} 9386$ <br> January 2021 <br> Project-Team Serena

# Lebesgue integration. Detailed proofs to be formalized in Coq 

François Clément*, Vincent Martin ${ }^{\dagger}$<br>Project-Team Serena<br>Research Report n ${ }^{\circ} 9386$ - version $3^{\ddagger}$ - initial version January 2021 revised version December 2022 - 567 pages


#### Abstract

To obtain the highest confidence on the correction of numerical simulation programs implementing the finite element method, one has to formalize the mathematical notions and results that allow to establish the soundness of the method. Sobolev spaces are the mathematical framework in which most weak formulations of partial derivative equations are stated, and where solutions are sought. These functional spaces are built on integration and measure theory. Hence, this chapter in functional analysis is a mandatory theoretical cornerstone for the definition of the finite element method. The purpose of this document is to provide the formal proof community with very detailed pen-and-paper proofs of the main results from integration and measure theory.


Key-words: measure theory, Lebesgue integration, detailed mathematical proof, formal proof in real analysis

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$\ddagger$ Version 3 is a major revision of the document (see Foreword).


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## Intégrale de Lebesgue. Preuves détaillées en vue d'une formalisation en Coq

Résumé : Pour obtenir la plus grande confiance en la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il faut formaliser les notions et résultats mathématiques qui permettent d'établir la justesse de la méthode. Les espaces de Sobolev sont le cadre mathématique dans lequel la plupart des formulations faibles pour résoudre les équations aux dérivées partielles sont posées, et où les solutions sont recherchées. La construction de ces espaces fonctionnels repose sur le calcul intégral et la théorie de la mesure. Ce chapitre de l'analyse fonctionnelle est donc un fondement théorique nécessaire à la définition de la méthode des éléments finis. L'objectif de ce document est de fournir à la communauté des chercheurs en preuve formelle des preuves papiers très détaillées des principaux résultats du calcul intégral et de la théorie de la mesure.
Mots-clés : théorie de la mesure, intégrale de Lebesgue, preuve mathématique détaillée, preuve formelle en analyse réelle

## Foreword

This document is intended to evolve over time. Last version is release 2.0 (i.e. version 3).
It is available at https://hal.inria.fr/hal-03105815/.
Version 3 (release 2.0, 2022/12/16) is a major revision.
Main changes are:

- new Sections 3.1.4 on Bochner integral, and 4.5 on Bochner scheme;
- new Section 4.1 on subset systems;
- new Sections 5.1, 5.2 and 5.6 on the sketches of proofs of the Riesz-Fischer theorem, the Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$, and the existence and uniqueness of the tensor product measure;
- new Figures in Sections 1.2 (finite element method), 4.1 (subset systems), 4.4 (Lebesgue scheme), 5.5 (Tonelli), and 5.6 (product measure);
- new Lemmas 212, 213, 217, 219, and 221, and generalized Lemma 215 in Section 7.1 on (naive) set theory;
- new Lemmas 231, and 238 in Section 7.2 on order theory;
- new Lemmas 245, 247, 249, 250, and 259, and generalized Lemmas 248, 252, 265, and 269 in Section 7.3.1 on topological basis;
new Lemmas 264, 268, and 266 in Section 7.3 .3 on metric spaces;
- new Sections 7.4.3.1, and 7.4.3.2 in Section 7.4 on algebraic structures;
- new Lemmas 300, 307, 308, 310, and 311 in Section 7.5.1 on real numbers;

Section 7.5.2 on extended real numbers has been seriously expanded: new Definition 374, new Lemmas 314, 321, 323, 324, 328, 333, 335, 338, 343, 344, 345, 348, 350, 351, 357, 358, $359,360,361,362,363,364,365,366,367,368,369,370,300,373,375,377,378,379,381$, $382,383,385,386,390,391,392,396,399,400,335,412,413,414,420,422,429,431,432$, $433,434,435,436,437,438,439,441,442,443,444,445,446,448,449,455$, and generalized Lemmas 380, 398, 417, 418, 419;
new Lemma 457, and generalized Lemma 466 in Section 7.5 . 3 on the second countability of real numbers;
new Section 7.5 .4 on the second countability of extended real numbers (the proof of Lemma 479 is now completed);
new Lemmas 483, 498, 499, 500, 501, 502, and 503, and generalized Lemmas 496, and 497 in Section 7.5.5 on infimum and supremum;
new Lemmas 524, and 525 in Section 7.5.7 on masking numeric functions;

- Chapter 8 on generic subset systems is partly rewritten;
new Remark 537 provides a mini-language to express properties of subset systems in a compact form;
new Lemmas $544,546,550,551,552,553,554,555,556,557,561,562,563,572,574,575$, 576 , and 577, and generalized Lemmas 569, 570, and 571, in Section 8.1 on basic properties of subset systems;
new Section 8.2 implements the Remark 430 p. 75 of Version 2, and even goes a step further, this allows to factor many proofs in the remainder of the chapter (some statements are grouped together);
new Lemmas 604, 607, and 609 in Section 8.3 on pi-systems;
new Section 8.4 on rings of sets;
new Lemmas $638,640,643,645,646,647,648,649,650,651$, new part on semi-algebras of sets (impacting the proof of Lemma 662), and generalized Lemmas 639, 652, and 662 in Section 8.5 on algebras of sets (previously called "set algebras");
new Lemmas 667, and 669 in Section 8.6 on monotone classes;
new Lemmas 679, 683, 685, 686, 687, 688, and generalized Lemma 680 in Section 8.7 on lambda-systems;
new Lemmas 703, 705, 706, 709, 722, 723, 724, and generalized Lemmas 696, 700, 714, 717, 720, 726, and 728, in Section 8.8 on sigma-algebras;
- new Definitions 739, and 748, and new Lemmas 740, and 742 in Section 9.1 on measurable spaces and Borel subsets;
new Lemma 760, and generalized Definition 758, and Lemmas 753, and 759 in Section 9.2 on measurable functions;
generalized Lemma 766 in Section 9.3 on measurable subspaces; generalized Lemma 780 in Section 9.4 on products of measurable spaces;
- new Lemmas 794, 800, and 806, and generalized Lemma 803 in Section 10.1 on Borel subsets of number spaces;
new Lemmas 808, 810, 837, 839, and 840, and generalized Definition 851, and Lemmas 809, 811, 813, 814, 841, and 853 in Section 10.2 on measurable numeric functions;
- new Chapter 11 generalizes the old Section 11.1 on measures (Version 2, pp. 117-122), it is similar to Chapter 8 on subset systems, but for numeric functions over subset systems, such as measures; it includes an extension of the mini-language in Remark 854 providing compact forms of statements;
- new Lemmas 915, 917, 918, 928, 929, and 932 in Section 12.1 on measure spaces;
new Lemmas 947, 955, 956, 957, 960, 963, 965, 966, 967, 970, 972, and 977, and generalized Lemma 962 in Section 12.3 on negligible subsets;
- new Lemmas 993, 994, 995, 996, 1001, 1006, 1007, 1010, 1011, 1012, 1013, 1014, 1015, and 1016, and generalized Lemmas 998, and 1008 in Section 13.1 on negligibility in number spaces;
new Section 13.2 on essential suprema, including the specific case of nonnegative functions; generalized Lemma 1081 in Section 13.3 on the construction of the Lebesgue measure;
- new Chapter 14 on simple functions has been extracted from the old Chapter 13 on integration of nonnegative functions (Version 2, pp. 145-184);
generalized Section 14.1 on indicator functions from the only case of real numbers (Version 2, Section 13.1, pp. 145-146) to the case of measurable number spaces $\mathbb{R}, \mathbb{C}$, and $\overline{\mathbb{R}}$, including new Lemma 1107;
generalized Section 14.2 on simple functions from the only case of real-values functions (Version 2, Section 13.2.1, pp. 149-153) to the case of vector-valued functions, including new Lemma 1121;
new Lemmas 1130, 1131, 1132, and 1134 in Section 14.3 on nonnegative simple functions; new Lemmas 1137, 1138, 1139, and 1141 in Section 14.4 on adapted sequences;
- new Lemmas 1147, and 1150, and generalized Lemmas 1146, and 1149 in Section 15.1 on the integration of indicator functions;
new Lemmas 1158, 1165, 1166, 1167, 1168, and 1169, generalized Lemma 1164, and modified Lemma 1163 in Section 15.2 on the integration of nonnegative simple functions;
new Lemmas 1192, 1193, 1194, 1201, 1203, 1204, 1205, 1206, and 1210, generalized Lemmas 1189, 1195, 1198, and modified Lemma 1175, and 1199 in Section 15.3 on the integration of nonnegative measurable functions;
new Definition 1216 in Section 15.4 on measure and integration over product spaces;
- new Lemmas 1253, 1254, 1255, 1260, 1263, 1264, 1270, 1271, 1272, and 1273, and modified Lemmas 1267, and 1269 in Section 16.1 on the definition of the integral (for functions with possibly varying sign);
new Lemmas 1276, and 1280, and modified Lemmas 1274, and 1275 in Section 16.2 on notations for specific cases;
- generalized Section 16.3 on the seminormed vector space $\mathcal{L}^{p}$ from the only case of $p=1$ (Version 2, Section 14.3, pp. 192-199) to the case of any $p \in(0, \infty]$ (for the definition and basic properties of the seminorm $N_{p}$ and $\mathcal{L}^{p}$ ), and to the case of any $p \in[1, \infty]$ for the Hölder and Minkowski inequalities and their variants, and for the resulting property of seminormed vector space of $\mathcal{L}^{p}$;
new Lemmas 1288, 1289, 1290, 1291, 1293, 1294, 1295, 1298, 1299 in Section 16.3.1 on the p-seminorm;
new Section 16.3.2 on Hölder inequality;
new Lemmas 1310, 1311, and 1312 in Section 16.3.3 on Minkowski inequality;
new Lemmas 1317, 1318, 1320, 1321, 1324, 1326, 1334, 1335, 1340, and 1341, new Theorems 1343 (Beppo Levi monotone convergence in $\mathcal{L}^{1}$ ), and 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ), and generalized Lemmas 1322, 1329, and 1331 in Section 16.3.4 on the seminormed vector space $\mathcal{L}^{p}$;
new Theorems 1351, and 1352, and fixed Theorem 1349 in Section 16.3.5 on Lebesgue's dominated convergence theorem;
new Section 16.3.6 on the Riesz-Fischer theorem (in $\mathcal{L}^{p}$, thus without uniqueness);
- new Remarks 207, 208, 216, 218, 220, 294, 297, 237, 251, 253, 301, 309, 356, 384, 440, 447, $450,467,470,472,473,481,523,537,549,560,573,636,663,673,676,767,793,795,801$, $805,807,812,827,846,852,950,961,964,971,973,976,1096,1097,1099,1122,1133,1140$, $1155,1173,1222,1226,1257,1265,1313,1325,1330,1332,1333,1338,1342,1344,1350$, and modified Remarks $255,263,376,536,698,781,968,1112,1143,1153,1215,1220,1230$, 1232, 1287;
- and possibly other slight modifications to improve the proof contents and their legibility.

Version 2 (release 1.1, 2021/04/01) is a minor revision.
Main changes are:

- addition of this foreword;
- reordering and sectioning:
sections have become chapters and gathered into parts to ease the reading;
all "complements" are gathered into a single chapter;
Section 7.2 about algebraic structures is moved right after Section 7.1 about set theory; Chapter 8 on subset systems is separated from Chapter 9 on measurability;
- some color modifications to increase legibility after grayscale printing;
- new contents in the introduction (Chapter 1), and minor corrections in Chapters 2-5 (including a bunch of hyperlinks in Chapters 4 and 5);
- fix proofs of uniqueness in Lemma 668 (statement changed too) and Theorem 724;
- fix statement and proof of Theorem 846 and Lemma 847, with new Definition 844;
- fix statement of Theorem 817;
- alternate proof of additivity for the integral of nonnegative simple functions in Lemma 774 based on the new disjoint representation of simple functions of Lemma 754; this involves new Definition 207, new Lemmas 734, 735, 756, 757, 764, 766, 768, and 772 new Remarks 208, 750, 753, 755, 758, and 773, addition of uniqueness in Lemmas 752 and 765, thus impacting proofs of Lemmas 742, 744, 771, 779, 784, and 814;
- factor the proof on layers into new Lemmas 446, 480, and 771 thus impacting proofs of Lemmas 617, 715, and 719;
- new Lemma 644 and Remark 645 about reasoning with properties satisfied almost everywhere, now used in the proofs of Lemmas 651-654, 659, and 664 (statement changed too in the latter);
- new Lemmas 376, 377, 600, and 601 shortening proofs of Lemmas 827 and 828;
- additional Remarks $214,430,473,507,512,689,723,730,795,816,845,850,896,898$, and modification of Remarks 407, 436, 504, and 674;
- Lemmas 552 and 553 are moved from Section 13.4 to Section 9.4;
- and possibly other slight modifications to improve the proof contents and their legibility.

Version 1 (release 1.0, 2021/01/14) is the first release.
It covers:

- the main basic results in measure theory, including the construction of Lebesgue measure in $\mathbb{R}$ via Carathéodory's extension scheme;
- integration of nonnegative measurable functions, including the Beppo Levi (monotone convergence) theorem and Fatou's lemma;
- the integral over a product space, including the Tonelli theorem;
- integration of measurable functions with possibly changing sign, including the seminormed vector space $\mathcal{L}^{1}$, and Lebesgue's dominated convergence theorem.


## Contents

I Overview ..... 11
1 Introduction ..... 13
1.1 Formal proof ..... 13
1.2 Objective ..... 13
1.3 Integration theories ..... 14
1.4 Contents ..... 14
1.5 Teaching ..... 15
1.6 Disclaimer ..... 15
1.7 Organization ..... 15
2 Notations ..... 17
3 State of the art ..... 21
3.1 A zoology of integrals ..... 21
3.1.1 Leibniz-Newton integral (late 17th century) ..... 21
3.1.2 Riemann integral (early 19th century) ..... 21
3.1.3 Lebesgue integral (early 20th century) ..... 22
3.1.4 Bochner integral (early mid-20th century) ..... 22
3.1.5 Henstock-Kurzweil integral (mid-20th century) ..... 22
3.1.6 Daniell integral (early 20th century) ..... 23
3.2 Lebesgue integral and Lebesgue measure ..... 23
3.3 Our main sources ..... 24
4 Proof techniques ..... 25
4.1 Subset systems ..... 25
4.2 Dynkin $\pi-\lambda$ theorem / monotone class theorem schemes ..... 26
4.3 Carathéodory's extension scheme ..... 26
4.4 Lebesgue scheme ..... 27
4.5 Bochner scheme ..... 27
5 Statements and sketches of proofs ..... 31
5.1 Sketch of the proof of the Riesz-Fischer theorem ..... 31
5.2 Sketch of the proof of Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$ ..... 32
5.3 Sketch of the proof of Lebesgue's extended dominated convergence theorem ..... 33
5.4 Sketch of the proof of Lebesgue's dominated convergence theorem ..... 33
5.5 Sketch of the proof of the Tonelli theorem ..... 34
5.6 Sketch of the proof of existence and uniqueness of the tensor product measure ..... 35
5.7 Sketch of the proof of Fatou's lemma ..... 36
5.8 Sketch of the proof of the Beppo Levi (monotone convergence) theorem ..... 37
5.9 Sketch of the proof of Carathéodory's extension theorem ..... 38
5.10 Sketch of the proof of the Dynkin $\pi-\lambda$ theorem / monotone class theorem ..... 38
II Detailed proofs ..... 40
6 Introduction ..... 41
7 Complements ..... 43
7.1 Complements on set theory ..... 44
7.2 Complements on order theory ..... 49
7.3 Complements on general topology ..... 51
7.3.1 Topological basis ..... 51
7.3.2 Second axiom of countability ..... 54
7.3.3 Complements on metric space ..... 55
7.4 Complements on algebraic structures ..... 57
7.4.1 Vector space ..... 57
7.4.2 Algebra over a field ..... 58
7.4.3 Seminormed vector space ..... 60
7.4.3.1 Banach space ..... 60
7.4.3.2 Semi-inner product vector space ..... 61
7.5 Complements on real numbers ..... 62
7.5.1 Real numbers ..... 62
7.5.2 Extended real numbers ..... 66
7.5.2.1 Nonnegative extended real numbers ..... 77
7.5.2.2 In the context of measure theory ..... 81
7.5.3 Second countability and real numbers ..... 88
7.5.4 Second countability and extended real numbers ..... 90
7.5.5 Infimum, supremum ..... 93
7.5.6 Limit inferior, limit superior ..... 99
7.5.7 Truncating a numeric function ..... 104
8 Subset system ..... 107
8.1 Basic properties ..... 108
8.1.1 Finite properties ..... 110
8.1.2 Countable properties ..... 112
8.2 Generated subset system ..... 118
8.3 Pi-system ..... 122
8.4 Ring of sets ..... 124
8.5 Algebra of sets ..... 129
8.6 Monotone class ..... 134
8.7 Lambda-system ..... 137
8.8 Sigma-algebra ..... 141
8.9 Dynkin $\pi-\lambda$ theorem ..... 147
8.10 Monotone class theorem ..... 148
9 Measurability ..... 149
9.1 Measurable space and Borel subset ..... 150
9.2 Measurable function ..... 152
9.3 Measurable subspace ..... 155
9.4 Product of measurable spaces ..... 158
10 Measurability and numbers ..... 163
10.1 Borel subset of numbers ..... 164
10.1.1 Borel subset of numbers ..... 164
10.1.2 Borel subset of real numbers ..... 164
10.1.3 Borel subset of extended real numbers ..... 165
10.1.4 Borel subset of nonnegative numbers ..... 165
10.1.5 Borel subset of complex numbers ..... 166
10.2 Measurable numeric function ..... 167
10.2.1 Measurable numeric function ..... 167
10.2.2 Measurable numeric function to $\mathbb{R}$ ..... 168
10.2.3 Measurable numeric function to $\overline{\mathbb{R}}$ ..... 168
10.2.4 Nonnegative measurable numeric function ..... 173
10.2.5 Tensor product of measurable numeric functions ..... 175
11 Numeric function over subset system ..... 177
11.1 Basic properties ..... 178
11.1.1 Finite properties ..... 178
11.1.2 Countable properties ..... 180
11.2 Content ..... 186
11.3 Premeasure ..... 188
12 Measure ..... 193
12.1 Measure space ..... 194
12.2 Some measures ..... 198
12.3 Negligible subset ..... 200
12.4 Uniqueness condition ..... 208
13 Measure and numbers ..... 211
13.1 Negligibility and numbers ..... 212
13.2 Essential supremum ..... 218
13.2.1 Essential supremum ..... 218
13.2.2 Nonnegative essential supremum for nonnegative functions ..... 221
13.3 The Lebesgue measure ..... 227
14 Simple function ..... 235
14.1 Indicator function ..... 236
14.2 Simple function ..... 238
14.3 Nonnegative simple function ..... 245
14.4 Adapted sequence ..... 248
15 Lebesgue integration of nonnegative function ..... 251
15.1 Integration of indicator function ..... 252
15.2 Integration of nonnegative simple function ..... 254
15.3 Integration of nonnegative measurable function ..... 260
15.4 Measure and integration over product space ..... 273
15.4.1 Measure over product space ..... 273
15.4.2 Lebesgue measure over product space ..... 280
15.4.3 Integration over product space of nonnegative function ..... 281
16 Lebesgue integration of real function ..... 285
16.1 Definition of the integral ..... 286
16.2 Notations for specific cases ..... 293
16.3 The seminormed vector space $\mathcal{L}^{p}$ ..... 297
16.3.1 The $p$-seminorm ..... 297
16.3.2 Hölder inequality ..... 302
16.3.3 Minkowski inequality ..... 305
16.3.4 The seminormed vector space $\mathcal{L}^{p}$ ..... 309
16.3.5 Lebesgue's dominated convergence theorem ..... 318
16.3.6 The Riesz-Fischer theorem ..... 323
17 Conclusions, perspectives ..... 327
Bibliography ..... 331
A Lists of statements ..... 337
List of Definitions ..... 337
List of Lemmas ..... 340
List of Theorems ..... 356
B The proof cites explicitly... ..... 357
C Is explicitly cited in the proof of... ..... 461

## Part I

## Overview

## Chapter 1

## Introduction

### 1.1 Formal proof

A formal proof is conducted in a logical framework that provides dedicated computer programs to mechanically check the validity of the proof, the so-called formal proof assistants. Such formal proofs may concern known mathematical theorems, but also properties of some piece of other computer programs, e.g. see [40], and [8, Glossary p. 343]. This field of computer science is extremely popular as it allows to certify with no doubt the behavior of critical programs.

Interactive theorem provers are now known to be able to tackle real analysis. For instance, in the field of Ordinary Differential Equations (ODEs) with Isabelle/HOL [43, 42, 44], and Coq [50], or in the field of Partial Differential Equations (PDEs), again with Isabelle/HOL [1], and Coq [7, 8]. In the latter example, the salient aspect is that the round-off error due to the use of IEEE-754 floating-point arithmetic can also be fully taken into account. But the price to pay is that all the details of the proofs has to be dealt with, and thus the availability of very detailed pen-and-paper proofs is a major asset.

### 1.2 Objective

Our current long-term purpose is to formally prove programs implementing the Finite Element Method (FEM), see Figure 1.1. The FEM is widely used to solve a broad class of PDEs, mainly because it has a sound mathematical foundation, e.g. see [61, 19, 53, 13, 27, 28, 29, 30]. The LaxMilgram theorem, one of the key ingredients to establish the FEM, was already addressed in [20] for a detailed pen-and-paper proof, and in [9] for a formal proof in Coq. The present document is a further contribution to our ultimate goal.

The Lax-Milgram theorem claims existence and uniqueness of the solution to the weak formulation of a PDE problem, such as the Poisson problem, and its discrete approximation; it is stated on a Hilbert space, i.e. a complete inner product space. The simplest Hilbert functional spaces relevant for the resolution of PDEs are the spaces $L^{2}$ and $H^{1}$. More generally, when stronger results such as the Banach-Nečas-Babuška theorem are involved, one may be interested in the Sobolev spaces $W^{m, p}$ that are Banach spaces, i.e. complete normed vector spaces.

Let $d \in\{1,2,3\}$ be the spatial dimension. Let $\Omega$ be an open subset of $\mathbb{R}^{d}$. We consider realvalued functions defined almost everywhere over $\Omega$. For $0<p \leqslant \infty$, the Lebesgue space $L^{p}(\Omega)$ is the space of measurable functions for which the $p$-th power of the absolute value (or the essential supremum when $p=\infty$ ) has a finite integral over $\Omega$. For any natural number $m$, the Sobolev space $W^{m, p}(\Omega)$ is the vector subspace of $L^{p}(\Omega)$ of functions for which all weak derivatives up to order $m$ also belong to $L^{p}(\Omega)$. Thus, the Lebesgue spaces correspond to the case $m=0$, $L^{p}(\Omega)=W^{0, p}(\Omega)$. And the Hilbert spaces correspond to the case $p=2, H^{m}(\Omega) \stackrel{\text { def. }}{=} W^{m, 2}(\Omega)$.


Figure 1.1: Brief recall of the various components of a typical (stationary) FEM formulation.
The black thick arrows depict the usual construction steps when solving a problem via the FEM. The black thick dashed arrow shows the possible feedback loop that a posteriori analysis allows, to change the mesh, the finite element (i.e. the order of polynomial approximation, the quadrature formula or the numerical scheme). The red dashed arrows depict when a theorem or a numerical tool takes place.

### 1.3 Integration theories

There is a huge variety of concepts of integral and integrability in the literature, e.g. see [18, 15], and one may wonder which one to use. Some are overridden by others, some are equivalent, and some have been developed for specific situations, such as vector-valued functions or functions defined on an infinite-dimensional domain. But a few of them have gained popularity, be it for their appropriateness for teaching, or their general-purpose nature: namely Riemann, Lebesgue, and Henstock-Kurzweil integrations.

Lebesgue integral (with Lebesgue measure) is the traditional framework in which the Sobolev functional spaces are defined. Indeed, Riemann integral is disqualified because of its poor results on limit and integral exchange making completeness unreachable, and so is Henstock-Kurzweil integral, because of its not so obvious extension to the multidimensional case and construction of a complete normed vector space of HK-integrable functions $[35,52]$.

### 1.4 Contents

The present version of this document covers all material up to the first main properties of the seminormed vector space $\mathcal{L}^{p}$ of integrable functions (before taking the quotient to obtain the normed vector space $L^{p}$ ). Among the rich literature on Lebesgue integral theory, it was mostly derived from the textbooks [49, 34, 55, 59].

It includes results on the general concepts of measurability, measure and negligibility, and integration of nonnegative measurable functions culminating with the Beppo Levi (monotone convergence) theorem allowing to exchange limit and integral for nondecreasing sequences of measurable functions, and Fatou's lemma that only gives an upper bound when the sequence is not monotone. The formalization in Coq of all these aspects is presented in [10].

It also includes product $\sigma$-algebras, product measures and their uniqueness, the construction of iterated integrals, up to the Tonelli theorem (for nonnegative measurable functions, the Fubini theorem which deals with integrable functions is not yet covered). The formalization in Coq of all these aspects is presented in [11].

In addition, this document also covers the building of the Lebesgue measure, the integral for measurable functions with possibly changing sign, the $p$-seminorms and the Lebesgue spaces $\mathcal{L}^{p}$ for $p \in(0, \infty]$, up to Lebesgue's dominated convergence theorem and its variants, and the RieszFischer theorem.

It is planned to add more results in a forthcoming version.

### 1.5 Teaching

This document is not primarily meant for teaching usage. The objective was to be as comprehensive as possible in the proofs. This led to very detailed demonstrations, and to a compact style of writing that is not common, and may seem daunting to the uninformed reader.

However, the authors tried to give some insights on the integration theory and on the proofs and theorems in the next introductory sections. They also strove to give some indications in the proofs when they felt it necessary. They believe that this document could be useful for interested teachers, and dedicated students.

### 1.6 Disclaimer

Note that the manuscript itself is not formally proved (and will never be). Indeed, $\mathrm{A}_{\mathrm{E}} \mathrm{X}$ compilers are not formal proof tools!

Moreover, formalization is not just straightforward translation of mathematical texts and formulas. Some design choices have to be made and proof paths may differ, mainly to favor usability of Coq theorems and ease formal developments. Thus, there exist various differences between the mathematical setting presented here, and the formal setting developed in Coq [10, 11]. For instance, the Coq definition of measurability of subsets (i.e. of $\sigma$-algebra) takes the generators as parameter, and not only the basic axioms. The ubiquitous use of total functions (defined for all values of their arguments) may be surprising at first: e.g. the addition in $\overline{\mathbb{R}}$ (" $\infty-\infty$ " is defined and takes the value 0 ), or the measure (one can take the measure of any subset, even nonmeasurable ones). Of course, just as in mathematical statements, unwanted behaviors are somehow prevented, for instance by adding an hypothesis stating the legality of the addition or the measurability of the subset (see [10] for details).

Hence, despite the care taken in its writing, this document might still be prone to errors or holes in the demonstrations. There could also exist simpler paths in the proofs. Please, feel free to inform the authors of any such issue, and to share any comments or suggestions...

### 1.7 Organization

Part I of this document is organized as follows. The notations are first collected in Chapter 2. Then, the literature on the subject is briefly reviewed in Chapter 3. Chapter 4 gathers some proof techniques. The chosen proof paths of the main results are then sketched in Chapter 5.

Part II (Chapters 6 to 16 ) is the core of this document. In this part, the definitions are presented, and the lemmas and theorems are stated with their detailed proofs. Its organization is described at the end of the introductory Chapter 6.

Chapter 17 concludes and gives some perspectives.
Finally, an appendix gathers the list of statements in Chapter A, and explicit dependencies (both ways) in Chapters B and C. The appendix is not intended for printing!

## Chapter 2

## Notations

In this chapter (as in most of the document), we use the following conventions:

- capital letters $X$ and $Y$ denote "surrounding" sets, typically the domain and codomain of functions, $A$ and $B$ are subsets of $X$, and $I$ denotes a set of indices;
- calligraphic letter $\mathcal{R}$ denotes a binary relation on $X$ (e.g. equality or inequality);
- the capital letter $P$ denotes a predicate, i.e. a function taking Boolean values;
- small letters $x, y$ and $i$ denote elements of the set using the matching capital letter;
- small letters $a$ and $b$ are (extended) real numbers (such that $a \leqslant b$ ), and $n$ and $p$ are (extended) natural numbers (such that $n \leqslant p$ );
- small letter $f$ denotes a function, e.g. from set $X$ to set $Y$;
- the capital Greek letter $\Sigma$ denotes a $\sigma$-algebra on $X$, and the small Greek letter $\mu$ denotes a measure on the measurable space $(X, \Sigma)$;
- positive/negative, and increasing/decreasing are meant in their strict sense, i.e. without the possibility of equality. Otherwise, we use nonnegative/nonpositive, and nondecreasing/nonincreasing.

The following notations and conventions are used throughout this document.

- Logic:
- Using a compound (tuple of elements of $X$, or subset of $X$ ) in an expression at a location where only a single element makes sense is a shorthand for the same expression expanded for all elements of the compound; for instance, " $\forall x, x^{\prime} \in X$ " means " $\forall x \in X, \forall x^{\prime} \in X$ ", " $\forall\left(x_{i}\right)_{i \in I} \in X$ " means " $\forall i \in I, x_{i} \in X^{\prime}$ ", and " $x, x^{\prime} \mathcal{R} x^{\prime \prime \prime}$ means $x \mathcal{R} x^{\prime \prime}$ and $x^{\prime} \mathcal{R} x^{\prime \prime}$;
$-x_{0} \mathcal{R}_{1} x_{1} \ldots \mathcal{R}_{m} x_{m}$ is a shorthand for $x_{0} \mathcal{R}_{1} x_{1} \wedge x_{1} \mathcal{R}_{2} x_{2} \wedge \ldots \wedge x_{m-1} \mathcal{R}_{m} x_{m}$;
- "iff" is a shorthand for "if and only if".
- Naive set theory:
- $\mathcal{P}(X)$ denotes the power set of $X$, i.e. the set of its subsets;
- subsets are denoted using $\subset$, and proper subsets by $\subsetneq$;
- big set operators ( $n$-ary or more) based on union and intersection, have precedence over binary set operations (no need for big parentheses); for instance, $\bigcup_{i \in I} A_{i} \cap B$ means $\left(\bigcup_{i \in I} A_{i}\right) \cap B ;$
- the definition provides the nullary union law $\bigcup_{\varnothing} \stackrel{\text { def. }}{=} \varnothing$ (no need for a convention), but unless specified otherwise, intersections are not considered with no operand; however, when a superset $X$ is not ambiguous, we may use the nullary intersection convention $\bigcap_{\varnothing} \stackrel{\text { def. }}{=} X$;
- superscript ${ }^{c}$ denotes the absolute complement;
$-\backslash$ denotes the set difference (or relative complement): $A \backslash B \stackrel{\text { def. }}{=} A \cap B^{c}$; when $B \subset A, A \backslash B$ is also called the local complement of $B$ (in $A$ );
- $\triangle$ denotes the symmetric (set) difference: $A \triangle B \stackrel{\text { def. }}{=}(A \backslash B) \cup(B \backslash A)=(A \cup B) \backslash(A \cap B)$;
$-\uplus$ denotes the disjoint union: $A \uplus B$ means $A \cup B$ with the assumption that $A \cap B=\varnothing$;
$-\bar{\Pi}$ is used to denote a set of traces of subsets, see Definition 226;
$-\bar{X}$ and $\bar{\Pi}$ are used to denote a set of Cartesian products of subsets, see Definition 227;
$-\operatorname{card}(X)$ is the cardinal of $X$, i.e. the number of its elements, with the convention $\operatorname{card}(X) \stackrel{\text { def. }}{=} \infty$ when $X$ is infinite;
- countable means either finite, or infinitely countable: $X$ is countable iff there exists an injection from $X$ to $\mathbb{N}$ iff there exists $I \subset \mathbb{N}$ and a bijection between $I$ and $X$;
$-\mathbb{1}_{A}$, or $\mathbb{1}_{\underline{A}}^{X}$, denotes the indicator function of the subset $A$ of $X$. It is the function from $X$ to $\mathbb{R}$, or $\overline{\mathbb{R}}$, that takes the value 1 for all elements of $A$, and the value 0 elsewhere;
- the set of functions from $X$ to $Y$ is either denoted $Y^{X}$, or through the type annotation " $X \rightarrow Y$ ". Both compact expressions "let $f \in Y^{X}$ " and "let $f: X \rightarrow Y$ " mean "let $f$ be a function from $X$ to $Y^{\prime \prime}$;
- to avoid double parentheses/braces, the expressions involving inverse images of singletons are simplified: $f^{-1}(y)$ is a shorthand for $f^{-1}(\{y\})$;
$-\{P(f)\}$ is a shorthand for $\{x \in X \mid P(f(x))\}$, or $f^{-1}(\{y \in Y \mid P(y)\})$; for instance, $\{f<a\}$ means either $f^{-1}[-\infty, a)$ or $f^{-1}(-\infty, a)$, see Lemma 816, and Lemma 820;
$-f^{+}$and $f^{-}$denote the nonnegative and nonpositive parts of a numeric function $f$, see Definition 528.
- Totally ordered set:
- when the nonempty set $X$ is totally ordered, intervals follow the notations used for real numbers; for instance, $\left(x_{1}, x_{2}\right]$ represents the subset $\left\{x \in X \mid x_{1}<x \leqslant x_{2}\right\}$;
- the square-and-round bracket notation is used for not specifying whether the bound is included or not: $\left(x_{1}, x_{2}\right)$ means either $\left(x_{1}, x_{2}\right)$ or $\left(x_{1}, x_{2}\right]$;
- the lower and greater bounds of $X$ may be denoted $-\infty$ and $+\infty$ (as usual, the plus sign may be omitted); they may belong to $X$ or not; $\bar{X}$ denotes $X \cup\{-\infty, \infty\}$.
- rays, or half-lines, are denoted using the square-and-round notation: whether $\infty$ belongs to $X$ or not, $(x, \infty)$ represents $\left\{x^{\prime} \in X \mid x<x^{\prime}\right\}$, i.e. either $(x, \infty]$ or $(x, \infty)$, see Definition 229;
$-\mathcal{I}_{X}^{o, p}$ denotes the set of open proper intervals of $X$, see Definition 229;
- $\mathcal{R}_{X}^{o}$ denotes the set of open rays of $X$, see Definition 229.
$-\mathcal{I}_{X}^{o}$ denotes the set of open intervals of $X$, see Definition 229.
- extrema of the empty set may be defined as $\inf \varnothing$ equals the greater bound (usually denoted $\infty$ ), and $\sup \varnothing$ equals the lower bound (usually denoted $-\infty$, but it is 0 for $\mathbb{R}_{+}$ and $\overline{\mathbb{R}}_{+}$); this fulfills the monotonicity property, and the law $\inf A \leqslant \sup A$ characterizes the nonempty subsets;
- Numbers:
$-\lfloor\cdot\rfloor$ denotes the floor function from $\mathbb{R}$ to $\mathbb{N}:\lfloor a\rfloor \leqslant a<\lfloor a\rfloor+1$;
$-[n . . p)]$ denotes the integer interval $(n, p) \backslash \mathbb{N}$ (with the natural variants using parentheses and/or brackets when exclusion or inclusion of bounds is specified);
- $\ell((a, b))$ denotes the length of the interval $[(a, b)]$, see Definition 1057;
- $\mathbb{K}$ denotes a number space: either the fields $\mathbb{R}$, and $\mathbb{C}$, or $\overline{\mathbb{R}}$.
- General topology:
- $\mathcal{T}_{X}(G)$ denotes the topology generated by $G \subset \mathcal{P}(X)$, see Definition 242;
- Measure theory:
- $\Pi_{X}(G)$ denotes the $\pi$-system generated by $G \subset \mathcal{P}(X)$, see Definition 606 ;
$-\mathcal{R}_{X}(G)$ denotes the ring of sets generated by $G \subset \mathcal{P}(X)$, see Definition 617;
$-\mathcal{A}_{X}(G)$ denotes the algebra of sets generated by $G \subset \mathcal{P}(X)$, see Definition 642;
$-\mathcal{C}_{X}(G)$ denotes the monotone class generated by $G \subset \mathcal{P}(X)$, see Definition 666;
- $\Lambda_{X}(G)$ denotes the $\lambda$-system generated by $G \subset \mathcal{P}(X)$, see Definition 682;
- $\Sigma_{X}(G)$ denotes the $\sigma$-algebra generated by $G \subset \mathcal{P}(X)$, see Definition 708;
$-\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra generated by the open subsets, see Definition 743;
- $\bigotimes_{i \in[1 . . m]} \Sigma_{i}$ denotes the tensor product of the $\sigma$-algebras $\left(\Sigma_{i}\right)_{i \in[1 . . m]}$, see Definition 774 ;
- $\mathbf{N}$ denotes the set of negligible subsets, see Definition 942;
- the annotation " $\mu$ a.e." specifies that the proposition is only considered $\mu$-almost everywhere for some measure $\mu$, e.g. $\stackrel{\mu \text { a.e. }}{=}, \forall_{\mu \text { a.e. }},(X \rightarrow Y)_{\mu \text { a.e. }}$, or $\mathcal{R}_{\mu \text { a.e., }}$, see Definition 953 ;
- $\delta_{Y}$ denotes the counting measure associated with $Y, \delta_{a}$ denotes the Dirac measure at $a$, see Lemma 935 and Definition 939;
- $\mathcal{L}$ denotes the Lebesgue $\sigma$-algebra on $\mathbb{R}$ and $\lambda$ denotes the (Borel-)Lebesgue measure on $\mathbb{R}$, see Definitions 1063 and 1071, and Theorem 1090;
$-\mu_{1} \otimes \mu_{2}$ denotes the tensor product of measures $\mu_{1}$ and $\mu_{2}$, see Definition 1224, Definition 1225, and Lemma 1233;
$-\lambda^{\otimes 2}$ denotes the Lebesgue measure on $\mathbb{R}^{2}$, see Lemma 1235;
- Lebesgue integral:
$-\mathcal{I} \mathcal{F}_{\mathbb{K}}$ denotes the set of measurable indicator functions on the number space $\mathbb{K}$, see Definition 1098;
$-\mathcal{S F _ { E }}$ denotes the vector space of simple functions to the vector space $E$, see Lemma 1108;
$-\mathcal{M}_{Y}$ denotes the set of measurable functions $X \rightarrow Y$, see Definition 758;
$-\mathcal{S F}+$ denotes the set of nonnegative simple functions, see Definition 1123;
$-\mathcal{M}_{+}$denotes the set of nonnegative measurable functions $X \rightarrow \overline{\mathbb{R}}_{+}$, see Definition 835 ;
$-\mathcal{M}$ denotes the set of measurable functions $X \rightarrow \overline{\mathbb{R}}$, see Definition 817;
$-\mathcal{M}_{\mathbb{R}}$ denotes the vector space of measurable functions $X \rightarrow \mathbb{R}$, see Lemma 815;
- $\int f d \mu$ denotes the (Lebesgue) integral of $f$ for the measure $\mu$, whether the function belongs to $\mathcal{I F}, \mathcal{S} \mathcal{F}_{+}, \mathcal{M}_{+}$, or $\mathcal{M}$, see Section 4.4, Definition 1144, Lemmas 1154 and 1172, and Lemma 1256;
$-\int_{A} f d \mu$ denotes the integral of the restriction of $f$ to $A$, see Lemmas 1148, 1163, 1199, and 1274;
- $\int_{a}^{b} f(x) d \mu(x)$ denotes the integral of $f$ over the interval ( $a, b$ ), see Lemma 1278;
$-\int f(x) d x$ denotes the integral of $f$ for the Lebesgue measure, see Definition 1285;
- $\left(\mathcal{L}^{p}, N_{p}\right)$ denotes the seminormed vector space of $p$-powered integrable functions for $p$ in $(0, \infty)$, and of essentially bounded measurable functions for $p=\infty$, see Lemma 1286 and Definition 1314;
- I denotes the integral operator for integrable functions, see Lemma 1319;
$-f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{p}}{\longrightarrow}} f$ denotes the convergence of $\left(f_{n}\right)_{n \in \mathbb{N}}$ towards $f$ in $p$-seminorm $N_{p}$ for $p$ in $[1, \infty]$, see Lemma 1339 .


## Chapter 3

## State of the art

After a brief survey of the wide variety of integrals, and of the means to build Lebesgue integral and Lebesgue measure, we review some works of a few authors, partly from the mathematical French school, that provide some details about results in measure theory, Lebesgue integration, and basics of functional analysis.

As usual, proofs provided in the literature are not comprehensive, and we have to cover a series of sources to collect all the details necessary for a formalization in a formal proof tool such as Coq. Usually, Lecture Notes in undergraduate mathematics are very helpful and we selected [37, 38, 39] among many other possible choices.

### 3.1 A zoology of integrals

The history of integral calculus dates back at least from the Greeks, with the method of exhaustion for the evaluation of areas, or volumes. Since then, mathematicians have constantly endeavored to develop new techniques. The objective is mainly to be able to integrate a wider class of functions. But also to establish more powerful results, such as convergence results, or the Fundamental Theorems of Calculus (expressing that derivation and integration are each other inverse operations). And possibly to fit specific contexts, e.g. driven by applications in physics, or for teaching purposes. For a wide panel, and pros and cons of different approaches, see for instance [21, 18, 15], in which more than a hundred named integrals are listed. But, as pointed out in [57], building a universal integral "will probably be unnecessarily complicated when restricted to a simple setting".

Let us review some of the most popular ones, mainly for the case of numeric functions defined on some interval.

### 3.1.1 Leibniz-Newton integral (late 17 th century)

The Leibniz-Newton (LN-)integral is defined in a descriptive way as the difference of any primitive evaluated at the bounds of the integration interval. All continuous functions are LN-integrable.

This integral possesses interesting properties such as formulas for integration by parts or by substitution, and a uniform convergence theorem. But since the definition is not constructive, there is no regular process to build the primitive of a function, even if it is continuous. Moreover, other monotone or dominated convergence results must assume that the limit is LN -integrable. And it raises the question of integrability of functions with no primitives.

### 3.1.2 Riemann integral (early 19 th century)

The Riemann (R-)integral [54] consists in cutting the area "under" the graph of the function into vertical rectangular strips by choosing a subdivision of the integration interval, and in increasing the number of strips by making the step of the subdivisions go down to 0 . Equivalently, it might
be more convenient to consider the Darboux integral [24], which checks that both upper and lower so-called Darboux sums have the same limit.

This simple approach has similar properties than the LN-integral, with for instance a uniform convergence result, and R-integrable functions are the piecewise continuous function which set of discontinuity points has null measure. But again, monotone convergence and dominated convergence theorems must assume R-integrability of the limit.

### 3.1.3 Lebesgue integral (early 20th century)

The key idea of the Lebesgue (L-) approach [46] is to consider subdivisions of the codomain of the function, i.e. cutting the area under the graph into horizontal pieces, which are no longer continuous strips when the function is not concave, and to define a measure for these pieces that generalizes the length of intervals.

This has the great advantage of providing powerful monotone and dominated convergence theorems, and to allow for an abstract setting in which one can handle functions defined on more general spaces than the Euclidean spaces $\mathbb{R}^{n}$. And this paves the road for probability theory. For instance, the Wiener measure used for the study of stochastic processes such as Brownian motion is defined on the infinite-dimensional Banach space of continuous functions from a compact interval to $\mathbb{R}^{n}$. Moreover, the concept of property satisfied almost everywhere opens the way to the $L^{p}$ Lebesgue spaces as (complete) normed vector spaces (Banach spaces).

The main difficulty is the necessity to develop the measure theory to be able to associate a length to the "horizontal pieces". In general, this is not possible for all subsets of the domain of the function, and leads to the concepts of measurable subset and measurable function.

Going back to the geometrical interpretation of the area under the graph of the function, it is interesting to note that the opposition Riemann (subdivision of the domain) versus Lebesgue (subdivision of the codomain) reflects in the field of ordinary differential equations where the classical numerical schemes based on a discretization of the time domain are now opposed to the recent quantized state system solvers that are based on the quantization of the state variables, see [17, 31, 32]. Moreover, we may also cite [36] in the field of signal processing where the Lebesgue geometrical scheme is seen as a nonlinear sampling qualified of "noise-free quantization", that could have impact on electronic design.

### 3.1.4 Bochner integral (early mid-20th century)

The Bochner (B-) integral $[6,51,59]$ is a generalization of the L-integral to the case of functions taking their values in a Banach space.

Similarly to the L-integral, its construction is based on the limit of integrals of simple functions (see Sections 4.4 and 4.5), that are measurable functions taking only a finite number of values. The main difference here is the absence of order in Banach spaces, which makes infinite values impracticable and prevents the use of the least-upper-bound property. Instead, the existence of the limit is ensured by completeness, and by the additional assumption of separability which provides the building of Cauchy sequences.

From the L-integral of nonnegative measurable functions, the B-integral has the great advantage to allow the handling of functions to $\mathbb{R}$ (with possibly changing sign) or $\mathbb{C}$, to the Euclidean space $\mathbb{R}^{n}$ or the Hermitian space $\mathbb{C}^{n}$, and to any separable Banach space. For instance, to $L^{p}(\Omega)$ (with $p \in[1, \infty)$ ), where $\Omega$ is a regular enough domain in $\mathbb{R}^{n}$, or $\mathbb{C}^{n}$. Given a real number $T>0$, this allows to define the space $L^{2}\left([0, T], L^{2}(\Omega)\right)$ of square-integrable functions from the time interval $[0, T]$ to the square-integrable functions defined on the (space) domain $\Omega$. Such a cascading construction can be useful for the study of some evolutionary PDEs.

### 3.1.5 Henstock-Kurzweil integral (mid-20th century)

The Henstock-Kurzweil (HK-)integral, or gauge integral [45, 41, 4] is a generalization of the Rintegral for which the subdivision of the integration interval is no longer uniform, but is driven by
the variations of the function to integrate through a so-called gauge function.
It is more powerful than the L-integral in the sense that a real-valued function is L-integrable if and only if the function and its absolute value are HK-integrable. Unlike the L-integral, this leads to a smoother formulation of the fundamental theorem of calculus, without the need for the concept of improper integral. The monotone and dominated convergence theorems are also valid. Moreover, the HK-integral is usually considered more suited for teaching, for it is hardly more complex than the R-integral.

Although less straightforward than with the L-integral, recent developments make some extensions feasible. We may cite the extension to functions defined on the Euclidean spaces $\mathbb{R}^{n}$ [47], and the completion of HK-integrable functions into a Banach space [3, 56, 35, 52, 58].

### 3.1.6 Daniell integral (early 20th century)

The Daniell (D-) approach $[23,33]$ is a general scheme that extends an elementary integral defined for elementary functions to a much wider class of functions by checking that upper and lower elementary integrals of elementary functions share the same limit, in a way that is similar to the Darboux approach. When applied to the integral of simple functions, this is somehow equivalent to steps 3 and 4 of the Lebesgue scheme described later in Section 4.4. But it is enough to consider the R-integral for the compactly supported continuous functions (a.k.a. the Cauchy integral) to get back the L-integral for the Lebesgue measure on the Euclidean spaces $\mathbb{R}^{n}$.

This Daniell approach to the L-integral has the great advantage of not needing the concept of measure, which can be reconstructed afterwards by taking the integral of indicator functions of measurable subsets. Moreover, in the mid-20th century, this approach was shown in [5] to be compatible with constructive analysis.

In the present document, we choose the Lebesgue approach.

### 3.2 Lebesgue integral and Lebesgue measure

The Lebesgue integral was originally built using measure theory following the so-called Lebesgue scheme, described in Section 4.4. We have seen that it may also be built in a constructive way by applying the Daniell extension scheme to some elementary integral defined for elementary functions, such as the Riemann integral for compactly supported continuous functions. Another alternative using the same basic ingredients consists in the completion of the normed vector space of compactly supported continuous functions and the extension of the Riemann integral which is uniformly continuous [12, 25].

The Lebesgue measure on $\mathbb{R}$ is a generalization of the length of bounded intervals to a much wider class of subsets. This extension is actually unique and complete. It is defined on the Lebesgue $\sigma$-algebra which is generated by the Borel subsets and the negligible subsets. The restriction to the sole Borel subsets (the $\sigma$-algebra generated by the open subsets, or the closed subsets) is sometimes called the Borel(-Lebesgue) measure. In the very same way, the Lebesgue measure on the Euclidean spaces $\mathbb{R}^{n}$ is a generalization of the $n$-volume of rectangular boxes. It is the tensor product of the Lebesgue measure on $\mathbb{R}$.

There are three main techniques for the construction of the Lebesgue measure on $\mathbb{R}$. The most popular one for teaching is through Carathéodory's extension theorem [16, 26]. The process builds progressively the Lebesgue $\sigma$-algebra and the Lebesgue measure with their properties (see Section 4.3). A more abstract approach is based on the Riesz-Markov-Kakutani representation theorem, which associates any positive linear form on the space of compactly supported continuous functions with a unique measure, e.g. see [55]. The third way is not based on measure theory. It follows the Daniell approach to integration, which is also based on the Riemann integration of compactly supported continuous functions, and defines the Lebesgue measure of any measurable subset as the integral of its indicator function.

The construction of non-Lebesgue-measurable subsets requires the use the axiom of choice, as for instance for the Vitali subset [60].

In the present document, we build the Lebesgue integral using the Lebesgue scheme (see Section 4.4), and the Lebesgue measure through Carathéodory's extension theorem (see Section 4.3).

### 3.3 Our main sources

The main ingredients of measure theory and Lebesgue integration are presented in an elementary manner in the textbook [49], which is very well suited for our purpose. However, some results such as the construction of the Lebesgue measure and the Tonelli theorem are admitted.

Details for the construction of the Lebesgue measure (through the Carathéodory approach), and for the proofs of the Fubini-Tonelli theorems can be found in [34]. This other textbook is quite comprehensive. Many results are presented as exercises with their solution.

We may also cite $[55,59,2,14]$. The former for instance for a construction of the Lebesgue measure through the Riesz-Markov-Kakutani representation theorem. The second one for a fairly comprehensive presentation of the Bochner integral and the main results. And the latter two for the $L^{p}$ Lebesgue spaces and the $W^{m, p}$ Sobolev spaces.

Note that some original forms of statements have been developed for the present document, and even though most results are well known, their formulations are new to our knowledge. This includes:

- the individual treatment of constitutive properties (closedness under set operations) of subset systems in Section 8.1, and of numeric functions over subset systems in Section 11.1;
- the formalization of most concepts based on properties almost satisfied in Section 12.3 with abstract results such as Lemmas 959, 960, 962, 963, 970, 972, 984, 985, and 988, 1189 in Section 15.3, and 1264 in Section 16.1;
- the concept of almost sum in Section 13.1 with Lemmas 1002, 1003, and 1004, impacting Lemma 1187 in Section 15.3, Lemmas 1253, 1268, and 1269 in Section 16.1, Lemma 1274 in Section 16.2, and Lemmas 1309, and 1310 in Section 16.3;
- the concept of disjoint representation of simple functions in Section 14.2 with Lemmas 1114, 1116, and 1117.


## Chapter 4

## Proof techniques

In the framework of integration theory, some concepts and proof techniques are used repeatedly. This chapter describes the most important ones: "subset systems", the "Dynkin $\pi-\lambda$ theorem / monotone class theorem scheme", the "Carathéodory extension scheme", the "Lebesgue scheme", and the "Bochner scheme".

### 4.1 Subset systems

A $\pi$-system, a ring of sets, an algebra of sets, a monotone class, a $\lambda$-system and a $\sigma$-algebra are all different kinds of subset systems (i.e. subsets of the power set), see Figure 4.1. They are defined in Chapter 8, where a mini-language to represent the properties in a compact form is introduced.

In the present document, the richest subset system is the $\sigma$-algebra (the others have fewer properties). The choice of some $\sigma$-algebra defines the measurable subsets (see Chapter 9), for which measures are then defined (in Chapter 12).


Figure 4.1: Description of the hierarchy of subset systems introduced in this document.
In each box, a set of minimal properties to define the subset system is given: for instance, a ring is nonempty and closed under set difference and union.
The red dashed arrows express some necessary properties to "upgrade" a system to a richer one: for instance, a monotone class is a $\lambda$-system if it contains the full set $X$ and is compatible with local complement. We recall that $\Delta,{ }^{C}, \biguplus_{\mathbb{N}}, \bigcup_{\mathbb{N}, \nearrow}$ and $\bigcap_{\mathbb{N}, \searrow}$ represent closedness under respectively symmetric difference, complement, disjoint countable union, monotone countable union and intersection.

### 4.2 Dynkin $\pi-\lambda$ theorem / monotone class theorem schemes

Roughly speaking, being a $\lambda$-system (resp. a monotone class) is what is missing to a $\pi$-system (resp. an algebra of subsets) to be a $\sigma$-algebra.

To prove that some property $P$ holds on some $\sigma$-algebra $\Sigma$ generated by some subset system $G$, the idea is to proceed in two steps. First, show that the property $P$ holds on the simpler kind of subset system ( $\pi$-system, or algebra of sets) generated by $G$, and then establish that the subset of $\Sigma$ where $P$ holds is of the less simple kind of subset system ( $\lambda$-system, or monotone class). Indeed, the Dynkin $\pi-\lambda$ theorem states that the $\lambda$-system generated by a $\pi$-system is equal to the $\sigma$-algebra generated by the same $\pi$-system (in short $\Lambda_{X}(\Pi)=\Sigma_{X}(\Pi)$ ), and similarly, the monotone class theorem states that the monotone class generated by an algebra of sets is equal to the $\sigma$-algebra generated by the same algebra of sets (in short $\mathcal{C}_{X}(\mathcal{A})=\Sigma_{X}(\mathcal{A})$, see Section 5.10). Thus, $P$ is established on the whole $\sigma$-algebra.

The five steps of the scheme of the Dynkin $\pi-\lambda$ theorem, and of the monotone class theorem are the following:

1. first define $\mathcal{S} \stackrel{\text { def. }}{=}\{A \in \Sigma \mid P(A)\}$; obviously, we have $\mathcal{S} \subset \Sigma$;
2. then, show that $G \subset \mathcal{S}$;
3. then, show that the $\pi$-system (resp. algebra of sets) $\mathcal{U}_{X}^{2}(G)$ is a subset of $\mathcal{S}$;
4. then, show that $\mathcal{S}$ is a $\lambda$-system (resp. monotone class);
5. and finally, obtain the other inclusion $\Sigma \subset \mathcal{S}$, because from monotonicity of subset system generation, and from the Dynkin $\pi-\lambda$ theorem (resp. monotone class theorem), i.e. the equality in the middle, we have

$$
\Sigma=\Sigma_{X}(G) \subset \Sigma_{X}\left(\mathcal{U}_{X}^{2}(G)\right)=\mathcal{U}_{X}^{1}\left(\mathcal{U}_{X}^{2}(G)\right) \subset \mathcal{U}_{X}^{1}(\mathcal{S})=\mathcal{S}
$$

where $\mathcal{U}_{X}^{1}=\Lambda_{X}\left(\right.$ resp. $\left.\mathcal{C}_{X}\right)$, i.e. the generated $\lambda$-system (resp. monotone class), and $\mathcal{U}_{X}^{2}=\Pi_{X}$ (resp. $\mathcal{A}_{X}$ ), i.e. the generated $\pi$-system (resp. algebra of sets).

In the present document, the fifth step is embodied by Lemma 734 for the Dynkin $\pi-\lambda$ theorem, and by Lemma 738 for the monotone class theorem. The Dynkin $\pi-\lambda$ theorem scheme is used once when extending the equality of two measures on a generator $\pi$-system to the whole $\sigma$-algebra in Lemma 992, which is then used to establish uniqueness of the Lebesgue measure on $\mathbb{R}$ in Theorem 1090 (see also Section 5.9). The monotone class theorem scheme is used twice when building the tensor product measure in the context of the product of finite measure spaces: to establish first the measurability of the measure of sections in Lemma 1221, and then the uniqueness of the tensor product measure in Lemma 1231 (both in the finite case).

### 4.3 Carathéodory's extension scheme

The Carathéodory extension scheme consists in applying the eponymous theorem (see Section 5.9) to extend a premeasure (i.e. defined on a ring of subsets, and not on a $\sigma$-algebra) into a complete measure on a $\sigma$-algebra containing the initial ring of subsets.

Let $X$ be the ambient set, $\mu$ be the premeasure and $\mathcal{R}$ be the ring of subsets (of $X$ ). The two steps of the scheme are the following:

1. extend the premeasure to any subset of $X$ by taking the infimum of the sum of premeasures of all coverings with elements of the ring $\mathcal{R}$ (and the value $\infty$ when there is no such covering):

$$
\mu^{\star}(A) \stackrel{\text { def. }}{=} \inf \left\{\sum_{n \in \mathbb{N}} \mu\left(A_{n}\right) \mid\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_{n}\right\} ;
$$

the theorem ensures that $\mu^{\star}$ is an outer measure on $X$, and that it is an extension of the premeasure $\mu$;
2. define the set of Carathéodory-measurable subsets as

$$
\Sigma \stackrel{\text { def. }}{=}\left\{E \subset X \mid \forall A \subset X, \mu^{\star}(A)=\mu^{\star}(A \cap E)+\mu^{\star}(A \backslash E)\right\} ;
$$

the theorem ensures that $\Sigma$ is a $\sigma$-algebra on $X$ containing the ring $\mathcal{R}$, and that the restriction $\mu^{\star}{ }_{{ }_{\Sigma \Sigma}}$ is a complete measure on $(X, \Sigma)$.
The Carathéodory extension scheme is a popular tool to build measures. The prominent example is the Lebesgue measure on $\mathbb{R}$ that extends the length of open intervals (see Section 13.3). But for instance, the extension scheme may also be used to build the Lebesgue-Stieltjes measure associated with its cumulative distribution function, and Loeb measures in a nonstandard analysis framework, e.g. see [48, 22].

### 4.4 Lebesgue scheme

Consider a set $X$ equipped with a $\sigma$-algebra $\Sigma$ of measurable subsets, and a measure $\mu$ that associates a "length" (in $\overline{\mathbb{R}}_{+}$) to any measurable subset. The set of extended real numbers $\overline{\mathbb{R}}$ is equipped with the Borel $\sigma$-algebra $\mathcal{B}(\overline{\mathbb{R}})$ that is generated by all open subsets. The purpose is to build, or prove properties of, the integral of "measurable" functions from $X$ to $\overline{\mathbb{R}}$, i.e. for which the inverse image of measurable subsets (of $\overline{\mathbb{R}}$ ) are measurable subsets (of $X$ ).

The Lebesgue scheme consists in establishing facts about the integral by working successively inside three embedded functional subsets: the set $\mathcal{I F}$ of indicator functions of measurable subsets, the vector space $\mathcal{S F}$ of simple functions (the linear span of $\mathcal{I F}$ ), and the set $\mathcal{M}$ of measurable functions, see Figure 4.2.

The four steps of the scheme are the following:

1. establish the fact in the case of indicator functions (in $\mathcal{I F}$ ) for which the integral is simply the measure of the support;
2. then, generalize the fact to the case of nonnegative simple functions (in $\mathcal{S \mathcal { F } _ { + }}$ ) by nonnegative (finite) linear combination of indicator functions;
3. then again, generalize the fact to the case of nonnegative measurable functions (in $\mathcal{M}_{+}$) by taking the supremum for all lower nonnegative simple functions;
4. finally, generalize the fact to the case of all measurable functions (in $\mathcal{M}$ ) by taking the difference between the expressions involving nonnegative and nonpositive parts (integrable functions are those for which this difference is well-defined).

Actually, a fifth step can be added to generalize the fact to the case of numeric functions taking their values in $\mathbb{R}^{n}, \mathbb{C}$, or $\mathbb{C}^{n}$ by considering separately all components.

In the present document, the steps of Lebesgue scheme are mainly used to build the Lebesgue integral (steps 1 to 4, see Chapter 15 and Section 16.1), but also to establish some properties such as the Beppo Levi (monotone convergence) theorem (step 3, see Section 5.8 and Theorem 1180) and the Tonelli theorem (steps 1 to 3, see Section 5.5 and Theorem 1242).

### 4.5 Bochner scheme

Let us also consider now a separable Banach space $E$ equipped with its Borel $\sigma$-algebra $\mathcal{B}(E)$. The Bochner integral extends the Lebesgue integral to the case of "integrable" functions from $X$ to $E$.

The Bochner scheme is similar to the Lebesgue scheme. It also consists in establishing facts about the integral by working successively inside two embedded functional subsets: the subset of Bochner-integrable simple functions, and the subset of (Bochner-)integrable functions. The main difference here is the absence of order in the codomain $E$ that prevents the consideration of


Figure 4.2: Description of the Lebesgue scheme to construct the Lebesgue integral, and insight into the construction of the Bochner integral.
In the boxes filled in light pink, the functions have values in $\overline{\mathbb{R}}$, whereas in the boxes filled in light blue, they have values in $\mathbb{R}$, and in the boxes filled in brown, they have values in a Banach space $E$. In the right column, the main lemmas over the integral are recalled (the blue boxes are stated and demonstrated in the present document, the red ones are not). The red dashed arrows show some instances of a general definition. Note that the construction of the Bochner integral is not included in the present version of the document.
nonnegative values, of supremum, and thus of (nonnegative) infinite values. However, the Bochner integral is based on the Lebesgue integral for nonnegative measurable functions.

The two steps of the scheme are the following:

1. establish the fact in the case of Bochner-integrable simple functions;
2. finally, generalize the fact to the case of all (Bochner-)integrable functions by taking the limit of integrable simple functions, using separability and completeness.

Note that the Bochner scheme is actually an alternative to step 4 of Section 4.4, and the informal fifth step for the extension of the integral to functions taking their values in $\mathbb{R}, \mathbb{R}^{n}, \mathbb{C}$, and $\mathbb{C}^{n}$.

## Chapter 5

## Statements and sketches of proofs

This chapter gathers the sketches of the proofs of the main results that are detailed in Part II. Namely: the Riesz-Fischer theorem, Lebesgue's dominated convergence theorem and its extended versions for integrable functions and in $\mathcal{L}^{p}$, the Tonelli theorem, the Existence and uniqueness of the tensor product measure suite of lemmas, Fatou's lemma, the Beppo Levi (monotone convergence) theorem, Carathéodory's extension theorem, the Dynkin $\pi-\lambda$ theorem, and the monotone class theorem.

### 5.1 Sketch of the proof of the Riesz-Fischer theorem

## Riesz-Fischer theorem.

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{p}$ be a Cauchy sequence. Then, there exists $f \in \mathcal{L}^{p}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$.

Moreover, there exists $\left(k_{n}\right)_{n \in \mathbb{N}} \in \mathbb{N}$ increasing and $h \in \mathcal{L}^{p}$ such that for all $n \in \mathbb{N},\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$, and $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} f$. When $p=\infty$, one can take $k_{n} \stackrel{\text { def. }}{=} n$, for all $n \in \mathbb{N}$.

See Theorem 1354. The proof of the Riesz-Fischer theorem uses the arithmetic of $\mathbb{R}$, the Beppo Levi (monotone convergence) theorem set in $\mathcal{L}^{p}$, and Lebesgue's extended dominated convergence theorem also set in $\mathcal{L}^{p}$ (see Section 5.2). The proof of the Riesz-Fischer theorem goes as follows:

- for $p \in[1, \infty)$
- from the Cauchy assumption, build $\left(k_{n}\right)_{n \in \mathbb{N}}$ increasing such that $N_{p}\left(f_{k_{n}+j}-f_{k_{n}}\right) \leqslant \frac{1}{2^{n}}$;
- apply the Beppo Levi (monotone convergence) theorem set in $\mathcal{L}^{p}$ to the sequence

$$
S_{n} \stackrel{\text { def. }}{=} \sum_{i \in[0 . . n)}\left|f_{k_{i+1}}-f_{k_{i}}\right|
$$

to obtain the existence of $S \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$such that $S_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} S$ and $\left|S_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} S$, then establish $\left|f_{k_{n+q}}-f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} S-S_{n} \leqslant S$ to show that $\left(f_{k_{n}}\right)$ is a Cauchy sequence (pointwise), and obtain the existence of $f \in \mathcal{M}_{\mathbb{R}}$ such that $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$;

- then, let $h_{n} \stackrel{\text { def. }}{=} \mid f_{k_{n}}-f$ and $g_{q}^{n} \stackrel{\text { def. }}{=}\left|f_{k_{n+q}}-f_{k_{n}}\right| \in \mathcal{L}^{p}$, and show successively

$$
h_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} 0, \quad g_{q}^{n} \xrightarrow[\rightarrow \rightarrow \infty]{\mu \text { a.e. }} h_{n}, \quad h_{n} \stackrel{\mu \text { a.e. }}{\leqslant} S, \quad f \in \mathcal{L}^{p}, \quad \text { and } \quad\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h \stackrel{\text { def. }}{=} S+|f| \in \mathcal{L}^{p},
$$

and finally apply the Lebesgue's extended dominated convergence theorem set in $\mathcal{L}^{p}$ to $\left(h_{n}\right)_{n \in \mathbb{N}}$ to obtain $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{p}}{\rightarrow}} f$, and then $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{p}}{\rightarrow}} f$;

- for $p=\infty$
- from the Cauchy assumption, establish $\left|f_{n+q}-f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} 1$ to show that $\left(f_{n}\right)$ is a Cauchy sequence (pointwise), and obtain the existence of $f \in \mathcal{M}_{\mathbb{R}}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$;
- then, let $h_{n} \stackrel{\text { def. }}{=}\left|f_{n}-f\right|$ and $g_{q}^{n} \stackrel{\text { def. }}{=}\left|f_{K_{0}+q}-f_{K_{0}+n}\right| \in \mathcal{L}^{\infty}$, and show successively

$$
\begin{gathered}
h_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} 0, \quad g_{q}^{n} \xrightarrow[q \rightarrow \infty]{\mu \text { a.e. }} h_{K_{0}+n}, \quad h_{K_{0}+n} \stackrel{\mu \text { a.e. }}{\leqslant} 1, \quad f \in \mathcal{L}^{\infty}, \\
\text { and }\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h \stackrel{\text { def. }}{=} \max _{k<K_{0}}\left\{\left|f_{k}\right|, 1+|f|\right\} \in \mathcal{L}^{\infty},
\end{gathered}
$$

and finally obtain $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{\infty}}{\longrightarrow}} f$.

For instance, the Riesz-Fischer theorem may be used to eventually apply the theory of Banach spaces to the Lebesgue spaces $L^{p}$ (after taking the quotient with respect to the subspace of functions vanishing almost everywhere).

### 5.2 Sketch of the proof of Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$

Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$.
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty)$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{M}$. Let $g \in \mathcal{L}^{p}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, and for all $n \in \mathbb{N},\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, there exists $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^{p}$ such that we have

$$
\begin{align*}
\forall n \in \mathbb{N}, & \tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n} \wedge N_{p}\left(\tilde{f}_{n}\right)=N_{p}\left(f_{n}\right) \leqslant N_{p}(g)<\infty,  \tag{5.1}\\
& \tilde{f}^{\mu \text { a.e. }} f  \tag{5.2}\\
= & N_{p}(\tilde{f})=N_{p}(f) \leqslant N_{p}(g)<\infty  \tag{5.3}\\
& \tilde{f}_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} \tilde{f},  \tag{5.4}\\
\int|f|^{p} d \mu=\int|\tilde{f}|^{p} d \mu= & \left.\int \lim _{n \rightarrow \infty}\left|\tilde{f}_{n}\right|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|\tilde{f}_{n}\right|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|f_{n}\right|^{p} d \mu \quad \text { (in } \mathbb{R}\right) .
\end{align*}
$$

See Theorem 1351. The proof of Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$ uses the arithmetic of $\overline{\mathbb{R}}$ (but with functions that are almost everywhere finite), and Lebesgue's extended dominated convergence theorem (see Section 5.3). The proof goes as follows:

- prove that $\left(h_{n} \stackrel{\text { def. }}{=}\left|f_{n}\right|^{p}\right)_{n \in \mathbb{N}}, h \stackrel{\text { def. }}{=}|f|^{p},|g|^{p} \in \mathcal{M}, h_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} h,|g|^{p}$ is $\mu$-integrable, and for all $n \in \mathbb{N},\left|h_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant}|g|^{p}$, then apply Lebesgue's extended dominated convergence theorem to $\left(h_{n}\right)_{n \in \mathbb{N}}, h$ and $|g|^{p}$ to obtain (5.4), and finally pick finite representatives $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^{p}$ such that (5.1), (5.2), and $\tilde{f}_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} \tilde{f}$ hold;
- prove that $\left(k_{n} \stackrel{\text { def. }}{=}\left|\tilde{f}_{n}-\tilde{f}\right|^{p}\right)_{n \in \mathbb{N}}, k \stackrel{\text { def. }}{=} 0,2^{p}|g|^{p} \in \mathcal{M}, k_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} k, 2^{p}|g|^{p}$ is $\mu$-integrable, and for all $n \in \mathbb{N},\left|k_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} 2^{p}|g|^{p}$, and then apply Lebesgue's extended dominated convergence theorem to $\left(k_{n}\right)_{n \in \mathbb{N}}, k$ and $2^{p}|g|^{p}$ to obtain the $\mathcal{L}^{p}$ convergence (5.3).

Lebesgue's extended dominated convergence theorem in $\mathcal{L}^{p}$ is used in the present document to prove the variant of the Beppo Levi (monotone convergence) theorem set in $\mathcal{L}^{p}$ (for $p \in[1, \infty)$, see Theorem 1352), and the Riesz-Fischer theorem (see Section 5.1 and Theorem 1354).

### 5.3 Sketch of the proof of Lebesgue's extended dominated convergence theorem

## Lebesgue's extended dominated convergence theorem.

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f, g$ is $\mu$ integrable, and for all $n \in \mathbb{N},\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, $\left(f_{n}\right)_{n \in \mathbb{N}}$ and $f$ are $\mu$-integrable, and there exist $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{1}$ such that, for all $n \in \mathbb{N}, \tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n}, \tilde{f}^{\mu} \stackrel{\text { a.e. }}{=} f, \tilde{f}_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{1}} \tilde{f}$, and we have

$$
\begin{equation*}
\int f d \mu=\int \tilde{f} d \mu=\int \lim _{n \rightarrow \infty} \tilde{f}_{n} d \mu=\lim _{n \rightarrow \infty} \int \tilde{f}_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad(\text { in } \mathbb{R}) . \tag{5.5}
\end{equation*}
$$

See Theorem 1349. The proof of Lebesgue's extended dominated convergence theorem uses the arithmetic of $\overline{\mathbb{R}}$ (but with functions that are almost everywhere finite), and Lebesgue's dominated convergence theorem (see Section 5.4). The proof goes as follows:

- $A \xlongequal{\text { def. }}\left\{f=\liminf _{n \rightarrow \infty} f_{n}\right\} \cap\left\{f=\lim \sup _{n \rightarrow \infty} f_{n}\right\} \cap \bigcap_{n \in \mathbb{N}}\left\{\left|f_{n}\right| \leqslant g\right\} \cap g^{-1}\left(\mathbb{R}_{+}\right) \subset X$ is first shown to be measurable with $\mu\left(A^{c}\right)=0$; thus $\tilde{f}_{n} \stackrel{\text { def. }}{=} f_{n} \mathbb{1}_{A} \stackrel{\mu \text { a.e. }}{=} f_{n}, \tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A} \stackrel{\mu \text { a.e. }}{=} f$, and $\tilde{g} \stackrel{\text { def. }}{=} g \mathbb{1}_{A} \stackrel{\mu \text { a.e. }}{=} g ;$ moreover, $\tilde{g} \in \mathcal{L}^{1}$;
- then, since $\lim _{n \rightarrow \infty} \tilde{f}_{n}=\tilde{f}$ and $\left|\tilde{f}_{n}\right| \leqslant \tilde{g}$, Lebesgue's dominated convergence theorem provides $\tilde{f}_{n}, \tilde{f} \in \mathcal{L}^{1}$, and the equality

$$
\int \tilde{f} d \mu=\int \lim _{n \rightarrow \infty} \tilde{f}_{n} d \mu=\lim _{n \rightarrow \infty} \int \tilde{f}_{n} d \mu
$$

- finally, the integrability of $f_{n}$ and $f$, and identity (5.5) follow from the compatibility of the integral in $\mathcal{M}$ with almost equality.

Lebesgue's extended dominated convergence theorem is used in the present document to prove the variant set in $\mathcal{L}^{p}$ (for $p \in[1, \infty)$, see Section 5.2 and Theorem 1351).

For instance, Lebesgue's extended dominated convergence theorem may be used to prove the Leibniz integral rule (differentiation under the integral sign), e.g. for the study of integrals function of their upper bound.

### 5.4 Sketch of the proof of Lebesgue's dominated convergence theorem

Lebesgue's dominated convergence theorem.
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $g \in \mathcal{L}^{1}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, and for all $n \in \mathbb{N},\left|f_{n}\right| \leqslant g$. Then, $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{1}, f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{1}} f$, and we have

$$
\begin{equation*}
\int f d \mu=\int \lim _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad(\text { in } \mathbb{R}) \tag{5.6}
\end{equation*}
$$

See Theorem 1347. The proof of Lebesgue's dominated convergence theorem uses the arithmetic of $\mathbb{R}$, and Fatou's lemma (see Section 5.7). The proof goes as follows:

- monotonicity of integral in $\mathcal{M}_{+}$provides $f_{n}, f \in \mathcal{L}^{1}$;
- the sequence $g_{n} \stackrel{\text { def. }}{=}\left|f_{n}-f\right|$ is shown to be in $\mathcal{L}^{1} \cap \mathcal{M}_{+}$with limit 0 ;
- the sequence $2 g-g_{n}$ is also shown to be in $\mathcal{L}^{1} \cap \mathcal{M}_{+}$, with limit inferior $2 g$;
- then, Fatou's lemma and linearity of the integral in $\mathcal{L}^{1}$ provide the inequality

$$
2 \int g d \mu \leqslant 2 \int g d \mu-\underset{n \rightarrow \infty}{\limsup } \int g_{n} d \mu ;
$$

thus, a nonnegativeness argument provides the nullity of the limit of the integral of $g_{n}$ 's, i.e. the convergence of the $f_{n}$ 's towards $f$ in $\mathcal{L}^{1}$;

- finally, identity (5.6) follows from nondecreasingness of the integral in $\mathcal{L}^{1}$, the squeeze theorem, and linearity of the limit.

Lebesgue's dominated convergence theorem is used in the present document to prove an extended variant with lighter assumptions (see Section 5.3 and Theorem 1349).

Lebesgue's dominated convergence theorem admits another variant set in $\mathcal{L}^{p}$ (for $p \in[1, \infty$ ), see Theorem 1351).

For instance, it may be used to establish the Fourier inversion formula, or to study the Gibbs phenomenon.

### 5.5 Sketch of the proof of the Tonelli theorem



Figure 5.1: Path of the proof of Tonelli theorem for $i=1$.
The dashed arrows represent the main arguments for the proof.

## Tonelli theorem.

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $f \in \mathcal{M}_{+}\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $\psi$ be the permutation $\left(\left(x_{i}, x_{j}\right) \mapsto\left(x_{1}, x_{2}\right)\right)$.
For all $x_{i} \in X_{i}$, let $f_{x_{i}} \stackrel{\text { def. }}{=}\left(x_{j} \mapsto f \circ \psi\left(x_{i}, x_{j}\right)\right)$. Let $I_{f, i} \stackrel{\text { def. }}{=}\left(x_{i} \mapsto \int f_{x_{i}} d \mu_{j}\right)$. Then, for all $x_{i} \in X_{i}, f_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), I_{f, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$, and we have

$$
\begin{equation*}
\int f d\left(\mu_{1} \otimes \mu_{2}\right)=\int I_{f, i} d \mu_{i} \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) . \tag{5.7}
\end{equation*}
$$

See Theorem 1242. The proof of the Tonelli theorem uses the arithmetic of $\overline{\mathbb{R}}_{+}$, and it follows steps 1 to 3 of the Lebesgue scheme (see Section 4.4). The proof goes as follows (see Figure 5.1):

- the result is first established for indicator functions from properties of the measure (see Section 5.6);
- then, the result is extended to nonnegative simple functions by taking (positive) linear combination of indicator functions and applying (positive) linearity of the integral in $\mathcal{M}_{+}$;
- and finally, the result is extended to nonnegative measurable functions by taking the supremum of adapted sequences and applying the Beppo Levi (monotone convergence) theorem (see Section 5.8).

The Tonelli theorem is used in the present document to establish identities: for an integral over a subset in Lemma 1243, and for an integral of a tensor product function in Lemma 1244.

### 5.6 Sketch of the proof of existence and uniqueness of the tensor product measure



Figure 5.2: Description of the construction of the product measure induced by two $\sigma$-finite measures $\mu_{1}$ and $\mu_{2}$. The dashed arrows represent the main arguments for the proof.

## Existence and uniqueness of the tensor product measure.

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Then, there exists a unique tensor product measure on $(X, \Sigma)$.

This measure is denoted $\mu_{1} \otimes \mu_{2}$, and for all $i \in\{1,2\}$ with $j \stackrel{\text { def. }}{=} 3-i$, we have

$$
\begin{equation*}
\forall A \in \Sigma, \quad\left(\mu_{1} \otimes \mu_{2}\right)(A)=\int \mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) d \mu_{i} . \tag{5.8}
\end{equation*}
$$

It is called the (tensor) product measure of $\mu_{1}$ and $\mu_{2}$, and $\left(X, \Sigma, \mu_{1} \otimes \mu_{2}\right)$ is called (tensor) product measure space.

See Lemmas 1221, $1223,1227,1231$, and 1233. The proof of the existence and uniqueness of the tensor product measure suite of lemmas uses the arithmetic of $\overline{\mathbb{R}}_{+}$, the concepts of algebra of sets and monotone class (see Section 4.1), and the monotone class theorem (see Section 5.10). The proof goes as follows (see Figure 5.2):

- the tensor product algebra $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$ is the $\sigma$-algebra generated by $\Sigma_{1} \overline{\times} \Sigma_{2} \stackrel{\text { def. }}{=}\left\{A_{1} \times A_{2} \mid A_{1} \in \Sigma_{1} \wedge A_{2} \in \Sigma_{2}\right\} ;$
- existence:
- first, the sections of measurable sets of $\Sigma$ are defined as

$$
s_{x_{1}}(A) \stackrel{\text { def. }}{=}\left\{x_{2} \in X_{2} \mid\left(x_{1}, x_{2}\right) \in A\right\}, \text { see Figure } 5.3 ;
$$

- then, the function $\left(x_{1} \longmapsto \mu_{2}\left(s_{x_{1}}(A)\right)\right)$ is proved to be measurable over $\left(X_{1}, \Sigma_{1}\right)$, in two stages:

1. if $\mu_{2}$ is finite, the set $\mathcal{S}:=\left\{A \in \Sigma \mid\left(x_{1} \longmapsto \mu_{2}\left(s_{x_{1}}(A)\right)\right) \in \mathcal{M}_{+}\left(X_{1}, \Sigma_{1}\right)\right\}$ is proved to be an algebra of set, and a monotone class (using continuity from below and from above of the measure); hence, thanks to the monotone class theorem, $\mathcal{S}=\Sigma$;
2. when $\mu_{2}$ is $\sigma$-finite, the previous point is used with a restricted finite measure, and the result is extended by taking the supremum in $\mathcal{M}_{+}\left(X_{1}, \Sigma_{1}\right)$;

- then, the tensor measure candidate can be defined as $\mu(A) \stackrel{\text { def. }}{=} \int_{X_{1}} \mu_{2}\left(s_{x_{1}}(A)\right) d \mu_{1}$;
- it is proved that $\mu$ is indeed a product measure (i.e. such that for all $A_{1} \in \Sigma_{1}$ and $\left.A_{2} \in \Sigma_{2}, \mu\left(A_{1} \times A_{2}\right)=\mu_{1}\left(A_{1}\right) \mu_{2}\left(A_{2}\right)\right) ;$
- uniqueness: the proof is done in two stages, that are similar to the ones for the existence.


Figure 5.3: Examples of two sections of a set $A \in \Sigma_{1} \otimes \Sigma_{2}$.
The Existence and uniqueness of the tensor product measure suite of lemmas is used in the present document to prove a property of measurable sections in Lemma 1234, the uniqueness of the Lebesgue measure on $\mathbb{R}^{2}$ in Lemma 1235, and the Tonelli theorem (see Section 5.5 and Theorem 1242).

### 5.7 Sketch of the proof of Fatou's lemma

## Fatou's lemma.

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$. Then, $\liminf _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \liminf _{n \rightarrow \infty} f_{n} d \mu \leqslant \liminf _{n \rightarrow \infty} \int f_{n} d \mu \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) . \tag{5.9}
\end{equation*}
$$

See Theorem 1208. The proof of Fatou's lemma uses the arithmetic of $\overline{\mathbb{R}}_{+}$, and the Beppo Levi (monotone convergence) theorem (see Section 5.8). The proof goes as follows:

- the sequence $\left(\inf _{p \in \mathbb{N}} f_{n+p}\right)_{n \in \mathbb{N}}$ is first shown to be pointwise nondecreasing in $\mathcal{M}_{+}$;
- then, the Beppo Levi (monotone convergence) theorem provides the equality

$$
\int \liminf _{n \in \mathbb{N}} f_{n} d \mu=\lim _{n \in \mathbb{N}} \int \inf _{p \in \mathbb{N}} f_{n+p} d \mu ;
$$

- and monotonicity of integral, and infimum and limit inferior definitions provide the inequality

$$
\lim _{n \in \mathbb{N}} \int \inf _{p \in \mathbb{N}} f_{n+p} d \mu \leqslant \liminf _{n \in \mathbb{N}} \int f_{n} d \mu .
$$

Fatou's lemma is used in the present document to prove properties of the integral in $\mathcal{M}_{+}$: identity for pointwise convergent sequences in Lemma 1209, Fatou's lemma variant set in $\mathcal{L}^{1}$
(see Theorem 1345), and Lebesgue's dominated convergence theorem (see Section 5.4 and Theorem 1347).

The Beppo Levi (monotone convergence) theorem and Fatou's lemma can be established independently of one another, or each one can be obtained as a consequence of the other. Of course, those independent proofs do share a common ingredient to somehow obtain an upper bound for the integral of the limit by the limit of the integrals. We chose to establish the Beppo Levi (monotone convergence) theorem first.

### 5.8 Sketch of the proof of the Beppo Levi (monotone convergence) theorem

Beppo Levi (monotone convergence) theorem.
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$.
Assume that the sequence is pointwise nondecreasing. Then, $\lim _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \lim _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) \tag{5.10}
\end{equation*}
$$

See Theorem 1180. The proof of the Beppo Levi (monotone convergence) theorem uses the arithmetic of $\overline{\mathbb{R}}_{+}$, and it follows step 3 of the Lebesgue scheme by taking the "supremum" of quantities involving simple functions (see Section 4.4). The proof goes as follows:

- let $f \stackrel{\text { def. }}{=} \lim _{n \rightarrow \infty} f_{n}$; monotonicity and completeness arguments provide that $f$ is also $\sup _{n \rightarrow \infty} f_{n}$, that it belongs to $\mathcal{M}_{+}$, and the "easy" inequality

$$
\lim _{n \rightarrow \infty} \int f_{n} d \mu \leqslant \int f d \mu
$$

- let $\varphi \in \mathcal{S F} \mathcal{F}_{+}$such that $\varphi \leqslant f$; using the nondecreasing sequence of measurable subsets $\left(\left\{a \varphi \leqslant f_{n}\right\}\right)_{n \in \mathbb{N}}$ for $a \in(0,1)$, continuity from below of $\mu$, linearity, monotonicity and continuity arguments, allows to show the inequality

$$
\int \varphi d \mu \leqslant \lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

- then, taking the supremum over $\varphi$ 's provides the other inequality

$$
\int f d \mu \leqslant \lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

The Beppo Levi (monotone convergence) theorem is used in the present document to prove properties of the integral in $\mathcal{M}_{+}$(possibly through Lemma 1182): homogeneity at $\infty$ in Lemma 1181, additivity in Lemma 1183, $\sigma$-additivity in Lemma 1185, identity for the integral over a subset in Lemma 1199, Fatou's lemma (see Section 5.7 and Theorem 1208), identities for the integral with the trivial and counting measures in Lemmas 1210 and 1211, the Tonelli theorem (see Section 5.5 and Theorem 1242), and the Beppo Levi (monotone convergence) theorem variant set in $\mathcal{L}^{1}$ (see Theorem 1343).

The Beppo Levi (monotone convergence) theorem admits another variant set in $\mathcal{L}^{p}$ (for $p$ in $[1, \infty)$, see Theorem 1352).

### 5.9 Sketch of the proof of Carathéodory's extension theorem

## Carathéodory's extension theorem.

Let $X$ be a set. Let $\mathcal{R}$ be a ring of subsets on $X$ (i.e. closed under complement and union). Let $\mu$ be a premeasure defined on $\mathcal{R}$ (i.e. null on the empty set, and $\sigma$-additive). Then, $\mu$ can be extended into a complete measure on a $\sigma$-algebra containing $\mathcal{R}$ (i.e. for which all negligible subsets are measurable, and of measure 0 ). Moreover, this extension is unique when $\mu$ is $\sigma$-finite.

See Theorem 1090 (in the specific case of $X \stackrel{\text { def. }}{=} \mathbb{R}$ and $\mathcal{R}$ is the ring generated by the intervals). The proof of Carathéodory's extension theorem uses the arithmetic of $\overline{\mathbb{R}}_{+}$, the concepts of $\pi$ systems and $\lambda$-systems (see Section 4.1), and the Dynkin $\pi-\lambda$ theorem (see Section 5.10); the existence part may also be known as the Carathéodory extension scheme (see Section 4.3). The proof goes as follows:

- the function $\mu^{\star}$ defined on $\mathcal{P}(X)$ by (with the convention inf $\varnothing=\infty$ )

$$
\forall A \subset X, \quad \mu^{\star}(A) \stackrel{\text { def. }}{=} \inf \left\{\sum_{n \in \mathbb{N}} \mu\left(A_{n}\right) \mid\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{R} \wedge A \subset \bigcup_{n \in \mathbb{N}} A_{n}\right\}
$$

is first shown to be an outer measure on $X$ (i.e. null on the empty set, nonnegative, monotone, and $\sigma$-subadditive) whose restriction to $\mathcal{R}$ is the premeasure $\mu$;

- then, the set of subsets

$$
\Sigma \stackrel{\text { def. }}{=}\left\{E \subset X \mid \forall A \subset X, \mu^{\star}(A)=\mu^{\star}(A \cap E)+\mu^{\star}(A \backslash E)\right\}
$$

is shown to be a $\sigma$-algebra on $X$ containing the $\operatorname{ring} \mathcal{R}$, while the restriction $\mu^{\star}{ }_{\mid \Sigma}$ is shown to be a complete measure on $(X, \Sigma)$ (closedness of $\Sigma$ under countable union comes from additivity of $\mu^{\star}$ on $\Sigma$ );

- moreover, when $\mu$ is $\sigma$-finite (with $X=\biguplus_{n \in \mathbb{N}} X_{n}$ and $\mu\left(X_{n}\right)<\infty$ ), two Carathéodory extensions $\mu_{1}$ and $\mu_{2}$ are first restricted to $X_{n}$, then the set $\left\{\mu_{1}\left(A \cap X_{n}\right)=\mu_{2}\left(A \cap X_{n}\right)\right\}$ is shown to be equal to $\Sigma$, and finally uniqueness follows from $\sigma$-additivity of $\mu_{1}$ and $\mu_{2}$.

In the present document, Carathéodory's extension theorem is stated (and proved) in the specific case of the building of the Lebesgue measure on $\mathbb{R}$ (see Section 13.3).

### 5.10 Sketch of the proof of the Dynkin $\pi-\lambda$ theorem / monotone class theorem

Both theorems take advantage of the complementarity of being a $\lambda$-system and being a $\pi$-system (for the Dynkin $\pi-\lambda$ theorem), or being a monotone class and being an algebra of sets (for the monotone class theorem). Using the notation of Section 4.2 where $\mathcal{U}_{X}^{1}$ represents the generated $\lambda$-system $\Lambda_{X}$ (resp. the generated monotone class $\mathcal{C}_{X}$ ), and $\mathcal{U}_{X}^{2}$ represents the generated $\pi$-system $\Pi_{X}$ (resp. the generated algebra of sets $\mathcal{A}_{X}$ ), they both share the following abstract form.

## Dynkin $\pi-\lambda$ theorem / monotone class theorem.

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{U}_{X}^{2}(G)=G$. Then, $\mathcal{U}_{X}^{1}(G)=\Sigma_{X}(G)$.
See Theorems 732 and 736 . The proofs use the concepts of $\pi$-system and $\lambda$-system for the Dynkin $\pi-\lambda$ theorem, and of algebra of sets and monotone class for the monotone class theorem (see Section 4.1). Their common proof goes as follows:

- first, prove that $\mathcal{U}_{X}^{2}\left(\mathcal{U}_{X}^{1}(G)\right)=\mathcal{U}_{X}^{1}(G)$ (i.e. the property of $G$ is transmitted to $\left.\mathcal{U}_{X}^{1}(G)\right)$;
- then, prove that $\mathcal{U}_{X}^{1}(G)=\Sigma_{X}(G)$.

In both cases, the most technical part is the second point. Note that we actually prove that for all $G^{\prime}, \mathcal{U}_{X}^{1}\left(G^{\prime}\right)=G^{\prime}$ and $\mathcal{U}_{X}^{2}\left(G^{\prime}\right)=G^{\prime}$ is equivalent to $\Sigma_{X}\left(G^{\prime}\right)=G^{\prime}$.

The Dynkin $\pi-\lambda$ theorem is used in the present document (through Lemma 734) to extend the equality of two measures on a generator $\pi$-system to the whole $\sigma$-algebra in Lemma 992 , which is used to establish uniqueness of the Lebesgue measure on $\mathbb{R}$ in Theorem 1090 (see also Section 5.9). The monotone class theorem is used in the present document (through Lemma 738) to prove measurability of the measure of sections (in the case of finite measure spaces) in Lemma 1221, and uniqueness of the tensor product measure (also in the finite case) in Lemma 1231. All these three proofs follow the Dynkin $\pi-\lambda$ theorem / monotone class theorem scheme (see Section 4.2).

## Part II

## Detailed proofs

## Chapter 6

## Introduction

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

| light gray is for remarks |
| :--- |
| light blue for lemmas, |
| light green for definitions |
| , |

Definitions and results have a number and a name. Inside the bodies of proofs, pertinent statements are referenced using both their number and name. When appropriate, some hints are given about the application, either to specify arguments, or to provide justification or consequences; they are underlined. Some useful definitions and results were already stated in [20], which was devoted to the detailed proof of the Lax-Milgram theorem. Those are numbered up to 206, and the statements in the present document are numbered starting from 207.

Furthermore, as in [20], the most basic results are supposed to be known and are not detailed further; they are displayed in bold dark red. These include:

- Logic: tautologies from propositional calculus.
- Set theory:
- definition and properties of inclusion, intersection, (disjoint) union, complement, set difference, Cartesian product, power set, cardinality, and indicator function (that takes values 0 and 1 ), such as De Morgan's laws, monotonicity of intersection and (partial) union, distributivity of intersection over (disjoint) union and set difference, distributivity of the Cartesian product over union, compatibility of intersection with Cartesian product, $\sigma$-additivity of the cardinality (with $\infty$ absorbing element for addition in $\overline{\mathbb{N}}$ );
- definition and properties of equivalence and order binary relations;
- definition and properties of function, composition of functions, inverse image (compatibility with set operations), injective and bijective functions, restriction and extension;
- countability of finite Cartesian products of countable sets $\left(\mathbb{N}^{2}, \mathbb{Q}, \mathbb{Q} \times \mathbb{Q}_{+}^{\star}\right)$.
- Algebraic structures: results from group theory.
- Topology: definition and properties of continuous functions.
- Real analysis:
- properties of the ordered and valued field $\mathbb{R}$ such as the Archimedean property, density of rational numbers, completeness, sum of the first terms of a geometric series;
- definition and properties of basic numeric analytic functions such as square root, power function, exponential function, natural logarithm function, and exponentiation;
- properties of limits in $\mathbb{R}$ (cluster point), compactness of closed and bounded intervals, compatibility of limit with arithmetic operations, the squeeze theorem.

This part is organized as follows. Chapter 7 contains some results from various fields of mathematics (set theory, order theory, general topology, algebraic structures, real numbers and extended real numbers including second-countability), that are needed in the proofs of measure and integration theories. We recall that the material stated in [20] may also be used.

Then, Chapter 8 is devoted to subset systems, from $\pi$-system to $\sigma$-algebra. Measurability and measurable space are presented in Chapter 9, and the specific case of sets of real numbers and of extended real numbers is treated in Chapter 10. Chapter 11 is devoted to numeric functions over subset systems. Chapter 12 is dedicated to measure and measure space, and Chapter 13 to the specific cases involving sets of numbers, including the construction of the Lebesgue measure on $\mathbb{R}$ through Carathéodory's extension theorem. Chapter 14 is devoted to simple functions, both in the scalar and vector cases. Then, the integral of nonnegative scalar functions is addressed in Chapter 15 with for instance the Beppo Levi (monotone convergence) theorem, Fatou's lemma, and the Tonelli theorem. Finally, Chapter 16 is dedicated to the integral of functions with arbitrary sign, with the seminormed vector spaces $\mathcal{L}^{p}$, Lebesgue's dominated convergence theorem and its variants, and the Riesz-Fischer theorem.

## Chapter 7

## Complements

## Contents

7.1 Complements on set theory ..... 44
7.2 Complements on order theory ..... 49
7.3 Complements on general topology ..... 51
7.3.1 Topological basis ..... 51
7.3.2 Second axiom of countability ..... 54
7.3.3 Complements on metric space ..... 55
7.4 Complements on algebraic structures ..... 57
7.4.1 Vector space ..... 57
7.4.2 Algebra over a field ..... 58
7.4.3 Seminormed vector space ..... 60
7.4.3.1 Banach space ..... 60
7.4.3.2 Semi-inner product vector space ..... 61
7.5 Complements on real numbers ..... 62
7.5.1 Real numbers ..... 62
7.5.2 Extended real numbers ..... 66
7.5.2.1 Nonnegative extended real numbers ..... 77
7.5.2.2 In the context of measure theory ..... 81
7.5.3 Second countability and real numbers ..... 88
7.5.4 Second countability and extended real numbers ..... 90
7.5.5 Infimum, supremum ..... 93
7.5.6 Limit inferior, limit superior ..... 99
7.5.7 Truncating a numeric function ..... 104

### 7.1 Complements on set theory

Remark 207. We recall that the definition of union provides the nullary union law $\bigcup_{\varnothing} \stackrel{\text { def. }}{=} \varnothing$. However, unless specified otherwise and provided a superset $X$ is not ambiguous, we choose not to use the nullary intersection convention (" $\bigcap_{\varnothing} \stackrel{\text { def. }}{=} X$ " is generally not used).

Remark 208. We recall that $\uplus$ denotes disjoint union, that $A \backslash B \stackrel{\text { def. }}{=} A \cap B^{c}$ denotes the set difference, and that when $B \subset A, A \backslash B$ is the local complement of $B$ (in $A$ ), see Chapter 2.

## Definition 209 (pseudopartition).

Let $X$ be a set. Let $I \subset \mathbb{N}$. Subsets $\left(X_{i}\right)_{i \in I}$ of $X$ are said to form a pseudopartition (of $X$ ) iff

$$
\begin{equation*}
\forall i, j \in I, i \neq j \Rightarrow X_{i} \cap X_{j}=\varnothing, \quad \text { and } \quad X=\biguplus_{i \in I} X_{i} . \tag{7.1}
\end{equation*}
$$

Remark 210. Note that, in contrast to proper partitions, parts of a pseudopartition may be empty.
Lemma 211 (compatibility of pseudopartition with intersection). Let $X$ be a set.
Let $I \subset \mathbb{N}$. Let $A,\left(X_{i}\right)_{i \in I} \subset X$. Assume that $\left(X_{i}\right)_{i \in I}$ form a pseudopartition of $X$.
Then, $\left(A \cap X_{i}\right)_{i \in I}$ form a pseudopartition of $A$, i.e. $A=\biguplus_{i \in I}\left(A \cap X_{i}\right)$.
Proof. Direct consequence of Definition 209 (pseudopartition), and distributivity of intersection over disjoint union.

Lemma 212 (countable union is monotone).
Let $X$ be a set. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I},\left(B_{i}\right)_{i \in I} \subset X$. Assume that for all $i \in I, \bigcup_{j \in[0 . . i]} A_{j} \subset \bigcup_{j \in[0 . . i]} B_{j}$. Then, we have $\bigcup_{i \in I} A_{i} \subset \bigcup_{i \in I} B_{i}$.

Proof. Let $x \in \bigcup_{i \in I} A_{i}$. Then, from the definition of (countable) union, there exists $i_{A, x} \in I$ such that $x \in A_{i_{A, x}} \subset \bigcup_{j \in\left[0 . . i_{A, x}\right]} A_{j}$. Hence, there exists $i_{B, x} \in\left[0 . . i_{A, x}\right]$ such that $x \in B_{i_{B, x}}$.

Therefore, we have $\bigcup_{i \in I} A_{i} \subset \bigcup_{i \in I} B_{i}$.

## Lemma 213 (equality of countable union).

Let $X$ be a set. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I},\left(B_{i}\right)_{i \in I} \subset X$. Assume that for all $i \in I, \bigcup_{j \in[0 . . i]} A_{j}=\bigcup_{j \in[0 . . i]} B_{j}$.
Then, we have $\bigcup_{i \in I} A_{i}=\bigcup_{i \in I} B_{i}$.
Proof. Direct consequence of Lemma 212 (countable union is monotone, applied twice, with $A_{i}$ 's and $B_{i}$ 's, then with $B_{i}$ 's and $A_{i}$ 's).

Remark 214. The next lemma builds a pairwise disjoint sequence from any nonempty sequence of subsets. It is achieved by adding the set differences of the subsets layer by layer.

When the input sequence is nondecreasing, layers can be seen as onion peels.

## Lemma 215 (disjoint union of union).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \subset X$. Let $B_{\varphi(0)} \stackrel{\text { def. }}{=} A_{\varphi(0)}$, and for all $n \in J$ such that $n+1 \in J$, let $B_{\varphi(n+1)} \stackrel{\text { def. }}{=} A_{\varphi(n+1)} \backslash \bigcup_{j \in[0 . . n]} B_{\varphi(j)}$. Then, $\left(B_{i}\right)_{i \in I}$ is pairwise disjoint, for all $i \in I, B_{i} \subset A_{i}$, and we have

$$
\begin{equation*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 . . n]} B_{\varphi(j)}, \tag{7.2}
\end{equation*}
$$

$$
\begin{equation*}
\bigcup_{i \in I} A_{i}=\biguplus_{i \in I} B_{i} \tag{7.3}
\end{equation*}
$$

Moreover, if $\left(A_{i}\right)_{i \in I}$ is nondecreasing, the $B_{\varphi(n+1)}$ is the local complement of $\biguplus_{j \in[0 . . n]} B_{\varphi(j)}$ in $A_{\varphi(n+1)}$.

Proof. Pairwise disjunction. Let $p \in J$. Assume that $p+1 \in J$. Then, from the definition of set difference, and De Morgan's laws, we have

$$
B_{\varphi(p+1)}=A_{\varphi(p+1)} \cap\left(\bigcup_{j \in[0 . . p]} B_{\varphi(j)}\right)^{c}=A_{\varphi(p+1)} \cap \bigcap_{j \in[0 . . p]} B_{\varphi(j)}^{c}
$$

Let $j \in[0 . . p]$. Then, from properties of set operations, $B_{\varphi(p+1)}$ and $B_{\varphi(j)}$ are disjoint.
Let $m, n \in J$. Assume that $m \neq n$, and let $j \stackrel{\text { def. }}{=} \min (m, n)$ and $p \stackrel{\text { def. }}{=} \max (m, n)-1$. Then, we have $p+1 \in J$ and $j \in[0 . . p]$. Hence, from commutativity of intersection, we have

$$
B_{\varphi(m)} \cap B_{\varphi(n)}=B_{\varphi(p+1)} \cap B_{\varphi(j)}=\varnothing
$$

Inclusion. Direct consequence of the definition of the $B_{i}$ 's, and the definition of set difference. (7.2). For all $n \in \mathbb{N}$, let

$$
P(n) \stackrel{\text { def. }}{=}\left(\bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 . . n]} B_{\varphi(j)}\right), \quad \text { and } \quad Q(n) \stackrel{\text { def. }}{=}(n \in J \Rightarrow P(n)) \text {. }
$$

Induction: $\boldsymbol{Q ( 0 )}$. Trivial.
Induction: $\boldsymbol{Q ( n )}$ implies $\boldsymbol{Q}(\boldsymbol{n}+\mathbf{1})$. Let $n \in \mathbb{N}$. Assume that $Q(n)$ and $n+1 \in J$ hold.
Then, $n \in J$ since $J$ is an interval starting at 0 . Thus, $P(n)$ holds. Let $B \stackrel{\text { def. }}{=} \biguplus_{j \in[0 . . n]} B_{\varphi(j)}$. Then, from $P(n)$, and properties of set operations (e.g., $\boldsymbol{A} \cup \boldsymbol{B}=\boldsymbol{A} \uplus(B \backslash \boldsymbol{A})$ ), we have

$$
\bigcup_{j \in[0 . . n+1]} A_{\varphi(j)}=B \cup A_{\varphi(n+1)}=B \uplus\left(A_{\varphi(n+1)} \backslash B\right)=B \uplus B_{\varphi(n+1)}=\biguplus_{j \in[0 . . n+1]} B_{\varphi(j)} .
$$

Thus, for all $n \in \mathbb{N}, Q(n)$ holds. Hence, for all $n \in J, P(n)$ holds.
(7.3). Direct consequence of the definition of (countable) union, the bijective property of $\varphi$, and Lemma 213 (equality of countable union, with $I \stackrel{\text { def. }}{=} J$ ).
With $\left(\boldsymbol{A}_{\boldsymbol{i}}\right)_{i \in I}$ nondecreasing. Let $n \in J$. Assume that $n+1 \in J$.
Then, from monotonicity of $\varphi$, we have $\biguplus_{j \in[0 . . n]} B_{\varphi(j)}=\bigcup_{j \in[0 . . n]} A_{\varphi(j)}=A_{\varphi(n)} \subset A_{\varphi(n+1)}$.
Hence, $B_{\varphi(n+1)}$ is the local complement of $\biguplus_{j \in[0 . . n]} B_{\varphi(j)}$ in $A_{\varphi(n+1)}$.
Therefore, all five properties hold.
Remark 216. The next lemma builds a nondecreasing sequence from any sequence of subsets. It is achieved by taking the partial unions.

## Lemma 217 (monotone union of union).

Let $X$ be a set. Let $I \subset \mathbb{N}$. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the (bijective) increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \subset X$. For all $n \in J$, let $B_{\varphi(n)} \stackrel{\text { def. }}{=} \bigcup_{j \in[0 . . n]} A_{\varphi(j)}$.
Then, $\left(B_{i}\right)_{i \in I}$ is nondecreasing, for all $i \in I, A_{i} \subset B_{i}$, and we have

$$
\begin{align*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\bigcup_{j \in[0 . . n]} B_{\varphi(j)}\left(=B_{\varphi(n)}\right)  \tag{7.4}\\
\bigcup_{i \in I} A_{i} & =\bigcup_{i \in I} B_{i} . \tag{7.5}
\end{align*}
$$

Moreover, if $\left(A_{i}\right)_{i \in I}$ is pairwise disjoint, then the $B_{i}$ 's are computed from the $A_{i}$ 's using disjoint union.

Proof. Nondecreasingness. Let $n \in J$. Assume that $n+1 \in J$. Then, from associativity of union, we have $B_{\varphi(n+1)}=B_{\varphi(n)} \cup A_{\varphi(n+1)}$. Hence, we have $B_{\varphi(n)} \subset B_{\varphi(n+1)}$.
Inclusion. Direct consequence of the definition of the $B_{i}$ 's, and the definition of union.
(7.4). For all $n \in J$, let $P(n) \stackrel{\text { def. }}{=}\left(\bigcup_{j \in[0 . . n]} B_{\varphi(j)}=B_{\varphi(n)}\right)$.

Induction: $\boldsymbol{P}(\mathbf{0})$. Trivial. Induction: $\boldsymbol{P}(\boldsymbol{n})$ and $\boldsymbol{n}+1 \in \boldsymbol{J}$ implies $\boldsymbol{P}(\boldsymbol{n}+1)$. Let $n \in J$. Assume that $P(n)$ holds and that $n+1 \in J$. Then, from associativity of union, and (7.4), we have $\bigcup_{j \in[0 . . n+1]} B_{\varphi(j)}=B_{\varphi(n)} \cup B_{\varphi(n+1)}=B_{\varphi(n+1)}$. Hence, for all $n \in J, P(n)$ holds.
(7.5). Direct consequence of the definition of (countable) union, the bijective property of $\varphi$,

With $\left(\boldsymbol{A}_{\boldsymbol{i}}\right)_{\boldsymbol{i} \in \boldsymbol{I}}$ pairwise disjoint. Let $n \in J$. Then, we have $B_{\varphi(n)}=\biguplus_{j \in[0 . . n]} A_{\varphi(j)}$.
Therefore, all five properties hold.
Remark 218. The next lemma builds an empty nonincreasing sequence from any nonempty sequence of subsets. It is achieved by taking local complements in the union.

Note that (7.7) cannot be compatible with the nullary intersection convention, hence the nonempty assumption on $I$. Moreover, note that in (7.8), the integer interval ( $n . . n_{I}$ ) is empty only when $I$ is finite and $n=n_{I}-1$, in which case we have $B_{n_{I}-1}=\varnothing$.

## Lemma 219 (empty monotone intersection of union).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \subset X$. Let $A \stackrel{\text { def. }}{=} \bigcup_{i \in I} A_{i}$. Let $B_{\varphi(0)} \stackrel{\text { def. }}{=} A \backslash A_{\varphi(0)}$, and for all $n \in J$ such that $n+1 \in J$, let $B_{\varphi(n+1)} \stackrel{\text { def. }}{=} B_{\varphi(n)} \backslash A_{\varphi(n+1)}$.
Then, $\left(B_{i}\right)_{i \in I}$ is nonincreasing, for all $i \in I, B_{i} \subset A$ and $A_{i} \cap B_{i}=\varnothing$, and we have

$$
\begin{align*}
\forall n \in J, \quad A \backslash \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\bigcap_{j \in[0 . . n]} B_{\varphi(j)}\left(=B_{\varphi(n)}\right),  \tag{7.6}\\
\varnothing & =\bigcap_{i \in I} B_{i} . \tag{7.7}
\end{align*}
$$

Moreover, if $\left(A_{i}\right)_{i \in I}$ is pairwise disjoint, then we have

$$
\begin{equation*}
\forall n \in J, \quad A=\biguplus_{j \in[0 . . n]} A_{\varphi(j)} \uplus B_{\varphi(n)} \quad\left(\text { i.e. } B_{\varphi(n)}=\biguplus_{j \in\left(n . . n_{I}\right)} A_{\varphi(j)}\right) . \tag{7.8}
\end{equation*}
$$

## Proof. (1). Nonincreasingness, inclusion, and disjunction.

Direct consequence of the definition of the $B_{i}$ 's, and the definition of local complement.
(7.6). Direct consequence of the definition of the $B_{i}$ 's, the identity $(A \backslash B) \backslash C=A \backslash(B \cup C)$, (1), and induction on $n \in J$.
(7.7). Direct consequence of the lower bound property of intersection $\left(\bigcap_{i \in I} B_{i} \subset B_{i}\right)$, (1) $\left(B_{i} \subset A, A_{i}^{c}\right)$, the greatest lower bound property of intersection $\left(\bigcap_{i \in I} B_{i} \subset \bigcap_{i \in I} A_{i}^{c}\right)$, and De Morgan's laws $\left(\bigcap_{i \in I} A_{i}^{c}=A^{c}\right)$.
(7.8). Direct consequence of (7.6), and the definition of local complement.

Therefore, all six properties hold.
Remark 220. The next lemma builds a nondecreasing sequence from any nonempty nonincreasing sequence of subsets. It is achieved by taking local complements.

Lemma 221 (monotone union of monotone intersection). Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of I. Let $\left(A_{i}\right)_{i \in I} \subset X$. Let $n_{0} \in J$. For all $n \in J_{n_{0}} \stackrel{\text { def. }}{=}\left[n_{0} . . n_{I}\right) \neq \varnothing$, let $B_{n} \stackrel{\text { def. }}{=} A_{\varphi\left(n_{0}\right)} \backslash A_{\varphi(n)}$. Then, we have

$$
\begin{equation*}
\bigcup_{n \in J_{n_{0}}} B_{n}=A_{\varphi\left(n_{0}\right)} \backslash \bigcap_{n \in J_{n_{0}}} A_{\varphi(n)} \tag{7.9}
\end{equation*}
$$

Moreover, if $\left(A_{i}\right)_{i \in I}$ is nonincreasing, then the $B_{n}$ 's are computed from the $A_{i}$ 's using local complement, and $\left(B_{n}\right)_{n \in J_{n_{0}}}$ is nondecreasing.

Proof. Direct consequence of left distributivity of set difference over intersection, monotonicity of $\varphi$, and monotonicity of set difference (right nonincreasingness).

Lemma 222 (technical inclusion for countable union).
Let $X$ be a set. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I} \subset X$. Let $\varphi: I \rightarrow I$. Then, we have $\bigcup_{i \in I} A_{\varphi(i)} \subset \bigcup_{i \in I} A_{i}$.
Proof. Let $x \in \bigcup_{i \in I} A_{\varphi(i)}$. then, from the definition of union, there exists $i \in I$ such that $x \in A_{\varphi(i)}$. Thus, there exists $j=\varphi(i) \in I$ such that $x \in A_{j}$. Hence, from the definition of union, we have $x \in \bigcup_{j \in I} A_{j}$.

Therefore, we have $\bigcup_{i \in I} A_{\varphi(i)} \subset \bigcup_{i \in I} A_{i}$.
Lemma 223 (order is meaningless in countable union). Let $X$ be a set. Let $I \subset \mathbb{N}$. $\operatorname{Let}\left(A_{i}\right)_{i \in I} \subset X$. Let $\varphi: I \rightarrow I$. Assume that $\varphi$ is bijective. Then, we have $\bigcup_{i \in I} A_{i}=\bigcup_{i \in I} A_{\varphi(i)}$.

Proof. Direct consequence of Lemma 222 (technical inclusion for countable union, with $\left(A_{i}\right)_{i \in I}$ and $\varphi$, then $\left(A_{\varphi(i)}\right)_{i \in I}$ and $\varphi^{-1}$ which satisfies $\left.\varphi \circ \varphi^{-1}=\operatorname{Id}_{I}\right)$.

Lemma 224 (definition of double countable union). Let $X$ be a set. Let $I, J, K \subset \mathbb{N}$. For all $(i, j) \in I \times J$, let $A_{i, j} \subset X$. Let $\varphi, \psi: K \rightarrow I \times J$. Assume that $\varphi$ and $\psi$ are bijective. Then, $\bigcup_{k \in K} A_{\varphi(k)}=\bigcup_{k \in K} A_{\psi(k)}$. This union is denoted $\bigcup_{(i, j) \in I \times J} A_{i, j}$.

Proof. Direct consequence of closedness of bijectivity by inverse and composition, and Lemma 223 (order is meaningless in countable union, with $I \stackrel{\text { def. }}{=} K,\left(A_{i}\right)_{i \in I} \stackrel{\text { def. }}{=}\left(A_{\varphi(k)}\right)_{k \in K}$ and $\left.\varphi \stackrel{\text { def. }}{=} \psi \circ \varphi^{-1}\right)$.

Lemma 225 (double countable union).
Let $X$ be a set. Let $I, J \subset \mathbb{N}$. For all $(i, j) \in I \times J$, let $A_{i, j} \subset X$. Then, we have $\bigcup_{i, j \in I \times J} A_{i, j}=\bigcup_{i \in I}\left(\bigcup_{j \in J} A_{i, j}\right)$.

Proof. From countability of $\mathbb{N}^{2}$, hence of $I \times J$, let $K \subset \mathbb{N}$ and $\varphi: K \rightarrow I \times J$ be a bijection. Then, from Lemma 224 (definition of double countable union), we have

$$
\bigcup_{i, j \in I \times J} A_{i, j}=\bigcup_{k \in K} A_{\varphi(k)}
$$

(and this does not depend on the choice for $\varphi$ ).
Let $x \in \bigcup_{k \in K} A_{\varphi(k)}$. Then, from the definition of union, there exists $k_{0} \in K$ such that $x \in A_{\varphi\left(k_{0}\right)}$. Let $\left(i_{0}, j_{0}\right) \stackrel{\text { def. }}{=} \varphi\left(k_{0}\right)$. Thus, we have $x \in A_{i_{0}, j_{0}}$. Hence, from the definition of union, we have $x \in \bigcup_{j \in J} A_{i_{0}, j} \subset \bigcup_{i \in I}\left(\bigcup_{j \in J} A_{i, j}\right)$.

Conversely, let $x \in \bigcup_{i \in I}\left(\bigcup_{j \in J} A_{i, j}\right)$. Then, from the definition of union, there exists $i_{0} \in I$ such that $x \in \bigcup_{j \in J} A_{i_{0}, j}$, and there exists $j_{0} \in J$ such that $x \in A_{i_{0}, j_{0}}$.

Let $k_{0} \stackrel{\text { def. }}{=} \varphi^{-1}\left(i_{0}, j_{0}\right)$. Thus, we have $x \in A_{\varphi\left(k_{0}\right)}$. Hence, from the definition of union, we have $x \in \bigcup_{k \in K} A_{\varphi(k)}$.

Therefore, we have $\bigcup_{(i, j) \in I \times J} A_{i, j}=\bigcup_{i \in I}\left(\bigcup_{j \in J} A_{i, j}\right)$.
Definition 226 (trace of subset system).
Let $X$ be a set. Let $Y \subset X$. Let $i$ be the canonical injection from $Y$ to $X$. Let $G \subset \mathcal{P}(X)$. The notation $G \bar{\cap} Y$ denotes the set
(7.10)

$$
G \bar{\cap} Y \stackrel{\text { def. }}{=} i^{-1}(G)=\{A \cap Y \mid A \in G\} .
$$

## Definition 227 (product of subset systems).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $X_{i}$ be a set, and $G_{i} \subset \mathcal{P}\left(X_{i}\right)$. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$. The notations $G_{1} \overline{\times} G_{2}$ and $\bar{\prod}_{i \in[1 . . m]} G_{i}$ respectively denote the sets

$$
\begin{align*}
G_{1} \overline{\times} G_{2} & \stackrel{\text { def. }}{=}\left\{A_{1} \times A_{2} \in \mathcal{P}\left(X_{1} \times X_{2}\right) \mid A_{1} \in G_{1} \wedge A_{2} \in G_{2}\right\},  \tag{7.11}\\
\bar{\prod}_{i \in[1 . . m]} G_{i} & \stackrel{\text { def. }}{=}\left\{\prod_{i \in[1 . . m]} A_{i} \in \mathcal{P}(X) \mid \forall i \in[1 . . m], A_{i} \in G_{i}\right\} . \tag{7.12}
\end{align*}
$$

When all the $X_{i}$ 's are identical to some set $Y$, and all the $G_{i}$ 's are identical to some $G \subset \mathcal{P}(Y)$, then the latter may be simply written $G^{\overline{\times} m}$; it is a subset of $\mathcal{P}\left(Y^{m}\right)$.

### 7.2 Complements on order theory

Remark 228. In the next definition, the strict inequality " $<$ " naturally means " $\leqslant$ and not equal" (equality is related to the set object).

We recall the notations $\pm \infty$ to represent the infinite extrema of a totally ordered set; they may belong to the set, or not. See the notations in Chapter 2.

Definition 229 (interval). Let $(X, \leqslant)$ be a totally ordered nonempty set. Let $a, b \in X$.
The subset $(a, b) \stackrel{\text { def. }}{=}\{x \in X \mid a<x<b\}$ is called the open proper interval from $a$ to $b$ (with excluded bounds $a$ and $b$ ). The set of open proper intervals for all $a, b \in X$ is denoted $\mathcal{I}_{X}^{o, p}$.

The subsets $\{x \in X \mid x<b\}$ and $\{x \in X \mid a<x\}$ are called open left ray and open right ray (with excluded bounds $a$ and $b$ ) (or half-lines); they are denoted $(a, \infty)]$ and $(-\infty, b)$. The set of open rays for all $a, b \in X$ is denoted $\mathcal{R}_{X}^{o}$.

The notation $(-\infty, \infty)$ may be used to represent the whole set $X$.
In all cases, square brackets "[,]" are used to specify that bounds are included, and square-and-round brackets " $[()$,$] " are used to avoid specifying inclusion or exclusion of the bounds.$

Let $a, b \in X \cup\{ \pm \infty\}$. The subset $[a, b)]$ is either a proper interval, a ray, or the whole set; it is called interval. The set of open intervals is denoted $\mathcal{I}_{X}^{o} \stackrel{\text { def. }}{=} \mathcal{I}_{X}^{o, p} \cup \mathcal{R}_{X}^{o} \cup\{X\}$.

Remark 230. Note that if $X$ contains at least two elements, $\mathcal{I}_{X}^{o, p}, \mathcal{R}_{X}^{o}$ and $\{X\}$ are pairwise disjoint. In particular, $(a, \infty)$ is an open ray when $\infty \notin X$, and an open proper interval when $\infty \in X$. More generally, open rays are never open proper intervals. For instance, when $\infty \in X$, $(a, \infty)$ is actually $(a, \infty]$ (an open subset), which is distinct from the open proper interval $(a, \infty)$.

## Lemma 231 (equivalent definition of interval).

Let $(X, \leqslant)$ be a totally ordered nonempty set. Let $A \subset X$.
Then, $A$ is an interval iff for all $a, b \in A,(a, b) \subset A$.

## Proof. "Left" implies "right".

Direct consequence of Definition 229 (interval), and transitivity of order.
"Right" implies "left". From the existence of the Dedekind-MacNeille completion of $\boldsymbol{X}$ (i.e. the smallest complete lattice that contains $X$, in which all subsets have both a supremum and an infimum), let $\bar{X}$ be the Dedekind-MacNeille completion of $X$. Let $\alpha \stackrel{\text { def. }}{=} \inf A$ and $\beta \stackrel{\text { def. }}{=} \sup A$ in $\bar{X}$. Let us show that $A=[\alpha, \beta)$, or equivalently that $(\alpha, \beta) \subset A$ (in $X$ ), and $A \subset[\alpha, \beta]$ (in $\bar{X}$ ).
Let $x \in X$. Assume that $\alpha<x<\beta$. Then, from the definition of infimum and supremum (greatest lower bound and least upper bound, contrapositive) (e.g. see Definition 9 (infimum), and Definition 2 (supremum) in the case of $X \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ ), there exists $a$ and $b$ in $A$ such that we have $a<x<b$. Thus, $x \in A$. Hence, we have $(\alpha, \beta) \subset A$.
Let $x \in A$. Then, from the definition of infimum and supremum (lower and upper bounds), we have $\alpha \leqslant x \leqslant \beta$. Hence, we have $A \subset[\alpha, \beta]$.

Therefore, we have the equivalence.

Lemma 232 (empty open interval).
Let $(X, \leqslant)$ be a totally ordered nonempty set. Assume that $X$ is dense-in-itself:

$$
\begin{equation*}
\forall x, y \in X, \quad x<y \quad \Longrightarrow \quad \exists z \in X, x<z<y \tag{7.13}
\end{equation*}
$$

Let $a, b \in X \cup\{ \pm \infty\}$. Then, the open interval $(a, b)$ is empty iff $b \leqslant a$.

Proof. Then, from Definition 229 (interval), property (7.13), transitivity of order (which provides the other implication), and the definition of strict inequality, we have

$$
\begin{aligned}
(a, b)=\varnothing & \Longleftrightarrow \forall x \in X, \neg(a<x<b) \\
& \Longleftrightarrow \neg(\exists x \in X, a<x<b) \quad \Longleftrightarrow \quad \neg(a<b) \quad \Longleftrightarrow \quad b \leqslant a .
\end{aligned}
$$

Remark 233. Of course, the previous statement is wrong in the presence of isolated points. For instance, $(0,1)$ is empty in the discrete sets $\mathbb{N}$ and $\mathbb{Z}$.

Remark 234. In the next lemma, the left and right square-and-round brackets must remain the same on the sides of each interval. For instance "( 1 " denotes either "[" or "(", but it remains identical in the three intervals of Equation (7.14).

## Lemma 235 (intervals are closed under finite intersection).

Let $(X, \leqslant)$ be a totally ordered nonempty set. Then, the intersection of two proper intervals is a proper interval, the intersection of two rays is either a ray (if they both point towards the same direction), or a proper interval, and the intersection of a proper interval and a ray is a proper interval. In particular, for all $a, b, c, d \in X \cup\{ \pm \infty\}$, we have (see remark above)

$$
\begin{equation*}
\left.\left.\left({ }_{1} a, b\right)\right)_{2} \cap\left({ }_{1} c, d\right)\right)_{2}=\left({ }_{1} \max (a, c), \min (b, d)\right)_{2} . \tag{7.14}
\end{equation*}
$$

Hence, the closure of $\mathcal{R}_{X}^{o} \cup\{X\}$ under finite intersection is $\mathcal{I}_{X}^{o}$. Moreover, $\mathcal{I}_{X}^{o, p}$ and $\mathcal{I}_{X}^{o}$ are closed under finite intersection.

Proof. Direct consequence of Definition 229 (interval), totally ordered set properties of $\boldsymbol{X}$, and induction on the number of operands of the finite intersection.

## Lemma 236 (empty intersection of open intervals).

Let $(X, \leqslant)$ be a totally ordered nonempty set. Assume that $X$ is dense-in-itself:

$$
\begin{equation*}
\forall x, y \in X, \quad x<y \quad \Longrightarrow \quad \exists z \in X, x<z<y \tag{7.15}
\end{equation*}
$$

Let $a, b, c, d \in X \cup\{ \pm \infty\}$. Then, $(a, b) \cap(c, d)$ is empty iff $b \leqslant a, d \leqslant c, d \leqslant a$ or $b \leqslant c$.
Proof. Direct consequence of Lemma 235 (intervals are closed under finite intersection), Lemma 232 (empty open interval), and totally ordered set properties of $\boldsymbol{X}$.

Remark 237. We recall that $\bar{\Pi}$ denotes a set of traces of subsets, see Definition 226.
Lemma 238 (trace interval on a subset). Let $(X, \leqslant)$ be a totally ordered nonempty set. Let $Y \subset X$. Assume that $Y$ is equipped with the induced total order. Then, traces on $Y$ of intervals of $X$ coincide with intervals of $Y$, and in particular we have

$$
\begin{equation*}
\mathcal{I}_{X}^{o, p} \bar{\cap} Y=\mathcal{I}_{Y}^{o, p}, \quad \mathcal{R}_{X}^{o} \bar{\cap} Y=\mathcal{R}_{Y}^{o}, \quad \text { and } \quad \mathcal{I}_{X}^{o} \bar{\cap} Y=\mathcal{I}_{Y}^{o} . \tag{7.16}
\end{equation*}
$$

Proof. Direct consequence of Definition 229 (interval), Definition 226 (trace of subset system), and the definition of induced order.

### 7.3 Complements on general topology

### 7.3.1 Topological basis

Remark 239. In [20], we have only covered topology for metric spaces in which a subset is open iff it contains a nonempty open ball centered in each of its points. In the present document, we deal with the general case of topological spaces for which the collection of open subsets is given, e.g. via a subbase, see Definition 242, or a topological basis, see Definition 246.

Definition 240 (topological space). Let $X$ be a set. A subset $\mathcal{T}$ of $\mathcal{P}(X)$ is called topology of $X$ iff $\varnothing, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under any union and finite intersection.

In this case, $(X, \mathcal{T})$ (or simply $X$ ) is called topological space, elements of $\mathcal{T}$ are called open subsets of $X$, and the complement of elements of $\mathcal{T}$ are called closed subsets of $X$.

## Lemma 241 (intersection of topologies).

Let $X$ and $I$ be sets. Let $\left(\mathcal{T}_{i}\right)_{i \in I}$ be topologies on $X$. Then, $\bigcap_{i \in I} \mathcal{T}_{i}$ is a topology on $X$.

Proof. Direct consequence of Definition 240 (topological space), and the definition and properties of union and intersection of subsets.

Definition 242 (generated topology).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. The topology on $X$ generated by $G$ is the intersection of all topologies on $X$ containing $G$; it is denoted $\mathcal{T}_{X}(G)$. The generator $G$ is also called subbase of the topology.

Lemma 243 (generated topology is minimum).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{T}_{X}(G)$ is the smallest topology on $X$ containing $G$.

Proof. Direct consequence of Definition 242 (generated topology), Lemma 241 (intersection of topologies), and properties of the intersection.

Lemma 244 (equivalent definition of generated topology). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Let $O \subset X$. Then, $O \in \mathcal{T}_{X}(G)$ iff $O$ is the union of finite intersections of elements of $G \cup\{X\}$.

Proof. Direct consequence of Definition 240 (topological space), associativity and commutativity of union, distributivity of union and intersection (both ways, thus the set of unions of finite intersections of elements of $G \cup\{X\}$ is a topology of $X$ containing $G$, and contained in all topologies of $\boldsymbol{X}$ containing $\boldsymbol{G}$ ), and Lemma 243 (generated topology is minimum).

Lemma 245 (topology generation is monotone).
Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Assume that $G_{1} \subset G_{2}$. Then, we have $\mathcal{T}_{X}\left(G_{1}\right) \subset \mathcal{T}_{X}\left(G_{2}\right)$.
Proof. Direct consequence of transitivity of inclusion, Lemma 243 (generated topology is minimum, with $G_{2}$, thus $G_{2} \subset \mathcal{T}_{X}\left(G_{2}\right)$, then with $G_{1}$ and $\mathcal{T}_{X}\left(G_{2}\right)$ being a topology containing $G_{1}$ ).

## Definition 246 (topological basis).

Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B}$ of $\mathcal{T}$ is called topological basis of $(X, \mathcal{T})$ iff

$$
\begin{equation*}
\forall O \in \mathcal{T}, \quad \exists J, \quad \exists\left(B_{j}\right)_{j \in J} \in \mathcal{B}, \quad O=\bigcup_{j \in J} B_{j} \tag{7.17}
\end{equation*}
$$

Lemma 247 (equivalent definition of topological basis).
Let $X$ and I be sets.
Let $\mathcal{B} \stackrel{\text { def. }}{=}\left\{B_{i} \mid i \in I\right\} \subset \mathcal{P}(X)$. Then, $\mathcal{B}$ is a topological basis of $\left(X, \mathcal{T}_{X}(\mathcal{B})\right)$ iff $\mathcal{B}$ covers $X$, and any intersection of two elements of $\mathcal{B}$ is the union of elements of $\mathcal{B}$ :

$$
\begin{align*}
& X=\bigcup_{i \in I} B_{i},  \tag{7.18}\\
& \forall B, B^{\prime} \in \mathcal{B}, \quad \exists J \subset I, \quad B \cap B^{\prime}=\bigcup_{j \in J} B_{j} . \tag{7.19}
\end{align*}
$$

If so, $\mathcal{T}_{X}(\mathcal{B})$ reduces to the set of unions of elements of $\mathcal{B}$.
Moreover, if $\mathcal{B} \cup\{\varnothing\}$ is closed under intersection, then (7.19) is satisfied.
Proof. "Left" implies "right". Direct consequence of Lemma 243 (generated topology is minimum, $\mathcal{T}_{X}(\mathcal{B})$ is a topology on $X$ containing $\mathcal{B}$ ), Definition 240 (topological space, $X \in \mathcal{T}_{X}(\mathcal{B})$ and closedness under finite intersection, thus $B \cap B^{\prime} \in \mathcal{T}_{X}(\mathcal{B})$ )), and Definition 246 (topological basis, with $O \stackrel{\text { def. }}{=} X$, then $\left.O \stackrel{\text { def. }}{=} B \cap B^{\prime}\right)$.
"Right" implies "left". Let $\mathcal{T} \stackrel{\text { def. }}{=}\left\{\bigcup_{j \in J} B_{j} \mid J \subset I\right\}$. Then, by construction, we have $\mathcal{B} \subset \mathcal{T}$ $($ with $\operatorname{card}(J)=1), \varnothing \in \mathcal{T}($ with $\operatorname{card}(J)=0), X \in \mathcal{T}$ (with $J \stackrel{\text { def. }}{=} I$ and (7.18)).

Let $K$ be a set. Let $\left(A_{k}\right)_{k \in K} \in \mathcal{T}$. Then, by construction, for each $k \in K$, there exists $J_{k} \subset I$ such that $A_{k}=\bigcup_{j_{k} \in J_{k}} B_{j_{k}}$. Let $L \stackrel{\text { def. }}{=} \bigcup_{k \in K} J_{k}$. Then, from the definition of union, we have $L \subset I$, and $\bigcup_{k \in K} A_{k}=\bigcup_{l \in L} B_{l} \in \mathcal{T}$, i.e. $\mathcal{T}$ is closed under any union.

Hence, from distributivity of intersection over union, (7.19), associativity of union (thus $\mathcal{T}$ is closed under intersection), induction on the number of operands (thus $\mathcal{T}$ is closed under finite intersection), Definition 240 (topological space, $\mathcal{T}$ is a topology on $X$ ), Lemma 243 (generated topology is minimum, $\mathcal{T}_{X}(\mathcal{B}) \subset \mathcal{T}$ ), Lemma 244 (equivalent definition of generated topology, $\mathcal{T} \subset \mathcal{T}_{X}(\mathcal{B})$ ), and Definition 246 (topological basis), $\mathcal{B}$ is a topological basis of $(X, \mathcal{T})$ with $\mathcal{T}=\overline{\mathcal{T}_{X}(\mathcal{B})}$.
Therefore, we have the equivalence, and the characterization of $\mathcal{T}_{X}(\mathcal{B})$.

## Additional property.

Trivial (with $\operatorname{card}(J)=1$ when $B \cap B^{\prime} \in \mathcal{B}$, and $\operatorname{card}(J)=0$ when $B \cap B^{\prime}=\varnothing$ ).

## Lemma 248 (augmented topological basis).

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Let $\mathcal{B}^{\prime} \subset \mathcal{P}(X)$ such that $\mathcal{B} \subset \mathcal{B}^{\prime} \subset \mathcal{T}$.
Then, $\mathcal{B}^{\prime}$ is a topological basis of $(X, \mathcal{T})$.

Proof. Direct consequence of Definition 246 (topological basis, with $\mathcal{B}^{\prime} \stackrel{\text { def. }}{=}\left\{B_{i^{\prime}} \mid i^{\prime} \in I^{\prime}\right\}$ such that $\left.I \subset I^{\prime}\right)$, and transitivity of inclusion $\left(J \subset I \Rightarrow J \subset I^{\prime}\right)$.

Lemma 249 (topological basis is generator). Let $(X, \mathcal{T})$ be a topological space.
Let $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Then, we have $\mathcal{T}=\mathcal{T}_{X}(\mathcal{B})$.
Proof. Direct consequence of Definition 246 (topological basis), and Lemma 244 (equivalent definition of generated topology).

Lemma 250 (topological subbasis). Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Let $\mathcal{B}^{\prime} \subset \mathcal{P}(X)$. Assume that elements of $\mathcal{B}$ are unions of elements of $\mathcal{B}^{\prime}$,

$$
\begin{equation*}
\forall B \in \mathcal{B}, \quad \exists J, \exists\left(B_{j}^{\prime}\right)_{j \in J} \in \mathcal{B}^{\prime}, \quad B=\bigcup_{j \in J} B_{j}^{\prime} . \tag{7.20}
\end{equation*}
$$

Then, $\mathcal{B}^{\prime}$ is a topological basis of $(X, \mathcal{T})$, and $\mathcal{T}=\mathcal{T}_{X}(\mathcal{B})=\mathcal{T}_{X}\left(\mathcal{B}^{\prime}\right)$.

Proof. Let $O \in \mathcal{T}$. Then, from Definition 246 (topological basis), let $J$ be a set, and let $\left(B_{j}\right)_{j \in J} \in \mathcal{B}$ such that $O=\bigcup_{j \in J} B_{j}$. Then, for all $j \in J$, let $K_{j}$ be a set, and let $\left(B_{k_{j}}^{\prime}\right)_{k_{j} \in K_{j}} \in \mathcal{B}^{\prime}$ such that $B_{j}=\bigcup_{k_{j} \in K_{j}} B_{k_{j}}^{\prime}$. Let $K \stackrel{\text { def. }}{=} \bigcup_{j \in J} K_{j}$. Hence, $\left(B_{k}^{\prime}\right)_{k \in K} \in \mathcal{B}^{\prime}$, and from associativity of union, we have $O=\bigcup_{j \in J}\left(\bigcup_{k_{j} \in K_{j}} B_{k_{j}}^{\prime}\right)=\bigcup_{k \in K} B_{k}^{\prime}$.

Therefore, from Definition 246 (topological basis), and Lemma 249 (topological basis is generator $), \mathcal{B}^{\prime}$ is a topological basis of $(X, \mathcal{T})$, and $\mathcal{T}=\mathcal{T}_{X}(\mathcal{B})=\mathcal{T}_{X}\left(\mathcal{B}^{\prime}\right)$.

Remark 251. We recall that $\bar{\Pi}$ denotes a set of traces of subsets, see Definition 226.
Lemma 252 (trace topology on subset).
Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}$ be a topological basis of $(X, \mathcal{T})$. Let $Y \subset X$.
Then, $\mathcal{T}_{Y} \stackrel{\text { def. }}{=} \mathcal{T} \bar{\cap} Y$ is a topology of $Y$, and $\mathcal{B} \bar{\cap} Y$ is a topological basis of $\left(Y, \mathcal{T}_{Y}\right)$.
Moreover, if $Y \in \mathcal{T}$ is an open subset of $X$, then we have $\mathcal{T}_{Y} \subset \mathcal{T}$.
$\mathcal{T}_{Y}$ is called trace topology (induced by $\left.\mathcal{T}\right)$, and $\left(Y, \mathcal{T}_{Y}\right)$ is said topological subspace of $(X, \mathcal{T})$.
Proof. Direct consequence of Definition 240 (topological space), Definition 246 (topological basis), Definition 226 (trace of subset system), distributivity of intersection over union, and commutativity of intersection.

Remark 253. We recall that $\bar{\Pi}$ denotes a set of products of subsets, see Definition 227.
The next lemma defines a topology on a finite Cartesian product of topological spaces based on the products of open subsets in each space. In the finite case, the box topology and the product topology are the same. However, when the set of indices is infinite, the former becomes too fine, and the latter is usually preferred as it has more "desirable" properties. The difference is that the product topology is generated by products where only a finite number of operands are distinct from the corresponding full space; it is the coarsest topology making projection maps continuous.

## Lemma 254 (box topology on Cartesian product).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \mathcal{T}_{i}\right)$ be a topological space. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$. Let $\overline{\mathcal{T}} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[1 . . m]} \mathcal{T}_{i}$. Let $\mathcal{T} \stackrel{\text { def. }}{=} \mathcal{T}_{X}(\overline{\mathcal{T}})$. Then, $\overline{\mathcal{T}}$ is a topological basis of $(X, \mathcal{T})$. $\mathcal{T}$ is called the box topology on $X$ (induced by the $\mathcal{T}_{i}$ 's), or the product topology on $X$.

Moreover, for all $i \in[1 . . m]$, let $\mathcal{B}_{i}$ be a topological basis of $\left(X_{i}, \mathcal{T}_{i}\right)$. Let $\overline{\mathcal{B}} \stackrel{\text { def. }}{=} \bar{\Pi}_{i \in[1 . . m]} \mathcal{B}_{i}$. Then, $\overline{\mathcal{B}}$ is a topological basis of $(X, \mathcal{T})$.

Proof. From Definition 240 (topological space), for all $i \in[1 . . m]$, we have $X_{i} \in \mathcal{T}_{i}$ and $\mathcal{T}_{i}$ is closed under intersection. Then, from compatibility of intersection with Cartesian product $\left(\prod_{i \in I} B_{i} \cap \prod_{i \in[1 . . m]} B^{\prime}{ }_{i}=\prod_{i \in[1 . . m]} B_{i} \cap B^{\prime}{ }_{i}\right), \overline{\mathcal{T}}$ is closed under intersection. Hence, from Lemma 247 (equivalent definition of topological basis, with $X, \varnothing \in \overline{\mathcal{T}}$ ), $\overline{\mathcal{T}}$ is a topological basis of $(X, \mathcal{T})$.

Let $O \in \mathcal{T}$. Then, from Definition 227 (product of subset systems), and Definition 246 (topological basis, with $\overline{\mathcal{T}}$, then each $\mathcal{B}_{i}$ ), we have

$$
\begin{aligned}
\exists K, \exists\left(O^{k}\right)_{k \in K} \in \overline{\mathcal{T}}, & O=\bigcup_{k \in K} O^{k}, \\
\forall k \in K, \exists\left(O_{i}^{k}\right)_{i \in[1 . . m]} \in \mathcal{T}_{i}, & O^{k}=\prod_{i \in[1 . . m]} O_{i}^{k}, \\
\forall k \in K, \forall i \in[1 . . m], \exists J_{i}^{k}, \exists\left(B_{i}^{k, j_{i}}\right)_{j_{i} \in J_{i}^{k}} \in \mathcal{B}_{i}, & O_{i}^{k}=\bigcup_{j_{i} \in J_{i}^{k}} B_{i}^{k, j_{i}} .
\end{aligned}
$$

For all $k \in K$, let $J^{k} \stackrel{\text { def. }}{=} J_{1}^{k} \times \ldots \times J_{m}^{k}$. Let $L \stackrel{\text { def. }}{=}\left\{(k, j) \mid k \in K \wedge j \in J^{k}\right\}$. Then, for all $l \stackrel{\text { def. }}{=}\left(k,\left(j_{1}, \ldots, j_{m}\right)\right) \in L$, let $C^{l} \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} B_{i}^{k, j_{i}}$. Then, from distributivity of Cartesian product over union, Definition 227 (product of subset systems, $C^{l} \in \overline{\mathcal{B}}$ ), associativity of union, and Definition 246 (topological basis), we have

$$
o=\bigcup_{k \in K}\left(\prod_{i \in[1 . m]}\left(\bigcup_{j_{i} \in J_{V}^{k}} B_{i}^{k, j_{i}}\right)\right)=\bigcup_{k \in K} \bigcup_{j \in J^{k}}\left(\prod_{i \in[1 . . m]} B_{i}^{k, j_{i}}\right)=\bigcup_{l \in L} C^{l},
$$

and $\overline{\mathcal{B}}$ is a topological basis of $(X, \mathcal{T})$.
Remark 255. In the next definition, see Definition 229 for the definition of $\mathcal{R}_{X}^{o}$.
Note that the order topology is the standard topology on the totally ordered sets of numbers $\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$, see Lemmas 307 and 308 .

Definition 256 (order topology).
Let $(X, \leqslant)$ be a totally ordered set. The topology $\mathcal{T}_{X}\left(\mathcal{R}_{X}^{o}\right)$ is called order topology on $X$.
Remark 257. In the next lemma, see Definition 229 (interval) for the definition of $\mathcal{I}_{X}^{o}$.

## Lemma 258 (topological basis of order topology).

Let $(X, \leqslant)$ be a totally ordered set. Then, $\mathcal{I}_{X}^{o}$ is a topological basis for the order topology on $X$.
Proof. Direct consequence of Definition 256 (order topology), Lemma 244 (equivalent definition of generated topology), Lemma 235 (intervals are closed under finite intersection), and Definition 246 (topological basis).

Lemma 259 (trace of order topology). Let $(X, \leqslant)$ be a totally ordered set. Let $Y \subset X$. Assume that $X$ is equipped with the order topology $\mathcal{T}$, and that $Y$ is equipped with the induced total order. Then, the trace topology $\mathcal{T}_{Y}$ and the order topology $\mathcal{T}_{Y}\left(\mathcal{R}_{Y}^{o}\right)$ on $Y$ are the same, $\mathcal{I}_{Y}^{o}$ is a topological basis of both, and when $Y \in \mathcal{T}$, we have $\mathcal{T}_{Y} \subset \mathcal{T}$.

Proof. Direct consequence of Lemma 258 (topological basis of order topology, $\mathcal{I}_{X}^{o}$ is a topological basis of $(X, \mathcal{T})$ and $\mathcal{I}_{Y}^{o}$ is a topological basis of $\left(Y, \mathcal{T}_{Y}\left(\mathcal{R}_{Y}^{o}\right)\right)$ ), Lemma 252 (trace topology on subset, $\overline{\mathcal{I}}_{X}^{o} \bar{\cap} Y$ is a topological basis of $\left(Y, \mathcal{T}_{Y}\right)$ and $\left.Y \in \mathcal{T} \Rightarrow \mathcal{T}_{Y} \subset \mathcal{T}\right)$, and Lemma 238 (trace interval on a subset, $\mathcal{I}_{X}^{o} \bar{\cap} Y=\mathcal{I}_{Y}^{o}$, thus both topologies on $Y$ share the same topological basis, they are the same).

### 7.3.2 Second axiom of countability

Definition 260 (second-countability). A topological space (X, $\mathcal{T})$ is said second-countable, or completely separable, iff it admits a countable topological basis,

## Lemma 261 (compatibility of second-countability with Cartesian product).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \mathcal{T}_{i}\right)$ be a second-countable topological space, and let $\mathcal{B}_{i}$ be a countable topological basis of $\left(X_{i}, \mathcal{T}_{i}\right)$. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$. Assume that $X$ is equipped with the box topology $\mathcal{T} \stackrel{\text { def. }}{=} \mathcal{T}_{X}\left(\bar{\prod}_{i \in[1 . . m]} \mathcal{T}_{i}\right)$.
Then, $\bar{\prod}_{i \in[1 . . m]} \mathcal{B}_{i}$ is a countable topological basis of $(X, \mathcal{T})$. Hence, $(X, \mathcal{T})$ is second-countable.
Proof. Direct consequence of Lemma 254 (box topology on Cartesian product), Definition 227 (product of subset systems), compatibility of finite Cartesian product with countability, and Definition 260 (second-countability).

### 7.3.3 Complements on metric space

## Definition 262 (pseudometric).

Let $X$ be a nonempty set. A function $d: X \times X \rightarrow \mathbb{R}$ is called pseudometric over $X$ iff it is nonnegative, symmetric, it is zero on the diagonal, and it satisfies the triangle inequality:

$$
\begin{align*}
\forall x, y \in X, & d(x, y) \geqslant 0  \tag{7.21}\\
\forall x, y \in X, & d(y, x)=d(x, y) \\
\forall x \in X, & d(x, x)=0 \\
\forall x, y, z \in X, & d(x, z) \leqslant d(x, y)+d(y, z)
\end{align*}
$$

If so, $(X, d)$ (or simply $X$ ) is called pseudometric space.
Remark 263. A pseudometric becomes a metric when it is also definite. Hence, most results from [20] on metric spaces are still valid on pseudometric spaces.

Note that a pseudometric does not separate points, hence there is no uniqueness of the limit (contrary to metric space, see Lemma 29).

Lemma 264 (pseudo-point separation). Let $(X, d)$ be a pseudometric space. Let $x, y \in X$. Assume that for all $\varepsilon>0, d(x, y) \leqslant \varepsilon$. Then, $d(x, y)=0$.

Proof. Assume that $d(x, y)>0$. Let $\varepsilon=\frac{d(x, y)}{2}$. Then, from ordered field properties of $\mathbb{R}$ (with $d(x, y)>0$ ), we have $\varepsilon>0$ and $0<d(x, y) \leqslant \varepsilon=\frac{d(x, y)}{2}$, i.e. $0<1 \leqslant \frac{1}{2}$, which is wrong. Therefore, from Definition 262 ( $p s e u d o m e t r i c, ~ \underline{d}$ is nonnegative), we have $d(x, y)=0$.

Lemma 265 (equivalent definition of convergent sequence).
Let $(X, d)$ be a pseudometric space. Let $\left(x_{n}\right)_{n \in \mathbb{N}} l \in X$. Then, $x_{n} \underset{n \rightarrow \infty}{\longrightarrow} l$ iff

$$
\begin{equation*}
\forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geqslant N \quad \Longrightarrow \quad d\left(x_{n}, l\right) \leqslant \frac{1}{k+1} \tag{7.25}
\end{equation*}
$$

## Proof. "Left" implies "right".

Direct consequence of Definition 27 (convergent sequence, with $\varepsilon \stackrel{\text { def. }}{=} \frac{1}{k+1}$ ).
"Right" implies "left". Assume that property (7.25) holds. Let $\varepsilon>0$. Then, from the Archimedean property of $\mathbb{R}$, and ordered field properties of $\mathbb{R}$, let $k \in \mathbb{N}$ such that $k \geqslant \frac{1}{\varepsilon}-1$, i.e. $\frac{1}{k+1} \leqslant \varepsilon$. Thus, from assumption, there exists $N \in \mathbb{N}$ such that for all $n \in[N . . \infty)$, we have $d\left(x_{n}, l\right) \leqslant \frac{1}{k+1} \leqslant \varepsilon$. Hence, from Definition 27 (convergent sequence), the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is convergent with limit $l$.

Therefore, we have the equivalence.

## Lemma 266 (compatibility of subsequence with convergence).

Let $(X, d)$ be a pseudometric space. Let $\left(x_{n}\right)_{n \in \mathbb{N}} l \in X$. Let $\left(n_{k}\right)_{k \in \mathbb{N}} \in \mathbb{N}$.
Assume that $x_{n} \underset{n \rightarrow \infty}{\longrightarrow} l$, and $\left(n_{k}\right)_{k \in \mathbb{N}}$ is increasing. Then, we also have $x_{n_{k}} \underset{k \rightarrow \infty}{\longrightarrow} l$.
Proof. Let $\varepsilon>0$. Then, from Definition 27 (convergent sequence), let $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $d\left(x_{n}, x\right) \leqslant \varepsilon$. Thus, since $\left(n_{k}\right)_{k \in \mathbb{N}}$ is increasing, let $K \in \mathbb{N}$ such that for all $k \geqslant K$, we have $n_{k}>N$. Hence, we have $d\left(x_{n_{k}}, x\right) \leqslant \varepsilon$. Therefore, from Definition 27 (convergent sequence), $\left(x_{n_{k}}\right)_{k \in \mathbb{N}}$ is convergent to $x$.

## Definition 267 (cluster point).

Let $X$ be a topological space. Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$.
A cluster point of the sequence is the limit $x$ of any convergent subsequence of $\left(x_{n}\right)_{n \in \mathbb{N}}$.
When $(X, d)$ is a pseudometric space, this can take the following form:

$$
\forall \varepsilon>0, \forall N \in \mathbb{N}, \exists n \geqslant N, \quad d\left(x_{n}, x\right) \leqslant \varepsilon .
$$

Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)).
Let $(X, d)$ be a pseudometric space. Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$. Then, $\left(x_{n}\right)_{n \in \mathbb{N}}$ is a Cauchy sequence iff

$$
\begin{equation*}
\forall r \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n, k \in \mathbb{N}, \quad n \geqslant N \quad \Longrightarrow \quad d\left(x_{n+k}, x_{n}\right) \leqslant \frac{1}{r+1} . \tag{7.27}
\end{equation*}
$$

Proof. Same proof as Lemma 265 (equivalent definition of convergent sequence), but using Definition 35 (Cauchy sequence).

## Lemma 269 (convergent subsequence of Cauchy sequence).

Let $(X, d)$ be a pseudometric space. Let $\left(x_{n}\right)_{n \in \mathbb{N}} l \in X$. Let $\left(n_{k}\right)_{k \in \mathbb{N}} \in \mathbb{N}$. Assume that $\left(x_{n}\right)_{n \in \mathbb{N}}$ is a Cauchy sequence, $\left(n_{k}\right)_{k \in \mathbb{N}}$ is increasing, and $x_{n_{k}} \underset{k \rightarrow \infty}{\longrightarrow}$. Then, we also have $x_{n} \underset{n \rightarrow \infty}{\longrightarrow} x$.

Proof. Let $\varepsilon>0$. Then, from Definition 35 (Cauchy sequence), let $N \in \mathbb{N}$ such that for all $p, q \geqslant N$, we have $d\left(x_{p}, x_{q}\right) \leqslant \frac{\varepsilon}{2}$. Thus, since $\left(n_{k}\right)_{k \in \mathbb{N}}$ is increasing, let $K^{\prime} \in \mathbb{N}$ such that for all $k \geqslant K^{\prime}$, we have $n_{k}>N$, and from Definition 27 (convergent sequence), let $K^{\prime \prime} \in \mathbb{N}$ such that for all $k \geqslant K^{\prime \prime}$, we have $d\left(x_{n_{k}}, x\right) \leqslant \frac{\varepsilon}{2}$. Let $K \stackrel{\text { def. }}{=} \max \left(K^{\prime}, K^{\prime \prime}\right)$. Then, from Definition 262 ( $p$ seudometric, triangle inequality), we have for all $n \geqslant N$,

$$
d\left(x_{n}, x\right) \leqslant d\left(x_{n}, x_{n_{K}}\right)+d\left(x_{n_{K}}, x\right) \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .
$$

Therefore, from Definition 27 (convergent sequence), $\left(x_{n}\right)_{n \in \mathbb{N}}$ is convergent with limit $x$.

### 7.4 Complements on algebraic structures

### 7.4.1 Vector space

## Definition 270 (relation compatible with vector operations).

Let $\mathbb{K}$ be a field. Let $\left(E,+_{E},{ }_{E}\right)$ be a $\mathbb{K}$-vector space. An equivalence relation $\mathcal{R}$ on $E$ is said compatible with the vector operations iff

$$
\begin{align*}
\forall u, u^{\prime}, v, v^{\prime} \in E, & u \mathcal{R} u^{\prime} \wedge v \mathcal{R} v^{\prime} \quad \Longrightarrow \quad\left(u+_{E} v\right) \mathcal{R}\left(u^{\prime}+E v^{\prime}\right),  \tag{7.28}\\
\forall \lambda \in \mathbb{K}, \forall u, u^{\prime} \in E, & u \mathcal{R} u^{\prime} \Longrightarrow \quad\left(\lambda \cdot_{E} u\right) \mathcal{R}\left(\lambda \cdot_{E} u^{\prime}\right) . \tag{7.29}
\end{align*}
$$

## Lemma 271 (quotient vector operations).

Let $\mathbb{K}$ be a field. Let $\left(E,+_{E}, \cdot_{E}\right)$ be a $\mathbb{K}-$ vector space. Let $\mathcal{R}$ be an equivalence relation on $E$. Assume that $\mathcal{R}$ is compatible with the vector operations. Then, the mappings $[+]: E / \mathcal{R} \times E / \mathcal{R} \rightarrow E / \mathcal{R}$ and $[\cdot]: \mathbb{K} \times E / \mathcal{R} \rightarrow E / \mathcal{R}$ defined by

$$
\begin{align*}
\forall u, v \in E, & {[u][+][v] \stackrel{\text { def. }}{=}\left[u+_{E} v\right], }  \tag{7.30}\\
\forall \lambda \in \mathbb{K}, \forall u \in E, & \lambda[\cdot][u] \stackrel{\text { def. }}{=}\left[\lambda \cdot_{E} u\right] . \tag{7.31}
\end{align*}
$$

are well-defined. These mappings are called quotient vector operations induced on $E / \mathcal{R}$.
Proof. Let $u, v \in E$. Let $\lambda \in \mathbb{K}$. Let $u^{\prime}, v^{\prime} \in E$ such that $u \mathcal{R} u^{\prime}$ and $v \mathcal{R} v^{\prime}$, i.e. such that $\left[u^{\prime}\right]=[u]$ and $\left[v^{\prime}\right]=[v]$. Then, from Definition 270 (relation compatible with vector operations), we have $\left(u+_{E} v\right) \mathcal{R}\left(u^{\prime}+_{E} v^{\prime}\right)$ and $\left(\lambda \cdot{ }_{E} u\right) \mathcal{R}\left(\lambda \cdot E u^{\prime}\right)$, i.e. $\left[u^{\prime}+_{E} v^{\prime}\right]=\left[u+_{E} v\right]$ and $\left[\lambda \cdot{ }_{E} u^{\prime}\right]=\left[\lambda \cdot{ }_{E} u\right]$. Therefore, the quotient vector operations [ + ] and [•] defined by (7.30) and (7.31) do not depend on the choice of the representative of classes, i.e. they are well-defined.

## Lemma 272 (quotient vector space, equivalence relation).

Let $\mathbb{K}$ be a field. Let $\left(E,+_{E},{ }^{-}\right)$be $a \mathbb{K}$-vector space. Let $\mathcal{R}$ be an equivalence relation on $E$. Assume that $\mathcal{R}$ is compatible with the vector operations. Let $[+]$ and $[\cdot]$ be the quotient vector operations induced on the quotient set $E / \mathcal{R}$. Then, $(E / \mathcal{R},[+],[\cdot])$ is a $\mathbb{K}$-vector space.

Proof. From group theory, $(E / \mathcal{R},[+])$ is an abelian group with identity element $\left[0_{E}\right]$ Distributivity of the quotient scalar multiplication over quotient vector addition and field addition, compatibility of the quotient scalar multiplication with field multiplication, and 1 is the identity element for the quotient scalar multiplication are direct consequences of Definition 61 (vector space), and Lemma 271 (quotient vector operations). Therefore, from Definition 61 (vector space), $(E / \mathcal{R},[+],[\cdot])$ is a vector space.

Lemma 273 (quotient vector space). Let $\mathbb{K}$ be a field. Let $E$ be $a \mathbb{K}$-vector space. Let $F$ be a vector subspace of $E$. Let $\mathcal{R}$ be the relation defined on $E$ by for all $u, v \in E, u \mathcal{R} v \Leftrightarrow v-u \in F$. Then, $\mathcal{R}$ is an equivalence relation compatible with the vector operations of $E$.

The quotient set $E / \mathcal{R}$ equipped with the quotient vector operations is thus $a \mathbb{K}$-vector space called quotient vector space of $E$ by $F$; it is denoted $E / F$. For all $u \in E$, the class of $u$ is denoted $u+F$. The quotient vector operations are still denoted + and $\cdot$ (the latter may be omitted).

Proof. Let $u, u^{\prime}, v, v^{\prime} \in E$. Let $\lambda \in \mathbb{K}$. Assume that $u \mathcal{R} u^{\prime}$ and $v \mathcal{R} v^{\prime}$, i.e. $u^{\prime}-u, v^{\prime}-v \in F$. Then, from Definition 61 (vector space, vector addition is associative, and scalar multiplication is distributive over field addition), and Lemma 81 (closed under vector operations is subspace), we have $\left(u^{\prime}+v^{\prime}\right)-(u+v)=\left(u^{\prime}-u\right)+\left(v^{\prime}-v\right) \in F$ and $\left(\lambda u^{\prime}\right)-(\lambda u)=\lambda\left(u^{\prime}-u\right) \in F$, i.e. $(u+v) \mathcal{R}\left(u^{\prime}+v^{\prime}\right)$ and $(\lambda u) \mathcal{R}\left(\lambda u^{\prime}\right)$. Hence, from Definition 270 (relation compatible with vector operations), the equivalence relation $\mathcal{R}$ is compatible with the vector operations. Therefore, from Lemma 272 (quotient vector space, equivalence relation), the quotient set $E / \mathcal{R}$ equipped with the induced quotient vector operations is a vector space.

Lemma 274 (linear map on quotient vector space).
Let $\mathbb{K}$ be a field. Let $E, F$ be $\mathbb{K}$-vector spaces. Let $G$ be a vector subspace of $E$. Let $f$ be a linear map from $E$ to $F$. Assume that $G \subset \operatorname{ker}(f)$. Then, the function $[f] \stackrel{\text { def. }}{=}(u+G \mapsto f(u))$ is a linear map from $E / G$ to $F$.

Proof. Let $u \in E$. Let $u^{\prime} \in u+G$. Then, from hypotheses, we have $u^{\prime}-u \in G \subset \operatorname{ker}(f)$. Thus, from Definition 64 (linear map), and Definition 101 (kernel), we have $f\left(u^{\prime}\right)=f(u)$. Hence, the function $[f]$ does not depend on the choice of the representative of equivalence classes, it is well-defined.

Let $(u+G),(v+G) \in E / G$. Let $\lambda \in \mathbb{K}$. Then, from Lemma 273 (quotient vector space), Lemma 271 (quotient vector operations), and Definition 64 (linear map), we have

$$
\begin{gathered}
{[f](\lambda(u+G))=[f]((\lambda u)+G)=f(\lambda u)=\lambda f(u)=\lambda[f](u+G)} \\
{[f]((u+G)+(v+G))=[f]((u+v)+G)=f(u+v)=f(u)+f(v)=[f](u+G)+[f](v+G)}
\end{gathered}
$$

Therefore, from Definition 64 (linear map), $[f]$ is a linear map from $E / G$ to $F$.

### 7.4.2 Algebra over a field

Remark 275. The following algebraic structure "algebra over a field" is not to be confused with the concept of "algebra of sets" defined in Section 8.5.

Remark 276. Most results on algebras over a field are valid on algebras over a ring, for which the property of vector space over the field is replaced by that of module over the ring; especially when the ring is an integral domain, i.e. a nonzero commutative ring with no nonzero zero divisors, or a subring of a field.

Definition 277 (algebra over a field).
Let $\mathbb{K}$ be a field. A set $E$ equipped with three algebra operations (a vector addition + , a scalar multiplication $\cdot$, and a vector multiplication $*$ ), is called algebra (over $\mathbb{K}$ ), or $\mathbb{K}$-algebra, iff $(E,+, \cdot)$ is a $\mathbb{K}$-vector space, and vector multiplication is bilinear (or left and right distributive over vector addition, and compatible with scalars):

$$
\begin{align*}
\forall u, v, w \in E, & (u+v) * w=(u * w)+(v * w),  \tag{7.32}\\
\forall u, v, w \in E, & u *(v+w)=(u * v)+(u * w),  \tag{7.33}\\
\forall \lambda, \mu \in \mathbb{K}, \forall u, v \in E, & (\lambda \cdot u) *(\mu \cdot v)=(\lambda \mu) \cdot(u * v) . \tag{7.34}
\end{align*}
$$

Remark 278. The • and $*$ infix signs in the scalar and vector multiplications may be omitted.

## Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra). <br> Let $\mathbb{K}$ be a field. It is an algebra over itself.

Proof. Direct consequence of Definition 277 (algebra over a field), and field properties of $\mathbb{K}$.

## Definition 280 (inherited algebra operations).

Let $X$ be a nonempty set. Let $\mathbb{K}$ be a field. Let $\left(E,+_{E}, \cdot_{E}, *_{E}\right)$ be an algebra over $\mathbb{K}$. The algebra operations inherited on $E^{X}$ are the mappings $+_{E^{X}}$ and $\cdot_{E^{X}}$ of Definition 91 (inherited vector operations), and the mapping $*_{E^{X}}: E^{X} \times E^{X} \rightarrow E^{X}$ defined by

$$
\begin{equation*}
\forall f, g \in E^{X}, \forall x \in X, \quad\left(f *_{E^{x}} g\right)(x) \stackrel{\text { def. }}{=} f(x) *_{E} g(x) . \tag{7.35}
\end{equation*}
$$

Remark 281. Usually, inherited algebra operations are denoted the same way as the algebra operations of the target algebra.

## Lemma 282 (algebra of functions to algebra).

Let $\mathbb{K}$ be a field. Let $X$ be a nonempty set. Let $\left(E,+_{E}, \cdot_{E}, *_{E}\right)$ be a $\mathbb{K}$-algebra. Let $+_{E^{x}}, \cdot_{E^{X}}$ and $*_{E^{X}}$ be the algebra operations inherited on $E^{X}$. Then, $\left(E^{X},+{ }_{E^{X}}, \cdot{ }_{E^{x}}, *_{E^{X}}\right)$ is a $\mathbb{K}$-algebra.

Proof. From Lemma 93 (space of functions to space), $\left(E^{X},+_{E^{x}}, \cdot_{E^{x}}\right)$ is a $\mathbb{K}$-vector space. Then, bilinearity of the inherited vector multiplication is a direct consequence of Definition 277 (algebra over a field), and Definition 280 (inherited algebra operations). Therefore, from Definition 277 (algebra over a field), $E^{X}$ equipped with $+_{E^{X}}, \cdot_{E^{X}}$ and $*_{E^{X}}$ is a $\mathbb{K}$-algebra.

Lemma 283 ( $\mathbb{K}^{X}$ is algebra).
Let $\mathbb{K}$ be a field. Let $X$ be a nonempty set. Then, $\mathbb{K}^{X}$ is a $\mathbb{K}$-algebra.

Proof. Direct consequence of Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra), and Lemma 282 (algebra of functions to algebra).

Definition 284 (subalgebra).
Let $\mathbb{K}$ be a field. Let $(E,+, \cdot, *)$ be a $\mathbb{K}$-algebra. A subset $F$ of $E$ equipped with the restrictions to $F+\left.\right|_{F}, \cdot{ }_{\mid F}$ and $*_{\left.\right|_{F}}$ of the algebra operations is called ( $\mathbb{K}$-) subalgebra of $E \mathrm{iff}\left(F,+_{\left.\right|_{F}}, \cdot_{\left.\right|_{F}}, *_{\left.\right|_{F}}\right)$ is a $\mathbb{K}$-algebra.

Remark 285. Usually, restrictions $+_{\left.\right|_{F}}, \cdot_{F}$, and $*_{\left.\right|_{F}}$ are still denoted,$+ \cdot$ and $*$.

Lemma 286 (vector subspace and closed under multiplication is subalgebra).
Let $\mathbb{K}$ be a field. Let $(E,+, \cdot, *)$ be a $\mathbb{K}$-algebra. Let $F \subset E$. Then, $F$ is a $\mathbb{K}$-subalgebra of $E$ iff $F$ is a $\mathbb{K}$-vector subspace of $E$, and $F$ is closed under vector multiplication:
(7.36)
$\forall u, v \in F, \quad u * v \in F$.

Proof. "Left" implies "right". Assume first that $F$ is a $\mathbb{K}$-subalgebra of $E$. Then, from Definition 284 (subalgebra, $F$ is a $\mathbb{K}$-algebra), Definition 277 (algebra over a field), $\left(F,+\left.\right|_{\left.\right|_{F}},\left.\cdot\right|_{F}\right)$ is a $\mathbb{K}$-vector space, and $F$ is closed under the restriction to $F$ of the three operations. Thus, from Definition 77 (subspace), $F$ is a $\mathbb{K}$-vector subspace of $E$, and $F$ is closed under vector multiplication.
"Right" implies "left". Conversely, assume now that $F$ is a $\mathbb{K}$-vector subspace of $E$, and that it is closed under vector multiplication. Then, from Definition 77 (subspace), $\left(F,+_{\left.\right|_{F}}, \cdot_{\mid F}\right)$ is a $\mathbb{K}$-vector space. Moreover, from Lemma 81 (closed under vector operations is subspace), $F$ is also closed under vector addition and scalar multiplication. Thus, since $F$ is a subset of $E$, and $E$ is a $\mathbb{K}$-algebra, properties (7.32) to (7.34) are trivially satisfied over $F$ with the restrictions of the three operations. Hence, from Definition 277 (algebra over a field), and Definition 284 (subalgebra), $F$ is a $\mathbb{K}$-subalgebra of $E$.

Therefore, we have the equivalence.

## Lemma 287 (closed under algebra operations is subalgebra).

Let $\mathbb{K}$ be a field. Let $E$ be a $\mathbb{K}$-algebra. Let $F \subset E$. Then, $F$ is a $\mathbb{K}$-subalgebra of $E$ iff $0_{E} \in F$, and $F$ is closed under vector addition, scalar and vector multiplications:

$$
\begin{align*}
\forall u, v \in F, & u+v \in F,  \tag{7.37}\\
\forall \lambda \in \mathbb{K}, \forall u \in F, & \lambda u \in F,  \tag{7.38}\\
\forall u, v \in F, & u * v \in F . \tag{7.39}
\end{align*}
$$

Proof. Direct consequence of Lemma 286 (vector subspace and closed under multiplication is subalgebra), and Lemma 81 (closed under vector operations is subspace).

### 7.4.3 Seminormed vector space

## Definition 288 (seminorm).

Let $\mathbb{K}$ be a valued field. Let $E$ be a $\mathbb{K}$-vector space. A function $\|\cdot\|: E \rightarrow \mathbb{R}$ is called seminorm over $E$ iff it is absolutely homogeneous of degree 1 , and it satisfies the triangle inequality:

$$
\begin{align*}
\forall \lambda \in \mathbb{K}, \quad \forall u \in E, & \|\lambda u\|=|\lambda|\|u\|,  \tag{7.40}\\
\forall u, v \in E, & \|u+v\| \leqslant\|u\|+\|v\| . \tag{7.41}
\end{align*}
$$

If so, $(E,\|\cdot\|)$ (or simply $E$ ) is called seminormed ( $(\mathbb{K}$-) vector space.
Remark 289. Most results from [20] on normed vector spaces can be generalized to the case of seminormed vector space, sometimes with slight modifications of the statement. In particular, the associated distance becomes a pseudometric.

## Lemma 290 (seminorm is homogeneous).

Let $(E,\|\cdot\|)$ be a seminormed vector space. Then, we have $\|\left(\| 0_{E}\right)=0_{\mathbb{R}}$.
Proof. From Lemma 71 (zero times yields zero), Definition 288 (seminorm, absolute homogeneity), definiteness of absolute values on a field, and field properties of $\mathbb{R}$, we have

$$
\left\|0_{E}\right\|=\left\|0_{\mathfrak{K}} \cdot 0_{E}\right\|=\left|0_{\mathfrak{K}}\right|\left\|0_{E}\right\|=0_{\mathbb{R}}\left\|0_{E}\right\|=0_{\mathbb{R}} .
$$

Lemma 291 (seminorm is nonnegative). Let $\mathbb{K}$ be a field of characteristic distinct from 2. Let $(E,\|\cdot\|)$ be a seminormed $\mathbb{K}$-vector space. Then, $\|\cdot\|$ is nonnegative.

Proof. Let $x \in E$. Then, from Lemma 290 (seminorm is homogeneous), Definition 61 (vector space), Definition 288 (seminorm, triangle inequality, then absolute homogeneity with $\lambda \stackrel{\text { def. }}{=}-1$ ), we have $0_{\mathbb{R}}=\left\|0_{E}\right\|=\|x+(-x)\| \leqslant\|x\|+\|-x\|=2\|x\|$. Hence, from field properties of $\mathbb{K}$ (2 is invertible), we have $\|x\| \geqslant 0$.

Lemma 292 (seminormed vector space is pseudometric space).
Let $(E,\|\cdot\|)$ be a seminormed vector space. Let $d:=((x, y) \in E \mapsto\|x-y\|)$. Then, $(E, d)$ is a pseudometric space.

Proof. Direct consequence of Lemma 291 (seminorm is nonnegative, thus $d$ is nonnegative), Definition 61 (vector space, thus $x-x=0_{E}$ and $x-z=(x-y)+(y-\bar{z})$ ), Lemma 290 (seminorm is homogeneous, thus $d$ is zero on the diagonal), Definition 288 (seminorm, with $\lambda \stackrel{\text { def. }}{=}-1$, thus $d$ is symmetric and satisfies the triangle inequality), and Definition 262 (pseudometric).

Lemma 293 (definite seminorm is norm). Let $(E,\|\cdot\|)$ be a seminormed vector space. Then, $(E,\|\cdot\|)$ is a normed vector space iff $\|\cdot\|$ is definite iff for all $u \in E,\|u\|=0 \Leftrightarrow u=0$.

Proof. Direct consequence of Definition 288 (seminorm), Definition 106 (normed vector space), and Definition 105 (norm).

### 7.4.3.1 Banach space

Remark 294. The next definition depends on the definitions of complete metric space (see Definition 39), and of distance associated with a norm (see Definition 112).

Definition 295 (Banach space). A normed vector space $\left(E,\|\cdot\|_{E}\right)$ is called Banach space iff $\left(E, d_{E}\right)$ is a complete metric space, where $d_{E}$ is the distance induced by the norm $\|\cdot\|_{E}$.

### 7.4.3.2 Semi-inner product vector space

## Definition 296 (semi-inner product).

Let $E$ be a real vector space. A function $(\cdot, \cdot)_{E}: E \times E \rightarrow \mathbb{R}$ is called semi-inner product on $E$ iff it is a bilinear form on $E$ that is symmetric and nonnegative:
(7.43)

$$
\begin{align*}
\forall u, v \in E, \quad(u, v)_{E}=(v, u)_{E}  \tag{7.42}\\
\forall u \in E, \quad(u, u)_{E} \geqslant 0 .
\end{align*}
$$

If so, $\left(E,(\cdot, \cdot)_{E}\right)$ (or simply $E$ ) is called semi-inner product vector space.
Remark 297. Most results from [20] on inner product vector spaces are still valid on semi-inner product vector spaces.

## Lemma 298 (definite semi-inner product is inner product).

Let $\left(E,(\cdot, \cdot)_{E}\right)$ be a semi-inner product vector space. Then, $\left(E,(\cdot, \cdot)_{E}\right)$ is an inner product vector space iff $(\cdot, \cdot)_{E}$ is definite, i.e. for all $u \in E,(u, u)_{E}=0$ implies $u=0$.

Proof. Direct consequence of Definition 296 (semi-inner product), Definition 168 (inner product space), and Definition 166 (inner product).

### 7.5 Complements on real numbers

### 7.5.1 Real numbers

## Lemma 299 (finite cover of compact interval).

Let $a, b,\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}} \in \mathbb{R}$. Assume that $a \leqslant b$ and $[a, b] \subset \bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right)$. Then, there exists $q \in \mathbb{N}$ and $\left(i_{p}\right)_{p \in[0 . . q]} \in \mathbb{N}$ pairwise distinct such that $[a, b] \subset \bigcup_{p \in[0 . . q]}\left(a_{i_{p}}, b_{i_{p}}\right)$ with $a_{i_{0}}<a$, $b<b_{i_{q}}$, and for all $p \in[0 . . q-1]$, $a_{i_{p+1}}<b_{i_{p}}$.

Proof. From the definition of compactness, and compactness of $[a, b]$, there exists $n \in \mathbb{N}$ such that $[a, b] \subset \bigcup_{j \in[0 . . n]}\left(a_{j}, b_{j}\right)$.

Let $J \subset[0 . . n]$ and $x \in \mathbb{R}$. Assume that $[x, b] \subset \bigcup_{j \in J}\left(a_{j}, b_{j}\right)$ and $x \leqslant b$.
(1). Next index: $\exists i \stackrel{\text { def. }}{=} i_{J}(x) \in J, a_{i}<x<b_{i}$.

Direct consequence of the definition of union (with $x \in[x, b]$ ).
(2). Next cover: $\left[b_{i}, b\right] \subset \bigcup_{j \in J \backslash\{i\}}\left(a_{j}, b_{j}\right)$.

Case $\boldsymbol{b}<\boldsymbol{b}_{\boldsymbol{i}}$. Trivial. Case $\boldsymbol{b}_{\boldsymbol{i}} \leqslant \boldsymbol{b}$. Then, from (1), we have $b_{i} \in(x, b]$. Let $y \in\left[b_{i}, b\right] \subset[x, b]$. Then, from the definition of union, there exists $j \in J$ such that $a_{j}<y<b_{j}$. Hence, from transitivity of order in $\mathbb{R}$, we have $b_{i}<b_{j}$, i.e. $j \neq i$, and $\left[b_{i}, b\right] \subset \bigcup_{j \in J \backslash\{i\}}\left(a_{j}, b_{j}\right)$.

Let $i_{-1} \stackrel{\text { def. }}{=}-1, J_{-1} \stackrel{\text { def. }}{=}[0 . . n]$, and $b_{i_{-1}}=b_{-1} \stackrel{\text { def. }}{=} a$. Let $\left(i_{p}\right)_{p \in[0 . . q]}$ be the sequence of integers computed by the following algorithm:

```
\(p \stackrel{\text { def. }}{=}-1 ;\)
repeat
    \(i_{p+1} \stackrel{\text { def. }}{=} i_{J_{p}}\left(b_{i_{p}}\right) ;\)
    \(J_{p+1} \stackrel{\text { def. }}{=} J_{p} \backslash\left\{i_{p+1}\right\} ;\)
    \(p \stackrel{\text { def. }}{=} p+1 ;\)
until \(b<b_{i_{p}}\);
return \(\left(i_{0}, \ldots, i_{p}\right)\);
```

For all $p \in\{-1\} \cup \mathbb{N}$, let $I_{p} \stackrel{\text { def. }}{=}\left\{i_{0}, \ldots, i_{p}\right\}$ (with $I_{-1} \stackrel{\text { def. }}{=} \varnothing$ ), and $P(p)$ be the property:

$$
\begin{aligned}
& \operatorname{card}\left(I_{p}\right)=p+1 \wedge[0 . . n]=I_{p} \uplus J_{p} \wedge\left(\forall m \in[0 . . p], a_{i_{m}}<b_{i_{m-1}}\right) \wedge \\
& {\left[a, b_{i_{p}}\right) \subset \bigcup_{j \in I_{p}}\left(a_{j}, b_{j}\right) \wedge \quad\left[b_{i_{p}}, b\right] \subset \bigcup_{j \in J_{p}}\left(a_{j}, b_{j}\right) .}
\end{aligned}
$$

Let us show that there exists $q \in[0 . . n]$ such that $P(q) \wedge b<b_{i_{q}}$ holds.
(3). Initialization: $P(-1) \wedge b_{i_{-1}} \leqslant b$. Trivial.
(4). Iterations: $\forall p \in\{-1\} \cup \mathbb{N}, P(p) \wedge b_{i_{p}} \leqslant b$ implies $P(p+1)$.

Let $p \in\{-1\} \cup \mathbb{N}$. Assume that $P(p) \wedge b_{i_{p}} \leqslant b$ holds.
Let $J \stackrel{\text { def. }}{=} J_{p}$ and $x \stackrel{\text { def. }}{=} b_{i_{p}}$. Then, we have $[x, b] \subset \bigcup_{j \in J}\left(a_{j}, b_{i}\right)$ and $x \leqslant b$. Thus, from (1), there exists $i_{p+1}=i_{J}(x) \in J=J_{p}$ (i.e. $i_{p+1} \notin I_{p}$ ) such that $a_{i_{p+1}}<b_{i_{p}}<b_{i_{p+1}}$. Moreover, from (2),
we have $\left[b_{i_{p+1}}, b\right] \subset \bigcup_{j \in J_{p+1}}\left(a_{j}, b_{j}\right)$. Then, we have

$$
\begin{gathered}
\operatorname{card}\left(I_{p+1}\right)=\operatorname{card}\left(I_{p}\right)+1=p+2, \\
{[0 . . n]=I_{p} \uplus\left\{i_{p+1}\right\} \uplus\left(J_{p} \backslash\left\{i_{p+1}\right\}\right)=I_{p+1} \uplus J_{p+1} .} \\
a_{i_{p+1}}<b_{i_{p}}, \\
{\left[a, b_{i_{p+1}}\right)=\left[a, b_{i_{p}}\right) \uplus\left[b_{i_{p}}, b_{i_{p+1}}\right) \subset \bigcup_{j \in I_{p}}\left(a_{j}, b_{j}\right) \cup\left(a_{i_{p+1}}, b_{i_{p+1}}\right)=\bigcup_{j \in I_{p+1}}\left(a_{j}, b_{j}\right),} \\
{\left[b_{i_{p+1}}, b\right] \subset \bigcup_{j \in J_{p+1}}\left(a_{j}, b_{j}\right) .}
\end{gathered}
$$

Hence, $P(p+1)$ holds.
(5). Termination: $\exists \boldsymbol{q} \in[\mathbf{0} . . \boldsymbol{n}], \boldsymbol{P}(\boldsymbol{q}) \wedge \boldsymbol{b}<\boldsymbol{b}_{\boldsymbol{i}_{\boldsymbol{q}}}$. Assume that $P(p) \wedge b_{i_{p}} \leqslant b$ holds for all $p$ in $[0 . . n]$. Then, from (4), we have $P(n+1)$. Thus, from additivity of the cardinality, we have

$$
n+1=\operatorname{card}([0 . . n])=\operatorname{card}\left(I_{n+1}\right)+\operatorname{card}\left(J_{p+1}\right) \geqslant \operatorname{card}\left(I_{n+1}\right)=n+2 .
$$

Which is impossible. Hence, there exists $q \in[0 . . n]$ such that $P(q) \wedge b<b_{i_{q}}$ holds.
Moreover, we have $[a, b] \subset\left[a, b_{i_{q}}\right) \subset \bigcup_{j \in I_{q}}\left(a_{j}, b_{j}\right)$.
Therefore, there exists $q \in \mathbb{N}$ and $\left(i_{p}\right)_{p \in[0 . . q]} \in \mathbb{N}$ pairwise distinct such that $[a, b]$ is included in $\bigcup_{p \in[0 . . q]}\left(a_{i_{p}}, b_{i_{p}}\right)$ with $a_{i_{0}}<a, b<b_{i_{q}}$, and for all $p \in[0 . . q-1], a_{i_{p+1}}<b_{i_{p}}$.

## Lemma 300 (power function satisfies relaxed triangle inequality).

Let $p \in[1, \infty)$. Then, the power function of degree $p$ satisfies the relaxed triangle inequality:

$$
\begin{equation*}
\forall a, b \in \mathbb{R}, \quad|a+b|^{p} \leqslant 2^{p-1}\left(|a|^{p}+|b|^{p}\right) \tag{7.44}
\end{equation*}
$$

Moreover, there is equality when $a, b \geqslant 0$ for $p=1$, and when $a=b$ (for all $p \geqslant 1$ ).
Proof. Let $a, b \in \mathbb{R}$. From valued field properties of $\mathbb{R}$, and convexity of the power function ( $x \mapsto|x|^{p}$ ) for $p \geqslant 1$, we have

$$
|a+b|^{p}=\left|2 \frac{a+b}{2}\right|^{p}=2^{p}\left|\frac{a}{2}+\frac{b}{2}\right|^{p} \leqslant 2^{p}\left(\frac{|a|^{p}}{2}+\frac{|b|^{p}}{2}\right)=2^{p-1}\left(|a|^{p}+|b|^{p}\right)
$$

Moreover, assume first that $p=1$, and $a, b \geqslant 0$. Then, from valued field properties, we have $a+b \geqslant 0$, and $|a+b|=a+b=2^{0} \times(|a|+|b|)$. Assume now that $a=b$. Then, from valued field properties of $\mathbb{R}$, we have $|a+b|^{p}=|2 a|^{p}=2^{p}|a|^{p}=2^{p-1}\left(2|a|^{p}\right)=2^{p-1}\left(|a|^{p}+|b|^{p}\right)$.

## Remark 301.

The equality cases mentioned in Lemma 300 show that the constant $2^{p-1}$ is actually optimal.
Definition 302 (Hölder conjugates in $\mathbb{R}$ ).
Real numbers $p, q \in(1, \infty)$ are said Hölder conjugates in $\mathbb{R}$ iff $\frac{1}{p}+\frac{1}{q}=1$.

## Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ).

The real number 2 is Hölder conjugate in $\mathbb{R}$ with itself.
Proof. Direct consequence of Definition 302 (Hölder conjugates in $\mathbb{R}$ ), and since $\frac{1}{2}+\frac{1}{2}=1$.
Lemma 304 (Young's inequality for products in $\mathbb{R}$ ). $\quad$ Let $p, q \in(1, \infty)$. Assume that $p$ and $q$ are Hölder conjugates. Let $a, b \in \mathbb{R}_{+}$. Then, we have $a b \leqslant \frac{a^{p}}{p}+\frac{b^{q}}{q}$.

Proof. Case $\boldsymbol{a b}=\mathbf{0}$. Then, from nonnegativeness of exponentiation in $\mathbb{R}$, closedness of multiplicative inverse in $\mathbb{R}_{+}^{\star}$, and of the multiplication and addition in $\mathbb{R}_{+}$, The right-hand side is nonnegative. Hence, the inequality holds.

Case $\boldsymbol{a b}>0$. Then, from the zero-product property in $\mathbb{R}_{+}$(contrapositive), we have $a, b>0$. Thus, from algebraic properties of the natural logarithm function, Definition 302 (Hölder conjugates in $\mathbb{R}$ ), and concavity of the natural logarithm function (with $\frac{1}{p}+\frac{1}{q}=1$ ), we have

$$
\ln (a b)=\ln a+\ln b=\frac{1}{p} \ln \left(a^{p}\right)+\frac{1}{q} \ln \left(b^{q}\right) \leqslant \ln \left(\frac{a^{p}}{p}+\frac{b^{q}}{q}\right) .
$$

Hence, from monotonicity of the exponential function in $\mathbb{R}$, and since the exponential and logarithm functions in $\mathbb{R}$ are each other inverse, we have $a b \leqslant \frac{a^{p}}{p}+\frac{b^{q}}{q}$.

Therefore, the inequality always holds.
Lemma 305 (Young's inequality for products in $\mathbb{R}$, case $p=2$ ).
Let $a, b \in \mathbb{R}_{+}$. Let $\varepsilon>0$. Then, we have $a b \leqslant \frac{a^{2}}{2 \varepsilon}+\frac{\varepsilon b^{2}}{2}$.
Proof. Direct consequence of Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ), Lemma 304 (Young's inequality for products in $\mathbb{R}$, with $p=q \stackrel{\text { def. }}{=} 2$ and $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon} b \in \mathbb{R}_{+}$), and properties of square root and multiplicative inverse in $\mathbb{R}_{+}^{\star}$, and of multiplication in $\mathbb{R}_{+}$.

Remark 306. Note that a similar result also holds in the general case of Hölder conjugate numbers $p, q \in(1, \infty)$ : for all $a, b \in \mathbb{R}_{+}$, for all $\varepsilon>0$, we have $a b \leqslant \frac{a^{p}}{p \varepsilon^{\frac{2}{q}}}+\frac{\varepsilon^{\frac{2}{p}} b^{q}}{q}$.

Lemma 307 (topology on $\mathbb{R}$ ). The order topology on $\mathbb{R}$ is the same as the usual Euclidean topology in dimension 1. They admit the set $\mathcal{I}_{\mathbb{R}}^{o}$ of open intervals as topological basis.

Proof. Direct consequence of Lemma 258 (topological basis of order topology, the set $\mathcal{I}_{\mathbb{R}}^{o}$ of open intervals is a topological basis of the order topology), the definition of the topology induced by a metric (the set $\left\{(x-r, x+r) \mid x \in \mathbb{R} \wedge r \in \mathbb{R}_{+}^{\star}\right\}$ of open balls is a topological basis of the usual Euclidean topology on $\mathbb{R}$ ), and Lemma 249 (topological basis is generator, both topological bases are identical, thus both topologies are the same).

## Lemma 308 (trace topology on subset of $\mathbb{R}$ ).

Let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}$, or the order topology. Let $Y \subset \mathbb{R}$.
Then, $\mathcal{I}_{Y}^{o}$ is a topological basis of the trace topology $\mathcal{T}_{Y}$, and when $Y$ is open, we have $\mathcal{T}_{Y} \subset \mathcal{T}$.
Proof. Direct consequence of Lemma 307 (topology on $\mathbb{R}$ ), and Lemma 252 (trace topology on subset).

Remark 309. We recall that $G^{\overline{\times}}{ }^{m}$ denotes a set of products of subsets, see Definition 227.

## Lemma 310 (topology on $\mathbb{R}^{m}$ ).

Let $m \in[2 . . \infty)$. Let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}$, or the order topology.
Then, the box topology $\mathcal{T}_{\mathbb{R}^{m}}\left(\mathcal{T}^{\bar{x}}{ }^{m}\right)$ on $\mathbb{R}^{m}$ is the same as the usual Euclidean topology in dimension $m$. They admit the set $\left(\mathcal{I}_{\mathbb{R}}^{o}\right) \overline{\times} m$ of open $m$-boxes, or the set of open balls for the metric induced by any p-norm (for $p \in[1, \infty]$ ), as topological bases.

Proof. Let $\mathcal{T}^{\square}$ be the box topology on $\mathbb{R}^{m}$, and let $\mathcal{B}^{\square} \stackrel{\text { def. }}{=}\left(\mathcal{I}_{\mathbb{R}}^{o}\right)^{\overline{\times} m}$ be the set of $m$-boxes.
For all $p \in[1 . . \infty]$, let $\mathcal{T}^{p}$ be the topology on $\mathbb{R}^{m}$ induced by the $p$-norm, and let $\mathcal{B}^{p}$ be the set of open balls for the same norm.

From Lemma 307 (topology on $\mathbb{R}$ ), and Lemma 254 (box topology on Cartesian product), $\mathcal{B}^{\square}$ is a topological basis of $\mathcal{T}^{\square}$.

From the equivalence of $p$-norms in finite-dimensional real vector space, the usual Euclidean topology on $\mathbb{R}^{m}$ is the same as the topology induced by any $p$-norm, thus they share the same bases, e.g. with the maximum norm (thus $\mathcal{T}^{2}=\mathcal{T}^{\infty}$ ). Hence, from the definition of the topology induced by a norm, $\mathcal{B}^{\infty}$ is a topological basis of $\mathcal{T}^{2}$.

Let $B \in \mathcal{B}^{\square}$ be an open $m$-box. Case $\boldsymbol{B}=\varnothing$. Then, from the nullary union law, $B$ is a (nullary) union of elements of $\mathcal{B}^{\infty}$. Case $\boldsymbol{B} \neq \varnothing$. Then, from Lemma 254 (box topology on Cartesian product), and Definition 227 (product of subset systems), let $\left(a_{n}\right)_{n \in[1 . . m]}$ and $\left(b_{n}\right)_{n \in[1 . . m]}$ be in $\mathbb{R}^{m}$ such that, for all $n \in[1 . . m], a_{n}<b_{n}$, and $B=\prod_{n \in[1 . . m]}\left(a_{n}, b_{n}\right)$. Without any loss of generality, assume that the smaller side of the $m$-box corresponds to the first dimension, and let $r \stackrel{\text { def. }}{=} \frac{b_{1}-a_{1}}{2}$. Then, we have $r>0$, and for all $n \in[2 . . m], b_{n}-r-\left(a_{n}+r\right)=b_{n}-a_{n}-\left(b_{1}-a_{1}\right) \geqslant 0$ (i.e. $\left[a_{n}+r, b_{n}-r\right]$ is nonempty). For all $x \in \mathbb{R}^{m}$, let $B_{x} \stackrel{\text { def. }}{=} \prod_{n \in[1 . . m]}\left(x_{n}-r, x_{n}+r\right)$, and let

$$
J \stackrel{\text { def. }}{=}\left\{\left.\left(\frac{a_{1}+b_{1}}{2}, x_{2}, \ldots, x_{m}\right) \in \mathbb{R}^{m} \right\rvert\, \forall n \in[2 . . m], x_{n} \in\left[a_{n}+r, b_{n}-r\right]\right\} .
$$

Then, from the definition of the maximum norm, for all $x \in \mathbb{R}^{m}$, we have $B_{x} \in \mathcal{B}^{\infty}$, and $B=\bigcup_{j \in J} B_{j}$.

Therefore, from Lemma 250 (topological subbasis), we have $\mathcal{T}^{\square}=\mathcal{T}^{\infty}=\mathcal{T}^{2}$, and all topologies share their respective bases.

Lemma 311 (trace topology on subset of $\mathbb{R}^{m}$ ).
Let $m \in[2 . . \infty)$. Let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}^{m}$, or the box topology on $\mathbb{R}^{m}$ induced by the usual Euclidean topology on $\mathbb{R}$, or by the order topology. Let $Y \subset \mathbb{R}^{m}$. Then, the set $\left(\mathcal{I}_{\mathbb{R}}^{o}\right)^{\bar{x}} m \bar{\cap} Y$ of traces of open m-boxes on $Y$ is a topological basis of the trace topology $\mathcal{T}_{Y}$, and when $Y \in \mathcal{T}$, we have $\mathcal{T}_{Y} \subset \mathcal{T}$.

Proof. Direct consequence of Lemma 310 (topology on $\mathbb{R}^{m}$ ), Lemma 252 (trace topology on subset, with $\left.X \stackrel{\text { def. }}{=} \mathbb{R}^{m}\right)$,

### 7.5.2 Extended real numbers

Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
The set of extended real numbers is $\overline{\mathbb{R}} \stackrel{\text { def. }}{=} \mathbb{R} \uplus\{-\infty, \infty\}$, and the order in $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ with the following rule:

1. $\forall a \in \mathbb{R}, \quad-\infty<a<\infty$,
and, in this case, $a$ is said to be finite.

## Lemma 313 (order in $\overline{\mathbb{R}}$ is total). <br> $(\overline{\mathbb{R}}, \leqslant)$ is a totally ordered set.

Proof. Direct consequence of the definition of total order, Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and totality of order in $\mathbb{R}$.

Lemma 314 ( $\overline{\mathbb{R}}$ is complete). $\quad(\overline{\mathbb{R}}, \leqslant)$ is the Dedekind-MacNeille completion of $\mathbb{R}$ : every subset has both an infimum and a supremum.

Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), completeness of $\mathbb{R}$ (bounded subsets have infimum and/or supremum), and the definition of DedekindMacNeille completion (unbounded subsets have infimum $-\infty$ and/or supremum $\infty$ ).

Remark 315. The goal of this section is to clarify some properties of basic operations such as addition, multiplication, and exponentiation, when extended to $\overline{\mathbb{R}}$.

Remark 316. Note that, as with regular real numbers, all results on operations on extended real numbers can be lifted into similar results on functions taking their values in $\overline{\mathbb{R}}$.

## Definition 317 (addition in $\overline{\mathbb{R}}$ ).

Addition and subtraction in $\mathbb{R}$ are extended to $\overline{\mathbb{R}}$ with the following rules:

$$
\begin{cases}\text { 1. } & \forall a>-\infty, \quad a+\infty=\infty+a \stackrel{\text { def. }}{=} \infty, \\ \text { 2. } & \forall a<\infty, \quad a+(-\infty)=-\infty+a \stackrel{\text { def. }}{=}-\infty, \\ \text { 3. } & \infty+(-\infty) \text { and }-\infty+\infty \text { are undefined, } \\ \text { 4. } & -( \pm \infty)=\mp \infty, \\ 5 . & \forall a, b \in \overline{\mathbb{R}}, \quad a+(-b) \text { defined } \Longrightarrow a-b \stackrel{\text { def. }}{=} a+(-b) .\end{cases}
$$

## Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ).

Let $a \in \overline{\mathbb{R}}$. Then, we have $a+0=0+a=a$.
Proof. Direct consequence of abelian additive group properties of $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, new rules 1 , and 2 are compatible with the property).

Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined).
Let $a, b, c \in \overline{\mathbb{R}}$. Then, $a+(b+c)$ and $(a+b)+c$ are either equal or both undefined.
Proof. Direct consequence of associativity of addition in $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, new rules 1,2 , and 3 are compatible with associativity).

Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined).
Let $a, b \in \overline{\mathbb{R}}$. Then, $a+b$ and $b+a$ are either equal or both undefined.
Proof. Direct consequence of commutativity of addition in $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, new rules 1,2 , and 3 are compatible with commutativity).

Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined). Let $a, b, c \in \overline{\mathbb{R}}$. Assume that $a \leqslant b$, and that $a+c$ and $b+c$ are well-defined. Then, we have $a+c \leqslant b+c$.

Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and compatibility of addition with order in $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, new rules 1, 2, and 3 are compatible with order).

Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ ).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\left.\begin{array}{rl}
a+b=\infty & \Longleftrightarrow(a=\infty \wedge \quad b>-\infty) \\
a+b=-\infty & \Longleftrightarrow(a>-\infty \wedge b=\infty)  \tag{7.48}\\
(a=-\infty \wedge b<\infty) & \vee \\
(a<\infty \wedge & \wedge
\end{array}\right)
$$

Moreover, if we assume that the sum $a+b$ is well-defined, then we have

$$
\begin{equation*}
a+b= \pm \infty \quad \Longleftrightarrow \quad a= \pm \infty \quad \vee \quad b= \pm \infty . \tag{7.49}
\end{equation*}
$$

Proof. Direct consequence of Definition 317 (addition in $\overline{\mathbb{R}}$, rules 1, 2 and 3).
Lemma 323 (finite-sum property in $\overline{\mathbb{R}}$ ). Let $a, b \in \overline{\mathbb{R}}$. Then, we have $a+b \in \mathbb{R}$ iff $a, b \in \mathbb{R}$.
Proof. Direct consequence of Definition 317 (addition in $\overline{\mathbb{R}}$, new rules are not finite-sum rules).
Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution). Let $a, b \in \overline{\mathbb{R}}$. Then, we have $-(-a)=a$.
Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), involution of additive inverse in $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, rule 4).

Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have $a<b \Leftrightarrow-b<-a$, and $a \leqslant b \Leftrightarrow-b \leqslant-a$.
Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), monotonicity of additive inverse in $\mathbb{R}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, rule 4).

Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
Multiplication and division by nonzero in $\mathbb{R}$ are extended to $\overline{\mathbb{R}}$ with the following rules:

1. $\forall a>0, \quad a \times( \pm \infty)= \pm \infty \times a \stackrel{\text { def. }}{=} \pm \infty$,
2. $\forall a<0, \quad a \times( \pm \infty)= \pm \infty \times a \stackrel{\text { def. }}{=} \mp \infty$,
3. $0 \times( \pm \infty)$ and $\pm \infty \times 0$ are undefined,
4. $\frac{1}{0}$ is undefined, and $\frac{1}{ \pm \infty} \stackrel{\text { def. }}{=} 0$,
5. $\forall a, b \in \overline{\mathbb{R}}, \quad \frac{1}{b}$ and $a \times \frac{1}{b}$ defined $\quad \Longrightarrow \quad \frac{a}{b} \stackrel{\text { def. }}{=} a \times \frac{1}{b}$.

Remark 327. The 5th rule in the previous definition implies that $\frac{\infty}{ \pm \infty}, \frac{-\infty}{ \pm \infty}$, and $\frac{a}{0}$ (for all $a \in \overline{\mathbb{R}}$ ) are undefined. Note that rule 3 is modified in the context of measure theory in Definition 415.

## Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ).

Let $a \in \overline{\mathbb{R}}$. Then, we have $a \times 1=1 \times a=a$.

Proof. Direct consequence of field properties of $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$, new rule 1 is compatible with the property).

Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined).
Let $a, b, c \in \overline{\mathbb{R}}$. Then, $a \times(b \times c)$ and $(a \times b) \times c$ are either equal or both undefined.
Proof. Direct consequence of associativity of multiplication in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$, new rules 1,2 , and 3 are compatible with associativity).

Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined).
Let $a, b \in \overline{\mathbb{R}}$. Then, $a \times b$ and $b \times a$ are either equal or both undefined.
Proof. Direct consequence of commutativity of multiplication in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$, new rules 1,2 , and 3 are compatible with commutativity).

Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined).
Let $a, b, c \in \overline{\mathbb{R}}$. Then, $a \times(b+c)$ and $(a \times b)+(a \times c)$ are either equal or both undefined.
Proof. Direct consequence of left distributivity of multiplication over addition in $\mathbb{R}$, Definition 317 (addition in $\overline{\mathbb{R}}$ ), and Definition 326 (multiplication in $\overline{\mathbb{R}}$ ) (new rules are compatible with left distributivity of multiplication over addition).

Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined). Let $a, b, c \in \overline{\mathbb{R}}$. Then, $(a+b) \times c$ and $(a \times c)+(b \times c)$ are either equal or both undefined.

Proof. Direct consequence of Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined, used twice), and Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined).

Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined). Let $a, b, c \in \overline{\mathbb{R}}$. Assume that $a \leqslant b$ and $c \geqslant 0$, and that $a \times c$ and $b \times c$ are well-defined. Then, we have $a \times c \leqslant b \times c$.

Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), compatibility of multiplication with order in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$, new rules 1,2 , and 3 are compatible with order).

Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have
(7.51)

$$
a b=0 \quad \Longleftrightarrow \quad a b \text { is defined } \wedge \quad(a=0 \quad \vee \quad b=0)
$$

Proof. Direct consequence of the zero-product property in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$, rule 3).

Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
a b=1 \quad \Longleftrightarrow \quad\left(a=\frac{1}{b} \quad \vee \quad b=\frac{1}{a}\right) \quad \wedge \quad a, b \notin\{0, \pm \infty\} \tag{7.52}
\end{equation*}
$$

Proof. Direct consequence of the one-product property in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).

Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{align*}
& a b=\infty \quad \Longleftrightarrow \quad(a=\infty \wedge b>0) \vee(a=-\infty \wedge b<0) \vee  \tag{7.53}\\
& (a>0 \wedge b=\infty) \vee(a<0 \wedge b=-\infty), \\
& a b=-\infty \quad \Longleftrightarrow(a=-\infty \quad \wedge \quad b>0) \vee(a=\infty \wedge b<0) \vee  \tag{7.54}\\
& (a>0 \wedge b=-\infty) \quad \vee \quad(a<0 \quad \wedge \quad b=\infty) .
\end{align*}
$$

Proof. "Left" implies "right". Assume that $a b= \pm \infty$. Then, from Definition 326 (multiplication in $\overline{\mathbb{R}}$, rules 1 and 2), we have $a= \pm \infty$ or $b= \pm \infty$. Assume that the other operand is zero. Then, from Definition 326 (multiplication in $\overline{\mathbb{R}}$, rule 3), the product $a b$ is undefined. Which is impossible.

Assume that $a b=\infty$. Then, from Definition 326 (multiplication in $\overline{\mathbb{R}}$, rules 1 and 2), we have either $a=\infty$ and $b>0, a=-\infty$ and $b<0, b=\infty$ and $a>0$, or $b=-\infty$ and $a<0$.

Similarly, assume now that $a b=-\infty$. Then, again from Definition 326 (multiplication in $\overline{\mathbb{R}}$, rules 1 and 2), we have either $a=-\infty$ and $b>0, a=\infty$ and $b<0, b=-\infty$ and $a>0$, or $b=\infty$ and $a<0$.
"Right" implies "left". Direct consequence of Definition 326 (multiplication in $\overline{\mathbb{R}}$, rules 1 and 2 ).
Therefore, we have the equivalences.
Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ ). Let $a, b \in \overline{\mathbb{R}}$. Then, we have $a b \in \mathbb{R}$ iff $a, b \in \mathbb{R}$.
Proof. Direct consequence of Definition 326 (multiplication in $\overline{\mathbb{R}}$, new rules are not finite-product rules).

Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined).
Let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{align*}
& a<b<0 \quad \Longrightarrow \quad \frac{1}{b}<\frac{1}{a},  \tag{7.55}\\
& a<0<b \quad \Longrightarrow \quad \frac{1}{a}<\frac{1}{b},  \tag{7.56}\\
& 0<a<b \quad \Longrightarrow \quad \frac{1}{b}<\frac{1}{a} . \tag{7.57}
\end{align*}
$$

Proof. Direct consequence of monotonicity of the multiplicative inverse in $\mathbb{R}$, and Definition 326 (multiplication in $\overline{\mathbb{R}}$ )new rule 4 is compatible with monotonicity.

## Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).

The absolute value in $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ with the value $\infty$ at $\pm \infty$.

```
Lemma 340 (equivalent definition of absolute value in \(\overline{\mathbb{R}}\) ).
Let \(a \in \overline{\mathbb{R}}\). Then, we have \(|a|=\max (-a, a)\).
```

Proof. Direct consequence of the definition of the absolute value in $\mathbb{R}$, and Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).

Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ ). $\quad$ Let $a \in \overline{\mathbb{R}}$. Then, we have $|a| \in \mathbb{R}_{+}$iff $a \in \mathbb{R}$.
Proof. Direct consequence of finiteness of the absolute value in $\mathbb{R}$, and Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).

Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative). $\quad$ Let $a \in \overline{\mathbb{R}}$. Then, we have $|a| \geqslant 0$.
Proof. Direct consequence of nonnegativeness of the absolute value in $\mathbb{R}$, and Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).

Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative).
Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|=a$ iff $a \geqslant 0$.
Proof. Direct consequence of valued field properties of $\mathbb{R}$ (thus the property holds in $\mathbb{R}$ ), and Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).

Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ). $\quad$ Let $a \in \overline{\mathbb{R}}$, and $b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{align*}
& |a|=b \quad \Longleftrightarrow \quad a= \pm b,  \tag{7.58}\\
& |a|<b \quad \Longleftrightarrow \quad-b<a<b,  \tag{7.59}\\
& |a| \leqslant b \quad \Longleftrightarrow \quad-b \leqslant a \leqslant b \tag{7.60}
\end{align*}
$$

Proof. (7.58). Direct consequence of Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ), and Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution).
(7.59). Direct consequence of Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$, thus $|a|<b$ is equivalent to $-a<b \wedge a<b$ ), Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone, thus $-a<b$ is equivalent to $-b<-(-a)$ ), and Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution).
(7.60). Direct consequence of (7.58) and (7.59).

Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent). The absolute value in $\overline{\mathbb{R}}$ is idempotent.
Proof. Direct consequence of Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative), and Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, with $a \stackrel{\text { def. }}{=}|a|$ ).
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even). The absolute value in $\overline{\mathbb{R}}$ is even.
Proof. Direct consequence of evenness of the absolute value in $\mathbb{R}$. and Definition 339 (absolute value in $\overline{\mathbb{R}}$, new rule is compatible with evenness).

## Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite). $\quad$ The absolute value in $\overline{\mathbb{R}}$ is definite.

Proof. Direct consequence of definiteness of the absolute value in $\mathbb{R}$, and Definition 339 (absolute value in $\overline{\mathbb{R}}$, new rule is not a zero-absolute-value rule).

## Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative).

The absolute value in $\overline{\mathbb{R}}$ is multiplicative (when defined):
(7.61) $\quad \forall a, b \in \overline{\mathbb{R}}, \quad a b$ is well-defined $\Longrightarrow \quad|a||b|$ is well-defined $\wedge \quad|a b|=|a||b|$.

Proof. Direct consequence of multiplicativity of the absolute value in $\mathbb{R}$, Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite, contrapositive), and Definition 339 (absolute value in $\overline{\mathbb{R}}$, new rule is compatible with the property).

## Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality).

Let $a, b \in \overline{\mathbb{R}}$. Assume that $a+b$ is well-defined. Then, we have $|a+b| \leqslant|a|+|b|$.
Proof. Direct consequence of Definition 339 (absolute value in $\overline{\mathbb{R}}$ ), the triangle inequality for the absolute value in $\mathbb{R}$, Definition 317 (addition in $\overline{\mathbb{R}}, \infty$ is absorbing for addition in $\overline{\mathbb{R}}_{+}$), and Definition 312 (extended real numbers, $\overline{\mathbb{R}}, \infty$ is the maximal element).

Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality).
Let $a, b \in \overline{\mathbb{R}}$. Assume that $a$ and $b$ are not both infinite (i.e. $(a, b) \notin\{( \pm \infty, \pm \infty)\})$.
Then, we have $||a|-|b|| \leqslant|a-b|$.
Proof. Case $a, \boldsymbol{b} \in \mathbb{R}$. Direct consequence of the reverse triangle inequality for the absolute value in $\mathbb{R}$.
Case $\boldsymbol{a} \in \mathbb{R}, \boldsymbol{b}= \pm \infty$. Direct consequence of Definition 317 (addition in $\overline{\mathbb{R}}, \underline{a- \pm \infty=\mp \infty \text {, }}$ and rule 3 does not occur (both sides of the inequality)), Definition 339 (absolute value in $\overline{\mathbb{R}}$, thus $|a- \pm \infty|=\infty)$, and Definition 312 (extended real numbers, $\overline{\mathbb{R}}, \infty$ is the maximal element).
Case $\boldsymbol{a}= \pm \infty, \boldsymbol{b} \in \mathbb{R}$. Direct consequence of the previous case (swapping $a$ and $b$ ), and Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).

Lemma 351 (absolute value in $\overline{\mathbb{R}}$ satisfies countable triangle inequality). Let $I \subset \mathbb{N}$. Let $\left(a_{i}\right)_{i \in I} \in \overline{\mathbb{R}}$. Assume that $\sum_{i \in I} a_{n}$ is well-defined. Then, we have $\left|\sum_{i \in I} a_{i}\right| \leqslant \sum_{i \in I}\left|a_{i}\right|$.

Proof. Direct consequence of Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality), and induction on the number of operands.

## Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ).

The exponential function in $\mathbb{R}$ and the natural logarithm function in $\mathbb{R}_{+}^{\star}$ are respectively extended to $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_{+}$with the following rules:

$$
\begin{equation*}
\exp (-\infty)=0, \quad \exp \infty=\infty, \quad \ln 0=-\infty, \quad \text { and } \quad \ln \infty=\infty \tag{7.62}
\end{equation*}
$$

## Lemma 353 (exponential and logarithm in $\overline{\mathbb{R}}$ are inverse).

The exponential function in $\overline{\mathbb{R}}$ and the natural logarithm function in $\overline{\mathbb{R}}_{+}$are each other inverse.
Proof. Direct consequence of properties of the exponential and natural logarithm functions in $\mathbb{R}$, and Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$, new rules are compatible with the property).

Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ). Exponentiation in $\mathbb{R}$ (with either positive base and any exponent, or zero base and positive exponent) is extended to $\overline{\mathbb{R}}$ with the following rule:

$$
\begin{equation*}
\forall a \in \overline{\mathbb{R}}_{+}, \forall b \in \overline{\mathbb{R}}, \quad b \ln a \text { defined } \quad \Longrightarrow \quad a^{b} \stackrel{\text { def. }}{=} \exp (b \ln a) . \tag{7.63}
\end{equation*}
$$

## Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ).

Exponentiation in $\overline{\mathbb{R}}$ is the function $\overline{\mathbb{R}}_{+} \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_{+}$defined by

$$
\left\{\begin{array}{l}
\text { 1. } \forall a \in \mathbb{R}_{+}^{\star}, \forall b \in \mathbb{R}, \quad a^{b} \stackrel{\text { def. }}{=} \exp (b \ln a) \in \mathbb{R}_{+}^{\star}, \\
\text { 2. }  \tag{7.64}\\
\text { 洨 } \\
\text { 3. } \\
\text { 4. } \\
0^{0}, \infty^{0}, \quad \text { and } 1^{ \pm \infty} \text { are undefined. }
\end{array}\right.
$$

Proof. Direct consequence of the definition of exponentiation in $\mathbb{R}$, Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ), and Definition 326 (multiplication in $\overline{\mathbb{R}})$.

Remark 356. Note that in the previous lemma, rules 2 and 3 are overlapping for the definition of $0^{-\infty}$ and $\infty^{\infty}$, but they are consistent as both are equal to $\infty$ for both rules.

Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 ). Let $a \in \overline{\mathbb{R}}_{+}$. Then, we have $a^{1}=a$.
Proof. Direct consequence of Lemma 355 (exponentiation in $\overline{\mathbb{R}}$, with $b \stackrel{\text { def. }}{=} 1$ ).
Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ ). Let $a \in \overline{\mathbb{R}}_{+}$. Let $b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{align*}
a^{b}=0 \Longleftrightarrow & (a=0 \quad \wedge \quad b>0) \vee(a<1 \wedge \quad b=\infty) \vee  \tag{7.65}\\
& (a=\infty \quad \wedge \quad b<0) \vee(a>1 \wedge \quad b=-\infty)
\end{align*}
$$

Proof. Direct consequence of Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ), positiveness of the exponential function in $\mathbb{R}$, Lemma 336 (infinity-product property in $\overline{\mathbb{R}},(7.54)$ ), increasingness of the natural logarithm function in $\mathbb{R}$ (thus it is negative on $(0,1)$ and positive on $(1, \infty)$ ).

Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ). Let $a \in \overline{\mathbb{R}}_{+}$. Let $b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
a^{b}=1 \quad \Longleftrightarrow \quad b \ln a \text { is defined } \wedge(a=1 \quad \vee \quad b=0) . \tag{7.66}
\end{equation*}
$$

Proof. Direct consequence of Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ), the one-exponential property in $\mathbb{R}$, Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ), the 0-natural logarithm property in $\mathbb{R}$.

Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ). Let $a \in \overline{\mathbb{R}}_{+}$. Let $b \in \overline{\mathbb{R}}$. Then,

$$
\begin{align*}
& a^{b}=\infty \Longleftrightarrow(a=0 \quad \wedge \quad b<0) \vee(a<1 \wedge b=-\infty) \vee  \tag{7.67}\\
&(a=\infty \quad \wedge \quad b>0) \vee \\
&(a>1 \quad \wedge b=\infty)
\end{align*}
$$

Proof. Direct consequence of Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ), Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ), positiveness of the exponential function in $\mathbb{R}$, Lemma 336 (infinity-product property in $\overline{\mathbb{R}},(7.53)$ ), increasingness of the natural logarithm function in $\mathbb{R}$ (thus it is negative on $(0,1)$ and positive on $(1, \infty)$ ).

Lemma 361 (power function in $\overline{\mathbb{R}}$ ). Let $p \in(0, \infty)$. Then, the power function of degree $p$, $\left(x \mapsto|x|^{p}\right)$, is defined on $\overline{\mathbb{R}}$; it is the same function in $\mathbb{R}$ extended with the value $\infty$ at $\pm \infty$.

Proof. Direct consequence of the definition of the power function in $\mathbb{R}$, Definition 339 (absolute value in $\overline{\mathbb{R}}$ ), and Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ).

Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value).
The power function of degree 1 in $\overline{\mathbb{R}}$ is the absolute value.
Proof. Direct consequence of Lemma 361 (power function in $\overline{\mathbb{R}}$ ), and Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 ).

Lemma 363 (finite power function in $\overline{\mathbb{R}}$ ).
Let $p \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p} \in \mathbb{R}_{+}$iff $a \in \mathbb{R}$.
Proof. Direct consequence of finiteness of the power function in $\mathbb{R}$, and Lemma 361 (power function in $\overline{\mathbb{R}}$ ).

Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative).
Let $p \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p} \geqslant 0$.
Proof. Direct consequence of nonnegativeness of the power function in $\mathbb{R}$, and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with nonnegativeness).

Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition).
Let $p, q \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p+q}=|a|^{p}|a|^{q}$.
Proof. Direct consequence of compatibility of the power function in $\mathbb{R}$ with addition, and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with the property).

Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication).
Let $p, q \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p q}=\left(|a|^{p}\right)^{q}$.
Proof. Direct consequence of compatibility of the power function in $\mathbb{R}$ with multiplication, and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with the property).

## Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone). $\quad$ Let $p \in(0, \infty)$. Let $a, b \in \overline{\mathbb{R}}$.

 Then, we have $|a|<|b| \Rightarrow|a|^{p}<|b|^{p}$, and thus $|a| \leqslant|b| \Rightarrow|a|^{p} \leqslant|b|^{p}$.Proof. Direct consequence of monotonicity of the power function in $\mathbb{R}_{+}$, and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with monotonicity).

## Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ).

Let $p \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}$, and $b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{align*}
& |a|^{p}=b \quad \Longleftrightarrow \quad a= \pm b^{\frac{1}{p}},  \tag{7.68}\\
& |a|^{p}<b \quad \Longleftrightarrow \quad-b^{\frac{1}{p}}<a<b^{\frac{1}{p}},  \tag{7.69}\\
& |a|^{p} \leqslant b \quad \Longleftrightarrow \quad-b^{\frac{1}{p}} \leqslant a \leqslant b^{\frac{1}{p}} . \tag{7.70}
\end{align*}
$$

Proof. Direct consequence of field properties of $\mathbb{R}$ (thus $\frac{1}{p} \in(0, \infty)$ and $\left.p \times \frac{1}{p}=\frac{1}{p} \times p=1\right)$, Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, thus $b=|b|$ ), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication, with $\frac{1}{p}$ and $p$, thus $|a|^{p}=b \Leftrightarrow|a|=b^{\frac{1}{p}}$, Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, with $\frac{1}{p}$ and $p$, thus $|a|^{p}<b \Leftrightarrow|a|<b^{\frac{1}{p}}$ and $|a|^{p} \leqslant b \Leftrightarrow|a| \leqslant b^{\frac{1}{p}}$ ), Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative, thus $b^{\frac{1}{p}} \geqslant 0$ ), and Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ).

## Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite).

Let $p \in(0, \infty)$. Then, the power function of degree $p$ in $\overline{\mathbb{R}}$ is definite.
Proof. Direct consequence of definiteness of the power function in $\mathbb{R}$, and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is not a zero-power-function rule).

Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative). $\quad$ Let $p \in(0, \infty)$.
Then, the power function of degree $p$ is multiplicative (when defined):

$$
\begin{equation*}
\forall a, b \in \overline{\mathbb{R}}, \quad a b \text { is well-defined } \quad \Longrightarrow \quad|a|^{p}|b|^{p} \text { is well-defined } \wedge \quad|a b|^{p}=|a|^{p}|b|^{p} . \tag{7.71}
\end{equation*}
$$

Proof. Direct consequence of multiplicativity of the power function in $\mathbb{R}$, Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite, contrapositive), and Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with the property).

## Lemma 371 (power function satisfies extended relaxed triangle inequality).

Let $p \in[1, \infty)$. Let $a, b \in \overline{\mathbb{R}}$. Assume that $a+b$ is well-defined. Then, $|a+b|^{p}$ is well-defined, and the power function of degree $p$ satisfies the extended relaxed triangle inequality:

$$
\begin{equation*}
|a+b|^{p} \leqslant 2^{p-1}\left(|a|^{p}+|b|^{p}\right) \tag{7.72}
\end{equation*}
$$

Moreover, there is equality when $a, b \geqslant 0$ for $p=1$, and when $a=b$ (for all $p \geqslant 1$ ).
Proof. Direct consequence of Lemma 361 (power function in $\overline{\mathbb{R}}$, new rule is compatible with the property: when $a$ or $b$ are infinite and not opposite, $|a+b|^{p}=\mid \overline{\left.\left.a\right|^{p} \text { or }|b|^{p}=\infty\right) \text {, Definition } 317}$ (addition in $\overline{\mathbb{R}}$, rule 1), Definition 326 (multiplication in $\overline{\mathbb{R}}$, rule 1), and Lemma 300 (power function satisfies relaxe triangle inequality).

## Lemma 372 (topology on $\overline{\mathbb{R}}$ ).

The totally ordered set $(\overline{\mathbb{R}}, \leqslant)$ is equipped with the order topology $\mathcal{T}_{\overline{\mathbb{R}}}\left(\mathcal{R}_{\overline{\mathbb{R}}}^{o}\right)$ generated by the open rays, for which the open intervals $\mathcal{I}_{\mathbb{R}}^{o}$ constitute a topological basis.

Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), Definition 256 (order topology), and Lemma 258 (topological basis of order topology).

Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ). Let $\overline{\mathcal{T}}$ be the order topology on $\overline{\mathbb{R}}$. Let $Y \subset \overline{\mathbb{R}}$. Then, $\mathcal{I}_{Y}^{o}$ is a topological basis of the trace topology $\overline{\mathcal{T}}_{Y}$, and when $Y \in \overline{\mathcal{T}}$, we have $\overline{\mathcal{T}}_{Y} \subset \overline{\mathcal{T}}$. Moreover, let $\mathcal{T}$ be the usual topology on $\mathbb{R}$, and assume that $Y \subset \mathbb{R}$.
Then, the trace topology $\overline{\mathcal{T}}_{Y}$ is equal to $\mathcal{T}_{Y}$. In particular, we have $\overline{\mathcal{T}}_{\mathbb{R}}=\mathcal{T} \subset \overline{\mathcal{T}}$.
Proof. From Lemma 259 (trace of order topology, with $X \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ and $\mathcal{T} \stackrel{\text { def. }}{=} \overline{\mathcal{T}}$ ), $\mathcal{T}_{Y}$ is a topology on $Y$ with topological basis $\mathcal{I}_{Y}^{o}$, and $Y \in \overline{\mathcal{T}}$ implies $\overline{\mathcal{T}}_{Y} \subset \overline{\mathcal{T}}$.

Moreover, from Lemma 307 (topology on $\mathbb{R}$, thus $\overline{\mathcal{T}}_{\mathbb{R}}=\mathcal{T}$ ), Lemma 252 (trace topology on subset, with $Y \subset \overline{\mathbb{R}}, \mathbb{R} \subset \overline{\mathbb{R}}$ and $Y \subset \mathbb{R}$ ), and Definition 226 (trace of subset system, with $Y \subset \mathbb{R} \subset \overline{\mathbb{R}}$ ), we have $\overline{\mathcal{T}}_{Y}=\overline{\mathcal{T}} \bar{\cap} Y=(\overline{\mathcal{T}} \bar{\cap} \mathbb{R}) \bar{\cap} Y=\mathcal{T} \bar{\cap} Y=\mathcal{T}_{Y}$. Hence, since $\mathbb{R} \in \overline{\mathcal{T}}$, we have $\overline{\mathcal{T}}_{\mathbb{R}}=\mathcal{T}_{\mathbb{R}}=\mathcal{T} \subset \overline{\mathcal{T}}$.

## Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$ ).

The tangent function from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$, and the arctangent function from $\mathbb{R}$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are respectively extended with the following rules:

$$
\begin{equation*}
\tan \left( \pm \frac{\pi}{2}\right)= \pm \infty \quad \text { and } \quad \arctan ( \pm \infty)= \pm \frac{\pi}{2} \tag{7.73}
\end{equation*}
$$

Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms). The tangent and arctangent functions extended to $\overline{\mathbb{R}}$ are each other inverse increasing homeomorphisms.

Proof. Direct consequence of the fact that the tangent function on $\mathbb{R}$ restricted to an open interval of the form $\left(-\frac{\pi}{2}+k \pi, \frac{\pi}{2}+k \pi\right)$ for $k \in \mathbb{Z}$ (here with $k \stackrel{\text { def. }}{=} 0$ ) is an increasing homeomorphism which (increasing) inverse is the arctangent function, Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$, the topology on $\overline{\mathbb{R}}$ is an extension of the topology on $\mathbb{R}$ ), and Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$, extension rules (7.73) are compatible with the property).

Remark 376. Note that $\overline{\mathbb{R}}$ may also be metrized using the hyperbolic tangent, extended into an homeomorphism from $\overline{\mathbb{R}}$ to $[-1,1]$.

But the usual Euclidean metric on $\mathbb{R}$ cannot be extended into a metric on $\overline{\mathbb{R}}$. Indeed, the extended metric would be a continuous function (e.g. using the box topology on $\overline{\mathbb{R}}^{2}$, see Lemma 385), and it would be bounded on $\overline{\mathbb{R}}$ compact, but the usual Euclidean metric on $\mathbb{R}$ is unbounded.

Lemma 377 (metric on $\overline{\mathbb{R}}$ ). Let $\bar{d}: \overline{\mathbb{R}}^{2} \rightarrow[0, \pi]$ be the function defined by

$$
\begin{equation*}
\forall x, y \in \overline{\mathbb{R}}, \quad \bar{d}(x, y) \stackrel{\text { def. }}{=}|\arctan y-\arctan x| . \tag{7.74}
\end{equation*}
$$

Then, $(\overline{\mathbb{R}}, \bar{d})$ is a metric space. The function $\bar{d}$ is called the arctangent metric on $\overline{\mathbb{R}}$.
Proof. From Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$ ), we have $\arctan (\overline{\mathbb{R}})=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, the subtraction and absolute value in (7.74) are those of $\mathbb{R}$, i.e. $\bar{d}$ is well-defined on $\overline{\mathbb{R}}^{2}$, and we have $\bar{d}(\overline{\mathbb{R}} \times\{\infty\})=[0, \pi]$. Moreover, from the triangle inequality for the absolute value in $\mathbb{R}$, we have $\bar{d}\left(\overline{\mathbb{R}}^{2}\right) \subset[0, \pi]$. Hence, $\bar{d}\left(\overline{\mathbb{R}}^{2}\right)=[0, \pi]$.

Then, from nonnegativeness of the absolute value in $\mathbb{R}$ (thus $\bar{d}$ is nonnegative), evenness of the absolute value in $\mathbb{R}$ (thus $\bar{d}$ is symmetric), definiteness of the absolute value in $\mathbb{R}$, Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms, arctan is bijective), the definition of bijectivity (arctan is injective, thus $\bar{d}$ is definite), the triangle inequality for the absolute value in $\mathbb{R}$ (thus $\bar{d}$ satisfies the triangle inequality), and Definition 17 (distance), $\bar{d}$ is a metric over $\overline{\mathbb{R}}$.

Therefore, from Definition 18 (metric space), $(\overline{\mathbb{R}}, \bar{d})$ is a metric space.

## Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ).

The order topology on $\overline{\mathbb{R}}$ is the same as the topology induced by the arctangent metric. They admit the set $\mathcal{I}_{\mathbb{R}}^{o}$ of open intervals as topological basis.

Proof. Direct consequence of Lemma 372 (topology on $\overline{\mathbb{R}}$, the set $\mathcal{I}_{\overline{\mathbb{R}}}^{o}$ of open intervals is a topological basis of the order topology), Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms, $\arctan ^{-1}=\tan$ is increasing), the definition of the topology induced by a metric (the set $\left\{(\tan (\arctan x-r), \tan (\arctan x+r)) \mid x \in \overline{\mathbb{R}} \wedge r \in \mathbb{R}_{+}^{\star}\right\}$ of open balls is a topological basis of the topology induced by the arctangent metric on $\overline{\mathbb{R}}$ ), and Lemma 249 (topological basis is generator, both topological bases are identical, thus both topologies are the same).

## Lemma 379 (convergence towards $\pm \infty$ ).

Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$. Then, we have $\lim _{n \rightarrow \infty} x_{n}=-\infty\left(\right.$ resp. $\left.\lim _{n \rightarrow \infty} x_{n}=\infty\right)$ iff

$$
\begin{equation*}
\forall l \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geqslant N, \quad x_{n} \leqslant l \quad\left(\text { resp } . l \leqslant x_{n}\right) \tag{7.75}
\end{equation*}
$$

If so, the sequence is said convergent towards $-\infty$ (resp. $\infty$ ).
Proof. Direct consequence of Lemma 372 (topology on $\overline{\mathbb{R}}$ ), and the definition of the limit using neighborhoods.

## Lemma 380 (convergence towards $\pm \infty$ (discrete)).

Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$. Then, the sequence is convergent towards $-\infty$ (resp. $\infty$ ) iff

$$
\begin{equation*}
\forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geqslant N, \quad x_{n} \leqslant-k \quad \text { (resp. } k \leqslant x_{n} \text { ). } \tag{7.76}
\end{equation*}
$$

Proof. Direct consequence of Lemma 379 (convergence towards $\pm \infty$ ), the Archimedean property of $\mathbb{R}$, and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity).

Lemma 381 (limit is monotone). Let $\left(u_{n}\right)_{n \in \mathbb{N}},\left(v_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$. Assume that both sequence converge in $\overline{\mathbb{R}}$, and satisfy $\exists N \in \mathbb{N}, \forall n \geqslant N, u_{n} \leqslant v_{n}$. Then, we have $\lim _{n \rightarrow \infty} u_{n} \leqslant \lim _{n \rightarrow \infty} v_{n}$.

Proof. Let $u=\lim _{n \rightarrow \infty} u_{n}$, and $v=\lim _{n \rightarrow \infty} v_{n}$.
Case $\boldsymbol{v}=\infty$, or $\boldsymbol{u}=-\infty$. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
Case $\boldsymbol{u}=\infty$, or $\boldsymbol{v}=-\infty$. Direct consequence of Definition 380 (convergence towards $\pm \infty$ (discrete), thus, we respectively have $v=\infty$, or $u=-\infty$, and there is equality).

Case $\boldsymbol{u}, \boldsymbol{v}$ are finite. Assume that $u>v$. Then, from Definition 27 (convergent sequence, with the distance induced by $|\cdot|$ in $\mathbb{R}$, and $\varepsilon=\frac{u-v}{3}>0$ ), there exists $N_{u}, N_{v}>0$ such that,

$$
\begin{array}{ll}
\forall n \in \mathbb{N}, \quad n \geqslant N_{u} \quad \Longrightarrow \quad\left|u_{n}-u\right| \leqslant \frac{u-v}{3} \\
\forall n \in \mathbb{N}, \quad n \geqslant N_{v} \quad \Longrightarrow \quad\left|v_{n}-v\right| \leqslant \frac{u-v}{3}
\end{array}
$$

Hence, using the hypothesis $u>v$, and from boundedness of the absolute value in $\mathbb{R}$, we have for all $n \geqslant \max \left(N, N_{u}, N_{v}\right)$,

$$
v_{n} \leqslant v+\frac{u-v}{3}=\frac{u+2 v}{3}<\frac{2 u+v}{3}=u-\frac{u-v}{3} \leqslant u_{n}
$$

which contradicts the definition of $N$. Therefore, we have $u \leqslant v$.

## Lemma 382 (cluster point at $\pm \infty$ (discrete)).

Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$. The sequence admits a cluster point at $-\infty$ (resp. $\infty$ ) iff

$$
\forall k, N \in \mathbb{N}, \exists n \geqslant N, \quad x_{n}<-k\left(\text { resp. } x_{n}>k\right) .
$$

Proof. Direct consequence of Definition 267 (cluster point), and Lemma 380 (convergence towards $\pm \infty$ (discrete)).

## Lemma 383 (limit is invariant by translation).

Let $\left(x_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$. Let $k \in \mathbb{N}$.
Then, $\left(x_{k+n}\right)_{n \in \mathbb{N}}$ is convergent in $\overline{\mathbb{R}}$ iff $\left(x_{n}\right)_{n \in \mathbb{N}}$ is convergent in $\overline{\mathbb{R}}$. If so, both limits are equal.
Proof. "Left" implies "right". Assume that $\left(x_{k+n}\right)_{n \in \mathbb{N}}$ converges towards some $l \in \overline{\mathbb{R}}$. Then, from the definition of the limit using neighborhoods (denoted by $\mathcal{V}$ ), we have

$$
\forall U \in \mathcal{V}(l), \exists N \in \mathbb{N}, \forall n \geqslant N, \quad x_{k+n} \in U
$$

Let $N^{\prime} \stackrel{\text { def. }}{=} k+N \in \mathbb{N}$. Then, for all $n \geqslant N^{\prime}$, we have $x_{n} \in U$. Hence, from the definition of the limit using neighborhoods, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is convergent with the same limit $l$.
"Right" implies "left". Assume that $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges towards some $l \in \overline{\mathbb{R}}$. Then, from the definition of the limit using neighborhoods, we have

$$
\forall U \in \mathcal{V}(l), \exists N \in \mathbb{N}, \forall n \geqslant N, \quad x_{n} \in U
$$

Let $N^{\prime} \stackrel{\text { def. }}{=} \max (0,-k+N) \in \mathbb{N}$. Then, for all $n \geqslant N^{\prime}$, we have $x_{k+n} \in U$. Hence, from the definition of the limit using neighborhoods, the sequence $\left(x_{k+n}\right)_{n \in \mathbb{N}}$ is convergent with the same limit $l$.

Therefore, we have the equivalence.
Remark 384. We recall that $G^{\overline{\times}}{ }^{m}$ denotes a set of products of subsets, see Definition 227.
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ).
Let $m \in[2 . . \infty)$.
Let $\overline{\mathcal{T}}$ be the order topology on $\overline{\mathbb{R}}$, or the topology induced by the arctangent metric. Then, the box topology $\mathcal{T}_{\overline{\mathbb{R}}^{m}}\left(\overline{\mathcal{T}}^{\overline{\times} m}\right)$ on $\overline{\mathbb{R}}^{m}$ admits the set $\left(\mathcal{I}_{\overline{\mathbb{R}}}^{o}\right)^{\overline{\times} m}$ of open m-boxes as a topological basis.

Proof. Direct consequence of Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ), and Lemma 254 (box topology on Cartesian product).

Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ). Let $m \in[2 . . \infty)$.
Let $\overline{\mathcal{T}}$ be the box topology on $\overline{\mathbb{R}}^{m}$. Let $Y \subset \overline{\mathbb{R}}^{m}$. Then, the set $\left(\mathcal{I}_{\overline{\mathbb{R}}}^{o}\right) \bar{x} m \bar{\cap} Y$ of traces of open m-boxes on $Y$ is a topological basis of the trace topology $\overline{\mathcal{T}}_{Y}$, and when $Y \in \overline{\mathcal{T}}$, we have $\overline{\mathcal{T}}_{Y} \subset \overline{\mathcal{T}}$. Moreover, let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}^{m}$, and assume that $Y \subset \mathbb{R}^{m}$.
Then, the trace topology $\overline{\mathcal{T}}_{Y}$ is equal to $\mathcal{T}_{Y}$. In particular, we have $\overline{\mathcal{T}}_{\mathbb{R}^{m}}=\mathcal{T} \subset \overline{\mathcal{T}}$.
Proof. From Lemma 252 (trace topology on subset, with $X \stackrel{\text { def. }}{=} \overline{\mathbb{R}}^{m}$ and $\mathcal{T} \stackrel{\text { def. }}{=} \overline{\mathcal{T}}$ ), and Lemma 385 (topology on $\left.\overline{\mathbb{R}}^{m}\right), \overline{\mathcal{T}}_{Y}$ is a topology on $Y$ with topological basis $\left(\mathcal{I}_{\mathbb{R}}^{o}\right) \overline{\times} m \bar{\cap} Y$, and $Y \in \overline{\mathcal{T}}$ implies $\overline{\mathcal{T}}_{Y} \subset \overline{\mathcal{T}}$.

Moreover, from Lemma 310 (topology on $\mathbb{R}^{m}$, thus $\overline{\mathcal{T}}_{\mathbb{R}}=\mathcal{T}$ ), Lemma 252 (trace topology on subset, with $Y \subset \overline{\mathbb{R}}^{m}, \mathbb{R}^{m} \subset \overline{\mathbb{R}}^{m}$ and $Y \subset \mathbb{R}^{m}$ ), and Definition 226 (trace of subset system, with $\underline{\left.Y \subset \mathbb{R}^{m} \subset \overline{\mathbb{R}}^{m}\right)}$, we have $\overline{\overline{\mathcal{T}}_{Y}}=\overline{\mathcal{T}} \bar{\cap} Y=\left(\overline{\mathcal{T}} \bar{\cap} \mathbb{R}^{m}\right) \bar{\cap} Y=\mathcal{T} \bar{\cap} Y=\mathcal{T}_{Y}$. Hence, since $\mathbb{R}^{m} \in \overline{\mathcal{T}}$, we have $\overline{\mathcal{T}}_{\mathbb{R}^{m}}=\mathcal{T}_{\mathbb{R}^{m}}=\mathcal{T} \subset \overline{\mathcal{T}}$.

## Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous).

Addition in $\overline{\mathbb{R}}$ is continuous when defined, i.e. on $\overline{\mathbb{R}}^{2} \backslash\{(-\infty, \infty),(\infty,-\infty)\}$.
Proof. Direct consequence of continuity of addition in $\mathbb{R}$, unboundedness of addition when an operand tends towards $\pm \infty$ and the other is not $\mp \infty$ (using neighborhoods), and Definition 317 (addition in $\overline{\mathbb{R}}$, rules 1 and 2).

## Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous).

Multiplication in $\overline{\mathbb{R}}$ is continuous when defined, i.e. on $\overline{\mathbb{R}}^{2} \backslash\{(0, \pm \infty),( \pm \infty, 0)\}$.
Proof. Direct consequence of continuity of multiplication in $\mathbb{R}$, unboundedness of multiplication when an operand tends towards $\pm \infty$ and the other is nonzero (using neighborhoods), and Definition 326 (multiplication in $\overline{\mathbb{R}}$, rules 1 and 2).

Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous). The absolute value in $\overline{\mathbb{R}}$ is continuous.
Proof. Direct consequence of continuity of the absolute value in $\mathbb{R}$, and Definition 339 (absolute value in $\overline{\mathbb{R}}$, absolute value is closed in $\{ \pm \infty\}$ ).

## Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous).

Let $p \in(0, \infty)$. Then, the power function of degree $p$ is continuous on $\overline{\mathbb{R}}$.
Proof. Direct consequence of continuity of the power function over $\mathbb{R}$ for $p \geqslant 0$, unboundedness of the power function in $\mathbb{R}$ for $p>0$, and Lemma 361 (power function in $\overline{\mathbb{R}}$, the value at $\pm \infty$ is $\infty$ ).

## Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear). $\quad$ Limit in $\overline{\mathbb{R}}$ is linear when defined.

Proof. Direct consequence of Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous), Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous), and Lemma 45 (compatibility of limit with continuous functions).

### 7.5.2.1 Nonnegative extended real numbers

Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete $) . \quad\left(\overline{\mathbb{R}}_{+}, \leqslant\right)$is closed and complete.
Proof. Direct consequence of Lemma 372 (topology on $\overline{\mathbb{R}}, \overline{\mathbb{R}}_{+}^{c}=[-\infty, 0$ ) is an open ray), and Lemma 314 ( $\overline{\mathbb{R}}$ is complete, for all $A \subset \overline{\mathbb{R}}_{+}$, inf $A$ and $\sup A$ exist, and belong to $\overline{\mathbb{R}}_{+}$).

Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed). $\quad$ Addition in $\overline{\mathbb{R}}_{+}$is closed.
Proof. Direct consequence of closedness of addition in $\mathbb{R}_{+}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur).

Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative). $\quad$ Addition in $\overline{\mathbb{R}}_{+}$is associative.
Proof. Direct consequence of Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined), and Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur).

Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative). $\quad$ Addition in $\overline{\mathbb{R}}_{+}$is commutative.
Proof. Direct consequence of Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined), and Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur).

Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order).
Let $a, b, c \in \overline{\mathbb{R}}_{+}$. Assume that $a \leqslant b$. Then, we have $a+c \leqslant b+c$.
Proof. Direct consequence of Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined), and Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur).

Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$). Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a+b=\infty \quad \Longleftrightarrow \quad a=\infty \quad \vee \quad b=\infty \tag{7.78}
\end{equation*}
$$

Proof. Direct consequence of Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ ).
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$).
Let $I \subset \mathbb{N}$. Let $\left(a_{i}\right)_{i \in I} \in \overline{\mathbb{R}}_{+}$. Then, $\sup \left\{\sum_{j \in J} a_{j} \in \overline{\mathbb{R}}_{+} \mid J \subset I \wedge J\right.$ is finite $\}$ exists, and belongs to $\overline{\mathbb{R}}_{+}$.

It is called the sum of the series of terms $\left(a_{i}\right)_{i \in I}$; it is denoted $\sum_{i \in I} a_{i}$.
Proof. Direct consequence of Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed), induction on the number of operands of the finite sum (thus $\left.\sum_{j \in J} a_{j} \in \overline{\mathbb{R}}_{+}\right)$, and Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete, thus the supremum exists, and belongs to $\overline{\mathbb{R}}_{+}$).

Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition).
Let $\left(a_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Let $I, J \subset \mathbb{N}$. Let $\left(I_{j}\right)_{j \in J}$ be a pseudopartition of $I$.
Then, we have $\sum_{i \in I} a_{i}=\sum_{j \in J}\left(\sum_{i \in I_{j}} a_{i}\right)$.
Proof. From Definition 209 (pseudopartition, thus $I=\biguplus_{j \in J} I_{j}$ ), and Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$, with $I$, the $I_{j}$ 's and $J$ ), we have

$$
\begin{gathered}
\sum_{i \in I} a_{i}=\sup \left\{\sum_{k_{i} \in K_{I}} a_{k_{i}} \mid K_{I} \subset I \wedge K_{I} \text { is finite }\right\} \\
\sum_{j \in J}\left(\sum_{i \in I_{j}} a_{i}\right)=\sup \left\{\sum_{k_{j} \in K_{J}}\left(\sum_{k_{i} \in K_{I_{k_{j}}}} a_{k_{i}}\right) \mid K_{J} \subset J \wedge K_{I_{k_{j}}} \subset I_{k_{j}} \wedge K_{J}, K_{I_{k_{j}}} \text { are finite }\right\}
\end{gathered}
$$

Let $K_{J} \subset J$, and for all $k_{j} \in K_{J}$, let $K_{I_{k_{j}}} \subset I_{k_{j}}$, all finite. Let $K_{I} \stackrel{\text { def. }}{=} \biguplus_{k_{j} \in K_{J}} K_{I_{k_{j}}}$. Then, obviously, $K_{I} \subset I, K_{I}$ is finite, and from Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative), and Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative), we have

$$
\sum_{k_{j} \in K_{J}}\left(\sum_{k_{i} \in K_{I_{k_{j}}}} a_{k_{i}}\right)=\sum_{k_{i} \in K_{I}} a_{k_{i}}
$$

Hence, from Definition 2 (supremum, upper bound for the sum on $I$, and the least upper bound for the sums on the $I_{j}$ 's and $\left.J\right)$, we have $\sum_{j \in J}\left(\sum_{i \in I_{j}} a_{i}\right) \leqslant \sum_{i \in I} a_{i}$.

Let $K_{I} \subset I$ finite. For all $j \in J$, let $K_{I_{j}} \stackrel{\text { def. }}{=} K_{I} \cap I_{j}$. Let $K_{J} \stackrel{\text { def. }}{=}\left\{j \in J \mid K_{I_{j}} \neq \varnothing\right\}$. Then, we have $K_{J} \subset J$, for all $k_{j} \in K_{J}, K_{I_{k_{j}}} \subset I_{k_{j}}$, from Definition 209 (pseudopartition), they are all finite, and from Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative), and Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative), we have

$$
\sum_{k_{i} \in K_{I}} a_{k_{i}}=\sum_{k_{j} \in K_{J}}\left(\sum_{k_{i} \in K_{I_{k_{j}}}} a_{k_{i}}\right)
$$

Hence, from Definition 2 (supremum, upper bound for the sums on the $I_{j}$ 's and $J$, and the least upper bound for the sum on $I)$, we have $\sum_{i \in I} a_{i} \leqslant \sum_{j \in J}\left(\sum_{i \in I_{j}} a_{i}\right)$.

Therefore, we have the equality.
Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone).
Let $\left(a_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Let $I \subset J \subset \mathbb{N}$. Then, we have $\sum_{i \in I} a_{i} \leqslant \sum_{j \in J} a_{j}$.

Proof. Direct consequence of Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition, with $J \stackrel{\text { def. }}{=}\{0,1\}, I_{0} \stackrel{\text { def. }}{=} I$ and $I_{1} \stackrel{\text { def. }}{=} J \backslash I$ ), and Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}, \sum_{j \in J \backslash I} a_{j} \geqslant 0$ ).

## Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds).

Let $\left(a_{n}\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$. Assume that $\varphi$ is injective. Then, $\sum_{j \in \mathbb{N}} a_{\varphi(j)} \leqslant \sum_{n \in \mathbb{N}} a_{n}$.
Proof. Direct consequence of Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$), and Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone, with $I \stackrel{\text { def. }}{=} \varphi(\mathbb{N})$ and $\left.J \stackrel{\text { def. }}{=} \mathbb{N}\right)$.

Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$).
Let $\left(a_{p}\right)_{p \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$. Assume that $\varphi$ is bijective. Then, $\sum_{p \in \mathbb{N}} a_{p}=\sum_{j \in \mathbb{N}} a_{\varphi(j)}$.
Proof. Direct consequence of Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds, first used with the sequence $\left(a_{p}\right)_{p \in \mathbb{N}}$ and the function $\varphi$, and then with the sequence $\left(b_{j}\right)_{j \in \mathbb{N}} \stackrel{\text { def. }}{=} \overline{\left.\left(a_{\varphi(j)}\right)\right)_{j \in \mathbb{N}} \text { and the }}$ function $\varphi^{-1}$ which satisfies $\varphi \circ \varphi^{-1}=\operatorname{Id}_{\mathbb{N}}$ ).

## Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$).

Let $\left(a_{p, q}\right)_{p, q \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Let $\varphi, \psi: \mathbb{N} \rightarrow \mathbb{N}^{2}$. Assume that $\varphi$ and $\psi$ are bijections.
Then, we have $\sum_{j \in \mathbb{N}} a_{\varphi(j)}=\sum_{j \in \mathbb{N}} a_{\psi(j)}$. This sum is denoted $\sum_{p, q \in \mathbb{N}} a_{p, q}$.
Proof. Direct consequence of Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$, with $\left(a_{\varphi(j)}\right)_{j \in \mathbb{N}}$ and $\varphi^{-1} \circ \psi$ ).

## Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$).

Let $\left(a_{p, q}\right)_{p, q \in \mathbb{N}} \in \overline{\mathbb{R}}_{+}$. Then, we have $\sum_{p, q \in \mathbb{N}} a_{p, q}=\sum_{p \in \mathbb{N}}\left(\sum_{q \in \mathbb{N}} a_{p, q}\right)$.
Proof. From countability of $\mathbb{N}^{2}$, let $\varphi: \mathbb{N} \rightarrow \mathbb{N}^{2}$ be a bijection. Then, from Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$), let $A \stackrel{\text { def. }}{=} \sum_{p, q \in \mathbb{N}} a_{p, q}=\sum_{j \in \mathbb{N}} a_{\varphi(j)}$ (the sum does not depend on the choice for $\varphi$ ). Let $B \stackrel{\text { def. }}{=} \sum_{p \in \mathbb{N}}\left(\sum_{q \in \mathbb{N}} a_{p, q}\right)$. Then, from Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$), we have $A, B \in \overline{\mathbb{R}}_{+}$.

Let $i \in \mathbb{N}$. Let $n \stackrel{\text { def. }}{=} \max \left\{\pi_{1}(\varphi(j)) \mid j \in[0 . . i]\right\}$ and $m \stackrel{\text { def. }}{=} \max \left\{\pi_{2}(\varphi(j)) \mid j \in[0 . . i]\right\}$ where $\pi_{1}$ and $\pi_{2}$ are the first and second canonical projections from $\mathbb{N}^{2}$ onto $\mathbb{N}$. Then, from Definition 7 ( maximum), we have $\varphi([0 . . i]) \subset[0 . . n] \times[0 . . m] \subset \mathbb{N}^{2}$. Thus, from Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), we have

$$
\sum_{j \in[0 . . i]} a_{\varphi(j)}=\sum_{j \in \varphi([0 . . i])} a_{j} \leqslant \sum_{p \in[0 . . n]}\left(\sum_{q \in[0 . . m]} a_{p, q}\right) \leqslant B .
$$

Hence, from Lemma 381 (limit is monotone, when $i \rightarrow \infty$, with $v_{i} \stackrel{\text { def. }}{=} B$ ), we have $A \leqslant B$.
Let $n, m \in \mathbb{N}$. Let $i \stackrel{\text { def. }}{=} \max \left\{\varphi^{-1}(p, q) \mid(p, q) \in[0 . . n] \times[0 . . m]\right\}$. Then, from Definition 7 ( maximum), we have $[0 . . n] \times[0 . . m] \subset \varphi([0 . . i])=\mathbb{N}^{2}$. Thus, from Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), we have

$$
\sum_{p \in[0 . . n]}\left(\sum_{q \in[0 . . m]} a_{p, q}\right) \leqslant \sum_{j \in[0 . . i]} a_{\varphi(j)} \leqslant A .
$$

Hence, from Lemma 381 (limit is monotone, when $m \rightarrow \infty$, then $n \rightarrow \infty$ ), we have $B \leqslant A$. Therefore, we have $\sum_{p, q \in \mathbb{N}}=\sum_{p \in \mathbb{N}}\left(\sum_{q \in \mathbb{N}} a_{p, q}\right)$.

## Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).

Multiplication and division in $\overline{\mathbb{R}}$ are restricted to $\overline{\mathbb{R}}_{+}$by replacing the fourth rule in (7.50) by

$$
\begin{equation*}
4^{\prime} . \quad \frac{1}{0} \stackrel{\text { def. }}{=} \infty, \text { and } \frac{1}{\infty} \stackrel{\text { def. }}{=} 0 . \tag{7.79}
\end{equation*}
$$

Remark 406. When restricting to nonnegative extended numbers, making multiplicative inverse a bijection from $\overline{\mathbb{R}}_{+}$onto itself through Definition 405 implies that for all $a>0, \frac{a}{0}=\infty$. The expressions $0 \times \infty, \frac{\infty}{\infty}$ and $\frac{0}{0}$ remain undefined.

## Lemma 407 (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined).

When they are defined, multiplication and division in $\overline{\mathbb{R}}_{+}$are closed.
Proof. Direct consequence of closedness of multiplication and division by nonzero in $\mathbb{R}_{+}$, Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).

| Lemma 408 (zero-product property in $\overline{\mathbb{R}}_{+}$). |  | Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have |
| :--- | :--- | :--- |
| $(7.80)$ | $a b=0 \Longleftrightarrow a b$ is defined $\wedge \quad(a=0 \quad \vee \quad b=0)$. |  |

Proof. Direct consequence of Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, new rule is not a zero-product rule).

Lemma 409 (one-product property in $\overline{\mathbb{R}}_{+}$). Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a b=1 \quad \Longleftrightarrow \quad\left(a=\frac{1}{b} \quad \vee \quad b=\frac{1}{a}\right) \quad \wedge \quad a, b \notin\{0, \infty\} . \tag{7.81}
\end{equation*}
$$

Proof. Direct consequence of Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ), and Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, new rule is not a one-product rule).

| Lemma 410 (infinity-product property in $\left.\overline{\mathbb{R}}_{+}\right)$. | Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have |  |
| :--- | :--- | :--- | :--- |
| $(7.82)$ | $a b=\infty \quad \Longleftrightarrow \quad(a=\infty \quad \wedge \quad b \neq 0)$ | $\vee \quad(a \neq 0 \quad \wedge \quad b=\infty)$. |

Proof. Direct consequence of Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ), and Definition 405 ( multiplication in $\overline{\mathbb{R}}_{+}$, new rule is not an infinity-product rule).

## Lemma 411 (finite-product property in $\overline{\mathbb{R}}_{+}$).

Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have $a b \in \mathbb{R}_{+}$iff $a, b \in \mathbb{R}_{+}$.
Proof. Direct consequence of Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ ), closedness of multiplication in $\mathbb{R}_{+}$, and Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, new rule is not a finite-product rule).

Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone).
Multiplicative inverse in $\overline{\mathbb{R}}_{+}$is nonincreasing.
Proof. Direct consequence of Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined), and Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, new rule is compatible with monotonicity).

Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous). $\quad$ Addition in $\overline{\mathbb{R}}_{+}$is continuous.
Proof. Direct consequence of Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous), and Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur).

Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear). Limit in $\overline{\mathbb{R}}_{+}$is positive linear when defined.
Proof. Direct consequence of Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear).

### 7.5.2.2 In the context of measure theory

Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory (and probability), the third rule in (7.50) is replaced by

$$
\begin{equation*}
3^{\prime} . \quad 0 \times( \pm \infty)= \pm \infty \times 0 \stackrel{\text { def. }}{=} 0 \tag{7.83}
\end{equation*}
$$

Remark 416. In the context of measure theory, making 0 an absorbing element for multiplication in $\overline{\mathbb{R}}$ through the previous definition implies that $\frac{\infty}{ \pm \infty}=\frac{-\infty}{ \pm \infty}=0$. The expression $\frac{a}{0}$ (for all $a \in \overline{\mathbb{R}}$ ) remains undefined. Note that multiplication in $\overline{\mathbb{R}}$ is always well-defined in this context.

## Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)).

In the context of measure theory, multiplication in $\overline{\mathbb{R}}$ is associative.
Proof. Direct consequence of Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), no longer undefined forms).

Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)).
In the context of measure theory, multiplication in $\overline{\mathbb{R}}$ is commutative.
Proof. Direct consequence of Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), no longer undefined forms).

Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)).
In the context of measure theory, multiplication is (left and right) distributive over addition in $\overline{\mathbb{R}}$.
Proof. Direct consequence of Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined), Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), no longer undefined forms).

Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)).
In the context of measure theory, let $a, b, c \in \overline{\mathbb{R}}$. Assume that $a \leqslant b$ and $c \geqslant 0$.
Then, we have $a \times c \leqslant b \times c$.
Proof. Direct consequence of Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule $3^{\prime}$ is compatible with the property).

Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
a b=0 \quad \Longleftrightarrow \quad a=0 \quad \vee \quad b=0 . \tag{7.84}
\end{equation*}
$$

Proof. Direct consequence of Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule $3^{\prime}$ is compatible with the property).

Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
a b=1 \quad \Longleftrightarrow \quad\left(a=\frac{1}{b} \quad \vee \quad b=\frac{1}{a}\right) \quad \wedge \quad a, b \notin\{0, \pm \infty\} \tag{7.85}
\end{equation*}
$$

Proof. Direct consequence of Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule is not a one-product rule).

Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{align*}
a b=\infty \Longleftrightarrow & (a=\infty \wedge b>0) \vee(a=-\infty \wedge b<0) \vee \\
& (a>0 \wedge b=\infty) \vee(a<0 \wedge b=-\infty)  \tag{7.86}\\
a b=-\infty \Longleftrightarrow & (a=-\infty \wedge b>0) \vee(a=\infty \wedge b<0) \vee \\
& (a>0 \wedge b=-\infty) \vee(a<0 \wedge b=\infty) \tag{7.87}
\end{align*}
$$

Proof. Direct consequence of Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule is not an infinity-product rule).

Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
a b \in \mathbb{R} \quad \Longleftrightarrow \quad a, b \in \mathbb{R} \quad \vee \quad a=0 \quad \vee \quad b=0 . \tag{7.88}
\end{equation*}
$$

Proof. Direct consequence of Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory), contrapositive), De Morgan's laws, and distributivity of logical conjunction over logical disjunction.

Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)).
In the context of measure theory, multiplication and division are total functions $\overline{\mathbb{R}}_{+} \times \overline{\mathbb{R}}_{+} \rightarrow \overline{\mathbb{R}}_{+}$.
In particular, we have $\frac{\infty}{\infty}=\frac{0}{0}=0$, and for all $a>0, \frac{a}{0}=\infty$.
Proof. Direct consequence of Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).

Remark 426. Of course, in the context of measure theory, multiplication and division are no longer continuous on the whole boundary of $\overline{\mathbb{R}}_{+} \times \overline{\mathbb{R}}_{+}$.

Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a b=\infty \quad \Longleftrightarrow \quad(a=\infty \quad \wedge \quad b>0) \quad \vee \quad(b=\infty \quad \wedge \quad a>0) \tag{7.89}
\end{equation*}
$$

Proof. Direct consequence of Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule is not an infinity-product rule).

Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a b \in \mathbb{R}_{+} \quad \Longleftrightarrow \quad a, b \in \mathbb{R}_{+} \quad \vee \quad a=0 \quad \vee \quad b=0 \tag{7.90}
\end{equation*}
$$

Proof. Direct consequence of Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)), and closedness of multiplication in $\mathbb{R}_{+}$.

Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)). In the context of measure theory, the absolute value in $\overline{\mathbb{R}}$ is multiplicative:

$$
\begin{equation*}
\forall a, b \in \overline{\mathbb{R}}, \quad|a||b| \text { is well-defined } \quad \wedge \quad|a b|=|a||b| . \tag{7.91}
\end{equation*}
$$

Proof. Direct consequence of Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), multiplication is always well-defined), and Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative).

## Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)).

In the context of measure theory, the fourth rule in (7.64) is replaced by

$$
\begin{equation*}
4^{\prime} . \quad 0^{0}=\infty^{0}=1^{ \pm \infty} \stackrel{\text { def. }}{=} 1 \tag{7.92}
\end{equation*}
$$

Thus, for all $a \in \overline{\mathbb{R}}_{+}$, for all $b \in \overline{\mathbb{R}}$, we have

$$
\begin{equation*}
a^{0}=1^{b}=1, \quad a<1 \Rightarrow a^{\infty}=0, \quad \text { and } \quad a>1 \Rightarrow a^{\infty}=\infty . \tag{7.93}
\end{equation*}
$$

Proof. Direct consequence of Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ), Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)), and Lemma 355 (exponentiation in $\overline{\mathbb{R}}$, new rule only affects rule 4).

Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)).
In the context of measure theory, let $a \in \overline{\mathbb{R}}_{+}$, and $b \in \overline{\mathbb{R}}$. Then, we have $a^{b}=1$ iff $a=1$ or $b=0$.
Proof. Direct consequence of Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ), and Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), new rule is a compatible one-exponentiation rule).

Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a^{b}=0 \quad \Longleftrightarrow \quad(a=0 \quad \wedge \quad b>0) \quad \vee \quad(a<1 \quad \wedge \quad b=\infty) . \tag{7.94}
\end{equation*}
$$

Proof. Direct consequence of Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$, with $b \geqslant 0$ ).
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{align*}
a^{b}=0 & \Longleftrightarrow(a=0 \wedge b>0) \vee(a<1 \wedge b=\infty)  \tag{7.95}\\
0<a^{b}<1 & \Longleftrightarrow 0<a<1 \wedge 0<b<\infty  \tag{7.96}\\
a^{b}=1 & \Longleftrightarrow a=1 \vee b=0  \tag{7.97}\\
1<a^{b}<\infty & \Longleftrightarrow 1<a<\infty \wedge 0<b<\infty  \tag{7.98}\\
a^{b}=\infty & \Longleftrightarrow(a>1 \wedge b=\infty) \vee(a=\infty \wedge \wedge>0)
\end{align*}
$$

Proof.
Direct consequence of Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)), Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)), Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ), and values of the exponentiation in $\mathbb{R}_{+}$(for values in $(0,1) \uplus(1, \infty)$ ).

Lemma 434 (finite-positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{align*}
0<a^{b}<\infty \Longleftrightarrow & (0<a<\infty \wedge b<\infty) \vee  \tag{7.100}\\
& (a=1 \wedge b=\infty) \vee(a \in\{0, \infty\} \wedge b=0)
\end{align*}
$$

Proof. Direct consequence of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory), with (7.96) to (7.98)), and ordered set properties of $\mathbb{R}_{+}$.

Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have
(7.101) $a^{b}<\infty \quad a \leqslant 1 \quad \vee \quad(1 \leqslant a<\infty \quad \wedge \quad b<\infty) \quad \vee \quad(a=\infty \quad \wedge \quad b=0)$.

Proof. Direct consequence of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory), with (7.95) to (7.98)), and ordered set properties of $\mathbb{R}_{+}$.

Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have
(7.102)

$$
0<a^{b} \quad \Longleftrightarrow \quad a=b=0 \quad \vee \quad(0<a \leqslant 1 \quad \wedge \quad b<\infty) \quad \vee \quad a \geqslant 1
$$

Proof. Direct consequence of Direct consequence of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$ (measure theory), with (7.96) to (7.99)), and ordered set properties of $\mathbb{R}_{+}$.

Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)). In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a^{b}<1 \Longleftrightarrow a<1 \quad \wedge \quad b>0 \tag{7.103}
\end{equation*}
$$

Proof. Direct consequence of Direct consequence of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$ (measure theory), with (7.95) and (7.96)), and ordered set properties of $\mathbb{R}_{+}$.

Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)). In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
a^{b}>1 \quad \Longleftrightarrow \quad a>1 \quad \wedge \quad b>0 \tag{7.104}
\end{equation*}
$$

Proof. Direct consequence of Direct consequence of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$ (measure theory), with (7.98) and (7.99)), and ordered set properties of $\mathbb{R}_{+}$.

Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)). In the context of measure theory, let $p \in \overline{\mathbb{R}}_{+}$. Let $a \in \overline{\mathbb{R}}$. Then, $|a|^{p}$ is well-defined and nonnegative.

Proof. Direct consequence of Lemma 361 (power function in $\overline{\mathbb{R}}$, for $p \in(0, \infty)$ ), Lemma 355 (exponentiation in $\overline{\mathbb{R}}$, for $p=0$ and $|x| \notin\{0, \infty\}$, and for $p=\infty$ and $|x| \neq 1$ ), and Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), for $p=0$ and $|x| \in\{0, \infty\}$, and for $p=\infty$ and $|x|=1$ ).

Remark 440. Note that, in the context of measure theory, the previous lemma also states that the power function of any degree $p \in \overline{\mathbb{R}}_{+}$is a total function $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_{+}$. Of course, the power function is no longer continuous on the whole $\overline{\mathbb{R}}$ when $p \stackrel{\text { def. }}{=} \infty$.

Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)). In the context of measure theory, let $p, q \in \overline{\mathbb{R}}_{+}$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p+q}=|a|^{p}|a|^{q}$.

Proof. Case $\boldsymbol{p}, \boldsymbol{q} \in(0, \infty)$.
Direct consequence of Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition).
Case $\boldsymbol{p}=\mathbf{0}$ or $\boldsymbol{q}=\mathbf{0}$ or $|\boldsymbol{a}|=\mathbf{1}$. Direct consequence of Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ), Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), thus $|a|^{0}=1$ or $1^{p+q}=1=1^{p}=1^{q}$ ), and Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ).
Case $\boldsymbol{p}=\infty$ or $\boldsymbol{q}=\infty$, and $|\boldsymbol{a}|<\mathbf{1}$. Direct consequence of Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$, thus $p+q=\infty$ ), Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), thus $|a|^{\infty}=0$ ), and Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory), thus $|a|^{p}|a|^{q}=0$ ).
Case $\boldsymbol{p}=\infty$ or $\boldsymbol{q}=\boldsymbol{\infty}$, and $|\boldsymbol{a}|>\mathbf{1}$. Direct consequence of Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$, thus $p+q=\infty$ ), Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), thus $|a|^{\infty}=\infty$ ), Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $|a|^{p},|a|^{q}>0$ ), and Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$, thus $|a|^{p}|a|^{q}=\infty$ ).

Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)). In the context of measure theory, let $p, q \in \overline{\mathbb{R}}_{+}$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|a|^{p q}=\left(|a|^{p}\right)^{q}$.

Proof. Case $p, q \in(0, \infty)$.
Direct consequence of Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication).
Case $p=0$ or $q=0$ or $|a|=1$.
Direct consequence of Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), and Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), thus $|a|^{p q}=|a|^{0}=1$ or $1^{p q}=1=1^{p}=1^{q}$ ).
Case $p=\infty$ and $q>0$, or $q=\infty$ and $p>0$, and $|a|<1$.
Direct consequence of Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $p q=\infty$ ), Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $\overline{|a|^{p}<1}$ ), and Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory), value 0, thus $|a|^{\infty}=0=\left(|a|^{p}\right)^{q}$ ).

Case $p=\infty$ and $q>0$, or $q=\infty$ and $p>0$, and $|a|>1$. Direct consequence of Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $p q=\infty$ ), Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $|a|^{p}>1$ ), and Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory), value $\infty$, thus $\left.|a|^{\infty}=\infty=\left(|a|^{p}\right)^{q}\right)$.

Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)). In the context of measure theory, let $p \in \overline{\mathbb{R}}_{+}$. Let $a \in \overline{\mathbb{R}}$, and $b \in \overline{\mathbb{R}}_{+}$. Then, we have $\left||a|^{p}\right|=|a|^{p}=||a||^{p}$, and $\left|b^{p}\right|=b^{p}=|b|^{p}$.

Proof. Direct consequence of Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative), Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent), and Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative).

Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)).
In the context of measure theory, let $p \in \overline{\mathbb{R}}_{+}$. Let $a, b \in \overline{\mathbb{R}}$.
Then, we have $|a|<|b| \Rightarrow|a|^{p}<|b|^{p}$, and thus $|a| \leqslant|b| \Rightarrow|a|^{p} \leqslant|b|^{p}$.
Proof. Direct consequence of Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, for $p \in(0, \infty)$ ), and Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), new rule is compatible with monotonicity (for $p \in\{0, \infty\})$ ).

Lemma 445 (power function in $\overline{\mathbb{R}}$ is even (measure theory)).
In the context of measure theory, let $p \in \overline{\mathbb{R}}_{+}$. Let $a \in \overline{\mathbb{R}}$. Then, we have $|-a|^{p}=|a|^{p}$.
Proof. Direct consequence of Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)).
In the context of measure theory, let $p \in \mathbb{R}_{+}$. Let $a, b \in \overline{\mathbb{R}}$. Then, we have $|a b|^{p}=|a|^{p}|b|^{p}$.
Proof. Direct consequence of Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative, for $p \in(0, \infty)$ ),
 is identity element for multiplication in $\overline{\mathbb{R}}$ ).

Remark 447. Note that the previous lemma does not hold when $p \stackrel{\text { def. }}{=} \infty$ and logarithms of $a$ and $b$ are of opposite sign. Indeed, we have $\left(\frac{1}{2} \times 2\right)^{\infty}=1^{\infty}=1 \neq 0=0 \times \infty=\frac{1}{2}^{\infty} \times 2^{\infty}$.

Lemma 448 (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)).
In the context of measure theory, multiplication in $\overline{\mathbb{R}}_{+}$is continuous, except at $(0, \infty)$ and $(\infty, 0)$.

Proof. Direct consequence of Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory), new rule $3^{\prime}$ is not compatible with continuity: for instance $\left(\frac{1}{n} \times \infty\right)_{n \in \mathbb{N}}$ is a constant sequence of value $\infty$, which is distinct from 0 , the value of the constant sequence $\left.(0 \times n)_{n \in \mathbb{N}}\right)$.

Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)). In the context of measure theory, let $p \in \mathbb{R}_{+}$. Then, the power function of degree $p$ is continuous on $\overline{\mathbb{R}}$.

Proof. Direct consequence of Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous, for $p \in(0, \infty)$ ), Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory), the power function of degree 0 is a constant function), and continuity of constant functions.

Remark 450. Note that the previous lemma does not hold when $p \stackrel{\text { def. }}{=} \infty$ since the power function of degree $\infty$ only takes the three values 0,1 and $\infty$.

## Definition 451 (Hölder conjugates). <br> Extended numbers $p, q \in[1, \infty]$ are said Hölder conjugates iff $\frac{1}{p}+\frac{1}{q}=1$.

Remark 452. The previous definition implies that extended numbers 1 and $\infty$ are Hölder conjugate, since from rule $4^{\prime}$ of Definition 405, we have $\frac{1}{\infty}=0$.

Lemma 453 (Young's inequality for products (measure theory)).
In the context of measure theory, let $p, q \in(1, \infty)$. Assume that $p$ and $q$ are Hölder conjugates. Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have $a b \leqslant \frac{a^{p}}{p}+\frac{b^{q}}{q}$.

Proof. From Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), Lemma 355 (exponentiation in $\overline{\mathbb{R}}$, with $p, q \neq 0, \pm \infty)$, closedness of multiplicative inverse in $\mathbb{R}_{+}^{\star}$, Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), Definition 326 (multiplication in $\overline{\mathbb{R}}$, rule 5 with $b=p, q \in \mathbb{R}_{+}^{\star}$ ), and Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed), ab, $a^{p}, b^{q}, \frac{a^{p}}{p}, \frac{b^{q}}{q}$, and $\frac{a^{p}}{p}+\frac{b^{q}}{q}$ are well-defined, and belong to $\overline{\mathbb{R}}_{+}$.

Case $\boldsymbol{a b}=\boldsymbol{b}$. Then, from Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory), $a=\infty$ or $b=\infty$ ), Lemma 355 (exponentiation in $\overline{\mathbb{R}}$, rule 2 with $b=p, q>0$ ), Definition 326 (multiplication in $\overline{\mathbb{R}}$, rule 1 with $a \stackrel{\text { def. }}{=} \frac{1}{p}, \frac{1}{q}>0$ ), and Definition 317 (addition in $\overline{\mathbb{R}}$, rule 1 , the other operand is not $-\infty$ ), the right-hand side equals $\infty$. Thus, the (in)equality holds.

Case $\boldsymbol{a b}=\mathbf{0}$. Then, the inequality holds since the right-hand side is nonnegative.
Case $\boldsymbol{a b} \in \mathbb{R}_{+}^{\star}$. Direct consequence of Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory), contrapositive), Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$, contrapositive), and Lemma 304 (Young's inequality for products in $\mathbb{R}$ ).

Therefore, the inequality always holds.
Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)).
In the context of measure theory, let $a, b \in \overline{\mathbb{R}}_{+}$. Let $\varepsilon \in \mathbb{R}_{+}^{\star}$. Then, we have $a b \leqslant \frac{a^{2}}{2 \varepsilon}+\frac{\varepsilon b^{2}}{2}$.
Proof. Direct consequence of Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ), Lemma 453 (Young's inequality for products (measure theory), with $p=q \stackrel{\text { def. }}{=} 2$ and $\frac{a}{\sqrt{\varepsilon}}, \sqrt{\varepsilon} b \in \overline{\mathbb{R}}_{+}$), properties of square root and multiplicative inverse in $\mathbb{R}_{+}^{\star}$, Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory) , Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)), and Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)).

Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}})$. Let $p, q \in(0, \infty]$. Assume that $p<q$. Let $r \in[p, q]$. Let $\alpha \stackrel{\text { def. }}{=} \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}$. Then, $\alpha \in[0,1]$, and $\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}$.

Proof. Direct consequence of Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, all multiplicative inverses exist), Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone, $\frac{1}{q} \leqslant \frac{1}{r} \leqslant \frac{1}{p} \overline{\text { belong to }[0,1] \text {, thus are finite) }}$, and ordered field properties of $\mathbb{R}$.

### 7.5.3 Second countability and real numbers

## Definition 456 (connected component in $\mathbb{R}$ ).

Let $A \subset \mathbb{R}$. Let $x \in \mathbb{R}$. The connected component of $A$ containing $x$ is the union of all open intervals $I$ containing $x$ and included in $A$; it is denoted $I_{x}^{A}$.

```
Lemma 457 (empty connected component).
Let \(A \subset \mathbb{R}\). Let \(x \in \mathbb{R}\). Assume that \(x \notin A\). Then, we have \(I_{x}^{A}=\varnothing\).
```

Proof.
Direct consequence of Definition 456 (connected component in $\mathbb{R}$ ), and the nullary union law.
Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval).
Let $O \subset \mathbb{R}$ be open. Let $x \in \mathbb{R}$. Assume that $x \in O$. Then, $I_{x}^{O}$ is an open interval contained in $O$.
Proof. Note that from Definition 22 (open subset), there exists $\varepsilon>0$ such that $(x-\varepsilon, x+\varepsilon) \subset O$. Then, from Definition 456 (connected component in $\mathbb{R}$ ) $(x-\varepsilon, x+\varepsilon) \subset I_{x}^{O}$. Thus, $I_{x}^{O}$ is nonempty, contains $x$. Moreover, from Definition 240 (topological space, closedness under union), $I_{x}^{O}$ is open.

Let $a, b, c \in \mathbb{R}$ such that $a<c<b$ and $a, b \in I_{x}^{O}$. Then, from Definition 456 (connected component in $\mathbb{R}$ ), and the definition of union, there exist intervals $I_{a}(x)$ and $I_{b}(x)$ such that $x, a \in I_{a}(x) \subset I_{x}^{O} \subset O$ and $x, b \in I_{b}(x) \subset I_{x}^{O} \subset O$. Case $\boldsymbol{c}=\boldsymbol{x}$. Then, we have $c=x \in I_{x}^{O}$. Case $\boldsymbol{c}<\boldsymbol{x}$. Then, we have $a<c<x$. Thus, since $a, x \in I_{a}(x)$, and $I_{a}(x)$ is an interval, we have successively $c \in(a, x) \subset I_{a}(x) \subset I_{x}^{O}$. Case $\boldsymbol{x}<\boldsymbol{c}$. Then, we have $x<c<b$. Thus, since $x, b \in I_{b}(x)$, and $I_{b}(x)$ is an interval, we have successively $c \in(x, b) \subset I_{b}(x) \subset I_{x}^{O}$. Hence, we always have $c \in I_{x}^{O}$. Thus, from Definition 229 (interval, with $X \stackrel{\text { def. }}{=} \mathbb{R}$ ), $I_{x}^{O}$ is an interval.

Therefore, $I_{x}^{O}$ is an open interval contained in $O$.
Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal).
Let $O \subset \mathbb{R}$ be open. Let $x, y \in \mathbb{R}$. Assume that $x, y \in O$, and $y \in I_{x}^{O}$. Then, we have $I_{y}^{O}=I_{x}^{O}$.
Proof. From Definition 456 (connected component in $\mathbb{R}$, union), there exists an open interval $I$ such that $y \in I$ and $x \in I \subset O$. Thus, we also have $x \in \overline{I_{y}^{O}}$. But, from Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval), $I_{x}^{O}$ is an open interval such that $y \in I_{x}^{O} \subset O$, and $I_{y}^{O}$ is an open interval such that $x \in I_{y}^{O} \subset O$. Thus, from Definition 456 (connected component in $\mathbb{R}$ ), we have $I_{x}^{O} \subset I_{y}^{O}$ and $I_{y}^{O} \subset I_{x}^{O}$. Therefore, we have $I_{x}^{O}=I_{y}^{O}$.

Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint). Let $O \subset \mathbb{R}$ be open. Let $x, y \in \mathbb{R}$. Assume that $x, y \in O$.
Then, we have either $I_{x}^{O}=I_{y}^{O}$, or $I_{x}^{O} \cap I_{y}^{O}=\varnothing$.
Proof. Assume that $I_{x}^{O} \cap I_{y}^{O} \neq \varnothing$. Then, there exists $z \in I_{x}^{O} \cap I_{y}^{O}$. Thus, from Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal), we have $I_{z}^{O}=I_{x}^{O}$ and $I_{z}^{O}=I_{y}^{O}$. Therefore, $I_{x}^{O}$ and $I_{y}^{O}$ are equal or disjoint.

Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ). Let $O \subset \mathbb{R}$ be open. Then, $O$ is a countable union of disjoint open intervals.

Proof. Case $\boldsymbol{O}$ empty. Direct consequence of Definition 229 (interval, thus $O$ is an open interval).
Case $\boldsymbol{O}$ nonempty. Let $x \in O$. Then, from Definition 22 (open subset), there exists $\varepsilon>0$ such that the open interval $(x-\varepsilon, x+\varepsilon)$ is included in $O$. From density of rational numbers in $\mathbb{R}$, there exists $q \in \mathbb{Q} \cap(x-\varepsilon, x+\varepsilon)$. Then, by construction, we have $x \in(x-\varepsilon, x+\varepsilon) \subset I_{q}^{O}$. Thus, we have the inclusion $O \subset \bigcup_{q \in \mathbb{Q} \cap O} I_{q}^{O}$.

Conversely, let $q \in \mathbb{Q} \cap O$. Then, from Definition 456 (connected component in $\mathbb{R}$ ), we have $I_{q}^{O} \subset O$. Thus, we also have the other inclusion $\bigcup_{q \in \mathbb{Q} \cap O} I_{q}^{O} \subset O$, and the equality.

Hence, from countability of $\mathbb{Q}$, Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint), and after eliminating doubles in the union, $O$ is a countable union of disjoint open intervals.
Therefore, we always have the result.
Lemma 462 (rational approximation of lower bound of open interval).
Let $a, b \in \overline{\mathbb{R}}$. Assume that $a<b$.
Then, there exists a sequence $\left(a_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q} \cap(a, b)$ that is nonincreasing with limit $a$.
Proof. Note that $-\infty \leqslant a<b \leqslant \infty$. From density of rational numbers in $\mathbb{R}$, let $a_{0} \in(a, b) \cap \mathbb{Q}$.
Case $\boldsymbol{a}=-\infty$. Then, from density of rational numbers in $\mathbb{R}$, for all $n \in \mathbb{N}$, let $a_{n+1}$ be in $\left(-\infty,-2\left|a_{n}\right|\right) \cap \mathbb{Q}$. Hence, from ordered field properties of $\mathbb{R}$, and the Archimedean property of $\mathbb{R}$, the sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ belongs to $(a, b)$, and is nonincreasing with limit $a=-\infty$.
Case $\boldsymbol{a}$ finite. Then, from density of rational numbers in $\mathbb{R}$, for all $n \in \mathbb{N}$, let $a_{n+1}$ be in $\left(a, \frac{a+a_{n}}{2}\right) \cap \mathbb{Q} \subset(a, b) \cap \mathbb{Q}$. Let $n \in \mathbb{N}$. Then, from ordered field properties of $\mathbb{R}$, we have $a_{n+1}<a_{n}$ and $0<a_{n+1}-a<\frac{a_{n}-a}{2}$. Thus, a trivial finite induction on index $i$ shows that

$$
\forall i \in[0 . . n+1], \quad 0<a_{n+1}-a<\frac{a_{i}-a}{2^{n+1-i}}<\frac{a_{0}-a}{2^{n+1}}
$$

Hence, from the squeeze theorem, the sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ is nonincreasing with limit $a$.

## Lemma 463 (rational approximation of upper bound of open interval).

Let $a, b \in \overline{\mathbb{R}}$. Assume that $a<b$.
Then, there exists a sequence $\left(b_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q} \cap(a, b)$ that is nondecreasing with limit $b$.
Proof. Direct consequence of Lemma 462 (rational approximation of lower bound of open interval, with $a \stackrel{\text { def. }}{=}-b, b \stackrel{\text { def. }}{=}-a$, and $\left.b_{n} \stackrel{\text { def. }}{=}-a_{n}\right)$.

Lemma 464 (open intervals with rational bounds cover open interval). Let $a, b \in \overline{\mathbb{R}}$. Assume that $a<b$. Then, there exist $\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q}$ such that $(a, b)=\bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right)$.
Proof. Direct consequence of Lemma 462 (rational approximation of lower bound of open interval, there exists $\left(a_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q} \cap(a, b)$ ), Lemma 463 (rational approximation of upper bound of open interval, there exists $\left(b_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q} \cap(a, b)$ ), Definition 27 (convergent sequence, with $\varepsilon \stackrel{\text { def. }}{=} x-a$, then $\varepsilon \stackrel{\text { def. }}{=} b-x$ for all $x \in(a, b)$ (hence, $(a, b) \subset \bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right)$ )), and Definition 240 (topological space, closedness under union $\left.\left(\bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right) \subset(a, b)\right)\right)$.

## Theorem 465 ( $\mathbb{R}$ is second-countable).

Let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}$, or the order topology. Then, $\{(a, b) \mid a, b \in \mathbb{Q} \wedge a<b\}$ is a topological basis of $(\mathbb{R}, \mathcal{T})$. Hence, $(\mathbb{R}, \mathcal{T})$ is second-countable.

Proof. Direct consequence of Lemma 307 (topology on $\mathbb{R}$, both topologies are equal), Lemma 464 (open intervals with rational bounds cover open interval), Lemma 250 (topological subbasis, with $\underline{\mathcal{B}} \stackrel{\text { def. }}{=} \mathcal{I}_{\mathbb{R}}^{o}$ ), countability of $\mathbb{Q} \times \mathbb{Q}_{+}^{\star}$, and Definition 260 (second-countability).

Lemma 466 ( $\mathbb{R}^{m}$ is second-countable). Let $m \in[2 . . \infty)$. Let $\mathcal{T}$ be the usual Euclidean topology on $\mathbb{R}^{m}$, or the box topology on $\mathbb{R}^{m}$ induced by the usual Euclidean topology on $\mathbb{R}$, or by the order topology. Then, $\left\{\prod_{n \in[1 . . m]}\left(a_{n}, b_{n}\right) \mid \forall n \in[1 . . m], a_{n}, b_{n} \in \mathbb{Q} \wedge a_{n}<b_{n}\right\}$ is a topological basis of $\left(\mathbb{R}^{m}, \mathcal{T}\right)$. Hence, $\left(\mathbb{R}^{m}, \mathcal{T}\right)$ is second-countable.

Proof. Direct consequence of Lemma 310 (topology on $\mathbb{R}^{m}$, all three topologies are equal), Theorem $465(\mathbb{R}$ is second-countable), and Lemma 261 (compatibility of second-countability with Cartesian product).

### 7.5.4 Second countability and extended real numbers

## Remark 467.

Subsets of $\overline{\mathbb{R}}$ are subsets of $\mathbb{R}$, or the union of subsets of $\mathbb{R}$ with $\{-\infty\},\{\infty\}$, or $\{-\infty, \infty\}$.
Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ).
The canonical projection from $\mathcal{P}(\overline{\mathbb{R}})$ onto $\mathcal{P}(\mathbb{R})$ is denoted $\downarrow$, the canonical injection of $\mathcal{P}(\mathbb{R})$ into $\mathcal{P}(\overline{\mathbb{R}})$ is denoted $\uparrow$, and other injections are denoted $\uparrow_{-}, \uparrow_{+}$, and $\uparrow_{ \pm}$.

$$
\begin{gather*}
\forall B \subset \overline{\mathbb{R}}, \quad \downarrow B \stackrel{\text { def. }}{=} B \cap \mathbb{R} \text { and } \forall A \subset \mathbb{R}, \quad \uparrow A \stackrel{\text { def. }}{=} A,  \tag{7.105}\\
\forall A \subset \mathbb{R}, \quad \uparrow-A \stackrel{\text { def. }}{=} A \uplus\{-\infty\}, \quad \uparrow+A \stackrel{\text { def. }}{=} A \uplus\{\infty\}, \quad \uparrow_{ \pm} A \stackrel{\text { def. }}{=} A \uplus\{-\infty, \infty\} . \tag{7.106}
\end{gather*}
$$

The notation $\uparrow_{\star}$ denote any of the four injections.
Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}$ ).
We have the following identities:

$$
\begin{gather*}
\forall A \subset \mathbb{R}, \quad \downarrow \uparrow_{\star} A=A,  \tag{7.107}\\
\forall B \subset \overline{\mathbb{R}}, \quad \begin{cases}-\infty \notin B \wedge \infty \notin B & \Longleftrightarrow \\
-\infty \in B \wedge \infty \notin B & \Longleftrightarrow \\
-\downarrow-\downarrow B=B, \\
-\infty \notin B \wedge \infty \in B & \Longleftrightarrow \\
-\infty \in \downarrow \wedge B=B, \\
-\infty \in B & \Longleftrightarrow \\
\hline \downarrow \downarrow=B .\end{cases}
\end{gather*}
$$

Proof. Direct consequence of Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ).

## Remark 470.

Note that in (7.110) in the next lemma, the complements are either in $\mathbb{R}$ or in $\overline{\mathbb{R}}$, as appropriate.
For instance, it is in $\overline{\mathbb{R}}$ in the left-hand side $\downarrow B^{c}$, and in $\mathbb{R}$ in the right-hand side $(\downarrow B)^{c}$. And this is the converse for equalities on injections.

Lemma 471 (compatibility of projection and injections in $\overline{\mathbb{R}}$ with subset operations). We have the following identities:

$$
\begin{align*}
& \downarrow \varnothing=\varnothing, \quad \uparrow \varnothing=\varnothing,  \tag{7.109}\\
& \forall B \subset \overline{\mathbb{R}}, \downarrow B^{c}=(\downarrow B)^{c}, \quad \forall A \subset \mathbb{R}, \begin{cases}\uparrow A^{c}=\left(\uparrow_{ \pm} A\right)^{c}, & \uparrow_{ \pm} A^{c}=(\uparrow A)^{c}, \\
\uparrow-A^{c}=(\uparrow+A)^{c}, & \uparrow+A^{c}=(\uparrow-A)^{c},\end{cases}  \tag{7.110}\\
& \forall I \subset \mathbb{N},\left\{\begin{array}{ll}
\forall\left(B_{i}\right)_{i \in I} \subset \overline{\mathbb{R}}, & \downarrow\left(\bigcup_{i \in I} B_{i}\right)=\bigcup_{i \in I} \downarrow B_{i}, \\
\downarrow\left(\bigcap_{i \in I} B_{i}\right)=\bigcap_{i \in I} \downarrow B_{i}, \\
\forall\left(A_{i}\right)_{i \in I} \subset \mathbb{R}, & \uparrow_{\star}\left(\bigcup_{i \in I} A_{i}\right)=\bigcup_{i \in I} \uparrow_{\star} A_{i},
\end{array} \uparrow_{\star}\left(\bigcap_{i \in I} A_{i}\right)=\bigcap_{i \in I} \uparrow_{\star} A_{i} .\right. \tag{7.111}
\end{align*}
$$

Proof. Direct consequence of Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ).
Remark 472. This remark deploys Definition 229 for the case of $\overline{\mathbb{R}}$.
Note that in $\overline{\mathbb{R}}$, inequalities with infinite values are useless since they either represent the full set, or the empty set. Thus, they are not taken into account.

The proper intervals of $\overline{\mathbb{R}}$ are defined with two inequalities. They only contain finite values, thus they are the intervals of $\mathbb{R}$ (either proper intervals, rays, or the full set), and they are open at their infinite ends (if ever).

The rays of $\overline{\mathbb{R}}$ are defined with only one inequality. They contain exactly one infinite value, at which end they are closed, and the other end is open if infinite.

The full set $\overline{\mathbb{R}}$ is the only interval with two infinite values.

Remark 473. In the sequel, let $\mathcal{T}$ denote the usual topology of $\mathbb{R}$ (i.e. either the order topology $\mathcal{T}_{\mathbb{R}}\left(\mathcal{R}_{\mathbb{R}}^{o}\right)$, or equivalently the Euclidean topology, see Lemma 307), and let $\overline{\mathcal{T}}$ denote the usual topology of $\overline{\mathbb{R}}$ (i.e. either the order topology $\mathcal{T}_{\overline{\mathbb{R}}}\left(\mathcal{R}_{\overline{\mathbb{R}}}^{o}\right)$, or equivalently the topology induced by the arctangent metric, see Lemma 378).

Lemma 474 (open down).
Let $B \subset \overline{\mathbb{R}}$. Then, $B \in \overline{\mathcal{T}}$ implies $\downarrow B \in \mathcal{T}$.
Proof. Direct consequence of Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ).

## Lemma 475 (open up). $\quad$ Let $A \subset \mathbb{R}$. Then, $A \in \mathcal{T}$ iff $\uparrow A \in \overline{\mathcal{T}}$.

Proof. "Left" implies "right". Direct consequence of Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ).
"Right" implies "left". Direct consequence of Lemma 474 (open down), and Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}, \downarrow \uparrow A=A$ ).

## Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).

The open subsets of $\overline{\mathbb{R}}$ with finite values only are the open subsets of $\mathbb{R}$.
The open subsets of $\overline{\mathbb{R}}$ with exactly one infinite value are the union of open subsets of $\mathbb{R}$ with an open ray (either left or right).

The open subsets of $\overline{\mathbb{R}}$ with two infinite values are the union of open subsets of $\mathbb{R}$ with a left open ray and a right open ray.

Proof. "Left" implies "right". Let $\bar{O} \in \overline{\mathcal{T}}$.
Then, from Lemma 474 (open down), we have $\downarrow \bar{O} \in \mathcal{T}$.
Case $\overline{\boldsymbol{O}} \subset \mathbb{R}$.
Then, from Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ), we have $\downarrow \bar{O}=\bar{O} \in \mathcal{T}$.
Case $-\infty \in \overline{\boldsymbol{O}}$ and $\infty \notin \overline{\boldsymbol{O}}$. Then, from Lemma 372 (topology on $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exists $b \in \mathbb{R}$ such that $[-\infty, b) \subset \bar{O}$, and we have $\bar{O}=\downarrow \bar{O} \cup[-\infty, b)$.
Case $-\infty \notin \overline{\boldsymbol{O}}$ and $\infty \in \overline{\boldsymbol{O}}$. Then, from Lemma 372 (topology on $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exists $a \in \mathbb{R}$ such that ( $a, \infty] \subset \bar{O}$, and we have $\bar{O}=\downarrow \bar{O} \cup(a, \infty]$.
Case $\{-\infty, \infty\} \subset \overline{\boldsymbol{O}}$. Then, from Lemma 372 (topology on $\overline{\mathbb{R}}$ ), and Definition 246 (topological basis), there exist $a, b \in \mathbb{R}$ such that $[-\infty, b),(a, \infty] \subset \bar{O}$, and we have $\bar{O}=[-\infty, b) \cup \downarrow \bar{O} \cup(a, \infty]$.
"Right" implies "left". Let $O \in \mathcal{T}$. Then, from Lemma 475 (open up), and closedness of open subsets under union, $\uparrow O$, and the union of $\uparrow O$ with open rays are open subsets of $\overline{\mathbb{R}}$.

## Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ).

All open subsets of $\overline{\mathbb{R}}$ are countable disjoint unions of open intervals of $\overline{\mathbb{R}}$.
As a consequence, the unions in the previous lemma can be specified as disjoint unions.
Proof. Let $\bar{O} \in \overline{\mathcal{T}}$. Then, from Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ), there exists $O \in \mathcal{T}$, and $a, b \in \mathbb{R}$ such that $\bar{O}$ equals either $O,[-\infty, b) \cup O, O \cup(a, \infty]$, or $[-\infty, b) \cup O \cup(a, \infty]$. Moreover, from Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ), there exists $I \subset \mathbb{N}$, and disjoint open intervals $\left(A_{i}\right)_{i \in I} \in \mathcal{I}^{o}(\mathbb{R})$ such that $O=\biguplus_{i \in I} A_{i}$.

Let $J_{a} \stackrel{\text { def. }}{=}\left\{i \in I \mid A_{i} \cap(a, \infty]=\varnothing\right\}$, and $J_{b} \stackrel{\text { def. }}{=}\left\{i \in I \mid A_{i} \cap[-\infty, b)=\varnothing\right\}$. If there exists $i_{0} \in I$ such that $a \in A_{i_{0}} \stackrel{\text { def. }}{=}\left(a_{i_{0}}, b_{i_{0}}\right)$, then let $a^{\prime} \stackrel{\text { def. }}{=} a_{i_{0}}$, otherwise let $a^{\prime} \stackrel{\text { def. }}{=} a$. Symmetrically, if there exists $i_{1} \in I$ such that $a \in A_{i_{1}} \stackrel{\text { def. }}{=}\left(a_{i_{1}}, b_{i_{1}}\right)$, then let $b^{\prime} \stackrel{\text { def. }}{=} b_{i_{1}}$, otherwise let $b^{\prime} \stackrel{\text { def. }}{=} b$.

Therefore, we have either

$$
\begin{aligned}
\bar{O}=\biguplus_{i \in I} A_{i}, & \bar{O}=\left[-\infty, b^{\prime}\right) \uplus \biguplus_{j \in J_{b}} A_{j}, \quad \bar{O}=\biguplus_{j \in J_{a}} A_{j} \uplus\left(a^{\prime}, \infty\right], \text { or } \\
& \bar{O}=\left[-\infty, b^{\prime}\right) \uplus \biguplus_{j \in J_{a} \cap J_{b}} A_{j} \uplus\left(a^{\prime}, \infty\right] .
\end{aligned}
$$

Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ).
Let $a, b \in \overline{\mathbb{R}}$. Assume that $a<b$. Then, there exist $\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}} \in \mathbb{Q}$ such that

$$
\begin{equation*}
(a, b)=\bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right), \quad[-\infty, b)=\bigcup_{n \in \mathbb{N}}\left[-\infty, b_{n}\right), \quad \text { and } \quad(a, \infty]=\bigcup_{n \in \mathbb{N}}\left(a_{n}, \infty\right] . \tag{7.112}
\end{equation*}
$$

Proof. Direct consequence of Lemma 464 (open intervals with rational bounds cover open interval), Lemma 462 (rational approximation of lower bound of open interval), and Lemma 463 (rational approximation of upper bound of open interval).

Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable). The open intervals with rational bounds constitute $a$ topological basis of $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$. Hence, $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$ is second-countable.

Proof. Direct consequence of Lemma 372 (topology on $\overline{\mathbb{R}}$ ), Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ), Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ), Lemma 250 (topological subbasis, with $\mathcal{B} \stackrel{\text { def. }}{=} \mathcal{I}_{\mathbb{R}}^{o}$ ), countability of $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}_{+}^{\star}$, compatibility of union with countability, and Definition 260 (second-countability).

Lemma $480\left(\overline{\mathbb{R}}^{m}\right.$ is second-countable $)$. Let $m \in[2 . . \infty)$. Let $\overline{\mathcal{T}}^{\bar{x} m}$ be the box topology on $\overline{\mathbb{R}}^{m}$ induced by the usual topology on $\overline{\mathbb{R}}$. Then, the open $m$-boxes with rational bounds constitute a topological basis of $\left(\overline{\mathbb{R}}^{m}, \overline{\mathcal{T}}^{\overline{\times}}\right)$. Hence, $\left(\overline{\mathbb{R}}^{m}, \overline{\mathcal{T}}^{\overline{\times} m}\right)$ is second-countable.

Proof. Direct consequence of Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ), Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable), and Lemma 261 (compatibility of second-countability with Cartesian product).

### 7.5.5 Infimum, supremum

## Remark 481.

Note that in this section, functions $X \rightarrow \overline{\mathbb{R}}$ can actually be replaced by functions $X \times Y \rightarrow \overline{\mathbb{R}}$ for any set $Y$. Then, extrema over $X$ have to be understood as functions $Y \rightarrow \overline{\mathbb{R}}$, and (in)equalities are pointwise. This is used at the end of the section for extrema of sequence of functions.

Lemma 482 (extrema of constant function). Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $a \in \overline{\mathbb{R}}$. Assume that $f$ is constant of value $a$. Then,

$$
\begin{equation*}
\sup (f(X))=\max (f(X))=\inf (f(X))=\min (f(X))=a \tag{7.113}
\end{equation*}
$$

Proof. Direct consequence of the definition of constant function, Lemma 8 (equivalent definition of maximum), Definition 7 (maximum), Lemma 15 (equivalent definition of minimum), and Definition 14 (minimum).

## Lemma 483 (compatibility of extrema with affine transformation).

Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $a, b \in \mathbb{R}$. Assume that $a \geqslant 0$. Then, we have

$$
\begin{gather*}
\sup ((a f+b)(X))=a \sup (f(X))+b, \quad \inf ((a f+b)(X))=a \inf (f(X))+b,  \tag{7.114}\\
\sup ((-a f+b)(X))=-a \inf (f(X))+b, \quad \inf ((-a f+b)(X))=-a \sup (f(X))+b
\end{gather*}
$$

Proof. (1). Case $a=1: \sup ((f+b)(X))=\sup (f(X))+b$. Direct consequence of Definition 2 (supremum), and Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined, with $b \in \mathbb{R})$.
(2). $\sup ((a f+b)(X))=a \sup (f(X))+b$.

Direct consequence of (1), and Lemma 5 (supremum is positive scalar multiplicative).
(3). $\inf ((\boldsymbol{a} \boldsymbol{f}+\boldsymbol{b})(\boldsymbol{X}))=\boldsymbol{a} \inf (\boldsymbol{f}(\boldsymbol{X}))+\boldsymbol{b}$. Direct consequence of Lemma 10 (duality infimumsupremum, first with $a f+b$, then with $f$ ), Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined, with $a \stackrel{\text { def. }}{=}-1$, everything is well-defined), and (2).
(4). $\sup ((-\boldsymbol{a} \boldsymbol{f}+\boldsymbol{b})(\boldsymbol{X}))=-\boldsymbol{a} \inf (\boldsymbol{f}(\boldsymbol{X}))+\boldsymbol{b}$. Direct consequence of Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined, with $a \stackrel{\text { def. }}{=}-1$, everything is well-defined), Lemma 10 (duality infimum-supremum, with $a f-b$ ), and (3).
(5). $\inf ((-\boldsymbol{a} \boldsymbol{f}+\boldsymbol{b})(\boldsymbol{X}))=-\boldsymbol{a} \sup (\boldsymbol{f}(\boldsymbol{X}))+\boldsymbol{b}$. Direct consequence of Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined, with $a \stackrel{\text { def. }}{=}-1$, everything is well-defined), Lemma 10 (duality infimum-supremum, with $-a f+b$ ), and (2).

Lemma 484 (equivalent definition of finite infimum). Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $l \in \mathbb{R}$. Then, we have $l=\inf (f(X))$ iff $l$ is a lower bound of $f(X)$, and there exists $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit $l$.

Proof. "Left" implies "right". Assume first that $l=\inf (f(X))$. Then, from Definition 9 (infimum), $l$ is a (finite) lower bound for $f(X)$. Thus, from Lemma 13 (finite infimum discrete), for all $n \in \mathbb{N}$, there exists $x_{n}^{\prime} \in X$ such that $f\left(x_{n}^{\prime}\right)<l+\frac{1}{n+1}$. Let $x_{0} \stackrel{\text { def. }}{=} x_{0}^{\prime}$. For all $n \in \mathbb{N}$, let $x_{n+1} \stackrel{\text { def. }}{=} \arg \min \left(f\left(x_{n}\right), f\left(x_{n}^{\prime}\right)\right)$. Then, for all $n \in \mathbb{N}$, we have

$$
f\left(x_{n+1}\right) \leqslant f\left(x_{n}\right) \leqslant f\left(x_{n}^{\prime}\right)<l+\frac{1}{n+1}
$$

Thus, from the definition of monotone sequence, the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is nonincreasing. Moreover, let $k \in \mathbb{N}$. Let $N \stackrel{\text { def. }}{=} k$. Let $n \in \mathbb{N}$ such that $n \geqslant N$. Then, from ordered field
properties of $\mathbb{R}$, we have $f\left(x_{n}\right) \leqslant f\left(x_{N}\right)=f\left(x_{k}\right)<l+\frac{1}{k+1}$. Hence, from Lemma 265 (equivalent definition of convergent sequence), the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is convergent with limit $l$.
"Right" implies "left". Conversely, assume now that $l$ is a (finite) lower bound of $f(X)$, and that there exists $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is nonincreasing with limit $l$. Let $n \in \mathbb{N}$, and let $\varepsilon_{n} \stackrel{\text { def. }}{=} \frac{1}{2(n+1)}>0$. Then, from Definition 27 (convergent sequence), let $N \in \mathbb{N}$ such that for all $p \geqslant N,\left|f\left(x_{p}\right)-l\right|<\varepsilon_{n}$. Let $x_{n}^{\prime} \stackrel{\text { def. }}{=} x_{N}$. Then, from ordered field properties of $\mathbb{R}$, we have $f\left(x_{n}^{\prime}\right) \leqslant l+\varepsilon_{n}<l+\frac{1}{n+1}$. Hence, from Lemma 13 (finite infimum discrete), we have $l=\inf (f(X))$.
Therefore, we have the equivalence.

Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ). Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $l \in \mathbb{R}$. Then, we have $l=\inf (f(X))$ iff $l$ is a lower bound of $f(X)$, and there exists $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit $l$.

Proof. Case $-\infty \in \boldsymbol{f}(\boldsymbol{X})$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and Definition 9 (infimum, lower bound), $-\infty$ is the only lower bound of $f(X)$. Thus, since $l \in \mathbb{R}$, both propositions " $l=\inf (f(X))$ " and " $l$ is a lower bound of $f(X)$ " are wrong. Hence, we have the equivalence.

Case $\boldsymbol{f}(\boldsymbol{X}) \subset \mathbb{R}$. Direct consequence of Lemma 484 (equivalent definition of finite infimum).
Case $-\infty \notin \boldsymbol{f}(\boldsymbol{X})$, and $\infty \in \boldsymbol{f}(\boldsymbol{X})$. Then, from Lemma 482 (extrema of constant function, contrapositive with $\inf (f(X)) \neq \infty)$, the function cannot be constant of value $\infty$. Thus, since $-\infty \notin f(X)$, let $\tilde{x} \in X$ such that $f(\tilde{x}) \in \mathbb{R}$. Let $\pi_{f}: X \rightarrow X$ be the projection defined by

$$
\pi_{f}(x) \stackrel{\text { def. }}{=} \begin{cases}x & \text { when } f(x) \neq \infty \\ \tilde{x} & \text { otherwise }\end{cases}
$$

Let $\tilde{f} \stackrel{\text { def. }}{=} f \circ \pi_{f}$. Then, by construction, we have $\tilde{f}(X) \subset \mathbb{R}$ and $\tilde{f} \leqslant f$.
Let $l$ be a lower bound of $\tilde{f}(X)$. Then, $l$ is also a lower bound of $f(X)$. Conversely, let $l$ be a lower bound of $f(X)$. Let $x \in X$. Then, from the definitions of $\pi_{f}$ and $\tilde{f}$, we have

$$
l \leqslant f\left(\pi_{f}(x)\right)=\tilde{f}(x)
$$

Thus, $l$ is also a lower bound of $\tilde{f}(X)$. Hence, $f(X)$ and $\tilde{f}(X)$ have the same lower bounds.
Moreover, from Definition 9 (infimum, lower bound), $\inf (f(X))$ is also a lower bound of $\tilde{f}(X)$, and $\inf (\tilde{f}(X))$ is also a lower bound of $f(X)$. Thus, from Definition 9 (infimum, greatest lower bound), we have $\inf (f(X)) \leqslant \inf (\tilde{f}(X))$, and $\inf (\tilde{f}(X)) \leqslant \inf (f(X))$. Hence, from Lemma 313 $\overline{\text { (order }}$ in $\overline{\mathbb{R}}$ is total, antisymmetry), we have $\inf (\tilde{f}(X))=\inf (f(X))$.

Furthermore, let $\left(\tilde{x}_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $X$. Then, from the definition of $\pi_{f}$ and $\tilde{f}$, the sequence $\left(x_{n} \stackrel{\text { def. }}{=} \pi_{f}\left(\tilde{x}_{n}\right)\right)_{n \in \mathbb{N}} \in X$ is such that both sequences $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ and $\left(\tilde{f}\left(\tilde{x}_{n}\right)\right)_{n \in \mathbb{N}}$ are identical. Conversely, assume that $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ is such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is in $\mathbb{R}$. Then, from the definition of $\pi_{f}$ and $\tilde{f}$, both sequences $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ and $\left(\tilde{f}\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ are identical.

Therefore, from what precedes, we have the equivalences $l=\inf (f(X))$ iff $l=\inf (\tilde{f}(X))$ iff $l$ is a lower bound of $\tilde{f}(X)$, and there exists a sequence $\left(\tilde{x}_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(\tilde{f}\left(\tilde{x}_{n}\right)\right)_{n \in \mathbb{N}}$ is nonincreasing with limit $l$ iff $l$ is a lower bound of $f(X)$, and there exists a sequence $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \mathbb{R}$ is nonincreasing with limit $l$

## Lemma 486 (equivalent definition of infimum).

Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $l \in \overline{\mathbb{R}}$. Then, we have $l=\inf (f(X))$ iff $l$ is a lower bound of $f(X)$, and there exists $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit $l$.

Proof. Case $\inf (f(X))=-\infty$.
"Left" implies "right". Assume first that $l=\inf (f(X))=-\infty$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), $l$ is a lower bound of $f(X)$. Moreover, from Lemma 11 (finite infimum, contrapositive), for all $m \in \mathbb{R}$, there exists $x \in X$ such that $f(x)<m$. For all $n \in \mathbb{N}$, let $x_{n}^{\prime} \in X$ such that $f\left(x_{n}^{\prime}\right)<-n$. Let $x_{0} \stackrel{\text { def. }}{=} x_{0}^{\prime}$. For all $n \in \mathbb{N}$, let $x_{n+1} \stackrel{\text { def. }}{=} \arg \min \left(f\left(x_{n}\right), f\left(x_{n+1}^{\prime}\right)\right)$. Then, for all $n \in \mathbb{N}$, we have $f\left(x_{n+1}\right) \leqslant f\left(x_{n}\right) \leqslant f\left(x_{n}^{\prime}\right)<-n$. Thus, from the definition of monotone sequence, the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is nonincreasing. Moreover, let $k \in \mathbb{N}$. Let $N \stackrel{\text { def. }}{=} k$. Let $n \in \mathbb{N}$ such that $n \geqslant N$. Then, from Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity), we have

$$
f\left(x_{n}\right) \leqslant f\left(x_{N}\right)=f\left(x_{k}\right)<-k .
$$

Hence, from Lemma 380 (convergence towards $\pm \infty$ (discrete)), the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is convergent in $\overline{\mathbb{R}}$ with limit $-\infty=l$.
"Right" implies "left". Conversely, assume now that $l$ is a lower bound of $f(X)$. Then, from Definition 9 (infimum, greatest lower bound), and Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), we have $-\infty \leqslant l \leqslant \inf (f(X))=-\infty$. Hence, we have $l=-\infty=\inf (f(X))$.
Case $\inf (f(X))=\infty$.
Then, from Definition 9 (infimum, lower bound), and Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), we have for all $x \in X, \infty=\inf (f(X)) \leqslant f(x) \leqslant \infty$. Which means that $f$ is a constant function of value $\infty$. Thus, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), every extended number is a lower bound of $f(X)=\{\infty\}$. Moreover, for all sequence $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$, the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}}$ is stationary of value $\infty$. Thus, from the definition of monotone sequence, and Lemma 34 (stationary sequence is convergent), $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit $\infty$. For instance, the stationary sequence $(\tilde{x})_{n \in \mathbb{N}}$ for some $\tilde{x} \in X \neq \varnothing$ is such a sequence. Hence, we have the equivalence $l=\infty(=\inf (f(X)))$ iff there exists a sequence $\left(x_{n}\right)_{n \in \mathbb{N}} \in X$ such that the sequence $\left(f\left(x_{n}\right)\right)_{n \in \mathbb{N}} \in \overline{\mathbb{R}}$ is nonincreasing with limit $l$.
Case $\inf (f(X)) \in \mathbb{R}$.
Direct consequence of Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ).
Therefore, we always have the equivalence.

## Lemma 487 (infimum is smaller than supremum).

Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $\inf (f(X)) \leqslant \sup (f(X))$.
Proof. Direct consequence of Definition 9 (infimum, lower bound), Definition 2 (supremum, upper bound), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity).

Lemma 488 (infimum is monotone (set)).
Let $X$ and $Y$ be nonempty sets.
Assume that $Y \subset X$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $\inf (f(X)) \leqslant \inf (f(Y))$.
Proof. Direct consequence of Definition 9 (infimum, lower bound with $X$, then greatest lower bound with $Y$ ), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity).

Lemma 489 (supremum is monotone (set)).
Let $X$ and $Y$ be nonempty sets.
Assume that $Y \subset X$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $\sup (f(Y)) \leqslant \sup (f(X))$.
Proof. Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 488 (infimum is monotone (set)), and Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone).

## Lemma 490 (infimum is monotone (function)). <br> Let $X$ be a nonempty set.

Let $f, g: X \rightarrow \overline{\mathbb{R}}$. Assume that $f \leqslant g$. Then, we have $\inf (f(X)) \leqslant \inf (g(X))$.
Proof. Direct consequence of Definition 9 (infimum, lower bound with $f$, and greatest lower bound with $g$ ), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity).

Lemma 491 (supremum is monotone (function)). Let $X$ be a nonempty set. Let $f, g: X \rightarrow \overline{\mathbb{R}}$. Assume that $f \leqslant g$. Then, we have $\sup (f(X)) \leqslant \sup (g(X))$.

Proof. Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone), and Lemma 490 (infimum is monotone (function), with $-g \leqslant-f$ ).

Lemma 492 (infimum of bounded is bounded). Let $X$ be a nonempty set. Let $a, b \in \overline{\mathbb{R}}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. If $f \leqslant b$, then we have $\inf (f(X)) \leqslant b$. If $a \leqslant f$, then we have $a \leqslant \inf (f(X))$.

Proof. Direct consequence of Lemma 482 (extrema of constant function), and Lemma 490 (infimum is monotone (function)).

Lemma 493 (supremum of bounded is bounded). Let $X$ be a nonempty set. Let $a, b \in \overline{\mathbb{R}}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. If $f \leqslant b$, then we have $\sup (f(X)) \leqslant b$. If $a \leqslant f$, then we have $a \leqslant \sup (f(X))$.

Proof. Direct consequence of Lemma 482 (extrema of constant function), and Lemma 491 (supremum is monotone (function)).

Lemma 494 (compatibility of infimum with absolute value).
Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $|\inf (f(X))| \leqslant \sup (|f(X)|)$.
 fimum is monotone (function)), Lemma 487 (infimum is smaller than supremum), and Lemma 491 (supremum is monotone (function)), we have

$$
\inf (f(X)) \leqslant \inf |f(X)| \leqslant \sup |f(X)|, \quad \text { and } \quad \sup ((-f)(X)) \leqslant \sup |f(X)|
$$

Case $\inf (\boldsymbol{f}(\boldsymbol{X})) \geqslant \mathbf{0}$. Then, from Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ), we have $|\inf (f(X))|=\inf (f(X)) \leqslant \sup |f(X)|$. Case $\inf (\boldsymbol{f}(\boldsymbol{X}))<\mathbf{0}$. Then, from Lemma 10 (duality infimum-supremum), we have $|\inf (f(X))|=-\inf (f(X))=\sup ((-f)(X)) \leqslant \sup |f(X)|$.

Lemma 495 (compatibility of supremum with absolute value).
Let $X$ be a nonempty set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $|\sup (f(X))| \leqslant \sup (|f(X)|)$.
Proof. Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 494 (compatibility of infimum with absolute value, with $-f$ ), and Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).

## Lemma 496 (compatibility of translation with infimum of sequence).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(f_{i}\right)_{i \in I}: X \rightarrow \overline{\mathbb{R}}$. Let $p \in J$. Let $J_{p} \stackrel{\text { def. }}{=}\left[p . . n_{I}\right) \neq \varnothing$. Then, we have $\inf _{i \in I} f_{i} \leqslant \inf _{j \in J_{p}} f_{\varphi(j)}$.

Proof. Direct consequence of Lemma 488 (infimum is monotone (set), with $\varnothing \neq \varphi\left(J_{p}\right) \subset I$ ).

## Lemma 497 (compatibility of translation with supremum of sequence).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(f_{i}\right)_{i \in I}: X \rightarrow \overline{\mathbb{R}}$. Let $p \in J$. Let $J_{p} \stackrel{\text { def. }}{=}\left[p . . n_{I}\right) \neq \varnothing$. Then, we have $\sup _{j \in J_{p}} f_{\varphi(j)} \leqslant \sup _{i \in I} f_{i}$.

Proof. Direct consequence of Lemma 489 (supremum is monotone (set), with $\varnothing \neq \varphi\left(J_{p}\right) \subset I$ ).
Lemma 498 (infimum of nonincreasing sequence).
Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(f_{i}\right)_{i \in I}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{i}\right)_{i \in I}$ is nonincreasing. Let $p \in J$. Let $J_{p} \stackrel{\text { def. }}{=}\left[p . . n_{I}\right) \neq \varnothing$. Then, we have $\inf _{i \in I} f_{i}=\inf _{j \in J_{p}} f_{\varphi(j)}$.

Proof. Let $i \in I$. Let $n \stackrel{\text { def. }}{=} \max \left(\varphi^{-1}(i), p\right)$ (i.e. $n \in J_{p}$ and $\left.i \leqslant \varphi(n)\right)$. Let $x \in X$. Then, from Definition 9 (infimum, lower bound with $X \stackrel{\text { def. }}{=} J_{p}$ and $f \stackrel{\text { def. }}{=}\left(j \mapsto f_{\varphi(j)}(x)\right)$ ), and monotonicity of the $f_{i}$ 's, we have $\inf _{j \in J_{p}} f_{\varphi(j)}(x) \leqslant f_{\varphi(n)}(x) \leqslant f_{i}(x)$. Hence, from Definition 9 (infimum, greatest lower bound with $X \stackrel{\text { def. }}{=} I$ and $\left.f \stackrel{\text { def. }}{=}\left(i \mapsto f_{i}(x)\right)\right)$, we have $\inf _{j \in J_{p}} f_{\varphi(j)}(x) \leqslant \inf _{i \in I} f_{i}(x)$.

Therefore, from Lemma 496 (compatibility of translation with infimum of sequence), we have the equality.

## Lemma 499 (supremum of nondecreasing sequence).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(f_{i}\right)_{i \in I}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{i}\right)_{i \in I}$ is nondecreasing. Let $p \in J$. Let $J_{p} \stackrel{\text { def. }}{=}\left[p . . n_{I}\right) \neq \varnothing$. Then, we have $\sup _{i \in I} f_{i}=\sup _{j \in J_{p}} f_{\varphi(j)}$.

Proof. Direct consequence of Lemma 10 (duality infimum-supremum), and Lemma 498 (infimum of nonincreasing sequence, with $\left.\left(-f_{i}\right)_{i \in I}\right)$.

## Lemma 500 (infimum is limit).

Let $X$ be a set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}}$ is nonincreasing.
Then, $\left(f_{n}\right)_{n \in \mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}$, and we have $\lim _{n \rightarrow \infty} f_{n}=\inf _{n \in \mathbb{N}} f_{n}$.
Proof. Let $x \in X$. Let $m \stackrel{\text { def. }}{=} \inf _{n \in \mathbb{N}} f_{n}(x)$.
Case $\left(f_{n}(x)\right) \equiv \infty$.
Direct consequence of Lemma 34 (stationary sequence is convergent, thus $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is convergent with limit $\infty$ ), and Lemma 482 (extrema of constant function, thus $m=\infty$ ).

Case $\left(f_{n}(x)\right)_{n \in \mathbb{N}} \not \equiv \infty$ is bounded from below in $\mathbb{R}$. Then, $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is finite from some rank, say $N_{1} \in \mathbb{N}$ (and equals to $\infty$ before). Let $\varepsilon \in \mathbb{R}$ such that $\varepsilon>0$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}, \infty$ is the greatest element), and Lemma 11 (finite infimum, with $X \stackrel{\text { def. }}{=} \mathbb{N}$ ), we have $m=\inf _{k \in \mathbb{N}} f_{N_{1}+k}(x)$ is finite, for all $n \in \mathbb{N}, m \leqslant f_{n}(x)$, and there exists $N_{2} \in \mathbb{N}$ such that $f_{N_{2}}(x)<m+\varepsilon$. Let $N \stackrel{\text { def. }}{=} \max \left(N_{1}, N_{2}\right)$. Let $n \in \mathbb{N}$ such that $n \geqslant N$. Then, since $\left(f_{n}(x)\right)$ is nonincreasing, and from Definition 27 (convergent sequence), we have

$$
0 \leqslant f_{n}(x)-m \leqslant f_{N_{2}}-m<\varepsilon
$$

and thus $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is convergent with limit $m$.
Case $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is unbounded from below in $\mathbb{R}$. Then, we have

$$
\forall l \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geqslant N \quad \Longrightarrow \quad f_{N}(x) \leqslant l .
$$

Let $l \in \mathbb{R}$, and $n \in \mathbb{N}$ such that $n \geqslant N$. Then, from nonincreasingness of the sequence, and Lemma 379 (convergence towards $\pm \infty$ ), we have $f_{n}(x) \leqslant f_{N}(x) \leqslant l$, i.e. the sequence is convergent towards $-\infty$. Let $l \in \overline{\mathbb{R}}$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and Definition 9 (infimum, with $X \stackrel{\text { def. }}{=} \mathbb{N}$ ), for all $n \in \mathbb{N}$, we have $-\infty \leqslant f_{n}(x)$, and $l \leqslant f_{n}(x)$ for all $n \in \mathbb{N}$ implies $l \notin \overline{\mathbb{R}}$ and $l \neq \infty$, thus $l=\infty$, i.e. $m=-\infty$.

Therefore, the sequence is always convergent, and the limit is the infimum.

## Lemma 501 (supremum is limit).

Let $X$ be a set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}}$ is nondecreasing.
Then, $\left(f_{n}\right)_{n \in \mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}$, and we have $\lim _{n \rightarrow \infty} f_{n}=\sup _{n \in \mathbb{N}} f_{n}$.
Proof. Direct consequence of Lemma 10 (duality infimum-supremum), Lemma 500 (infimum is limit, with $\left(-f_{n}\right)_{n \in \mathbb{N}}$ ), and Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous, with left operand equals the constant function of value -1 ).

Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum). Let $I \subset \mathbb{N}$. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(a_{i}\right)_{i \in I} \in \overline{\mathbb{R}}_{+}$. Then, $\sum_{i \in I} a_{i}=\sup _{n \in J} \sum_{j \in[0 . . n]} a_{\varphi(j)}$.

Proof. Let $A \stackrel{\text { def. }}{=}\left\{\sum_{j \in[0 . . n]} a_{\varphi(j)} \mid n<\operatorname{card}(I)\right\}$, and $B \stackrel{\text { def. }}{=}\left\{\sum_{k \in K} a_{k} \mid K \subset I \wedge \operatorname{card}(K)<\infty\right\}$.
From Lemma 488 (infimum is monotone (set), with $A \subset B$ ), we have sup $A \leqslant \sup B$.
Let $K$ be a finite subset of $I$. Let $k_{M} \stackrel{\text { def. }}{=} \max (K) \in I$. Then, from the fact that an injective function between two sets of the same cardinality is bijective (thus $\varphi$ is bijective), let $n \stackrel{\text { def. }}{=} \varphi^{-1}\left(k_{M}\right) \in J$. Thus, from Definition 2 (supremum, upper bound for $\sup A$ ), we have $K \subset[0, n]$, i.e. $\sum_{k \in K} a_{k} \leqslant \sum_{j \in[0 . . n]} a_{\varphi(j)} \leqslant \sup A$. Hence, from Definition 2 (supremum, least upper bound for $\sup B$ ), we have $\sup B \leqslant \sup A$.

Therefore, from Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$), we have $\sum_{i \in I} a_{i}=\sup B=\sup A$.

## Lemma 503 (constant sum of monotone nonnegative functions).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(f_{i}\right)_{i \in I},\left(g_{i}\right)_{i \in I}: X \rightarrow \overline{\mathbb{R}}_{+}$. Let $\alpha \in \overline{\mathbb{R}}_{+}$. Assume that $\left(f_{i}\right)_{i \in I}$ is nondecreasing, $\left(g_{i}\right)_{i \in I}$ is nonincreasing, and that for all $i \in I$, we have $\alpha=f_{i}+g_{i}$. Then, we have $\alpha=\sup _{i \in I} f_{i}+\inf _{i \in I} g_{i}$.

Proof. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. We have $n_{I}>0$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$.

Case $\operatorname{card}(\boldsymbol{I})=\infty$. Direct consequence of Lemma 501 (supremum is limit, with $\left.\left(f_{\varphi(n)}\right)_{n \in \mathbb{N}}\right)$, Lemma 500 (infimum is limit, with $\left(g_{\varphi(n)}\right)_{n \in \mathbb{N}}$ ), and Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous).
Case $\boldsymbol{\operatorname { c a r d }}(\boldsymbol{I})<\infty$. Direct consequence of Lemma 8 (equivalent definition of maximum, with $X \stackrel{\text { def. }}{=} I$, for each $x \in X$ ), Lemma 15 (equivalent definition of minimum, with $X \stackrel{\text { def. }}{=} I$, for each $x \in X$ ), Definition 7 (maximum, $\underline{\sup }_{i \in I} f_{i}=\max _{i \in I} f_{i}=f_{\varphi\left(n_{I}-1\right)}$ ), and Definition 14 (minimum, $\left.\underline{\inf }_{i \in I} g_{i}=\min _{i \in I} g_{i}=g_{\varphi\left(n_{I}-1\right)}\right)$.

### 7.5.6 Limit inferior, limit superior

Lemma 504 (limit inferior). Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, for all $x \in X,\left(\inf _{p \in \mathbb{N}} f_{n+p}(x)\right)_{n \in \mathbb{N}}$ is nondecreasing, and we have

$$
\begin{equation*}
\forall x \in X, \quad \lim _{n \rightarrow \infty}\left(\inf _{p \in \mathbb{N}} f_{n+p}(x)\right)=\sup _{n \in \mathbb{N}}\left(\inf _{p \in \mathbb{N}} f_{n+p}(x)\right) \in \overline{\mathbb{R}} \tag{7.116}
\end{equation*}
$$

The limit inferior of the sequence is the function $X \rightarrow \overline{\mathbb{R}}$ defined by

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x) \stackrel{\text { def. }}{=} \lim _{n \rightarrow \infty}\left(\inf _{p \in \mathbb{N}} f_{n+p}(x)\right) \tag{7.117}
\end{equation*}
$$

Proof. Direct consequence of Lemma 488 (infimum is monotone (set)), and Lemma 314 ( $\overline{\mathbb{R}}$ is complete, a nondecreasing sequence is convergent and its limit is its least upper bound).

Lemma 505 (limit inferior is $\infty$ ). Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Let $x \in X$. Assume that $\liminf _{n \rightarrow \infty} f_{n}(x)=\infty$. Then, we have $\lim _{n \rightarrow \infty} f_{n}(x)=\infty$

Proof. Let $a \in \mathbb{R}$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), we have

$$
a<\liminf _{n \rightarrow \infty} f_{n}(x)
$$

Thus, from Lemma 504 (limit inferior), and the definition of the limit, there exists $N \in \mathbb{N}$ such that $a<\inf _{p \in \mathbb{N}} f_{N+p}(x)$. Hence, from Definition 9 (infimum, lower bound), we have for all $p \in \mathbb{N}, a<\inf _{p \in \mathbb{N}} f_{N+p}(x) \leqslant f_{N+p}(x)$. Therefore, from the definition of the limit, we have $\lim _{n \rightarrow \infty} f_{n}(x)=\infty$.

Lemma 506 (equivalent definition of the limit inferior).
Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$.
Then, for all $x \in X, \liminf _{n \rightarrow \infty} f_{n}(x)$ is the smallest cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Proof. Let $x \in X$. For all $n \in \mathbb{N}$, let $F_{n}^{-}(x) \stackrel{\text { def. }}{=} \inf _{p \in \mathbb{N}} f_{n+p}(x)$.
Let $\underline{f}(x) \stackrel{\text { def. }}{=} \liminf _{n \rightarrow \infty} f_{n}(x)=\lim _{n \rightarrow \infty} F_{n}^{-}(x) \in \overline{\mathbb{R}}$.
Let us first show that $\underline{f}(x)$ is a cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Case $\underline{\boldsymbol{f}}(\boldsymbol{x})$ is finite. Let $\varepsilon>0$. Let $M \in \mathbb{N}$. Then, from Lemma 504 (limit inferior, $\left(F_{n}^{-}(x)\right)_{n \in \mathbb{N}}$ is nondecreasing), and Definition 27 (convergent sequence, with $\left(F_{n}^{-}(x)\right)_{n \in \mathbb{N}}$ ), there exists $N \in \mathbb{N}$ such that we have

$$
\forall k \in[N . . \infty), \quad \underline{f}(x)-\varepsilon \leqslant F_{k}^{-}(x) \leqslant \underline{f}(x) .
$$

Let $K \stackrel{\text { def. }}{=} \max (M, N)$. Then, from Lemma 11 (finite infimum, for $F_{K}^{-}$), there exists $k \geqslant K \geqslant M$ such that

$$
\underline{f}(x)-\varepsilon \leqslant F_{K}^{-}(x) \leqslant f_{k}(x)<F_{K}^{-}(x)+\varepsilon \leqslant \underline{f}(x)+\varepsilon .
$$

Hence, from Definition 267 (cluster point), $\underline{f}(x)$ is a cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Case $\underline{f}(\boldsymbol{x})=-\infty$. From Lemma 504 (limit inferior), and Definition 2 (supremum, upper bound), we have for all $n \in \mathbb{N}, F_{n}^{-}(x)=-\infty$. Let $n \in \mathbb{N}$. Let $a \in \mathbb{R}$. Then, $F_{n}^{-}(x)<a$. Thus, from Definition 9 (infimum, greatest lower bound, contrapositive), there exists $P \in \mathbb{N}$ such that $f_{n+P}(x)<a$. Hence, from Lemma 382 (cluster point at $\pm \infty$ (discrete)), $\underline{f}(x)=-\infty$ is a cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Case $\underline{\boldsymbol{f}}(\boldsymbol{x})=\infty$. Then, from Lemma 505 (limit inferior is $\infty$ ), we have $\lim _{n \rightarrow \infty} f_{n}(x)=\infty$, and from uniqueness of the cluster point of a convergent sequence, $\underline{f}(x)=\infty$ is a cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.

Now, let $f(x)$ be a cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$. Let us show that $\underline{f}(x) \leqslant f(x)$. Case $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{\infty}$. Then, from Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), we have $\underline{f}(x) \leqslant f(x)$.

Case $\boldsymbol{f}(\boldsymbol{x})<\infty$. Let $a \in \mathbb{R}$ such that $f(x)<a$. Let $\varepsilon \stackrel{\text { def. }}{=} \frac{a-f(x)}{2}>0$. Let $n \in \mathbb{N}$. Then, from Definition 267 (cluster point), there exists $k \geqslant n$ such that $f_{k}(x) \leqslant f(x)+\varepsilon<a$. Thus, from Definition 9 (infimum, lower bound), $F_{n}^{-}(x)<a$. Hence, from Lemma 381 (limit is monotone, with the stationary sequence $v_{n}=a, \forall n \in \mathbb{N}$ ), Lemma 34 (stationary sequence is convergent), and Lemma 504 (limit inferior, $\underline{\lim }_{n \rightarrow \infty} F_{n}^{-}(x)$ exists), we have $\underline{f}(x)=\lim _{n \rightarrow \infty} F_{n}^{-}(x) \leqslant a$. Since this is true for all $a>f(x)$, we also have $\underline{f}(x) \leqslant f(x)$.
Therefore, for all $x \in X, \underline{f}(x)$ is the smallest cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Lemma 507 (limit inferior is invariant by translation).
Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
\forall k \in \mathbb{N}, \quad \forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{k+n}(x)=\liminf _{n \rightarrow \infty} f_{n}(x) \tag{7.118}
\end{equation*}
$$

Proof. Direct consequence of Lemma 506 (equivalent definition of the limit inferior), Definition 267 (cluster point), and Lemma 383 (limit is invariant by translation).

## Lemma 508 (limit inferior is monotone).

Let $X$ be a nonempty set.
Let $\left(f_{n}\right)_{n \in \mathbb{N}},\left(g_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}} \leqslant\left(g_{n}\right)_{n \in \mathbb{N}}$ from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad f_{n}(x) \leqslant g_{n}(x) \tag{7.119}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x) \leqslant \liminf _{n \rightarrow \infty} g_{n}(x) \tag{7.120}
\end{equation*}
$$

Proof. Direct consequence of Lemma 507 (limit inferior is invariant by translation, with $k \stackrel{\text { def. }}{=} N$ ), Lemma 504 (limit inferior), Lemma 490 (infimum is monotone (function), with $X \stackrel{\text { def. }}{=} \mathbb{N}$ ), and Lemma 491 (supremum is monotone (function), with $X \stackrel{\text { def. }}{=} \mathbb{N}$ ).

## Lemma 509 (limit superior).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, for all $x \in X,\left(\sup _{p \in \mathbb{N}} f_{n+p}(x)\right)_{n \in \mathbb{N}}$ is nonincreasing, and we have

$$
\begin{equation*}
\forall x \in X, \quad \lim _{n \rightarrow \infty}\left(\sup _{p \in \mathbb{N}} f_{n+p}(x)\right)=\inf _{n \in \mathbb{N}}\left(\sup _{p \in \mathbb{N}} f_{n+p}(x)\right) \quad \text { in } \overline{\mathbb{R}} . \tag{7.121}
\end{equation*}
$$

The limit superior of the sequence is the function $X \rightarrow \overline{\mathbb{R}}$ defined by

$$
\begin{equation*}
\forall x \in X, \quad \limsup _{n \rightarrow \infty} f_{n}(x) \stackrel{\text { def. }}{=} \lim _{n \in \mathbb{N}}\left(\sup _{p \in \mathbb{N}} f_{n+p}(x)\right) . \tag{7.122}
\end{equation*}
$$

Proof. Let $x \in X$. For all $n \in \mathbb{N}$, let $F_{n}(x) \stackrel{\text { def. }}{=}\left\{f_{n+p}(x) \mid p \in \mathbb{N}\right\}$. Then, we have

$$
\forall n \in \mathbb{N}, \quad \sup F_{n}(x)=\sup _{p \in \mathbb{N}} f_{n+p}(x) \quad \wedge \quad F_{n}(x) \subset F_{n+1}(x) \subset \overline{\mathbb{R}}
$$

Thus, the sequence $\left(\sup F_{n}(x)\right)_{n \in \mathbb{N}}$ is nonincreasing in $\overline{\mathbb{R}}$. Therefore, from Lemma $314(\overline{\mathbb{R}}$ is complete), the sequence is convergent and its limit is its greatest lower bound $\inf _{n \in \mathbb{N}}\left(\sup F_{n}(x)\right)$.

## Lemma 510 (duality limit inferior-limit superior).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \limsup _{n \rightarrow \infty} f_{n}(x)=-\liminf _{n \rightarrow \infty}\left(-f_{n}(x)\right) \tag{7.123}
\end{equation*}
$$

Proof. Direct consequence of Lemma 509 (limit superior), Lemma 504 (limit inferior), and Lemma 10 (duality infimum-supremum, with $X \stackrel{\text { def. }}{=} \mathbb{N}$ ).

## Lemma 511 (equivalent definition of limit superior).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$.
Then, for all $x \in X, \lim \sup _{n \rightarrow \infty} f_{n}(x)$ is the largest cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.
Proof. Let $x \in X$. Let $\bar{f}(x) \stackrel{\text { def. }}{=} \lim \sup _{n \rightarrow \infty} f_{n}(x)$. Then, from Lemma 510 (duality limit inferiorlimit superior), and Lemma 506 (equivalent definition of the limit inferior), $-\bar{f}(x)$ is the smallest cluster point of the sequence $\left(-f_{n}(x)\right)_{n \in \mathbb{N}}$. Therefore, from Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous, with -1 as the first operand), and Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone), $\bar{f}(x)$ is the largest cluster point of the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$.

Lemma 512 (limit inferior is smaller than limit superior).
Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x) \leqslant \limsup _{n \rightarrow \infty} f_{n}(x) \tag{7.124}
\end{equation*}
$$

Proof. Direct consequence of Lemma 506 (equivalent definition of the limit inferior), Lemma 511 (equivalent definition of limit superior), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity).

## Lemma 513 (limit superior is monotone).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}},\left(g_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}} \leqslant\left(g_{n}\right)_{n \in \mathbb{N}}$ from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad f_{n}(x) \leqslant g_{n}(x) \tag{7.125}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \limsup _{n \rightarrow \infty} f_{n}(x) \leqslant \limsup _{n \rightarrow \infty} g_{n}(x) . \tag{7.126}
\end{equation*}
$$

Proof. Direct consequence of Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone), Lemma 508 (limit inferior is monotone, with $-g_{n} \leqslant-f_{n}$ ), and Lemma 510 (duality limit inferior-limit superior).

## Lemma 514 (compatibility of limit inferior with absolute value).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
\forall x \in X, \quad\left|\liminf _{n \rightarrow \infty} f_{n}(x)\right| \leqslant \limsup _{n \rightarrow \infty}\left|f_{n}(x)\right| . \tag{7.127}
\end{equation*}
$$

Proof. From Lemma 494 (compatibility of infimum with absolute value, with $X \stackrel{\text { def. }}{=}[n . . \infty) \times X$ ), for all $x \in X$, for all $n \in \mathbb{N}$, we have

$$
\left|\inf _{p \in \mathbb{N}} f_{n+p}(x)\right| \leqslant \sup _{p \in \mathbb{N}}\left|f_{n+p}(x)\right| .
$$

Therefore, from Lemma 504 (limit inferior), Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous), Lemma 381 (limit is monotone, with $u_{n} \stackrel{\text { def. }}{=}\left|\inf _{p \in \mathbb{N}} f_{n+p}(x)\right|$ and $v_{n} \stackrel{\text { def. }}{=} \sup _{p \in \mathbb{N}}\left|f_{n+p}(x)\right|$, and $\left(u_{n}\right)_{n \in \mathbb{N}}$ and $\left(v_{n}\right)_{n \in \mathbb{N}}$ are convergent in $\overline{\mathbb{R}}$ ), and Lemma 509 (limit superior), we have
$\left|\liminf _{n \rightarrow \infty} f_{n}(x)\right|=\left|\lim _{n \rightarrow \infty} \inf _{p \in \mathbb{N}} f_{n+p}(x)\right|=\lim _{n \rightarrow \infty}\left|\inf _{p \in \mathbb{N}} f_{n+p}(x)\right| \leqslant \lim _{n \rightarrow \infty} \sup _{p \in \mathbb{N}}\left|f_{n+p}(x)\right|=\limsup _{n \rightarrow \infty}\left|f_{n}(x)\right|$.

Lemma 515 (compatibility of limit superior with absolute value).
Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
\forall x \in X, \quad\left|\limsup _{n \rightarrow \infty} f_{n}(x)\right| \leqslant \limsup _{n \rightarrow \infty}\left|f_{n}(x)\right| . \tag{7.128}
\end{equation*}
$$

Proof. Direct consequence of Lemma 510 (duality limit inferior-limit superior), Lemma 514 (compatibility of limit inferior with absolute value, with $-f_{n}$ ), and Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).

Definition 516 (pointwise convergence).
Let $X$ be a nonempty set. A sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ of functions $X \rightarrow \overline{\mathbb{R}}$ is said pointwise convergent iff for all $x \in X,\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is convergent in $\mathbb{R}$; it is denoted $f_{n} \underset{n \rightarrow \infty}{\longrightarrow} f$.

## Lemma 517 (limit inferior and limit superior of pointwise convergent).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$.
Assume that the sequence is pointwise convergent. Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x)=\limsup _{n \rightarrow \infty} f_{n}(x)=\lim _{n \rightarrow \infty} f_{n}(x) \tag{7.129}
\end{equation*}
$$

Proof. Direct consequence of Definition 516 (pointwise convergence), Lemma 506 (equivalent definition of the limit inferior), Lemma 511 (equivalent definition of limit superior), and uniqueness of the cluster point of a convergent sequence.

## Lemma 518 (limit inferior bounded from below).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Let $m \in \overline{\mathbb{R}}$.
Assume that the sequence is bounded by $m$ from below from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad m \leqslant f_{n}(x) \tag{7.130}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad m \leqslant \liminf _{n \rightarrow \infty} f_{n}(x) \tag{7.131}
\end{equation*}
$$

Proof. Direct consequence of Lemma 508 (limit inferior is monotone), Lemma 34 (stationary sequence is convergent), Definition 33 (stationary sequence, constant sequence is stationary ( $N=0$ ), , and Lemma 517 (limit inferior and limit superior of pointwise convergent, with the constant sequence of value $m$ ).

## Lemma 519 (limit inferior bounded from above).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Let $M \in \overline{\mathbb{R}}$.
Assume that the sequence is bounded by $M$ from above from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad f_{n}(x) \leqslant M \tag{7.132}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x) \leqslant M \tag{7.133}
\end{equation*}
$$

Proof. Direct consequence of Lemma 508 (limit inferior is monotone), Lemma 34 (stationary sequence is convergent), Definition 33 (stationary sequence, constant sequence is stationary $(N=0)$ ), and Lemma 517 (limit inferior and limit superior of pointwise convergent, with the constant sequence of value $M$ ).

## Lemma 520 (limit superior bounded from below)

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Let $m \in \overline{\mathbb{R}}$.
Assume that the sequence is bounded by $m$ from below from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad m \leqslant f_{n}(x) \tag{7.134}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad m \leqslant \limsup _{n \rightarrow \infty} f_{n}(x) . \tag{7.135}
\end{equation*}
$$

Proof. Direct consequence of Lemma 519 (limit inferior bounded from above, with $f_{n} \stackrel{\text { def. }}{=}-f_{n}$ and $M \stackrel{\text { def. }}{=}-m$ ), and Lemma 510 (duality limit inferior-limit superior).

## Lemma 521 (limit superior bounded from above).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Let $M \in \overline{\mathbb{R}}$.
Assume that the sequence is bounded by $M$ from above from some rank:

$$
\begin{equation*}
\exists N \in \mathbb{N}, \forall n \in[N . . \infty), \forall x \in X, \quad f_{n}(x) \leqslant M \tag{7.136}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\forall x \in X, \quad \limsup _{n \rightarrow \infty} f_{n}(x) \leqslant M \tag{7.137}
\end{equation*}
$$

Proof. Direct consequence of Lemma 518 (limit inferior bounded from below, with $f_{n} \stackrel{\text { def. }}{=}-f_{n}$ and $m \stackrel{\text { def. }}{=}-M$ ), and Lemma 510 (duality limit inferior-limit superior).

## Lemma 522 (limit inferior, limit superior and pointwise convergence).

Let $X$ be a nonempty set. Let $\left(f_{n}\right)_{n \in \mathbb{N}}: X \rightarrow \overline{\mathbb{R}}$. Assume that

$$
\begin{equation*}
\forall x \in X, \quad \limsup _{n \rightarrow \infty} f_{n}(x) \leqslant \liminf _{n \rightarrow \infty} f_{n}(x) \tag{7.138}
\end{equation*}
$$

Then, the sequence is pointwise convergent and

$$
\begin{equation*}
\forall x \in X, \quad \liminf _{n \rightarrow \infty} f_{n}(x)=\limsup _{n \rightarrow \infty} f_{n}(x)=\lim _{n \rightarrow \infty} f_{n}(x) . \tag{7.139}
\end{equation*}
$$

Proof. Let $x \in X$. Then, from Lemma 512 (limit inferior is smaller than limit superior), we have $\liminf _{n \rightarrow \infty} f_{n}(x)=\lim \sup _{n \rightarrow \infty} f_{n}(x) \stackrel{\text { def. }}{=} f(x) \in \overline{\mathbb{R}}$.
Case $\boldsymbol{f}(\boldsymbol{x})=\infty$. Then, from Lemma 505 (limit inferior is $\infty$ ), we have

$$
\lim _{n \rightarrow \infty} f_{n}(x)=\infty=f(x)
$$

Case $\boldsymbol{f}(\boldsymbol{x})=-\infty$. Then, from Lemma 510 (duality limit inferior-limit superior), and Lemma 505 (limit inferior is $\infty$ ), we have $\lim _{n \rightarrow \infty} f_{n}(x)=-\infty=f(x)$.
Case $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}$. Let $\varepsilon>0$. Then, we have

$$
f(x)-\varepsilon<\liminf _{n \rightarrow \infty} f_{n}(x)=f(x)=\limsup _{n \rightarrow \infty} f_{n}(x)<f(x)+\varepsilon .
$$

Thus, from Lemma 506 (equivalent definition of the limit inferior), there exists $N^{-} \in \mathbb{N}$ such that for all $n \geqslant N^{-}$, we have $f(x)-\varepsilon<f_{n}(x)$, and from Lemma 511 (equivalent definition of limit superior), there exists $N^{+} \in \mathbb{N}$ such that for all $n \geqslant N^{+}$, we have $f_{n}(x)<f(x)+\varepsilon$.
Let $N \stackrel{\text { def. }}{=} \max \left(N^{-}, N^{+}\right)$. Then, for all $n \geqslant N$, we have $f(x)-\varepsilon<f_{n}(x)<f(x)+\varepsilon$. Therefore, from Definition 27 (convergent sequence), the sequence $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is convergent with limit $f(x)$.

### 7.5.7 Truncating a numeric function

Remark 523 . Note that in the next lemma, masking must be denoted acting on the left $\left(\mathbb{1}_{A} f\right)$ when $E$ is of dimension greater than 1 , but it is often on the right $\left(f \mathbb{1}_{A}\right)$ in the scalar case.

## Lemma 524 (restriction is masking).

Let $X$ be a set. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$, and $E$ be a vector space over $\mathbb{K}$, or let $E \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$.
Let $A \subset Y \subset X$. Let $f: Y \rightarrow E$ and $\hat{f}: X \rightarrow E$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $\left(\mathbb{1}_{A} \hat{f}\right)_{\left.\right|_{A}}=f_{\left.\right|_{A}}$ and $\left(\mathbb{1}_{A} f\right)_{\left.\right|_{A^{c}}} \equiv 0_{E}$.
Proof. Direct consequence of the definition of the indicator function, and the definition of restriction of function.

## Lemma 525 (masking is $\sigma$-additive).

Let $X$ be a set. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$, and $E$ be a vector space over $\mathbb{K}$, or let $E \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$.
Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \subset X$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow E$.
Then, we have $\mathbb{1}_{A}=\sum_{i \in I} \mathbb{1}_{A_{i}}$, and $\mathbb{1}_{A} f=\sum_{i \in I} \mathbb{1}_{A_{i}} f$.
Proof. Direct consequence of the formula for the indicator of disjoint union (sum), Definition 209 ( $p s e u d o p a r t i t i o n, ~ e i t h e r ~ x \notin A$, or $x$ belongs to a single $A_{i}$ ), the definition of indicator function (either the identity is $0=0$, or the sums reduce to the single term $\left.\mathbb{1}_{A_{i}}(x) f(x)=f(x)\right)$.

## Definition 526 (finite part).

Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. The finite part of $f$ is the function $f \mathbb{1}_{f^{-1}(\mathbb{R})}$.

## Lemma 527 (finite part is finite).

Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, the finite part of $f$ is finite.
Proof. Direct consequence of Definition 526 (finite part), Definition 312 (extended real numbers, $\overline{\mathbb{R}}, \overline{\mathbb{R}}=\{ \pm \infty\} \uplus \mathbb{R})$, properties of inverse image $\left(X=f^{-1}(\overline{\mathbb{R}})=f^{-1}( \pm \infty) \uplus f^{-1}(\mathbb{R})\right)$, and the definition of the indicator function.

## Definition 528 (nonnegative and nonpositive parts).

Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. The functions $f^{+} \stackrel{\text { def. }}{=} \max (f, 0)$ and $f^{-} \stackrel{\text { def. }}{=} \max (-f, 0)$ are respectively called the nonnegative part of $f$ and the nonpositive part of $f$.

## Lemma 529 (equivalent definition of nonnegative and nonpositive parts).

Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f^{+}=f \mathbb{1}_{f^{-1}\left(\overline{\mathbb{R}}_{+}\right)}$and $f^{-}=-f \mathbb{1}_{f^{-1}\left(\overline{\mathbb{R}}_{-}\right)}$.
Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts), Definition 7 (maximum), and the definition of the indicator function.

Lemma 530 (nonnegative and nonpositive parts are nonnegative).
Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f^{+}, f^{-} \geqslant 0$.
Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts), and Definition 7 (maximum).

Lemma 531 (nonnegative and nonpositive parts are orthogonal).
Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $x \in X$. Then, we have

$$
\begin{equation*}
f^{+}(x)=0\left(\text { and } f^{-}(x)=-f(x)\right) \quad \vee \quad f^{-}(x)=0\left(\text { and } f^{+}(x)=f(x)\right) . \tag{7.140}
\end{equation*}
$$

Proof. Direct consequence of Definition 7 (maximum), Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and the partition of $\mathbb{R}$ into negative, zero, and positive numbers.

Lemma 532 (decomposition into nonnegative and nonpositive parts).
Let $X$ be a set. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f=f^{+}-f^{-}$and $|f|=f^{+}+f^{-}$.
Proof. Direct consequence of Lemma 531 (nonnegative and nonpositive parts are orthogonal, $\underline{f}^{+}$ and $f^{-}$cannot take value $\infty$ at the same point), Definition 317 (addition in $\overline{\mathbb{R}}$ ), and Definition $\overline{339}$ (absolute value in $\overline{\mathbb{R}}$ ).

Lemma 533 (compatibility of nonpositive and nonnegative parts with addition).
Let $X$ be a set. Let $f, g: X \rightarrow \overline{\mathbb{R}}$ such that their sum is well-defined. Then, we have

$$
\begin{equation*}
(f+g)^{+}+f^{-}+g^{-}=(f+g)^{-}+f^{+}+g^{+} . \tag{7.141}
\end{equation*}
$$

Proof. Let $x \in X$.
Case $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ finite. From Lemma 532 (decomposition into nonnegative and nonpositive parts, with $f+g$, then $f$ and $g$ ), we have

$$
(f+g)^{+}(x)-(f+g)^{-}(x)=f(x)+g(x)=\left(f^{+}(x)-f^{-}(x)\right)+\left(g^{+}(x)-g^{-}(x)\right)
$$

Then, from Definition 528 (nonnegative and nonpositive parts), $(f+g)^{+}(x),(f+g)^{-}(x), f^{+}(x)$, $f^{-}(x), g^{+}(x)$ and $g^{-}(x)$, are finite. Hence, from abelian additive group properties of $\mathbb{R}$, we have

$$
(f+g)^{+}(x)+f^{-}(x)+g^{-}(x)=(f+g)^{-}(x)+f^{+}(x)+g^{+}(x) .
$$

Case $\boldsymbol{f}(\boldsymbol{x})$ or $\boldsymbol{g}(\boldsymbol{x})$ is $\boldsymbol{\infty}$. Then, from Definition 317 (addition in $\overline{\mathbb{R}}$, rule 3 cannot occur), and Lemma 531 (nonnegative and nonpositive parts are orthogonal), we have

$$
(f+g)^{+}(x)=f(x)+g(x)=\infty, \quad \text { and } \quad\left(f^{+}(x)=\infty \quad \vee \quad g^{+}(x)=\infty\right)
$$

Thus, from Lemma 530 (nonnegative and nonpositive parts are nonnegative), Lemma 397 (infinitysum property in $\overline{\mathbb{R}}_{+}$), Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative), and Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$ is commutative), we have

$$
(f+g)^{+}(x)+f^{-}(x)+g^{-}(x)=\infty, \quad \text { and } \quad(f+g)^{-}(x)+f^{+}(x)+g^{+}(x)=\infty .
$$

Hence, property (7.141) is satisfied.
Case $f(x)$ or $\boldsymbol{g}(\boldsymbol{x})$ is $-\infty$. Same reasoning with the nonpositive parts.
Therefore, we always have the identity.
Lemma 534 (compatibility of nonpositive and nonnegative parts with mask).
Let $X$ be a set. Let $A \subset X$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $\left(f \mathbb{1}_{A}\right)^{ \pm}=f^{ \pm} \mathbb{1}_{A}$.
Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts), and nonnegativeness of the indicator function.

Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction). Let $X$ be a set. Let $A \subset X$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $\left(f_{\left.\right|_{A}}\right)^{ \pm}=f^{ \pm}{ }_{\left.\right|_{A}}$.

Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts), and compatibility of restriction of function with $\mathrm{min} / \mathrm{max}$.

## Chapter 8

## Subset system

## Contents

8.1 Basic properties ..... 108
8.1.1 Finite properties ..... 110
8.1.2 Countable properties ..... 112
8.2 Generated subset system ..... 118
8.3 Pi-system ..... 122
8.4 Ring of sets ..... 124
8.5 Algebra of sets ..... 129
8.6 Monotone class ..... 134
8.7 Lambda-system ..... 137
8.8 Sigma-algebra ..... 141
8.9 Dynkin $\pi-\lambda$ theorem ..... 147
8.10 Monotone class theorem ..... 148

Remark 536. Subset systems of a set $X$ are subsets of its power set $\mathcal{P}(X)$.

Interesting subset systems are those closed under a family of set operations such as complement, union or intersection. The simplest one is $\pi$-system (see Section 8.3) that is nonempty and only closed under intersection. The most elaborate one is $\sigma$-algebra (see Section 8.8) that is somehow closed under all set operations, including countable union and intersection.

Measure theory and Lebesgue integration are usually based on $\sigma$-algebra, that is the most general subset system concept with which one can build desirable properties such as $\sigma$-additivity for measures, and linearity and powerful convergence theorems for the integral. Thus, the last two sections of this chapter are dedicated to ways to extend weaker subset systems, such as algebra of sets or $\lambda$-system, into a $\sigma$-algebra.

Note that when the full set is not required to be measurable, $\sigma$-rings (i.e. rings that are also closed under countable union) can be used instead of $\sigma$-algebras, and $\delta$-rings (i.e. rings that are also closed under countable intersection) when subsets of infinite measure are not allowed.

### 8.1 Basic properties

Remark 537. Mini-language to represent properties of subset systems under a compact form. All the following keywords contain an uppercase letter.
$N e, w E, w F$ are for nonemptiness, content of the empty set, and of the full set;
$C, D^{\prime}, D, s D$ are for closedness under complement, local complement, set difference (or relative complement), and symmetric difference (see definitions in Chapter 2);
$U d, U, I$ are for closedness under (binary) disjoint union, union, and intersection;
$P t f C, P t f D^{\prime}, P t f D$ are for the capabilities of expressing any complement, local complement, or set difference, as a finite disjoint union (i.e. a pseudopartition) of elements of the same subset system;
$U d f, U f$, If are for closedness under finite disjoint union, and finite union and intersection;
$U d c, U m c, I m c, U c, I c$ are for closedness under countable disjoint union, countable monotone union and intersection, and countable union and intersection;
$e_{1} ; e_{2} \ldots$ denotes a sequence of expressions;
[ $e_{1} \mid e_{2} \ldots$ ] denotes syntactic alternative: the complete expression is true for each case; for instance, the expression $e_{1}\left(\left[e_{2} \mid e_{3}\right]\right)$ is equivalent to $e_{1}\left(e_{2}\right) ; e_{1}\left(e_{3}\right)$.

Compact formulas express properties of subset systems, they should be understood as universally quantified. For instance, the first item of the compact formula of Lemma 538 should be read as for all $\mathcal{S} \subset \mathcal{P}(X), w E(\mathcal{S})$ implies $N e(\mathcal{S})$, and $w F(\mathcal{S})$ implies $N e(\mathcal{S})$.

## Lemma 538 (nonempty and with empty or full).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
If $\mathcal{S}$ contains either the empty set or the full set $X$, then it is nonempty.
Assume that $\mathcal{S}$ is nonempty and closed under complement. Then, it contains the empty set if it is closed under intersection, and it contains the full set if it is closed under union.
Assume that $\mathcal{S}$ is nonempty and closed under set difference. Then, it contains the empty set.
Compact form: $[w E \mid w F] \Rightarrow N e ;[C \wedge I \mid D] \Rightarrow N e \Rightarrow w E ; C \Rightarrow U \Rightarrow N e \Rightarrow w F$.
Proof. Direct consequence of the identities $A \cap A^{c}=A \backslash A=\varnothing$, and $A \cup A^{c}=X$.

## Lemma 539 (with empty and full).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it contains the empty set iff it contains the full set.
Compact form: $C \Rightarrow w F \Leftrightarrow w E$.
Proof. Direct consequence of the identities $\varnothing^{c}=X$, and $X^{c}=\varnothing$.
Remark 540.
The local complement is the set difference when the first operand contains the second one.

Lemma 541 (closedness under local complement and complement). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set and is closed under local complement. Then, it is closed under complement.
Compact form: $w F \Rightarrow D^{\prime} \Rightarrow C$.
Proof. Direct consequence of the identity $A^{c}=\boldsymbol{X} \backslash \boldsymbol{A}$.

## Lemma 542 (closedness under disjoint union and local complement).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it is closed under disjoint union iff it is closed under local complement.
Compact form: $C \Rightarrow U d \Leftrightarrow D^{\prime}$.
Proof. Direct consequence of the identity $A \backslash B=\left(A^{c} \cup B\right)^{c}$, monotonicity of complement (then $A \cap B=\varnothing \Leftrightarrow B \subset A^{c}$ ), and the identity $A \cup B=\left(A^{c} \backslash B\right)^{c}$.

Lemma 543 (closedness under set difference and local complement).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
If $\mathcal{S}$ is closed under set difference, then it is closed under local complement.
If $\mathcal{S}$ is closed under intersection and local complement, then it is closed under set difference.
Compact form: $D \Rightarrow D^{\prime} ; I \Rightarrow D^{\prime} \Rightarrow D$.
Proof. Direct consequence of the identity $A \backslash B=A \backslash(A \cap B)$ with $A \cap B \subset A$.
Lemma 544 (closedness under symmetric difference, local complement, and disjoint union). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. If $\mathcal{S}$ is closed under symmetric difference, then it is closed under local complement and disjoint union. If $\mathcal{S}$ is closed under intersection, local complement, and disjoint union, then it is closed under symmetric difference.

Compact form: $s D \Rightarrow\left[D^{\prime} \mid U d\right] ; I \Rightarrow D^{\prime} \Rightarrow U d \Rightarrow s D$.
Proof. Direct consequence of the identities

$$
\begin{gathered}
B \subset A \quad \Longrightarrow \quad A \backslash B=A \triangle B, \quad A \cap B=\varnothing \quad \Longrightarrow \quad A \uplus B=A \triangle B \\
\text { and } \quad A \triangle B=(A \backslash(A \cap B)) \uplus(B \backslash(A \cap B)) \quad \text { with } A \cap B \subset A, B .
\end{gathered}
$$

## Lemma 545 (closedness under intersection and set difference).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
If $\mathcal{S}$ is closed under complement and intersection, then it is closed under set difference.
If $\mathcal{S}$ is closed under set difference, then it is closed under intersection.
Compact form: $C \Rightarrow I \Rightarrow D ; D \Rightarrow I$.
Proof. Direct consequence of the identity $A \cap B=A \backslash(A \backslash B)$.

## Lemma 546 (closedness under union and disjoint union).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
If $\mathcal{S}$ is closed under set difference and disjoint union, then it is closed under union.
If $\mathcal{S}$ is closed under union, then it is closed under disjoint union.
Compact form: $D \Rightarrow U d \Rightarrow U ; U \Rightarrow U d$.
Proof. Direct consequence of the identity $A \cup B=(A \backslash B) \uplus B$.

## Lemma 547 (closedness under union and intersection).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it is closed under union iff it is closed under intersection.
Compact form: $C \Rightarrow U \Leftrightarrow I$.
Proof. Direct consequence of De Morgan's laws.

## Lemma 548 (closedness under union and set difference).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it is closed under union iff it is closed under set difference.
Compact form: $C \Rightarrow U \Leftrightarrow D$.
Proof. Direct consequence of Lemma 547 (closedness under union and intersection, $C \Rightarrow U \Leftrightarrow I$ ), and Lemma 545 (closedness under intersection and set difference, $C \Rightarrow I \Leftrightarrow D$ ).

## Remark 549.

We recall that $\triangle$ denotes the symmetric difference of subsets, see its definition in Chapter 2.
Lemma 550 (closedness under symmetric difference and set difference).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under union.
Then, it is closed under symmetric difference iff it is closed under set difference.
Compact form: $U \Rightarrow s D \Leftrightarrow D$.
Proof. Direct consequence of the identity $A \backslash B=(A \cup B) \triangle B$, and the definition of symmetric difference.

Lemma 551 (closedness under symmetric difference and intersection). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under symmetric difference and intersection. Then, it is closed under union.
Compact form: $s D \Rightarrow I \Rightarrow U$.
Proof. Direct consequence of the identity $A \cup B=(A \triangle B) \triangle(A \cap B)$.

### 8.1.1 Finite properties

Lemma 552 (nonempty is with empty). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, $\mathcal{S}$ contains the empty set if it is nonempty.
Compact form: PtfD $\Rightarrow N e \Rightarrow w E$.
Proof. Direct consequence of the identity $A \backslash A=\varnothing$, and the upper bound property of union $\left(B_{i} \subset \biguplus_{i \in[0 . . n]} B_{i}=\varnothing\right.$ with $\left.B_{i} \in \mathcal{S}\right)$.

Lemma 553 (with full is with empty). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that the complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, $\mathcal{S}$ contains the empty set if it contains the full set.

Compact form: $P t f C \Rightarrow w F \Rightarrow w E$.
Proof. Direct consequence of the identity $X^{c}=\varnothing$, and the upper bound property of union $\left(B_{i} \subset \biguplus_{i \in[0 . . n]} B_{i}=\varnothing\right.$ with $\left.B_{i} \in \mathcal{S}\right)$.

## Lemma 554 (disjoint decomposition).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. If $\mathcal{S}$ is closed under complement (resp. local complement, or set difference), then the complement (resp. local complement, or set difference) of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.

Compact form: $C \Rightarrow$ PtfC $; D^{\prime} \Rightarrow P t f D^{\prime} ; D \Rightarrow P t f D$.
Proof. Direct consequence of the identity $A \backslash B=A \cap B^{c}$, and distributivity of intersection over disjoint union.

## Lemma 555 (disjoint decomposition of complement and set difference).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under intersection, and that the complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.

Compact form: $I \Rightarrow P t f C \Rightarrow P t f D$.
Proof. Direct consequence of the identity $A \backslash B=A \cap B^{c}$, and distributivity of intersection over disjoint union.

## Lemma 556 (disjoint decomposition of local complement and complement).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set, and that the local complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, the complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.

Compact form: $w F \Rightarrow P t f D^{\prime} \Rightarrow P t f C$.
Proof. Direct consequence of the identity $A^{c}=X \backslash A$.
Lemma 557 (disjoint decomposition of set difference and local complement).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, the local complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Compact form: PtfD $\Rightarrow P t f D^{\prime}$.
Proof. Direct consequence of the definitions of set difference and local complement.
Lemma 558 (closedness under finite operations).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
$\mathcal{S}$ is closed under finite intersection iff it is closed under intersection.
$\mathcal{S}$ is closed under finite union iff it is closed under union.
$\mathcal{S}$ is closed under finite disjoint union iff it is closed under disjoint union.
Compact form: If $\Leftrightarrow I ; U f \Leftrightarrow U ; U d f \Leftrightarrow U d$.
Proof. Direct consequence of induction on the number of operands.

## Lemma 559 (closedness under finite union and intersection).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement.
Then, it is closed under finite union iff it is closed under finite intersection.
Compact form: $C \Rightarrow U f \Leftrightarrow I f$.
Proof. Direct consequence of Lemma 547 (closedness under union and intersection, $\underline{C \Rightarrow U \Leftrightarrow I)}$, and Lemma 558 (closedness under finite operations, $\underline{U f} \Leftrightarrow U$ and If $\Leftrightarrow I$ ).

Remark 560. Note that the compact formula in the next lemma should actually be read as for all subset system $\mathcal{S}, I(\mathcal{S})$ implies $I\left(\mathcal{U}^{+}(\mathcal{S})\right)$.

Lemma 561 (closedness under intersection and finite disjoint union closure).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under intersection.
Then, the set $\mathcal{U}^{+}$of finite disjoint unions of elements of $\mathcal{S}$ is closed under intersection.
Compact form: $I \Rightarrow I\left(\mathcal{U}^{+}\right)$.
Proof. Let $A^{0}, A^{1} \in \mathcal{U}^{+}$. Let $\alpha \in\{0,1\}$. Let $n^{\alpha} \in \mathbb{N}$ and let $\left(A_{i^{\alpha}}^{\alpha}\right)_{i^{\alpha} \in\left[0 . . n^{\alpha}\right]}$ be pairwise disjoint elements of $\mathcal{S}$ such that $A^{\alpha}=\biguplus_{i^{\alpha} \in\left[0 . . n^{\alpha}\right]} A_{i^{\alpha}}^{\alpha}$.

For all $\left(i^{0}, i^{1}\right) \in\left[0 . . n^{0}\right] \times\left[0 . . n^{1}\right]$, let $B_{i^{0}, i^{1}} \stackrel{\text { def. }}{=} A_{i^{0}}^{0} \cap A_{i^{1}}^{1}$. Then, from hypothesis, $B_{i^{0}, i^{1}}$ belongs to $\mathcal{S}$. Moreover, from distributivity of intersection over disjoint union, we have

$$
A^{0} \cap A^{1}=\biguplus_{i^{0} \in\left[0 . . n^{0}\right]} A_{i^{0}}^{0} \cap \biguplus_{i^{1} \in\left[0 . . n^{1}\right]} A_{i^{1}}^{1}=\biguplus_{\substack{i^{0} \in\left[0 . . n^{0}\right] \\ i^{1} \in\left[0 . . n^{1}\right]}} B_{i^{0}, i^{1}} .
$$

Hence, from the definition of $\mathcal{U}^{+}$, we have $A^{0} \cap A^{1} \in \mathcal{U}^{+}$.
Therefore, $\mathcal{U}^{+}$is closed under intersection.
Lemma 562 (finite disjoint union closure is closed under finite disjoint union).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Then, the set $\mathcal{U}^{+}$of finite disjoint unions of elements of $\mathcal{S}$ is closed under disjoint union.
Compact form: $U d\left(\mathcal{U}^{+}\right)$.
Proof. Direct consequence of properties of disjoint union (disjoint union of disjoint union is disjoint union).

## Lemma 563 (closedness under set difference and finite disjoint union closure).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under intersection, and that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Then, the set $\mathcal{U}^{+}$of finite disjoint unions of elements of $\mathcal{S}$ is closed under set difference.
Compact form: $I \Rightarrow$ PtfD $\Rightarrow D\left(\mathcal{U}^{+}\right)$.
Proof. Let $A^{0}, A^{1} \in \mathcal{U}^{+}$. Let $\alpha \in\{0,1\}$. Let $n^{\alpha} \in \mathbb{N}$ and let $\left(A_{i^{\alpha}}^{\alpha}\right)_{i^{\alpha} \in\left[0 . . n^{\alpha}\right]}$ be pairwise disjoint elements of $\mathcal{S}$ such that $A^{\alpha}=\biguplus_{i^{\alpha} \in\left[0 . . n^{\alpha}\right]} A_{i^{\alpha}}^{\alpha}$.

For all $\left(i^{0}, i^{1}\right) \in\left[0 . . n^{0}\right] \times\left[0 . . n^{1}\right]$, let $B_{i^{0}, i^{1}} \stackrel{\text { def. }}{=} A_{i^{0}}^{0} \backslash A_{i^{1}}^{1}$. Then, from hypothesis, $B_{i^{0}, i^{1}}$ is a finite disjoint union of elements of $\mathcal{S}$, and thus belongs to $\mathcal{U}^{+}$. Moreover, from left and right distributivity of set difference over disjoint union $(A \backslash(B \cup C)=(A \backslash B) \cap(A \backslash C)$ and $(A \uplus B) \backslash C=(A \backslash C) \uplus(B \backslash C))$, we have
$A^{0} \backslash A^{1}=\biguplus_{i^{0} \in\left[0 . . n^{0}\right]} A_{i^{0}}^{0} \backslash \biguplus_{i^{1} \in\left[0 . . n^{1}\right]} A_{i^{1}}^{1}=\bigcap_{i^{1} \in\left[0 . . n^{1}\right]}\left(\biguplus_{i^{0} \in\left[0 \ldots n^{0}\right]} A_{i^{0}}^{0} \backslash A_{i^{1}}^{1}\right)=\bigcap_{i^{1} \in\left[0 . . n^{1}\right]}\left(\biguplus_{i^{0} \in\left[0 . . n^{0}\right]} B_{i^{0}, i^{1}}\right)$.
Hence, from Lemma 562 (finite disjoint union closure is closed under finite disjoint union), and Lemma 561 (closedness under intersection and finite disjoint union closure), we have $A^{0} \backslash A^{1} \in \mathcal{U}^{+}$.

Therefore, $\mathcal{U}^{+}$is closed under set difference.

### 8.1.2 Countable properties

## Remark 564.

Note that obviously, closedness under finite or countable union implies closedness under finite or countable disjoint union. In the same way, closedness under countable union (resp. intersection) implies closedness under countable monotone union (resp. intersection).
Compact form: $U f \Rightarrow U d f ; U c \Rightarrow[U d c \mid U m c] ; I c \Rightarrow I m c$.

Remark 565. Note that closedness under a countable subset operation actually means that the subset system is closed under this operation with at most a countable number of operands, i.e. it is also valid for a finite number (larger than or equal to two).

Compact form: $U d c \Rightarrow[U d f \mid U d] ; U c \Rightarrow[U f \mid U] ; I c \Rightarrow[I f \mid I]$.

Lemma 566 (closedness under countable disjoint union and local complement).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement and countable disjoint union. Then, it is closed under local complement.

Compact form: $C \Rightarrow U d c \Rightarrow D^{\prime}$.
Proof. Direct consequence of Lemma 542 (closedness under disjoint union and local complement, $\left.C \Rightarrow(U d c \Rightarrow) U d \Rightarrow D^{\prime}\right)$.

## Lemma 567 (closedness under countable union and intersection).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under complement. Then, it is closed under countable (monotone) union iff it is closed under countable (monotone) intersection.

Compact form: $C \Rightarrow U m c \Leftrightarrow \operatorname{Imc} ; C \Rightarrow U c \Leftrightarrow I c$.
Proof. Direct consequence of De Morgan's laws, and monotonicity of complement.
Lemma 568 (closedness under countable disjoint and monotone union).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement and countable disjoint union. Then, it is closed under countable monotone union.

Compact form: $D^{\prime} \Rightarrow U d c \Rightarrow U m c$.
Proof. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that $\left(A_{i}\right)_{i \in I}$ is nondecreasing.
Then, from Lemma 215 (disjoint union of union), there exists $\left(B_{i}\right)_{i \in I} \subset X$ pairwise disjoint, that is computed from the $A_{i}$ 's using local complement and disjoint union (thus belonging to $\mathcal{S}$ ), and such that $\bigcup_{i \in I} A_{i}=\biguplus_{i \in I} B_{i} \in \mathcal{S}$.

Therefore, $\mathcal{S}$ is closed under countable monotone union.

## Lemma 569 (closedness under countable monotone and disjoint union).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under disjoint union and countable monotone union. Then, it is closed under countable disjoint union.

Compact form: $U d \Rightarrow U m c \Rightarrow U d c$.
Proof. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that $\left(A_{i}\right)_{i \in I}$ is pairwise disjoint. Then, from Lemma 217 (monotone union of union), there exists $\left(B_{i}\right)_{i \in I} \subset X$ nondecreasing, that is computed from the $A_{i}$ 's using disjoint union (thus belonging to $\mathcal{S}$ ), and such that $\biguplus_{i \in I} A_{i}=\bigcup_{i \in I} B_{i} \in \mathcal{S}$.

Therefore, $\mathcal{S}$ is closed under countable disjoint union.

## Lemma 570 (closedness under countable disjoint union and countable union).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under set difference and countable disjoint union. Then, it is closed under countable union.

Compact form: $D \Rightarrow U d c \Rightarrow U c$.
Proof. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Then, from Lemma 215 (disjoint union of union), there exists $\left(B_{i}\right)_{i \in I} \subset X$ pairwise disjoint, that is computed from the $A_{i}$ 's using set difference and disjoint union (thus belonging to $\mathcal{S}$ ), and such that $\bigcup_{i \in I} A_{i}=\biguplus_{i \in I} B_{i} \in \mathcal{S}$.

Therefore, $\mathcal{S}$ is closed under countable union.

## Lemma 571 (closedness under countable monotone union and countable union).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under set difference, disjoint union and countable monotone union. Then, it is closed under countable union.

Compact form: $D \Rightarrow U d \Rightarrow U m c \Rightarrow U c$.
Proof. Direct consequence of Lemma 569 (closedness under countable monotone and disjoint union, $U d \wedge U m c \Rightarrow U d c$ ), and Lemma 570 (closedness under countable disjoint union and countable union, $\underline{D \wedge U d c \Rightarrow U c}$ ).

## Lemma 572 (disjoint union of union in subset system).

Let $X$ be a set.
Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under set difference, and disjoint union. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Then, there exists $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ pairwise disjoint such that we have

$$
\begin{align*}
\forall i \in I, \quad B_{i} & \subset A_{i}  \tag{8.1}\\
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\biguplus_{j \in[0 . . n]} B_{\varphi(j)} \in \mathcal{S},  \tag{8.2}\\
\bigcup_{i \in I} A_{i} & =\biguplus_{i \in I} B_{i} . \tag{8.3}
\end{align*}
$$

Moreover, when the sequence $\left(A_{i}\right)_{i \in I}$ is nondecreasing, it is sufficient to assume that $\mathcal{S}$ is closed under local complement (instead of set difference); and if $\mathcal{S}$ is additionally closed under countable disjoint union, then equality (8.3) also occurs in $\mathcal{S}$.
Compact form: $H y p \stackrel{\text { def. }}{=} D\left(\right.$ or $\left.D^{\prime}\right) \wedge U d$.

Proof. Direct consequence of Lemma 215 (disjoint union of union), and Lemma 558 (closedness under finite operations, $U d \Rightarrow U d f)$.

Remark 573. The previous lemma is intended to be applied in rings of sets (or in ( $\sigma$-)algebras, see next sections). The next one is dedicated to semirings of sets (or to semi-algebras of sets).

Lemma 574 (disjoint union of union in subset system (variant)). Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is nonempty, closed under intersection, and that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Then, there exist $n_{K} \in \overline{\mathbb{N}}, K \stackrel{\text { def. }}{=}\left[0 . . n_{K}\right), \psi:\left[0 . . n_{I}\right] \rightarrow\left[0 . . n_{K}\right]$ nondecreasing (with $\psi(0)=0$ and $\psi\left(n_{I}\right)=n_{K}$ ), and $\left(C_{k}\right)_{k \in K} \in \mathcal{S}$ pairwise disjoint such that we have

$$
\begin{equation*}
\forall n \in J, \quad \biguplus_{k \in[\psi(n) . . \psi(n+1))} C_{k} \subset A_{\varphi(n)}, \tag{8.4}
\end{equation*}
$$

$$
\begin{align*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\biguplus_{\substack{j \in[0 . . n] \\
k \in[\psi(j) \ldots \psi(j+1))}} C_{k} \tag{8.5}
\end{align*}=\biguplus_{k \in[0 . . \psi(n+1))} C_{k},
$$

Moreover, when the sequence $\left(A_{i}\right)_{i \in I}$ is nondecreasing, it is sufficient to assume that the local complement (instead of the set difference) of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.

Compact form: Hyp $\stackrel{\text { def. }}{=} N e \wedge I \wedge P t f D\left(\right.$ or PtfD $\left.{ }^{\prime}\right)$.

Proof. From Lemma 552 (nonempty is with empty), we have $\varnothing \in \mathcal{S}$,
From Lemma 215 (disjoint union of union), let $B_{\varphi(0)} \stackrel{\text { def. }}{=} A_{\varphi(0)}$, and for all $n \in J$ such that $n+1 \in J$, let $B_{\varphi(n+1)} \stackrel{\text { def. }}{=} A_{\varphi(n+1)} \backslash \bigcup_{j \in[0 . . n]} B_{\varphi(j)}$. Then, $\left(B_{i}\right)_{i \in I}$ is pairwise disjoint, for all $i \in I$, $B_{i} \subset A_{i}$, and we have

$$
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 \ldots n]} B_{\varphi(j)}, \quad \text { and } \quad \bigcup_{i \in I} A_{i}=\biguplus_{i \in I} B_{i} .
$$

For all $n \in \mathbb{N}$, let $P(n)$ be the property:

$$
\begin{aligned}
\forall j \in[0 . . n], \exists k_{j} \in \mathbb{N}, \exists\left(C_{k}^{j}\right)_{k \in\left[0 . . k_{j}\right)} \in \mathcal{S}, & \left(\forall k \in\left[0 . . k_{j}\right), C_{k}^{j} \subset A_{\varphi(j)}\right) \\
& \wedge \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{\substack{j \in[0 . . n] \\
k \in\left[0 . . k_{j}\right)}} C_{k}^{j} .
\end{aligned}
$$

Let $Q(n) \stackrel{\text { def. }}{=}(n \in J \Rightarrow P(n))$.
Induction: $\boldsymbol{Q} \mathbf{( 0 )}$. Trivial (with $k_{0} \stackrel{\text { def. }}{=} 1$ and $C_{0}^{0} \stackrel{\text { def. }}{=} A_{\varphi(0)}$ ).
Induction: $\boldsymbol{Q}(\boldsymbol{n})$ implies $\boldsymbol{Q}(\boldsymbol{n}+\mathbf{1})$. Let $n \in \mathbb{N}$. Assume that $Q(n)$ and $n+1 \in J$ hold.
Then, $n \in J$ since $J$ is an interval starting at 0 . Thus, $P(n)$ holds. Hence, from left distributivity of set difference over union $(A \backslash(B \cup C)=(A \backslash B) \cap(A \backslash C)$ ), and assumption (PtfD), let $l_{n+1}^{j, k} \in \mathbb{N}$, and $\left(D_{l}^{j, k}\right)_{l \in\left[0 . . l_{n+1}^{j, k}\right)} \in \mathcal{S}$ such that we have

$$
B_{\varphi(n+1)}=\bigcap_{\substack{j \in[0 . . . n] \\ k \in\left[0 . . k_{j}\right)}}\left(A_{\varphi(n+1)} \backslash C_{k}^{j}\right)=\bigcap_{\substack{j \in[0 . . n] \\ k \in\left[0 . k_{j}\right)}} \biguplus_{\substack{\left.\hline 0 . . l_{n+1}^{j, k}\right)}} D_{l}^{j, k} .
$$

Let $k_{n+1} \stackrel{\text { def. }}{=} \max _{j \in[0 . . n], k \in\left[0 . . k_{j}\right]} l_{n+1}^{j, k}$, and

$$
\begin{aligned}
& \forall j \in[0 . . n], \forall k \in\left[0 . . k_{j}\right], \forall l \in\left[l_{n+1}^{j, k} . . k_{n+1}\right), \text { let } D_{l}^{j, k} \stackrel{\text { def. }}{=} \varnothing, \\
& \forall l \in\left[0 . . k_{n+1}\right), \quad \text { let } C_{l}^{n+1} \stackrel{\text { def. }}{=} \bigcap_{\substack{j \in[0 . . n] \\
k \in\left[0 . . k_{j}\right)}} D_{l}^{j, k} .
\end{aligned}
$$

Then, since the empty set is the identity element for union, and from distributivity of intersection over disjoint union, we have

$$
B_{\varphi(n+1)}=\bigcap_{\substack{j \in[0 . . n] \\ k \in\left[0 . . k_{j}\right)}} \biguplus_{l \in\left[0 . . k_{n+1}\right)} D_{l}^{j, k}=\biguplus_{k \in\left[0 . . k_{n+1}\right)} C_{k}^{n+1}
$$

Let $k \in\left[0 . . k_{n+1}\right)$. Then, since $\varnothing \in \mathcal{S}$, and from transitivity of inclusion, associativity of union, and $P(n)$, we have $C_{k}^{n+1} \in \mathcal{S}, C_{k}^{n+1} \subset A_{\varphi(n+1)}$, and

$$
\bigcup_{j \in[0 . . n+1)} A_{\varphi(j)}=\biguplus_{\substack{j \in[0 . . n) \\ k \in\left[0 . . k_{j}\right)}} C_{k}^{j} \uplus B_{\varphi(n+1)}=\biguplus_{\substack{j \in[0 . . n+1) \\ k \in\left[0 . . k_{j}\right)}} C_{k}^{j}
$$

Thus, for all $n \in \mathbb{N}, Q(n)$ holds. Hence, for all $n \in J, P(n)$ holds.
Let $\psi(0) \stackrel{\text { def. }}{=} 0$. For all $n \in\left(0 . . n_{I}\right]$, let $\psi(n) \stackrel{\text { def. }}{=} \sum_{j \in[0 . . n)} k_{j}$. Let $n_{K} \stackrel{\text { def. }}{=} \psi\left(n_{I}\right)$. For all $n \in J$, for all $k \in[\psi(n) . . \psi(n+1))$ (i.e. we have $\left.k-\psi(n) \in\left[0 . . k_{n}\right)\right)$, let $C_{k} \stackrel{\text { def. }}{=} C_{k-\psi(n)}^{n}$. Then, the $C_{k}$ 's belong to $\mathcal{S}$, are pairwise disjoint, and properties (8.1), (8.2), and (8.3) are satisfied.

## Lemma 575 (monotone union of union in subset system).

Let $X$ be a set. Let $I \subset \mathbb{N}$. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under union. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$.
Then, there exists $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing such that for all $i \in I, A_{i} \subset B_{i}$, and we have

$$
\begin{align*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\bigcup_{j \in[0 . . n]} B_{\varphi(j)}\left(=B_{\varphi(n)}\right) \in \mathcal{S},  \tag{8.7}\\
\bigcup_{i \in I} A_{i} & =\bigcup_{i \in I} B_{i} . \tag{8.8}
\end{align*}
$$

Moreover, when the sequence $\left(A_{i}\right)_{i \in I}$ is pairwise disjoint, it is sufficient to assume that $\mathcal{S}$ is closed under disjoint union (instead of union); and if $\mathcal{S}$ is additionally closed under countable monotone union, then equality (8.8) also occurs in $\mathcal{S}$.

Compact form: Hyp $\stackrel{\text { def. }}{=} U($ or $U d)$.

Proof. Direct consequence of Lemma 217 (monotone union of union), and Lemma 558 (closedness under finite operations, $U \Rightarrow U f$ and $U d \Rightarrow U d f$ ).

## Lemma 576 (empty monotone intersection of union in subset system).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that $A \stackrel{\text { def. }}{=} \bigcup_{i \in I} A_{i} \in \mathcal{S}$. Then, there exists $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nonincreasing such that for all $i \in I, B_{i} \subset A$ and $A_{i} \cap B_{i}=\varnothing$, and we have

$$
\begin{align*}
\forall n \in J, \quad A \backslash \bigcup_{j \in[0 . . n]} A_{\varphi(j)} & =\bigcap_{j \in[0 . . n]} B_{\varphi(j)}\left(=B_{\varphi(n)}\right) \in \mathcal{S},  \tag{8.9}\\
\varnothing & =\bigcap_{i \in I} B_{i} . \tag{8.10}
\end{align*}
$$

Moreover, if $\left(A_{i}\right)_{i \in I}$ is pairwise disjoint, then we have

$$
\begin{equation*}
\forall n \in J, \quad A=\biguplus_{j \in[0 . . n]} A_{\varphi(j)} \uplus B_{\varphi(n)} \quad\left(\text { i.e. } B_{\varphi(n)}=\biguplus_{j \in\left(n . . n_{I}\right)} A_{\varphi(j)} \in \mathcal{S}\right) . \tag{8.11}
\end{equation*}
$$

And if $\mathcal{S}$ contains the empty set, then equality (8.10) also occurs in $\mathcal{S}$.
Compact form: $H y p \stackrel{\text { def. }}{=} D^{\prime}$.

## Lemma 577 (monotone union of monotone intersection in subset system).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under set difference. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Let $n_{0} \in J$. Let $J_{n_{0}} \stackrel{\text { def. }}{=}\left[n_{0} . . n_{I}\right) \neq \varnothing$. For all $n \in J_{n_{0}}$, let $B_{n} \stackrel{\text { def. }}{=} A_{\varphi\left(n_{0}\right)} \backslash A_{\varphi(n)}$. Then, $\left(B_{n}\right)_{n \in J_{n_{0}}} \in \mathcal{S}$, and we have

$$
\begin{equation*}
\bigcup_{n \in J_{n_{0}}} B_{n}=A_{\varphi\left(n_{0}\right)} \backslash \bigcap_{n \in J_{n_{0}}} A_{\varphi(n)} \tag{8.12}
\end{equation*}
$$

Moreover, when the sequence $\left(A_{i}\right)_{i \in I}$ is nonincreasing, it is sufficient to assume that $\mathcal{S}$ is closed under local complement (instead of set difference), and then $\left(B_{n}\right)_{n \in J_{n_{0}}}$ is nondecreasing; and if $\mathcal{S}$ is additionally closed under countable (monotone) union or intersection, then equality (8.12) occurs in $\mathcal{S}$.
Compact form: Hyp $\stackrel{\text { def. }}{=} D\left(\right.$ or $\left.D^{\prime}\right)$.
Proof. Direct consequence of Lemma 221 (monotone union of monotone intersection).

### 8.2 Generated subset system

## Definition 578 (subset system).

Let $X$ be a set. Subsets of the power set $\mathcal{P}(X)$ are called subset systems (on $X$ ).
Remark 579 . Note that subset systems are elements of $\mathcal{P}(\mathcal{P}(X))$.
Definition 580 (subset system class). Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$, i.e. a function from $\mathcal{P}(\mathcal{P}(X))$ to a Boolean domain. The subset system class (on $X$ ) corresponding to predicate $P$ (in short the subset systems of class $P$, or $P$-class subset systems) is denoted $\mathfrak{C}_{P, X}$ (or simply $\mathfrak{C}_{P}$ ); it is the collection of subset systems on $X$ satisfying $P$ :

$$
\begin{equation*}
\mathfrak{C}_{P, X} \stackrel{\text { def. }}{=}\{\mathcal{S} \subset \mathcal{P}(X) \mid P(\mathcal{S})\} \tag{8.13}
\end{equation*}
$$

Remark 581. Note that subset system classes are elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.
Lemma 582 (intersection of subset system classes). Let $X$ be a set.
Let $P$ and $Q$ be predicates on subset systems on $X$. Then, we have $\mathfrak{C}_{P \wedge Q}=\mathfrak{C}_{P} \cap \mathfrak{C}_{Q}$.
Proof. Direct consequence of Definition 580 (subset system class), and compatibility of logical conjunction with intersection.

Lemma 583 (subset system classes are monotone). Let $X$ be a set. Let $P$ and $Q$ be predicates on subset systems on $X$. Assume that $P$ implies $Q$. Then, we have $\mathfrak{C}_{P} \subset \mathfrak{C}_{Q}$.

Proof. Direct consequence of Definition 580 (subset system class), and compatibility of implication with inclusion.

## Definition 584 (generated subset system).

Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Let $G \subset \mathcal{P}(X)$. The $P$-class subset system (on $X$ ) generated by $G$ is denoted $\mathfrak{G}_{P, X}(G)$ (or simply $\mathfrak{G}_{P}(G)$ ); it is the intersection of all $P$-class subset systems on $X$ containing $G$ :

$$
\begin{equation*}
\mathfrak{G}_{P, X}(G) \stackrel{\text { def. }}{=} \bigcap_{G \subset \mathcal{S} \in \mathfrak{C}_{P}} \mathcal{S} \tag{8.14}
\end{equation*}
$$

Lemma 585 (subset system generation preserves class).
Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Then, generated $P$-class subset systems are $P$-class subset systems containing the generator:

$$
\begin{equation*}
\forall G \subset \mathcal{P}(X), \quad G \subset \mathfrak{G}_{P}(G) \in \mathfrak{C}_{P} \tag{8.15}
\end{equation*}
$$

Proof. Direct consequence of Definition 584 (generated subset system), Definition 580 (subset system class), Definition 578 (subset system), and the lower bound property of intersection.

## Lemma 586 (properties of subset system generation).

Let $X$ be a set.
Let $P$ be a predicate on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Then, $P$-class subset system generation is minimum, monotone, and idempotent:

$$
\begin{align*}
& \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P}, \quad G \subset \mathcal{S} \quad \Longrightarrow \quad \mathfrak{G}_{P}(G) \subset \mathcal{S}  \tag{8.16}\\
& \forall G_{1}, G_{2} \subset \mathcal{P}(X), \quad G_{1} \subset G_{2} \Longrightarrow \mathfrak{G}_{P}\left(G_{1}\right) \subset \mathfrak{G}_{P}\left(G_{2}\right)  \tag{8.17}\\
& \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P} \quad \Longleftrightarrow \mathfrak{G}_{P}(\mathcal{S})=\mathcal{S} . \tag{8.18}
\end{align*}
$$

Proof. (8.16). Direct consequence of Definition 584 (generated subset system), Definition 580 (subset system class), Definition 578 (subset system), and properties of the intersection.
(8.17). Direct consequence of Lemma 585 (subset system generation preserves class, with $G_{2}$, thus $\underline{G_{2} \subset \mathfrak{G}_{P}\left(G_{2}\right) \in \mathfrak{C}_{P}}$ ), and (8.16) (with $G_{1}$ and $\mathcal{S} \stackrel{\text { def. }}{=} \mathfrak{G}_{P}\left(G_{2}\right)$ ).
(8.18). Direct consequence of Lemma 585 (subset system generation preserves class, we always have $\mathcal{S} \subset \mathfrak{G}_{P}(\mathcal{S}) \in \mathfrak{C}_{P}$, thus $\mathfrak{G}_{P}(\mathcal{S}) \subset \mathcal{S} \Rightarrow \mathcal{S} \in \mathfrak{C}_{P}$ ), reflexivity of inclusion, and (8.16) (with $G \stackrel{\text { def. }}{=} \mathcal{S}$, thus $\left.\mathcal{S} \in \mathfrak{C}_{P} \Rightarrow \mathfrak{G}_{P}(\mathcal{S}) \subset \mathcal{S}\right)$.

## Lemma 587 (other properties of subset system generation).

Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
G_{1} \subset \mathfrak{G}_{P}\left(G_{2}\right) \wedge G_{2} \subset \mathfrak{G}_{P}\left(G_{1}\right) & \Longrightarrow \mathfrak{G}_{P}\left(G_{1}\right)=\mathfrak{G}_{P}\left(G_{2}\right),  \tag{8.19}\\
G_{2} \subset \mathfrak{G}_{P}\left(G_{1}\right) & \Longrightarrow \mathfrak{G}_{P}\left(G_{1} \cup G_{2}\right)=\mathfrak{G}_{P}\left(G_{1}\right) .
\end{align*}
$$

Proof. (8.19). Direct consequence of Lemma 586 (properties of subset system generation, monotonicity and idempotence: $\mathfrak{G}_{P}\left(G_{i}\right) \subset \mathfrak{G}_{P}\left(\mathfrak{G}_{P}\left(G_{j}\right)\right)=\mathfrak{G}_{P}\left(G_{j}\right)$ with $i \in\{1,2\}$ and $\left.j \stackrel{\text { def. }}{=} 3-i\right)$, and Lemma 585 (subset system generation preserves class, $\left.\mathfrak{G}_{P}\left(G_{j}\right) \in \mathfrak{C}_{P}\right)$.
(8.20). Direct consequence of Lemma 585 (subset system generation preserves class, $G_{1}$ is included in $\mathfrak{G}_{P}\left(G_{1}\right)$, monotonicity of union $\left(G_{1} \cup G_{2} \subset \mathfrak{G}_{P}\left(G_{1}\right) \cup G_{2}=\mathfrak{G}_{P}\left(G_{1}\right)\right)$, Lemma 586 $\overline{(\text { properties }}$ of subset system generation, monotonicity: $G_{1} \subset \mathfrak{G}_{P}\left(G_{1}\right) \subset \mathfrak{G}_{P}\left(G_{1} \cup G_{2}\right)$ ), and (8.19) (with $G_{1} \stackrel{\text { def. }}{=} G_{1} \cup G_{2}$ and $G_{2} \stackrel{\text { def. }}{=} G_{1}$ ).

Lemma 588 (generated subset systems are monotone). $\quad$ Let $X$ be a set. Let $P$ and $Q$ be predicates on subset systems on $X$. Assume that $P$ implies $Q$. Let $G \subset \mathcal{P}(X)$.
Then, we have $\mathfrak{G}_{Q}(G) \subset \mathfrak{G}_{P}(G)$.
Proof. Direct consequence of Definition 584 (generated subset system), and Lemma 583 (subset system classes are monotone).

Remark 589 . Note that the previous lemma provides $\mathfrak{G}_{P}(G), \mathfrak{G}_{Q}(G) \subset \mathfrak{G}_{P \wedge Q}(G)$, but in general, we do not have the inclusion the other way around.

Hence, $\mathfrak{G}_{P \wedge Q}(G)$ is generally distinct from $\mathfrak{G}_{P}(G) \cap \mathfrak{G}_{Q}(G)$.

## Lemma 590 (upgrade generated subset system).

Let $X$ be a set. Let $P, Q$ and $R$ be predicates on subset systems on $X$. Let $G \subset \mathcal{P}(X)$. Assume that $\mathfrak{C}_{P}$ and $\mathfrak{C}_{Q}$ are closed under any intersection, that $Q \wedge R$ implies $P$, and that $R$ holds on $\mathfrak{G}_{Q}(G)$. Then, we have $\mathfrak{G}_{P}(G) \subset \mathfrak{G}_{Q}(G)$.

Proof. Direct consequence of Lemma 585 (subset system generation preserves class, $\underline{\mathfrak{G}}_{Q}(G)$ is a $Q$-class subset system containing $G$ ), Definition 580 (subset system class, $\mathfrak{G}_{Q}(G)$ is a $P$-class subset system), Lemma 586 (properties of subset system generation, minimality with $P$ ).

## Lemma 591 (cascading generated subset systems).

Let $X$ be a set. Let $P$ and $Q$ be predicates on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ and $\mathfrak{C}_{Q}$ are closed under any intersection, and that $P$ implies $Q$. Let $G \subset \mathcal{P}(X)$.
Then, we have $\mathfrak{G}_{P}(G)=\mathfrak{G}_{P}\left(\mathfrak{G}_{Q}(G)\right)$.
Proof. $\mathfrak{G}_{P}(\boldsymbol{G}) \subset \mathfrak{G}_{P}\left(\mathfrak{G}_{Q}(\boldsymbol{G})\right)$. Direct consequence of Lemma 585 (subset system generation preserves class, with $Q$, thus $G \subset \mathfrak{G}_{Q}(G)$ ), and Lemma 586 (properties of subset system generation, $\underline{\left.\text { monotonicity with } P, G_{1} \stackrel{\text { def. }}{=} G \text { and } G_{2} \stackrel{\text { def. }}{=} \mathfrak{G}_{Q}(G)\right) \text {. } . . . . . ~}$
$\mathfrak{G}_{P}\left(\mathfrak{G}_{Q}(\boldsymbol{G})\right) \subset \mathfrak{G}_{P}(\boldsymbol{G})$. Direct consequence of Lemma 585 (subset system generation preserves class, with $P$, thus $G \subset \mathfrak{G}_{P}(G) \in \mathfrak{C}_{P}$ ), Lemma 583 (subset system classes are monotone, $\mathfrak{C}_{P} \subset \mathfrak{C}_{Q}$ ), Lemma 586 (properties of subset system generation, minimality, first with $Q, G$ and $\mathcal{S} \stackrel{\text { def. }}{=} \mathfrak{G}_{P}(G) \in \mathfrak{C}_{Q}$, thus $\mathfrak{G}_{Q}(G) \subset \mathfrak{G}_{P}(G)$, and then with $P, G \stackrel{\text { def. }}{=} \mathfrak{G}_{Q}(G)$ and $\left.\mathcal{S} \stackrel{\text { def. }}{=} \mathfrak{G}_{P}(G) \in \mathfrak{C}_{P}\right)$. Therefore, we have the equality.

Lemma 592 (double cascading generated subset systems).
Let $X$ be a set. Let $P, Q$ and $R$ be predicates on subset systems on $X$. Assume that

$$
\forall G \subset \mathcal{P}(X), \quad\left\{\begin{array}{l}
\mathfrak{G}_{P}(G)=\mathfrak{G}_{P}\left(\mathfrak{G}_{Q}(G)\right)  \tag{8.21}\\
\mathfrak{G}_{Q}(G)=\mathfrak{G}_{Q}\left(\mathfrak{G}_{R}(G)\right)
\end{array}\right.
$$

Let $G \subset \mathcal{P}(X)$. Then, we have $\mathfrak{G}_{P}(G)=\mathfrak{G}_{P}\left(\mathfrak{G}_{R}(G)\right)$.
Proof. Direct consequence of the identity

$$
\mathfrak{G}_{P}(G)=\mathfrak{G}_{P}\left(\mathfrak{G}_{Q}(G)\right)=\mathfrak{G}_{P}\left(\mathfrak{G}_{Q}\left(\mathfrak{G}_{R}(G)\right)\right)=\mathfrak{G}_{P}\left(\mathfrak{G}_{R}(G)\right) .
$$

## Remark 593.

Note that when the predicate $P$ involves nonemptiness (denoted as $N e$, see Section 8.1), the premise on closedness under intersection cannot be met, and one has to add nonemptiness of the generator as expressed in the next lemmas. This is useful for $\pi$-systems (see Section 8.3).

## Lemma 594 (class of nonempty subset systems).

Let $X$ be a set.
Let $P$ be a predicate on subset systems on $X$. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is nonempty. Then, we have $\mathcal{S} \in \mathfrak{C}_{P \wedge N e} \Leftrightarrow \mathcal{S} \in \mathfrak{C}_{P}$.

Proof. Direct consequence of Lemma 582 (intersection of subset system classes), and Definition 580 (subset system class, thus $\mathcal{S} \in \mathfrak{C}_{N e}$ ).

## Lemma 595 (generated nonempty subset systems).

Let $X$ be a set.
Let $P$ be a predicate on subset systems on $X$. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is nonempty. Then, we have $\mathfrak{G}_{P \wedge N e}(G)=\mathfrak{G}_{P}(G)$.

Proof. Direct consequence from Definition 584 (generated subset system), transitivity of (strict) inclusion ( $\varnothing \subsetneq G \subset \mathcal{S}$ implies $\varnothing \subsetneq \mathcal{S}$ ), and Lemma 594 (class of nonempty subset systems).
Lemma 596 (subset system generation preserves class (with nonemptiness)).
Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Let $P^{\prime} \stackrel{\text { def. }}{=} P \wedge N e$. Then, generated $P^{\prime}$-class subset systems are $P^{\prime}$-class subset systems containing the nonempty generator:

$$
\begin{equation*}
\forall G \subset \mathcal{P}(X), \quad G \neq \varnothing \quad \Longrightarrow \quad G \subset \mathfrak{G}_{P^{\prime}}(G) \in \mathfrak{C}_{P^{\prime}} \tag{8.22}
\end{equation*}
$$

Proof. Direct consequence of Lemma 585 (subset system generation preserves class, with $P$, thus $G \subset \mathfrak{G}_{P}(G) \in \mathfrak{C}_{P}$ ), transitivity of (strict) inclusion $\left(\mathfrak{G}_{P}(G) \neq \varnothing\right.$ ), Lemma $\overline{594 \text { (class of }}$ nonempty subset systems, thus $\mathfrak{G}_{P}(G) \in \mathfrak{C}_{P^{\prime}}$ ), and Lemma 595 (generated nonempty subset systems, $\left.\underline{\mathfrak{G}}_{P^{\prime}}(G)=\mathfrak{G}_{P}(G)\right)$.

Lemma 597 (properties of generated subset system (with nonemptiness)).
Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Let $P^{\prime} \stackrel{\text { def. }}{=} P \wedge N e$. Then, $P^{\prime}$-class subset system generation (or $P$-class subset system generation with nonemptiness) is minimum, monotone, and idempotent:

$$
\begin{align*}
& \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P^{\prime}}, \quad \varnothing \neq G \subset \mathcal{S} \Longrightarrow \mathfrak{G}_{P^{\prime}}(G) \subset \mathcal{S}  \tag{8.23}\\
& \forall G_{1}, G_{2} \subset \mathcal{P}(X), \quad \varnothing \neq G_{1} \subset G_{2} \Longrightarrow \mathfrak{G}_{P^{\prime}}\left(G_{1}\right) \subset \mathfrak{G}_{P^{\prime}}\left(G_{2}\right),  \tag{8.24}\\
& \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \neq \varnothing \quad \Longrightarrow \quad\left(\mathcal{S} \in \mathfrak{C}_{P^{\prime}} \quad \Longleftrightarrow \quad \mathfrak{G}_{P^{\prime}}(\mathcal{S})=\mathcal{S}\right) . \tag{8.25}
\end{align*}
$$

Proof. Direct consequence of transitivity of (strict) inclusion ( $\varnothing \neq G \subset G^{\prime} \Rightarrow G^{\prime} \neq \varnothing$ ), Lemma 594 (class of nonempty subset systems), Lemma 595 (generated nonempty subset systems, with $G, G_{1}, G_{2}$, and $\mathcal{S}$ ), and Lemma 586 (properties of subset system generation).

Lemma 598 (other properties of subset system generation (with nonemptiness)).
Let $X$ be a set. Let $P$ be a predicate on subset systems on $X$. Let $P^{\prime} \stackrel{\text { def. }}{=} P \wedge N e$. Assume that $\mathfrak{C}_{P}$ is closed under any intersection. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Assume that $G_{1}$ and $G_{2}$ are nonempty. Then, we have

| $(8.26)$ | $G_{1} \subset \mathfrak{G}_{P^{\prime}}\left(G_{2}\right) \wedge G_{2} \subset \mathfrak{G}_{P^{\prime}}\left(G_{1}\right)$ | $\Longrightarrow \mathfrak{G}_{P^{\prime}}\left(G_{1}\right)=\mathfrak{G}_{P^{\prime}}\left(G_{2}\right)$, |  |
| :--- | :--- | :--- | :--- |
| $(8.27)$ |  | $G_{2} \subset \mathfrak{G}_{P^{\prime}}\left(G_{1}\right)$ | $\Longrightarrow \mathfrak{G}_{P^{\prime}}\left(G_{1} \cup G_{2}\right)=\mathfrak{G}_{P^{\prime}}\left(G_{1}\right)$. |

Proof. Direct consequence of the upper bound property of union $\left(\varnothing \neq G_{1}, G_{2} \subset G_{1} \cup G_{2}\right.$, thus $G_{1} \cup G_{2} \neq \varnothing$ ), Lemma 595 (generated nonempty subset systems, with $G_{1}, G_{2}$, and $G_{1} \cup G_{2}$ ), and Lemma 587 (other properties of subset system generation).

Lemma 599 (generated subset systems are monotone (with nonemptiness)).
Let $X$ be a set. Let $P$ and $Q$ be predicates on subset systems on $X$. Assume that $P$ implies $Q$. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is nonempty. Then, we have $\mathfrak{G}_{Q \wedge N e}(G) \subset \mathfrak{G}_{P}(G)$.

Proof. Direct consequence of Lemma 595 (generated nonempty subset systems, with $Q$ ), and Lemma 588 (generated subset systems are monotone).

## Lemma 600 (upgrade generated subset system (with nonemptiness)).

Let $X$ be a set. Let $P, Q$ and $R$ be predicates on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ and $\mathfrak{C}_{Q}$ are closed under any intersection, and that $Q \wedge R$ implies $P$. Let $G \subset \mathcal{P}(X)$. Assume that $R$ holds on $\mathfrak{G}_{Q}(G)$, and that $G$ is nonempty. Then, we have $\mathfrak{G}_{P}(G) \subset \mathfrak{G}_{Q \wedge N e}(G)$.

Proof. Direct consequence of Lemma 596 (subset system generation preserves class (with nonemptiness), with $Q$, thus $\mathfrak{G}_{Q \wedge N e}(G) \neq \varnothing$ ), Lemma 595 (generated nonempty subset systems, with $G \neq \varnothing$ and $Q$, thus $\mathfrak{G}_{Q \wedge N e}(G)=\mathfrak{G}_{Q}(G)$ ), and Lemma 590 (upgrade generated subset system, with $\underline{Q \wedge(N e \wedge R) \Rightarrow P)}$.

Lemma 601 (cascading generated subset systems (with nonemptiness)).
Let $X$ be a set. Let $P$ and $Q$ be predicates on subset systems on $X$. Assume that $\mathfrak{C}_{P}$ and $\mathfrak{C}_{Q}$ are closed under any intersection, and that $P$ implies $Q$. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is nonempty. Then, we have $\mathfrak{G}_{P}(G)=\mathfrak{G}_{P}\left(\mathfrak{G}_{Q \wedge N e}(G)\right)$.

Proof. Direct consequence of Lemma 595 (generated nonempty subset systems, with $G \neq \varnothing$ and $Q$, thus $\mathfrak{G}_{Q \wedge N e}(G)=\mathfrak{G}_{Q}(G)$ ), and Lemma 591 (cascading generated subset systems, with $P \Rightarrow Q$ ).

### 8.3 Pi-system

Definition 602 ( $\pi$-system).
Let $X$ be a set.
Let $P_{\Pi}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of nonemptiness and closedness under finite intersection. The class of $\pi$-systems on $X$ is $\mathfrak{C}_{P_{\Pi}}$ : a subset $\Pi$ of $\mathcal{P}(X)$ is a $\pi$-system on $X$ iff

$$
\begin{equation*}
\forall n \in \mathbb{N}, \forall\left(A_{i}\right)_{i \in[0 . . n]} \in \Pi, \quad \bigcap_{i \in[0 . . n]} A_{i} \in \Pi . \tag{8.28}
\end{equation*}
$$

Compact form: $P_{\Pi} \stackrel{\text { def. }}{=} N e \wedge I f$.
Remark 603. Note that adding the empty set to a $\pi$-system keeps it a $\pi$-system. We may replace in the previous definition the nonempty condition (8.28) by the "contain-the-empty-set" condition, and still maintain the Dynkin $\pi-\lambda$ theorem (see Theorem 732).

Lemma 604 (equivalent definition of $\pi$-system).
Let $X$ be a set. Let $\Pi \subset \mathcal{P}(X)$. Then, $\Pi$ is a $\pi$-system iff it is nonempty and closed under (binary) intersection.

Compact form: $P_{\Pi} \Leftrightarrow N e \wedge I$.
Proof. Direct consequence of Definition $602\left(\pi\right.$-system, $\underline{\left.\left.P_{\Pi}=N e \wedge I f\right) \text {, and Lemma } 558 \text { (closedness }\right) ~}$ under finite operations, If $\Leftrightarrow I$ ).

Lemma 605 (intersection of $\pi$-systems). Let $X$ and $I$ be sets. Let $\left(\Pi_{i}\right)_{i \in I}$ be $\pi$-systems on $X$. Then, $\bigcap_{i \in I} \Pi_{i}$ is closed under intersection, i.e. it is a $\pi$-system iff it is nonempty.

Proof. Direct consequence of Lemma 604 (equivalent definition of $\pi$-system), and the definition of intersection.

## Definition 606 (generated $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. The $\pi$-system generated by $G$ is denoted $\Pi_{X}(G) \stackrel{\text { def. }}{=} \mathfrak{G}_{P_{\Pi}, X}(G)$; it is the intersection of all $\pi$-systems on $X$ containing $G$.

Lemma 607 (generated $\pi$-system is $\pi$-system).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$.
Assume that $G \neq \varnothing$. Then, $\Pi_{X}(G)$ is a $\pi$-system on $X$ containing $G$.
Proof. Direct consequence of Definition 606 (generated $\pi$-system), Lemma 605 (intersection of $\pi$-systems), and Lemma 596 (subset system generation preserves class (with nonemptiness)).

## Lemma 608 (properties of $\pi$-system generation).

Let $X$ be a set. Then, $\pi$-system generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
& \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\Pi}, X}, \varnothing \neq G \subset \mathcal{S}  \tag{8.30}\\
& \forall G_{1}, G_{2} \subset \mathcal{P}(X), \quad \varnothing \neq G_{1} \subset G_{2} \Longrightarrow \Pi_{X}(G) \subset \mathcal{S}  \tag{8.31}\\
& \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \neq \varnothing \quad \Pi_{X}\left(G_{1}\right) \subset \Pi_{X}\left(G_{2}\right)  \tag{8.32}\\
&\left(\mathcal{S} \in \mathfrak{C}_{P_{\Pi}, X} \quad \Longleftrightarrow \Pi_{X}(\mathcal{S})=\mathcal{S}\right)
\end{align*}
$$

Proof. Direct consequence of Definition 606 (generated $\pi$-system), Lemma 605 (intersection of $\pi$-systems), and Lemma 597 (properties of generated subset system (with nonemptiness)).

## Lemma 609 (other properties of $\pi$-system generation).

Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Assume that $G_{1}, G_{2} \neq \varnothing$. Then, we have

$$
\begin{align*}
G_{1} \subset \Pi_{X}\left(G_{2}\right) \wedge G_{2} \subset \Pi_{X}\left(G_{1}\right) & \Longrightarrow \Pi_{X}\left(G_{1}\right)=\Pi_{X}\left(G_{2}\right)  \tag{8.33}\\
G_{2} \subset \Pi_{X}\left(G_{1}\right) & \Longrightarrow \Pi_{X}\left(G_{1} \cup G_{2}\right)=\Pi_{X}\left(G_{1}\right) . \tag{8.34}
\end{align*}
$$

Proof. Direct consequence of Definition 606 (generated $\pi$-system), Lemma 605 (intersection of $\pi$ systems), and Lemma 598 (other properties of subset system generation (with nonemptiness)).

### 8.4 Ring of sets

## Remark 610.

The next concept of "ring of sets" is not to be confused with the algebraic structure "ring".

## Definition 611 (ring of sets).

Let $X$ be a set. Let $P_{\mathcal{R}}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of nonemptiness, and closedness under set difference and finite union. The class of rings (of (sub)sets) on $X$ is $\mathfrak{C}_{P_{\mathcal{R}}}$ : a subset $\mathcal{R}$ of $\mathcal{P}(X)$ is a ring of sets on $X$ iff

$$
\begin{align*}
& \mathcal{R} \neq \varnothing  \tag{8.35}\\
& \forall A, B \in \mathcal{R}, A \backslash B \in \mathcal{R},  \tag{8.36}\\
& \forall n \in \mathbb{N}, \forall\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{R}, \bigcup_{i \in[0 . . n]} A_{i} \in \mathcal{R} . \tag{8.37}
\end{align*}
$$

Compact form: $P_{\mathcal{R}} \stackrel{\text { def. }}{=} N e \wedge D \wedge U f$.

## Lemma 612 (equivalent definition of ring of sets).

Let $X$ be a set. Let $\mathcal{R} \subset \mathcal{P}(X)$. Then, $\mathcal{R}$ is a ring of sets on $X$ iff

$$
\begin{equation*}
\mathcal{R} \neq \varnothing, \quad \text { or } \quad \varnothing \in \mathcal{R}, \tag{8.38}
\end{equation*}
$$

$$
\begin{align*}
& \left\{\begin{array} { l l } 
{ \forall A , B \in \mathcal { R } , } & { A \backslash B \in \mathcal { R } , } \\
{ \forall A , B \in \mathcal { R } , } & { A \cup B \in \mathcal { R } , }
\end{array} \quad \text { or } \quad \left\{\begin{array}{ll}
\forall A, B \in \mathcal{R}, & A \backslash B \in \mathcal{R}, \\
\forall A, B \in \mathcal{R}, & A \cap B=\varnothing \quad \Longrightarrow \quad A \uplus B \in \mathcal{R},
\end{array}\right.\right.  \tag{8.39}\\
& \text { or } \quad\left\{\begin{array} { l l } 
{ \forall A , B \in \mathcal { R } , } & { A \triangle B \in \mathcal { R } , } \\
{ \forall A , B \in \mathcal { R } , } & { A \cap B \in \mathcal { R } , }
\end{array} \quad \text { or } \quad \left\{\begin{array}{ll}
\forall A, B \in \mathcal{R}, & B \subset A \quad \Longrightarrow \quad A \backslash B \in \mathcal{R}, \\
\forall A, B \in \mathcal{R}, & A \cap B=\varnothing \\
\forall A, B \in \mathcal{R}, & A \cap B \in \mathcal{R} .
\end{array} \quad A \uplus B \in \mathcal{R},\right.\right.
\end{align*}
$$

Compact form: $P_{\mathcal{R}} \Leftrightarrow[N e \mid w E] \wedge\left[D \wedge U|D \wedge U d| s D \wedge I \mid D^{\prime} \wedge U d \wedge I\right]$.
Proof. Direct consequence of Definition 611 (ring of sets, $P_{\mathcal{R}}=N e \wedge D \wedge U f$ ), Lemma 538 (nonempty and with empty or full, $N e \wedge D \Rightarrow w E$ and $w E \Rightarrow N e$ ), Lemma 558 (closedness under finite operations, $U f \Leftrightarrow U$ ), Lemma 546 (closedness under union and disjoint union, $U \Rightarrow U d$ and $D \wedge U d \Rightarrow U$ ), Lemma 545 (closedness under intersection and set difference, $D \Rightarrow I$ ), Lemma 550 (closedness under symmetric difference and set difference, $U \Rightarrow D \Leftrightarrow s D$ ), Lemma 551 (closedness
 symmetric difference, local complement, and disjoint union, $s D \Rightarrow D^{\prime}, s D \Rightarrow U d$ and $D^{\prime} \wedge U d \wedge I$ implies $s D$ ).

Remark 613. Note that the previous lemma means that the predicate $P_{\mathcal{R}}$ characterizing rings of sets is equivalent to any of the eight alternatives summarized by the conjunction (8.38) and (8.39).

## Lemma 614 (ring of sets is $\pi$-system).

Let $X$ be a set. Then, rings of sets on $X$ are $\pi$-systems on $X$.
Compact form: $P_{\mathcal{R}} \Rightarrow P_{\Pi}$.
Proof. Direct consequence of Lemma 612 (equivalent definition of ring of sets, $\underline{P_{\mathcal{R}}} \Rightarrow[N e \mid I]$ ), and Lemma 604 (equivalent definition of $\pi$-system, $P_{\Pi}=N e \wedge I$ ).

Lemma 615 (other properties of ring of sets). Let $X$ be a set. Let $\mathcal{R}$ be a ring of sets on $X$.
Then, $\mathcal{R}$ is closed under finite disjoint union and finite intersection.
Compact form: $P_{\mathcal{R}} \Rightarrow[U d f \mid I f]$.

Proof. Direct consequence of Lemma 612 (equivalent definition of ring of sets, $\underline{P_{\mathcal{R}} \Rightarrow U d}$ ), Lemma 558 (closedness under finite operations, $U d \Rightarrow U f$ ), Lemma 614 (ring of sets is $\pi$-system, $\underline{\left.P_{\mathcal{R}} \Rightarrow P_{\Pi}\right) \text {, and Definition } 602\left(\pi \text {-system, } P_{\Pi} \Rightarrow I f\right) . ~ . ~ . ~ . ~}$

Lemma 616 (intersection of rings of sets).
Let $X$ and $I$ be sets. Let $\left(\mathcal{R}_{i}\right)_{i \in I}$ be rings of sets on $X$. Then, $\bigcap_{i \in I} \mathcal{R}_{i}$ is a ring of sets on $X$.
Proof. Direct consequence of Lemma 612 (equivalent definition of ring of sets, with $w E$ ), and the definition of intersection.

## Definition 617 (generated ring of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. The ring of sets generated by $G$ is denoted $\mathcal{R}_{X}(G) \stackrel{\text { def. }}{=} \mathfrak{G}_{P_{\mathcal{R}}, X}(G)$; it is the intersection of all rings of sets on $X$ containing $G$.

## Lemma 618 (generated ring of sets is ring of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{R}_{X}(G)$ is a ring of sets on $X$ containing $G$.

Proof. Direct consequence of Definition 617 (generated ring of sets), Lemma 616 (intersection of rings of sets), and Lemma 585 (subset system generation preserves class).

## Lemma 619 (properties of ring of sets generation).

Let $X$ be a set. Then, ring of sets generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
& \forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{R}}, X}, G \subset \mathcal{S} \quad \Longrightarrow \mathcal{R}_{X}(G) \subset \mathcal{S}  \tag{8.40}\\
& \forall G_{1}, G_{2} \subset \mathcal{P}(X), G_{1} \subset G_{2}  \tag{8.41}\\
& \forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{R}}, X} \Longleftrightarrow \mathcal{R}_{X}\left(G_{1}\right) \subset \mathcal{R}_{X}\left(G_{2}\right),  \tag{8.42}\\
&
\end{align*}
$$

Proof. Direct consequence of Definition 617 (generated ring of sets), Lemma 612 (equivalent definition of ring of sets, with wE), Lemma 616 (intersection of rings of sets), and Lemma 586 (properties of subset system generation).

## Lemma 620 (other properties of ring of sets generation).

Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
G_{1} \subset \mathcal{R}_{X}\left(G_{2}\right) \wedge G_{2} \subset \mathcal{R}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{R}_{X}\left(G_{1}\right)=\mathcal{R}_{X}\left(G_{2}\right)  \tag{8.43}\\
G_{2} \subset \mathcal{R}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{R}_{X}\left(G_{1} \cup G_{2}\right)=\mathcal{R}_{X}\left(G_{1}\right) \tag{8.44}
\end{align*}
$$

Proof. Direct consequence of Definition 617 (generated ring of sets), Lemma 616 (intersection of rings of sets), and Lemma 587 (other properties of subset system generation).

## Lemma 621 (ring of sets contains $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Pi_{X}(G) \subset \mathcal{R}_{X}(G)$.
Proof. Direct consequence of Definition 617 (generated ring of sets), Definition 606 (generated $\pi$-system), Lemma 614 (ring of sets is $\pi$-system), and Lemma 599 (generated subset systems are monotone (with nonemptiness), with $P_{\mathcal{R}} \Rightarrow P_{\Pi}$ ).

Lemma 622 (upgrade $\pi$-system to ring of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$, and that $\Pi_{X}(G)$ is closed under symmetric difference.
Then, we have $\Pi_{X}(G)=\mathcal{R}_{X}(G)$.
Similar compact form: $P_{\Pi} \wedge s D \Leftrightarrow P_{\mathcal{R}}$.

Proof. Direct consequence of Lemma 621 (ring of sets contains $\pi$-system, $\underline{\Pi_{X}(G) \subset \mathcal{R}_{X}(G) \text { ), Def- }}$ inition 617 (generated ring of sets), Definition 606 (generated $\pi$-system), Lemma 616 (intersection of rings of sets), Lemma 605 (intersection of $\pi$-systems), Lemma 612 (equivalent definition of ring of sets, $P_{\mathcal{R}}=N e \wedge s D \wedge I$ ), Lemma 604 (equivalent definition of $\pi$-system, $P_{\Pi}=N e \wedge I$ ), and Lemma 600 (upgrade generated subset system (with nonemptiness), with $P_{\Pi} \wedge s D=P_{\mathcal{R}}$, thus $\left.\mathcal{R}_{X}(G) \subset \Pi_{X}(G)\right)$.

## Lemma 623 (ring of sets generated by $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\mathcal{R}_{X}\left(\Pi_{X}(G)\right)=\mathcal{R}_{X}(G)$.
Proof. Direct consequence of Definition 617 (generated ring of sets), Definition 606 (generated $\pi$-system), Lemma 616 (intersection of rings of sets), Lemma 605 (intersection of $\pi$-systems), Lemma 614 (ring of sets is $\pi$-system), and Lemma 601 (cascading generated subset systems (with nonemptiness), with $P_{\mathcal{R}} \Rightarrow P_{\Pi}$ ).

## Lemma 624 (disjoint union of union in ring of sets).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{R}$ be a ring of sets on $X$. Let $\left(A_{i}\right)_{i \in I} \subset \mathcal{R}$.
Then, there exists $\left(B_{i}\right)_{i \in I} \in \mathcal{R}$ pairwise disjoint such that for all $i \in I, B_{i} \subset A_{i}$, and we have

$$
\begin{equation*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 . . n]} B_{\varphi(j)} \in \mathcal{R} . \tag{8.45}
\end{equation*}
$$

Proof. Direct consequence of Lemma 612 (equivalent definition of ring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow[D \mid U d]$ ), and Lemma 572 (disjoint union of union in subset system, $\underline{H y p}=D \wedge U d$ ).

Remark 625. The next concept of "semiring of sets" is not to be confused with the algebraic structure "semiring".

## Definition 626 (semiring of sets).

Let $X$ be a set. Let $P_{\mathcal{R} / 2}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ expressing nonemptiness, closedness under intersection, and that any set difference is a finite disjoint union. The class of semirings (of (sub)sets) on $X$ is $\mathfrak{C}_{P_{\mathcal{R} / 2}}$ : a subset $\mathcal{R}$ of $\mathcal{P}(X)$ is a semiring of sets on $X$ iff

$$
\begin{array}{ll} 
& \mathcal{R} \neq \varnothing \\
\forall A, B \in \mathcal{R}, & A \cap B \in \mathcal{R} \tag{8.47}
\end{array}
$$

$$
\forall A, B \in \mathcal{R}, \quad \exists n \in \mathbb{N}, \exists\left(C_{i}\right)_{i \in[0 . . n]} \in \mathcal{R}, \quad\left\{\begin{array}{l}
\left(C_{i}\right)_{i \in[0 . . n]} \text { is pairwise disjoint }  \tag{8.48}\\
A \backslash B=\biguplus_{i \in[0 . . n]} C_{i}
\end{array}\right.
$$

Compact form: $P_{\mathcal{R} / 2} \stackrel{\text { def. }}{=} N e \wedge I \wedge P t f D$.
Lemma 627 (equivalent definition of semiring of sets). Let $X$ be a set. Let $\mathcal{R} \subset \mathcal{P}(X)$. Then, $\mathcal{R}$ is a semiring of sets on $X$ iff (8.47), and (8.48) hold and

$$
\begin{equation*}
\mathcal{R} \neq \varnothing, \quad \text { or } \quad \varnothing \in \mathcal{R} \tag{8.49}
\end{equation*}
$$

Compact form: $P_{\mathcal{R} / 2} \Leftrightarrow[N e \mid w E] \wedge I \wedge P t f D$.
Proof. Direct consequence of Definition 626 (semiring of sets, $\underline{P}_{\mathcal{R} / 2}=N e \wedge I \wedge P t f D$ ), Lemma 552 (nonempty is with empty, $N e \wedge P t f D \Rightarrow w E$ ), and Lemma 538 (nonempty and with empty or full, $\underline{w E \Rightarrow N e}$.

Remark 628. Note that (8.48) is not a property compatible with intersection: the intersection of two semirings of sets on $X$ is not necessarily a semiring of sets on $X$. Thus, the concept of generated semiring of sets is not practical. The main use of semirings of sets is to ease the building of rings of sets, e.g. through Lemma 632.

## Lemma 629 (semiring of sets is $\pi$-system).

Let $X$ be a set. Then, semirings of sets on $X$ are $\pi$-systems on $X$.
Compact form: $P_{\mathcal{R} / 2} \Rightarrow P_{\Pi}$.
Proof. Direct consequence of Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow[N e \mid I] \text { ), and Lemma } 604}$ (equivalent definition of $\pi$-system, $P_{\Pi}=N e \wedge I$ ).

## Lemma 630 (other properties of semiring of sets).

Let $X$ be a set. Let $\mathcal{R}$ be a semiring of sets on $X$. Then, any local complement of elements of $\mathcal{R}$ is a finite disjoint union of elements of $\mathcal{R}$, and $\mathcal{R}$ is closed under finite intersection.

Compact form: $P_{\mathcal{R} / 2} \Rightarrow\left[P t f D^{\prime} \mid I f\right]$.

Proof. Direct consequence of Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow \text { PtfD }}$ ), Lemma 557 (disjoint decomposition of set difference and local complement, PtfD $\Rightarrow \overline{P t f D^{\prime}}$ ), Lemma 629 (semiring of sets is $\pi$-system, $\underline{\left.P_{\mathcal{R} / 2} \Rightarrow P_{\Pi}\right)}$, and Definition $602\left(\pi\right.$-system, $\left.\underline{P_{\Pi} \Rightarrow I f}\right)$.

## Lemma 631 (ring of sets is semiring of sets).

Let $X$ be a set. Then, rings of sets on $X$ are semirings of sets on $X$.
Compact form: $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R} / 2}$.
Proof. Direct consequence of Lemma 612 (equivalent definition of ring of sets, $\mathcal{P}_{\mathcal{R}} \Rightarrow[N e|I| D]$ ), Lemma 554 (disjoint decomposition, $\underline{D \Rightarrow P t f D}$ ), and Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2} \text { equals }}$ $N e \wedge I \wedge P t f D)$.

## Lemma 632 (explicit ring of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is a semiring of sets on $X$.
Then, $\mathcal{R}_{X}(G)$ is the set of finite (disjoint) unions of elements of $G$ :

$$
\begin{align*}
\mathcal{R}_{X}(G) & =\left\{\bigcup_{i \in[0 . . n]} A_{i} \mid n \in \mathbb{N} \wedge\left(A_{i}\right)_{i \in[0 . . n]} \in G\right\}  \tag{8.50}\\
& =\left\{\biguplus_{i \in[0 . . n]} A_{i} \mid n \in \mathbb{N} \wedge\left(A_{i}\right)_{i \in[0 . . n]} \in G \text { is pairwise disjoint }\right\} .
\end{align*}
$$

Proof. Let $\mathcal{R} \stackrel{\text { def. }}{=} \mathcal{R}_{X}(G)$. Let $\mathcal{U}$ be the set of finite unions of elements of $G$, and $\mathcal{U}^{+}$be the set of finite disjoint unions of elements of $G$.
(0a). $\mathcal{U}^{+} \subset \mathcal{U}$. Direct consequence of the definitions of $\mathcal{U}^{+}$and $\mathcal{U}$.
(0b). $\boldsymbol{U} \subset \mathcal{R}$. Direct consequence of Lemma 618 (generated ring of sets is ring of sets, $\underline{G \subset \mathcal{R}}$ ), and Definition 611 (ring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow U f$ (for $\mathcal{R}$ )).
(1a). $\boldsymbol{G} \subset \mathcal{U}^{+}$. Direct consequence of the definition of $\mathcal{U}^{+}$(with $n \stackrel{\text { def. }}{=} 0$ ).
(1b). $\varnothing \in \mathcal{U}^{+}$. Direct consequence of Lemma 627 (equivalent definition of semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow w E \text {, thus } \varnothing \in G}$ ), and (1a).
(1c). $\mathcal{U}^{+}$is closed under set difference. Direct consequence of Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2}} \Rightarrow[I \mid P t f D]($ for $G)$ ), and Lemma 563 (closedness under set difference and finite disjoint union closure, $I \wedge P t f D \Rightarrow D\left(\mathcal{U}^{+}\right)$).
(1d). $\mathcal{U}^{+}$is closed under disjoint union. Direct consequence of Lemma 562 (finite disjoint union closure is closed under finite disjoint union, $\left.\underline{U d}\left(\mathcal{U}^{+}\right)\right)$.
(2). $\boldsymbol{\mathcal { R }} \subset \mathcal{U}^{+}$. Direct consequence of (1b), (1c), (1d), Lemma 612 (equivalent definition of ring of sets, $P_{\mathcal{R}}=w E \wedge D \wedge U d$, thus $\mathcal{U}^{+}$is ring of sets), (1a), and Lemma 619 (properties of ring of sets generation, minimality).

Therefore, from (0a), (0b) and (2), we have $\mathcal{R}=\mathcal{U}=\mathcal{U}^{+}$.

### 8.5 Algebra of sets

Remark 633. The next concept of "algebra of sets" is not to be confused with the algebraic structure "algebra over a field" defined in Section 7.4.2.

Note that the same concept is sometimes called "field of sets", which is of course not to be confused with the algebraic structure either.

Definition 634 (algebra of sets). Let $X$ be a set. Let $P_{\mathcal{A}}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of containing the empty set, and of closedness under complement and finite union. The class of algebras (of (sub)sets) on $X$ is $\mathfrak{C}_{P_{\mathcal{A}}}$ : a subset $\mathcal{A}$ of $\mathcal{P}(X)$ is an algebra of sets on $X$ iff

$$
\begin{equation*}
\forall n \in \mathbb{N}, \forall\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{A}, \quad \bigcup_{i \in[0 . . n]} A_{i} \in \mathcal{A} . \tag{8.52}
\end{equation*}
$$

Compact form: $P_{\mathcal{A}} \stackrel{\text { def. }}{=} w E \wedge C \wedge U f$.

## Lemma 635 (equivalent definition of algebra of sets).

Let $X$ be a set. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then, $\mathcal{A}$ is an algebra of sets on $X$ iff (8.52) holds and

$$
\begin{gather*}
\mathcal{A} \neq \varnothing, \quad \text { or } \quad \varnothing \in \mathcal{A}, \quad \text { or } \quad X \in \mathcal{A},  \tag{8.54}\\
\forall A, B \in \mathcal{A}, A \cup B \in \mathcal{A} \quad \text { or } \quad \forall A, B \in \mathcal{A}, A \cap B \in \mathcal{A} . \tag{8.55}
\end{gather*}
$$

Compact form: $P_{\mathcal{A}} \Leftrightarrow[N e|w E| w F] \wedge C \wedge[U \mid I]$.
Proof. Direct consequence of Definition 634 (algebra of sets, $P_{\mathcal{A}}=w E \wedge C \wedge U f$ ), Lemma 558 (closedness under finite operations, Uf $\Leftrightarrow U$ ). Lemma 547 (closedness under union and intersection, $C \Rightarrow U \Leftrightarrow I$ ), Lemma $538(\overline{\text { nonempty }}$ and with empty or full, wE $\Rightarrow N e$ and $C \wedge I \wedge N e$ implies $\overline{w E}$ ), and Lemma 539 (with empty and full, $C \Rightarrow w E \Leftrightarrow w F$ ).

## Remark 636.

Note that the previous lemma means that the predicate $P_{\mathcal{A}}$ characterizing algebras of sets is equivalent to any of the six alternatives summarized by the conjunction (8.54), (8.52), and (8.55).

## Lemma 637 (other equivalent definition of algebra of sets).

Let $X$ be a set. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then, $\mathcal{A}$ is an algebra of sets on $X$ iff it contains the full set, and it is closed under set difference:

$$
\begin{align*}
& X \in \mathcal{A} .  \tag{8.56}\\
\forall A, B \in \mathcal{A}, & A \backslash B \in \mathcal{A} . \tag{8.57}
\end{align*}
$$

Compact form: $P_{\mathcal{A}} \Leftrightarrow w F \wedge D$.
Proof. Direct consequence of Lemma 635 (equivalent definition of algebra of sets, $\underline{P_{\mathcal{A}}}$ equals $w F \wedge C \wedge I$ ), Lemma 545 (closedness under intersection and set difference, $C \wedge I \Rightarrow D$ and $\overline{D \Rightarrow I}$ ), Lemma 543 (closedness under set difference and local complement, $D \Rightarrow \overline{D^{\prime}}$ ), and Lemma 541 (closedness under local complement and complement, wF $\wedge D^{\prime} \Rightarrow C$ ).

## Lemma 638 (algebra of sets is ring of sets).

Let $X$ be a set. Then, algebras of sets on $X$ are rings of sets on $X$.
Compact form: $P_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$.

Proof. Direct consequence of Lemma 635 (equivalent definition of algebra of sets, $\underline{P_{\mathcal{A}}} \Rightarrow[w E \mid U]$ ), Lemma 637 (other equivalent definition of algebra of sets, $\underline{P}_{\mathcal{A}} \Rightarrow D$ ), and Lemma 612 (equivalent definition of ring of sets, $\left.P_{\mathcal{R}}=w E \wedge D \wedge U\right)$.

## Lemma 639 (other properties of algebra of sets).

Let $X$ be a set. Let $\mathcal{A}$ be an algebra of sets on $X$. Then, $\mathcal{A}$ is closed under local complement, symmetric difference, disjoint union, finite disjoint union, and finite intersection.
Compact form: $P_{\mathcal{A}} \Rightarrow\left[D^{\prime}|s D| U d|U d f| I f\right]$.
Proof. Direct consequence of Lemma 638 (algebra of sets is ring of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$ ), Lemma 612 (equivalent definition of ring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow\left[D^{\prime} \mid s D\right]$ ), Lemma 615 (other properties of ring of sets, $\underline{P_{\mathcal{R}}} \Rightarrow I f$ ), Lemma 635 (equivalent definition of algebra of sets, $\underline{P} \mathcal{A}^{\Rightarrow U}$ ), Lemma 546 (closedness under union and disjoint union, $\underline{U} \Rightarrow U d$ ), and Lemma 558 (closedness under finite operations, $\underline{U d \Rightarrow U d f})$.

## Lemma 640 (algebra of sets is $\pi$-system).

Let $X$ be a set. Then, algebras of sets on $X$ are $\pi$-systems on $X$.
Compact form: $P_{\mathcal{A}} \Rightarrow P_{\Pi}$.
Proof. Direct consequence of Lemma 638 (algebra of sets is ring of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$ ), and Lem-


Lemma 641 (intersection of algebras of sets). Let $X$ and $I$ be sets. Let $\left(\mathcal{A}_{i}\right)_{i \in I}$ be algebras of sets on $X$. Then, $\bigcap_{i \in I} \mathcal{A}_{i}$ is an algebra of sets on $X$.

Proof. Direct consequence of Definition 634 (algebra of sets), and the definition of intersection.

Definition 642 (generated algebra of sets).
Let $X$ be a set.
Let $G \subset \mathcal{P}(X)$. The algebra of sets generated by $G$ is denoted $\mathcal{A}_{X}(G) \stackrel{\text { def. }}{=} \mathfrak{G}_{P_{\mathcal{A}}, X}(G)$; it is the intersection of all algebras of sets on $X$ containing $G$.

## Lemma 643 (generated algebra of sets is algebra of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{A}_{X}(G)$ is an algebra of sets on $X$ containing $G$.
Proof. Direct consequence of Definition 642 (generated algebra of sets), Lemma 641 (intersection of algebras of sets), and Lemma 585 (subset system generation preserves class).

## Lemma 644 (properties of algebra of sets generation).

Let $X$ be a set. Then, algebra of sets generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
\forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{A}}, X}, & G \subset \mathcal{S} \quad \Longrightarrow \mathcal{A}_{X}(G) \subset \mathcal{S},  \tag{8.58}\\
\forall G_{1}, G_{2} \subset \mathcal{P}(X), & G_{1} \subset G_{2}  \tag{8.59}\\
\forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{A}}, X} & \Longleftrightarrow \mathcal{A}_{X}\left(G_{1}\right) \subset \mathcal{A}_{X}\left(G_{2}\right),  \tag{8.60}\\
& \Longleftrightarrow \mathcal{A}_{X}(\mathcal{S})=\mathcal{S}
\end{align*}
$$

Proof. Direct consequence of Definition 642 (generated algebra of sets), Lemma 641 (intersection of algebras of sets), and Lemma 586 (properties of subset system generation).

## Lemma 645 (other properties of algebra of sets generation).

Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
G_{1} \subset \mathcal{A}_{X}\left(G_{2}\right) \wedge G_{2} \subset \mathcal{A}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{A}_{X}\left(G_{1}\right)=\mathcal{A}_{X}\left(G_{2}\right)  \tag{8.61}\\
G_{2} \subset \mathcal{A}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{A}_{X}\left(G_{1} \cup G_{2}\right)=\mathcal{A}_{X}\left(G_{1}\right) . \tag{8.62}
\end{align*}
$$

Proof. Direct consequence of Definition 642 (generated algebra of sets), Lemma 641 (intersection of algebras of sets), and Lemma 587 (other properties of subset system generation).

## Lemma 646 (algebra of sets contains ring of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{R}_{X}(G) \subset \mathcal{A}_{X}(G)$.
Proof. Direct consequence of Definition 642 (generated algebra of sets), Definition 617 (generated ring of sets), Lemma 638 (algebra of sets is ring of sets), and Lemma 588 (generated subset systems are monotone, with $\left.P_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}\right)$.

Lemma 647 (upgrade ring of sets to algebra of sets). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{R}_{X}(G)$ either contains the full set, or is closed under complement.
Then, we have $\mathcal{R}_{X}(G)=\mathcal{A}_{X}(G)$.
Similar compact form: $P_{\mathcal{R}} \wedge[w F \mid C] \Leftrightarrow P_{\mathcal{A}}$.
Proof. Direct consequence of Lemma 646 (algebra of sets contains ring of sets, $\mathcal{R}_{X}(G) \subset \mathcal{A}_{X}(G)$ ), Definition 642 (generated algebra of sets), Definition 617 (generated ring of sets), Lemma 641 (intersection of algebras of sets), Lemma 616 (intersection of rings of sets), Lemma 637 (other equivalent definition of algebra of sets, $\underline{P} \mathcal{A}=w F \wedge D$ ), Lemma 635 (equivalent definition of algebra of sets, $\left.\underline{P}_{\mathcal{A}}=N e \wedge C \wedge U\right)$, Lemma 612 (equivalent definition of ring of sets, $\left.\underline{P}_{\mathcal{R}}=N e \wedge D \wedge U\right)$, Lemma 590 (upgrade generated subset system, with $P_{\mathcal{R}} \wedge[w F \mid C] \Rightarrow P_{\mathcal{A}}$, thus $\mathcal{A}_{X}(G) \subset \mathcal{R}_{X}(G)$ ), and Lemma 638 (algebra of sets is ring of sets, $\underline{P A}_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$, thus $P_{\mathcal{A}} \Rightarrow\left[P_{\mathcal{R}}|w F| C\right]$ ).

## Lemma 648 (algebra of sets generated by ring of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{A}_{X}\left(\mathcal{R}_{X}(G)\right)=\mathcal{A}_{X}(G)$.
Proof. Direct consequence of Definition 642 (generated algebra of sets), Definition 617 (generated ring of sets), Lemma 641 (intersection of algebras of sets), Lemma 616 (intersection of rings of sets), Lemma 638 (algebra of sets is ring of sets), and Lemma 591 (cascading generated subset systems, with $\left.P_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}\right)$.

## Lemma 649 (algebra of sets contains $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Pi_{X}(G) \subset \mathcal{A}_{X}(G)$.
Proof. Direct consequence of Lemma 621 (ring of sets contains $\pi$-system, $\Pi_{X}(G) \subset \mathcal{R}_{X}(G)$ ), Lemma 646 (algebra of sets contains ring of sets, $\mathcal{R}_{X}(G) \subset \mathcal{A}_{X}(G)$ ), and transitivity of inclusion.

Lemma 650 (upgrade $\pi$-system to algebra of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$, and that $\Pi_{X}(G)$ is closed under complement. Then, we have $\Pi_{X}(G)=\mathcal{A}_{X}(G)$.
Similar compact form: $P_{\Pi} \wedge C \Leftrightarrow P_{\mathcal{A}}$.
Proof. Direct consequence of Lemma 604 (equivalent definition of $\pi$-system, $P_{\Pi} \Rightarrow I$ ), Lemma 547 (closedness under union and intersection, $I \wedge C \Rightarrow U$ ), Lemma 545 (closedness under intersection and set difference, $I \wedge C \Rightarrow D)$, Lemma 550 (closedness under symmetric difference and set difference, $U \wedge D \Rightarrow s D$ ), Lemma 622 (upgrade $\pi$-system to ring of sets, $P_{\Pi} \wedge C \Rightarrow P_{\Pi} \wedge s D=P_{\mathcal{R}}$, thus $\Pi_{X}(G)=\mathcal{R}_{X}(G)$ ), Lemma 614 (ring of sets is $\pi$-system, $P_{\mathcal{R}} \Rightarrow \overline{P_{\Pi}}$ ), and Lemma 647 (upgrade ring of sets to algebra of sets, $P_{\Pi} \wedge C \Leftrightarrow P_{\mathcal{R}} \wedge C \Leftrightarrow P_{\mathcal{A}}$, thus $\left.\mathcal{R}_{X}(G)=\mathcal{A}_{X}(G)\right)$.

## Lemma 651 (algebra of sets generated by $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\mathcal{A}_{X}\left(\Pi_{X}(G)\right)=\mathcal{A}_{X}(G)$.

Proof. Direct consequence of Definition 642 (generated algebra of sets), Definition 617 (generated ring of sets), Definition 606 (generated $\pi$-system), Lemma 648 (algebra of sets generated by ring of sets), Lemma 623 (ring of sets generated by $\pi$-system), and Lemma 592 (double cascading generated subset systems).

## Lemma 652 (disjoint union of union in algebra of sets).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\mathcal{A}$ be an algebra of sets on $X . \operatorname{Let}\left(A_{i}\right)_{i \in I} \subset \mathcal{A}$.
Then, there exists $\left(B_{i}\right)_{i \in I} \in \mathcal{A}$ pairwise disjoint such that for all $i \in I, B_{i} \subset A_{i}$, and we have

$$
\begin{equation*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 . . n]} B_{\varphi(j)} \in \mathcal{A} . \tag{8.63}
\end{equation*}
$$

Proof. Direct consequence of Lemma 638 (algebra of sets is ring of sets), and Lemma 624 (disjoint union of union in ring of sets).

## Remark 653.

Note that the next concept of "semi-algebra ${ }^{a}$ of sets" is sometimes called "semifield of sets".
${ }^{a}$ We use a hyphen here because "semialgebra" looks awful (thus applying the algorithm from https://www. grammar-monster.com/lessons/hyphens_in_prefixes.htm).

## Definition 654 (semi-algebra of sets).

Let $X$ be a set.
Let $P_{\mathcal{A} / 2}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of being a semiring of sets and containing the full set. The class of semi-algebras (of (sub)sets) on $X$ is $\mathfrak{C}_{P_{\mathcal{A} / 2}}$.

Compact form: $P_{\mathcal{A} / 2} \stackrel{\text { def. }}{=} P_{\mathcal{R} / 2} \wedge w F$.

## Lemma 655 (equivalent definition of semi-algebra of sets).

Let $X$ be a set. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then, $\mathcal{A}$ is a semi-algebra of sets on $X$ iff

$$
\begin{equation*}
\forall A, B \in \mathcal{A}, \quad A \cap B \in \mathcal{A} \tag{8.64}
\end{equation*}
$$



Compact form: $P_{\mathcal{A} / 2} \Leftrightarrow w F \wedge I \wedge[P t f C \mid P t f D]$.
Proof. Direct consequence of Definition 654 (semi-algebra of sets, $P_{\mathcal{A} / 2}=P_{\mathcal{R} / 2} \wedge w F$ ), Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2}}=N e \wedge I \wedge P t f D$ ), Lemma 557 (disjoint decomposition of set difference and local complement, $\underline{P t f D} \Rightarrow P t f D^{\prime}$ ), Lemma 556 (disjoint decomposition of local complement and complement, wF $\wedge \overline{P t f D^{\prime}} \Rightarrow \operatorname{PtfC}$ ). Lemma 538 (nonempty and with empty or full, $w F \Rightarrow N e$ ), and Lemma $5 \overline{55 \text { (disjoint decomposition of complement and set difference, } I \wedge P t f C}$ implies PtfD).

Remark 656. Note that (8.66) is not a property compatible with intersection: the intersection of two semi-algebras of sets on $X$ is not necessarily a semi-algebra of sets on $X$. Thus, the concept of generated semi-algebra of sets is not practical. The main use of semi-algebras of sets is to ease the building of algebras of sets, e.g. through Lemma 662 .

## Lemma 657 (semi-algebra of sets is semiring of sets).

Let $X$ be a set. Then, semi-algebras of sets on $X$ are semirings of sets on $X$.
Compact form: $P_{\mathcal{A} / 2} \Rightarrow P_{\mathcal{R} / 2}$.


## Lemma 658 (other properties of semi-algebra of sets).

Let $X$ be a set. Let $\mathcal{A}$ be a semi-algebra of sets on $X$. Then, any local complement of elements of $\mathcal{A}$ is a finite disjoint union of elements of $\mathcal{A}$, and $\mathcal{A}$ is closed under finite intersection.

Compact form: $P_{\mathcal{A} / 2} \Rightarrow\left[P t f D^{\prime} \mid I f\right]$.
Proof. Direct consequence of Lemma 657 (semi-algebra of sets is semiring of sets, $\underline{P}_{\mathcal{A} / 2} \Rightarrow P_{\mathcal{R} / 2}$ ), and Lemma 630 (other properties of semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow\left[P t f D^{\prime} \mid I f\right] \text { ). }}$

## Lemma 659 (semi-algebra of sets is $\pi$-system).

Let $X$ be a set. Then, semi-algebras of sets on $X$ are $\pi$-systems on $X$.
Compact form: $P_{\mathcal{A} / 2} \Rightarrow P_{\Pi}$.
Proof. Direct consequence of Lemma 657 (semi-algebra of sets is semiring of sets, $\underline{P_{\mathcal{A} / 2} \Rightarrow P_{\mathcal{R} / 2} \text { ), }}$ and Lemma 629 (semiring of sets is $\pi$-system, $\underline{P_{\mathcal{R} / 2}} \Rightarrow P_{\Pi}$ ).

## Lemma 660 (algebra of sets is semi-algebra of sets).

Let $X$ be a set. Then, algebras of sets on $X$ are semi-algebras of sets on $X$.
Compact form: $P_{\mathcal{A}} \Rightarrow P_{\mathcal{A} / 2}$.
Proof. Direct consequence of Lemma 635 (equivalent definition of algebra of sets, $P_{\mathcal{A}}$ implies $[w F|I| C]$ ), Lemma 554 (disjoint decomposition, $C \Rightarrow P t f C$ ), and Lemma 655 (equivalent definition of semi-algebra of sets, $\left.\underline{P_{\mathcal{A} / 2}=w F \wedge I \wedge P t f C}\right)$.

Lemma 661 (algebra of sets is semiring of sets).
Let $X$ be a set. Then, algebras of sets on $X$ are semirings of sets on $X$.
Compact form: $P_{\mathcal{A}} \Rightarrow P_{\mathcal{R} / 2}$.
Proof. Direct consequence of Lemma 660 (algebra of sets is semi-algebra of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{A} / 2}$ ), and Lemma 657 (semi-algebra of sets is semiring of sets, $\underline{P_{\mathcal{A} / 2} \Rightarrow P_{\mathcal{R} / 2}}$ ).

## Lemma 662 (explicit algebra of sets).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is a semi-algebra of sets on $X$.
Then, $\mathcal{A}_{X}(G)$ is the set of finite (disjoint) unions of elements of $G$ :

$$
\begin{align*}
\mathcal{A}_{X}(G) & =\left\{\bigcup_{i \in[0 . . n]} A_{i} \mid n \in \mathbb{N} \quad \wedge \quad \forall i \in[0 . . n], A_{i} \in G\right\}  \tag{8.67}\\
& =\left\{\biguplus_{p \in[0 . . n]} A_{p} \mid n \in \mathbb{N} \wedge\left(A_{i}\right)_{i \in[0 . . n]} \in G \text { is pairwise disjoint }\right\}
\end{align*}
$$

Proof. Direct consequence of Definition 654 (semi-algebra of sets, $\underline{P}_{\mathcal{A} / 2}=P_{\mathcal{R} / 2} \wedge w F$ ), Lemma 632 (explicit ring of sets, $\mathcal{R}_{X}(G)=\mathcal{U}=\mathcal{U}^{+}$), Lemma 618 (generated ring of sets is ring of sets, thus $\underline{X \in G \subset \mathcal{R}_{X}(G)}$ ), and Lemma 647 (upgrade ring of sets to algebra of sets, $\underline{\left.\mathcal{A}_{X}(G)=\mathcal{R}_{X}(G)\right) .}$

### 8.6 Monotone class

Remark 663. Note that in the next definition, nonemptiness of the set of indices $I$ is important since a monotone class is not required to contain the empty set, or the full set.

## Definition 664 (monotone class).

Let $X$ be a set. Let $P_{\mathcal{C}}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of closedness under countable monotone union and intersection. The class of monotone classes on $X$ is $\mathfrak{C}_{P_{\mathcal{C}}}$ : a subset $\mathcal{C}$ of $\mathcal{P}(X)$ is a monotone class on $X$ iff

$$
\begin{align*}
& \forall I \subset \mathbb{N} \text { nonempty, } \forall\left(A_{i}\right)_{i \in I} \in \mathcal{C}, \quad\left(A_{i}\right)_{i \in I} \text { nondecreasing }  \tag{8.68}\\
& \forall I \subset \mathbb{N} \text { nonempty, } \forall\left(A_{i}\right)_{n \in I} \in \mathcal{C}, \quad\left(A_{i}\right)_{i \in I} \text { nonincreasing } \tag{8.69}
\end{align*} \quad \Longrightarrow \bigcup_{i \in I} A_{n} \in \mathcal{C}, ~ \bigcap_{i \in I} A_{n} \in \mathcal{C} .
$$

Compact form: $P_{\mathcal{C}} \stackrel{\text { def. }}{=} U m c \wedge I m c$.
Lemma 665 (intersection of monotone classes).
Let $X$ and $I$ be sets.
Let $\left(\mathcal{C}_{i}\right)_{i \in I}$ be a family of monotone classes on $X$. Then, $\bigcap_{i \in I} \mathcal{C}_{i}$ is a monotone class on $X$.
Proof.
Direct consequence of Definition 664 (monotone class), and the definition of intersection.

## Definition 666 (generated monotone class).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. The monotone class generated by $G$ is denoted $\mathcal{C}_{X}(G) \stackrel{\text { def }}{=}$ $\mathfrak{G}_{P_{\mathcal{C}}, X}(G)$; it is the intersection of all monotone classes on $X$ containing $G$.

## Lemma 667 (generated monotone class is monotone class).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\mathcal{C}_{X}(G)$ is an monotone class on $X$ containing $G$.
Proof. Direct consequence of Definition 666 (generated monotone class), Lemma 665 (intersection of monotone classes), and Lemma 585 (subset system generation preserves class).

## Lemma 668 (properties of monotone class generation).

Let $X$ be a set. Then, monotone class generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
\forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\mathcal{C}}, X}, & G \subset \mathcal{S} \quad \Longrightarrow \mathcal{C}_{X}(G) \subset \mathcal{S},  \tag{8.70}\\
\forall G_{1}, G_{2} \subset \mathcal{P}(X), & G_{1} \subset G_{2} \tag{8.71}
\end{align*} \quad \Longrightarrow \mathcal{C}_{X}\left(G_{1}\right) \subset \mathcal{C}_{X}\left(G_{2}\right), ~ 子 \mathcal{P} \subset \mathcal{C}_{X}(\mathcal{S})=\mathcal{S} .
$$

Proof. Direct consequence of Definition 666 (generated monotone class), Lemma 665 (intersection of monotone classes), and Lemma 586 (properties of subset system generation).

## Lemma 669 (other properties of monotone class generation).

Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
G_{1} \subset \mathcal{C}_{X}\left(G_{2}\right) \wedge G_{2} \subset \mathcal{C}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{C}_{X}\left(G_{1}\right)=\mathcal{C}_{X}\left(G_{2}\right)  \tag{8.73}\\
G_{2} \subset \mathcal{C}_{X}\left(G_{1}\right) & \Longrightarrow \mathcal{C}_{X}\left(G_{1} \cup G_{2}\right)=\mathcal{C}_{X}\left(G_{1}\right) \tag{8.74}
\end{align*}
$$

Proof. Direct consequence of Definition 666 (generated monotone class), Lemma 665 (intersection of monotone classes), and Lemma 587 (other properties of subset system generation).

Definition 670 (monotone class and symmetric set difference). Let $X$ be a set. Let $\mathcal{C}$ be a monotone class on $X$. For all $A \subset X$, let $\mathcal{C}_{A}$ be the subset system defined by

$$
\begin{equation*}
\mathcal{C}_{A}^{\backslash} \stackrel{\text { def. }}{=}\{B \subset X \mid A \backslash B \in \mathcal{C} \wedge B \backslash A \in \mathcal{C}\} \tag{8.75}
\end{equation*}
$$

Lemma 671 (C $\backslash$ is symmetric).
Let $X$ be a set.
Let $\mathcal{C}$ be a monotone class on $X$. Then, for all $A, B \subset X$, we have $B \in \mathcal{C}_{A}^{\backslash}$ iff $A \in \mathcal{C}_{B}$.
Proof. Direct consequence of Definition 670 (monotone class and symmetric set difference), and commutativity of logical conjunction.

## Lemma 672 ( $\mathcal{C}$ \is monotone class).

Let $X$ be a set.
Let $\mathcal{C}$ be a monotone class on $X$. Then, for all $A \subset X, \mathcal{C}_{A}$ is a monotone class on $X$.
Compact form: $P_{\mathcal{C}} \Rightarrow P_{\mathcal{C}}\left(\mathcal{C}_{A}^{\backslash}\right)$.
Proof. Let $A \subset X$. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(B_{i}\right)_{i \in I} \in \mathcal{C}_{A}$. Let $i \in I$. Then, from Definition 670 (monotone class and symmetric set difference), we have $A \backslash B_{i}, B_{i} \backslash A \in \mathcal{C}$.

Assume first that $\left(B_{i}\right)_{i \in I}$ is nondecreasing. Then, from monotonicity of set difference, $\left(A \backslash B_{i}\right)_{i \in I}$ is nonincreasing, and $\left(B_{i} \backslash A\right)_{i \in I}$ is nondecreasing. Let $B \stackrel{\text { def. }}{=} \bigcup_{i \in I} B_{i}$. Then, from the definition of set difference, De Morgan's laws, Definition 670 (monotone class and symmetric set difference), and Definition 664 (monotone class), we have $A \backslash B=\bigcap_{i \in I}\left(A \backslash B_{i}\right) \in \mathcal{C}$ and $B \backslash A=\bigcup_{i \in I}\left(B_{i} \backslash A\right) \in \mathcal{C}$. Thus, from Definition 670 (monotone class and symmetric set difference), we have $B \in \mathcal{C}_{A}$. Hence, $\mathcal{C}_{A}^{\backslash}$ is closed under countable monotone union.

Assume now that $\left(B_{i}\right)_{i \in I}$ is nonincreasing. Then, from monotonicity of set difference, $\left(A \backslash B_{i}\right)_{i \in I}$ is nondecreasing, and $\left(B_{i} \backslash A\right)_{i \in I}$ is nonincreasing. Let $B \stackrel{\text { def. }}{=} \bigcap_{i \in I} B_{i}$. Then, from the definition of set difference, De Morgan's laws, Definition 670 (monotone class and symmetric set difference), and Definition 664 (monotone class), we have $A \backslash B=\bigcup_{n \in \mathbb{N}}\left(A \backslash B_{n}\right) \in \mathcal{C}$ and $B \backslash A=\bigcap_{n \in \mathbb{N}}\left(B_{n} \backslash A\right) \in \mathcal{C}$. Thus, from Definition 670 (monotone class and symmetric set difference), we have $B \in \mathcal{C}_{A}^{\backslash}$. Hence, $\mathcal{C}_{A}$ is closed under countable monotone intersection.

Therefore, from Definition 664 (monotone class), $\mathcal{C}_{A}^{\backslash}$ is a monotone class on $X$.
Remark 673 . Note that the compact formula in the next lemma should actually be read as for all subset system $G, D(G)$ implies $I\left(\mathcal{C}_{X}(G)\right)$.

## Lemma 674 (monotone class is closed under set difference).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is closed under set difference.
Then, $\mathcal{C}_{X}(G)$ is closed under set difference.
Compact form: $D \Rightarrow D\left(\mathcal{C}_{X}\right)$.
Proof. For all $A \subset X$, we use the simplified notation $\mathcal{C}_{A} \backslash$ def. $\left(\mathcal{C}_{X}(G)\right)_{A}^{\}$.
(1). $\forall A \in G, \mathcal{C}_{\boldsymbol{X}}(\boldsymbol{G}) \subset \mathcal{C}_{A}$. Let $A \in G$.

Let $B \in G$. Then, we have $A \backslash B, B \backslash A \in G$. Thus, from Lemma 667 (generated monotone class is monotone class, $G \subset \mathcal{C}_{X}(G)$ ), and Definition 670 (monotone class and symmetric set difference), we have $B \in \mathcal{C}_{A}^{\backslash}$, i.e. $G \subset \mathcal{C}_{A} \backslash$. Hence, from Lemma 672 ( $\mathcal{C} \backslash$ is monotone class), and Lemma 668 (properties of monotone class generation, minimality), we have $\mathcal{C}_{X}(G) \subset \mathcal{C}_{A}$.
(2). $\forall \boldsymbol{B} \in \mathcal{C}_{\boldsymbol{X}}(\boldsymbol{G}), \mathcal{C}_{\boldsymbol{X}}(\boldsymbol{G}) \subset \mathcal{C}_{B}$. Let $B \in \mathcal{C}_{X}(G)$.

Let $A \in G$. Then, from (1), we have $B \in \mathcal{C}_{A}^{\backslash}$, and from Lemma 671 ( $\mathcal{C} \backslash$ is symmetric), we have $A \in \mathcal{C}_{B}^{\backslash}$, i.e. $G \subset \mathcal{C}_{B}$. Hence, from Lemma 672 ( $\mathcal{C} \backslash$ is monotone class), and Lemma 668 (properties of monotone class generation, minimality), we have $\mathcal{C}_{X}(G) \subset \mathcal{C}_{B}$.
Let $A, B \in \mathcal{C}_{X}(G)$. Then, from (2), we have $B \in \mathcal{C}_{A}$, and from Definition 670 (monotone class and symmetric set difference), we have $A \backslash B \in \mathcal{C}_{X}(G)$.
Therefore, $\mathcal{C}_{X}(G)$ is closed under set difference.

Lemma 675 (monotone class generated by algebra of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is an algebra of sets on $X$.
Then, $\mathcal{C}_{X}(G)$ is an algebra of sets on $X$.
Compact form: $P_{\mathcal{A}} \Rightarrow P_{\mathcal{A}}\left(\mathcal{C}_{X}\right)$.
Proof. Direct consequence of Lemma 637 (other equivalent definition of algebra of sets, $\underline{P_{\mathcal{A}} \text { equals }}$ $w F \wedge D$ (for $G$, then for $\left.\mathcal{C}_{X}(G)\right)$ ), Lemma 667 (generated monotone class is monotone class, $\overline{G \subset \mathcal{C}_{X}(G), ~ t h u s ~} w F\left(\mathcal{C}_{X}(G)\right)$ ), and Lemma 674 (monotone class is closed under set difference, thus $\left.D\left(\mathcal{C}_{X}(G)\right)\right)$.

### 8.7 Lambda-system

Remark 676. Note that the next concept of " $\lambda$-system" is sometimes called "Dynkin-system".
Definition 677 ( $\lambda$-system).
Let $X$ be a set. Let $P_{\Lambda}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of containing the full set, and of closedness under complement and countable disjoint union.
The class of $\lambda$-systems on $X$ is $\mathfrak{C}_{P_{\Lambda}}$ : a subset $\Lambda$ of $\mathcal{P}(X)$ is a $\lambda$-system on $X$ iff

$$
\begin{align*}
& X \in \Lambda  \tag{8.76}\\
\forall A \in \Lambda, & A^{c} \in \Lambda,  \tag{8.77}\\
\forall I \subset \mathbb{N}, \forall\left(A_{i}\right)_{i \in I} \in \Lambda, & \left(A_{i}\right)_{i \in I} \text { pairwise disjoint } \quad \Longrightarrow \quad \biguplus_{i \in I} A_{i} \in \Lambda . \tag{8.78}
\end{align*}
$$

Compact form: $P_{\Lambda} \stackrel{\text { def. }}{=} w F \wedge C \wedge U d c$.

## Lemma 678 (equivalent definition of $\lambda$-system).

Let $X$ be a set. Let $\Lambda \subset \mathcal{P}(X)$. Then, $\Lambda$ is a $\lambda$-system on $X$ iff (8.76) holds, and it is closed under local complement and countable monotone union:

$$
\begin{align*}
& \forall A, B \in \Lambda, \quad B \subset A \quad \Longrightarrow \quad A \backslash B \in \Lambda,  \tag{8.79}\\
& \forall I \subset \mathbb{N}, \forall\left(A_{i}\right)_{i \in I} \in \Lambda, \quad\left(A_{i}\right)_{i \in I} \text { nondecreasing } \Longrightarrow \bigcup_{i \in I} A_{i} \in \Lambda . \tag{8.80}
\end{align*}
$$

Compact form: $P_{\Lambda} \Leftrightarrow w F \wedge D^{\prime} \wedge U m c$.

## Proof. "Left" implies "right".

Direct consequence of Definition 677 ( $\lambda$-system, $P_{\Lambda} \Rightarrow[C \mid U d c]$ ), Lemma 566 (closedness under countable disjoint union and local complement, $C \wedge U d c \Rightarrow D^{\prime}$ ), and Lemma 568 (closedness under countable disjoint and monotone union, $U d c \wedge \overline{\left.D^{\prime} \Rightarrow U m c\right)}$.

## "Right" implies "left".

Direct consequence of Lemma 541 (closedness under local complement and complement, wF $\wedge D^{\prime}$ implies $C$ ), Lemma 542 (closedness under disjoint union and local complement, $\underline{\left.D^{\prime} \wedge C \Rightarrow U d\right)}$, Lemma 569 (closedness under countable monotone and disjoint union, $U d \wedge U m c \Rightarrow U d c$ ), and Definition $677\left(\lambda\right.$-system, $\left.\underline{P}_{\Lambda}=w F \wedge C \wedge U d c\right)$.

## Lemma 679 ( $\lambda$-system is monotone class).

Let $X$ be a set. Then, $\lambda$-systems on $X$ are monotone classes on $X$.
Compact form: $P_{\Lambda} \Rightarrow P_{\mathcal{C}}$.
Proof. Direct consequence of Definition $677\left(\lambda\right.$-system, $P_{\Lambda} \Rightarrow C$ ), Lemma 678 (equivalent definition of $\lambda$-system,,$P_{\Lambda} \Rightarrow U m c$ ), Lemma 567 (closedness under countable union and intersection, $\underline{C \wedge U m c \Rightarrow I m c)}$, and Definition 664 (monotone class, $\underline{\left.P_{\mathcal{C}}=U m c \wedge I m c\right) .}$

## Lemma 680 (other properties of $\lambda$-system).

Let $X$ be a set. Let $\Lambda$ be a $\lambda$-system on $X$. Then, $\Lambda$ is nonempty, contains the empty set, and is closed under disjoint union, finite disjoint union, and countable monotone intersection.

Compact form: $P_{\Lambda} \Rightarrow[N e|w E| U d|U d f| I m c]$.
Proof. Direct consequence of Definition 677 ( $\lambda$-system, $P_{\Lambda} \Rightarrow[w F|C| U d c]$ ), Lemma 538 (nonempty and with empty or full, $w F \Rightarrow N e$ ), Lemma 539 (with empty and full, $w F \wedge C \Rightarrow w E$ ), Lemma 558 (closedness under finite operations, $(U d c \Rightarrow) U d f \Rightarrow U d)$, Lemma $679(\lambda$-system $i s$ monotone class, $\underline{P_{\Lambda} \Rightarrow P_{\mathcal{C}}}$ ), and Definition 664 (monotone class, $\underline{\left.P_{\mathcal{C}} \Rightarrow I m c\right) . ~}$

Lemma 681 (intersection of $\lambda$-systems).
Let $X$ and $I$ be sets. Let $\left(\Lambda_{i}\right)_{i \in I}$ be $\lambda$-systems on $X$. Then, if $\bigcap_{i \in I} \Lambda_{i}$ is a $\lambda$-system on $X$.
Proof. Direct consequence of Definition 677 ( $\lambda$-system), and the definition of intersection.

## Definition 682 (generated $\lambda$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. The $\lambda$-system generated by $G$ is denoted $\Lambda_{X}(G) \stackrel{\text { def. }}{=} \mathfrak{G}_{P_{\Lambda}, X}(G)$; it is the intersection of all $\lambda$-systems on $X$ containing $G$.

## Lemma 683 (generated $\lambda$-system is $\lambda$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\Lambda_{X}(G)$ is a $\lambda$-system on $X$ containing $G$.
Proof. Direct consequence of Definition 682 (generated $\lambda$-system), Lemma 681 (intersection of $\lambda$-systems), and Lemma 585 (subset system generation preserves class).

## Lemma 684 (properties of $\lambda$-system generation).

Let $X$ be a set. Then, $\lambda$-system generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
\forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\Lambda}, X}, & G \subset \mathcal{S} \quad \Longrightarrow \quad \Lambda_{X}(G) \subset \mathcal{S},  \tag{8.81}\\
\forall G_{1}, G_{2} \subset \mathcal{P}(X), & G_{1} \subset G_{2} \tag{8.82}
\end{align*} \quad \Longrightarrow \quad \Lambda_{X}\left(G_{1}\right) \subset \Lambda_{X}\left(G_{2}\right), ~\left(X \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P_{\Lambda}, X}\right) \Longrightarrow \quad \Lambda_{X}(\mathcal{S})=\mathcal{S} .
$$

Proof. Direct consequence of Definition 682 (generated $\lambda$-system), Lemma 681 (intersection of $\lambda$-systems), and Lemma 586 (properties of subset system generation).

Lemma 685 (other properties of $\lambda$-system generation).
Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
G_{1} \subset \Lambda_{X}\left(G_{2}\right) \wedge G_{2} \subset \Lambda_{X}\left(G_{1}\right) & \Longrightarrow \Lambda_{X}\left(G_{1}\right)=\Lambda_{X}\left(G_{2}\right)  \tag{8.84}\\
G_{2} \subset \Lambda_{X}\left(G_{1}\right) & \Longrightarrow \Lambda_{X}\left(G_{1} \cup G_{2}\right)=\Lambda_{X}\left(G_{1}\right) \tag{8.85}
\end{align*}
$$

Proof. Direct consequence of Definition 682 (generated $\lambda$-system), Lemma 681 (intersection of $\lambda$-systems), and Lemma 587 (other properties of subset system generation).

## Lemma 686 ( $\lambda$-system contains monotone class).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{C}_{X}(G) \subset \Lambda_{X}(G)$.
Proof. Direct consequence of Definition 682 (generated $\lambda$-system), Definition 666 (generated monotone class), Lemma 679 ( $\lambda$-system is monotone class), and Lemma 599 (generated subset systems are monotone (with nonemptiness), with $P_{\Lambda} \Rightarrow P_{\mathcal{C}}$ ).

Lemma 687 (upgrade monotone class to $\lambda$-system). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{C}_{X}(G)$ contains the full set, and is closed under local complement.
Then, we have $\mathcal{C}_{X}(G)=\Lambda_{X}(G)$.
Similar compact form: $P_{\mathcal{C}} \wedge w F \wedge D^{\prime} \Leftrightarrow P_{\Lambda}$.
Proof. Direct consequence of Lemma 686 ( $\lambda$-system contains monotone class, $\mathcal{C}_{X}(G) \subset \Lambda_{X}(G)$ ), Definition 682 (generated $\lambda$-system), Definition 666 (generated monotone class), Lemma 681 (intersection of $\lambda$-systems), Lemma 665 (intersection of monotone classes), Lemma 678 (equivalent
 Lemma 590 (upgrade generated subset system, with $P_{\mathcal{C}} \wedge w F \wedge D^{\prime} \Rightarrow P_{\Lambda}$, thus $\overline{\Lambda_{X}(G) \subset \mathcal{C}_{X}(G)}$ ), Lemma 541 (closedness under local complement and complement, $w F \wedge D^{\prime} \Rightarrow C$ ), and Lemma 567 (closedness under countable union and intersection, $C \wedge U m c \Rightarrow \overline{I m c, \text { thus } P_{\Lambda}} \Rightarrow\left[P_{\mathcal{C}}|w F| D^{\prime}\right]$ ).

## Lemma 688 ( $\lambda$-system generated by monotone class).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Lambda_{X}\left(\mathcal{C}_{X}(G)\right)=\Lambda_{X}(G)$.
Proof. Direct consequence of Definition 682 (generated $\lambda$-system), Definition 666 (generated monotone class), Lemma 681 (intersection of $\lambda$-systems), Lemma 665 (intersection of monotone classes), Lemma 679 ( $\lambda$-system is monotone class), and Lemma 591 (cascading generated subset systems, with $\left.P_{\Lambda} \Rightarrow P_{\mathcal{C}}\right)$.

Definition 689 ( $\lambda$-system and intersection).
Let $X$ be a set.
Let $\Lambda$ be a $\lambda$-system on $X$. For all $A \subset X$, let $\Lambda_{A}^{\cap}$ be the subset system defined by

$$
\begin{equation*}
\Lambda_{A}^{\cap} \stackrel{\text { def. }}{=}\{B \in \Lambda \mid A \cap B \in \Lambda\} . \tag{8.86}
\end{equation*}
$$

## Lemma 690 ( $\Lambda^{\cap}$ is symmetric).

Let $X$ be a set. Let $\Lambda$ be a $\lambda$-system on $X$. Then, for all $A, B \in \Lambda$, we have $B \in \Lambda_{A}^{\cap}$ iff $A \in \Lambda_{B}^{\cap}$.
Proof. Direct consequence of Definition 689 ( $\lambda$-system and intersection), and commutativity of intersection.

Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system).
Let $X$ be a set. Let $\Lambda$ be a $\lambda$-system on $X$. Then, for all $A \in \Lambda, \Lambda_{A}^{\cap}$ is a $\lambda$-system on $X$.
Compact form: $P_{\Lambda} \Rightarrow P_{\Lambda}\left(\Lambda_{A}^{\cap}\right)$.
Proof. Let $A \in \Lambda$.
 Let $B, C \in \Lambda_{A}^{\cap}$, i.e. $B, C, A \cap B, A \cap C \in \Lambda$. Assume that $C \subset B$.
Then, from monotonicity of intersection, distributivity of intersection over set difference, and Lemma 678 (equivalent definition of $\lambda$-system, closedness under local complement), we have $(A \cap C) \subset(A \cap B), A \cap(B \backslash C)=(A \cap B) \backslash(A \cap C) \in \Lambda$, and $B \backslash C \in \Lambda$. Hence, $\Lambda_{A}^{\cap}$ is closed under local complement.

Let $I \subset \mathbb{N}$. Let $\left(B_{i}\right)_{i \in I} \in \Lambda_{A}^{\cap}$, i.e. for all $i \in I, B_{i}, A \cap B_{i} \in \Lambda$. Assume that $\left(B_{i}\right)_{i \in I}$ is nondecreasing. Then, from monotonicity of intersection, distributivity of intersection over union, and Lemma 678 (equivalent definition of $\lambda$-system, closedness under countable monotone union), $\left(A \cap B_{i}\right)_{i \in I}$ is nondecreasing, and we have $A \cap \bigcup_{i \in I} B_{i}=\bigcup_{i \in I}\left(A \cap B_{i}\right) \in \Lambda$, and $\bigcup_{i \in I} B_{i} \in \Lambda$. Hence, $\Lambda_{A}^{\cap}$ is closed under countable monotone union.

Therefore, from Lemma 678 (equivalent definition of $\lambda$-system), $\Lambda_{A}^{\cap}$ is a $\lambda$-system on $X$.
Lemma 692 ( $\lambda$-system with intersection). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$.
Assume that $G$ is closed under intersection. Then, $\forall A \in \Lambda_{X}(G),\left(\Lambda_{X}(G)\right)_{A}^{\cap}=\Lambda_{X}(G)$.
Proof. For all $A \subset X$, we use the simplified notation $\Lambda_{A}^{\cap} \stackrel{\text { def. }}{=}\left(\Lambda_{X}(G)\right)_{A}^{\cap}$.
(0). From Definition 689 ( $\lambda$-system and intersection), we have $\forall A \in \Lambda_{X}(G), \Lambda_{A}^{\cap} \subset \Lambda_{X}(G)$.
(1). $\forall A \in G, \Lambda_{X}(G) \subset \Lambda_{A}^{\cap}$. Let $A \in G$.

Let $B \in G$. Then, from Lemma 683 (generated $\lambda$-system is $\lambda$-system, $G \subset \Lambda_{X}(G)$ ), and Definition 689 ( $\lambda$-system and intersection), we have $B \in \Lambda_{A}^{\cap}$, i.e. $G \subset \Lambda_{A}^{\cap}$. Hence, from Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system, with $A \in G \subset \Lambda_{X}(G)$ ), and Lemma 684 (properties of $\lambda$-system generation, $\underline{\text { minimality }}$, we have $\Lambda_{X}(G) \subset \Lambda_{A}^{\cap}$.
(2). $\forall \boldsymbol{B} \in \boldsymbol{\Lambda}_{\boldsymbol{X}}(\boldsymbol{G}), \boldsymbol{\Lambda}_{\boldsymbol{X}}(\boldsymbol{G}) \subset \boldsymbol{\Lambda}_{\boldsymbol{B}}^{\cap}$. Let $B \in \Lambda_{X}(G)$.

Let $A \in G$. Then, from (1), we have $B \in \Lambda_{X}(G) \subset \Lambda_{A}^{\cap}$. Thus, from Lemma 683 (generated $\lambda$-system is $\lambda$-system, $G \subset \Lambda_{X}(G)$ ), and Lemma 690 ( $\Lambda^{\cap}$ is symmetric, with $A, B \in \Lambda_{X}(G)$ ), we
 of $\lambda$-system generation, minimality), we have $\Lambda_{X}(G) \subset \Lambda_{B}^{\cap}$.
Therefore, from (0) and (2), we have the equality.

Lemma 693 ( $\lambda$-system is closed under intersection). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$.
Assume that $G$ is closed under intersection. Then, $\Lambda_{X}(G)$ is closed under intersection.
Compact form: $I \Rightarrow I\left(\Lambda_{X}\right)$.
Proof. Let $A, B \in \Lambda_{X}(G)$. Then, from Lemma 692 ( $\lambda$-system with intersection, $B$ belongs to

Therefore, $\Lambda_{X}(G)$ is closed under intersection.
Lemma 694 ( $\lambda$-system generated by $\pi$-system). Let $X$ be a set. Let $G \subset \mathcal{P}(X)$.
Assume that $G$ is a $\pi$-system on $X$. Then, $\Lambda_{X}(G)$ is a $\pi$-system on $X$.
Compact form: $P_{\Pi} \Rightarrow P_{\Pi}\left(\Lambda_{X}\right)$.
Proof. Direct consequence of Lemma 604 (equivalent definition of $\pi$-system, $P_{\Pi}=N e \wedge I$ (for $G$, then for $\left.\Lambda_{X}(G)\right)$ ), Lemma 683 (generated $\lambda$-system is $\lambda$-system, $G \subset \Lambda_{X}(G)$, thus $N e\left(\Lambda_{X}(G)\right)$ ), and Lemma 693 ( $\lambda$-system is closed under intersection, thus $I\left(\Lambda_{X}(G)\right)$ ).

### 8.8 Sigma-algebra

Remark 695. Note that the next concept of " $\sigma$-algebra" is sometimes called " $\sigma$-field".
Lemma 696 ( $\sigma$-algebra). Let $X$ be a set. Let $P_{\Sigma}$ be the predicate on $\mathcal{P}(\mathcal{P}(X))$ of being an algebra of sets, and of closedness under countable union.
The class of $\sigma$-algebras on $X$ is $\mathfrak{C}_{P_{\Sigma}}$ : a subset $\Sigma$ of $\mathcal{P}(X)$ is a $\sigma$-algebra on $X$ iff

$$
\begin{align*}
& \varnothing \in \Sigma  \tag{8.87}\\
& \forall A \in \Sigma, A^{c} \in \Sigma  \tag{8.88}\\
& \forall I \subset \mathbb{N}, \forall\left(A_{i}\right)_{i \in I} \in \Sigma, \bigcup_{i \in I} A_{i} \in \Sigma . \tag{8.89}
\end{align*}
$$

Compact form: $P_{\Sigma} \stackrel{\text { def. }}{=} P_{\mathcal{A}} \wedge U c ; P_{\Sigma} \Leftrightarrow w E \wedge C \wedge U c$.
Proof. Direct consequence of Definition 634 (algebra of sets).

## Lemma 697 (equivalent definition of $\sigma$-algebra).

Let $X$ be a set. Let $\Sigma \subset \mathcal{P}(X)$. Then, $\Sigma$ is a $\sigma$-algebra on $X$ iff (8.88) holds and

$$
\begin{equation*}
\Sigma \neq \varnothing, \quad \text { or } \quad \varnothing \in \Sigma, \quad \text { or } \quad X \in \Sigma, \tag{8.90}
\end{equation*}
$$

$$
\begin{equation*}
\forall I \subset \mathbb{N}, \forall\left(A_{i}\right)_{i \in I} \in \Sigma, \bigcup_{i \in I} A_{i} \in \Sigma, \quad \text { or } \quad \forall I \subset \mathbb{N}, \forall\left(A_{i}\right)_{i \in I} \in \Sigma, \bigcap_{i \in I} A_{i} \in \Sigma \tag{8.91}
\end{equation*}
$$

Compact form: $P_{\Sigma} \Leftrightarrow[N e|w E| w F] \wedge C \wedge[U c \mid I c]$.
Proof. Direct consequence of Lemma 696 ( $\sigma$-algebra, $P_{\Sigma}=w E \wedge C \wedge U c$ ), Lemma 538 (nonempty and with empty or full, wE $\Rightarrow N e$ and $C \wedge I c \wedge N e \Rightarrow C \wedge I \wedge N e \Rightarrow w E$ ), Lemma 539 (with empty and full, $C \Rightarrow w E \Leftrightarrow w F$ ), and Lemma 567 (closedness under countable union and intersection, $C \Rightarrow U c \Leftrightarrow I c$.

## Remark 698.

Note that the previous lemma means that the predicate $P_{\Sigma}$ characterizing $\sigma$-algebras is equivalent to any of the six alternatives summarized by the conjunction (8.90), (8.88), and (8.91).

## Lemma 699 ( $\sigma$-algebra is algebra of sets).

Let $X$ be a set. Then, $\sigma$-algebras of sets on $X$ are algebras of sets on $X$.
Compact form: $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$.
Proof. Direct consequence of Lemma 696 ( $\sigma$-algebra, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ).

## Lemma 700 (other properties of $\sigma$-algebra).

Let $X$ be a set. Let $\Sigma$ be a $\sigma$-algebra on $X$. Then, $\Sigma$ is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, and countable monotone union and intersection.
Compact form: $P_{\Sigma} \Rightarrow\left[D^{\prime}|D| s D|U d| U|I| U d f|U f| I f|U d c| U m c \mid I m c\right]$.
 (other properties of algebra of sets, $\underline{P} \mathcal{A} \Rightarrow\left[D^{\prime}|s D| U d|U d f| I f\right]$ ). Lemma 637 (other equivalent definition of algebra of sets, $\underline{P} \mathcal{A}^{\Rightarrow D}$ ), Lemma 635 (equivalent definition of algebra of sets, $\left.\underline{P_{\mathcal{A}}} \Rightarrow[U \mid I]\right)$, Definition 634 (algebra of sets, $\underline{P_{\mathcal{A}} \Rightarrow U f}$ ), and Lemma 697 (equivalent definition


## Lemma 701 ( $\sigma$-algebra is $\lambda$-system).

Let $X$ be a set. Then, $\sigma$-algebras on $X$ are $\lambda$-systems on $X: P_{\Sigma} \Rightarrow P_{\Lambda}$.

## Proof.

Direct consequence of Lemma 697 (equivalent definition of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow[w F \mid C] \text { ), Lemma } 700}$ (other properties of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow U d c}$ ), and Definition $677\left(\lambda\right.$-system, $\underline{\left.P_{\Lambda}=w F \wedge C \wedge U d c\right)}$.

Lemma 702 ( $\sigma$-algebra is monotone class).
Let $X$ be a set. Then, $\sigma$-algebras on $X$ are monotone classes on $X: P_{\Sigma} \Rightarrow P_{\mathcal{C}}$.
 system is monotone class, $\underline{P_{\Lambda} \Rightarrow P_{\mathcal{C}}}$.

Lemma 703 ( $\sigma$-algebra is ring of sets).
Let $X$ be a set. Then, $\sigma$-algebras on $X$ are rings of sets on $X: P_{\Sigma} \Rightarrow P_{\mathcal{R}}$.
Proof. Direct consequence of Lemma 699 ( $\sigma$-algebra is algebra of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ), and Lemma 638 (algebra of sets is ring of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{R}}$ ).

## Lemma 704 ( $\sigma$-algebra is $\pi$-system).

Let $X$ be a set. Then, $\sigma$-algebras on $X$ are $\pi$-systems on $X: P_{\Sigma} \Rightarrow P_{\Pi}$.
Proof. Direct consequence of Lemma 699 ( $\sigma$-algebra is algebra of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ), and Lemma 640


Lemma 705 ( $\sigma$-algebra is semi-algebra of sets).
Let $X$ be a set. Then, $\sigma$-algebras of sets on $X$ are semi-algebras of sets on $X$.
Compact form: $P_{\Sigma} \Rightarrow P_{\mathcal{A} / 2}$.
 (algebra of sets is semi-algebra of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{A} / 2}$ ).

## Lemma 706 ( $\sigma$-algebra is semiring of sets).

Let $X$ be a set. Then, $\sigma$-algebras of sets on $X$ are semirings of sets on $X$.
Compact form: $P_{\Sigma} \Rightarrow P_{\mathcal{R} / 2}$.
Proof. Direct consequence of Lemma 699 ( $\sigma$-algebra is algebra of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ), and Lemma 661 (algebra of sets is semiring of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{R} / 2}$ ).

## Lemma 707 (intersection of $\sigma$-algebras).

Let $X$ and $I$ be sets. Let $\left(\Sigma_{i}\right)_{i \in I}$ be $\sigma$-algebras on $X$. Then, $\bigcap_{i \in I} \Sigma_{i}$ is a $\sigma$-algebra on $X$.
Proof. Direct consequence of Lemma 696 ( $\sigma$-algebra), and the definition of intersection.

## Definition 708 (generated $\sigma$-algebra).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. The $\sigma$-algebra generated by $G$ is denoted $\Sigma_{X}(G) \stackrel{\text { def. }}{=} \mathfrak{G}_{P_{\Sigma}, X}(G)$; it is the intersection of all $\sigma$-algebras on $X$ containing $G$.

## Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, $\Sigma_{X}(G)$ is a $\sigma$-algebra on $X$ containing $G$.
Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Lemma 707 (intersection of $\sigma$-algebras), and Lemma 585 (subset system generation preserves class).

## Lemma 710 (properties of $\sigma$-algebra generation).

Let $X$ be a set. Then, $\sigma$-algebra generation on $X$ is minimum, monotone, and idempotent:

$$
\begin{align*}
\forall G \subset \mathcal{P}(X), \forall \mathcal{S} \in \mathfrak{C}_{P_{\Sigma}, X}, & G \subset \mathcal{S} \quad \Longrightarrow \quad \Sigma_{X}(G) \subset \mathcal{S},  \tag{8.92}\\
\forall G_{1}, G_{2} \subset \mathcal{P}(X), \quad G_{1} \subset G_{2} & \Longrightarrow \quad \Sigma_{X}\left(G_{1}\right) \subset \Sigma_{X}\left(G_{2}\right),  \tag{8.93}\\
\forall \mathcal{S} \subset \mathcal{P}(X), \quad \mathcal{S} \in \mathfrak{C}_{P_{\Sigma}, X} & \Longleftrightarrow \quad \Sigma_{X}(\mathcal{S})=\mathcal{S} \tag{8.94}
\end{align*}
$$

Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Lemma 707 (intersection of $\sigma$-algebras), and Lemma 586 (properties of subset system generation).

## Lemma 711 (other properties of $\sigma$-algebra generation).

Let $X$ be a set. Let $G_{1}, G_{2} \subset \mathcal{P}(X)$. Then, we have

$$
\begin{align*}
& G_{1} \subset \Sigma_{X}\left(G_{2}\right) \wedge G_{2} \subset \Sigma_{X}\left(G_{1}\right) \Longrightarrow \Sigma_{X}\left(G_{1}\right)=\Sigma_{X}\left(G_{2}\right)  \tag{8.95}\\
& G_{2} \subset \Sigma_{X}\left(G_{1}\right) \Longrightarrow \Sigma_{X}\left(G_{1} \cup G_{2}\right)=\Sigma_{X}\left(G_{1}\right)  \tag{8.96}\\
& \hline
\end{align*}
$$

Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Lemma 707 (intersection of $\sigma$-algebras), and Lemma 587 (other properties of subset system generation).

## Lemma 712 (countable union $\sigma$-algebra generator).

Let $X$ be a set. Let $G_{1} \subset G_{2} \subset \mathcal{P}(X)$. Assume that all elements of $G_{2}$ are countable unions of elements of $G_{1}$. Then, we have $\Sigma_{X}\left(G_{1}\right)=\Sigma_{X}\left(G_{2}\right)$.

Proof. Direct consequence of Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, $G_{1} \subset G_{2} \subset \Sigma_{X}\left(G_{2}\right)$ ), Lemma 696 ( $\sigma$-algebra, closedness under countable union, thus $G_{2} \subset \Sigma_{X}\left(G G_{1}\right)$ ), and Lemma 711 ( other properties of $\sigma$-algebra generation, with $G_{1} \cup G_{2}=G_{2}$ ).

## Lemma 713 ( $\sigma$-algebra contains $\lambda$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Lambda_{X}(G) \subset \Sigma_{X}(G)$.
Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 682 (generated $\lambda$ system), Lemma 701 ( $\sigma$-algebra is $\lambda$-system), and Lemma 588 (generated subset systems are monotone, with $P_{\Sigma} \Rightarrow P_{\Lambda}$ ).

Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$.
Assume that $\Lambda_{X}(G)$ is closed under intersection. Then, we have $\Lambda_{X}(G)=\Sigma_{X}(G)$.
Similar compact form: $P_{\Lambda} \wedge I \Leftrightarrow P_{\Sigma}$.
Proof. Direct consequence of Lemma 713 ( $\sigma$-algebra contains $\lambda$-system, $\Lambda_{X}(G) \subset \Sigma_{X}(G)$ ), Definition 708 (generated $\sigma$-algebra), Definition 682 (generated $\lambda$-system), Lemma 707 (intersection of $\sigma$-algebras), Lemma 681 (intersection of $\lambda$-systems), Lemma 697 (equivalent definition of $\sigma$-algebra, $P_{\Sigma}=w F \wedge C \wedge U c$ ), Definition 677 ( $\lambda$-system, $\underline{P}_{\Lambda}=w F \wedge C \wedge U d c$ ), Lemma 545 (closedness un-
 joint union and countable union, $D \wedge \overline{U d c} \Rightarrow U c$ ), Lemma 590 (upgrade generated subset system, with $P_{\Lambda} \wedge I \Rightarrow P_{\Sigma}$, thus $\left.\Sigma_{X}(G) \subset \Lambda_{X}(G)\right)$, Lemma 701 ( $\sigma$-algebra is $\lambda$-system, $P_{\Sigma} \Rightarrow P_{\Lambda}$ ), and


## Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_{X}\left(\Lambda_{X}(G)\right)=\Sigma_{X}(G)$.
Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 682 (generated $\lambda$ system), Lemma 707 (intersection of $\sigma$-algebras), Lemma 681 (intersection of $\lambda$-systems), Lemma 701 ( $\sigma$-algebra is $\lambda$-system), and Lemma 591 (cascading generated subset systems, with $P_{\Sigma}$ implies $P_{\Lambda}$ ).

Lemma 716 ( $\sigma$-algebra contains monotone class).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{C}_{X}(G) \subset \Sigma_{X}(G)$.

Proof. Direct consequence of Lemma 713 ( $\sigma$-algebra contains $\lambda$-system, $\Lambda_{X}(G) \subset \Sigma_{X}(G)$ ), Lemma 686 ( $\lambda$-system contains monotone class, $\mathcal{C}_{X}(G) \subset \Lambda_{X}(G)$ ), and transitivity of inclusion.

Lemma 717 (upgrade monotone class to $\sigma$-algebra). $\quad$ Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{C}_{X}(G)$ contains the empty set, and is closed under complement and union.
Then, we have $\mathcal{C}_{X}(G)=\Sigma_{X}(G)$.
Similar compact form: $P_{\mathcal{C}} \wedge w E \wedge C \wedge U \Leftrightarrow P_{\Sigma}$.
Proof. Direct consequence of Lemma 539 (with empty and full, $w E \wedge C \Rightarrow w F$ and $w F \wedge C \Rightarrow w E$ ), Lemma 546 (closedness under union and disjoint union, $U \Rightarrow U d$ ), Lemma 542 (closedness under disjoint union and local complement, $C \wedge U d \Rightarrow D^{\prime}$ ), Lemma 687 (upgrade monotone class to
 ness under union and intersection, $C \wedge U \Rightarrow I$ and $C \wedge I \Rightarrow U)$, Definition 677 ( $\lambda$-system, $P_{\Lambda}$ equals $w F \wedge C \wedge U d c$ ), and Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra, $\underline{P_{\mathcal{C}} \wedge w E \wedge C \wedge U \Leftrightarrow}$ $\left.\underline{P_{\Lambda} \wedge I \Leftrightarrow P_{\Sigma}, \text { thus } \Lambda_{X}}(G)=\Sigma_{X}(G)\right)$.

Lemma 718 ( $\sigma$-algebra generated by monotone class).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_{X}\left(\mathcal{C}_{X}(G)\right)=\Sigma_{X}(G)$.
Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 682 (generated $\lambda$ system), Definition 666 (generated monotone class), Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system), Lemma 688 ( $\lambda$-system generated by monotone class), and Lemma 592 (double cascading generated subset systems).

Lemma 719 ( $\sigma$-algebra contains algebra of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{A}_{X}(G) \subset \Sigma_{X}(G)$.
Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 642 (generated algebra of sets), Lemma 699 ( $\sigma$-algebra is algebra of sets), and Lemma 588 (generated subset systems are monotone, with $\left.P_{\Sigma} \Rightarrow P_{\mathcal{A}}\right)$.

Lemma 720 (upgrade algebra of sets to $\sigma$-algebra). $\quad$ Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{A}_{X}(G)$ is closed under countable disjoint or monotone union.
Then, we have $\mathcal{A}_{X}(G)=\Sigma_{X}(G)$.
Similar compact form: $P_{\mathcal{A}} \wedge[U d c \mid U m c] \Leftrightarrow P_{\Sigma}$.
Proof. Direct consequence of Lemma 719 ( $\sigma$-algebra contains algebra of sets, $\mathcal{A}_{X}(G) \subset \Sigma_{X}(G)$ ), Definition 708 (generated $\sigma$-algebra), Definition 642 (generated algebra of sets), Lemma 707 (intersection of $\sigma$-algebras), Lemma 641 (intersection of algebras of sets), Lemma 696 ( $\sigma$-algebra, $P_{\Sigma}=P_{\mathcal{A}} \wedge U c$ ), Lemma 637 (other equivalent definition of algebra of sets, $P_{\mathcal{A}} \Rightarrow D$ ), Lemma 570 $\overline{(\text { closedness under countable disjoint union and countable union, } D \wedge U d c \Rightarrow U c)}$, Lemma 639 (other properties of algebra of sets, $P_{\mathcal{A}} \Rightarrow U d$ ), Lemma 571 (closedness under countable monotone union and countable union, $D \widehat{\wedge U d \wedge U m c \Rightarrow U c}$ ), Lemma 590 (upgrade generated subset system, with $P_{\mathcal{A}} \wedge[U d c \mid U m c] \Rightarrow P_{\Sigma}$, thus $\Sigma_{X}(G) \subset \mathcal{A}_{X}(G)$ ), Lemma 699 ( $\sigma$-algebra is algebra of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ), and Lemma 700 (other properties of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow[U d c \mid U m c] \text {, thus }}$ $\left.P_{\Sigma} \Rightarrow\left[P_{\mathcal{A}} \mid \overline{U d c \mid U m c}\right]\right)$.

Lemma 721 ( $\sigma$-algebra generated by algebra of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_{X}\left(\mathcal{A}_{X}(G)\right)=\Sigma_{X}(G)$.

Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 642 (generated algebra of sets), Lemma 707 (intersection of $\sigma$-algebras), Lemma 641 (intersection of algebras of sets), Lemma 699 ( $\sigma$-algebra is algebra of sets), and Lemma 591 (cascading generated subset systems, with $\left.P_{\Sigma} \Rightarrow P_{\mathcal{A}}\right)$.

Lemma 722 ( $\sigma$-algebra contains ring of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\mathcal{R}_{X}(G) \subset \Sigma_{X}(G)$.

Proof. Direct consequence of Lemma 719 ( $\sigma$-algebra contains algebra of sets), Lemma 646 (algebra of sets contains ring of sets), and transitivity of inclusion.

Lemma 723 (upgrade ring of sets to $\sigma$-algebra).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $\mathcal{R}_{X}(G)$ either contains the full set or is closed under complement, and is closed under countable disjoint or monotone union. Then, we have $\mathcal{R}_{X}(G)=\Sigma_{X}(G)$.
Similar compact form: $P_{\mathcal{R}} \wedge[w F \mid C] \wedge[U d c \mid U m c] \Leftrightarrow P_{\Sigma}$.
Proof. Direct consequence of Lemma 647 (upgrade ring of sets to algebra of sets, $\mathcal{R}_{X}(G)=\mathcal{A}_{X}(G)$ and $P_{\mathcal{R}} \wedge[w F \mid C] \Leftrightarrow P_{\mathcal{A}}$ ), and Lemma 720 (upgrade algebra of sets to $\sigma$-algebra, $\overline{\mathcal{A}_{X}(G)=\Sigma_{X}(G)}$ and $\left.P_{\mathcal{A}} \wedge[U d c \mid U m c] \Leftrightarrow P_{\Sigma}\right)$.

Lemma 724 ( $\sigma$-algebra generated by ring of sets).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_{X}\left(\mathcal{R}_{X}(G)\right)=\Sigma_{X}(G)$.

Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 642 (generated algebra of sets), Definition 617 (generated ring of sets), Lemma 721 ( $\sigma$-algebra generated by algebra of sets), Lemma 648 (algebra of sets generated by ring of sets), and Lemma 592 (double cascading generated subset systems).

## Lemma 725 ( $\sigma$-algebra contains $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Pi_{X}(G) \subset \Sigma_{X}(G)$.

Proof. Direct consequence of Lemma 722 ( $\sigma$-algebra contains ring of sets), Lemma 621 (ring of sets contains $\pi$-system), and transitivity of inclusion.

Lemma 726 (upgrade $\pi$-system to $\sigma$-algebra).
Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$, and that $\Pi_{X}(G)$ is closed under complement and countable disjoint or monotone union. Then, we have $\Pi_{X}(G)=\Sigma_{X}(G)$.

Similar compact form: $P_{\Pi} \wedge C \wedge[U d c \mid U m c] \Leftrightarrow P_{\Sigma}$.

Proof. Direct consequence of Lemma 650 (upgrade $\pi$-system to algebra of sets, $\Pi_{X}(G)=\mathcal{A}_{X}(G)$ and $P_{\Pi} \wedge C \Leftrightarrow P_{\mathcal{A}}$ ), and Lemma 720 (upgrade algebra of sets to $\sigma$-algebra, $\underline{\mathcal{A}_{X}(G)=\Sigma_{X}(G) \text { and }}$ $\left.\overline{P_{\mathcal{A}} \wedge[U d c \mid U m c]} \Leftrightarrow P_{\Sigma}\right)$.

## Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Assume that $G \neq \varnothing$. Then, we have $\Sigma_{X}\left(\Pi_{X}(G)\right)=\Sigma_{X}(G)$.

Proof. Direct consequence of Definition 708 (generated $\sigma$-algebra), Definition 617 (generated ring of sets), Definition 606 (generated $\pi$-system), Lemma 724 ( $\sigma$-algebra generated by ring of sets), Lemma 623 (ring of sets generated by $\pi$-system), and Lemma 592 (double cascading generated subset systems).

## Lemma 728 (disjoint union of union in $\sigma$-algebra).

Let $X$ be a set. Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\Sigma$ be a $\sigma$-algebra on $X$. Let $\left(A_{i}\right)_{i \in I} \subset \Sigma$.
Then, there exists $\left(B_{i}\right)_{i \in I} \in \Sigma$ pairwise disjoint such that for all $i \in I, B_{i} \subset A_{i}$, and we have

$$
\begin{align*}
\forall n \in J, & \bigcup_{j \in[0 . . n]} A_{\varphi(j)}=\biguplus_{j \in[0 . . n]} B_{\varphi(j)} \in \Sigma  \tag{8.97}\\
& \bigcup_{i \in I} A_{i}=\biguplus_{i \in I} B_{i} \in \Sigma \tag{8.98}
\end{align*}
$$

Proof. Direct consequence of Lemma 700 (other properties of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow[D|U d| U d c] \text { ), }}$ and Lemma 572 (disjoint union of union in subset system, $\underline{H y p=D \wedge U d \wedge U d c \text { ). }}$

Remark 729. Note that in the next proof, the point (1) does not depend on the hypotheses. It could be an independent lemma that characterizes $\Sigma_{1} \overline{\times} \Sigma_{2}$.

Note also that $\Sigma_{1} \overline{\times} \Sigma_{2}$ may not be a $\sigma$-algebra. See Section 9.4 for a definition of a $\sigma$-algebra on the product of measurable spaces.

## Lemma 730 (algebra of sets generated by product of $\sigma$-algebras).

Let $\left(X_{1}, \Sigma_{1}\right)$ and $\left(X_{2}, \Sigma_{2}\right)$ be measurable spaces. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$, and $\bar{\Sigma} \stackrel{\text { def. }}{=} \Sigma_{1} \overline{\times} \Sigma_{2}$.
Then, $\mathcal{A}_{X}(\bar{\Sigma})$ is the set of finite (disjoint) unions of elements of $\bar{\Sigma}$.
Proof. (1). $\boldsymbol{X} \in \overline{\boldsymbol{\Sigma}}$. Direct consequence of Definition 227 (product of subset systems), and

(2). $\bar{\Sigma}$ is closed under intersection.

Direct consequence of Definition 227 (product of subset systems), compatibility of intersection with Cartesian product $\left(\left(\boldsymbol{A}_{1} \times \boldsymbol{A}_{2}\right) \cap\left(\boldsymbol{B}_{1} \times \boldsymbol{B}_{2}\right)=\left(\boldsymbol{A}_{1} \cap \boldsymbol{B}_{1}\right) \times\left(\boldsymbol{A}_{2} \cap \boldsymbol{B}_{2}\right)\right)$, and Lemma 700 (other properties of $\sigma$-algebra, $\underline{P}_{\Sigma} \Rightarrow I$, for $\Sigma_{1}$ and $\Sigma_{2}$ ).
(3). The complement of elements of $\bar{\Sigma}$ is the disjoint union of elements of $\bar{\Sigma}$.

Direct consequence of Definition 227 (product of subset systems), the identity $\left(\boldsymbol{A}_{1} \times \boldsymbol{A}_{\mathbf{2}}\right)^{\text {c }}$ equals $\left(\boldsymbol{X}_{1} \times \boldsymbol{A}_{2}^{c}\right) \uplus\left(\boldsymbol{A}_{1}^{c} \times \boldsymbol{A}_{\mathbf{2}}\right)$, and Lemma 697 (equivalent definition of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow w F \text {, for } \Sigma_{1} \text {, }, ~, ~, ~}$ and $P_{\Sigma} \Rightarrow C$, for $\Sigma_{1}$ and $\Sigma_{2}$ ),
Therefore, from (1), (2), (3), Lemma 655 (equivalent definition of semi-algebra of sets, $\underline{P}_{\mathcal{A} / 2}$ equals $w F \wedge I \wedge P t f C($ with $n \stackrel{\text { def. }}{=} 1)$ ), and Lemma 662 (explicit algebra of sets, with $G \stackrel{\text { def. }}{=} \bar{\Sigma}), \bar{\Sigma}$ is a semialgebra on $X$, and $\mathcal{A}_{X}(\bar{\Sigma})$ is the set of finite (disjoint) unions of elements of $\bar{\Sigma}$.

### 8.9 Dynkin $\pi-\lambda$ theorem

Remark 731. See the sketch of next proof in Section 5.10.

Theorem 732 (Dynkin $\pi-\lambda$ theorem).
Let $X$ be a set. Let $\Pi$ be a $\pi$-system on $X$. Then, we have $\Lambda_{X}(\Pi)=\Sigma_{X}(\Pi)$.
Similar compact form: $P_{\Pi} \wedge P_{\Lambda} \Leftrightarrow P_{\Sigma}$.
Proof. Direct consequence of Lemma 694 ( $\lambda$-system generated by $\pi$-system, $P_{\Pi} \Rightarrow P_{\Pi}\left(\Lambda_{X}\right)$ ), Lemma 604 (equivalent definition of $\pi$-system, $\underline{P}_{\Pi} \Rightarrow I$, for $\Lambda_{X}(\Pi)$ ), Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra, $\Lambda_{X}(\Pi)=\Sigma_{X}(\Pi)$ and $P_{\Pi} \wedge P_{\Lambda} \Rightarrow P_{\Sigma}$, for $\Lambda_{X}(\Pi)$ ), Lemma 704 ( $\sigma$-algebra is $\pi$-system,


Remark 733. Note that the Dynkin $\pi-\lambda$ theorem may take the following form: if a $\lambda$-system contains a $\pi$-system, then it also contains the $\sigma$-algebra generated by the $\pi$-system.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemma 992 in Section 12.4, itself later used to prove uniqueness of the Lebesgue measure.

## Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem).

Let $(X, \Sigma)$ be a measurable space. Let $P$ be a predicate over $\mathcal{P}(X)$, and let $\mathcal{S} \stackrel{\text { def. }}{=}\{A \in \Sigma \mid P(A)\}$. Let $G \subset \mathcal{P}(X)$ be a nonempty generator of $\Sigma$. Assume that $\Pi_{X}(G) \subset \mathcal{S}$, and that $\mathcal{S}$ is a $\lambda$-system. Then, we have $\mathcal{S}=\Sigma$, i.e. $P$ holds for all subsets in $\Sigma$.

Proof. From the definition of $\mathcal{S}$, we have $\mathcal{S} \subset \Sigma$.
Let $\Pi \stackrel{\text { def. }}{=} \Pi_{X}(G)$. Then, from Definition 708 (generated $\sigma$-algebra, $\left.\Sigma=\Sigma_{X}(G)\right)$, Lemma 607 (generated $\pi$-system is $\pi$-system, $\Pi$ is a $\pi$-system containing $G$ ), Lemma 710 (properties of $\sigma$ algebra generation, monotonicity: $\overline{\left.\Sigma_{X}(G) \subset \Sigma_{X}(\Pi)\right) \text {, Theorem } 732 \text { (Dynkin } \pi-\lambda \text { theorem, } \Sigma_{X}(\Pi)}$ equals $\Lambda_{X}(\Pi)$ ), and Lemma 684 (properties of $\lambda$-system generation, monotonicity: $\Lambda_{X} \frac{\Lambda_{X}(\Pi) \text { is }}{}$ included in $\Lambda_{X}(\mathcal{S})$, and idempotence: $\left.\Lambda_{X}(\mathcal{S})=\mathcal{S}\right)$, we have $\Sigma \subset \mathcal{S}$.

Therefore, we have the equality.

### 8.10 Monotone class theorem

Remark 735. See the sketch of next proof in Section 5.10.

```
Theorem 736 (monotone class).
Let \(X\) be a set. Let \(\mathcal{A}\) be an algebra of sets on \(X\). Then, \(\mathcal{C}_{X}(\mathcal{A})=\Sigma_{X}(\mathcal{A})\).
Similar compact form: \(P_{\mathcal{A}} \wedge P_{\mathcal{C}} \Leftrightarrow P_{\Sigma}\).
```

Proof. Direct consequence of Lemma 675 (monotone class generated by algebra of sets, $\underline{P}_{\mathcal{A}}$ implies $\underline{P_{\mathcal{A}}\left(\mathcal{C}_{X}\right)}$ ), Lemma 635 (equivalent definition of algebra of sets, $P_{\mathcal{A}} \Rightarrow[w E|C| U]$, for $\mathcal{C}_{X}(\mathcal{A})$ ), Lemma 717 (upgrade monotone class to $\sigma$-algebra, $\mathcal{C}_{X}(\mathcal{A})=\overline{\Sigma_{X}(\mathcal{A}) \text { and } P_{\mathcal{A}} \wedge P_{\mathcal{C}} \Rightarrow P_{\Sigma} \text {, for }}$ $\underline{\mathcal{C}_{X}(\mathcal{A})}$ ), Lemma 699 ( $\sigma$-algebra is algebra of sets, $\underline{P}_{\Sigma} \Rightarrow P_{\mathcal{A}}$, for $\mathcal{C}_{X}(\mathcal{A})$ ), and Lemma $702(\sigma-$ algebra is monotone class, $P_{\Sigma} \Rightarrow P_{\mathcal{C}}$, for $\mathcal{C}_{X}(\mathcal{A})$ ).

Remark 737. Note that the monotone class theorem may take the following form: if a monotone class contains an algebra of sets, then it also contains the $\sigma$-algebra generated by the algebra.

Similarly, the next statement is an application lemma for the previous theorem. It is used to prove Lemmas 1221 and 1231 in Section 15.4 in the context of product spaces.

## Lemma 738 (usage of monotone class theorem).

Let $(X, \Sigma)$ be a measurable space. Let $P$ be a predicate over $\mathcal{P}(X)$, and let $\mathcal{S} \stackrel{\text { def. }}{=}\{A \in \Sigma \mid P(A)\}$. Let $G \subset \mathcal{P}(X)$ be a generator of $\Sigma$. Assume that $\mathcal{A}_{X}(G) \subset \mathcal{S}$, and that $\mathcal{S}$ is a monotone class. Then, we have $\mathcal{S}=\Sigma$, i.e. $P$ holds for all subsets in $\Sigma$.

Proof. From the definition of $\mathcal{S}$, we have $\mathcal{S} \subset \Sigma$.
Let $\mathcal{A} \stackrel{\text { def. }}{=} \mathcal{A}_{X}(G)$. Then, from Definition 708 (generated $\sigma$-algebra, $\Sigma=\Sigma_{X}(G)$ ), Lemma 643 (generated algebra of sets is algebra of sets, $\mathcal{A}$ is algebra of sets containing $G$ ), Lemma 710 (properties of $\sigma$-algebra generation, monotonicity: $\Sigma_{X}(G) \subset \Sigma_{X}(\mathcal{A})$ ), Theorem 736 (monotone class, $\Sigma_{X}(\mathcal{A})=\mathcal{C}_{X}(\mathcal{A})$ ), and Lemma 668 (properties of monotone class generation, monotonicity: $\mathcal{C}_{X}(\mathcal{A}) \subset \mathcal{C}_{X}(\mathcal{S})$, and idempotence: $\left.\mathcal{C}_{X}(\mathcal{S})=\mathcal{S}\right)$, we have $\Sigma \subset \mathcal{S}$.

Therefore, we have the equality.

## Chapter 9

## Measurability

## Contents

9.1 Measurable space and Borel subset ..... 150
9.2 Measurable function ..... 152
9.3 Measurable subspace ..... 155
9.4 Product of measurable spaces ..... 158

### 9.1 Measurable space and Borel subset

Definition 739 (premeasurable space). Let $X$ be a set. Let $\mathcal{R}$ be a ring of sets on $X$. Then, $(X, \mathcal{R})$ is called premeasurable space, and elements of $\mathcal{R}$ are said ( $\mathcal{R}$-)premeasurable.

## Lemma 740 (properties of premeasurable space).

Let $(X, \mathcal{R})$ be a premeasurable space. Then, $\mathcal{R}$ is a (semi)ring a sets, and a $\pi$-system.
Moreover, $\mathcal{R}$ is nonempty, contains the empty set, is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, and finite union and intersection, and local complement and set difference of elements of $\mathcal{R}$ can be expressed as a finite disjoint union of elements of $\mathcal{R}$.

Compact form: Concl $\stackrel{\text { def. }}{=}\left[P_{\mathcal{R}}\left|P_{\mathcal{R} / 2}\right| P_{\Pi} \mid\right.$

$$
\text { Ne|wE|D} \left.|D| s D|U d| U|I| U d f|U f| I f\left|P t f D^{\prime}\right| P t f D\right] .
$$

Proof. Direct consequence of Definition 739 (premeasurable space, thus $P_{\mathcal{R}}$ ), Lemma 631 (ring of sets is semiring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow P_{\mathcal{R} / 2}$ ), Lemma 614 (ring of sets is $\pi$-system, $\underline{P}_{\mathcal{R}} \Rightarrow P_{\Pi}$ ), Lemma 612 (equivalent definition of ring of sets, $P_{\mathcal{R}} \Rightarrow\left[N e|w E| D^{\prime}|D| s D|U d| U \mid I\right]$ ), Lemma 615 (other properties of ring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow[U d f \mid I f]$ ), Definition 611 (ring of sets, $\underline{P_{\mathcal{R}} \Rightarrow U f}$ ), and Lemma 554 (disjoint decomposition, $\underline{D}^{\prime} \Rightarrow P t f D^{\prime}$ and $D \Rightarrow P t f D$ ).

Definition 741 (measurable space).
Let $X$ be a set. Let $\Sigma$ be a $\sigma$-algebra on $X$. Then, $(X, \Sigma)$ is called measurable space, and elements of $\Sigma$ are said ( $\Sigma$-) measurable.

## Lemma 742 (properties of measurable space).

Let $(X, \Sigma)$ be a measurable space. Then, $(X, \Sigma)$ is a premeasurable space, and $\Sigma$ is a $\lambda$-system, a monotone class, a (semi-)algebra of sets, a (semi)ring of sets, and a $\pi$-system.

Moreover, $\Sigma$ is nonempty, contains the empty set and the full set, is closed under complement, local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, countable monotone union and intersection, and countable union and intersection, and complement, local complement and set difference of elements of $\Sigma$ can be expressed as a finite disjoint union of elements of $\Sigma$.

$$
\begin{gathered}
\text { Compact form: Concl } \stackrel{\text { def. }}{=}\left[P_{\Sigma}\left|P_{\Lambda}\right| P_{\mathcal{C}}\left|P_{\mathcal{A}}\right| P_{\mathcal{A} / 2}\left|P_{\mathcal{R}}\right| P_{\mathcal{R} / 2}\left|P_{\Pi}\right|\right. \\
N e|w E| w F|C| D^{\prime}|D| s D|U d| U|I| U d f|U f| I f \mid \\
\left.U d c|U m c| \text { Imc }|U c| I c|P t f C| P t f D^{\prime} \mid P t f D\right] .
\end{gathered}
$$

Proof. Direct consequence of Definition 741 (measurable space, thus $P_{\Sigma}$ ), Lemma 701 ( $\sigma$-algebra is $\lambda$-system,,$P_{\Sigma} \Rightarrow P_{\Lambda}$ ), Lemma 702 ( $\sigma$-algebra is monotone class, $P_{\Sigma} \Rightarrow P_{\mathcal{C}}$ ), Lemma 699 ( $\sigma$-algebra is algebra of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{A}}$ ), Lemma 660 (algebra of sets is semi-algebra of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{A} / 2}$ ), Lemma 703 ( $\sigma$-algebra is ring of sets, $P_{\Sigma} \Rightarrow P_{\mathcal{R}}$ ), Lemma 631 (ring of sets is semiring of sets, $\underline{P}_{\mathcal{R}} \Rightarrow P_{\mathcal{R} / 2}$ ), Lemma 704 ( $\sigma$-algebra is $\pi$-system, $\underline{P}_{\Sigma} \Rightarrow P_{\Pi}$ ), Definition 739 (premeasurable space), Lemma 697 (equivalent definition of $\sigma$-algebra, $\overline{P_{\Sigma} \Rightarrow[N e|w E| w F|C| U c \mid I c] \text { ), Lemma } 700}$ (other properties of $\sigma$-algebra, $P_{\Sigma} \Rightarrow\left[D^{\prime}|D| s \overline{D|U d| U|I| U d f|U f| I f \mid U d c}|U m c|\right.$ Imc $]$ ) and Lemma 554 (disjoint decomposition, $C \Rightarrow P t f C, D^{\prime} \Rightarrow P t f D^{\prime}$ and $D \Rightarrow$ PtfD).

## Definition 743 (Borel $\sigma$-algebra).

Let $(X, \mathcal{T})$ be a topological space. The Borel $\sigma$-algebra of $X$ is the $\sigma$-algebra generated by the open subsets; it is denoted $\mathcal{B}(X) \stackrel{\text { def. }}{=} \Sigma_{X}(\mathcal{T})$.
$\mathcal{B}(X)$-measurable subsets are called Borel subsets of $X$.

[^0]Proof. From Definition 743 (Borel $\sigma$-algebra), and Lemma 696 ( $\sigma$-algebra), the Borel subsets are the elements of the Borel $\sigma$-algebra.

Let $Y$ be an open subset of $X$. Then, from Definition 743 (Borel $\sigma$-algebra), and Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $Y \in \mathcal{T} \subset \mathcal{B}(X)$.

Let $Y$ be a closed subset of $X$. Then, from Definition 240 (topological space, closed subset), Definition 743 (Borel $\sigma$-algebra), and Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $Y^{c} \in \mathcal{T} \subset \mathcal{B}(X)$. Hence, from Lemma 696 ( $\sigma$-algebra, closedness under complement), and from involutiveness of complement, we have $Y=\left(Y^{c}\right)^{c} \in \overline{\mathcal{B}}(X)$.

Assume now that $X$ is separable, i.e. $X$ contains a countable dense subset. Let $Y$ be a countable subset of $X$. Then, from the definition of countability, there exist $I \subset \mathbb{N}$, and $\left(x_{i}\right)_{i \in I} \in X$ such that $Y=\bigcup_{i \in I}\left\{x_{i}\right\}$. Let $i \in I$. Then, from closedness of singletons in a separable space, $\left\{x_{i}\right\}$ is closed, thus it belongs to $\mathcal{B}(X)$. Hence, from Lemma 696 ( $\sigma$-algebra, closedness under countable union), we have $Y \in \mathcal{B}(X)$.

Therefore, open, closed, and countable subsets (when $X$ is separable) are Borel subsets of $X$.

## Lemma 745 (countable union Borel $\sigma$-algebra generator).

Let $(X, \mathcal{T})$ be a topological space. Let $G \subset \mathcal{T}$. Assume that all open subsets of $X$ are countable unions of elements of $G$. Then, we have $\mathcal{B}(X)=\Sigma_{X}(G)$.

Proof. Direct consequence of Definition 743 (Borel $\sigma$-algebra, $\mathcal{B}(X)=\Sigma_{X}(\mathcal{T})$ ), and Lemma 712 (countable union $\sigma$-algebra generator, with $G_{1} \stackrel{\text { def. }}{=} G$ and $G_{2} \stackrel{\text { def. }}{=} \mathcal{T}$ ).

Remark 746. The previous lemma means that if $G$ contains a countable topological basis of $(X, \mathcal{T})$ (and possibly other open subsets) (see Definitions 246 and 260), then the Borel $\sigma$-algebra of $X$ is generated by $G$.

Remark 747. Note that metric spaces are separable topological spaces, thus both previous lemmas apply in metric spaces.

Definition 748 (measurability and structure).
Let $(X, \mathcal{T})$ be a topological space.
The measurable space $(X, \mathcal{B}(X)$ ) (or $X$ ), is called measurable topological space.
Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $\left(E,\|\cdot\|_{E}\right)$ be a normed vector space over $\mathbb{K}$ equipped with the induced topology. The measurable space $(E, \mathcal{B}(E))$ (or $E$ ) is called measurable normed vector space.

Let $\mathbb{K}$ be $\mathbb{R}, \mathbb{C}$, or $\overline{\mathbb{R}}$ equipped with the usual topology. The measurable space $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$ (or $\mathbb{K})$ is called measurable number space. For $\mathbb{R}$ and $\mathbb{C}$, it is also called measurable valued field.

### 9.2 Measurable function

Definition 749 (measurable function). Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. A function $f: X \rightarrow X^{\prime}$ is said measurable (for $\Sigma$ and $\Sigma^{\prime}$ ) iff

$$
\begin{equation*}
f^{-1}\left(\Sigma^{\prime}\right) \subset \Sigma \quad \text { i.e. } \forall A^{\prime} \in \Sigma^{\prime}, f^{-1}\left(A^{\prime}\right) \in \Sigma \tag{9.1}
\end{equation*}
$$

If $\Sigma$ and $\Sigma^{\prime}$ are the Borel $\sigma$-algebras of $X$ and $X^{\prime}$, then $f$ is called Borel function.

## Lemma 750 (inverse $\sigma$-algebra).

Let $X$ be a set. Let $\left(X^{\prime}, \Sigma^{\prime}\right)$ be a measurable space. Let $f: X \rightarrow X^{\prime}$. Then, $\Sigma=f^{-1}\left(\Sigma^{\prime}\right)$ is the smallest $\sigma$-algebra of $f^{-1}\left(X^{\prime}\right) \subset X$ such that $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

It is called the inverse $\sigma$-algebra of $\Sigma^{\prime}$ by $f$.
Proof. Direct consequence of Definition 741 (measurable space), Lemma 696 ( $\sigma$-algebra), homogeneity of inverse image, compatibility of inverse image with complement, union and intersection, and Definition 749 (measurable function).

Lemma 751 (image $\sigma$-algebra).
Let $(X, \Sigma)$ be a measurable space. Let $X^{\prime}$ be a set.
Let $f: X \rightarrow X^{\prime}$. Then, $\Sigma^{\prime}=\Sigma_{f} \stackrel{\text { def. }}{=}\left\{A^{\prime} \subset X^{\prime} \mid f^{-1}\left(A^{\prime}\right) \in \Sigma\right\}$ is the largest $\sigma$-algebra of $f(X)$ such that $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$. It is called the image $\sigma$-algebra of $\Sigma$ by $f$.

Proof. Direct consequence of Definition 741 (measurable space), Lemma 696 ( $\sigma$-algebra), compatibility of inverse image with complement, union and intersection, homogeneity of inverse image $\left(f^{-1}(\varnothing)=\varnothing\right)$, and Definition 749 (measurable function).

## Lemma 752 (identity function is measurable).

Let $(X, \Sigma)$ be a measurable space. Then, the identity function is measurable for $\Sigma$ (and $\Sigma$ ).
Proof. Direct consequence of the definition of the identity function, and Definition 749 (measurable function).

## Lemma 753 (countable range is measurable).

Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Assume that $f(X)$ is countable, and that for all $x^{\prime} \in f(X), f^{-1}\left(x^{\prime}\right) \in \Sigma$. Then, $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

Proof. Let $I \subset \mathbb{N}$, and $\left(x_{i}^{\prime}\right)_{i \in I} \in X^{\prime}$ be pairwise distinct such that $f(X)=\left\{x_{i}^{\prime} \mid i \in I\right\}$. For all $i \in I$, let $A_{i} \stackrel{\text { def. }}{=} f^{-1}\left(x_{i}^{\prime}\right) \in \Sigma$. Then, from properties of inverse image, we have $X=\biguplus_{i \in I} A_{i}$. Let $A^{\prime} \in \Sigma^{\prime}$. Let $I_{A^{\prime}} \stackrel{\text { def. }}{=}\left\{i \in I \mid x_{i}^{\prime} \in A^{\prime}\right\} \subset I \subset \mathbb{N}$. Then, from properties of inverse image, we have $f^{-1}\left(A^{\prime}\right)=\biguplus_{i \in I_{A^{\prime}}} A_{i}$ (which is empty when $I_{A^{\prime}}=\varnothing$ ). Therefore, from Definition 741 ( measurable space), Lemma 700 (other properties of $\sigma$-algebra, $P_{\Sigma} \Rightarrow$ Udc), Lemma 696 ( $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow w E}$ ), and Definition 749 (measurable function), $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

Lemma 754 (constant function is measurable). Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Assume that $f$ is constant. Then, $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

Proof. Direct consequence of Definition 741 (measurable space), Lemma 697 (equivalent definition of $\sigma$-algebra, $P_{\Sigma} \Rightarrow w F$ ), and Lemma 753 (countable range is measurable, with $f(X) \stackrel{\text { def. }}{=}\left\{x^{\prime}\right\}$ and $\underline{f^{-1}\left(x^{\prime}\right)}=X \in \Sigma$.

Lemma 755 (inverse image of generating family).
Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Then, we have

$$
\forall G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right), \quad \Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)=f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)
$$

Proof. "Left" included in "right". Let $G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right)$. Then, from Lemma 709 (generated $\sigma$ algebra is $\sigma$-algebra), we have $G^{\prime} \subset \Sigma_{X^{\prime}}\left(G^{\prime}\right)$. Thus, from monotonicity of inverse image, $f^{-1}\left(G^{\prime}\right)$ is a subset of $f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)$. Hence, from Lemma 710 (properties of $\sigma$-algebra generation, monotonicity and idempotence), and Lemma 750 (inverse $\sigma$-algebra, $\underline{f}^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)$ is $\sigma$-algebra), we have $\Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right) \subset \Sigma_{X}\left(f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)\right)=f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)$.
"Right" included in "left". Conversely, let $A^{\prime} \in G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right)$. Then, from monotonicity of inverse image, and Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have

$$
f^{-1}\left(A^{\prime}\right) \in f^{-1}\left(G^{\prime}\right) \subset \Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)
$$

Thus, from Lemma 751 (image $\sigma$-algebra), we have $G^{\prime} \subset\left(\Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)\right)_{f}$, and from Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $\Sigma_{X^{\prime}}\left(G^{\prime}\right) \subset\left(\Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)\right)_{f}$. Hence, from Lemma 751 (image $\sigma$-algebra), and the definition of inverse image, we have

$$
f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right) \subset \Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)
$$

Therefore, we have $\Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right)=f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)$.
Lemma 756 (equivalent definition of measurable function). Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Then, $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$ iff

$$
\begin{equation*}
\forall G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right), \quad \Sigma_{X^{\prime}}\left(G^{\prime}\right)=\Sigma^{\prime} \quad \Longrightarrow \quad f^{-1}\left(G^{\prime}\right) \subset \Sigma \tag{9.3}
\end{equation*}
$$

Proof. "Left" implies "right". Assume first that $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$. Let $G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right)$ such that $\Sigma_{X^{\prime}}\left(G^{\prime}\right)=\Sigma^{\prime}$. Then, from Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $G^{\prime} \subset \Sigma^{\prime}$. Hence, from monotonicity of inverse image, and Definition 749 (measurable function), we have $f^{-1}\left(G^{\prime}\right) \subset f^{-1}\left(\Sigma^{\prime}\right) \subset \Sigma$.
"Right" implies "left". Conversely, assume now that there exists $G^{\prime} \subset \mathcal{P}\left(X^{\prime}\right)$ such that $\Sigma_{X^{\prime}}\left(G^{\prime}\right)=\Sigma^{\prime}$ and $f^{-1}\left(G^{\prime}\right) \subset \Sigma$. Then, from Lemma 755 (inverse image of generating fam$i l y$ ), and Lemma 710 (properties of $\sigma$-algebra generation, monotonicity and idempotence), we have $f^{-1}\left(\Sigma^{\prime}\right)=f^{-1}\left(\Sigma_{X^{\prime}}\left(G^{\prime}\right)\right)=\Sigma_{X}\left(f^{-1}\left(G^{\prime}\right)\right) \subset \Sigma_{X}(\Sigma)=\Sigma$.
Therefore, we have the equivalence.
Lemma 757 (compatibility of measurability with composition).
Let $(X, \Sigma),\left(X^{\prime}, \Sigma^{\prime}\right)$, and $\left(X^{\prime \prime}, \Sigma^{\prime \prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$ and $g: X^{\prime} \rightarrow X^{\prime \prime}$. Assume that $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$, and that $g$ is measurable for $\Sigma^{\prime}$ and $\Sigma^{\prime \prime}$. Then, $g \circ f$ is measurable for $\Sigma$ and $\Sigma^{\prime \prime}$.

Proof. Direct consequence of the definition of composition, Definition 749 (measurable function, with $f$ and $g$ ), and monotonicity of inverse image.

## Definition 758 ( $\mathcal{M}_{Y}$, set of measurable functions to $Y$ ).

Let $(X, \Sigma)$ be a measurable space. Let $(Y, \mathcal{B}(Y))$ be a measurable topological space. The set of functions $X \rightarrow Y$ that are measurable for $\Sigma$ and $\mathcal{B}(Y)$ is called the set of measurable functions $($ from $X)$ to $Y$; it is denoted $\mathcal{M}_{Y}(X, \Sigma)$ (or simply $\left.\mathcal{M}_{Y}\right)$.

Lemma 759 ( $\mathcal{M}_{Y}$ inverse image of countable is measurable).
Let $(X, \Sigma)$ be a measurable space. Let $(Y, \mathcal{B}(Y))$ be a measurable separable topological space. Let $f \in \mathcal{M}_{Y}$. Then, we have

$$
\begin{equation*}
\forall I \subset \mathbb{N}, \forall\left(y_{i}\right)_{i \in I} \in Y, \quad\left\{f=\left(y_{i}\right)_{i \in I}\right\} \stackrel{\text { def. }}{=} f^{-1}\left\{y_{i} \mid i \in I\right\} \in \Sigma \tag{9.4}
\end{equation*}
$$

Proof. Direct consequence of Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$, Definition 749 (measurable function), Lemma 744 (some Borel subsets, countable subsets are measurable).

Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range). Let $(X, \Sigma)$ be a measurable space. Let $(Y, \mathcal{B}(Y))$ be a measurable separable topological space. Let $f: X \rightarrow Y$. Assume that $f(X)$ is countable. Then, we have $f \in \mathcal{M}_{Y}$ iff for all $y \in f(X), f^{-1}(y) \in \Sigma$.

Proof. Direct consequence of Lemma 759 ( $\mathcal{M}_{Y}$ inverse image of countable is measurable, with $\underline{\operatorname{card}(I)=1}$ ), and Lemma 753 (countable range is measurable).

Lemma 761 (continuous is measurable). Let $(X, \mathcal{B}(X))$ and $\left(X^{\prime}, \mathcal{B}\left(X^{\prime}\right)\right)$ be measurable topological spaces. Let $f: X \rightarrow X^{\prime}$. Assume that $f$ is continuous. Then, $f$ is a Borel function.

Proof. Let $\mathcal{T}$ and $\mathcal{T}^{\prime}$ be the topologies of $X$ and $X^{\prime}$. Let $O^{\prime} \in \mathcal{T}^{\prime}$. Then, from the definition of continuity, $f^{-1}\left(O^{\prime}\right) \in \mathcal{T}$. Thus, from Definition 743 (Borel $\sigma$-algebra), and Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $\mathcal{B}\left(X^{\prime}\right)=\Sigma_{X^{\prime}}\left(\mathcal{T}^{\prime}\right)$ and $f^{-1}\left(\mathcal{T}^{\prime}\right) \subset \mathcal{T} \subset \Sigma_{X}(\mathcal{T})=\mathcal{B}(X)$. Therefore, from Lemma 756 (equivalent definition of measurable function), and Definition 749 (measurable function), $f$ is a Borel function from $X$ to $X^{\prime}$.

### 9.3 Measurable subspace

Remark 762. We recall that $\bar{\cap}$ denotes a set of traces of subsets, see Definition 226.
Lemma 763 (trace $\sigma$-algebra).
Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $i$ be the canonical injection from $Y$ to $X$. Then, $\Sigma \bar{\cap} Y$ is a $\sigma$-algebra of $Y$, and $i$ is measurable for $\Sigma \bar{\cap} Y$ and $\Sigma$.
$\Sigma \bar{\cap} Y$ is called trace $\sigma$-algebra of $\Sigma$ on $Y$. The measurable space $(Y, \Sigma \bar{\cap} Y)$ is said measurable subspace of $(X, \Sigma)$.

Proof. From Definition 741 (measurable space, $\Sigma$ is a $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, $\varnothing \in \Sigma$ ), and since $\varnothing$ is absorbing for intersection, we have $\varnothing=\varnothing \cap Y \in \Sigma \bar{\cap} Y$.

Let $A \in \Sigma \bar{\cap} Y$. Then, from Definition 226 (trace of subset system), let $B \in \Sigma$ such that $A=B \cap Y$. Then, from the definition of set difference, De Morgan's laws, distributivity of intersection over union, the definition of complement, and commutativity of intersection, we have

$$
Y \backslash A=Y \backslash(B \cap Y)=Y \cap(B \cap Y)^{c}=Y \cap\left(B^{c} \cup Y^{c}\right)=\left(Y \cap B^{c}\right) \cup\left(Y \cap Y^{c}\right)=B^{c} \cap Y
$$

Hence, from Lemma 696 ( $\sigma$-algebra, closedness under complement), we have $Y \backslash A \in \Sigma \bar{\cap} Y$.
Let $I$ be a set. Let $\left(A_{i}\right)_{i \in I} \in \Sigma \overline{\bar{\Pi} Y}$. Then, from Definition 226 (trace of subset system), for all $i \in I$, let $B_{i} \in \Sigma$ such that $A_{i}=B_{i} \cap Y$. Then, from distributivity of intersection over union, we have

$$
\bigcup_{i \in I} A_{i}=\bigcup_{i \in I}\left(B_{i} \cap Y\right)=\bigcup_{i \in I} B_{i} \cap Y
$$

Hence, from Lemma 696 ( $\sigma$-algebra, closedness under countable union), we have $\bigcup_{i \in I} A_{i} \in \Sigma \bar{\cap} Y$.
Therefore, from Lemma $696(\sigma$-algebra $), \Sigma \bar{\cap} Y$ is a $\sigma$-algebra of $Y$. Moreover, from the definition of the canonical injection, and Definition 749 (measurable function), $i$ is measurable for $\Sigma \bar{\cap} Y$ and $\Sigma$.

Lemma 764 (measurability of measurable subspace).
Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Then, we have $Y \in \Sigma$ iff $\Sigma \bar{\cap} Y=\Sigma \cap \mathcal{P}(Y)$.
Proof. From Definition 226 (trace of subset system), we always have $\Sigma \cap \mathcal{P}(Y) \subset \Sigma \bar{\cap} Y$.
"Left" implies "right". Assume first that $Y \in \Sigma$. Let $A \in \Sigma \bar{\cap} Y$. Then, from Definition 226 (trace of subset system), there exists $B \in \Sigma$ such that $A=B \cap Y \subset Y$. Thus, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection), we have $A \in \Sigma$, and $\Sigma \bar{\cap} Y \subset \Sigma \cap \mathcal{P}(Y)$. Hence, we have $\Sigma \bar{\cap} Y=\Sigma \cap \mathcal{P}(Y)$.
"Right" implies "left". Conversely, assume now that $\Sigma \bar{\cap} Y=\Sigma \cap \mathcal{P}(Y)$. Then, from Lemma 763 (trace $\sigma$-algebra), and Lemma 697 (equivalent definition of $\sigma$-algebra, contains full set), we have $Y \in \Sigma \bar{\cap} Y \subset \Sigma$.

Therefore, we have the equivalence.
Lemma 765 (generating measurable subspace). Let $X$ be a set, $Y \subset X$, and $G \subset \mathcal{P}(X)$. Then, we have $\Sigma_{Y}(G \bar{\cap} Y)=\Sigma_{X}(G) \bar{\cap} Y$, and $Y \in \Sigma_{X}(G)$ iff $\Sigma_{Y}(G \bar{\cap} Y)=\Sigma_{X}(G) \cap \mathcal{P}(Y)$.

Proof. Let $i$ be the canonical injection from $Y$ to $X$. Then, from Definition 226 (trace of subset system), we have $i^{-1}(G)=G \bar{\cap} Y$ and $i^{-1}\left(\Sigma_{X}(G)\right)=\Sigma_{X}(G) \bar{\cap} Y$. Therefore, from Lemma 755 (inverse image of generating family), we have

$$
\Sigma_{Y}(G \bar{\cap} Y)=\Sigma_{Y}\left(i^{-1}(G)\right)=i^{-1}\left(\Sigma_{X}(G)\right)=\Sigma_{X}(G) \bar{\cap} Y
$$

And the equivalence directly follows from Lemma 764 (measurability of measurable subspace).

## Lemma 766 (generating on a partition).

Let $X$ be a set. Let $G \subset \mathcal{P}(X)$. Let $I \subset \mathbb{N}$. Let $\left(Y_{i}\right)_{i \in I} \in \Sigma_{X}(G)$. Assume that $X=\biguplus_{i \in I} Y_{i}$. Let $A \subset X$. Then, we have

$$
\begin{equation*}
A \in \Sigma_{X}(G) \quad \Longleftrightarrow \quad \forall i \in I, \quad A \cap Y_{i} \in \Sigma_{Y_{i}}\left(G \bar{\cap} Y_{i}\right) \tag{9.5}
\end{equation*}
$$

## Proof. "Left" implies "right".

Assume first that $A \in \Sigma_{X}(G)$. Let $i \in I$. Then, from Definition 226 (trace of subset system), and Lemma 765 (generating measurable subspace), we have $A \cap Y_{i} \in \Sigma_{X}(G) \bar{\cap} Y_{i}=\Sigma_{Y_{i}}\left(G \bar{\cap} Y_{i}\right)$.
"Right" implies "left". Conversely, assume now that for all $i \in I, A \cap Y_{i} \in \Sigma_{Y_{i}}\left(G \bar{\cap} Y_{i}\right)$. Let $i \in I$. Then, from Lemma 765 (generating measurable subspace, with $Y_{i} \in \Sigma_{X}(G)$ ), we have $A \cap Y_{i} \in \Sigma_{X}(G)$. Hence, from Lemma 211 (compatibility of pseudopartition with intersection), and Lemma 696 ( $\sigma$-algebra, closedness under countable union), we have $A=\biguplus_{i \in I}\left(A \cap Y_{i}\right) \in \Sigma_{X}(G)$.
Therefore, we have the equivalence.
Remark 767. We recall that $\mathcal{T}_{Y} \stackrel{\text { def. }}{=} \mathcal{T} \bar{\cap} Y$ denotes the trace topology, see Lemma 252.

## Lemma 768 (Borel sub- $\sigma$-algebra). Let $(X, \mathcal{T})$ be a topological space. Let $Y \subset X$.

 Then, we have $\mathcal{B}(Y) \stackrel{\text { def. }}{=} \Sigma_{Y}\left(\mathcal{T}_{Y}\right)=\mathcal{B}(X) \bar{\cap} Y$, and $Y \in \mathcal{B}(X)$ iff $\mathcal{B}(Y)=\mathcal{B}(X) \cap \mathcal{P}(Y)$.Proof. Direct consequence of Definition 743 (Borel $\sigma$-algebra), Lemma 252 (trace topology on sub-


Lemma 769 (Borel subsets of partition).
Let $(X, \mathcal{T})$ be a topology space.
Let $I \subset \mathbb{N}$. Let $\left(Y_{i}\right)_{i \in I} \in \mathcal{B}(X)$. Assume that $X=\biguplus_{i \in I} Y_{i}$. Let $A \subset X$. Then, we have

$$
\begin{equation*}
A \in \mathcal{B}(X) \quad \Longleftrightarrow \quad \forall i \in I, \quad A \cap Y_{i} \in \mathcal{B}\left(Y_{i}\right) \tag{9.6}
\end{equation*}
$$

Proof. Direct consequence of Lemma 766 (generating on a partition, with $G \stackrel{\text { def. }}{=} \mathcal{T}$ ).
Lemma 770 (source restriction of measurable function). $\quad$ Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Let $Y \subset X$. Let $f_{\left.\right|_{Y}}$ be the restriction of $f$ to the source $Y$. Assume that $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$. Then, $f_{\mid Y}$ is measurable for $\Sigma \bar{\cap} Y$ and $\Sigma^{\prime}$.

Proof. Let $i$ be the canonical injection from $Y$ to $X$. Then, from Definition 226 (trace of subset system), and Lemma 763 (trace $\sigma$-algebra), $i^{-1}(\Sigma)=\Sigma \bar{\cap} Y$ is a $\sigma$-algebra of $Y$. Thus, from Definition 749 (measurable function), $i$ is measurable for $\Sigma \bar{\cap} Y$ and $\Sigma$. Therefore, from Lemma 757 (compatibility of measurability with composition), $f_{\left.\right|_{Y}}=f \circ i$ is measurable for $\Sigma \bar{\cap} Y$ and $\Sigma^{\prime}$.

Lemma 771 (destination restriction of measurable function). $\quad$ Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $f: X \rightarrow X^{\prime}$. Let $Y^{\prime} \subset X^{\prime}$. Assume that $f(X) \subset Y^{\prime}$. Let $f^{\left.\right|^{Y^{\prime}}}$ be the restriction of $f$ to the destination $Y^{\prime}$. Then, $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$ iff $\left.f\right|^{Y^{\prime}}$ is measurable for $\Sigma$ and $\Sigma^{\prime} \bar{\cap} Y^{\prime}$.

Proof. Let $A^{\prime} \in \Sigma^{\prime}$. Then, from the definition of set difference, and monotonicity of inverse image and complement, we have

$$
f^{-1}\left(A^{\prime} \backslash Y^{\prime}\right)=f^{-1}\left(A^{\prime} \cap Y^{\prime c}\right) \subset f^{-1}\left(Y^{\prime c}\right) \subset f^{-1}\left(f(X)^{c}\right)=\varnothing
$$

Then, from compatibility of inverse image with disjoint union, and the definition of destination restriction, we have

$$
f^{-1}\left(A^{\prime}\right)=f^{-1}\left(\left(A^{\prime} \cap Y^{\prime}\right) \uplus\left(A^{\prime} \backslash Y^{\prime}\right)\right)=f^{-1}\left(A^{\prime} \cap Y^{\prime}\right)=\left(f^{\left.\right|^{Y^{\prime}}}\right)^{-1}\left(A^{\prime} \cap Y^{\prime}\right)
$$

Hence, $f^{-1}\left(\Sigma^{\prime}\right)=\left(f^{Y^{\prime}}\right)^{-1}\left(\Sigma^{\prime} \cap Y^{\prime}\right)$. Therefore, from Definition 749 (measurable function), $f^{\left.\right|^{\prime}}$ is measurable for $\Sigma$ and $\Sigma^{\prime} \bar{\cap} Y^{\prime}$.

Lemma 772 (measurability of function defined on a pseudopartition).
Let $(X, \Sigma)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $X_{i} \in \Sigma$ and $f_{i}: X \rightarrow X^{\prime}$. Assume that for all $i \in I, f_{i}$ is measurable for $\Sigma$ and $\Sigma^{\prime}$, and that $X=\biguplus_{i \in I} X_{i}$. Then, the function defined by $f_{i}$ on $X_{i}$ for all $i \in I$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

Proof. Let $f$ be the function defined by for all $i \in I$, for all $x \in X_{i}, f(x) \stackrel{\text { def. }}{=} f_{i}(x)$. Let $A^{\prime} \in \Sigma^{\prime}$. Let $i \in I$. Then, from Definition 749 (measurable function, with $f_{i}$ ), we have $f_{i}^{-1}\left(A^{\prime}\right) \in \Sigma$. Thus, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable union and intersection), we have $f^{-1}\left(A^{\prime}\right)=\bigcup_{i \in I}\left(X_{i} \cap f_{i}^{-1}\left(A^{\prime}\right)\right) \in \Sigma$. Hence, $f^{-1}\left(\Sigma^{\prime}\right) \subset \Sigma$.

Therefore, from Definition 749 (measurable function), $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.

### 9.4 Product of measurable spaces

Remark 773. We recall that $[n . . p]$ denotes an interval of integers, and that $\bar{\Pi}$ and $\overline{\times}$ denote a set of Cartesian products of subsets, see Definition 227.

The concepts presented in this section are used mainly in Section 15.4.

## Definition 774 (tensor product of $\sigma$-algebras).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$ and $\bar{\Sigma} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[1 . . m]} \Sigma_{i}$. The (tensor) product of the $\sigma$-algebras $\left(\Sigma_{i}\right)_{i \in[1 . . m]}$ on $X$ is the $\sigma$-algebra generated by $\bar{\Sigma}$; it is denoted $\bigotimes_{i \in[1 . . m]} \Sigma_{i} \stackrel{\text { def. }}{=} \Sigma_{X}(\bar{\Sigma})$.

## Lemma 775 (product of measurable subsets is measurable).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space, and $A_{i} \in \Sigma_{i}$.
Let $\bar{\Sigma} \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} \Sigma_{i}$ and $\Sigma \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]} \Sigma_{i}$. Then, $\prod_{i \in[1 . . m]} A_{i} \in \bar{\Sigma} \subset \Sigma$.
Proof. Direct consequence of Definition 227 (product of subset systems), Definition 774 (tensor product of $\sigma$-algebras), and Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, $\left.\overline{\bar{\Sigma} \subset \Sigma_{X}(\bar{\Sigma})}\right)$.

## Lemma 776 (measurability of function to product space).

Let $m \in[2 . . \infty)$. Let $(X, \Sigma)$ be a measurable space. For all $i \in[1 . . m]$, let $\left(X_{i}^{\prime}, \Sigma_{i}^{\prime}\right)$ be a measurable space, and $f_{i}: X \rightarrow X_{i}^{\prime}$. Then, $\left(f_{i}\right)_{i \in[1 . . m]}$ is measurable for $\Sigma$ and $\bigotimes_{i \in[1 . . m]} \Sigma_{i}^{\prime}$ iff for all $i \in[1 . . m], f_{i}$ is measurable for $\Sigma$ and $\Sigma_{i}^{\prime}$.

Proof. Let $\Sigma^{\prime} \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]} \Sigma_{i}^{\prime}$.
"Left" implies "right". Assume first that $\left(f_{i}\right)_{i \in[1 . . m]}$ is measurable for $\Sigma$ and $\Sigma$.
Let $i \in[1 . . m]$. Let $A_{i}^{\prime} \in \Sigma_{i}^{\prime}$. Let $A^{\prime} \stackrel{\text { def. }}{=} X_{1}^{\prime} \times \ldots \times A_{i}^{\prime} \times \ldots \times X_{m}^{\prime}$. Then, from Lemma 697 (equivalent
 (product of measurable subsets is measurable), we have $A^{\prime} \in \Sigma^{\prime}$. Thus, from Definition 749 (measurable function, with $f$ ), we have $f_{i}^{-1}\left(A_{i}^{\prime}\right)=f^{-1}\left(A^{\prime}\right) \in \Sigma$. Hence, from Definition 749 (measurable function), $f_{i}$ is measurable for $\Sigma$ and $\Sigma_{i}^{\prime}$.
"Right" implies "left". Assume now that for all $i \in[1 . . m], f_{i}$ is measurable for $\Sigma$ and $\Sigma_{i}^{\prime}$.
Let $\bar{\Sigma}^{\prime} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[1 . . m]} \Sigma_{i}^{\prime}$ and $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in[1 . . p]}$. Let $A^{\prime} \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} A_{i}^{\prime} \in \bar{\Sigma}^{\prime}$. From Definition 227 (product of subset systems), for all $i \in[1 . . m]$, we have $A_{i}^{\prime} \in \Sigma_{i}^{\prime}$. Let $x \in X$. Then, $f(x) \in A^{\prime}$ iff for all $i \in[1 . . m], f_{i}(x) \in A_{i}^{\prime}$. Thus, we have $f^{-1}\left(A^{\prime}\right)=\bigcap_{i \in[1 . . m]} f_{i}^{-1}\left(A_{i}^{\prime}\right)$. Hence, from Definition 749 (measurable function, with $f_{i}$ ), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection), we have $f^{-1}\left(\bar{\Sigma}^{\prime}\right)=\bigcap_{i \in[1 . . m]} f_{i}^{-1}\left(\Sigma_{i}^{\prime}\right) \subset \Sigma$. Thus, from Lemma 756 (equivalent definition of measurable function, with $G^{\prime} \stackrel{\text { def. }}{=} \bar{\Sigma}^{\prime}$ ), $f$ is measurable for $\Sigma$ and $\Sigma^{\prime}$.
Therefore, we have the equivalence.

## Lemma 777 (canonical projection is measurable).

Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space, and let $\pi_{i}$ be the canonical projection from $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$ onto $X_{i}$. Let $\Sigma \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]} \Sigma_{i}$.
Then, $\Sigma$ is the smallest $\sigma$-algebra on $X$ such that for all $i \in[1 . . m], \pi_{i}$ is measurable for $\Sigma$ and $\Sigma_{i}$.
Proof. Let $\Sigma^{\prime}$ be a $\sigma$-algebra on $X$. Then, from Lemma 776 (measurability of function to product space, with $\Sigma \stackrel{\text { def. }}{=} \Sigma^{\prime}, X_{i}^{\prime} \stackrel{\text { def. }}{=} X_{i}, \Sigma_{i}^{\prime} \stackrel{\text { def. }}{=} \Sigma_{i}$, and $f \stackrel{\text { def. }}{=} \operatorname{Id}_{X}=\left(\pi_{i}\right)_{i \in[1 . . m]}$ ), Definition 749 (measurable function), and reflexivity of inclusion we have

$$
\begin{aligned}
\forall i \in[1 . . m], \quad \pi_{i} \text { is measurable for } \Sigma^{\prime} \text { and } \Sigma_{i} & \Longleftrightarrow \operatorname{Id}_{X} \text { is measurable for } \Sigma^{\prime} \text { and } \Sigma \\
& \Longleftrightarrow \operatorname{Id}_{X}^{-1}(\Sigma)=\Sigma \subset \Sigma^{\prime}
\end{aligned}
$$

Lemma 778 (permutation is measurable).
Let $m \in[2 . . \infty)$. Let $\psi$ be a permutation of $[1 . . m]$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space, and let $\pi_{i}$ be the canonical projection from $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$ onto $X_{i}$. Let $\Sigma \stackrel{\text { def. }}{=} \otimes_{i \in[1 . . m]} \Sigma_{i}$, and $\Sigma^{\psi} \stackrel{\text { def. }}{=} \otimes_{i \in[1 . . m]} \Sigma_{\psi(i)}$. Then, the permutation of coordinates $\left(\pi_{\psi(i)}\right)_{i \in[1 . . m]}$ is measurable for $\Sigma$ and $\Sigma^{\psi}$.

Proof. Direct consequence of Lemma 776 (measurability of function to product space), and Lemma 777 (canonical projection is measurable, with $X_{i}^{\prime} \stackrel{\text { def. }}{=} X_{\psi(i)}$ and $\left.f_{i} \stackrel{\text { def. }}{=} \pi_{\psi(i)}\right)$.

Lemma 779 (generating product measurable space). Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space, $G_{i} \subset \mathcal{P}\left(X_{i}\right)$, and assume that $X_{i} \in G_{i}$ and $\Sigma_{i}=\Sigma_{X_{i}}\left(G_{i}\right)$. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}, \Sigma \stackrel{\text { def. }}{=} \otimes_{i \in[1 . . m]} \Sigma_{i}$, and $\bar{G} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[1 . . m]} G_{i}$. Then, we have $\Sigma=\Sigma_{X}(\bar{G})$.

Proof. $\boldsymbol{\Sigma} \subset \boldsymbol{\Sigma}_{\boldsymbol{X}}(\overline{\boldsymbol{G}})$. Let $i \in[1 . . m]$. Let $\pi_{i}$ be the canonical projection from $X$ onto $X_{i}$. Then, from Lemma 755 (inverse image of generating family), we have

$$
\pi_{i}^{-1}\left(\Sigma_{i}\right)=\pi_{i}^{-1}\left(\Sigma_{X_{i}}\left(G_{i}\right)\right)=\Sigma_{X}\left(\pi_{i}^{-1}\left(G_{i}\right)\right)
$$

Moreover, since $X_{i} \in G_{i}$, and from Definition 227 (product of subset systems), we have

$$
\pi_{i}^{-1}\left(G_{i}\right)=\left\{X_{1} \times \ldots \times A_{i} \times \ldots \times X_{m} \mid A_{i} \in G_{i}\right\} \subset \bar{G}
$$

Hence, from Lemma 710 (properties of $\sigma$-algebra generation, monotonicity, thus $\pi_{i}^{-1}\left(\Sigma_{i}\right)$ is a subset of $\Sigma_{X}(\bar{G})$ ), Definition 749 (measurable function, $\pi_{i}$ is measurable for $\Sigma_{X}(\bar{G})$ and $\Sigma_{i}$ ), and Lemma 777 (canonical projection is measurable, smallest $\sigma$-algebra), we have $\Sigma \subset \Sigma_{X}(\bar{G})$.
$\boldsymbol{\Sigma}_{\boldsymbol{X}}(\overline{\boldsymbol{G}}) \subset \boldsymbol{\Sigma}$. Let $\bar{\Sigma} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[1 . . m]} \Sigma_{i}$.
Direct consequence of Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, $G_{i} \subset \Sigma_{i}$ ), Definition 227 (product of subset systems, $\bar{G} \subset \bar{\Sigma}$ ), Lemma 710 (properties of $\sigma$-algebra $\overline{\text { generation, monotonicity, }}$ thus $\Sigma_{X}(\bar{G}) \subset \Sigma_{X}(\bar{\Sigma})$ ), and Definition 774 (tensor product of $\sigma$-algebras, $\underline{\Sigma=\Sigma_{X}(\bar{\Sigma}) \text { ), }}$
Therefore, we have the equality.
Lemma 780 (Borel subsets in product measurable subspace).
Let $m \in[2 . . \infty)$.
For all $i \in[1 . . m]$, let $\left(X_{i}, \mathcal{T}_{i}\right)$ be a topological space, and $Y_{i} \subset X_{i}$. Assume that the $Y_{i}$ 's are equipped with the trace topology, and $Y \stackrel{\text { def. }}{=} \prod_{i \in[0 . . m]} Y_{i}$ with the box topology. Then, we have

$$
\begin{equation*}
\mathcal{B}(Y)=\bigotimes_{i \in[1 . . m]} \mathcal{B}\left(X_{i}\right) \bar{\cap} Y_{i} . \tag{9.7}
\end{equation*}
$$

When the subspaces are all the same, we define the notation $\mathcal{B}(Z)^{m \otimes} \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]}(\mathcal{B}(X) \bar{\cap} Z)$.
Proof. From Lemma 252 (trace topology on subset), and Lemma 254 (box topology on Cartesian product), we have $\mathcal{T}_{Y} \stackrel{\text { def. }}{=} \bar{\prod}_{i \in[0 . . m]} \mathcal{T}_{i} \bar{\cap} Y_{i}$. Therefore, from Definition 743 (Borel $\sigma$-algebra, $\mathcal{B}(Y)=\Sigma_{Y}\left(\mathcal{T}_{Y}\right)$ and $\mathcal{B}\left(X_{i}\right)=\Sigma_{X_{i}}\left(\mathcal{T}_{i}\right)$ ), Lemma 779 (generating product measurable space), and Lemma 765 (generating measurable subspace), we have

$$
\mathcal{B}(Y)=\bigotimes_{i \in[0 \ldots m]} \Sigma_{Y_{i}}\left(\mathcal{T}_{i} \bar{\cap} Y_{i}\right)=\bigotimes_{i \in[0 . . m]} \mathcal{B}\left(X_{i}\right) \bar{\cap} Y_{i} .
$$

Remark 781.
For instance, the previous lemma states $\mathcal{B}\left(\mathbb{R}_{+}^{n}\right)=\left(\mathcal{B}\left(\mathbb{R}_{+}\right)\right)^{n \otimes}\left(\right.$ with $\left.Y_{i}=\mathbb{R}_{+} \subset \mathbb{R}=X_{i}\right)$.

Remark 782. For the sake of simplicity, we only present the remainder of this section in the case of the product of two measure spaces. When $i \in\{1,2\}$, the complement $\{1,2\} \backslash\{i\}$ is $\{3-i\}$.

Definition 783 (section in Cartesian product). Let $X_{1}$ and $X_{2}$ be sets. Let $A \subset X_{1} \times X_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $x_{i} \in X_{i}$. The $i$-th section of $A$ in $x_{i}$ is the subset

$$
\begin{equation*}
s_{i}\left(x_{i}, A\right) \stackrel{\text { def. }}{=}\left\{x_{j} \in X_{j} \mid\left(x_{1}, x_{2}\right) \in A\right\} . \tag{9.8}
\end{equation*}
$$

Lemma 784 (section of product). Let $X_{1}$ and $X_{2}$ be sets. Let $A_{1} \in X_{1}$ and $A_{2} \in X_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $x_{i} \in X_{i}$. Then, we have

$$
s_{i}\left(x_{i}, A_{1} \times A_{2}\right)=\left\{\begin{array}{cc}
A_{j} & \text { when } x_{i} \in A_{i},  \tag{9.9}\\
\varnothing & \text { otherwise } .
\end{array}\right.
$$

Proof. Direct consequence of Definition 783 (section in Cartesian product).
Lemma 785 (compatibility of section with set operations). Let $X_{1}$ and $X_{2}$ be sets. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$. Let $i \in\{1,2\}$. Let $x_{i} \in X_{i}$. Then, the sections are compatible with the empty set, the complement, countable union and intersection, and are monotone:

$$
\begin{gather*}
s_{i}\left(x_{i}, \varnothing\right)=\varnothing,  \tag{9.10}\\
\forall A \subset X,  \tag{9.11}\\
s_{i}\left(x_{i}, A^{c}\right)=s_{i}\left(x_{i}, A\right)^{c},  \tag{9.12}\\
\forall I \subset \mathbb{N}, \forall\left(A_{n}\right)_{n \in I} \subset X, \quad s_{i}\left(x_{i}, \bigcup_{n \in I} A_{n}\right)=\bigcup_{n \in I} s_{i}\left(x_{i}, A_{n}\right),  \tag{9.13}\\
\forall I \subset \mathbb{N}, \forall\left(A_{n}\right)_{n \in I} \subset X, \quad s_{i}\left(x_{i}, \bigcap_{n \in I} A_{n}\right)=\bigcap_{n \in I} s_{i}\left(x_{i}, A_{n}\right),  \tag{9.14}\\
\forall A, B \subset X,
\end{gather*} \quad A \subset B \Longrightarrow s_{i}\left(x_{i}, A\right) \subset s_{i}\left(x_{i}, B\right) . ~ \$
$$

Proof. Direct consequences of Definition 783 (section in Cartesian product).
Lemma 786 (measurability of section). Let $\left(X_{1}, \Sigma_{1}\right)$ and $\left(X_{2}, \Sigma_{2}\right)$ be measurable spaces. Let $A \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $x_{i} \in X_{i}$. Then, we have $s_{i}\left(x_{i}, A\right) \in \Sigma_{j}$.

Proof. Let $\bar{\Sigma} \stackrel{\text { def. }}{=} \Sigma_{1} \overline{\times} \Sigma_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Let $\mathcal{S}_{i} \stackrel{\text { def. }}{=}\left\{A \subset X \mid s_{i}\left(x_{i}, A\right) \in \Sigma_{j}\right\}$.
From Definition 741 (measurable space), $\Sigma_{j}$ is a $\sigma$-algebras.
Let $A_{1} \in \Sigma_{1}$, and $A_{2} \in \Sigma_{2}$. Then, from Lemma 784 (section of product, $s_{i}\left(x_{i}, A_{1} \times A_{2}\right)$ belongs to $\left.\left\{\varnothing, A_{j}\right\}\right)$, Lemma $696\left(\sigma\right.$-algebra, $\left.\varnothing, A_{j} \in \Sigma_{j}\right)$, and the definition of $\mathcal{S}_{i}$, we have $A_{1} \times A_{2} \in \mathcal{S}_{i}$. Hence, from Definition 227 (product of subset systems), we have $\bar{\Sigma} \subset \mathcal{S}_{i}$.

From Lemma 785 (compatibility of section with set operations), and Lemma 696 ( $\sigma$-algebra), $\mathcal{S}_{i}$ contains $\varnothing$, and is closed under complement and countable union. Thus, from Lemma 696 ( $\sigma$-algebra), $\mathcal{S}_{i}$ is a $\sigma$-algebra on $X$. Hence, from Definition 774 (tensor product of $\sigma$-algebras, $\Sigma$ is a generated by $\bar{\Sigma}$ ), Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra), we have $\Sigma \subset \mathcal{S}_{i}$.

Therefore, for all $A \in \Sigma$ we have $s_{i}\left(x_{i}, A\right) \in \Sigma_{j}$.

## Lemma 787 (countable union of sections is measurable).

Let $\left(X_{1}, \Sigma_{1}\right)$ and $\left(X_{2}, \Sigma_{2}\right)$ be measurable spaces. Let $I \subset \mathbb{N}$. Let $\left(A_{n}\right)_{n \in I} \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i$ be in $\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $x_{i} \in X_{i}$. Then, we have

$$
\begin{equation*}
s_{i}\left(x_{i}, \bigcup_{n \in I} A_{n}\right)=\bigcup_{n \in I} s_{i}\left(x_{i}, A_{n}\right) \in \Sigma_{j} . \tag{9.15}
\end{equation*}
$$

Proof. Direct consequence of Lemma 785 (compatibility of section with set operations, with countable union), Definition 741 (measurable space, $\Sigma_{j}$ is a $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, closedness under countable union), and Lemma 786 (measurability of section).

## Lemma 788 (countable intersection of sections is measurable).

Let $\left(X_{1}, \Sigma_{1}\right)$ and $\left(X_{2}, \Sigma_{2}\right)$ be measurable spaces. Let $\left(A_{n}\right)_{n \in \mathbb{N}} \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i$ be in $\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $x_{i} \in X_{i}$. Then, we have

$$
\begin{equation*}
s_{i}\left(x_{i}, \bigcap_{n \in \mathbb{N}} A_{n}\right)=\bigcap_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right) \in \Sigma_{j} . \tag{9.16}
\end{equation*}
$$

Proof. Direct consequence of Lemma 785 (compatibility of section with set operations, with countable intersection), Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection), and Lemma 786 (measurability of section).

Lemma 789 (indicator of section). Let $\left(X_{1}, \Sigma_{1}\right)$ and $\left(X_{2}, \Sigma_{2}\right)$ be measurable spaces. Let $A \subset X_{1} \times X_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Then, we have

$$
\begin{equation*}
\forall x_{1} \in X_{1}, \quad \forall x_{2} \in X_{2}, \quad \mathbb{1}_{A}\left(x_{1}, x_{2}\right)=\mathbb{1}_{s_{i}\left(x_{i}, A\right)}\left(x_{j}\right) . \tag{9.17}
\end{equation*}
$$

Proof. Direct consequence of Definition 783 (section in Cartesian product), and the definition of the indicator function.

## Lemma 790 (measurability of function from product space).

Let $\left(X_{1}, \Sigma_{1}\right),\left(X_{2}, \Sigma_{2}\right)$ and $\left(X^{\prime}, \Sigma^{\prime}\right)$ be measurable spaces. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $f: X_{1} \times X_{2} \rightarrow X^{\prime}$. Assume that $f$ is measurable for $\Sigma_{1} \otimes \Sigma_{2}$ and $\Sigma^{\prime}$.
Then, for all $x_{i} \in X_{i}$, the function $\left(x_{j} \mapsto f\left(x_{1}, x_{2}\right)\right)$ is measurable for $\Sigma_{j}$ and $\Sigma^{\prime}$.
Proof. Direct consequence of properties of inverse image $\left(\left(f_{x_{i}}^{j}\right)^{-1}\left(A^{\prime}\right)=s_{i}\left(x_{i}, f^{-1}\left(A^{\prime}\right)\right)\right.$ where $f_{x_{i}}^{j} \stackrel{\text { def. }}{=}\left(x_{j} \mapsto f\left(x_{1}, x_{2}\right)\right)$ and $\left.A^{\prime} \in \Sigma^{\prime}\right)$, Lemma 786 (measurability of section, $\left.\left(f_{x_{i}}^{j}\right)^{-1}\left(A^{\prime}\right) \in \Sigma_{j}\right)$, and Definition 749 (measurable function).

Remark 791. Note that the reciprocal of the previous lemma is false.
Indeed, let us build a counter-example. Let $X_{1}=X_{2}=X^{\prime} \stackrel{\text { def. }}{=} \mathbb{R}, \Sigma_{1}=\Sigma_{2} \stackrel{\text { def. }}{=} \Sigma_{\mathbb{R}}\left(\{\{x\}\}_{x \in \mathbb{R}}\right)$, and $\Sigma^{\prime} \stackrel{\text { def. }}{=} \mathcal{B}(\mathbb{R})$. Let $D \stackrel{\text { def. }}{=}\{(x, x) \mid x \in \mathbb{R}\} \subset X_{1} \times X_{2}$. Then, the tensor product $\sigma$-algebra $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$ is also generated by all singletons of $\mathbb{R}^{2}$, and $D$ is not measurable. Let $f=\mathbb{1}_{D}$. Then, $f$ is not measurable for $\Sigma$ and $\Sigma^{\prime}$, but for all $x \in X,(y \mapsto f(x, y))=(y \mapsto f(y, x))=\mathbb{1}_{\{x\}}$ is obviously measurable for $\Sigma$ and $\Sigma^{\prime}$.

## Chapter 10

## Measurability and numbers

## Contents

10.1 Borel subset of numbers ..... 164
10.1.1 Borel subset of numbers ..... 164
10.1.2 Borel subset of real numbers ..... 164
10.1.3 Borel subset of extended real numbers ..... 165
10.1.4 Borel subset of nonnegative numbers ..... 165
10.1.5 Borel subset of complex numbers ..... 166
10.2 Measurable numeric function ..... 167
10.2.1 Measurable numeric function ..... 167
10.2.2 Measurable numeric function to $\mathbb{R}$ ..... 168
10.2.3 Measurable numeric function to $\overline{\mathbb{R}}$ ..... 168
10.2.4 Nonnegative measurable numeric function ..... 173
10.2.5 Tensor product of measurable numeric functions ..... 175 $\mathbb{R}, \mathbb{C}$, or $\mathbb{R}$, see Definition 748), and of measurable functions to it.

### 10.1 Borel subset of numbers

### 10.1.1 Borel subset of numbers

## Remark 793. In the next lemma, $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$, or $\overline{\mathbb{R}}$ (see Definition 748).

## Lemma 794 (some Borel subsets of numbers). Let $\mathbb{K}$ be a measurable number space.

 Then, open subsets, closed subsets, and countable subsets are Borel subsets of $\mathbb{K}$.Proof. Direct consequence of Lemma 744 (some Borel subsets, with $Y \stackrel{\text { def. }}{=} \mathbb{R}=\mathbb{Q}, Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}=\mathbb{Q}$ and $\underline{Y} \stackrel{\text { def. }}{=} \mathbb{C}=\overline{\{a+i b \mid a, b \in \mathbb{Q}\}}$, all separable $).$

### 10.1.2 Borel subset of real numbers

Remark 795. In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

Lemma 796 (Borel subsets of $\mathbb{R}$ ).
$\mathcal{B}(\mathbb{R})$ is generated by any of the following sets of intervals: $\left.\{(a, b)\}_{a<b},\{(-\infty, b)\}\right\}_{b \in \mathbb{R}},\{(a, \infty)\}_{a \in \mathbb{R}}$.

Proof. From Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ), and Lemma 745 (countable union Borel $\sigma$-algebra generator), we have $\mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{(a, b)\}_{a<b}\right)$.

Let $a, b \in \mathbb{R}$ such that $a<b$. Then, we have the following countable unions:

$$
\begin{gathered}
{[a, b)=\{a\} \cup(a, b), \quad[a, b]=\{a\} \cup(a, b) \cup\{b\}, \quad(a, b]=(a, b) \cup\{b\}} \\
(a, b)=\bigcup_{n \in \mathbb{N}}\left[a+\frac{1}{n+1}, b\right)=\bigcup_{n \in \mathbb{N}}\left[a+\frac{1}{n+1}, b-\frac{1}{n+1}\right]=\bigcup_{n \in \mathbb{N}}\left(a, b-\frac{1}{n+1}\right]
\end{gathered}
$$

Thus, from Definition 708 (generated $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, closedness under countable union), and Lemma 794 (some Borel subsets of numbers, singletons of $\mathbb{R}$ are Borel subsets),

$$
\begin{gathered}
\{[a, b)\}_{a<b},\{[a, b]\}_{a<b},\{(a, b]\}_{a<b} \subset \mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{(a, b)\}_{a<b}\right), \\
\{(a, b)\}_{a<b} \subset \Sigma_{\mathbb{R}}\left(\{[a, b)\}_{a<b}\right), \Sigma_{\mathbb{R}}\left(\{[a, b]\}_{a<b}\right), \Sigma_{\mathbb{R}}\left(\{(a, b]\}_{a<b}\right) .
\end{gathered}
$$

Hence, from Lemma 711 (other properties of $\sigma$-algebra generation), we have

$$
\mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{(a, b)\}_{a<b}\right)=\Sigma_{\mathbb{R}}\left(\{[a, b)\}_{a<b}\right)=\Sigma_{\mathbb{R}}\left(\{[a, b]\}_{a<b}\right)=\Sigma_{\mathbb{R}}\left(\{(a, b]\}_{a<b}\right)
$$

Let $a, b \in \mathbb{R}$ such that $a<b$. Then, we have the following set differences:

$$
\begin{aligned}
& {[a, b)=(-\infty, b) \backslash(-\infty, a)=[a, \infty) \backslash[b, \infty)} \\
& (a, b]=(-\infty, b] \backslash(-\infty, a]=(a, \infty) \backslash(b, \infty)
\end{aligned}
$$

Thus, since $(-\infty, b)$ and $(a, \infty)$ are open, $(-\infty, b]$ and $[a, \infty)$ are closed, and from Definition 708 (generated $\sigma$-algebra), Lemma 700 (other properties of $\sigma$-algebra, closedness under set difference), and Lemma 794 (some Borel subsets of numbers, closed subsets of $\mathbb{R}$ are Borel subsets), we have

$$
\begin{aligned}
& \{(-\infty, b)\}_{b \in \mathbb{R}},\{[a, \infty)\}_{a \in \mathbb{R}} \subset \mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{[a, b)\}_{a<b}\right), \\
& \{(-\infty, b]\}_{b \in \mathbb{R}},\{(a, \infty)\}_{a \in \mathbb{R}} \subset \mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{(a, b]\}_{a<b}\right), \\
& \{[a, b)\}_{a<b} \subset \Sigma_{\mathbb{R}}\left(\{(-\infty, b)\}_{b \in \mathbb{R}}\right), \Sigma_{\mathbb{R}}\left(\{[a, \infty)\}_{a \in \mathbb{R}}\right), \\
& \\
& \{(a, b]\}_{a<b} \subset \Sigma_{\mathbb{R}}\left(\{(-\infty, b]\}_{b \in \mathbb{R}}\right), \Sigma_{\mathbb{R}}\left(\{(a, \infty)\}_{a \in \mathbb{R}}\right) .
\end{aligned}
$$

Hence, from Lemma 711 (other properties of $\sigma$-algebra generation), we have

$$
\begin{aligned}
& \mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{[a, b)\}_{a<b}\right)=\Sigma_{\mathbb{R}}\left(\{(-\infty, b)\}_{b \in \mathbb{R}}\right)=\Sigma_{\mathbb{R}}\left(\{[a, \infty)\}_{a \in \mathbb{R}}\right), \\
& \mathcal{B}(\mathbb{R})=\Sigma_{\mathbb{R}}\left(\{(a, b]\}_{a<b}\right)=\Sigma_{\mathbb{R}}\left(\{(-\infty, b]\}_{b \in \mathbb{R}}\right)=\Sigma_{\mathbb{R}}\left(\{(a, \infty)\}_{a \in \mathbb{R}}\right) .
\end{aligned}
$$

Therefore, all eight sets of intervals generate the Borel subsets of $\mathbb{R}$.

## Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ ).

In Lemma 796 (Borel subsets of $\mathbb{R}$ ), finite bounds of intervals, a and b, may be taken rational.
Proof. Same proof as for Lemma 796 (Borel subsets of $\mathbb{R}$ ), but with using the countability of the topological basis of $\mathbb{R}$ equipped with the Euclidean distance through Theorem 465 ( $\mathbb{R}$ is secondcountable) instead of Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ).

### 10.1.3 Borel subset of extended real numbers

Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ). $\quad$ Let $A \subset \overline{\mathbb{R}}$. Then, $A \in \mathcal{B}(\overline{\mathbb{R}})$ iff $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$.
Proof. Direct consequence of Definition 743 (Borel $\sigma$-algebra, $\mathbb{R}$ (open subset) belongs to $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 696 ( $\sigma$-algebra, $\{-\infty, \infty\}=\mathbb{R}^{c} \in \mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 769 (Borel subsets of partition, with $X \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0} \stackrel{\text { def. }}{=} \mathbb{R}$, and $Y_{1}=\{-\infty, \infty\}$ ), Lemma 768 (Borel sub- $\sigma$-algebra, with $X$ set to $\overline{\mathbb{R}}$ and $Y \stackrel{\text { def. }}{=}\{-\infty, \infty\}$, i.e. $\mathcal{B}(\{-\infty, \infty\})=\mathcal{P}(\{-\infty, \infty\})$, and thus $A \cap\{-\infty, \infty\}$ always belongs to $\mathcal{B}(\{-\infty, \infty\}))$.

Remark 799. In other words, Borel subsets of $\overline{\mathbb{R}}$ are Borel subsets of $\mathbb{R}$, or the union of Borel subsets of $\mathbb{R}$ with $\{-\infty\}$, $\{\infty\}$, or $\{-\infty, \infty\}$.

## Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ ).

Let $\bar{G} \subset \mathcal{P}(\overline{\mathbb{R}})$. Assume that $\bar{G} \bar{\cap}\{-\infty, \infty\}$ is one of the two singletons $\{ \pm \infty\}$, and that $\bar{G} \bar{\cap}$ is a generator of $\mathcal{B}(\mathbb{R})$. Then, $\bar{G}$ is a generator of $\mathcal{B}(\overline{\mathbb{R}})$.

Proof. Let $A \subset \overline{\mathbb{R}}$. Then, from Definition 708 (generated $\sigma$-algebra), we have

$$
A \cap\{-\infty, \infty\} \in \mathcal{P}(\{-\infty, \infty\})=\Sigma_{\{-\infty, \infty\}}(\bar{G} \bar{\cap}\{-\infty, \infty\})
$$

Hence, from Lemma 766 (generating on a partition, with $X \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0} \stackrel{\text { def. }}{=} \mathbb{R}$, and $\underline{Y_{1}} \stackrel{\text { def. }}{=}\{-\infty, \infty\}$ ), and Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ), we have

$$
A \in \Sigma_{\overline{\mathbb{R}}}(\bar{G}) \quad \Longleftrightarrow \quad A \cap \mathbb{R} \in \Sigma_{\mathbb{R}}(\bar{G} \bar{\cap} \mathbb{R})=\mathcal{B}(\mathbb{R}) \quad \Longleftrightarrow \quad A \in \mathcal{B}(\overline{\mathbb{R}})
$$

Remark 801. In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

## Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ).

$\mathcal{B}(\overline{\mathbb{R}})$ is generated by any of the following sets of intervals: $\{[-\infty, b)\}_{b \in \mathbb{R}},\{(a, \infty]\}_{a \in \mathbb{R}}$.
Proof. Direct consequence of Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$, with $\bar{G}$ being any of the four families of rays above), Definition 226 (trace of subset system, $\bar{G} \bar{\cap}\{-\infty, \infty\}$ is one of the two singletons $\{ \pm \infty\}$ ), and Lemma 796 (Borel subsets of $\mathbb{R}, \bar{G} \bar{\cap}$ is either $\{(-\infty, b)\}\}_{b \in \mathbb{R}}$, or $\{(a, \infty)\}_{a \in \mathbb{R}}$, i.e. a generator of $\left.\mathcal{B}(\mathbb{R})\right)$.

### 10.1.4 Borel subset of nonnegative numbers

Lemma 803 (Borel subsets of $\mathbb{R}_{+}$). $\mathcal{B}\left(\mathbb{R}_{+}\right)=\mathcal{B}(\mathbb{R}) \cap \mathcal{P}\left(\mathbb{R}_{+}\right)$is generated by any of the following sets of intervals: $\{[0, b)\}_{b>0},\{(a, \infty)\}_{a>0}$.

Proof. Direct consequence of Lemma 796 (Borel subsets of $\mathbb{R}, \underline{\mathbb{R}_{+}}=[0, \infty) \in \mathcal{B}(\mathbb{R})$ ), Lemma 768 (Borel sub- $\sigma$-algebra, with $Y \stackrel{\text { def. }}{=} \mathbb{R}_{+}$), and Lemma 765 (generating measurable subspace, with $\underline{Y} \stackrel{\text { def. }}{=} \mathbb{R}_{+}$and $G$ a generator of $\left.\mathcal{B}(\mathbb{R})\right)$.

Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$). $\quad \mathcal{B}\left(\overline{\mathbb{R}}_{+}\right)=\mathcal{B}(\overline{\mathbb{R}}) \cap \mathcal{P}\left(\overline{\mathbb{R}}_{+}\right)$is generated by any of the following sets of intervals: $\{[0, b)\}_{b>0},\{(a, \infty]\}_{a>0}$.

Proof. Direct consequence of Lemma 802 (Borel subsets of $\overline{\mathbb{R}}, \overline{\mathbb{R}}_{+}=[0, \infty] \in \mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 768 (Borel sub- $\sigma$-algebra, with $Y \stackrel{\text { def. }}{=} \mathbb{R}_{+}$), and Lemma 765 (generating measurable subspace, with $Y$ set to $\overline{\mathbb{R}}_{+}$and $G$ a generator of $\left.\mathcal{B}(\overline{\mathbb{R}})\right)$.

### 10.1.5 Borel subset of complex numbers

Remark 805. In the next lemma, the intent is that the left and right square-and-round brackets remains the same for each set of intervals, see Definition 229.

Of course, there are several other possibly interesting generators...

## Lemma 806 (Borel subsets of $\mathbb{C}$ ).

$\mathcal{B}(\mathbb{C})$ can be identified to $\mathcal{B}(\mathbb{R})^{2 \otimes}$, which is generated by any of the following sets of boxes:
(10.1)

$$
\left.\left.\{(a, b)] \times[(c, d)]\}_{a<b, c<d}, \quad\{(-\infty, b)] \times(-\infty, d)\right\}\right\}_{b, d \in \mathbb{R}}, \quad\left\{(a, \infty) \times[(c, \infty]\}_{a, c \in \mathbb{R}}\right.
$$

Proof. Direct consequence of the canonical isometry $((x, y) \mapsto x+i y): \mathbb{R}^{2} \rightarrow \mathbb{C}$, Lemma 780 (Borel subsets in product measurable subspace, with $Y_{i}=X_{i} \stackrel{\text { def. }}{=} \mathbb{R}$ and $m \stackrel{\text { def. } 2 \text { 2 }}{=}$, and Lemma 796 (Borel subsets of $\mathbb{R}$ ).

### 10.2 Measurable numeric function

### 10.2.1 Measurable numeric function

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Remark 807. In the next four lemmas, \mathbb{K}\mathrm{ is }\mathbb{R},\mathbb{C}\mathrm{ , or }\overline{\mathbb{R}}\mathrm{ (see Definition 748).}
```

Lemma 808 (inverse image of countable is measurable). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $f \in \mathcal{M}_{\mathbb{K}}$. Then, we have

$$
\begin{equation*}
\forall I \subset \mathbb{N}, \forall\left(y_{i}\right)_{i \in I} \in \mathbb{K}, \quad\left\{f=\left(y_{i}\right)_{i \in I}\right\} \stackrel{\text { def. }}{=} f^{-1}\left\{y_{i} \mid i \in I\right\} \in \Sigma . \tag{10.2}
\end{equation*}
$$

Proof. Direct consequence of Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable, with $\underline{\underline{\text { def. }}} \mathbb{R}=\mathbb{Q}, Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}=\mathbb{Q}$ and $Y \stackrel{\text { def. }}{=} \mathbb{C}=\overline{\{a+i b \mid a, b \in \mathbb{Q}\}}$, all separable $).$

## Lemma 809 (inverse image is measurable). Let $(X, \Sigma)$ be a measurable space.

Let $\mathbb{K}$ be a measurable number space. Let $f \in \mathcal{M}_{\mathbb{K}}$. Then, we have

$$
\begin{equation*}
\forall y \in \mathbb{K}, \quad\{f=y\} \stackrel{\text { def. }}{=} f^{-1}(y) \in \Sigma \tag{10.3}
\end{equation*}
$$

Proof.
Direct consequence of Lemma 808 (inverse image of countable is measurable, with card $(I)=1$ ).
Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$ countable range). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $f: X \rightarrow \mathbb{K}$. Assume that $f(X)$ is countable.
Then, we have $f \in \mathcal{M}_{\mathbb{K}}$ iff for all $y \in f(X), f^{-1}(y) \in \Sigma$.
Proof. Direct consequence of Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range, with $Y \stackrel{\text { def. }}{=} \mathbb{R}=\mathbb{Q}, Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}=\mathbb{Q}$ and $Y \stackrel{\text { def. }}{=} \mathbb{C}=\overline{\{a+i b \mid a, b \in \mathbb{Q}\}}$, all separable).

Lemma 811 (measurability of indicator function). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $A \subset X$. Then, we have $\mathbb{1}_{A} \in \mathcal{M}_{\mathbb{K}}$ iff $A \in \Sigma$.

Proof. Direct consequence of Definition 741 (measurable space), Lemma 697 (equivalent definition of $\sigma$-algebra, $P_{\Sigma} \Rightarrow C$ ), and Definition 748 (measurability and structure), Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$ countable range, with $\left.f(X)=\left\{0_{\mathfrak{K}}, 1_{\mathbb{K}}\right\}, f^{-1}\left(0_{\mathbb{K}}\right)=A^{c} \in \Sigma, f^{-1}\left(1_{\mathbb{K}}\right)=A \in \Sigma\right)$.

Remark 812. In the next two lemmas, the measurable valued field $\mathbb{K}$ is $\mathbb{R}$, or $\mathbb{C}$ (see Definition 748, thus $\overline{\mathbb{R}}$ is not involved here).

Note also that "(sub)algebra" is the algebraic structure.
Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable valued field. Then, $\mathcal{M}_{\mathbb{K}}$ is a subalgebra of $\mathbb{K}^{X}$, i.e. an algebra over $\mathbb{K}$ with the $0_{\mathbb{K}}$-function as zero.

Proof. Let $f, g \in \mathcal{M}_{\mathbb{K}}$. Then, from Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $\underline{Y} \stackrel{\text { def. }}{=} \mathbb{K}$ ), Lemma 776 (measurability of function to product space), and Lemma 780 (Borel subsets in product measurable subspace, with $Y_{i}=X_{i} \stackrel{\text { def. }}{=} \mathbb{K}$ and $m \stackrel{\text { def. }}{=} 2$, the function $(f, g): X \rightarrow \mathbb{K}^{2}$ is measurable for $\Sigma$ and $\mathcal{B}\left(\mathbb{K}^{2}\right)$. Moreover, from continuity of addition and multiplication from $\mathbb{K}^{2}$ to $\mathbb{K}$, Lemma 761 (continuous is measurable), and Lemma 757 (compatibility of measurability with composition), $f+g$ and $f g$ are measurable for $\Sigma$ and $\mathcal{B}(\mathbb{K})$.

Let $f: X \rightarrow \mathbb{K}$. Let $a \in \mathbb{K}$. Then, from Lemma 754 (constant function is measurable), and Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $\left.Y \stackrel{\text { def. }}{=} \mathbb{K}\right)$, we have $a \in \mathcal{M}_{\mathbb{K}}$. Thus, from the previous result, we have $a f \in \mathcal{M}_{\mathfrak{k}}$.

Therefore, from Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ or $\mathbb{C}$ ), Lemma 282 (algebra of functions to algebra), and Lemma 287 (closed under algebra operations is subalgebra), $\mathcal{M}_{\mathbb{K}}$ is a subalgebra of $\left(\mathbb{K}^{X},+_{\mathfrak{K}}, \cdot{ }_{\mathfrak{K}}, *_{\mathfrak{K}}\right)$.

Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$ is vector space).
Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable valued field.
Then, $\mathcal{M}_{\mathbb{K}}$ is a vector subspace of $\mathbb{K}^{X}$, i.e. a vector space over $\mathbb{K}$ with the $0_{\mathbb{K}}$-function as zero.
Proof. Direct consequence of Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra), Definition 284 (subalgebra), Definition 277 (algebra over a field, $\mathcal{M}_{\mathbb{K}}$ is a vector space), and Definition 77 (subspace).

### 10.2.2 Measurable numeric function to $\mathbb{R}$

Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$. Let $(X, \Sigma)$ be a measurable space. The set $\mathcal{M}_{\mathbb{R}}(X, \Sigma)$ (or simply $\mathcal{M}_{\mathbb{R}}$ ) of measurable functions from $X$ to $\mathbb{R}$ is a vector space; it is called the vector space of finite-valued measurable functions (over $X$ ).

Proof. Direct consequence of Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space, with $\left.\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and valued field properties of $\mathbb{R}$.

Lemma 816 (measurability of numeric function to $\mathbb{R}$ ). Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \mathbb{R}$. Then, $f \in \mathcal{M}_{\mathbb{R}}$ iff one of the following conditions is satisfied:

$$
\begin{array}{ll}
\forall a \in \mathbb{R}, & \{f<a\} \stackrel{\text { def. }}{=} f^{-1}(-\infty, a) \in \Sigma, \\
\forall a \in \mathbb{R}, & \{f \leqslant a\} \stackrel{\text { def. }}{=} f^{-1}(-\infty, a] \in \Sigma, \\
\forall a \in \mathbb{R}, \quad\{f>a\} \stackrel{\text { def. }}{=} f^{-1}(a, \infty) \in \Sigma, \\
\forall a \in \mathbb{R}, \quad\{f \geqslant a\} \stackrel{\text { def. }}{=} f^{-1}[a, \infty) \in \Sigma . \tag{10.7}
\end{array}
$$

Proof. "Left" implies "right". Direct consequence of Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$ ), Definition 749 (measurable function), Definition 743 (Borel $\sigma$ algebra), Lemma 794 (some Borel subsets of numbers), open intervals of $\mathbb{R}$ are open subsets, and closed intervals of $\mathbb{R}$, are closed subsets.
"Right" implies "left". Direct consequence of Lemma 796 (Borel subsets of $\mathbb{R}$ ), Lemma 756 (equivalent definition of measurable function), and Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R}$ ).

Therefore, we have the equivalence.

### 10.2.3 Measurable numeric function to $\overline{\mathbb{R}}$

## Definition 817 (M, set of measurable numeric functions).

Let $(X, \Sigma)$ be a measurable space. The set $\mathcal{M}_{\overline{\mathbb{R}}}(X, \Sigma)$ of measurable functions from $X$ to $\overline{\mathbb{R}}$ is called the set of measurable functions (over $X$ ); it is simply denoted $\mathcal{M}(X, \Sigma)$ (or $\mathcal{M}$ ).

Remark 818. We use the convention of identifying real-valued functions and their extension as functions $X \rightarrow \overline{\mathbb{R}}$. This allows us to consider $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ and $\mathbb{R}^{X} \subset \overline{\mathbb{R}}^{X}$ in the next lemma.

Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ).
Let $(X, \Sigma)$ be a measurable space. Then, we have $\mathcal{M}_{\mathbb{R}}=\mathcal{M} \cap \mathbb{R}^{X}$.
Proof. $\mathcal{M}_{\mathbb{R}} \subset \mathcal{M} \cap \mathbb{R}^{\boldsymbol{X}}$. Let $f \in \mathcal{M}_{\mathbb{R}}$. Let $A \in \mathcal{B}(\overline{\mathbb{R}})$. Then, from Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R})$, Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $Y \stackrel{\text { def. }}{=} \mathbb{R}$ ), Definition 312 (extended real numbers, $\overline{\mathbb{R}}, \overline{\mathbb{R}}=\mathbb{R} \uplus\{ \pm \infty\}$ ), compatibility of inverse image with union, Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}, A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ ), and Definition 749 (measurable function), we have $f \in \mathbb{R}^{X}$, and

$$
f^{-1}(A)=f^{-1}(A \cap \mathbb{R}) \uplus f^{-1}(A \cap\{ \pm \infty\})=f^{-1}(A \cap \mathbb{R}) \in \Sigma
$$

Hence, from Definition 749 (measurable function), Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}})$, and Definition $817\left(\mathcal{M}\right.$, set of measurable numeric functions), $f \in \mathcal{M} \cap \mathbb{R}^{X}$.
$\mathcal{M} \cap \mathbb{R}^{\boldsymbol{X}} \subset \mathcal{M}_{\mathbb{R}}$. Let $f \in \mathcal{M} \cap \mathbb{R}^{X}$. Let $A \in \mathcal{B}(\mathbb{R})$. Then, from Definition $817(\mathcal{M}$, set of measurable numeric functions), Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $\left.Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}\right)$, Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}, A \cap \mathbb{R}=A \in \mathcal{B}(\mathbb{R})$ ), and Definition 749 (measurable function), we have $A \in \mathcal{B}(\overline{\mathbb{R}})$, and $f^{-1}(A) \in \Sigma$. Hence, from Definition 749 (measurable function), Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $\left.Y \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R}$ ), we have $f \in \mathcal{M}_{\mathbb{R}}$.

Therefore, we have the equality.
Lemma 820 (measurability of numeric function). Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, $f \in \mathcal{M}$ iff one of the following conditions is satisfied:
$\forall a \in \mathbb{R}, \quad\{f<a\} \stackrel{\text { def. }}{=} f^{-1}[-\infty, a) \in \Sigma$,
(10.9) $\quad \forall a \in \mathbb{R}, \quad\{f \leqslant a\} \stackrel{\text { def. }}{=} f^{-1}[-\infty, a] \in \Sigma$,
(10.10) $\quad \forall a \in \mathbb{R}, \quad\{f>a\} \stackrel{\text { def. }}{=} f^{-1}(a, \infty] \in \Sigma$,
(10.11)

$$
\forall a \in \mathbb{R}, \quad\{f \geqslant a\} \stackrel{\text { def. }}{=} f^{-1}[a, \infty] \in \Sigma
$$

Proof. "Left" implies "right". Direct consequence of Definition 817 (M, set of measurable numeric functions), Definition 749 (measurable function), Definition 743 (Borel $\sigma$-algebra), Lemma 794 (some Borel subsets of numbers, with $\mathbb{K} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ ), Lemma 372 (topology on $\overline{\mathbb{R}}$, open intervals are open subsets and closed intervals are closed subsets).
"Right" implies "left". Direct consequence of Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ), Lemma 756 (equivalent definition of measurable function), and Definition 817 ( $\mathcal{M}$, set of measurable numeric functions).

Therefore, we have the equivalence.
Lemma 821 ( $\mathcal{M}$ is closed under finite part).
Let $(X, \Sigma)$ be a measurable space.
Let $f \in \mathcal{M}$. Let $A \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R})$. Then, $A \in \Sigma$ and (the finite part of $f$ ) $f \mathbb{1}_{A} \in \mathcal{M}_{\mathbb{R}}(\subset \mathcal{M})$.
Proof. From Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), we have $\mathbb{R}=\{ \pm \infty\}^{c}$. Let

$$
A \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R}), \quad \text { and } \quad B \stackrel{\text { def. }}{=} f^{-1}(\{ \pm \infty\})
$$

Then, from Lemma 808 (inverse image of countable is measurable, with $\operatorname{card}(I)=2$ ), compatibility of inverse image with complement, involutiveness of complement, Lemma 696 ( $\sigma$-algebra, closedness under complement), and Definition 209 (pseudopartition), we have $A=B^{c}$, $A, B \in \Sigma$, and $(A, B)$ forms a pseudopartition of $X$.

From Definition 526 (finite part), let $\hat{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A}$ be the finite part of $f$. Therefore, from Lemma 527 (finite part is finite), the definition of the indicator function, Lemma 754 (constant function is measurable, $0 \in \mathcal{M}$ ), Lemma 772 (measurability of function defined on a pseudopartition, with $\left(X^{\prime}, \Sigma^{\prime}\right) \stackrel{\text { def. }}{=}(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, and the pseudopartition $\left.X=A \uplus B\right)$, and Lemma 819 $\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$, we have $\hat{f} \in \mathcal{M} \cap \mathbb{R}^{X}=\mathcal{M}_{\mathbb{R}}$.

## Lemma 822 ( $\mathcal{M}$ is closed under addition when defined).

Let $(X, \Sigma)$ be a measurable space. Let $f, g \in \mathcal{M}$. Assume that $f$ and $g$ never take opposite infinite values at the same point. Then, $f+g$ is well-defined, and belongs to $\mathcal{M}$.

Proof. By hypothesis, the subsets $\{f=\infty\} \cap\{g=-\infty\}$ and $\{f=-\infty\} \cap\{g=\infty\}$ are empty. Thus, from Definition 209 (pseudopartition), the whole space reduces to the pseudopartition

$$
X=X_{ \pm \infty}^{c} \uplus X_{\infty} \uplus X_{-\infty}
$$

(we recall that the notation $\uplus$ is for disjoint union) where

$$
\begin{aligned}
X_{ \pm \infty}^{c} & \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) \\
X_{\infty} & \stackrel{\text { def. }}{=}\{f=\infty\} \cup\{g=\infty\} \\
X_{-\infty} & \stackrel{\text { def. }}{=}\{f=-\infty\} \cup\{g=-\infty\} .
\end{aligned}
$$

Then, from Definition 817 (M, set of measurable numeric functions), Definition 749 (measurable function, for $f$ and $g$ ), Definition 743 (Borel $\sigma$-algebra, $\mathbb{R}$ open in $\overline{\mathbb{R}}$ belongs to $\mathcal{B}(\overline{\mathbb{R}})$ ), Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable union and intersection), and Lemma 809 (inverse image is measurable, with $y \stackrel{\text { def. }}{=} \pm \infty$ ), we have $X_{ \pm \infty}^{c}, X_{\infty}, X_{-\infty} \in \Sigma$.

Let $\tilde{f}$ and $\tilde{g}$ be the finite parts of $f$ and $g$. Then, from Definition 526 (finite part), and Definition 317 (addition in $\overline{\mathbb{R}}$ ), the sum $f+g$ can be defined on the pseudopartition by

$$
f+g \stackrel{\text { def. }}{=} \begin{cases}\tilde{f}+\tilde{g} & \text { on } X_{ \pm \infty}^{c} \\ \infty & \text { on } X_{\infty} \\ -\infty & \text { on } X_{-\infty}\end{cases}
$$

Therefore, from Lemma $821(\mathcal{M}$ is closed under finite part, with $f$ and $g)$, Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra, sum with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ ), Lemma 754 (constant function is measurable), and Lemma 772 (measurability of function defined on a pseudopartition), we have $f+g \in \mathcal{M}$.

## Lemma 823 ( $\mathcal{M}$ is closed under finite sum when defined).

Let $(X, \Sigma)$ be a measurable space. Let $I$ be a finite set. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}$. Assume that the $f_{i}$ 's never take opposite infinite values at the same point, i.e. for all $i, j \in I, i \neq j$ implies $\left\{f_{i}=\infty\right\} \cap\left\{f_{j}=-\infty\right\}=\varnothing$. Then, $\sum_{i \in I} f_{i}$ is well-defined, and belongs to $\mathcal{M}$.

Proof. Direct consequence of Lemma $822(\mathcal{M}$ is closed under addition when defined), and induction on the cardinality of $I$.

## Lemma 824 ( $\mathcal{M}$ is closed under multiplication).

Let $(X, \Sigma)$ be a measurable space. Let $f, g \in \mathcal{M}$. Then, fg is well-defined, and belongs to $\mathcal{M}$.
Proof. From Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), and Lemma 410 (infinityproduct property in $\overline{\mathbb{R}}_{+}$), the whole space can be pseudopartitioned into $X=X_{ \pm \infty}^{c} \uplus X_{0} \uplus X_{\infty} \uplus X_{-\infty}$ (the notation $\uplus$ is for disjoint union) where

$$
\begin{aligned}
X_{ \pm \infty}^{c} & \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}), \\
X_{0} \stackrel{\text { def. }}{=} & (\{|f|=\infty\} \cap\{g=0\}) \cup(\{f=0\} \cap\{|g|=\infty\}), \\
X_{\infty} & \stackrel{\text { def. }}{=}(\{f=\infty\} \cap\{g>0\}) \cup(\{f=-\infty\} \cap\{g<0\}) \\
& \cup(\{f>0\} \cap\{g=\infty\}) \cup(\{f<0\} \cap\{g=-\infty\}), \\
X_{-\infty} & \stackrel{\text { def. }}{=}(\{f=-\infty\} \cap\{g>0\}) \cup(\{f=\infty\} \cap\{g<0\}) \\
& \cup(\{f>0\} \cap\{g=-\infty\}) \cup(\{f<0\} \cap\{g=\infty\}) .
\end{aligned}
$$

Then, from Definition $817(\mathcal{M}$, set of measurable numeric functions), Definition 749 (measurable
 (inverse image is measurable, with $y \stackrel{\text { def. }}{=} \pm \infty, 0$ ) Lemma 820 (measurability of numeric function,
$\underline{\text { with } a \stackrel{\text { def. }}{=} 0)}$, and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable union and intersection), we have $X_{ \pm \infty}^{c}, X_{0}, X_{\infty}, X_{-\infty} \in \Sigma$.

Let $\tilde{f}$ and $\tilde{g}$ be the finite parts of $f$ and $g$. Then, from Definition 526 (finite part), Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)), the product $f g$ can be defined on the pseudopartition by

$$
f g \stackrel{\text { def. }}{=} \begin{cases}\tilde{f} \tilde{g} & \text { on } X_{ \pm \infty}^{c} \\ 0 & \text { on } X_{0} \\ \infty & \text { on } X_{\infty} \\ -\infty & \text { on } X_{-\infty}\end{cases}
$$

Therefore, from Lemma $821(\mathcal{M}$ is closed under finite part, with $f$ and $g)$, Lemma $813\left(\mathcal{M}_{\mathfrak{K}}\right.$ is algebra, product with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ ), Lemma 754 (constant function is measurable), and Lemma 772 (measurability of function defined on a pseudopartition), we have $f g \in \mathcal{M}$.

Lemma 825 ( $\mathcal{M}$ is closed under finite product).
Let $(X, \Sigma)$ be a measurable space. Let $I$ be a finite set. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}$. Then, $\prod_{i \in I} f_{i} \in \mathcal{M}$.
Proof. Direct consequence of Lemma 824 ( $\mathcal{M}$ is closed under multiplication), and induction on the cardinality of $I$.

Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication).
Let $(X, \Sigma)$ be a measurable space. Let $a \in \overline{\mathbb{R}}$. Let $f \in \mathcal{M}$. Then, we have af $\in \mathcal{M}$.
Proof. Direct consequence of Lemma 824 ( $\mathcal{M}$ is closed under multiplication), and Lemma 754 (constant function is measurable).

Remark 827. Note that in the next two lemmas, the index set $I$ could also be the empty set, provided the use of the convention $\inf \varnothing \stackrel{\text { def. }}{=} \infty$ and $\sup \varnothing \stackrel{\text { def. }}{=}-\infty$, and Lemma 754 (constant function is measurable).

Lemma 828 ( $\mathcal{M}$ is closed under infimum).
Let $(X, \Sigma)$ be a measurable space.
Let $I \subset \mathbb{N}$ be nonempty. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}$. Then, we have $\inf _{i \in I} f_{i} \in \mathcal{M}$.
Proof. Let $a \in \mathbb{R}$. Then, from Definition 9 (infimum), Lemma 820 (measurability of numeric function, with $f_{i}$ ), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection), we have

$$
\left\{\inf _{i \in I} f_{i} \geqslant a\right\}=\bigcap_{i \in I}\left\{f_{i} \geqslant a\right\} \in \Sigma
$$

Therefore, from Lemma 820 (measurability of numeric function, with $\inf _{i \in I} f_{i}$ ), $\inf _{i \in I} f_{i}$ belongs to $\mathcal{M}$.

Lemma 829 ( $\mathcal{M}$ is closed under supremum).
Let $(X, \Sigma)$ be a measurable space.
Let $I \subset \mathbb{N}$ be nonempty. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}$. Then, we have $\sup _{i \in I} f_{i} \in \mathcal{M}$.
Proof. Let $a \in \mathbb{R}$. Then, from Definition 2 (supremum), Lemma 820 (measurability of numeric function, with all the $f_{i}$ 's), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection),

$$
\left\{\sup _{i \in I} f_{i} \leqslant a\right\}=\bigcap_{i \in I}\left\{f_{i} \leqslant a\right\} \in \Sigma .
$$

Therefore, from Lemma 820 (measurability of numeric function, with $\sup _{i \in I} f_{i}$ ), $\sup _{i \in I} f_{i}$ belongs to $\mathcal{M}$.

Lemma 830 ( $\mathcal{M}$ is closed under limit inferior).
Let $(X, \Sigma)$ be a measurable space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Then, we have $\liminf _{n \rightarrow \infty} f_{n} \in \mathcal{M}$.
Proof. Direct consequence of Lemma $828\left(\mathcal{M}\right.$ is closed under infimum, $\left.F_{n}^{-} \stackrel{\text { def. }}{=} \inf _{p \in \mathbb{N}} f_{n+p} \in \mathcal{M}\right)$,
 $\underline{\left.\liminf _{n \rightarrow \infty} f_{n}=\underline{f}\right)}$.

Lemma 831 ( $\mathcal{M}$ is closed under limit superior).
Let $(X, \Sigma)$ be a measurable space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Then, we have $\lim \sup _{n \rightarrow \infty} f_{n} \in \mathcal{M}$.
Proof. Direct consequence of Lemma $829\left(\mathcal{M}\right.$ is closed under supremum, $F_{n}^{+} \stackrel{\text { def. }}{=} \sup _{p \in \mathbb{N}} f_{n+p}$ belongs to $\mathcal{M})$, Lemma $828\left(\mathcal{M}\right.$ is closed under infimum, $\left.\bar{f} \stackrel{\text { def. }}{=} \inf _{n \in \mathbb{N}} F_{n}^{+} \in \mathcal{M}\right)$, and Lemma 509 (limit superior, $\left.\limsup _{n \rightarrow \infty} f_{n}=\bar{f}\right)$.

## Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent).

Let $(X, \Sigma)$ be a measurable space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Assume that $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}$. Then, we have $\lim _{n \rightarrow \infty} f_{n} \in \mathcal{M}$.

Proof. Direct consequence of Lemma 522 (limit inferior, limit superior and pointwise convergence), Lemma $830(\mathcal{M}$ is closed under limit inferior), or Lemma 831 ( $\mathcal{M}$ is closed under limit superior).

## Lemma 833 (measurability and masking).

Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, $f \in \mathcal{M}$ iff for all $A \in \Sigma, f \mathbb{1}_{A} \in \mathcal{M}$.
Proof. "Left" implies "right". Direct consequence of Lemma 811 (measurability of indicator function), Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}, \underline{\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}}$ ), and Lemma 824 ( $\mathcal{M}$ is closed under multiplication).
"Right" implies "left". Direct consequence of Definition 741 (measurable space, $\Sigma$ is a $\sigma$-algebra), Lemma $696(\sigma$-algebra, $X \in \Sigma)$, the definition of the indicator function ( $\mathbb{1}_{X} \equiv 1$ ), and Definition 326 (multiplication in $\overline{\mathbb{R}}, 1$ is unity).
Therefore, we have the equivalence.

## Lemma 834 (measurability of restriction).

Let $(X, \Sigma)$ be a measurable space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$. Then, we have $f_{\left.\right|_{A}} \in \mathcal{M}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{M}(X, \Sigma)$.

Proof. Let $B^{\prime} \in \mathcal{B}(\overline{\mathbb{R}})$. Case $\mathbf{0} \notin \boldsymbol{B}^{\prime}$. Then, from the definition of the indicator function, we have $\left(\hat{f} \mathbb{1}_{A}\right)^{-1}\left(B^{\prime}\right)=f^{-1}\left(B^{\prime}\right) \cap A$. Case $0 \in B^{\prime}$. Then, from compatibility of inverse image with disjoint union, the definition of the indicator function, and associativity of disjoint union, we have

$$
\begin{aligned}
\left(\hat{f} \mathbb{1}_{A}\right)^{-1}\left(B^{\prime}\right) & =\left(\hat{f} \mathbb{1}_{A}\right)^{-1}\left(B^{\prime} \backslash\{0\}\right) \uplus\left(\hat{f} \mathbb{1}_{A}\right)^{-1}(0) \\
& =\left(f^{-1}\left(B^{\prime} \backslash\{0\}\right) \cap A\right) \uplus\left(\left(f^{-1}(0) \cap A\right) \uplus A^{c}\right) \\
& =\left(f^{-1}\left(B^{\prime}\right) \cap A\right) \uplus A^{c} .
\end{aligned}
$$

Moreover, from the definition of restriction of function, we have $f_{\left.\right|_{A}}^{-1}\left(B^{\prime}\right)=f^{-1}\left(B^{\prime}\right) \cap A$. Hence, we have $\left(\hat{f} \mathbb{1}_{A}\right)^{-1}\left(B^{\prime}\right)=f_{\mid A}^{-1}\left(B^{\prime}\right) \uplus C$ with $C \in\left\{\varnothing, A^{c}\right\}$.

Let $B \subset A$. Assume first that $B \in \Sigma$. Then, from Definition 741 (measurable space, $\underline{\Sigma}$ is a $\sigma$-algebra), and Lemma 696 ( $\sigma$-algebra, closedness under complement and countable union), we have $B \uplus A^{c} \in \Sigma$. Conversely, assume now that $B \uplus A^{c} \in \Sigma$. Then, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection), we have ( $\left.B \uplus A^{c}\right) \cap A=B \in \Sigma$. Hence, we have the equivalence $B \in \Sigma$ iff $B \uplus A^{c} \in \Sigma$.

Therefore, from Definition 749 (measurable function), and Definition 817 (M, set of measurable numeric functions), we have the equivalence $f_{\left.\right|_{A}} \in \mathcal{M}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{M}(X, \Sigma)$.

### 10.2.4 Nonnegative measurable numeric function

Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
Let $(X, \Sigma)$ be a measurable space. The subset of nonnegative measurable functions (over $X$ ) is denoted $\mathcal{M}_{+}(X, \Sigma)\left(\right.$ or simply $\left.\mathcal{M}_{+}\right)$; it is defined by $\mathcal{M}_{+}(X, \Sigma) \stackrel{\text { def. }}{=}\left\{f \in \mathcal{M} \mid f(X) \subset \overline{\mathbb{R}}_{+}\right\}$.

## Lemma 836 (measurability of nonnegative and nonpositive parts).

Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f \in \mathcal{M}$ iff $f^{+}, f^{-} \in \mathcal{M}_{+}$.
Proof. "Left" implies "right". Direct consequence of Definition 528 (nonnegative and nonpositive parts), Lemma $826(\mathcal{M}$ is closed under scalar multiplication, $-f \in \mathcal{M}$ ), Lemma 754 (constant function is measurable, $0 \in \mathcal{M})$, Lemma $829(\mathcal{M}$ is closed under supremum, maximum is supremum), Lemma 530 (nonnegative and nonpositive parts are nonnegative), and Definition 835 $\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
"Right" implies "left". Direct consequence of Definition 835 ( $\mathcal{M}_{+}$, subset of nonnegative measurable numeric functions, $\left.\mathcal{M}_{+} \subset \mathcal{M}\right)$, Lemma $826(\mathcal{M}$ is closed under scalar multiplication, $\left.-f^{-} \in \mathcal{M}\right)$, Lemma 532 (decomposition into nonnegative and nonpositive parts, $\underline{f^{+}-f^{-} \text {is }}$ well-defined), and Lemma 822 ( $\mathcal{M}$ is closed under addition when defined).

Therefore, we have the equivalence.
Lemma 837 ( $\mathcal{M}_{+}$is closed under masking).
Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$. Then, $f \in \mathcal{M}_{+}$iff for all $A \in \Sigma, f \mathbb{1}_{A} \in \mathcal{M}_{+}$.
Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, thus $\mathcal{M}_{+}=\mathcal{M} \cap \overline{\mathbb{R}}_{+}^{X}$ ), Lemma 833 (measurability and masking), and the definition of the indicator function (thus $f \geqslant 0$ iff for all $A \in \Sigma, f \mathbb{1}_{A} \geqslant 0$ ).

Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part).
Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{M}_{+}$. Let $A \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R})$. Then, $A \in \Sigma$ and (the finite part of $f$ ) $f \mathbb{1}_{A} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.

Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Lemma 821 ( $\mathcal{M}$ is closed under finite part), nonnegativeness of the indicator function, and closedness of multiplication in $\mathbb{R}_{+}$.

Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part). Let ( $X, \Sigma$ ) be a measurable space. Let $f \in \mathcal{M}$. Let $A \in \Sigma$. Let $A_{f} \stackrel{\text { def. }}{=} A \cap f^{-1}\left(\mathbb{R}_{+}\right)$. Let $\tilde{f}_{A} \stackrel{\text { def. }}{=} f \mathbb{1}_{A_{f}}$. Then, $A_{f} \in \Sigma$ and $\tilde{f}_{A} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$.

Proof. Let $B \stackrel{\text { def. }}{=} f^{-1}\left(\overline{\mathbb{R}}_{+}\right)$and $C \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R})$. Then, from Lemma 820 (measurability of numeric function, with $\left.\overline{\mathbb{R}}_{+}=[0, \infty]\right)$, and Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part, thus $\left.C \in \Sigma\right)$, Definition 741 (measurable space, $\Sigma$ is a $\sigma$-algebra), Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under intersection), compatibility of inverse image with intersection, the definition of $A_{f}$, and properties of the indicator function, we have

$$
B, C \in \Sigma, \quad f^{-1}\left(\mathbb{R}_{+}\right)=B \cap C \in \Sigma, \quad A_{f}=A \cap B \cap C \in \Sigma, \quad \text { and } \quad \mathbb{1}_{A_{f}}=\mathbb{1}_{A} \mathbb{1}_{B} \mathbb{1}_{C}
$$

Hence, from Lemma 529 (equivalent definition of nonnegative and nonpositive parts), and Definition 526 (finite part), $\tilde{f}_{A}$ is the finite part of $\left(f \mathbb{1}_{A}\right)^{+}$. Therefore, from Lemma 833 (measurability and masking, $f \mathbb{1}_{A} \in \mathcal{M}$ ), Lemma 836 (measurability of nonnegative and nonpositive parts, with $\left.f \mathbb{1}_{A}\right)$, and Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part, with $\left.\left(f \mathbb{1}_{A}\right)^{+}\right)$, we have $\tilde{f}_{A} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.

Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).
Let $(X, \Sigma)$ be a measurable space. Let $p \in \overline{\mathbb{R}}_{+}$. Then, we have $\left(x \mapsto|x|^{p}\right) \in \mathcal{M}_{+}(\overline{\mathbb{R}}) \subset \mathcal{M}(\overline{\mathbb{R}})$.
Proof. Let $\varphi_{p} \stackrel{\text { def. }}{=}\left(x \in \overline{\mathbb{R}} \mapsto|x|^{p}\right)$.
Case $\boldsymbol{p} \in[\mathbf{0}, \boldsymbol{\infty}$ ). Direct consequence of Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)), and Lemma 761 (continuous is measurable).

Case $\boldsymbol{p}=\boldsymbol{\infty}$. Direct consequence of Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ), Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory), thus $\varphi_{\infty}(\overline{\mathbb{R}})=\{0,1, \infty\}$, and $\varphi_{\infty}^{-1}(\{0\})=(-1,1), \varphi_{\infty}^{-1}(\{1\})=\{ \pm 1\}$, and $\varphi_{\infty}^{-1}(\{\infty\})=\overline{\mathbb{R}} \backslash[-1,1]$ ), Lemma 796 (Borel subsets of $\mathbb{R}$ ), Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$, thus all previous preimages belong to $\mathcal{B}(\overline{\mathbb{R}})$ ), and Lemma 753 (countable range is measurable).

Therefore, from Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory), thus $\varphi_{p} \geqslant 0$ ), and Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), we always have $\varphi_{p} \in \mathcal{M}_{+}(\overline{\mathbb{R}}) \subset \mathcal{M}(\overline{\mathbb{R}})$.

Lemma 841 ( $\mathcal{M}$ is closed under power function). Let $(X, \Sigma)$ be a measurable space. Let $p \in \overline{\mathbb{R}}_{+}$. Let $f \in \mathcal{M}$, and $g \in \mathcal{M}_{+}$. Then, $|f|^{p}, g^{p} \in \mathcal{M}_{+} \subset \mathcal{M}$.

Moreover, if $p$ is finite and $f \in \mathcal{M}_{\mathbb{R}}$, then we have $|f|^{p} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.
Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, thus $g \geqslant 0$ ), Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, thus $|g|=g$ ), Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable), Lemma 757 (compatibility of measurability with composition), Definition $817\left(\mathcal{M}\right.$, set of measurable numeric functions, $\left.|f|^{p}, g^{p} \in \mathcal{M}\right)$, Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\left\lfloor\left. f\right|^{p}, g^{p} \in \mathcal{M}_{+}\right.$), closedness of the power function of nonnegative degree in $\mathbb{R}$ (for $p \in(0, \infty)$ ), Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory), the power function of degree 0 is finite $)$, and Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R},|f|^{p} \in \mathcal{M}_{\mathbb{R}}$ ).

Lemma 842 ( $\mathcal{M}$ is closed under absolute value).
Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{M}$. Then, we have $|f| \in \mathcal{M}_{+} \subset \mathcal{M}$. Moreover, if $f \in \mathcal{M}_{\mathbb{R}}$, then we have $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.

Proof. Direct consequence of Lemma 841 ( $\mathcal{M}$ is closed under power function, with $p \stackrel{\text { def. }}{=} 1)$, and Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value).

Lemma 843 ( $\mathcal{M}_{+}$is closed under addition). Let $(X, \Sigma)$ be a measurable space. Let $f, g \in \mathcal{M}_{+}$. Then, $f+g$ is well-defined, and belongs to $\mathcal{M}_{+}$.

Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Lemma 822 ( $\mathcal{M}$ is closed under addition when defined, $f$ and $g$ cannot take opposite infinite values), and Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed).

## Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication).

Let $(X, \Sigma)$ be a measurable space. Let $f, g \in \mathcal{M}_{+}$. Then, we have $f g \in \mathcal{M}_{+}$.
Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Lemma 824 ( $\mathcal{M}$ is closed under multiplication), and Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)).

Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication).
Let $(X, \Sigma)$ be a measurable space. Let $a \in \overline{\mathbb{R}}_{+}$. Let $f \in \mathcal{M}_{+}$. Then, we have af $\in \mathcal{M}_{+}$.

Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), and Lemma 425 ( multiplication in $\overline{\mathbb{R}}_{+}$ is closed (measure theory)).

Remark 846. Note that in the next two lemmas, the index set $I$ could also be the empty set, provided the use of the convention inf $\varnothing \stackrel{\text { def. }}{=} \infty$ and $\sup \varnothing \stackrel{\text { def. }}{=} 0$, and Lemma 754 (constant function is measurable).

Lemma 847 ( $\mathcal{M}_{+}$is closed under infimum).
Let $(X, \Sigma)$ be a measurable space.
Let $I \subset \mathbb{N}$ be nonempty. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}_{+}$. Then, we have $\inf _{i \in I} f_{i} \in \mathcal{M}_{+}$.
Proof. Direct consequence of Lemma 828 ( $\mathcal{M}$ is closed under infimum), and Lemma 492 (infimum of bounded is bounded, with $X \stackrel{\text { def. }}{=} I$ and $a \stackrel{\text { def. }}{=} 0)$.

Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum). Let $(X, \Sigma)$ be a measurable space. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}_{+}$. Then, we have $\sup _{i \in I} f_{i} \in \mathcal{M}_{+}$.

Proof. Direct consequence of Lemma 829 ( $\mathcal{M}$ is closed under supremum), and Lemma 493 (supremum of bounded is bounded, with $X \stackrel{\text { def. }}{=} I$ and $a \stackrel{\text { def. }}{=} 0)$.

## Lemma 849 ( $\mathcal{M}_{+}$is closed under limit when pointwise convergent).

Let $(X, \Sigma)$ be a measurable space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}_{+}$. Then, we have $\lim _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$.

Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Lemma $832\left(\mathcal{M}\right.$ is closed under limit when pointwise convergent), and Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete, .)

Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum). Let $(X, \Sigma)$ be a measurable space. Let $I \subset \mathbb{N}$. Let $\left(f_{i}\right)_{i \in I} \in \mathcal{M}_{+}$. Then, $\sum_{i \in I} f_{i}$ is well-defined, and belongs to $\mathcal{M}_{+}$.

Proof. Direct consequence of Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$), Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition), induction on $n \in \mathbb{N}$ (with $g_{n} \stackrel{\text { def. }}{=} \sum_{i \in I \cap[0 . . n]} f_{i} \in \mathcal{M}_{+}$, with the convention that a sum indexed by $\varnothing$ is the identity element 0 ), and Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent, with $f_{n} \stackrel{\text { def. }}{=} g_{n}$ ).

### 10.2.5 Tensor product of measurable numeric functions

## Definition 851 (tensor product of numeric functions).

Let $\mathbb{K}$ be a set equipped with a multiplication. Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $X_{i}$ be a set, and $f_{i}: X_{i} \rightarrow \mathbb{K}$. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$. The tensor product of $\left(f_{i}\right)_{i \in[1 . . m]}$ is the function $\bigotimes_{i \in[1 . . m]} f_{i}: X \rightarrow \mathbb{K}$ defined by

$$
\begin{equation*}
\forall x \stackrel{\text { def. }}{=}\left(x_{i}\right)_{i \in[1 . . m]} \in X, \quad\left(\bigotimes_{i \in[1 . . m]} f_{i}\right)(x) \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} f_{i}\left(x_{i}\right) . \tag{10.12}
\end{equation*}
$$

Remark 852. In the next lemma, the measurable number space $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$, or $\overline{\mathbb{R}}$ (see Definition 748).

## Lemma 853 (measurability of tensor product of numeric functions).

Let $\mathbb{K}$ be a measurable number space. Let $m \in[2 . . \infty)$. For all $i \in[1 . . m]$, let $\left(X_{i}, \Sigma_{i}\right)$ be a measurable space, and $f_{i} \in \mathcal{M}_{\mathbb{K}}\left(X_{i}, \Sigma_{i}\right)$. Then, we have $\bigotimes_{i \in[1 . . m]} f_{i} \in \mathcal{M}_{\mathbb{K}}\left(\prod_{i \in[1 . . m]} X_{i}, \bigotimes_{i \in[1 . . m]} \Sigma_{i}\right)$.

Proof. Let $X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}$ and $\Sigma \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]} \Sigma_{i}$. Let $i \in[1 . . m]$. Let $\pi_{i}$ be the canonical projection from $X$ onto $X_{i}$. Then, from Definition 748 (measurability and structure), Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $Y$, with $Y \stackrel{\text { def. }}{=} \mathbb{K}$ ), Lemma 777 (canonical projection is measurable), and Lemma 757 (compatibility of measurability with composition), $f_{i} \circ \pi_{i}$ belongs to $\mathcal{M}_{\mathfrak{K}}(X, \Sigma)$. Therefore, from Definition 851 (tensor product of numeric functions), Lemma 813 $\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra, multiplicativity, when $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ or $\left.\mathbb{C}\right)$, and Lemma $825(\mathcal{M}$ is closed under finite product, with $I \stackrel{\text { def. }}{=}[1 . . m]$, when $\mathbb{K} \stackrel{\text { def. }}{=} \overline{\mathbb{R}})$,

$$
\bigotimes_{i \in[1 . . m]} f_{i}=\prod_{i \in[1 . . m]}\left(f_{i} \circ \pi_{i}\right) \in \mathcal{M}_{\mathbb{K}}(X, \Sigma)
$$

## Chapter 11

## Numeric function over subset system

## Contents

11.1 Basic properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 178
11.1.1 Finite properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 178
11.1.2 Countable properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 180
11.2 Content . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 186
11.3 Premeasure . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 188

### 11.1 Basic properties

Remark 854.
Extension of the mini-language of Section 8.1 to represent properties of numeric functions over subset systems under a compact form. None of the following new keywords contain an uppercase letter ( $A$, below, is an argument to continuity).
pos, hom, diff, fin are for nonnegativeness, homogeneity, diffuseness, and finiteness;
$\underline{a d d}, \overline{a d d}, a d d$ are for subadditivity, superadditivity, and additivity;
sub, monot are for subtractivity, and monotonicity;
$\sigma \underline{a d d}, \sigma \overline{a d d}, \sigma a d d, \sigma$ fin are for $\sigma$-subadditivity, $\sigma$-superadditivity, $\sigma$-additivity, and $\sigma$-finiteness; $\operatorname{cont} \uparrow(A)$, cont $\downarrow(A)$ are for continuity from below (at $A$ ), and from above (at $A$ ).

When there is only one subset system and/or only one numeric function over a subset system in the context, it is not necessary to specify to which one the properties apply. For instance, the compact formula of Lemma 867 should be read as $\mathcal{S}$ is $P t f D^{\prime}$ and $\mu$ is both pos and add implies that $\mu$ is nondecreasing.

Definition 855 (homogeneity). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said homogeneous iff $\varnothing \in \mathcal{S}$ implies $\mu(\varnothing)=0$.

Definition 856 (diffuseness). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains all singletons of $X$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said diffuse iff for all $x \in X, \mu(\{x\})=0$.

## Definition 857 (finiteness).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said finite iff $X \in \mathcal{S}$ and $\mu(X) \in \mathbb{R}$.

## Lemma 858 (finite is bounded).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, nondecreasing, and finite. Then, $\mu$ is bounded.
Compact form: $H y p \stackrel{\text { def. }}{=} w F \wedge$ pos $\wedge \operatorname{monot} \wedge$ fin.
Proof. Direct consequence of Definition 857 (finiteness), and the definition of nondecreasingness $(\varnothing \subset A \subset X \Rightarrow 0 \leqslant \mu(\varnothing) \leqslant \mu(A) \leqslant \mu(X)<\infty)$.

### 11.1.1 Finite properties

Definition 859 (subadditivity).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said subadditive iff for all $n \in \mathbb{N}$, for all $\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$,

$$
\begin{equation*}
\bigcup_{i \in[0 . . n]} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu\left(\bigcup_{i \in[0 . . n]} A_{i}\right) \leqslant \sum_{i \in[0 . . n]} \mu\left(A_{i}\right) \tag{11.1}
\end{equation*}
$$

Remark 860. We recall that $\uplus$ denotes disjoint union.
Definition 861 (superadditivity). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said superadditive iff for all $n \in \mathbb{N}$, for all $\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$ pairwise disjoint,

$$
\begin{equation*}
\biguplus_{i \in[0 . . n]} A_{i} \in \mathcal{S} \Longrightarrow \mu\left(\biguplus_{i \in[0 . . n]} A_{i}\right) \geqslant \sum_{i \in[0 . . n]} \mu\left(A_{i}\right) . \tag{11.2}
\end{equation*}
$$

Definition 862 (additivity)
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said additive iff for all $n \in \mathbb{N}$, for all $\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$ pairwise disjoint,

$$
\begin{equation*}
\biguplus_{i \in[0 . . n]} A_{i} \in \mathcal{S} \Longrightarrow \mu\left(\biguplus_{i \in[0 . . n]} A_{i}\right)=\sum_{i \in[0 . . n]} \mu\left(A_{i}\right) . \tag{11.3}
\end{equation*}
$$

Lemma 863 (decomposition over finite pseudopartition).
Let $X$ be a set.
Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is additive. Let $n \in \mathbb{N}$. Let $A,\left(B_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$. Assume that for all $i \in[0 . . n], A \cap B_{i} \in \mathcal{S}$, and that $X=\biguplus_{i \in[0 . . n]} B_{i}$. Then, we have

$$
\begin{equation*}
\mu(A)=\sum_{i \in[0 . . n]} \mu\left(A \cap B_{i}\right)=\sum_{i \in[0 . . n]} \mu\left(B_{i} \cap A\right) . \tag{11.4}
\end{equation*}
$$

Compact form: Hyp $\stackrel{\text { def. }}{=}$ add.
Proof. Direct consequence of Lemma 211 (compatibility of pseudopartition with intersection), Definition 862 (additivity), and commutativity of intersection.

## Lemma 864 (binary inclusion-exclusion principle).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is additive.
Let $A, B \in \mathcal{S}$. Assume that $B \backslash A$ is a finite disjoint union of elements of $\mathcal{S}$.
Then, $\mu$ satisfies the binary inclusion-exclusion principle with $A$ and $B$ :

$$
\begin{equation*}
A \cup B, A \cap B \in \mathcal{S} \quad \Longrightarrow \quad \mu(A \cup B)+\mu(A \cap B)=\mu(A)+\mu(B) \tag{11.5}
\end{equation*}
$$

When this holds for all $A, B \in \mathcal{S}, \mu$ is said to satisfy the binary inclusion-exclusion principle. Compact form: Hyp $\stackrel{\text { def. }}{=} P t f D \wedge a d d$.

Proof. From the definition of set difference, we have

$$
A \cup B=A \uplus(B \backslash A), \quad \text { and } \quad B=(B \backslash A) \uplus(A \cap B) .
$$

Let $n \in \mathbb{N}$, and $\left(C_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$ such that $B \backslash A=\biguplus_{i \in[0 . . n]} C_{i}$. Assume that $A \cup B, A \cap B \in \mathcal{S}$. Then, from Definition 862 (additivity, with $n+2$ ), and Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),

$$
\mu(A \cup B)+\mu(A \cap B)=\mu(A)+\sum_{i \in[0 . . n]} \mu\left(C_{i}\right)+\mu(A \cap B)=\mu(A)+\mu(B)
$$

## Definition 865 (subtractivity).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said subtractive iff for all $A, B \in \mathcal{S}$,

$$
\begin{equation*}
A \subset B \quad \wedge \quad \mu(A) \in \mathbb{R} \quad \wedge \quad B \backslash A \in \mathcal{S} \quad \Longrightarrow \quad \mu(B \backslash A)=\mu(B)-\mu(A) \tag{11.6}
\end{equation*}
$$

## Lemma 866 (subtractivity).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is additive. Then, $\mu$ is subtractive.
Compact form: add $\Rightarrow$ sub.
Proof. Direct consequence of the definition of set difference $(B=A \uplus(B \backslash A)$ when $\boldsymbol{A} \subset \boldsymbol{B}$ ), Definition 862 (additivity, with $n \stackrel{\text { def. }}{=} 1$ ), Definition 317 (addition in $\overline{\mathbb{R}}$, rule 5 applies), and Definition 865 (subtractivity).

## Lemma 867 (monotonicity).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, and additive. Let $A, B \in \mathcal{S}$. Assume that $A \subset B$, and that $B \backslash A$ is a finite disjoint union of elements of $\mathcal{S}$. Then, we have $\mu(A) \leqslant \mu(B)$.
Compact form: PtfD ${ }^{\prime} \Rightarrow$ pos $\Rightarrow$ add $\Rightarrow$ monot.
Proof. Direct consequence of the definition of set difference $(B=(A \cap B) \uplus(B \backslash A)$ equals $\left.A \uplus \biguplus_{i \in[0 . . n]} C_{i}\right)$, Definition 862 (additivity, with $n+1$ ), and nonnegativeness $\left(\mu\left(C_{i}\right) \geqslant 0\right)$.

Remark 868. Note that in the next lemma, $\mathcal{S}$ is actually a semiring of sets.
This lemma is the core of the finite Boole inequality in Section 11.2.

## Lemma 869 (subadditivity).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is nonempty, closed under intersection, and that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, and additive. Then, $\mu$ is subadditive (see (11.1)).
Compact form: $N e \Rightarrow I \Rightarrow P t f D \Rightarrow$ pos $\Rightarrow$ add $\Rightarrow \underline{\text { add }}$.
Proof. Let $n \in \mathbb{N}$. Let $\left(A_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$. Assume that $\bigcup_{i \in[0 . . n]} A_{i} \in \mathcal{S}$.
Then, from Lemma 574 (disjoint union of union in subset system (variant), Hyp $=N e, I$, and PtfD $)$, let $n_{K} \in \mathbb{N}$, let $\psi:[0 . . n+1] \rightarrow\left[0 . . n_{K}+1\right]$ be nondecreasing with $\overline{\psi(0)=0 \text { and }}$ $\overline{\psi(n+1)}=n_{K}+1$, and $\left(C_{k}\right)_{k \in\left[0 . . n_{K}\right]} \in \mathcal{S}$ be pairwise disjoint such that we have

$$
\forall i \in[0 . . n], \quad \biguplus_{k \in[\psi(i) . . \psi(i+1))} C_{k} \subset A_{i}, \quad \text { and } \quad \bigcup_{i \in[0 . . n]} A_{i}=\biguplus_{\substack{i \in[0 . . n] \\ k \in[\psi(i) . . \psi(i+1))}} C_{k} \in \mathcal{S} .
$$

Hence, from Definition 862 (additivity), Lemma 557 (disjoint decomposition of set difference and local complement, PtfD $\Rightarrow$ PtfD'), Lemma 867 (monotonicity), and Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), we have

$$
\mu\left(\bigcup_{i \in[0 . . n]} A_{i}\right)=\sum_{i \in[0 . . n]} \mu\left(\biguplus_{k \in[\psi(i) . . \psi(i+1))} C_{k}\right) \leqslant \sum_{i \in[0 . . n]} \mu\left(A_{i}\right) .
$$

Therefore, from Definition 859 (subadditivity), $\mu$ is subadditive.

### 11.1.2 Countable properties

Definition 870 ( $\sigma$-subadditivity). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said $\sigma$-subadditive iff for all nonempty $I \subset \mathbb{N}$, for all $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$,

$$
\begin{equation*}
\bigcup_{i \in I} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu\left(\bigcup_{i \in I} A_{i}\right) \leqslant \sum_{i \in I} \mu\left(A_{i}\right) . \tag{11.7}
\end{equation*}
$$

Lemma 871 ( $\sigma$-subadditive is subadditive).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is $\sigma$-subadditive. Then, $\mu$ is subadditive (see(11.1)).
Compact form: $\sigma \underline{\text { add }} \Rightarrow \underline{\text { add }}$.

Proof. Direct consequence of Definition 870 ( $\sigma$-subadditivity, with $I \stackrel{\text { def. }}{=}[0 . . n]$ ), and Definition 859 (subadditivity).

Definition 872 ( $\sigma$-superadditivity). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said $\sigma$-superadditive iff for all nonempty $I \subset \mathbb{N}$, for all $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ pairwise disjoint,

$$
\begin{equation*}
\biguplus_{i \in I} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu\left(\biguplus_{i \in I} A_{i}\right) \geqslant \sum_{i \in I} \mu\left(A_{i}\right) . \tag{11.8}
\end{equation*}
$$

Lemma 873 ( $\sigma$-superadditive is superadditive). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is $\sigma$-superadditive. Then, $\mu$ is superadditive (see(11.2)).
Compact form: $\sigma \overline{a d d} \Rightarrow \overline{a d d}$.
Proof. Direct consequence of Definition 872 ( $\sigma$-superadditivity, with $I \stackrel{\text { def. }}{=}[0 . . n]$ ), and Definition 861 (superadditivity).

Definition 874 ( $\sigma$-additivity).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said $\sigma$-additive iff for all nonempty $I \subset \mathbb{N}$, for all $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ pairwise disjoint,

$$
\begin{equation*}
\biguplus_{i \in I} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu\left(\biguplus_{i \in I} A_{i}\right)=\sum_{i \in I} \mu\left(A_{i}\right) . \tag{11.9}
\end{equation*}
$$

## Lemma 875 ( $\sigma$-additive is additive).

 Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is $\sigma$-additive. Then, $\mu$ is additive (see(11.3)).Compact form: $\sigma a d d \Rightarrow a d d$.
Proof. Direct consequence of Definition 874 ( $\sigma$-additivity, with $I \stackrel{\text { def. }}{=}[0 . . n]$ ), and Definition 862 (additivity).

Lemma 876 (decomposition over countable pseudopartition). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is $\sigma$-additive. Let $I \subset \mathbb{N}$ be nonempty. Let $A,\left(B_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that for all $i \in I, A \cap B_{i} \in \mathcal{S}$, and that $X=\biguplus_{i \in I} B_{i}$. Then, we have

$$
\begin{equation*}
\mu(A)=\sum_{i \in I} \mu\left(A \cap B_{i}\right)=\sum_{i \in I} \mu\left(B_{i} \cap A\right) . \tag{11.10}
\end{equation*}
$$

Compact form: Hyp $\stackrel{\text { def. }}{=}$ $\sigma a d d$.
Proof. Direct consequence of Lemma 211 (compatibility of pseudopartition with intersection), Definition 874 ( $\sigma$-additivity), and commutativity of intersection.

Remark 877 . Note that in the next lemma, $\mathcal{S}$ is actually a semiring of sets.
The next lemma is the core of the Boole inequality in Sections 11.3 and 12.1.

Lemma 878 ( $\sigma$-subadditivity). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is nonempty, closed under intersection, and that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, and $\sigma$-additive. Then, $\mu$ is $\sigma$-subadditive (see(11.7)).

Compact form: $N e \Rightarrow I \Rightarrow P t f D \Rightarrow$ pos $\Rightarrow \sigma a d d \Rightarrow \sigma \underline{a d d}$.
Proof. Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I}=\operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that $\bigcup_{i \in I} A_{i} \in \mathcal{S}$.

Then, from Lemma 574 (disjoint union of union in subset system (variant), Hyp $=N e, I$, and PtfD , let $n_{K} \in \overline{\mathbb{N}}, \psi:\left[0 . . n_{I}\right] \rightarrow\left[0 . . n_{K}\right]$ be nondecreasing with $\psi(0)=0$ and $\psi\left(n_{I}\right)=n_{K}$, and $\left(C_{k}\right)_{k \in K} \in \mathcal{S}$ be pairwise disjoint such that we have

$$
\forall n \in J, \quad \biguplus_{k \in[\psi(n) . . \psi(n+1))} C_{k} \subset A_{\varphi(n)}, \quad \text { and } \quad \bigcup_{i \in I} A_{i}=\biguplus_{\substack{n \in J \\ k \in[\psi(n) . . \psi(n+1))}} C_{k} \in \mathcal{S} .
$$

Hence, from Definition 874 ( $\sigma$-additivity), Lemma 557 (disjoint decomposition of set difference and local complement, $\underline{P t f D} \Rightarrow P t f D^{\prime}$ ), Lemma 875 ( $\sigma$-additive is additive), Lemma 867 (monotonicity), and Lemma $396\left(\overline{\text { addition in } \overline{\mathbb{R}}_{+}}\right.$is compatible with order), we have

$$
\mu\left(\bigcup_{i \in I} A_{i}\right)=\sum_{n \in J} \mu\left(\biguplus_{k \in[\psi(n) . . \psi(n+1))} C_{k}\right) \leqslant \sum_{i \in I} \mu\left(A_{i}\right)
$$

Therefore, from Definition 870 ( $\sigma$-subadditivity), $\mu$ is $\sigma$-subadditive.

## Definition 879 (continuity from below).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $A \in \mathcal{S}$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said continuous from below at $A$ iff for all nonempty $I \subset \mathbb{N}$, for all $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing,

$$
\begin{equation*}
A=\bigcup_{i \in I} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu(A)=\sup _{i \in I} \mu\left(A_{i}\right) \tag{11.11}
\end{equation*}
$$

It is said continuous from below iff it is continuous from below at all $A \in \mathcal{S}$.

## Lemma 880 (continuity from below).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Assume that $\mathcal{S}$ is closed under local complement, and disjoint union. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, and $\sigma$-additive. Then, $\mu$ is continuous from below.

Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}$ nondecreasing, we have

$$
\begin{equation*}
A \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} A_{n} \in \mathcal{S} \quad \Longrightarrow \quad \mu(A)=\lim _{n \rightarrow \infty} \mu\left(A_{n}\right) . \tag{11.12}
\end{equation*}
$$

Compact form: $D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow \sigma a d d \Rightarrow$ cont $\uparrow$.
Proof. From Lemma 875 ( $\sigma$-additive is additive), $\mu$ is additive.
Let $A \in \mathcal{S}$. Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the (bijective) increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ be nondecreasing. Assume that $A=\bigcup_{i \in I} A_{i} \in \mathcal{S}$. Then, from Lemma 572 (disjoint union of union in subset system, Hyp $\left.=D^{\prime} \wedge U d\right)$, let $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ be pairwise disjoint such that

$$
\forall n \in J, \quad \biguplus_{j \in[0 . . n]} B_{\varphi(j)}=\bigcup_{j \in[0 . . n]} A_{\varphi(j)}=A_{\varphi(n)} \in \mathcal{S}, \quad \text { and } \quad \biguplus_{i \in I} B_{i}=\bigcup_{i \in I} A_{i}=A \in \mathcal{S} .
$$

Hence, from Definition 874 ( $\sigma$-additivity), Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum, with $a_{i} \stackrel{\text { def. }}{=} \mu\left(B_{i}\right)$ ), Definition 862 (additivity), and the bijective property of $\varphi$,

$$
\mu(A)=\sum_{i \in I} \mu\left(B_{i}\right)=\sup _{n \in J} \sum_{j \in[0 . . n]} \mu\left(B_{\varphi(j)}\right)=\sup _{n \in J} \mu\left(A_{\varphi(n)}\right)=\sup _{i \in I} \mu\left(A_{i}\right) .
$$

Therefore, from Definition 879 (continuity from below), $\mu$ is continuous from below.
Moreover, when $I \stackrel{\text { def. }}{=} \mathbb{N}=J$ (i.e. $\varphi$ is the identity map), property (11.12) is a direct consequence of Lemma 554 (disjoint decomposition, $D^{\prime} \Rightarrow P t f D^{\prime}$ ), Lemma 867 (monotonicity), and Lemma 501 (supremum is limit).

## Lemma 881 ( $\sigma$-additivity (from below)).

Let $X$ be a set.
Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement and disjoint union.
Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative.
Then, $\mu$ is $\sigma$-additive iff it is additive and continuous from below.
Compact form: $D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow \sigma a d d \Leftrightarrow(a d d \wedge \operatorname{cont} \uparrow \uparrow)$.

## Proof. "Left" implies "right".

Direct consequence of Lemma 875 ( $\sigma$-additive is additive), and Lemma 880 (continuity from below).
"Right" implies "left". Assume that $\mu$ is additive and continuous from below.
Let $I \subset \mathbb{N}$ be nonempty. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the (bijective) increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ be pairwise disjoint. Assume that $\biguplus_{i \in I} A_{i} \in \mathcal{S}$. Then, from Lemma 575 (monotone union of union in subset system, Hyp=Ud), let $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ be nondecreasing such that we have

$$
\forall n \in J, \quad B_{\varphi(n)}=\bigcup_{j \in[0 \ldots n]} B_{\varphi(j)}=\biguplus_{j \in[0 \ldots n]} A_{\varphi(j)} \in \mathcal{S}, \quad \text { and } \quad \bigcup_{i \in I} B_{i}=\biguplus_{i \in I} A_{i} \in \mathcal{S} .
$$

Thus, from Definition 879 (continuity from below), the bijective property of $\varphi$, Definition 862 (additivity), and Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum, with $a_{i} \stackrel{\text { def. }}{=} \mu\left(A_{i}\right)$ ), we have

$$
\mu\left(\biguplus_{i \in I} A_{i}\right)=\sup _{i \in I} \mu\left(B_{i}\right)=\sup _{n \in J} \mu\left(B_{\varphi(n)}\right)=\sup _{n \in J} \sum_{j \in[0 . . n]} \mu\left(A_{\varphi(j)}\right)=\sum_{i \in I} \mu\left(A_{i}\right) .
$$

Hence, from Definition 874 ( $\sigma$-additivity), $\mu$ is $\sigma$-additive.
Therefore, we have the equivalence.

## Definition 882 (continuity from above).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $A \in \mathcal{S}$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said continuous from above at $A$ iff for all nonempty $I \subset \mathbb{N}$, for all $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ nonincreasing,
(11.13) $\quad\left(\exists i_{0} \in I, \mu\left(A_{i_{0}}\right) \in \mathbb{R}\right) \wedge \quad A=\bigcap_{i \in I} A_{i} \in \mathcal{S} \quad \Longrightarrow \quad \mu(A)=\inf _{i \in I} \mu\left(A_{i}\right)<\infty$.

It is said continuous from above iff it is continuous from above at all $A \in \mathcal{S}$.

## Lemma 883 (continuity from above).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement, and disjoint union. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, and $\sigma$-additive. Then, $\mu$ is continuous from above.

Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}$ nonincreasing, we have
(11.14) $\quad\left(\exists n_{0} \in \mathbb{N}, \mu\left(A_{n_{0}}\right)<\infty\right) \wedge A \stackrel{\text { def. }}{=} \bigcap_{n \in \mathbb{N}} A_{n} \in \mathcal{S} \quad \Longrightarrow \quad \mu(A)=\lim _{n \rightarrow \infty} \mu\left(A_{n}\right)<\infty$.

Compact form: $D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow \sigma a d d \Rightarrow$ cont $\downarrow$.
Proof. From Lemma 875 ( $\sigma$-additive is additive), $\mu$ is additive. Then, from Lemma 554 (disjoint decomposition, $D^{\prime} \Rightarrow P t f D^{\prime}$ ), and Lemma 867 (monotonicity), $\mu$ is nondecreasing.

Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ be nonincreasing. Assume that there exists $i_{0} \in I$ such that $\mu\left(A_{i_{0}}\right)<\infty$, and that $A=\bigcap_{i \in I} A_{i} \in \mathcal{S}$. Let $n_{0} \stackrel{\text { def. }}{=} \varphi^{-1}\left(i_{0}\right) \in J$ (i.e. $i_{0}=\varphi\left(n_{0}\right)$ ). Let $J_{n_{0}} \stackrel{\text { def. }}{=}\left[n_{0} . . n_{I}\right) \neq \varnothing$.

Let $n \in J_{n_{0}}$. Then, from the lower bound property of intersection, and monotonicity of $\varphi$, we have $\bigcap_{j \in J_{n_{0}}} A_{\varphi(j)} \subset A_{\varphi(n)} \subset A_{\varphi\left(n_{0}\right)}$. Let $p \in\left[0 . . n_{0}\right]$. Then, from the right intersection law $(A \supset B \Leftrightarrow A \cap B=B)$, monotonicity of $\varphi$, and finite induction on $p$, we have $A=\bigcap_{j \in\left[p . . n_{I}\right)} A_{\varphi(j)}=\bigcap_{j \in J_{n_{0}}} A_{\varphi(j)} \in \mathcal{S}$. Hence, from monotonicity of $\mu$, we have

$$
0 \leqslant \mu(A)=\mu\left(\bigcap_{j \in J_{n_{0}}} A_{\varphi(j)}\right) \leqslant \mu\left(A_{\varphi(n)}\right) \leqslant \mu\left(A_{\varphi\left(n_{0}\right)}\right)<\infty
$$

For all $n \in J_{n_{0}}$, let $B_{n} \stackrel{\text { def. }}{=} A_{\varphi\left(n_{0}\right)} \backslash A_{\varphi(n)}$. Then, from Lemma 577 (monotone union of monotone intersection in subset system, Hyp $\left.=D^{\prime}\right),\left(B_{n}\right)_{n \in J_{n_{0}}} \in \mathcal{S}$ is nondecreasing, and such that we have $\bigcup_{n \in J_{n_{0}}} B_{n}=A_{\varphi\left(n_{0}\right)} \backslash \bigcap_{n \in J_{n_{0}}} A_{\varphi(n)} \in \mathcal{S}$. Thus, from Lemma 866 (subtractivity), Definition 865 (subtractivity, twice), Lemma 880 (continuity from below), Definition 879 (continuity from below), and Lemma 483 (compatibility of extrema with affine transformation, with $a \stackrel{\text { def. }}{=} 1$ and $\left.b \stackrel{\text { def. }}{=} \mu\left(A_{\varphi\left(n_{0}\right)}\right)\right)$, we have

$$
\begin{aligned}
\mu\left(A_{\varphi\left(n_{0}\right)}\right)-\mu\left(\bigcap_{n \in J_{n_{0}}} A_{\varphi(n)}\right)=\mu\left(\bigcup_{n \in J_{n_{0}}} B_{n}\right)=\sup _{n \in J_{n_{0}}} \mu\left(B_{n}\right) & =\sup _{n \in J_{n_{0}}}\left(\mu\left(A_{\varphi\left(n_{0}\right)}\right)-\mu\left(A_{\varphi(n)}\right)\right) \\
& =\mu\left(A_{\varphi\left(n_{0}\right)}\right)-\inf _{n \in J_{n_{0}}} \mu\left(A_{\varphi(n)}\right)
\end{aligned}
$$

Hence, from additive group properties of $\mathbb{R}$, Lemma 498 (infimum of nonincreasing sequence), and Definition 9 (infimum, lower bound),

$$
\mu(A)=\inf _{n \in J_{n_{0}}} \mu\left(A_{\varphi(n)}\right)=\inf _{i \in I} \mu\left(A_{i}\right) \leqslant \mu\left(A_{i_{0}}\right)<\infty .
$$

Therefore, from Definition 882 (continuity from above), $\mu$ is continuous from above.
Moreover, when $I \stackrel{\text { def. }}{=} \mathbb{N}=J$ (i.e. $\varphi$ is the identity map), property (11.14) is a direct consequence of monotonicity of $\mu$, and Lemma 500 (infimum is limit).

Lemma 884 ( $\sigma$-additivity (from above)).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the empty set, and is closed under local complement, and disjoint union. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative, homogeneous, and finite.
Then, $\mu$ is $\sigma$-additive iff it is additive and continuous from above at $\varnothing$.
Compact form: $w E \Rightarrow D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow$ hom $\Rightarrow$ fin $\Rightarrow \sigma a d d \Leftrightarrow($ add $\wedge$ cont $\downarrow \varnothing)$.

## Proof. "Left" implies "right".

Direct consequence of Lemma 875 ( $\sigma$-additive is additive), and Lemma 883 (continuity from above).
"Right" implies "left". Assume that $\mu$ is additive and continuous from above at $\varnothing$.
Let $I \subset \mathbb{N}$ be nonempty. Let $n_{I} \stackrel{\text { def. }}{=} \operatorname{card}(I)$. Let $J \stackrel{\text { def. }}{=}\left[0 . . n_{I}\right)$. Let $\varphi: J \rightarrow I$ be the (bijective) increasing numbering of $I$. Let $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ be pairwise disjoint. Assume that $A \stackrel{\text { def. }}{=} \biguplus_{i \in I} A_{i} \in \mathcal{S}$. Then, from Lemma 576 (empty monotone intersection of union in subset system, Hyp=$=D^{\prime}$ ), let $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ be nonincreasing such that we have

$$
\forall n \in J, \quad A=\biguplus_{j \in[0 . . n]} A_{\varphi(j)} \uplus B_{\varphi(n)}, \quad \text { and } \quad \bigcap_{i \in I} B_{i}=\varnothing \in \mathcal{S} .
$$

Thus, from Definition 862 (additivity, with $n+2$ ), Lemma 503 (constant sum of monotone nonnegative functions, with $\left.I \stackrel{\text { def. }}{=} J, f_{n} \stackrel{\text { def. }}{=} \sum_{j \in\lfloor-n]} \mu\left(A_{\varphi(j)}\right), g_{n} \stackrel{\text { def. }}{=} \mu\left(B_{\varphi(n)}\right)\right)$, the bijective property of $\varphi$, Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum, with $a_{i} \stackrel{\text { def. }}{=} \mu\left(A_{i}\right)$ ), finiteness of $\mu$, Definition 882
(continuity from above, with $\left(B_{i}\right)_{i \in I}$ and $i_{0} \stackrel{\text { def. }}{=} \varphi(0)$ ), homogeneity of $\mu$, and additive group properties of $\mathbb{R}$, we have for all $n \in J$,

$$
\mu(A)=\sum_{j \in[0 . . n]} \mu\left(A_{\varphi(j)}\right)+\mu\left(B_{\varphi(n)}\right)=\sup _{n \in J} \sum_{j \in[0 . . n]} \mu\left(A_{\varphi(j)}\right)+\inf _{n \in J} \mu\left(B_{\varphi(n)}\right)=\sum_{i \in I} \mu\left(A_{i}\right) .
$$

Hence, from Definition 874 ( $\sigma$-additivity), $\mu$ is $\sigma$-additive.
Therefore, we have the equivalence.

## Definition 885 ( $\sigma$-finiteness).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is said $\sigma$-finite iff

$$
\begin{equation*}
\exists I \subset \mathbb{N}, \exists\left(A_{i}\right)_{i \in I} \in \mathcal{S}, \quad\left(\forall i \in I, \mu\left(A_{i}\right) \in \mathbb{R}\right) \quad \wedge \quad X=\bigcup_{i \in I} A_{i} \tag{11.15}
\end{equation*}
$$

Lemma 886 (equivalent definition of $\sigma$-finiteness). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under union. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is subadditive.
Then, $\mu$ is $\sigma$-finite iff there exist $I \subset \mathbb{N}$ and $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing such that we have

$$
\begin{equation*}
\left(\forall i \in I, \mu\left(B_{i}\right) \in \mathbb{R}\right) \quad \wedge \quad X=\bigcup_{i \in I} B_{i} . \tag{11.16}
\end{equation*}
$$

Compact form: $H y p \stackrel{\text { def. }}{=} U \wedge \underline{a d d}$.

Proof. "Left" implies "right". From Definition 885 ( $\sigma$-finiteness), let $I \subset \mathbb{N}$ and $\left(A_{i}\right)_{i \in I} \in \mathcal{S}$ such that for all $i \in I, \mu\left(A_{i}\right) \in \mathbb{R}$, and $X=\bigcup_{i \in I} A_{i}$. Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Then, from Lemma 575 (monotone union of union in subset system, $\underline{H y p}=U)$, let $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ be nondecreasing such that we have

$$
\forall n \in J, \quad \bigcup_{i \in[0 . . n]} A_{\varphi(i)}=\bigcup_{i \in[0 . . n]} B_{\varphi(i)}=B_{\varphi(n)} \in \mathcal{S}, \quad \text { and } \quad \bigcup_{i \in I} A_{i}=\bigcup_{i \in I} B_{i}
$$

Let $i \in I$. Let $n \stackrel{\text { def. }}{=} \varphi^{-1}(i) \in J$. Then, from Definition 859 (subadditivity), and closedness of addition in $\mathbb{R}$, we have, $\mu\left(B_{i}\right) \leqslant \sum_{j \in[0 . . n]} \mu\left(A_{\varphi(j)}\right) \in \mathbb{R}$, and $X=\bigcup_{i \in I} B_{i}$.
"Right" implies "left". Trivial.
Therefore, we have the equivalence.
Lemma 887 (finite is $\sigma$-finite). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set. Let $\mu: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is finite. Then, $\mu$ is $\sigma$-finite.
Compact form: $w F \Rightarrow$ fin $\Rightarrow \sigma$ fin.
Proof. Direct consequence of Definition 857 (finiteness), and Definition 885 ( $\sigma$-finiteness, with $I \stackrel{\text { def. }}{=}\{0\}$ and $\left.A_{0} \stackrel{\text { def. }}{=} X\right)$.

### 11.2 Content

## Definition 888 (content).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\gamma: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is called content on $(X, \mathcal{S})$ iff it is nonnegative, homogeneous, and additive. Compact form: $P_{\gamma} \stackrel{\text { def. }}{=} \operatorname{pos} \wedge h o m \wedge a d d$.

Remark 889 . Note that the concept of content is mostly used when the subset system $\mathcal{S}$ is a (semi)ring of sets, or a (semi-)algebra of sets.

## Lemma 890 (content over finite pseudopartition).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\gamma$ be a content on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under intersection. Let $n \in \mathbb{N}$. Let $A,\left(B_{i}\right)_{i \in[0 . . n]} \in \mathcal{S}$. Assume that $X=\biguplus_{i \in[0 . . n]} B_{i}$. Then, we have
(11.17) $\forall i \in[0 . . n], \quad A \cap B_{i}=B_{i} \cap A \in \mathcal{S} \quad \wedge \quad \gamma(A)=\sum_{i \in[0 . . n]} \gamma\left(A \cap B_{i}\right)=\sum_{i \in[0 . . n]} \gamma\left(B_{i} \cap A\right)$.

For instance, this holds when $\mathcal{S}$ is a $\pi$-system.
Compact form: Hyp $\stackrel{\text { def. }}{=} I \wedge P_{\gamma}$.
Proof. Direct consequence of Definition 888 (content, $\boldsymbol{P}_{\gamma} \Rightarrow$ add), Lemma 863 (decomposition over


## Lemma 891 (content satisfies the binary inclusion-exclusion principle).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\gamma$ be a content on $(X, \mathcal{S})$. Assume that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Then, $\gamma$ satisfies the binary inclusion-exclusion principle (11.5).
For instance, this holds when $\mathcal{S}$ is a semiring of sets (and $A \cap B \in \mathcal{S}$ always holds).
Compact form: Hyp $\stackrel{\text { def. }}{=} P t f D \wedge P_{\gamma}$.
Proof. Direct consequence of Definition 888 (content, $P_{\gamma} \Rightarrow$ add), Lemma 864 (binary inclusionexclusion principle, $\underline{H y p=P t f D \wedge a d d}$ ), and Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow \operatorname{PtfD}}$ ).

## Lemma 892 (content is subtractive).

Let $X$ be a set.
Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\gamma$ be a content on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under local complement. Then, $\gamma$ is subtractive (and $B \backslash A \in \mathcal{S}$ always holds when $A \subset B$, see (11.6)).

For instance, this holds when $\mathcal{S}$ is a ring of sets.
Compact form: $D^{\prime} \Rightarrow P_{\gamma} \Rightarrow$ sub.
Proof. Direct consequence of Definition 888 (content, $\underline{P_{\gamma} \Rightarrow a d d}$ ), Lemma 866 (subtractivity, $\underline{\text { Hyp }}$ equals add), Definition 865 (subtractivity), and Lemma 612 (equivalent definition of ring of sets, $\underline{\left.P_{\mathcal{R}} \Rightarrow D^{\prime}\right)}$,

## Lemma 893 (content is monotone).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Let $\gamma$ be a content on $(X, \mathcal{S})$. Assume that the local complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Then, $\gamma$ is nondecreasing.

For instance, this holds when $\mathcal{S}$ is a semiring of sets.
Compact form: PtfD ${ }^{\prime} \Rightarrow P_{\gamma} \Rightarrow$ monot.

## Proof.

Direct consequence of Definition 888 (content, $\underline{\left.P_{\gamma} \Rightarrow[p o s \mid a d d]\right) \text {, Lemma } 867 \text { (monotonicity, Hyp }}$ equals $\left.P t f D^{\prime} \wedge \operatorname{pos} \wedge a d d\right)$, and Lemma 630 (other properties of semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow P t f D^{\prime}}$ ).

## Lemma 894 (content satisfies the finite Boole inequality).

Let $X$ be a set. Let $\mathcal{S}$ be a semiring of sets on $X$. Let $\gamma$ be a content on $(X, \mathcal{S})$.
Then, $\gamma$ satisfies the finite Boole inequality, i.e. it is subadditive (see (11.1)).
Compact form: $P_{\mathcal{R} / 2} \Rightarrow P_{\gamma} \Rightarrow \underline{\text { add }}$.
Proof. Direct consequence of Definition 626 (semiring of sets, $\underline{P}_{\mathcal{R} / 2} \Rightarrow[N e|I| P t f D]$ ), Definition 888 (content, $\underline{P}_{\gamma} \Rightarrow[$ pos $\mid$ add $]$ ), and Lemma 869 (subadditivity, Hyp $=N e \wedge I \wedge P t f D$ and pos $\wedge a d d)$.

## Lemma 895 (finite content is bounded).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set, and that the local complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Let $\gamma$ be a content on $(X, \mathcal{S})$. Assume that $\gamma$ is finite. Then, $\gamma$ is bounded.

For instance, this holds when $\mathcal{S}$ is a semi-algebra of sets.
Compact form: Hyp $\stackrel{\text { def. }}{=} w F \wedge P t f D^{\prime} \wedge P_{\gamma} \wedge$ fin .
Proof. Direct consequence of Definition 888 (content, $\underline{P_{\gamma} \Rightarrow \text { pos }}$ ), Lemma 893 (content is monotone, $\underline{P t f D^{\prime}} \wedge P_{\gamma} \Rightarrow$ monot), Lemma 858 ( finite is bounded, Hyp $\stackrel{\text { def. }}{=} w F \wedge$ pos $\wedge$ monot $\wedge$ fin), Definition 654 (semi-algebra of sets, $\underline{P}_{\mathcal{A} / 2} \Rightarrow w F$ ), and Lemma 658 (other properties of semi-algebra of sets, $\underline{P}_{\mathcal{A} / 2} \Rightarrow$ PtfD $\left.^{\prime}\right)$.

Lemma 896 ( $\sigma$-finite content).
Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Assume that $\mathcal{S}$ is a semiring of sets, and is closed under union. Let $\gamma$ be a content on $(X, \mathcal{S})$. Then, $\gamma$ is $\sigma$-finite iff there exist $I \subset \mathbb{N}$ and $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing such that we have

$$
\begin{equation*}
\left(\forall i \in I, \gamma\left(B_{i}\right) \in \mathbb{R}\right) \quad \wedge \quad X=\bigcup_{i \in I} B_{i} . \tag{11.18}
\end{equation*}
$$

For instance, this holds when $\mathcal{S}$ is a ring of sets.
Compact form: $H y p \stackrel{\text { def. }}{=} P_{\mathcal{R} / 2} \wedge U \wedge P_{\gamma}$.
Proof. Direct consequence of Lemma 894 (content satisfies the finite Boole inequality, $\underline{P}_{\mathcal{R} / 2} \wedge P_{\gamma}$ implies add) , Lemma 886 (equivalent definition of $\sigma$-finiteness, $\underline{H y p}=U \wedge \underline{a d d}$ ), Lemma 631 (ring of sets is semiring of sets, $P_{\mathcal{R}} \Rightarrow P_{\mathcal{R} / 2}$ ), and Lemma 612 (equivalent definition of ring of sets, $\underline{\left.P_{\mathcal{R}} \Rightarrow U\right)}$.

### 11.3 Premeasure

## Lemma 897 (premeasure).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. A function $\mu_{0}: \mathcal{S} \rightarrow \overline{\mathbb{R}}$ is called premeasure on $(X, \mathcal{S})$ iff it is a content on $(X, \mathcal{S})$ that is $\sigma$-additive, i.e. iff it is nonnegative, homogeneous, and $\sigma$-additive.

When $\mu_{0}$ is a premeasure on the premeasurable space $(X, \mathcal{R})$, i.e. when $\mathcal{R}$ is a ring of sets on $X$, then $\left(X, \mathcal{R}, \mu_{0}\right)$ is called premeasure space. Moreover, if the premeasure $\mu_{0}$ is diffuse (resp. finite, or $\sigma$-finite), then the premeasure space $\left(X, \mathcal{R}, \mu_{0}\right)$ is also said diffuse (resp. finite, or $\sigma$-finite).
Compact form: $P_{\mu_{0}} \stackrel{\text { def. }}{=} P_{\gamma} \wedge \sigma a d d ; P_{\mu_{0}} \Leftrightarrow \operatorname{pos} \wedge h o m \wedge \sigma a d d$.
Proof. Direct consequence of Lemma 875 ( $\sigma$-additive is additive, $\sigma a d d \Rightarrow$ add), Definition 888 (content, $\underline{P_{\gamma}}=$ pos $\wedge h o m \wedge a d d$ ), and Definition 739 (premeasurable space).

Remark 898. Note that the concept of premeasure is mostly used when the subset system $\mathcal{S}$ is a ring of sets, an algebra of sets, or a $\sigma$-algebra (in which case it is called measure, see Section 12.1).

## Lemma 899 (premeasure is additive).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Then, $\mu_{0}$ is additive.
Compact form: $P_{\mu_{0}} \Rightarrow a d d$.
Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}} \Rightarrow P_{\gamma}$ ), and Definition 888 (content, $\left.P_{\gamma} \Rightarrow a d d\right)$.

## Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that the set difference of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$.
Then, $\mu_{0}$ satisfies the binary inclusion-exclusion principle (11.5).
For instance, this holds when $\mathcal{S}$ is a ring of sets (i.e. on a premeasure space).
Compact form: Hyp $\stackrel{\text { def. }}{=} P t f D \wedge P_{\mu_{0}}$.
Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}} \Rightarrow P_{\gamma}$ ), Lemma 891 (content satisfies the binary inclusion-exclusion principle, Hyp $=P t f D \wedge \overline{P_{\gamma}}$, and $P_{\mathcal{R}} / 2$ ), and Lemma 740 (properties of premeasurable space, thus $P_{\mathcal{R} / 2}$ ).

## Lemma 901 (premeasure is subtractive).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$.
Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under local complement.
Then, $\mu_{0}$ is subtractive (and $B \backslash A \in \mathcal{S}$ always holds when $A \subset B$, see (11.6)).
For instance, this holds on a premeasure space.
Compact form: $D^{\prime} \Rightarrow P_{\mu_{0}} \Rightarrow$ sub.
Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}} \Rightarrow P_{\gamma}$ ), Lemma 892 (content is sub-


Lemma 902 (premeasure is monotone). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under local complement. Then, $\mu_{0}$ is nondecreasing. For instance, this holds on a premeasure space.
Compact form: $D^{\prime} \Rightarrow P_{\mu_{0}} \Rightarrow$ monot .
Proof. Direct consequence of Lemma 554 (disjoint decomposition, $D^{\prime} \Rightarrow$ PtfD ${ }^{\prime}$ ), Lemma 897 (premeasure, $P_{\mu_{0}} \Rightarrow P_{\gamma}$ ), Lemma 893 (content is monotone, Hyp $=P t f D^{\prime} \wedge P_{\gamma}$, and $P_{\mathcal{R} / 2}$ ), and Lemma 740 (properties of premeasurable space, thus $P_{\mathcal{R} / 2}$ ).

## Lemma 903 (finite premeasure is bounded).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the full set, and that the local complement of elements of $\mathcal{S}$ can be expressed as a finite disjoint union of elements of $\mathcal{S}$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mu_{0}$ is finite. Then, $\mu_{0}$ is bounded.

For instance, this holds when $\mathcal{S}$ is an algebra of sets, and on a measurable space.
Compact form: Hyp $\stackrel{\text { def. }}{=} w F \wedge P t f D^{\prime} \wedge P_{\mu_{0}} \wedge$ fin .
Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}} \Rightarrow P_{\gamma}$ ), Lemma 895 (finite content is bounded, Hyp $\stackrel{\text { def. }}{=} w F \wedge P t f D^{\prime} \wedge P_{\gamma} \wedge$ fin, and $P_{\mathcal{A} / 2}$ ), Lemma 660 (algebra of sets is semi-algebra of sets, $\underline{P}_{\mathcal{A}} \Rightarrow P_{\mathcal{A} / 2}$ ), and Lemma 742 (properties of measurable space, thus $P_{\mathcal{A}}$ ).

## Lemma 904 ( $\sigma$-finite premeasure).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is a semiring of sets, and is closed under union. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Then, $\mu_{0}$ is $\sigma$-finite iff there exist $I \subset \mathbb{N}$ and $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing such that we have

$$
\begin{equation*}
\left(\forall i \in I, \mu_{0}\left(B_{i}\right) \in \mathbb{R}\right) \quad \wedge \quad X=\bigcup_{i \in I} B_{i} . \tag{11.19}
\end{equation*}
$$

For instance, this holds on a premeasure space.
Compact form: Hyp $\stackrel{\text { def. }}{=} P_{\mathcal{R} / 2} \wedge U \wedge P_{\mu_{0}}$.
Proof. Direct consequence of Lemma 897 (premeasure, $P_{\mu_{0}} \Rightarrow P_{\gamma}$ ), Lemma 896 ( $\sigma$-finite content, Hyp $=P_{\mathcal{R} / 2} \wedge U \wedge P_{\gamma}$, and $P_{\mathcal{R}}$ ), and Lemma 740 (properties of premeasurable space, thus $P_{\mathcal{R}}$ ).

## Lemma 905 (premeasure over countable pseudopartition).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under intersection. Let $I \subset \mathbb{N}$. Let $A,\left(B_{i}\right)_{i \in I} \in \mathcal{S}$. Assume that $X=\biguplus_{i \in I} B_{i}$. Then, we have

$$
\begin{equation*}
\forall i \in I, \quad A \cap B_{i}=B_{i} \cap A \in \mathcal{S} \quad \wedge \quad \mu_{0}(A)=\sum_{i \in I} \mu_{0}\left(A \cap B_{i}\right)=\sum_{i \in I} \mu_{0}\left(B_{i} \cap A\right) . \tag{11.20}
\end{equation*}
$$

For instance, this holds when $\mathcal{S}$ is a $\pi$-system, and on a premeasure space.
Compact form: $H y p \stackrel{\text { def. }}{=} I \wedge P_{\mu_{0}}$.
Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}} \Rightarrow \sigma a d d$ ), Lemma 876 (decomposition
 (properties of premeasurable space, thus $P_{\Pi}$ ).

## Lemma 906 (premeasure satisfies the Boole inequality).

Let $X$ be a set. Let $\mathcal{S}$ be a semiring of sets on $X$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Then, $\mu_{0}$ satisfies the Boole inequality, i.e. it is $\sigma$-subadditive (see (11.7), and subadditive, see (11.1)).

For instance, this holds on a premeasure space.
Compact form: $P_{\mathcal{R} / 2} \Rightarrow P_{\mu_{0}} \Rightarrow[\sigma \underline{\text { add }} \mid \underline{\text { add }}]$.
Proof. Direct consequence of Definition 626 (semiring of sets, $\underline{P_{\mathcal{R} / 2} \Rightarrow[N e|I| P t f D]}$ ), Lemma 897 (premeasure, $\left.\underline{P}_{\mu_{0}} \Rightarrow[p o s \mid \sigma a d d]\right)$, Lemma 878 ( $\sigma$-subadditivity, Hyp $=N e \wedge I \wedge P t f D \wedge$ pos and
 premeasurable space, thus $P_{\mathcal{R} / 2}$ ).

## Lemma 907 (compatibility of null premeasure with countable union).

Let $X$ be a set. Let $\mathcal{S}$ be a semiring of sets on $X$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Let $I \subset \mathbb{N}$. For all $i \in I$, let $A_{i} \in \mathcal{S}$ such that $\mu_{0}\left(A_{i}\right)=0$, and $\bigcup_{i \in I} A_{i} \in \mathcal{S}$. Then, we have $\mu_{0}\left(\bigcup_{i \in I} A_{i}\right)=0$.

For instance, this holds on a premeasure space.
Compact form: Hyp $\stackrel{\text { def. }}{=} P_{\mathcal{R} / 2} \wedge P_{\mu_{0}}$.

Proof. Direct consequence of Lemma 897 (premeasure, nonnegativeness), Lemma 906 (premeasure satisfies the Boole inequality), additive group properties of $\mathbb{R}$, Lemma 34 (stationary sequence is convergent, countable sum of zero terms is zero), and Lemma 740 (properties of premeasurable space, thus $P_{\mathcal{R} / 2}$ ).

## Lemma 908 (premeasure is continuous from below).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under local complement, and disjoint union. Then, $\mu_{0}$ is continuous from below (see (11.11)).

Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}$ nondecreasing, we have

$$
\begin{equation*}
A \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} A_{n} \in \mathcal{S} \quad \Longrightarrow \quad \mu_{0}(A)=\lim _{n \rightarrow \infty} \mu_{0}\left(A_{n}\right) . \tag{11.21}
\end{equation*}
$$

For instance, this holds on a premeasure space.
Compact form: $D^{\prime} \Rightarrow U d \Rightarrow P_{\mu_{0}} \Rightarrow$ cont $\uparrow$.
Proof. Direct consequence of Lemma 897 (premeasure, ${\underline{P_{\mu_{0}}}} \Rightarrow[$ pos $\mid \sigma a d d]$ ), Lemma 880 (continuity from below, $\boldsymbol{H y p}=D^{\prime} \wedge U d \wedge \operatorname{pos} \wedge \sigma a d d$ ), and Lemma 740 (properties of premeasurable space, thus $\left[D^{\prime} \mid U d\right]$ ).

Lemma 909 (equivalent definition of premeasure (from below)). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ is closed under local complement, and disjoint union.
Let $\mu_{0}: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu_{0}$ is nonnegative and homogeneous.
Then, $\mu_{0}$ is a premeasure on $(X, \mathcal{S})$ iff it is additive and continuous from below.
For instance, this holds on a premeasure space.
Compact form: $D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow$ hom $\Rightarrow P_{\mu_{0}} \Leftrightarrow($ add $\wedge$ cont $\uparrow)$.
Proof. Direct consequence of Lemma 897 (premeasure, ${\underline{P_{\mu_{0}}}}=$ pos $\wedge$ hom $\wedge \sigma a d d$ ), Lemma 881 ( $\sigma-$ additivity (from below), $D^{\prime} \wedge U d \wedge$ pos $\Rightarrow \sigma a d d \Leftrightarrow(a d \bar{d} \wedge$ cont $\uparrow)$ ), and Lemma 740 (properties of premeasurable space, thus $\left.\left[D^{\prime} \mid U d\right]\right)$.

## Lemma 910 (premeasure is continuous from above).

Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Let $\mu_{0}$ be a premeasure on $(X, \mathcal{S})$. Assume that $\mathcal{S}$ is closed under local complement, and disjoint union. Then, $\mu_{0}$ is continuous from above (see (11.13)).

Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}$ nonincreasing, we have
(11.22) $\left(\exists n_{0} \in \mathbb{N}, \mu\left(A_{n_{0}}\right)<\infty\right) \wedge A \stackrel{\text { def. }}{=} \bigcap_{n \in \mathbb{N}} A_{n} \in \mathcal{S} \quad \Longrightarrow \quad \mu_{0}(A)=\lim _{n \rightarrow \infty} \mu_{0}\left(A_{n}\right)<\infty$.

For instance, this holds on a premeasure space.
Compact form: $D^{\prime} \Rightarrow U d \Rightarrow P_{\mu_{0}} \Rightarrow$ cont $\downarrow$.
Proof. Direct consequence of Lemma 897 (premeasure, ${\underline{P_{\mu_{0}}}} \Rightarrow[$ pos $\mid \sigma a d d]$ ), Lemma 883 (continuity from above, Hyp $=D^{\prime} \wedge U d \wedge \operatorname{pos} \wedge \sigma a d d$ ), and Lemma 740 (properties of premeasurable space, thus $\left[D^{\prime} \mid U d\right]$ ).

Lemma 911 (equivalent definition of premeasure (from above)). Let $X$ be a set. Let $\mathcal{S} \subset \mathcal{P}(X)$. Assume that $\mathcal{S}$ contains the empty set, and is closed under local complement, and disjoint union. Let $\mu_{0}: \mathcal{S} \rightarrow \overline{\mathbb{R}}$. Assume that $\mu_{0}$ is nonnegative, homogeneous, and finite.
Then, $\mu_{0}$ is a finite premeasure on $(X, \mathcal{S})$ iff it is additive and continuous from above at $\varnothing$.
For instance, this holds on a premeasure space.
Compact form: $w E \Rightarrow D^{\prime} \Rightarrow U d \Rightarrow$ pos $\Rightarrow h o m \Rightarrow$ fin $\Rightarrow P_{\mu_{0}} \Leftrightarrow($ add $\wedge \operatorname{cont} \downarrow \varnothing)$.

Proof. Direct consequence of Lemma 897 (premeasure, $\underline{P}_{\mu_{0}}=$ pos $\wedge$ hom $\wedge \sigma a d d$ ), Lemma 884 ( $\sigma$ additivity (from above), wE $\wedge D^{\prime} \wedge U d \wedge \operatorname{pos} \wedge h o m \wedge \overline{f i n} \Rightarrow \sigma a d d \Leftrightarrow($ add $\wedge$ cont $\downarrow \varnothing)$ ), and Lemma 740 (properties of premeasurable space, thus $\left[w E\left|D^{\prime}\right| U d\right]$ ).

Lemma 912 (properties of premeasure space). Let $\left(X, \mathcal{R}, \mu_{0}\right)$ be a premeasure space. Then, $\mu_{0}$ is a premeasure, and a content, and $\mathcal{R}$ is a (semi)ring a sets, and a $\pi$-system.

Moreover, $\mu_{0}$ is additive, subtractive, monotone, continuous from below (see (11.11)) and above (see (11.13)), and satisfies the binary inclusion-exclusion principle (see (11.5)), the equivalent $\sigma$-finiteness definition (see (11.19)), the formula of decomposition against a countable pseudopartition (see (11.20)), and the Boole inequality (see (11.7)); and $\mathcal{R}$ is nonempty, contains the empty set, is closed under local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, and finite union and intersection, and local complement and set difference of elements of $\mathcal{R}$ can be expressed as a finite disjoint union of elements of $\mathcal{R}$.

Compact form: Concl $\stackrel{\text { def. }}{=}\left[P_{\mu_{0}}\left|P_{\gamma}\right| P_{\mathcal{R}}\left|P_{\mathcal{R} / 2}\right| P_{\Pi} \mid\right.$ pos $\mid$ hom $\mid$ add $|\sigma a d d|$ sub $\mid$ monot $\mid$ cont $\uparrow \mid$ cont $\downarrow|\underline{\text { add }}|$ $\sigma \underline{\text { add }} \mid$ $N e|w E| D^{\prime}|D| s D|U d| U|I| U d f|U f| I f\left|P t f D^{\prime}\right|$ PtfD].

Proof. Direct consequence of Lemma 897 (premeasure, thus $\left[P_{\mu_{0}}\left|P_{\gamma}\right|\right.$ pos $\mid$ hom $\mid$ $\left.\sigma a d d\right]$ ), Lemma 740 (properties of premeasurable space, thus $\left[P_{\mathcal{R}}\left|P_{\mathcal{R} / 2}\right| P_{\Pi}|N e| w E\left|D^{\prime}\right| D|s D| U d|U|\right.$ $\left.I|U d f| U f|I f| P t f D^{\prime} \mid P t f D\right]$ ), Lemma 899 (premeasure is additive, thus add), Lemma 901 (premeasure is subtractive, thus sub), Lemma 902 (premeasure is monotone, thus monot), Lemma 908 (premeasure is continuous from below, thus cont $\uparrow$ ), Lemma 910 (premeasure is continuous from above, thus cont $\downarrow$ ), Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle), Lemma 904 ( $\sigma$-finite premeasure), Lemma 905 (premeasure over countable pseudopartition), and Lemma 906 (premeasure satisfies the Boole inequality).

## Chapter 12

## Measure

## Contents

12.1 Measure space ..... 194
12.2 Some measures ..... 198
12.3 Negligible subset ..... 200
12.4 Uniqueness condition ..... 208

### 12.1 Measure space

## Lemma 913 (measure).

Let $(X, \Sigma)$ be a measurable space. A function $\mu: \Sigma \rightarrow \overline{\mathbb{R}}$ is called measure on $(X, \Sigma)$ iff it is a premeasure on $(X, \Sigma)$, i.e. iff it is nonnegative, homogeneous, and $\sigma$-additive $\biguplus_{n \in I} A_{i} \in \mathcal{S} \stackrel{\text { def. }}{=} \Sigma$ always holds). If so, $(X, \Sigma, \mu)$ is called measure space.

Moreover, if the measure $\mu$ is diffuse (resp. finite, or $\sigma$-finite), then the measure space $(X, \Sigma, \mu)$ is also said diffuse (resp. finite, or $\sigma$-finite).

Compact form: $P_{\mu} \stackrel{\text { def. }}{=} P_{\Sigma} \wedge P_{\mu_{0}} ; P_{\mu} \Leftrightarrow P_{\Sigma} \wedge$ pos $\wedge$ hom $\wedge \sigma a d d$.
Proof. Direct consequence of Lemma 897 (premeasure), Lemma 700 (other properties of $\sigma$-algebra, $P_{\Sigma} \Rightarrow U d c$ ), and Definition 874 ( $\sigma$-additivity).

Remark 914. The previous definition is actually that of nonnegative measure, but as we do not consider "signed" measures in this document, we omit the qualifier "nonnegative".

Lemma 915 (measure space is premeasure space).
Let $(X, \Sigma, \mu)$ be a measure space. Then, it is a premeasure space.
Compact form: $P_{\mu} \Rightarrow P_{\mathcal{R}} \wedge P_{\mu_{0}}$.
Proof. Direct consequence of Lemma 913 (measure, $\underline{\mu}$ is a premeasure), Definition 741 (measurable space, $\Sigma$ is $\sigma$-algebra), Lemma 703 ( $\sigma$-algebra is ring of sets), and Lemma 897 (premeasure).

## Lemma 916 (measure is additive). Let $(X, \Sigma, \mu)$ be a measure space.

Then, $\mu$ is additive (see (11.3) where $\biguplus_{n \in[0 . . n]} A_{i} \in \Sigma$ always holds).
Compact form: $P_{\mu} \Rightarrow a d d$.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 899 (premeasure is additive), Definition 862 (additivity), and Lemma 700 (other properties of $\sigma$-algebra, $\left.P_{\Sigma} \Rightarrow U d f\right)$.

Lemma 917 (measure satisfies the binary inclusion-exclusion principle).
Let $(X, \Sigma, \mu)$ be a measure space. Then, $\mu$ satisfies the binary inclusion-exclusion principle (see (11.5) where $A \cup B, A \cap B \in \Sigma$ always holds).

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle), Lemma 864 (binary inclusion-exclusion principle), and Lemma 700 (other properties of $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow[I \mid U] \text { ). }}$

## Lemma 918 (measure is subtractive).

## Let $(X, \Sigma, \mu)$ be a measure space.

Then, $\mu$ is subtractive (see (11.6) where $B \backslash A \in \Sigma$ always holds).
Compact form: $P_{\mu} \Rightarrow$ sub.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 901 (premeasure is subtractive), Definition 865 (subtractivity), and Lemma 700 (other properties of $\sigma$ algebra, $\left.P_{\Sigma} \Rightarrow D^{\prime}\right)$.

```
Lemma 919 (measure is monotone).
Let \((X, \Sigma, \mu)\) be a measure space. Then, \(\mu\) is nondecreasing.
Compact form: \(P_{\mu} \Rightarrow\) monot.
```

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 902 (premeasure is monotone).

## Lemma 920 (finite measure is bounded).

Let $(X, \Sigma, \mu)$ be a measure space. Assume that $\mu$ is finite. Then, $\mu$ is bounded.
Compact form: $H y p \stackrel{\text { def. }}{=} P_{\mu} \wedge$ fin.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 903 (finite premeasure is bounded).

Lemma 921 ( $\sigma$-finite measure).
Let $(X, \Sigma, \mu)$ be a measure space.
Then, $\mu$ is $\sigma$-finite iff there exist $I \subset \mathbb{N}$ and $\left(B_{i}\right)_{i \in I} \in \mathcal{S}$ nondecreasing such that we have

$$
\begin{equation*}
\left(\forall i \in I, \mu\left(B_{i}\right) \in \mathbb{R}\right) \quad \wedge \quad X=\bigcup_{i \in I} B_{i} . \tag{12.1}
\end{equation*}
$$

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 904 ( $\sigma$-finite premeasure).

## Lemma 922 (measure over countable pseudopartition).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(B_{i}\right)_{i \in I} \in \Sigma$. Assume that $X=\biguplus_{i \in I} B_{i}$. Then, for all $i \in I, A \cap B_{i}=B_{i} \cap A \in \Sigma$, and we have

$$
\begin{equation*}
\mu(A)=\sum_{i \in I} \mu\left(A \cap B_{i}\right)=\sum_{i \in I} \mu\left(B_{i} \cap A\right) \tag{12.2}
\end{equation*}
$$

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 905 (premeasure over countable pseudopartition).

## Lemma 923 (measure satisfies the Boole inequality).

Let $(X, \Sigma, \mu)$ be a measure space. Then, $\mu$ satisfies the Boole inequality, i.e. it is $\sigma$-subadditive (see (11.7) where $\bigcup_{n \in I} A_{i} \in \Sigma$ always holds).
Compact form: $P_{\mu} \Rightarrow[\underline{\sigma a d d} \mid \underline{\text { add }}]$.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 906 (premeasure satisfies the Boole inequality), Definition 870 ( $\sigma$-subadditivity), and Lemma 696 ( $\sigma$-algebra, $\left.\underline{P_{\Sigma} \Rightarrow U c}\right)$.

## Lemma 924 (compatibility of null measure with countable union).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. For all $i \in I$, let $A_{i} \in \Sigma$ such that $\mu\left(A_{i}\right)=0$. Then, we have $\mu\left(\bigcup_{i \in I} A_{i}\right)=0$.

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 907 (compatibility of null premeasure with countable union), and Lemma 696 ( $\sigma$-algebra, $P_{\Sigma} \Rightarrow U c$ ).

## Lemma 925 (measure is continuous from below). Let $(X, \Sigma, \mu)$ be a measure space.

 Then, $\mu$ is continuous from below (see (11.11) where $\bigcup_{n \in I} A_{i} \in \Sigma$ always holds).Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \Sigma$ nondecreasing, we have

$$
\begin{equation*}
\bigcup_{n \in \mathbb{N}} A_{n} \in \Sigma \wedge \mu\left(\bigcup_{n \in \mathbb{N}} A_{n}\right)=\lim _{n \rightarrow \infty} \mu\left(A_{n}\right) \tag{12.3}
\end{equation*}
$$

Compact form: $P_{\mu} \Rightarrow$ cont $\uparrow$.

Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 908 (premeasure is continuous from below), and Lemma 696 ( $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow U c}$ ).

Lemma 926 (equivalent definition of measure (from below)).
Let $(X, \Sigma)$ be a measurable space. Let $\mu: \Sigma \rightarrow \overline{\mathbb{R}}$. Assume that $\mu$ is nonnegative and homogeneous. Then, $\mu$ is a measure on $(X, \Sigma)$ iff it is additive and continuous from below.
Compact form: $P_{\Sigma} \Rightarrow$ pos $\Rightarrow$ hom $\Rightarrow P_{\mu} \Leftrightarrow($ add $\wedge$ cont $\uparrow)$.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 909 (equivalent definition of premeasure (from below)).

Lemma 927 (measure is continuous from above). Let $(X, \Sigma, \mu)$ be a measure space. Then, $\mu$ is continuous from above (see (11.13) where $\bigcap_{n \in I} A_{i} \in \Sigma$ always holds). Moreover, for all $\left(A_{n}\right)_{n \in \mathbb{N}} \in \Sigma$ nonincreasing, we have

$$
\begin{equation*}
\bigcap_{n \in \mathbb{N}} A_{n} \in \Sigma \wedge\left[\left(\exists n_{0} \in \mathbb{N}, \mu\left(A_{n_{0}}\right)<\infty\right) \quad \Longrightarrow \quad \mu\left(\bigcap_{n \in \mathbb{N}} A_{n}\right)=\lim _{n \rightarrow \infty} \mu\left(A_{n}\right)<\infty\right] . \tag{12.4}
\end{equation*}
$$

Compact form: $P_{\mu} \Rightarrow$ cont $\downarrow$.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), Lemma 910 (premeasure is continuous from above), and Lemma 696 ( $\sigma$-algebra, $\underline{P_{\Sigma} \Rightarrow I c \text { ). } . ~ . ~}$

Lemma 928 (equivalent definition of measure (from above)).

Compact form: $P_{\Sigma} \Rightarrow$ pos $\Rightarrow$ hom $\Rightarrow$ fin $\Rightarrow P_{\mu} \Leftrightarrow($ add $\wedge$ cont $\downarrow \varnothing)$.
Proof. Direct consequence of Lemma 915 (measure space is premeasure space), and Lemma 911 (equivalent definition of premeasure (from above)).

## Lemma 929 (properties of measure space). Let $(X, \Sigma, \mu)$ be a measure space.

 Then, $\mu$ is a measure, a premeasure and a content, and $\Sigma$ is a $\lambda$-system, a monotone class, a (semi-)algebra of sets, a (semi)ring of sets, and a $\pi$-system.Moreover, $\mu$ is additive, subtractive, monotone, continuous from below (see (11.11)) and above (see (11.13)), and satisfies the binary inclusion-exclusion principle (see (11.5)), the equivalent $\sigma$ finiteness definition (see (11.19)), the formula of decomposition against a countable pseudopartition (see (11.20)), and the Boole inequality (see (11.7)); and $\Sigma$ is nonempty, contains the empty set and the full set, is closed under complement, local complement, set difference, symmetric difference, disjoint union, union and intersection, finite disjoint union, finite union and intersection, countable disjoint union, countable monotone union and intersection, and countable union and intersection, and complement, local complement and set difference of elements of $\Sigma$ can be expressed as a finite disjoint union of elements of $\Sigma$.

```
Compact form: Concl }\stackrel{\mathrm{ def. }}{=}[\mp@subsup{P}{\mu}{}|\mp@subsup{P}{\mp@subsup{\mu}{0}{}}{}|\mp@subsup{P}{\gamma}{}|\mp@subsup{P}{\Sigma}{}|\mp@subsup{P}{\Lambda}{}|\mp@subsup{P}{\mathcal{C}}{}|\mp@subsup{P}{\mathcal{A}}{}|\mp@subsup{P}{\mathcal{A}/2}{}|\mp@subsup{P}{\mathcal{R}}{}|\mp@subsup{P}{\mathcal{R}/2}{}|\mp@subsup{P}{\Pi}{}
    pos | hom | add | \sigmaadd | sub | monot | cont\uparrow | cont \downarrow \ add | \sigma\underline{add |}
    Ne|wE|wF|C| D'|D|sD|Ud|U|I|Udf|Uf|If |
    Udc|Umc|Imc|Uc|Ic|PtfC |PtfD'|PtfD].
```

Proof.
Direct consequence of Lemma 913 (measure, thus $P_{\mu}$ ), Lemma 912 (properties of premeasure space, thus $\left[P_{\mu_{0}}\left|P_{\gamma}\right|\right.$ pos $\mid$ hom $\mid$ add $\mid$ oadd $\mid$ sub $\mid$ monot $\mid$ cont $\uparrow \mid$ cont $\downarrow \mid \underline{\text { add } \mid \sigma \underline{\text { add }}] \text { ), and Lemma } 742}$ (properties of measurable space, thus $\left[P_{\Sigma}\left|P_{\Lambda}\right| P_{\mathcal{C}}\left|P_{\mathcal{A}}\right| P_{\mathcal{A} / 2}\left|P_{\mathcal{R}}\right| P_{\mathcal{R} / 2}\left|P_{\Pi}\right| N e|w E| w F \mid C\right.$


Lemma 930 (trace measure).
Let $(X, \Sigma, \mu)$ be a measure space. Let $Y \in \Sigma$. Then, $\mu_{I_{\Sigma \bar{\Pi}}}$ is a measure on $(Y, \Sigma \bar{\cap} Y)$.
The measure $\mu_{Y} \stackrel{\text { def. }}{=} \mu_{\left.\right|_{\Sigma \bar{\Pi} Y}}$ is called trace measure on $Y$. The measure space $\left(Y, \Sigma \bar{\cap} Y, \mu_{Y}\right)$ is called trace measure space on $Y$.

Proof. Direct consequence of Lemma 763 (trace $\sigma$-algebra), Lemma 764 (measurability of measurable subspace, $\underline{\Sigma \bar{\cap} Y \subset \Sigma \text { ), and Lemma } 913 \text { (measure). }}$

Lemma 931 (restricted measure). Let $(X, \Sigma, \mu)$ be a measure space. Let $Y \in \Sigma$. Then, the function $\mu_{Y}^{\prime} \stackrel{\text { def. }}{=}(A \in \Sigma \mapsto \mu(A \cap Y))$ is a measure on $(X, \Sigma)$.

Proof. Direct consequence of Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection (with $\operatorname{card}(I)=2$ ), properties of intersection, and Lemma 913 (measure).

Lemma 932 (trace measure is restricted measure). Let $(X, \Sigma, \mu)$ be a measure space. Let $Y \in \Sigma$. Let $B \in \Sigma \bar{\cap} Y$. Then, we have $\mu_{Y}(B)=\mu(B)=\mu_{Y}^{\prime}(B)$.

Proof. Direct consequence of Lemma 930 (trace measure), Lemma 931 (restricted measure), and Lemma 764 (measurability of measurable subspace, thus $B \in \Sigma$ and $B \subset Y$ ).

### 12.2 Some measures

## Lemma 933 (trivial measure).

Let $(X, \Sigma)$ be a measurable space. Then, the zero function on $\Sigma$ is a measure on $(X, \Sigma)$.
It is called the trivial measure on $(X, \Sigma)$, and $(X, \Sigma, 0)$ is called trivial measure space.
Proof.
Direct consequence of Lemma 913 (measure), and ordered additive group properties of $\mathbb{R}$.
Lemma 934 (equivalent definition of trivial measure).
Let $(X, \Sigma, \mu)$ be a measure space. Then, $\mu$ is the trivial measure iff $\mu(X)=0$.
Proof. Direct consequence of Lemma 933 (trivial measure), Lemma 913 (measure, nonnegativeness), and Lemma 919 (measure is monotone, $\underline{\mu}$ is nonpositive).

## Lemma 935 (counting measure).

Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $\delta_{Y}$ be the function defined by

$$
\begin{equation*}
\forall A \in \Sigma, \quad \delta_{Y}(A) \stackrel{\text { def. }}{=} \operatorname{card}(A \cap Y) \tag{12.5}
\end{equation*}
$$

with the convention that the cardinality of an infinite set is $\infty$. Then, $\delta_{Y}$ is a measure on $(X, \Sigma)$. It is called the counting measure (associated with $Y$ ).

Proof. From the definition of the cardinality, and Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), the function $\delta_{Y}$ is nonnegative. Moreover, from Lemma 913 (measure), Definition 741 (measurable


$$
\delta_{Y}(\varnothing)=\operatorname{card}(\varnothing)=0
$$

Let $I \subset \mathbb{N}$ be nonempty. Let $\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that the $A_{i}$ 's are pairwise disjoint. Then, from distributivity of intersection over disjoint union, and $\sigma$-additivity of the cardinality (with the convention that $\infty$ is absorbing element for addition in $\overline{\mathbb{N}}$ ),

$$
\delta_{Y}\left(\biguplus_{i \in I} A_{i}\right)=\operatorname{card}\left(\biguplus_{i \in I} A_{i} \cap Y\right)=\operatorname{card}\left(\biguplus_{i \in I}\left(A_{i} \cap Y\right)\right)=\sum_{i \in I} \operatorname{card}\left(A_{i} \cap Y\right)=\sum_{i \in I} \delta_{Y}\left(A_{i}\right) .
$$

Therefore, from Definition 874 ( $\sigma$-additivity), and Lemma 913 (measure), $\delta_{Y}$ is a measure on $(X, \Sigma)$.

## Lemma 936 (finiteness of counting measure).

Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Then, $\delta_{Y}$ is finite iff the set $Y$ is finite.
Proof. Direct consequence of Lemma 697 (equivalent definition of $\sigma$-algebra, contains full set), Lemma 935 (counting measure, $\delta_{Y}(X)=\operatorname{card}(Y)$ ), and Definition 857 (finiteness).

## Lemma 937 ( $\sigma$-finite counting measure).

Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Assume that $\Sigma$ contains all singletons of $X$. Then, $\delta_{Y}$ is $\sigma$-finite implies $Y$ is countable, and $X$ is countable implies $\delta_{Y}$ is $\sigma$-finite.

Proof. From Lemma 913 (measure), Definition 741 (measurable space), and Lemma 696 ( $\sigma$-algebra, closedness under countable union), the $\sigma$-algebra $\Sigma$ contains all countable subsets of $X$. Hence, if $X$ is countable, we have $\Sigma=\mathcal{P}(X)$.
First implication. Assume that $\delta_{Y}$ is $\sigma$-finite.
Then, from Lemma 886 (equivalent definition of $\sigma$-finiteness), let $\left(B_{n}\right)_{n \in \mathbb{N}} \in \Sigma$ such that for all $n \in \mathbb{N}, B_{n} \subset B_{n+1}, \delta_{Y}\left(B_{n}\right)=\operatorname{card}\left(B_{n} \cap Y\right)<\infty$, and $X=\bigcup_{n \in \mathbb{N}} B_{n}$. For all $n \in \mathbb{N}$,
let $A_{n} \stackrel{\text { def. }}{=} B_{n} \cap Y$. Let $n \in \mathbb{N}$. Then, from De Morgan's laws, and Lemma 935 (counting measure), we have $Y=\bigcup_{n \in \mathbb{N}} A_{n}$, and $\operatorname{card}\left(A_{n}\right)=\delta_{Y}\left(B_{n}\right)<\infty$. Hence, from countability of countable union of finite subsets, $Y$ is countable.

Second implication. Assume that $X$ is countable. Then, we have $\Sigma=\mathcal{P}(X)$.
From the definition of countability, let $\varphi$ be a bijection from $I \subset \mathbb{N}$ onto $X$. For all $n \in \mathbb{N}$, let $A_{n} \stackrel{\text { def. }}{=} \varphi([0 . . n] \cap I) \in \Sigma$. Let $n \in \mathbb{N}$. Then, from Lemma 935 (counting measure), monotonicity of the cardinality, preservation of the cardinality by bijection, compatibility of direct image with union, and De Morgan's laws, we have

$$
\begin{gathered}
\delta_{Y}\left(A_{n}\right)=\operatorname{card}\left(A_{n} \cap Y\right) \leqslant \operatorname{card}\left(A_{n}\right)=\operatorname{card}([0 . . n] \cap I) \leqslant n+1<\infty, \\
\bigcup_{n \in \mathbb{N}} A_{n}=\varphi\left(\bigcup_{n \in \mathbb{N}}([0 . . n] \cap I)\right)=\varphi\left(\bigcup_{n \in \mathbb{N}}[0 . . n] \cap I\right)=\varphi(I)=X .
\end{gathered}
$$

Hence, from Definition 885 ( $\sigma$-finiteness), the counting measure $\delta_{Y}$ is $\sigma$-finite.
Therefore, we have both implications.
Remark 938.
Note that in the previous lemmas, the set $Y$ is not required to be measurable. Note also that the previous lemmas are valid for any $\sigma$-algebra $\Sigma$, including the discrete $\sigma$-algebra $\mathcal{P}(X)$.

Counting measures are usually considered in the case where $Y=X$ is countable, and $\Sigma$ is the discrete $\sigma$-algebra $\mathcal{P}(X)$. A typical example is the $\sigma$-finite measure space $\left(\mathbb{N}, \mathcal{P}(\mathbb{N}), \delta_{\mathbb{N}}\right)$.

Definition 939 (Dirac measure).
Let $(X, \Sigma)$ be a measurable space. Let $a \in X$. The counting measure associated with $\{a\}$ is called Dirac measure (at a); it is also denoted $\delta_{a} \stackrel{\text { def. }}{=} \delta_{\{a\}}$.

## Lemma 940 (equivalent definition of Dirac measure).

Let $(X, \Sigma)$ be a measurable space. Let $a \in X$, and let $A \in \Sigma$. Then, we have $\delta_{a}(A)=\mathbb{1}_{A}(a)$.
Proof. Direct consequence of Definition 939 (Dirac measure), and Lemma 935 (counting measure, $\underline{\operatorname{card}(A \cap\{a\})=1}$ when $a \in A$ and 0 otherwise).

## Lemma 941 (Dirac measure is finite).

Let $(X, \Sigma)$ be a measurable space. Let $a \in X$. Then, $\delta_{a}$ is finite and $\delta_{a}(X)=1$.
Proof. Direct consequence of Definition 939 (Dirac measure), Lemma 936 (finiteness of counting measure, $\delta_{a}$ is finite), and Lemma 935 (counting measure, $\operatorname{card}(X \cap\{a\})=1$ ).

### 12.3 Negligible subset

Definition 942 (negligible subset).
Let $(X, \Sigma, \mu)$ be a measure space.
A subset $A$ of $X$ is said ( $\mu$-)negligible iff there exists $B \in \Sigma$ such that $A \subset B$ and $\mu(B)=0$.
The set of $\mu$-negligible subsets is denoted $\mathbf{N}(X, \Sigma, \mu)$ (or simply $\mathbf{N}$ ).

Definition 943 (complete measure). Let ( $X, \Sigma, \mu$ ) be a measure space. The measure $\mu$ is said complete iff $\mathbf{N}(X, \Sigma, \mu) \subset \Sigma$. If so, the measure space $(X, \Sigma, \mu)$ is also said complete.

Remark 944. In the next definition, "considerable" naturally means "non-negligible".
Definition 945 (considerable subset).
Let $(X, \Sigma, \mu)$ be a measure space. A subset $A$ of $X$ is said ( $\mu$-) considerable iff $A \notin \mathbf{N}(X, \Sigma, \mu)$.

Lemma 946 (equivalent definition of considerable subset). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $A \subset X$. Then, $A$ is $\mu$-considerable iff for all $B \in \Sigma, A \subset B$ implies $\mu(B)>0$.

Proof. Direct consequence of Definition 945 (considerable subset), Definition 942 (negligible subset), Lemma 913 (measure, nonnegativeness), and the tautology $\neg(P \wedge Q) \Leftrightarrow(P \Rightarrow \neg Q)$.

Lemma 947 (negligibility with trivial measure).
Let $(X, \Sigma)$ be a measurable space. Then, we have $\mathbf{N}(X, \Sigma, 0)=\mathcal{P}(X)$.
Proof. Direct consequence of Lemma 696 ( $\sigma$-algebra, $X \in \Sigma$ ), Lemma 933 (trivial measure, $X$ is of measure zero), and Definition 942 (negligible subset, with $B \stackrel{\text { def. }}{=} X$ ).

## Lemma 948 (negligibility of measurable subset).

Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Then, we have $A \in \mathbf{N}$ iff $\mu(A)=0$.
Proof. Direct consequence of Definition 942 (negligible subset, with $B=A$ ), Lemma 919 (measure is monotone, with $A \subset B$, in $\Sigma$ ), and Lemma 913 (measure, nonnegativeness).

Lemma 949 (empty set is negligible). $\quad$ Let $(X, \Sigma, \mu)$ be a measure space. Then, $\varnothing \in \mathbf{N}$.
Proof. Direct consequence of Lemma 948 (negligibility of measurable subset), Lemma 913 (measure), Definition 741 (measurable space, $\underline{\Sigma}$ is a $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, $\varnothing \in \Sigma$ ), and Lemma 913 (measure, homogeneity).

Remark 950. Note that from Lemma 913 (measure, homogeneity), and Lemma 949 (empty set is negligible), the next two lemmas are compliant with the nullary union law $\left(\bigcup_{\varnothing}=\varnothing\right)$.

Lemma 951 (subset of negligible is negligible).
Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \mathbf{N}$. Then, we have $\mathcal{P}(A) \subset \mathbf{N}$.
Proof. Direct consequence of Definition 942 (negligible subset), the definition of the power set, and transitivity of the inclusion.

Lemma 952 ( N is closed under countable union).
Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I} \in \mathbf{N}$. Then, we have $\bigcup_{i \in I} A_{i} \in \mathbf{N}$.
Proof. Direct consequence of Definition 942 (negligible subset), monotonicity of union, and Lemma 924 (compatibility of null measure with countable union).

## Definition 953 (property almost satisfied).

Let $(X, \Sigma, \mu)$ be a measure space. A predicate $P$ defined on $X$ is said satisfied ( $\mu$-) almost everywhere iff $\{\neg P\} \stackrel{\text { def. }}{=}\{x \in X \mid \neg P(x)\} \in \mathbf{N}$; this is denoted either " $P \mu$ a.e.", " $\forall x \in X$, $P(x) \mu$ a.e.", " $\forall_{\mu \text { a.e. }} x \in X, P(x)$ ", or "for $\mu$-almost all $x \in X, P(x)$ " (or simply without the mention of the measure $\mu$ ).

Remark 954. When a single binary relation (using infix notation) is involved in $P$, the annotation $" \mu$ a.e." may be put above the infix operator, as in " $\stackrel{\mu \text { a.e. } " " ~(o r ~ s i m p l y ~ a s ~ i n ~}{=} \stackrel{\text { a.e." }}{=}$ ).

## Lemma 955 (equivalent definition of property almost satisfied).

Let $(X, \Sigma, \mu)$ be a measure space. Let $P$ be a predicate on $X$. Then, we have

$$
\begin{equation*}
P \mu \text { a.e. } \Longleftrightarrow \exists A \in \Sigma, \quad A \subset\{P\} \quad \wedge \quad \mu\left(A^{c}\right)=0 . \tag{12.6}
\end{equation*}
$$

Proof. Direct consequence of Definition 953 (property almost satisfied, $\{P\}^{c} \in \mathbf{N}$ ), Definition 942 (negligible subset, there exists $B \in \Sigma$ such that $\{P\}^{c} \subset B$ and $\mu(B)=0$ ), involutiveness of complement (with $A \stackrel{\text { def. }}{=} \boldsymbol{B}^{c}$ ), Lemma 913 (measure), Definition 741 (measurable space), Lemma 696


## Lemma 956 (property almost satisfied with trivial measure). Let ( $X, \Sigma$ ) be a

 measurable space. Let $P$ be a predicate on $X$. Then, $P$ is satisfied 0-almost everywhere.Proof. Direct consequence of Lemma 947 (negligibility with trivial measure, $\{\neg P\} \in \mathbf{N}=\mathcal{P}(X)$ ), and Definition 953 (property almost satisfied).

Lemma 957 (measurable property almost satisfied). Let $(X, \Sigma, \mu)$ be a measure space. Let $P$ be a predicate on $X$. Assume that $\{P\} \in \Sigma$. Then, we have $P \mu$ a.e. iff $\mu(\{\neg P\})=0$.

Proof. Direct consequence of Definition 953 (property almost satisfied), and Lemma 948 (negligibility of measurable subset).

## Lemma 958 (everywhere is almost everywhere).

Let $(X, \Sigma, \mu)$ be a measure space. Let $P$ be a predicate on $X$. Then, we have

$$
(\forall x \in X, P(x)) \quad \Longrightarrow \quad P \mu \text { a.e. }
$$

Proof. Direct consequence of Definition 953 (property almost satisfied), and Lemma 949 (empty set is negligible).

## Lemma 959 (almost modus ponens).

Let $(X, \Sigma, \mu)$ be a measure space. Let $P$ and $Q$ be predicates on $X$. Then, we have

$$
\begin{equation*}
(\forall x \in X, P(x) \Rightarrow Q(x)) \quad \wedge \quad P \mu \text { a.e. } \quad \Longrightarrow \quad Q \mu \text { a.e. } \tag{12.8}
\end{equation*}
$$

Proof. Direct consequence of monotonicity of complement $\left(\{Q\}^{c} \subset\{P\}^{c}\right)$, Definition 953 (property almost satisfied), and Lemma 951 (subset of negligible is negligible, with $A \stackrel{\text { def. }}{=}\{P\}^{c}$ ).

## Lemma 960 (countable almost properties is almost property).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$, and let $\left(P_{i}\right)_{i \in I}$ be predicates on $X$. Then, we have

$$
\begin{align*}
\left(\forall i \in I, \quad P_{i} \mu \text { a.e. }\right) & \Longleftrightarrow \bigwedge_{i \in I} P_{i} \mu \text { a.e. }  \tag{12.9}\\
& \Longleftrightarrow \exists A \in \Sigma, \quad A \subset\left\{\bigwedge_{i \in I} P_{i}\right\}=\bigcap_{i \in I}\left\{P_{i}\right\} \quad \wedge \quad \mu\left(A^{c}\right)=0 . \tag{12.10}
\end{align*}
$$

Proof. Direct consequence of Definition 953 (property almost satisfied), Lemma 952 ( $\mathbf{N}$ is closed under countable union), Lemma 951 (subset of negligible is negligible), and Lemma 955 (equivalent definition of property almost satisfied).

Remark 961.
The previous lemma allows to split a property satisfied almost everywhere into all its components (each satisfied almost everywhere) when it is a countable conjunction of subproperties.

Lemma 962 (almost everywhere for almost the same). Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $P$ be a predicate on $Y$, lifted into a predicate on $Y^{X}$. Then, for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}, g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
P(f) \mu \text { a.e. } \wedge f^{\mu} \stackrel{\text { a.e. }}{=} g \Longrightarrow P(g) \mu \text { a.e. } \tag{12.11}
\end{equation*}
$$

In particular, we have $(\forall x \in X, P(f(x))) \wedge f \stackrel{\mu \text { a.e. }}{=} g \Rightarrow P(g) \mu$ a.e.

Proof. Direct consequence of the tautology $P(f) \wedge f=g \Rightarrow \boldsymbol{P}(g)$, Lemma 959 (almost modus ponens, with $P(x) \stackrel{\text { def. }}{=} P(f(x)) \wedge f(x)=g(x)$ ), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=} P$ and $P_{1} \stackrel{\text { def. }}{=}(f=g)$ ), and Lemma 958 (everywhere is almost everywhere).

## Lemma 963 (extended almost modus ponens).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $y \stackrel{\text { def. }}{=}\left(y_{i}\right)_{i \in I} \in Y$. Let $P$ and $Q$ be predicates on $Y$, lifted into predicates on $Y^{X}$. Let $R$ and $S$ be predicates on $Y^{X}$. Assume that $P(y)$ holds. Assume that $R$ and $S$ are compatible with almost equality ( $f \stackrel{\mu \text { a.e. }}{=} g \wedge R(f) \Rightarrow R(g)$, and the same for $S$ ). Assume that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}: X \rightarrow Y$, we have
(12.12) $(\forall x \in X, P(f(x))) \wedge R(f) \Longrightarrow Q(f) \mu$ a.e. $\wedge S(f)$.

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
P(g) \mu \text { a.e. } \wedge R(g) \Longrightarrow Q(g) \mu \text { a.e. } \wedge \quad S(g) \tag{12.13}
\end{equation*}
$$

Proof. Let $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$. Assume that $P(g) \mu$ a.e. and $R(g)$ hold.
Let $B \stackrel{\text { def. }}{=}\{P(g)\}$. Then, from Lemma 955 (equivalent definition of property almost satisfied), there exists $A \subset B$ such that $A, A^{c} \in \Sigma$, and $\mu\left(A^{c}\right)=0$.

For all $i \in I$, let $f_{i}(x) \stackrel{\text { def. }}{=} g_{i}(x)$ when $x \in A$, and let $f_{i}(x) \stackrel{\text { def. }}{=} y_{i}$ otherwise. Let $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}$. Then, by construction, $P(f)$ holds everywhere, and from Lemma 955 (equivalent definition of property almost satisfied, with $P \stackrel{\text { def. }}{=}(f=g)$ and $A \subset\{f=g\}$, thus $f \stackrel{\mu \text { a.e. }}{=} g$, we have $R(f)$.

Therefore, from (12.12), and Lemma 962 (almost everywhere for almost the same), we have $Q(f) \mu$ a.e. and $S(f)$, thus $Q(g) \mu$ a.e., and $S(g)$.

Remark 964. In the previous lemma, $P$ and $Q$ are pointwise predicates acting on $Y$, while $R$ and $S$ are "functionwise" predicates acting on functions $X \rightarrow Y$ (such as properties involving integrals when $Y_{i} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ ). Of course, any of them may be the always-true predicate, and thus disappears from the conjunctions.

## Lemma 965 (extended almost modus ponens (simplified 0)).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $y \stackrel{\text { def. }}{=}\left(y_{i}\right)_{i \in I} \in Y$. Let $P$ be a predicate on $Y$, lifted into predicates on $Y^{X}$. Let $S$ be a predicates on $Y^{X}$. Assume that $P(y)$ holds. Assume that $R$ is compatible with almost equality. Assume that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
(\forall x \in X, P(f(x))) \quad \Longrightarrow \quad S(f) \tag{12.14}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, we have $P(g) \mu$ a.e. $\Rightarrow S(g)$.

Proof. Direct consequence of Lemma 963 (extended almost modus ponens, with $Q=R \stackrel{\text { def. }}{=}$ True).

## Lemma 966 (extended almost modus ponens (simplified 1)).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $y \stackrel{\text { def. }}{=}\left(y_{i}\right)_{i \in I} \in Y$. Let $P$ and $Q$ be predicates on $Y$, lifted into predicates on $Y^{X}$. Assume that $P(y)$ holds. Assume that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
(\forall x \in X, P(f(x))) \quad \Longrightarrow \quad Q(f) \mu \text { a.e. } \tag{12.15}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, we have $P(g) \mu$ a.e. $\Rightarrow Q(g) \mu$ a.e.
Proof. Direct consequence of Lemma 963 (extended almost modus ponens, with $R=S \stackrel{\text { def. }}{=}$ True).

## Lemma 967 (extended almost modus ponens (simplified 2)).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $y \stackrel{\text { def. }}{=}\left(y_{i}\right)_{i \in I} \in Y$. Let $P$ and $Q$ be predicates on $Y$, lifted into predicates on $Y^{X}$. Assume that $P(y)$ holds. Assume that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
(\forall x \in X, P(f(x))) \quad \Longrightarrow \quad \forall x \in X, Q(f(x)) \tag{12.16}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, we have $P(g) \mu$ a.e. $\Rightarrow Q(g) \mu$ a.e.
Proof. Direct consequence of Lemma 966 (extended almost modus ponens (simplified 1)), and Lemma 958 (everywhere is almost everywhere, with $P \stackrel{\text { def. }}{=} Q$ ).

Remark 968. The four previous lemmas allow to still use modus ponens in reasoning when predicates are only valid almost everywhere.

## Definition 969 (almost definition).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. A function $f: X \rightarrow Y$ is said defined ( $\mu$-)almost everywhere in $X$ iff the property " $f(x)$ is defined" is satisfied $\mu$-almost everywhere.

The set of functions $X \rightarrow Y$ defined ( $\mu$-)almost everywhere is denoted $\left(Y^{X}\right)_{\mu \text { a.e. }}$, or through the type annotation $(X \rightarrow Y)_{\mu \text { a.e. }}$ (or simply without the mention of the measure $\mu$ ).

## Lemma 970 (extension to almost defined).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y$ be a nonempty set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $P$ be a predicate on $Y$, lifted into a predicate on $Y^{X}$. Then, we have

$$
\begin{equation*}
\left(\forall\left(f_{i}\right)_{i \in I}: X \rightarrow Y, P\left(f_{i}\right)_{i \in I}\right) \quad \Longrightarrow \quad \forall\left(g_{i}\right)_{i \in I}:(X \rightarrow Y)_{\mu \text { a.e. }}, P\left(g_{i}\right)_{i \in I} \mu \text { a.e. } \tag{12.17}
\end{equation*}
$$

Proof. Let $y \stackrel{\text { def. }}{=}\left(y_{i}\right)_{i \in I} \in Y$. Assume that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}: X \rightarrow Y, P(f)$ holds. Let $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}:(X \rightarrow Y)_{\mu \text { a.e. }}$.

Let $i \in I$. Let $B_{i} \stackrel{\text { def. }}{=}\left\{g_{i}\right.$ is defined $\}$. Let $f_{i}$ be the function defined by $f_{i}(x) \stackrel{\text { def. }}{=} g_{i}(x)$ when $x \in B_{i}$, and $f_{i}(x) \stackrel{\text { def. }}{=} y_{i}$ otherwise. Then, by construction, and from Definition 953 (property almost satisfied, with $\left.\left\{f_{i}=g_{i}\right\}=B_{i}\right), f_{i}$ is a total function on $X$, and we have $f_{i} \stackrel{\mu \text { a.e. }}{=} g_{i}$.

Therefore, from Lemma 960 (countable almost properties is almost property, with $P_{i}$ set to $\left(f_{i}=g_{i}\right)$, thus $f \stackrel{\mu \text { a.e. }}{=} g$ ), and Lemma 959 (almost modus ponens, with $P \stackrel{\text { def. }}{=}(f=g), Q \stackrel{\text { def. }}{=} P(g)$ and $(f=g) \Rightarrow P(g))$, we have $P(g) \mu$ a.e.

Remark 971. The previous lemma allows to replace a property on total functions by the same property satisfied almost everywhere for functions defined almost everywhere.

Lemma 972 (generalized almost modus ponens). Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. For all $i \in I$, let $Y_{i}$ be a nonempty set. Let $Y \stackrel{\text { def. }}{=} \prod_{i \in I} Y_{i}$. Let $P$ and $Q$ be predicates on $Y$, lifted into predicates on $Y^{X}$. Assume that for all $\left(f_{i}\right)_{i \in I}: X \rightarrow Y$, we have

$$
\begin{equation*}
\forall x \in X, \quad P\left(f_{i}(x)\right)_{i \in I} \quad \Longrightarrow \quad Q\left(f_{i}(x)\right)_{i \in I} \tag{12.18}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}:(X \rightarrow Y)_{\mu \text { a.e. }}$, we have

$$
\begin{equation*}
P(g) \mu \text { a.e. } \quad \Longrightarrow \quad Q(g) \mu \text { a.e. } \tag{12.19}
\end{equation*}
$$

Proof. Direct consequence of Lemma 970 (extension to almost defined), modus ponens ( $\boldsymbol{P}$ implies $Q$ and $P$, implies $Q$ ), Lemma 959 (almost modus ponens, with $P \stackrel{\text { def. }}{=}(P \Rightarrow Q) \wedge P)$, and Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=}(P \Rightarrow Q)$ and $\left.P_{1} \stackrel{\text { def. }}{=} P\right)$.

Remark 973. The previous lemma also allows to still use modus ponens in reasoning when predicates are only valid almost everywhere.

## Definition 974 (almost binary relation).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. The functions $f, g:(X \rightarrow Y)_{\mu \text { a.e. }}$ are said in relation ( $\mu$-) almost everywhere through $\mathcal{R}$ iff the property " $f(x) \mathcal{R} g(x)$ " is satisfied $\mu$-almost everywhere, i.e.

$$
\{x \in X \mid f(x) \mathcal{R} g(x)\}^{c} \in \mathbf{N} .
$$

This is denoted $f \mathcal{R}_{\mu \text { a.e. }} g$ (or simply $f \mathcal{R}_{\text {a.e. }} g$ ).

## Lemma 975 (compatibility of almost binary relation with reflexivity).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is reflexive. Then, $\mathcal{R}_{\mu \text { a.e. }}$ is also reflexive,

$$
\forall f:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad f \mathcal{R}_{\mu \text { a.e. }} f .
$$

Proof. Direct consequence of Definition 974 (almost binary relation), the definition of reflexivity, and Lemma 970 (extension to almost defined, with $\operatorname{card}(I)=1$ and $P(y) \stackrel{\text { def. }}{=} y \mathcal{R} y)$.

Remark 976. Note that reflexivity can also take a form symmetrical to that of antisymmetry:

$$
\forall x, y \in X, \quad x=y \quad \Longrightarrow \quad x \mathcal{R} y \quad \wedge \quad y \mathcal{R} x .
$$

This leads to the next lemma, which is actually stronger than the previous one.

Lemma 977 (compatibility of almost binary relation with extended reflexivity).
Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is reflexive.
Then, $\mathcal{R}_{\mu \text { a.e. }}$ is "almost" reflexive (where equality is replaced by almost equality), i.e.

$$
\forall f, g:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad f \stackrel{\mu \text { a.e. }}{=} g \Longrightarrow f \mathcal{R}_{\mu \text { a.e. }} g \wedge \quad g \mathcal{R}_{\mu \text { a.e. }} f .
$$

Proof. Direct consequence of the definition of reflexivity, Definition 974 (almost binary relation), Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} Y, P(f, g)$ is the predicate $(f=g)$ and $Q(f, g) \stackrel{\text { def. }}{=} f \mathcal{R} g \wedge g \mathcal{R} f$ ), and Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=} f \mathcal{R} g$ and $\left.P_{1} \stackrel{\text { def. }}{=} g \mathcal{R} f\right)$.

Lemma 978 (compatibility of almost binary relation with symmetry). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is symmetric. Then, $\mathcal{R}_{\mu \text { a.e. }}$ is also symmetric,

$$
\forall f, g:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad f \mathcal{R}_{\mu \text { a.e. }} g \quad \Longrightarrow \quad g \mathcal{R}_{\mu \text { a.e. }} f .
$$

Proof. Direct consequence of the definition of symmetry, Definition 974 (almost binary relation), and Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} Y$, $\underline{P(f, g)} \stackrel{\text { def. }}{=} f \mathcal{R} g$ and $Q(f, g) \stackrel{\text { def. }}{=} g \mathcal{R} f)$.

Lemma 979 (compatibility of almost binary relation with extended antisymmetry). Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is antisymmetric.
Then, $\mathcal{R}_{\mu \text { a.e. }}$ is "almost" antisymmetric (where equality is replaced by almost equality), i.e.

$$
\forall f, g:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad f \mathcal{R}_{\mu \text { a.e. }} g \quad \wedge \quad g \mathcal{R}_{\mu \text { a.e. }} f \Rightarrow \quad \Longrightarrow \stackrel{\mu \text { a.e. }}{=} g .
$$

Proof. Direct consequence of the definition of antisymmetry, Definition 974 (almost binary relation), Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} Y, P(f, g)$ is the predicate $f \mathcal{R} g \wedge g \mathcal{R} f$ and $Q(f, g) \stackrel{\text { def. }}{=}(f=g)$ ), and Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=} f \mathcal{R} g$ and $\left.P_{1} \stackrel{\text { def. }}{=} g \mathcal{R} f\right)$.

## Lemma 980 (compatibility of almost binary relation with transitivity).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is transitive. Then, $\mathcal{R}_{\mu \text { a.e. }}$ is also transitive,

$$
\forall f, g, h:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad f \mathcal{R}_{\mu \text { a.e. }} g \quad \wedge \quad g \mathcal{R}_{\mu \text { a.e. }} h \quad \Longrightarrow \quad f \mathcal{R}_{\mu \text { a.e. }} h .
$$

Proof. Direct consequence of the definition of transitivity, Definition 974 (almost binary relation), Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=}\{0,1,2\}, Y_{0}=Y_{1}=Y_{2} \stackrel{\text { def. }}{=} Y$, $\underline{P(f, g, h) \stackrel{\text { def. }}{=} f \mathcal{R} g \wedge g \mathcal{R} h \text { and } Q(f, g, h) \stackrel{\text { def. }}{=} f \mathcal{R} h \text { ), and Lemma } 960 \text { (countable almost proper- }}$ ties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=} f \mathcal{R} g$ and $\left.P_{1} \stackrel{\text { def. }}{=} g \mathcal{R} h\right)$.

Lemma 981 (almost equivalence is equivalence relation). Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is an equivalence relation. Then, $\mathcal{R}_{\mu \text { a.e. }}$ is also an equivalence relation.

Proof. Direct consequence of the definition of equivalence relation, Lemma 975 (compatibility of almost binary relation with reflexivity), Lemma 978 (compatibility of almost binary relation with symmetry), and Lemma 980 (compatibility of almost binary relation with transitivity).

Lemma 982 (almost equality is equivalence relation). Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Then, $\stackrel{\mu \text { a.e. }}{=}$ is an equivalence relation over $Y^{X}$.

Proof. Direct consequence of Lemma 981 (almost equivalence is equivalence relation, with equality over $Y$ ).

Lemma 983 (almost order is order relation).
Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a binary relation over $Y$, lifted into a binary relation over $Y^{X}$. Assume that $\mathcal{R}$ is an order relation. Then, $\mathcal{R}_{\mu \text { a.e. }}$ is an "almost" order relation (where equality is replaced by almost equality in antisymmetry).

Proof. Direct consequence of the definition of order relation, Lemma 975 (compatibility of almost binary relation with reflexivity), Lemma 979 (compatibility of almost binary relation with extended antisymmetry), and Lemma 980 (compatibility of almost binary relation with transitivity).

## Lemma 984 (compatibility of almost binary relation with operator).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $\mathcal{R}$ and $\mathcal{R}^{\prime}$ be binary relations over $Y$, lifted into binary relations over $Y^{X}$. Let $I \subset \mathbb{N}$ be nonempty. Let $\diamond: Y^{I} \rightarrow Y$, lifted into an operator $\left(Y^{X}\right)^{I} \rightarrow Y^{X}$. Assume that for all $\left(f_{i}\right)_{i \in I},\left(g_{i}\right)_{i \in I}: X \rightarrow Y$, for all $x \in X$, we have

$$
\begin{equation*}
\left(\forall i \in I, f_{i}(x) \mathcal{R} g_{i}(x)\right) \quad \Longrightarrow \quad \diamond\left(f_{i}(x)\right)_{i \in I} \mathcal{R}^{\prime} \diamond\left(g_{i}(x)\right)_{i \in I} . \tag{12.20}
\end{equation*}
$$

Then, for all $\left(f_{i}\right)_{i \in I},\left(g_{i}\right)_{i \in I}:(X \rightarrow Y)_{\mu \text { a.e., }}$, we have

$$
\begin{equation*}
\left(\forall i \in I, f_{i} \mathcal{R}_{\mu \text { a.e. }} g_{i}\right) \quad \Longrightarrow \quad \diamond\left(f_{i}\right)_{i \in I} \mathcal{R}_{\mu \text { a.e. }}^{\prime} \diamond\left(g_{i}\right)_{i \in I} \tag{12.21}
\end{equation*}
$$

Proof. Direct consequence of Definition 974 (almost binary relation), Lemma 972 (generalized almost modus ponens, with $f_{i} \stackrel{\text { def. }}{=}\left(f_{i}, g_{i}\right), P \stackrel{\text { def. }}{=}\left(\forall i, f_{i} \mathcal{R} g_{i}\right)$ and $\left.Q \stackrel{\text { def. }}{=}\left(\diamond\left(f_{i}\right)_{i \in I} \mathcal{R}^{\prime} \diamond\left(g_{i}\right)_{i \in I}\right)\right)$, and Lemma 960 (countable almost properties is almost property, with $P_{i} \stackrel{\text { def. }}{=} f_{i} \mathcal{R} g_{i}$ ).

## Lemma 985 (compatibility of almost equality with operator).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a nonempty set. Let $I \subset \mathbb{N}$ be nonempty.
Let $\diamond: Y^{I} \rightarrow Y$, lifted into an operator $\left(Y^{X}\right)^{I} \rightarrow Y^{X}$. Let $\left(f_{i}\right)_{i \in I},\left(g_{i}\right)_{i \in I}:(X \rightarrow Y)_{\mu \text { a.e. }}$.
Assume that for all $i \in I, f_{i} \stackrel{\mu \text { a.e. }}{=} g_{i}$. Then, we have $\diamond\left(f_{i}\right)_{i \in I} \stackrel{\mu \text { a.e. }}{=} \diamond\left(g_{i}\right)_{i \in I}$.
Proof. Direct consequence of Lemma 984 (compatibility of almost binary relation with operator, with $\mathcal{R}, \mathcal{R}^{\prime} \stackrel{\text { def. }}{=}$ equality), and the definition of function ((for all $i \in I, f_{i}=g_{i}$ ) implies $\widehat{\left.\diamond\left(f_{i}\right)_{i \in I}=\diamond\left(g_{i}\right)_{i \in I}\right) .}$

Remark 986 . The previous generic results apply to unary operators $\diamond$ such as the scalar multiplication, or the absolute value (for which $\operatorname{card}(I)=1$ ), as well as to binary (or $n$-ary) operators $\diamond$ such as the addition (when defined), the multiplication, the maximum, or the minimum (for which $\operatorname{card}(I) \geqslant 2$ is finite). But it also applies to operators $\diamond$ taking a countable number of arguments such as the infimum, the supremum, or the limit (assuming that the sequence is pointwise convergent).

Remark 987 . Note that Lemma 984 may be useful with $\mathcal{R}^{\prime}$ distinct from $\mathcal{R}$. For instance, when $\mathcal{R} \stackrel{\text { def. }}{=} \leqslant$ and $\diamond$ is the scalar multiplication by a negative number, we need $\mathcal{R}^{\prime}=\geqslant$.

## Lemma 988 (definiteness is almost definiteness).

Let $(X, \Sigma, \mu)$ be a measure space. Let $Y$ be a set. Assume that $0 \in Y$.
Let $\diamond: Y \rightarrow Y$, lifted into an operator $Y^{X} \rightarrow Y^{X}$. Assume that $\diamond$ is definite.
Then, $\diamond$ is also "almost definite" (where equality is replaced by almost equality), i.e.
(12.22)

$$
\forall f:(X \rightarrow Y)_{\mu \text { a.e. }}, \quad \diamond(f) \stackrel{\mu \text { a.e. }}{=} 0 \quad \Longrightarrow \quad f \stackrel{\mu \text { a.e. }}{=} 0 .
$$

Proof. Direct consequence of Lemma 972 (generalized almost modus ponens, with $\operatorname{card}(I)=1$, $P(f) \stackrel{\text { def. }}{=}(\diamond(f)=0)$ and $Q(f) \stackrel{\text { def. }}{=}(f=0))$.

Remark 989. Note that Lemma 985 provides the implication in the other direction.

### 12.4 Uniqueness condition

Remark 990. The next lemma is useful to establish in Section 13.3 uniqueness of the Lebesgue measure that generalizes the length of bounded intervals (see Theorem 1090).

Remark 991. The next proof follows the Dynkin $\pi-\lambda$ theorem scheme (see Section 4.2).

## Lemma 992 (uniqueness of measures extended from a $\pi$-system).

Let $(X, \Sigma)$ be a measurable space. Let $\mu_{1}$ and $\mu_{2}$ be measures on $(X, \Sigma)$. Let $G \subset \mathcal{P}(X)$. Assume that $G$ is a $\pi$-system, a generator of $\Sigma$, contains a countable pseudopartition $\left(X_{n}\right)_{n \in \mathbb{N}}$ of $X$ with finite measure $\mu_{1}$, and that both measures coincide on $G$. Then, we have $\mu_{1}=\mu_{2}$.

Proof. Let $A \in G$. Assume that $\mu_{1}(A)<\infty\left(\right.$ e.g. $\left.A \stackrel{\text { def. }}{=} X_{0}\right)$.
Let $B \in \Sigma$. Then, from Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, $A \in G \subset \Sigma$ ), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection with card $(I)$ equals 2), we have $B \cap A \in \Sigma$. Let $\mathcal{S}_{A} \stackrel{\text { def. }}{=}\left\{B \in \Sigma \mid \mu_{1}(B \cap A)=\mu_{2}(B \cap A)\right\}$.
(1). $G \neq \varnothing \wedge \Pi_{X}(G) \subset \mathcal{S}_{A}$.

Let $B \in G$. Then, from Definition 602 ( $\pi$-system, closedness under intersection), $B \cap A \in G$, i.e. $B \in \mathcal{S}_{A}$. Thus, we have $G \subset \mathcal{S}_{A}$. Hence, from Definition 602 ( $\pi$-system, nonemptiness), and Lemma 608 (properties of $\pi$-system generation, idempotence), we have $G \neq \varnothing$ and $\Pi_{X}(G)=G \subset$ $\mathcal{S}_{A}$.

## (2). $\mathcal{S}_{A}$ is $\lambda$-system.

From the identity $\boldsymbol{X} \cap \boldsymbol{A}=\boldsymbol{A}$, and Lemma 697 (equivalent definition of $\sigma$-algebra, contains full set), we have $X \in \mathcal{S}_{A}$.

Let $B_{1}, B_{2} \in \mathcal{S}_{A}$. Assume that $B_{1} \subset B_{2}$. Then, from distributivity of intersection over local complement, and Lemma 919 (measure is monotone, with $\mu_{1}$ and $\mu_{2}$ ), we have

$$
\begin{gathered}
\mu_{2}\left(B_{1} \cap A\right)=\mu_{1}\left(B_{1} \cap A\right) \leqslant \mu_{1}(A)<\infty, \quad \text { and } \\
\mu_{1}\left(\left(B_{2} \backslash B_{1}\right) \cap A\right)=\mu_{1}\left(B_{2} \cap A \backslash B_{1} \cap A\right)=\mu_{1}\left(B_{2} \cap A\right)-\mu_{1}\left(B_{1} \cap A\right) \\
=\mu_{2}\left(B_{2} \cap A\right)-\mu_{2}\left(B_{1} \cap A\right)=\mu_{2}\left(B_{2} \cap A \backslash B_{1} \cap A\right)=\mu_{2}\left(\left(B_{2} \backslash B_{1}\right) \cap A\right) .
\end{gathered}
$$

Then, $B_{2} \backslash B_{1} \in \mathcal{S}_{A}$. Thus, $\mathcal{S}_{A}$ is closed under local complement.
Let $\left(B_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}_{A}$. Assume that for all $n \in \mathbb{N}, B_{n} \subset B_{n+1}$. Then, from distributivity of intersection over union, Lemma 925 (measure is continuous from below, with $\mu_{1}$ and $\mu_{2}$, and nondecreasing $\left(B_{n} \cap A\right)_{n \in \mathbb{N}}$ ), and Definition 879 (continuity from below), we have

$$
\begin{aligned}
\mu_{1}\left(\bigcup_{n \in \mathbb{N}} B_{n} \cap A\right)=\mu_{1}\left(\bigcup_{n \in \mathbb{N}}\right. & \left.\left(B_{n} \cap A\right)\right)=\lim _{n \rightarrow \infty} \mu_{1}\left(B_{n} \cap A\right) \\
& =\lim _{n \rightarrow \infty} \mu_{2}\left(B_{n} \cap A\right)=\mu_{2}\left(\bigcup_{n \in \mathbb{N}}\left(B_{n} \cap A\right)\right)=\mu_{2}\left(\bigcup_{n \in \mathbb{N}} B_{n} \cap A\right) .
\end{aligned}
$$

Then, $\bigcup_{n \in \mathbb{N}} B_{n} \in \mathcal{S}_{A}$. Thus, $\mathcal{S}_{A}$ is closed under countable monotone union.
Hence, from Lemma 678 (equivalent definition of $\lambda$-system), $\mathcal{S}_{A}$ is a $\lambda$-system on $X$.
(3). $\mathcal{S}_{\boldsymbol{A}}=\boldsymbol{\Sigma}$. Direct consequence of (1), (2), and Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem, with $P$ being the function $\left.B \mapsto \mu_{1}(B \cap A)=\mu_{2}(B \cap A)\right)$.
Hence, for all $B \in \Sigma$, for all $A \in G$ such that $\mu_{1}(A)<\infty$, we have $\mu_{1}(B \cap A)=\mu_{2}(B \cap A)$.
(4). $\mu_{1}=\mu_{2}$. Let $B \in \Sigma$.

For all $n \in \mathbb{N}$, let $B_{n} \stackrel{\text { def. }}{=} B \cap X_{n}$. Then, from Definition 209 (pseudopartition), Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, $\underline{X}_{n} \in G \subset \Sigma$ ), Lemma 697 (equivalent definition of $\sigma$-algebra,
closedness under countable intersection with $\operatorname{card}(I)$ equals 2), compatibility of intersection with pairwise disjunction, left distributivity of intersection over union, and the identity law for intersection, $\left(B_{n}\right)_{n \in \mathbb{N}} \in \Sigma$ is pairwise disjoint, and we have $\biguplus_{n \in \mathbb{N}} B_{n}=B$.
Let $n \in \mathbb{N}$. Then, from (3) (with $B \in \Sigma$, and $A \stackrel{\text { def. }}{=} X_{n} \in G$ such that $\mu_{1}\left(X_{n}\right)<\infty$ ), we have $\mu_{1}\left(B_{n}\right)=\mu_{2}\left(B_{n}\right)$. Hence, from Lemma 913 (measure, $\sigma$-additivity with $\mu_{1}$ and $\mu_{2}$ ), and Definition 874 ( $\sigma$-additivity), we have

$$
\mu_{1}(B)=\sum_{n \in \mathbb{N}} \mu_{1}\left(B_{n}\right)=\sum_{n \in \mathbb{N}} \mu_{2}\left(B_{n}\right)=\mu_{2}(B)
$$

Therefore, we have the equality $\mu_{1}=\mu_{2}$.

## Chapter 13

## Measure and numbers

## Contents

13.1 Negligibility and numbers ..... 212
13.2 Essential supremum ..... 218
13.2.1 Essential supremum ..... 218
13.2.2 Nonnegative essential supremum for nonnegative functions ..... 221
13.3 The Lebesgue measure ..... 227

### 13.1 Negligibility and numbers

Lemma 993 (transitivity of almost nonnegativeness).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}$. Then, we have

$$
\begin{equation*}
f^{\mu \text { a.e. }} \stackrel{\wedge}{\leqslant} f^{\mu \text { a.e. }} \geqslant 0 \Rightarrow g^{\mu \text { a.e. }} \geqslant 0 \tag{13.1}
\end{equation*}
$$

Proof. Direct consequence of Lemma 980 (compatibility of almost binary relation with transitivity, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, f \stackrel{\text { def. }}{=}(x \mapsto 0), g \stackrel{\text { def. }}{=} f$ and $h \stackrel{\text { def. }}{=} g)$.

## Lemma 994 (transitivity of almost inequality and equality).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g, h:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}$. Then, we have

$$
\begin{align*}
f \stackrel{\mu \text { a.e. }}{\leqslant} g \stackrel{\mu \text { a.e. }}{=} h \quad \vee \quad f \stackrel{\mu \text { a.e. }}{=} g \stackrel{\mu \text { a.e. }}{\leqslant} h & \Longrightarrow f \stackrel{\mu \text { a.e. }}{\leqslant} h,  \tag{13.2}\\
f \leqslant g \stackrel{\mu \text { a.e. }}{=} h \quad \vee \quad f^{\mu} \stackrel{\text { a.e. }}{=} g \leqslant h & \Longrightarrow f^{\mu \text { a.e. }} \stackrel{\Rightarrow}{\leqslant} h . \tag{13.3}
\end{align*}
$$

Proof. Property (13.2). Direct consequence of Lemma 977 (compatibility of almost binary relation with extended reflexivity, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ and $\mathcal{R} \stackrel{\text { def. }}{=} \leqslant$, thus $f \stackrel{\mu \text { a.e. }}{\leqslant} g \stackrel{\mu \text { a.e. }}{\leqslant} h$, and Lemma 980 (compatibility of almost binary relation with transitivity, with $\mathcal{R} \stackrel{\text { def. }}{=} \leqslant$ ).
Property (13.3). Direct consequence of (13.2), and Lemma 958 (everywhere is almost everywhere).

Lemma 995 (compatibility of almost nonnegativeness with almost equality).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}$. Then, we have
(13.4)

$$
f^{\mu} \stackrel{\text { a.e. }}{=} g \quad \Longrightarrow \quad(f \stackrel{\mu \text { a.e. }}{\geqslant} 0 \quad \Longleftrightarrow \quad g \stackrel{\mu \text { a.e. }}{\geqslant} 0) .
$$

Proof. Direct consequence of Lemma 977 (compatibility of almost binary relation with extended reflexivity, thus $f \stackrel{\mu \text { a.e. }}{\leqslant} g$ and $g \stackrel{\mu \text { a.e. }}{\leqslant} f$ ), and Lemma 993 (transitivity of almost nonnegativeness, with $f, g$, then with $f \stackrel{\text { def. }}{=} g$ and $g \stackrel{\text { def. }}{=} f)$.

## Lemma 996 (almost finite).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, we have $|f| \stackrel{\mu \text { a.e. }}{<} \infty$ iff $\mu(\{f= \pm \infty\})=0$.
Proof. Direct consequence of Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$, thus $\{|f|<\infty\}=\{f \neq \pm \infty$ ), Lemma 808 (inverse image of countable is measurable, thus $\{f= \pm \infty\} \in \Sigma$ ), and Lemma 957 (measurable property almost satisfied).

## Definition 997 (summability domain).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g: X \rightarrow \overline{\mathbb{R}}$. The summability domain of $f$ and $g$ is denoted $\mathcal{D}^{+}(f, g)$; it is defined by

$$
\begin{equation*}
\mathcal{D}^{+}(f, g) \stackrel{\text { def. }}{=}[(\{f=\infty\} \cap\{g=-\infty\}) \cup(\{f=-\infty\} \cap\{g=\infty\})]^{c} \tag{13.5}
\end{equation*}
$$

Lemma 998 (summability on summability domain). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g: X \rightarrow \overline{\mathbb{R}}$. Then, $f(x)+g(x)$ is well-defined iff $x \in \mathcal{D}^{+}(f, g)$.

Proof. Direct consequence of Definition 317 (addition in $\overline{\mathbb{R}}$, rule 3 cannot occur), and Definition 997 (summability domain).

## Lemma 999 (measurability of summability domain).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}$. Then, $\mathcal{D}^{+}(f, g) \in \Sigma$.

Proof. Direct consequence of Lemma 809 (inverse image is measurable, with $f, g$, and $y \stackrel{\text { def. }}{=} \pm \infty$ ), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection, union and complement).

Lemma 1000 (negligibility and summability domain). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g: X \rightarrow \overline{\mathbb{R}}$. Then, $f+g$ is well-defined almost everywhere iff $\left[\mathcal{D}^{+}(f, g)\right]^{c} \in \mathbf{N}$.

Proof. Direct consequence of Lemma 998 (summability on summability domain), and Definition 953 (property almost satisfied).

Lemma 1001 (absolute value and almost inequality).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g, h:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}$. Then, we have

$$
\begin{equation*}
|f| \stackrel{\mu \text { a.e. }}{\leqslant} g \quad \Longleftrightarrow \quad-g \stackrel{\mu \text { a.e. }}{\leqslant} f \stackrel{\mu \text { a.e. }}{\leqslant} g . \tag{13.6}
\end{equation*}
$$

If $\mathcal{D}^{+}(g, h)^{c} \in \mathbf{N}$, then we have

$$
\begin{equation*}
|f| \stackrel{\mu \text { a.e. }}{\leqslant}|g+h| \quad \Longrightarrow|f| \stackrel{\mu \text { a.e. }}{\leqslant}|g|+|h|, \tag{13.7}
\end{equation*}
$$

and if $\mathcal{D}^{+}(f, \pm g)^{c}, \mathcal{D}^{+}(h, \pm g)^{c} \in \mathbf{N}$, then we have

$$
\begin{equation*}
|f-g| \stackrel{\mu \text { a.e. }}{\leqslant} h \Longrightarrow||f|-|g|| \stackrel{\mu \text { a.e. }}{\leqslant} h \quad \Longleftrightarrow \quad-h+|g| \stackrel{\mu \text { a.e. }}{\leqslant}|f| \stackrel{\mu \text { a.e. }}{\leqslant} h+|g| \text {. } \tag{13.8}
\end{equation*}
$$

Proof. Property (13.6). Direct consequence of Lemma 972 (generalized almost modus ponens, with $\operatorname{card}(I)=2$ ), and Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$, thus $|f| \leqslant g \Leftrightarrow-g \leqslant f \leqslant g$ ).
Property (13.7). Direct consequence of Lemma 1000 (negligibility and summability domain), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=}(g+h$ welldefined) and $P_{1} \stackrel{\text { def. }}{=}|f| \leqslant|g+h|$ ), Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity, thus $P_{0} \wedge P_{1} \Rightarrow|f| \leqslant|g+h|$ ), and Lemma 972 (generalized almost modus ponens, with $\operatorname{card}(I)=3$ and $\left.P \stackrel{\text { def. }}{=} P_{0} \wedge P_{1}\right)$.
Property (13.8). Direct consequence of Lemma 1000 (negligibility and summability domain), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}[0 . .4], P_{0} \stackrel{\text { def. }}{=}(f+g$ welldefined $), P_{1} \stackrel{\text { def. }}{=}(f-g$ well-defined $), P_{2} \stackrel{\text { def. }}{=}(h+g$ well-defined $), P_{3} \stackrel{\text { def. }}{=}(h-g$ well-defined $)$ and $P_{4} \stackrel{\text { def. }}{=}|f-g| \leqslant h$ ), Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality), Lemma 983 (almost order is order relation, transitivity), Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ), and Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined, thus $P_{0} \wedge P_{1} \wedge P_{2} \wedge P_{3} \wedge P_{4}$ implies $||f|-|g|| \leqslant h \Leftrightarrow-h+|g| \leqslant|f| \leqslant h+|g|)$, and Lemma 972 (generalized almost modus ponens, with $\operatorname{card}(I)=3$ ),
Therefore, all properties hold.
Lemma 1002 (almost sum).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $f, g \in \mathcal{M}$. Let $A \stackrel{\text { def. }}{=} \mathcal{D}^{+}(f, g)$. Let $\tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A}$ and $\tilde{g} \stackrel{\text { def. }}{=} g \mathbb{1}_{A}$. Assume that $f+g$ is welldefined almost everywhere. Then, we have $\tilde{f} \stackrel{\mu}{\mu} \stackrel{\text { a.e. }}{=} f, \tilde{g} \stackrel{\mu \text { a.e. }}{=} g, \mathcal{D}^{+}(\tilde{f}, \tilde{g})=X$, and $\tilde{f}, \tilde{g}, \tilde{f}+\tilde{g} \in \mathcal{M}$.

The expression $f \stackrel{\mu \text { a.e. }}{+} g \stackrel{\text { def. }}{=} \tilde{f}+\tilde{g}$ is called the almost sum of $f$ and $g$.
Proof. From Lemma 1000 (negligibility and summability domain), and Definition 953 (property almost satisfied, with $\left.\{\tilde{f} \neq f\}=\{\tilde{g} \neq g\}=A^{c}\right)$, we have $\tilde{f} \stackrel{\mu \text { a.e. }}{=} f$ and $\tilde{g} \stackrel{\mu \text { a.e. }}{=} g$. From Lemma 999 (measurability of summability domain), Lemma 811 (measurability of indicator function), and Lemma $824(\mathcal{M}$ is closed under multiplication), we have $\tilde{f}, \tilde{g} \in \mathcal{M}$. Hence, from Definition 317 (addition in $\overline{\mathbb{R}}$ ), and Lemma $822(\mathcal{M}$ is closed under addition when defined), $\tilde{f}+\tilde{g}$ is well-defined, and we have $\tilde{f}+\tilde{g} \in \mathcal{M}$.

## Lemma 1003 (compatibility of almost sum with almost equality).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g, f^{\prime}, g^{\prime} \in \mathcal{M}$. Assume that $f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime} \stackrel{\mu \text { a.e. }}{=} g$, and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$. Then, $f^{\prime}+g^{\prime} \in \mathcal{M}, f+g$ is well-defined almost everywhere, $f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and we have $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f^{\mu \text { a.e. }}+g$.

Proof. From Lemma 998 (summability on summability domain), and Lemma 822 ( $\mathcal{M}$ is closed under addition when defined), $f^{\prime}+g^{\prime}$ is well-defined, and we have $f^{\prime}+g^{\prime} \in \mathcal{M}$.

From Lemma 998 (summability on summability domain), and Lemma 962 (almost everywhere for almost the same, with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, f \stackrel{\text { def. }}{=}\left(f^{\prime}, g^{\prime}\right), P\left(f^{\prime}, g^{\prime}\right) \stackrel{\text { def. }}{=}\left(f^{\prime}+g^{\prime}\right.$ is well-defined) and $g \stackrel{\text { def. }}{=}(f, g)), f+g$ is well-defined almost everywhere.

Let $C \stackrel{\text { def. }}{=} \mathcal{D}^{+}(f, g), \tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{C}$ and $\tilde{g} \stackrel{\text { def. }}{=} g \mathbb{1}_{C}$. Then, from Lemma 1002 (almost sum), Lemma 982 (almost equality is equivalence relation, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$, transitivity and reflexivity), we have $f^{\prime \mu} \stackrel{\text { a.e. }}{=} \tilde{f}, g^{\prime} \stackrel{\mu \text { a.e. }}{=} \tilde{g}$, and $f \stackrel{\mu \text { a.e. }}{+} g=\tilde{f}+\tilde{g}$ is well-defined, and belongs to $\mathcal{M}$.

Therefore, from Lemma 985 (compatibility of almost equality with operator, with the binary


## Lemma 1004 (almost sum is sum).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}$. Assume that $\mathcal{D}^{+}(f, g)=X$.
Then, $f+g \in \mathcal{M}$, and we have $f+\quad{ }^{\mu \text { a.e. }}+f+g$. In particular, this is the case when $f, g \in \mathcal{M}_{\mathbb{R}}$.
Proof. Direct consequence of Definition 997 (summability domain, $\underline{f+g}$ is well-defined everywhere), Lemma 1002 (almost sum, with $\tilde{f}=f$ and $\tilde{g}=g)$, Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R}$, thus $f+g$ is well-defined everywhere), and Lemma 998 (summability on summability domain, thus $\left.\mathcal{D}^{+}(f, g)=X\right)$.

Lemma 1005 (absolute value is almost definite). Let $(X, \Sigma, \mu)$ be a measure space. Then, the absolute value, lifted into an operator $\overline{\mathbb{R}}^{X} \rightarrow \overline{\mathbb{R}}^{X}$, is almost definite, i.e.

$$
\begin{equation*}
\forall f:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}, \quad|f| \stackrel{\mu \text { a.e. }}{=} 0 \quad \Longleftrightarrow \quad f \stackrel{\mu \text { a.e. }}{=} 0 . \tag{13.9}
\end{equation*}
$$

Proof. Direct consequence of Lemma 988 (definiteness is almost definiteness, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ and the unary operator absolute value), Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite), and Lemma 985 (compatibility of almost equality with operator, with $Y \stackrel{\text { def. }}{=} \mathbb{\mathbb { R }}, I \stackrel{\text { def. }}{=}\{0\}$, the unary operator absolute value and $\left.g_{0} \stackrel{\text { def. }}{=} 0\right)$.

Lemma 1006 (absolute value is almost equal to almost nonnegative function).
Let $(X, \Sigma, \mu)$ be a measure space. Then, we have

$$
\begin{equation*}
\forall f:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}, \quad|f| \stackrel{\mu \text { a.e. }}{=} f \Longleftrightarrow f^{\mu \text { a.e. }} \geqslant 0 \tag{13.10}
\end{equation*}
$$

Proof. Direct consequence of Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative), and Lemma 972 (generalized almost modus ponens, with $\operatorname{card}(I)=1$ and $P, Q$ being $(a \mapsto|a|=a)$ or $(a \mapsto a \geqslant 0)$ ).

## Lemma 1007 (power function is almost definite).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty)$.
Then, the power function of degree $p$, lifted into an operator $\overline{\mathbb{R}}^{X} \rightarrow \overline{\mathbb{R}}^{X}$, is almost definite, i.e.

$$
\begin{equation*}
\forall f:(X \rightarrow \overline{\mathbb{R}})_{\mu \text { a.e. }}, \quad|f|^{p} \stackrel{\mu \text { a.e. }}{=} 0 \Longleftrightarrow f \stackrel{\mu \text { a.e. }}{=} 0 \tag{13.11}
\end{equation*}
$$

Proof. Direct consequence of Lemma 988 (definiteness is almost definiteness, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ and the unary operator power function of degree p), Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite), and Lemma 985 (compatibility of almost equality with operator, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, I \stackrel{\text { def. }}{=}\{0\}$, the unary operator power function of degree $p$ and $g_{0} \stackrel{\text { def. }}{=} 0$ ).
Lemma 1008 (masking almost nowhere). Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $A \in \Sigma$. Assume that $\mu\left(A^{c}\right)=0$. Then, we have $f \stackrel{\mu \text { a.e. }}{=} f \mathbb{1}_{A}$.
Proof. Direct consequence of Lemma 913 (measure), Definition 741 (measurable space, $\underline{\Sigma}$ is a $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, closedness under complement), the definition of the indicator function $\left(A \subset\left\{f=f \overline{\left.\mathbb{1}_{A}\right\}}\right)\right.$, monotonicity of complement $\left(\left\{f=f \mathbb{1}_{A}\right\}^{c}\right.$ is included in $A^{c}$ ), Definition 942 (negligible subset). and Definition 953 (property almost satisfied).

## Lemma 1009 (finite nonnegative part).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $A \in \Sigma$. Assume that $0 \stackrel{\mu \text { a.e. }}{\leqslant} f \stackrel{\mu \text { a.e. }}{<} \infty$, and $\mu\left(A^{c}\right)=0$. Let $A_{f} \stackrel{\text { def. }}{=} A \cap f^{-1}\left(\mathbb{R}_{+}\right)$. Let $\tilde{f}_{A} \stackrel{\text { def. }}{=} f \mathbb{1}_{A_{f}}$.
Then, $A_{f} \in \Sigma, \tilde{f}_{A} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$, and we have $\mu\left(A_{f}^{c}\right)=0$, i.e. $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}_{A}$.
Proof. From Lemma $839\left(\mathcal{M}\right.$ is closed under nonnegative finite part), we have $A_{f} \in \Sigma$ and $\tilde{f}_{A}$ belongs to $\mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$.

Assume that $0 \leqslant f<\infty$, i.e. $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$. Then, since $f^{-1}\left(\mathbb{R}_{+}\right)=X$, we have $A_{f}=A$, and from Lemma 955 (equivalent definition of property almost satisfied, with $A_{f} \subset\left\{f=\tilde{f}_{A}\right\}$ and $\left.\mu\left(A_{f}^{c}\right)=\mu\left(A^{c}\right)=0\right)$, we have $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}_{A}$.

Therefore, from Lemma 966 (extended almost modus ponens (simplified 1), with $\operatorname{card}(I)=1$, $P(f) \stackrel{\text { def. }}{=}(0 \leqslant f<\infty)$ and $\left.Q(f) \stackrel{\text { def. }}{=}\left(f=\tilde{f}_{A}\right)\right)$, Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0}(f) \stackrel{\text { def. }}{=}(0 \leqslant f)$ and $P_{1}(f) \stackrel{\text { def. }}{=}(f<\infty)$ ), and Lemma 955 (equivalent definition of property almost satisfied), we have $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}_{A}$, i.e. $\mu\left(A_{f}^{c}\right)=0$.

Lemma 1010 (compatibility of almost limit with subsequence).
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f: X \rightarrow \overline{\mathbb{R}} . \operatorname{Let}\left(k_{n}\right)_{n \in \mathbb{N}} \in \mathbb{N}$ be increasing. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\text { a.e. }}{\rightarrow}} f$. Then, we have $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$.
Proof. Direct consequence of Lemma 266 (compatibility of subsequence with convergence, thus $f_{n} \longrightarrow f$ implies $f_{k_{n}} \longrightarrow f$ ), and Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=} \mathbb{N}$,


## Lemma 1011 (compatibility of almost limit with continuous function).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f: X \rightarrow \overline{\mathbb{R}}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f$. For $\mu$-almost all $x \in X$, let $g_{x}: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ be a continuous function. Then, $\left(x \mapsto g_{x}\left(f_{n}(x)\right)\right) \underset{n \rightarrow \infty}{\substack{\mu \text { a.e. }}}\left(x \mapsto g_{x}(f(x))\right)$.

Proof. Let $x \in X$. Assume that $\lim _{n \rightarrow \infty} f_{n}(x)=f(x)$, and $g_{x}$ is continuous. Then, from Lemma 45 (compatibility of limit with continuous functions), we have $\lim g_{x}\left(f_{n}(x)\right)=g_{x}(f(x))$.

Therefore, from Lemma 517 (limit inferior and limit superior of pointwise convergent), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1,2\}, P_{0}(x)$ is the predicate $\left(\liminf _{n \rightarrow \infty} f_{n}(x)=f(x)\right), P_{1}(x) \stackrel{\text { def. }}{=}\left(\limsup _{n \rightarrow \infty} f_{n}(x)=f(x)\right)$ and $P_{2}(x) \stackrel{\text { def. }}{=}\left(g_{x}\right.$ continuous $\left.)\right)$, Lemma 972 (generalized almost modus ponens, with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0} \stackrel{\text { def. }}{=} \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}, Y_{1} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, f_{0} \stackrel{\text { def. }}{=} f$, $\left.f_{1}=g \stackrel{\text { def. }}{=}\left(x \mapsto g_{x}\right), \quad P(f, g) \stackrel{\text { def. }}{=} P_{0} \wedge P_{1} \wedge P_{2}, Q(f, g) \stackrel{\text { def. }}{=}\left(\lim _{n \rightarrow \infty} g_{x}\left(f_{n}(x)\right)=g_{x}(f(x))\right)\right)$, and Lemma 958 (everywhere is almost everywhere, thus we can have $f$ defined everywhere), we have $\lim _{n \rightarrow \infty} g_{x}\left(f_{n}(x)\right) \stackrel{\mu \text { a.e. }}{=} g_{x}(f(x))$.

Lemma 1012 (compatibility of almost limit with power function). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f: X \rightarrow \overline{\mathbb{R}}$. Let $g: X \rightarrow \mathbb{R}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f$. Then, for all $p \in \mathbb{R}_{+}$and $n \in \mathbb{N}, f_{n}-g$, and $f-g$ are well defined, and $\left|f_{n}-g\right|^{p} \underset{n \rightarrow \infty}{\mu \text { a.e. }}|f-g|^{p}$.

Proof. Direct consequence of Definition 317 (addition in $\overline{\mathbb{R}}$, rule 3 cannot occur, thus for all $n \in \mathbb{N}$ $f_{n}-g$ and $f-g$ are well defined), continuity of translation (with $\tau_{x}(z) \stackrel{\text { def. }}{=} z-g(x)$ ), Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory), $\varphi(y) \stackrel{\text { def. }}{=}|y|^{p}$ is continuous), compatibility of composition with continuity, and Lemma 1011 (compatibility of almost limit with continuous function, with $\left.g(x, z) \stackrel{\text { def. }}{=} \varphi \circ \tau_{x}(z)\right)$.

Lemma 1013 (almost limits are almost equal). Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f, g: X \rightarrow \overline{\mathbb{R}}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f, g$. Then, we have $f \stackrel{\mu \text { a.e. }}{=} g$.

Proof. Direct consequence of Lemma 517 (limit inferior and limit superior of pointwise convergent, thus limit is unique), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1,2,3\}, P_{0} \stackrel{\text { def. }}{=}\left(\liminf _{n \rightarrow \infty} f_{n}=f\right), P_{1} \stackrel{\text { def. }}{=}\left(\limsup _{n \rightarrow \infty} f_{n}=f\right), P_{2}$ is the predicate $\left(\lim _{\inf }^{n \rightarrow \infty} f_{n}=g\right)$ and $P_{3} \stackrel{\text { def. }}{=}\left(\limsup _{n \rightarrow \infty} f_{n}=g\right)$ ), and Lemma 967 (extended almost modus ponens (simplified 2), with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, f_{0} \stackrel{\text { def. }}{=} f, f_{1} \stackrel{\text { def. }}{=} g$ and $P(f, g)$ is the predicate $\left.\left(P_{0} \wedge P_{1} \wedge P_{2} \wedge P_{3}\right)\right)$.

Lemma 1014 (compatibility of almost limit with almost equality).
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}},\left(g_{n}\right)_{n \in \mathbb{N}}, f, g: X \rightarrow \overline{\mathbb{R}}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f$, $f \stackrel{\mu \text { a.e. }}{=} g$, and there exists $N \in \mathbb{N}$ such that for all $n \geqslant N, f_{n} \stackrel{\mu \text { a.e. }}{=} g_{n}$. Then, we have $g_{n} \xrightarrow[n \rightarrow \infty]{\substack{\mu \text { a.e. }}} g$. In particular, the last assumption holds when $\left(g_{n}\right)_{n \in \mathbb{N}}=\left(f_{n}\right)_{n \in \mathbb{N}}$. Then, we have $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} g$.

Proof. Direct consequence of Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=} N, P_{0} \stackrel{\text { def. }}{=}(f=g)$ and $\left.P_{n+1} \stackrel{\text { def. }}{=}\left(f_{n}=g_{n}\right)\right)$, Lemma 962 (almost everywhere for almost the same, with $I \stackrel{\text { def. }}{=} \mathbb{N}$ and $P\left(f,\left(f_{n}\right)_{n \in \mathbb{N}}\right) \stackrel{\text { def. }}{=}\left(\lim _{n \rightarrow \infty} f_{n}=f\right)$ ), and Lemma 958 (everywhere is almost everywhere).

## Lemma 1015 (compatibility of almost limit with almost inequality).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f, g: X \rightarrow \overline{\mathbb{R}}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f$, and there exists $N \in \mathbb{N}$ such that for all $n \geqslant N,\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, we have $|f| \stackrel{\mu \text { a.e. }}{\leqslant} g$.

Proof. Direct consequence of Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous), Lemma 45 (compatibility of limit with continuous functions, with the absolute value), Lemma 34 (stationary sequence is convergent, with stationary value $g$ ), Lemma 381 (limit is monotone, thus $P_{0} \stackrel{\text { def. }}{=}\left(\lim _{n \rightarrow \infty} f_{n}=f\right)$ and $P_{1} \stackrel{\text { def. }}{=}\left(\left|f_{n}\right| \leqslant g\right.$ from some rank) implies $|f| \leqslant g$ ), Lemma 967 (extended almost modus ponens (simplified 2), with $I \stackrel{\text { def. }}{=}\{0,1\}, Y_{0}=Y_{1} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}, f_{0} \stackrel{\text { def. }}{=} f, f_{1} \stackrel{\text { def. }}{=} g$ and $P(f, g) \stackrel{\text { def. }}{=}\left(P_{0} \wedge P_{1}\right)$ ), and Lemma 960 (countable almost properties is almost property, with $\operatorname{card}(I)=2$ ).

## Lemma 1016 (compatibility of almost limit with measurability).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f$. Then, there exists $\tilde{f} \in \mathcal{M}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} \tilde{f}$, and $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$.

Moreover, if $f$ is finite, then $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$, and if $f$ is nonnegative, then $\tilde{f} \in \mathcal{M}_{+}$.

Proof. Let $B \stackrel{\text { def. }}{=}\left\{f=\liminf _{n \rightarrow \infty} f_{n}\right\} \cap\left\{f=\lim \sup _{n \rightarrow \infty} f_{n}\right\}$. Then, from Lemma 517 (limit inferior and limit superior of pointwise convergent), Definition 953 (property almost satisfied, thus $B^{c} \in \mathbf{N}$ ), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}$, $P_{0} \stackrel{\text { def. }}{=}\left(f=\liminf _{n \rightarrow \infty} f_{n}\right)$ and $\left.P_{1} \stackrel{\text { def. }}{=}\left(f=\lim \sup _{n \rightarrow \infty} f_{n}\right)\right)$, there exists $A \in \Sigma$ such that $\mu\left(A^{c}\right)$ equals 0 , and $A \subset B$.

For all $n \in \mathbb{N}$, let $\tilde{f}_{n} \stackrel{\text { def. }}{=} f_{n} \mathbb{1}_{A}$, and let $\tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A}$. Let $x \in X$. Then, from the definition of indicator function, and Lemma 34 (stationary sequence is convergent, with stationary value 0 ), we have $\lim _{n \rightarrow \infty} \tilde{f}_{n}(x)=\lim _{n \rightarrow \infty} f_{n}(x)=f(x)=\tilde{f}(x)$ when $x \in A$, and $\lim _{n \rightarrow \infty} \tilde{f}_{n}(x)=0=\tilde{f}(x)$ otherwise. Hence, $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}$ is pointwise convergent towards $\tilde{f}$.

Let $n \in \mathbb{N}$. Then, from Lemma 833 (measurability and masking, with $\tilde{f}_{n}$ ), Lemma $832(\mathcal{M}$ is closed under limit when pointwise convergent), Lemma 1008 (masking almost nowhere), and Lemma 1014 (compatibility of almost limit with almost equality, with $g \stackrel{\text { def. }}{=} \tilde{f}$ ), we have $\tilde{f}_{n}, \tilde{f} \in \mathcal{M}$, $f^{\mu} \stackrel{\text { a.e. }}{=} \tilde{f}$, and $\left(f_{n}\right)_{n \in \mathbb{N}}$ is almost everywhere pointwise convergent towards $\tilde{f}$.

Moreover, from the definition of indicator function, Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ), and Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), if $f$ is finite, then $f$ is also finite and $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$, and if $f$ is nonnegative, then $\tilde{f}$ is also nonnegative and $\tilde{f} \in \mathcal{M}_{+}$.

### 13.2 Essential supremum

### 13.2.1 Essential supremum

Lemma 1017 (essential supremum).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, $B_{f} \stackrel{\text { def. }}{=}\{a \in \overline{\mathbb{R}} \mid \mu(\{f>a\})=0\}$, and $\operatorname{ess} \sup f \stackrel{\text { def. }}{=} \inf B_{f}$ are well-defined in $\overline{\mathbb{R}}$. Moreover, we have $\infty \in B_{f}$, and $a \in B_{f} \Rightarrow[a, \infty] \subset B_{f}$.
The extended number ess sup $f$ is called the essential supremum of $f$.
Proof. From Lemma 820 (measurability of numeric function), Lemma 913 (measure), and Lemma $314(\overline{\mathbb{R}}$ is complete $)$, the functions $\psi_{f} \stackrel{\text { def. }}{=}(a \mapsto\{f>a\}): \overline{\mathbb{R}} \rightarrow \Sigma$, and $\mu \circ \psi_{f}: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_{+}$are well-defined, $B_{f} \stackrel{\text { def. }}{=}\left(\mu \circ \psi_{f}\right)^{-1}(\{0\})$ is a well-defined subset of $\overline{\mathbb{R}}$, and $\inf B_{f}$ is well-defined in $\overline{\mathbb{R}}$.

From Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and Lemma 913 (measure, homogeneity), we have $\mu(\{f>\infty\})=\mu(\varnothing)=0$. Hence, we have $\infty \in B_{f}$.

Let $a \in B_{f}$, i.e. $\mu(\{f>a\})=0$. Let $b \in \overline{\mathbb{R}}$. Assume that $b>a$. Then, from monotonicity of inverse image (with $(b, \infty] \subset(a, \infty]$ ), we have $\{f>b\} \subset\{f>a\}$. Thus, from Lemma 913 (measure, nonnegativeness), and Lemma 919 (measure is monotone), we have

$$
0 \leqslant \mu(\{f>b\}) \leqslant \mu(\{f>a\})=0
$$

i.e. $b \in B_{f}$. Hence, we have $[a, \infty] \subset B_{f}$.

Remark 1018. Note that in the previous lemma, the essential supremum could be defined for any nonmeasurable function $f$ by specifying $B_{f} \stackrel{\text { def. }}{=}\{a \in \overline{\mathbb{R}} \mid\{f>a\} \in \mathbf{N}\}$.

Lemma 1019 (essential supremum with trivial measure). Let $(X, \Sigma, \mu)$ be a trivial measure space. Let $f \in \mathcal{M}$. Then, we have ess $\sup f=-\infty$, and $B_{f}=\overline{\mathbb{R}}$.

Proof. Direct consequence of Lemma 933 (trivial measure, $\mu(\{f>-\infty\})=0$ ), and Lemma 1017 (essential supremum, $-\infty \in B_{f}, B_{f}=\overline{\mathbb{R}}$ and ess $\sup f=-\infty$ ).

## Lemma 1020 (essential supremum is bounded by supremum).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, we have ess $\sup f \leqslant \sup f$.
Proof. Let $M \stackrel{\text { def. }}{=} \sup f \in \overline{\mathbb{R}}$. Then, from Definition 2 (supremum, upper bound), Lemma 913 (measure, homogeneity), Lemma 1017 (essential supremum, ess sup is infimum), and Definition 9 (infimum, lower bound), we have $f \leqslant M$ and $\mu(\{f>M\})=\mu(\varnothing)=0$, thus $M \in B_{f}$ and ess $\sup f \leqslant M$.

## Lemma 1021 (essential supremum is minimum).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, $\mu(\{f>\operatorname{ess} \sup f\})=0$, i.e. ess $\sup f \in B_{f}$ Moreover, we have $B_{f}=[\operatorname{ess} \sup f, \infty]$.

Proof. Let $M \stackrel{\text { def. }}{=}$ ess $\sup f \in \overline{\mathbb{R}}$.
Case $\boldsymbol{M}=\infty$. Then, from Lemma 1017 (essential supremum), we have $\infty \in B_{f}$.
Case $\boldsymbol{M}<\infty$. Then, from Lemma 1017 (essential supremum, $M$ is an infimum), Lemma 486 (equivalent definition of infimum), let $\left(a_{n}\right)_{n \in \mathbb{N}} \in B_{f}$ be a nonincreasing sequence of extended numbers such that $\lim _{n \rightarrow \infty} a_{n}=\inf _{n \in \mathbb{N}} a_{n}=M$. Then, from compatibility of inverse image with union (with $(M, \infty]=\bigcup_{n \in \mathbb{N}}\left(a_{n}, \infty\right]$ ), we have $\{f>M\}=\bigcup_{n \in \mathbb{N}}\left\{f>a_{n}\right\}$. Therefore, from Lemma 924 (compatibility of null measure with countable union), we have $\mu(\{f>M\})=0$, i.e. $M \in B_{f}$.

Moreover, from Lemma 1017 (essential supremum), we have $[M, \infty] \subset B_{f}$. Conversely, let $a$ be in $B_{f}$. Then, from Lemma 1017 (essential supremum, $M$ is infimum), and Definition 9 (infimum, $\underline{\text { lower bound }}$, we have $M \leqslant a$. Thus, $B_{f} \subset[M, \infty]$. Hence, we have $B_{f}=[M, \infty]$.

Lemma 1022 (essential supremum is almost upper bound). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $b \in B_{f}$. Then, we have $f \stackrel{\mu \text { a.e. }}{\leqslant} b$, and in particular $f \stackrel{\mu \text { a.e. }}{\leqslant}$ ess $\sup f$.

Proof. Direct consequence of Lemma 1017 (essential supremum, $\mu(\{f>b\})=0$ ), Lemma 948 (negligibility of measurable subset, $\{f \leqslant b\}^{c} \in \mathbf{N}$ ), Definition 974 (almost binary relation), and Lemma 1021 (essential supremum is minimum, ess $\sup f \in B_{f}$ ).

Lemma 1023 (essential supremum of constant function). Let $(X, \Sigma, \mu)$ be a nontrivial measure space. Let $y \in \overline{\mathbb{R}}$. Let $f$ be the constant function from $X$ to $\overline{\mathbb{R}}$ of value $y$. Then we have $f \in \mathcal{M}$, ess $\sup f=y$, and $B_{f}=[y, \infty]$.

Proof. From Lemma 754 (constant function is measurable), and Lemma 1017 (essential supremum), ess $\sup f$ is well-defined. Then, From the definition of complement, Lemma 913 ( measure), and Lemma 1017 (essential supremum), we have $\mu(\{f>y\})=\mu\left(\{f \leqslant y\}^{c}\right)=\mu(\varnothing)=0$, $y \in B_{f}$, and $[y, \infty] \subset B_{f}$.

Let $a<y$. Then, from Lemma 934 (equivalent definition of trivial measure, contrapositive), and Lemma 913 (measure, nonnegativeness), we have $\mu(\{f>a\})=\mu(X)>0$. Therefore, from Lemma 1017 (essential supremum), we have $a \notin B_{f}, B_{f}=[y, \infty]$, and ess $\sup f=\inf B_{f}=y$.

Lemma 1024 (essential supremum of extended simple functions).
Let $(X, \Sigma, \mu)$ be a nontrivial measure space. Let a triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \overline{\mathbb{R}}$ such that $f: X \rightarrow \overline{\mathbb{R}}$ is defined by

$$
\begin{align*}
\forall i \in[0 . . n], & A_{i} \subset\left\{f=a_{i}\right\},  \tag{13.12}\\
& X=\biguplus_{i \in[0 . . n]} A_{i},  \tag{13.13}\\
& f=\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}} . \tag{13.14}
\end{align*}
$$

Then, we have $f \in \mathcal{M}$, and ess sup $f=\max _{i \in[0 . . n] \mid \mu\left(A_{i}\right)>0} a_{i}$.
Proof. From Lemma 811 (measurability of indicator function), Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication, $a_{i} \mathbb{1}_{A_{i}} \in \mathcal{M}$ ), Definition 317 (addition in $\overline{\mathbb{R}}$, undefined forms of rule 3 cannot occur), and Lemma $823(\mathcal{M}$ is closed under finite sum when defined), we have $f \in \mathcal{M}$.

From Lemma 934 (equivalent definition of trivial measure, contrapositive), Definition 209 ( pseu dopartition), Lemma 916 (measure is additive), Definition 862 (additivity), and Lemma 913 (measure, nonnegativeness), we have $0<\mu(X)=\sum_{i \in[0 . . n]} \mu\left(A_{i}\right)$, and thus there exists $i_{0} \in[0 . . n]$ such that $\mu\left(A_{i_{0}}\right)>0$.

Let $J \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid \mu\left(A_{i}\right)>0\right\} \neq \varnothing$, and $j \stackrel{\text { def. }}{=} \arg \max _{i \in J} a_{i}$ (note that $j$ may not be unique). Then, we have $\forall x \in \biguplus_{i \in J} A_{i}, f(x) \leqslant a_{j}$, the pseudopartition $X=\biguplus_{i \in J} A_{i} \uplus \biguplus_{i \in[0 . . n] \backslash J} A_{i}$, and thus $\left\{f>a_{j}\right\} \subset \biguplus_{i \in[0 . . n] \backslash J} A_{i}$. Thus, from Lemma 913 (measure, nonnegativeness), Lemma 919 (measure is monotone), Lemma 916 (measure is additive), and Definition 862 (additivity), we have $0 \leqslant \mu\left(\left\{f>a_{j}\right\}\right) \leqslant \mu\left(\biguplus_{i \in I \backslash J} A_{i}\right)=\sum_{i \in I \backslash J} \mu\left(A_{i}\right)=0$. Hence, from Lemma 1017 (essential supremum), we have $a_{j} \in B_{f}$.
Case $\boldsymbol{a}_{\boldsymbol{j}}=-\infty$. Then, from Lemma 1017 (essential supremum, $\underline{B_{f}=\overline{\mathbb{R}}}$ ), ess $\sup f=-\infty=a_{j}$.
Case $-\infty<\boldsymbol{a}_{\boldsymbol{j}}$. Let $y<a_{j}$. Then, $A_{j} \subset\left\{f=a_{j}\right\} \subset\{f>y\}$, and from Lemma 919 (measure is monotone), we have $0<\mu\left(A_{j}\right) \leqslant \mu(\{f>y\})$. Hence, from Lemma 1017 (essential supremum), we have $y \notin B_{f}$, and ess $\sup f=a_{j}$.

Therefore, we always have ess sup $f=a_{j}$.

Remark 1025. Note that both previous lemmas do not hold when the measure is trivial, as all essential suprema are then equal to $-\infty$.

Lemma 1026 (essential supremum of infinite times function).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, $\infty f \in \mathcal{M}$, and we have

$$
\text { ess } \sup (\infty f)= \begin{cases}-\infty & \text { if } \mu(\{f \geqslant 0\})=0  \tag{13.15}\\ 0 & \text { if } \mu(\{f>0\})=0 \wedge \mu(\{f=0\})>0 \\ \infty & \text { if } \mu(\{f>0\})>0\end{cases}
$$

Proof. From Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), and Lemma 1017 (essential supremum ), we have $\infty f \in \mathcal{M}$, and ess $\sup (\infty f)$ is well-defined.

Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 933 (trivial measure, $\mu(\{f>0\})=0$ ), and Lemma 1019 (essential supremum with trivial measure).
Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 209 (pseudopartition, $X$ is the disjoint union of $\{f<0\},\{f=0\}$ and $\{f>0\}$ ), Lemma 820 (measurability of numeric function, $\{f>0\}$ and $\{f \geqslant 0\}$ belong to $\Sigma$ ), Lemma 809 (inverse image is measurable, $\{f=0\} \in \Sigma$ ), and Lemma 1024 (essential supremum of extended simple functions, with $\left.\infty f=-\infty \mathbb{1}_{\{f<0\}}+0 \mathbb{1}_{\{f=0\}}+\infty \mathbb{1}_{\{f>0\}}\right)$.

Lemma 1027 (essential supremum is positive homogeneous). Let $(X, \Sigma, \mu)$ be a nontrivial measure space. Let $\lambda \in \mathbb{R}_{+}$. Let $f \in \mathcal{M}$. Then, $\lambda f \in \mathcal{M}$, and we have ess $\sup (\lambda f)=\lambda$ ess $\sup f$.

Proof. From Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), we have $\lambda f \in \mathcal{M}$. Hence, from Lemma 1017 (essential supremum), ess sup $f$ and ess $\sup (\lambda f)$ are well-defined.
Let $M \stackrel{\text { def. }}{=}$ ess $\sup f \in \overline{\mathbb{R}}$, and let $M_{\lambda} \stackrel{\text { def. }}{=}$ ess $\sup (\lambda f) \in \overline{\mathbb{R}}$. Case $\boldsymbol{\lambda}=\mathbf{0}$. Then, from Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), and Lemma 1023 (essential supremum of constant function, $\mu$ is nontrivial), $0 f$ is the zero function, and we have $M_{0}=0=0 M$. Case $\boldsymbol{\lambda} \neq 0$. Let $a \in \overline{\mathbb{R}}$. Then, from Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), and Lemma 1017 (essential supremum), we have $\{\lambda f>a\}=\left\{f>\frac{a}{\lambda}\right\}$, and $a \in B_{\lambda f} \Leftrightarrow \frac{a}{\lambda} \in B_{f}$. Thus, from Lemma 1021 (essential supremum is minimum), we have

$$
\frac{M_{\lambda}}{\lambda} \in B_{f}=[M, \infty], \quad \text { and } \quad \lambda M \in B_{\lambda f}=\left[M_{\lambda}, \infty\right]
$$

Hence, from Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), and the definition of order (antisymmetry), we have $M_{\lambda}=\lambda M$.
Therefore, we always have $M_{\lambda}=\lambda M$.
Remark 1028. Note that the result of Lemma 1027 does not hold when $\mu \stackrel{\text { def. }}{=} 0$. Indeed, from Lemma 1019, and Definition 415, we have then ess $\sup (0 f)=-\infty \neq 0=0$ ess sup $f$.

Note also that the result does not hold either when $\lambda \stackrel{\text { def. }}{=} \infty$. Indeed, let $f \stackrel{\text { def. }}{=} \arctan x-\frac{\pi}{2}$. Then, with the Lebesgue measure, we have ess $\sup f=\sup f=0$, and $\mu(\{f \geqslant 0\})=0$. Hence, from Lemma 1026, we have ess $\sup (\infty f)=-\infty \neq 0=\infty$ ess $\sup f$.

Lemma 1029 (essential supremum is almost monotone). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, we have ess $\sup f \leqslant \operatorname{ess} \sup g$.

Proof. From Definition 974 (almost binary relation, with inequality), we have $A \stackrel{\text { def. }}{=}\{f \leqslant g\}^{c} \in \mathbf{N}$.
Let $a \in \overline{\mathbb{R}}$. Then, from distributivity of intersection over union, and the definition of order (transitivity), we have

$$
\{f>a\}=\left(\{f>a\} \cap A^{c}\right) \uplus(\{f>a\} \cap A) \subset\{g>a\} \cup A .
$$

Assume that $\mu(\{g>a\})=0$. Then, from Lemma 820 (measurability of numeric function, $\{g>a\},\{f>a\} \in \Sigma)$, Lemma 948 (negligibility of measurable subset, $\{g>a\} \in \mathbf{N}$ ), Lemma 952 ( $\mathbf{N}$ is closed under countable union), Lemma 951 (subset of negligible is negligible, $\{f>a\} \in \mathbf{N}$ ), we have $\mu(\{f>a\})=0$. Hence, from Lemma 1017 (essential supremum), we have $B_{g} \subset B_{f}$.

Therefore, from Lemma 1021 (essential supremum is minimum), we have ess $\sup f \leqslant \operatorname{ess} \sup g$.

Lemma 1030 (essential supremum bounded by constant).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$, and $a, b \in \overline{\mathbb{R}}$. Then, we have

$$
\begin{align*}
\mu \text { nontrivial } \wedge a \stackrel{\mu \text { a.e. }}{\leqslant} f & \Longrightarrow a \leqslant \operatorname{ess} \sup f  \tag{13.16}\\
f \stackrel{\mu \text { a.e. }}{\leqslant} b & \Longleftrightarrow \quad \text { ess } \sup f \leqslant b \tag{13.17}
\end{align*}
$$

Proof. "Left" implies "right". Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1019 (essential supremum with trivial measure, thus ess $\sup f=-\infty$ ), and Definition 312 (extended real numbers, $\overline{\mathbb{R}}$, thus $-\infty \leqslant b$ ). Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Lemma 1029 (essential supremum is almost monotone, with $f \stackrel{\text { def. }}{=} a$ and $g \stackrel{\text { def. }}{=} f$, then with $f$ and $g \stackrel{\text { def. }}{=}$ ), and Lemma 1023 (essential supremum of constant function, ess $\sup (a)=a$ and ess $\sup (b)=b)$.
"Right" implies "left". Direct consequence of Lemma 1022 (essential supremum is almost upper bound), and Lemma 983 (almost order is order relation, transitivity).

Lemma 1031 (essential supremum of absolute value is nonnegative). Let $(X, \Sigma, \mu)$ be a nontrivial measure space. Let $f \in \mathcal{M}$. Then, $|f| \in \mathcal{M}$, and we have ess sup $|f| \geqslant 0$.

Proof. Direct consequence of Lemma $842(\mathcal{M}$ is closed under absolute value, $|f| \in \mathcal{M}$ and $0 \leqslant|f|)$, Lemma 958 (everywhere is almost everywhere, $\underline{0}^{\mu \text { a.e. }} \leqslant|f|$ ), and Lemma 1030 (essential supremum bounded by constant, with $a \stackrel{\text { def. }}{=} 0$ ).

Lemma 1032 (compatibility of essential supremum with almost equality). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{=} g$. Then, we have ess $\sup f=\operatorname{ess} \sup g$.
Proof. Direct consequence of Lemma 977 (compatibility of almost binary relation with extended reflexivity, with $\leqslant$ ), Lemma 1029 (essential supremum is almost monotone), and the definition of order (antisymmetry).

### 13.2.2 Nonnegative essential supremum for nonnegative functions

Remark 1033. In this section, the definition of essential supremum is restricted to nonnegative functions. This provides a more regular treatment of extreme cases, such as when the measure is trivial, or when dealing with infinite values (e.g. for positive homogeneity).

This will be useful when defining the infinite seminorm $N_{\infty}$ in Section 16.3.1.
Definition 1034 (nonnegative essential supremum). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+} \subset \mathcal{M}$. The (nonnegative) essential supremum of $f$ is its usual essential supremum ess $\sup f$ when the measure $\mu$ is nontrivial, and 0 otherwise; it is denoted ess sup ${ }^{+} f$.

Remark 1035. Note that from the previous definition, the essential supremum and the nonnegative essential supremum coincide on nonnegative measurable functions when the measure is nontrivial. Thus, when no confusion is possible, we may drop the " + " superscript and simply write ess sup.

## Lemma 1036 (equivalent definition of nonnegative essential supremum).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Then, $f \in \mathcal{M}, C_{f} \stackrel{\text { def. }}{=}\left\{a \in \overline{\mathbb{R}}_{+} \mid \mu(\{f>a\})=0\right\}$ and $\inf C_{f}$ are well-defined in $\overline{\mathbb{R}}_{+}$, and we have ess sup ${ }^{+} f=\inf C_{f}$.

Moreover, we have $\infty \in C_{f}$, and $a \in C_{f} \Rightarrow[a, \infty] \subset C_{f}$.

Proof. From Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), and Lemma 1017 (essential supremum), we have $f \in \mathcal{M}, B_{f}$ is well-defined in $\overline{\mathbb{R}}$, and $C_{f}=B_{f} \cap \overline{\mathbb{R}}_{+}$ and $\inf C_{f}$ are well-defined in $\overline{\mathbb{R}}_{+}$.

Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 933 (trivial measure, thus $C_{f}=\overline{\mathbb{R}}_{+}$), and Definition 9 (infimum, thus ess sup ${ }^{+} f \stackrel{\text { def. }}{=} 0=\inf C_{f}, \infty \in C_{f}$ and $a \in C_{f} \Rightarrow[a, \infty] \subset C_{f}$ ).
Case $\boldsymbol{\mu}$ nontrivial. Let $a \in \overline{\mathbb{R}}_{-}^{\star}$. Then, from Lemma 913 (measure, nonnegativeness), and Lemma 934 (equivalent definition of trivial measure), we have $\mu(\{f>a\})=\mu(X)>0$. Hence, from Lemma 1017 (essential supremum), we have $a \notin B_{f}$, i.e. $B_{f} \subset \overline{\mathbb{R}}_{+}$, thus $C_{f}=B_{f}$,

$$
\begin{gathered}
\text { ess sup }^{+} f \stackrel{\text { def. }}{=} \operatorname{ess} \sup f=\inf B_{f}=\inf C_{f}, \\
\infty \in B_{f}=C_{f}, \quad \text { and } \quad a \in C_{f}=B_{f} \Rightarrow[a, \infty] \subset B_{f}=C_{f}
\end{gathered}
$$

Therefore, we always have the result.
Lemma 1037 (nonnegative essential supremum with trivial measure).
Let $(X, \Sigma, \mu)$ be a trivial measure space. Let $f \in \mathcal{M}_{+}$. Then, ess sup ${ }^{+} f=0$, and $C_{f}=\overline{\mathbb{R}}_{+}$.
Proof. Direct consequence of Definition 1034 (nonnegative essential supremum, ess sup ${ }^{+} f=0$ ), Lemma 933 (trivial measure, thus $\mu(\{f>a\})=0$ for all a), Lemma 1036 (equivalent definition of nonnegative essential supremum, thus $\left.C_{f}=\overline{\mathbb{R}}_{+}\right)$.

Lemma 1038 (nonnegative essential supremum is minimum). Let ( $X, \Sigma, \mu$ )
be a measure space. Let $f \in \mathcal{M}_{+}$. Then, we have $\mu\left(\left\{f>\operatorname{ess} \sup ^{+} f\right\}\right)=0$, i.e. ess sup ${ }^{+} f \in C_{f}$. Moreover, we have $C_{f}=\left[\operatorname{esssup}^{+} f, \infty\right]$.

## Proof. Case $\boldsymbol{\mu}$ trivial.

Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure).
Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1021 (essential supremum is minimum).

Therefore, we always have the result.

## Lemma 1039 (nonnegative essential supremum is almost upper bound).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Let $b \in C_{f}=\left[\operatorname{ess~sup}^{+} f, \infty\right]$.
Then, we have $f \stackrel{\mu \text { a.e. }}{\leqslant} b\left(e . g . f \stackrel{\mu \text { a.e. }}{\leqslant}\right.$ ess sup $\left.^{+} f\right)$.
In particular, for all $g \in \mathcal{M}$, we have $|g| \in \mathcal{M}_{+}$, and for all $b \in C_{|g|}=\left[\operatorname{esssup}^{+}|g|, \infty\right]$, we have $|g| \stackrel{\mu \text { a.e. }}{\leqslant} b\left(e . g .|g| \stackrel{\mu \text { a.e. }}{\leqslant} \operatorname{ess~sup}^{+}|g|\right)$.

Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 956 (property almost satisfied with trivial measure, with $P \stackrel{\text { def. }}{=}(f \leqslant b)$ ).

Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1022 (essential supremum is almost upper bound).
Therefore, we always have the result.
Let $g \in \mathcal{M}$. Then, from Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value, $\left.|g| \in \mathcal{M}_{+}\right)$, the previous result applies with $f \stackrel{\text { def. }}{=}|g|$.

Lemma 1040 (nonnegative essential supremum of constant function). Let ( $X, \Sigma, \mu$ ) be a nontrivial measure space. Let $y \in \overline{\mathbb{R}}_{+}$. Let $f$ be the constant function from $X$ to $\overline{\mathbb{R}}_{+}$of value $y$. Then, $f \in \mathcal{M}_{+}$, and we have ess sup ${ }^{+} f=y$ and $C_{f}=[y, \infty]$.

Proof.
Direct consequence of Lemma 754 (constant function is measurable, $\underline{f \in \mathcal{M}_{+}}$), Definition 1034 (nonnegative essential supremum), and Lemma 1023 (essential supremum of constant function).

Lemma 1041 (nonnegative essential supremum for extended simple functions).
Let $(X, \Sigma, \mu)$ be a nontrivial measure space. Let a triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \overline{\mathbb{R}}_{+}$ such that $f: X \rightarrow \overline{\mathbb{R}}_{+}$is defined by

$$
\begin{equation*}
\forall i \in[0 . . n], \quad A_{i} \subset\left\{f=a_{i}\right\}, \tag{13.18}
\end{equation*}
$$

$$
\begin{align*}
& X=\biguplus_{i \in[0 . . n]} A_{i},  \tag{13.19}\\
& f=\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}} . \tag{13.20}
\end{align*}
$$

Then, $f \in \mathcal{M}_{+}$, and we have ess sup ${ }^{+} f=\underset{i \in[0 . . n] \mid \mu\left(A_{i}\right)>0}{ } a_{i}$.
Proof. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1024 (essential supremum of extended simple functions).

Remark 1042. Note that both previous lemmas do not hold when the measure is trivial, as all nonnegative essential suprema are then equal to 0 .

Lemma 1043 (nonnegative essential supremum of infinite times function).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Then, $\infty f \in \mathcal{M}_{+}$, and we have

$$
\operatorname{ess}^{2} \sup ^{+}(\infty f)= \begin{cases}0 & \text { if } \mu(\{f>0\})=0  \tag{13.21}\\ \infty & \text { if } \mu(\{f>0\})>0\end{cases}
$$

Proof. From Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$, and Definition 1034 (nonnegative essential supremum), we have $\infty f \in \mathcal{M}_{+}$, and ess sup ${ }^{+}(\infty f)$ is well-defined.
Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 933 (trivial measure, thus $\mu(\{f>0\})=0$ ), and Lemma 1037 (nonnegative essential supremum with trivial measure).
Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), Lemma 934 (equivalent definition of trivial measure, thus $\mu(X) \neq 0$ ), and Lemma 1026 (essential supremum of infinite times function, $\mu(\{f \geqslant 0\})=0$ cannot occur $)$.
Therefore, we always have the result.
Lemma 1044 (nonnegative essential supremum is positive homogeneous).
Let $(X, \Sigma, \mu)$ be a measure space. Let $\lambda \in \overline{\mathbb{R}}_{+}$. Let $f \in \mathcal{M}_{+}$.
Then, $\lambda f \in \mathcal{M}_{+}$, and we have ess sup ${ }^{+}(\lambda f)=\lambda \operatorname{ess}^{\text {sup }}{ }^{+} f$.
Proof. From Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication, $\left.\lambda f \in \mathcal{M}_{+}\right)$, and Definition 1034 (nonnegative essential supremum), ess sup ${ }^{+} f$ and $\operatorname{ess}^{\text {sup }}{ }^{+}(\lambda f)$ are well-defined.
Let $M \stackrel{\text { def. }}{=} \operatorname{ess} \sup ^{+} f \in \overline{\mathbb{R}}_{+}$, and $M_{\lambda} \stackrel{\text { def. }}{=} \operatorname{ess} \sup ^{+}(\lambda f) \in \overline{\mathbb{R}}_{+}$.
Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, $M=M_{\lambda}=0$ ), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).

Case $\boldsymbol{\mu}$ nontrivial, and $\boldsymbol{\lambda} \in[\mathbf{0}, \infty$ ). Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1027 (essential supremum is positive homogeneous).
Case $\mu$ nontrivial, $\lambda=\infty$, and $\mu(\{f>0\})=0$.
Then, from Lemma 1036 (equivalent definition of nonnegative essential supremum), Lemma 1043
(nonnegative essential supremum of infinite times function), and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)), we have $0 \in C_{f}, C_{f}=\overline{\mathbb{R}}_{+}, M=0$, and $M_{\infty}=0=\infty M$.
Case $\boldsymbol{\mu}$ nontrivial, $\boldsymbol{\lambda}=\infty$, and $\boldsymbol{\mu}(\{\boldsymbol{f}>\mathbf{0}\})>\mathbf{0}$. Then, from Lemma 1036 (equivalent definition of nonnegative essential supremum), Lemma 1038 (nonnegative essential supremum is minimum), Lemma 1043 (nonnegative essential supremum of infinite times function), and Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)), we have $0 \notin C_{f}=[M, \infty]$, thus $M>0$ and $M_{\infty}=\infty=\infty M$.

Therefore, we always have $M_{\lambda}=\lambda M$.

## Lemma 1045 (nonnegative essential supremum is almost monotone).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Assume that $f \stackrel{\mu \text { a.e. }}{\leqslant} g$.
Then, we have esssup ${ }^{+} f \leqslant \operatorname{esssup}^{+} g$.
Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, thus ess sup ${ }^{+} f=\operatorname{esssup}^{+} g=0$ ), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, reflexivity).

Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1029 (essential supremum is almost monotone).

Therefore, we always have the result.
Lemma 1046 (nonnegative essential supremum bounded by constant).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Let $a, b \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{align*}
& \mu \text { nontrivial } \wedge a \stackrel{\mu \text { a.e. }}{\leqslant} f \quad \Longrightarrow \quad a \leqslant \operatorname{ess}^{*} \sup ^{+} f,  \tag{13.22}\\
& f \stackrel{\mu \text { a.e. }}{\leqslant} b \quad \Longleftrightarrow \quad \operatorname{ess}^{*} \sup ^{+} f \leqslant b .  \tag{13.23}\\
& \text { In particular, for all } g \in \mathcal{M} \text {, we have }|g| \in \mathcal{M}_{+} \text {, and }|g| \stackrel{\mu \text { a.e. }}{\leqslant} b \text { iff ess sup }{ }^{+}|g| \leqslant b \text {. }
\end{align*}
$$

Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, thus ess $\sup f=0$ ), Definition 312 (extended real numbers, $\overline{\mathbb{R}}$, thus $0 \leqslant b$ ), and Lemma 956 (property almost satisfied with trivial measure, thus $f \stackrel{\mu \text { a.e. }}{\leqslant} b$ always holds).

Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1030 (essential supremum bounded by constant).
Let $g \in \mathcal{M}$. Then, from Lemma 842 ( $\mathcal{M}$ is closed under absolute value, $\left.|g| \in \mathcal{M}_{+}\right)$, (13.23) applies.

Lemma 1048 (essentially bounded is almost finite). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Then, $f$ is essentially bounded iff $|f| \stackrel{\mu \text { a.e. }}{<} \infty$ iff $\mu(\{f= \pm \infty\})=0$.

Proof. Direct consequence of Definition 1047 (essentially bounded), Lemma 1046 (nonnegative essential supremum bounded by constant, with $g \stackrel{\text { def. }}{=} f$ and $b \stackrel{\text { def. }}{=} M \in \mathbb{R}_{+}$), Lemma 958 (everywhere is almost everywhere), and Lemma 983 ( almost order is order relation, transitivity).

Lemma 1049 (compatibility of nonnegative essential supremum with almost equality). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Assume that $f \stackrel{\mu \text { a.e. }}{=} g$. Then, we have esssup ${ }^{+} f=\operatorname{esssup}^{+} g$.

Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, thus ess sup ${ }^{+} f=0=\operatorname{esssup}^{+} g$ ).
Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Definition 1034 (nonnegative essential supremum), and Lemma 1032 (compatibility of essential supremum with almost equality).

Lemma 1050 (nonnegative essential supremum is almost definite). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
\operatorname{ess~sup}^{+} f=0 \quad \Longleftrightarrow \quad f^{\mu} \stackrel{\text { a.e. }}{=} 0 \quad \Longleftrightarrow \quad \mu(\{f>0\})=0 \tag{13.24}
\end{equation*}
$$

In particular, for all $g \in \mathcal{M}$, we have $|g| \in \mathcal{M}_{+}$, and

$$
\begin{equation*}
\operatorname{ess}_{\sup }+|g|=0 \quad \Longleftrightarrow \quad|g| \stackrel{\mu \text { a.e. }}{=} 0 \quad \Longleftrightarrow \quad \mu(\{|g|>0\})=0 \tag{13.25}
\end{equation*}
$$

Proof. From Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\left.\underline{f \geqslant 0}\right)$, Definition 209 (pseudopartition, $X=\{f=0\} \uplus\{f>0\}$ ), Lemma 809 (inverse image is measurable, $\{f=0\} \in \Sigma$ ), Lemma 820 (measurability of numeric function, $\{f>0\} \in \Sigma$ ), Definition 953 (property almost satisfied), and Lemma 948 (negligibility of measurable subset), we have the equivalence $f \stackrel{\mu \text { a.e. }}{=} 0 \Leftrightarrow \mu(\{f>0\})=0$.

## Case $\mu$ trivial.

Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, thus $\operatorname{ess}^{+}{ }^{+} f=0$ ), and Lemma 956 (property almost satisfied with trivial measure, thus $f \stackrel{\mu \text { a.e. } 0}{=}$ ).

Case $\boldsymbol{\mu}$ nontrivial. Direct consequence of Lemma 1038 (nonnegative essential supremum is minimum, thus ess sup ${ }^{+} f=0$ implies $\mu(\{f>0\})=0$ ), Lemma 1049 (compatibility of nonnegative essential supremum with almost equality, with $g \stackrel{\text { def. }}{=}$ ), and Lemma 1040 (nonnegative essential supremum of constant function, thus ess sup $\left.{ }^{+}(x \mapsto 0)=0\right)$.
Therefore, we always have the result.
Let $g \in \mathcal{M}$. Then, from Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value, $\left.|g| \in \mathcal{M}_{+}\right),(13.24)$ applies.

## Lemma 1051 (nonnegative essential supremum is subadditive).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$.
Then, $f+g \in \mathcal{M}_{+}$, and we have ess sup ${ }^{+}(f+g) \leqslant \operatorname{esssup}^{+} f+\operatorname{esssup}^{+} g$.
Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, all the ess sup ${ }^{+}$are 0 ), Definition 317 (addition in $\overline{\mathbb{R}}$, thus $0+0=0$ ), and the definition of order (reflexivity).

Case $\boldsymbol{\mu}$ nontrivial. Let $M_{f}=\operatorname{esssup}^{+} f$, and $M_{g}=\operatorname{esssup}^{+} g$.
Direct consequence of Lemma 1039 (nonnegative essential supremum is almost upper bound, thus $\underline{f^{\mu \text { a.e. }}} \leqslant M_{f}$ and $g \stackrel{\mu \text { a.e. }}{\leqslant} M_{q}$ ), Lemma 984 (compatibility of almost binary relation with operator, $\underline{\text { with }}$ $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}_{+}, \mathcal{R}, \mathcal{R}^{\prime} \stackrel{\text { def. }}{=} \leqslant$ and the binary operator addition, thus $f+g \stackrel{\mu \text { a.e. }}{\leqslant} M_{f}+M_{g}$ ), Lemma 843 $\left(\mathcal{M}_{+}\right.$is closed under addition, $\left.\underline{f+g \in \mathcal{M}_{+}}\right)$, and Lemma 1046 (nonnegative essential supremum bounded by constant, with $\left.b \stackrel{\text { def. }}{=} M_{f}+M_{q}\right)$.

Therefore, we always have the result.
Remark 1052. Note that a counterpart of Lemma 1051 probably holds over $\mathcal{M}$ with the essential supremum (instead of the nonnegative essential supremum), at the cost of handling almost sum of the functions. It is not stated.

## Lemma 1053 (nonnegative essential supremum is submultiplicative).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$.
Then, $f g \in \mathcal{M}_{+}$, and we have esssup ${ }^{+}(f g) \leqslant \operatorname{ess}^{+} f \times \operatorname{ess}^{+} \sup ^{+} g$.
Proof. Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, all the ess sup ${ }^{+}$are 0), Definition 326 (multiplication in $\overline{\mathbb{R}}$, thus $0 \times 0=0$ ), and the definition of order (reflexivity).
Case $\boldsymbol{\mu}$ nontrivial. Let $M_{f}=\operatorname{esssup}^{+} f$, and $M_{g}=\operatorname{esssup}{ }^{+} g$.
Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\frac{\text { thus } f, g \geqslant 0}{\mu \text { a.e. }}$ ), Lemma 1039 (nonnegative essential supremum is almost upper bound, thus $\underset{f^{\mu \text { a.e. }}}{\leqslant} M_{f}$ and $g \stackrel{\mu \text { a.e. }}{\leqslant} M_{q}$ ), Lemma 984 (compatibility of almost binary relation with operator, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}_{+}$, $\mathcal{R}, \mathcal{R}^{\prime} \stackrel{\text { def. }}{=} \leqslant$ and the binary operator multiplication, thus $\left.f g \stackrel{\mu \text { a.e. }}{\leqslant} M_{f} M_{g}\right)$, Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication, $f g \in \mathcal{M}_{+}$), and Lemma 1046 (nonnegative essential supremum bounded by constant, with $b \stackrel{\text { def. }}{=} M_{f} M_{g}$ ).
Therefore, we always have the result.
Lemma 1054 (nonnegative essential supremum is compatible with power function). Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in \overline{\mathbb{R}}_{+}$. Let $f \in \mathcal{M}_{+}$. Then, $f^{p} \in \mathcal{M}_{+}$, and we have

$$
\begin{align*}
\mu \text { trivial } \wedge 0<p & \Longrightarrow \quad \operatorname{esssup}^{+} f^{p}=\left(\operatorname{esssup}^{+} f\right)^{p},  \tag{13.26}\\
\mu \text { nontrivial } \wedge \quad p<\infty & \Longrightarrow \quad \operatorname{esssup}^{+} f^{p}=\left(\operatorname{esssup}^{+} f\right)^{p} . \tag{13.27}
\end{align*}
$$

Proof. Case $\boldsymbol{\mu}$ trivial, and $\mathbf{0}<\boldsymbol{p}$. Direct consequence of Lemma 1037 (nonnegative essential supremum with trivial measure, all the ess sup ${ }^{+}$are 0), and Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory), thus $\left.\left(\operatorname{ess~sup}^{+} f\right)^{p}=0\right)$.
Case $\boldsymbol{\mu}$ nontrivial, and $\boldsymbol{p}=\mathbf{0}$. Direct consequence of Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory), thus $f^{0}=\left(\operatorname{esssup}^{+} f\right)^{0}=1$ ), and Lemma 1040 (nonnegative essential supremum of constant function, with $y \stackrel{\text { def. }}{=} 1$, thus ess sup $f^{0}=1$ ).
Case $\boldsymbol{\mu}$ nontrivial, and $\boldsymbol{p} \in(\mathbf{0}, \boldsymbol{\infty})$. Let $M=\operatorname{ess} \sup ^{+} f$, and $N=\operatorname{ess} \sup ^{+}\left(f^{p}\right)$.
Then, from Lemma $841\left(\mathcal{M}\right.$ is closed under power function, thus $\left.f^{p} \in \mathcal{M}_{+}\right)$, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, thus $\overline{f, f^{p} \geqslant 0}$ ), and Lemma 1039 (nonnegative essential supremum is almost upper bound), we have $f \stackrel{\mu \text { a.e. }}{\leqslant} M$ and $f^{p} \stackrel{\mu \text { a.e. }}{\leqslant} N$. Thus, from Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory), with $q \stackrel{\text { def. }}{=} \frac{1}{p}$ ), Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory), thus $p \times \frac{1}{p}=1$ ), Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory), with $p$ and $\frac{1}{p}$ ), Lemma 984 (compatibility of almost binary relation with operator, with $Y \stackrel{\text { def. }}{=} \overline{\mathbb{R}}_{+}, \mathcal{R}, \mathcal{R}^{\prime} \stackrel{\text { def. }}{=} \leqslant$ and the unary operators power function of degrees $p$ and $\frac{1}{p}$, thus $f^{p} \stackrel{\mu \text { a.e. }}{\leqslant} M^{p}$ and $f^{\mu \text { a.e. }} \leqslant N^{\frac{1}{p}}$ ), and Lemma 1046 (nonnegative essential supremum bounded by constant), we have $N \leqslant M^{p}$ and $M \leqslant N^{\frac{1}{p}}$. Hence, from Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)), and Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory), with $(p, q) \stackrel{\text { def. }}{=}\left(\frac{1}{p}, p\right)$ ), we have the other inequality $M^{p} \leqslant N$, and the equality holds.

Therefore, we have the result in both cases.

### 13.3 The Lebesgue measure

Remark 1055. This section follows Carathéodory's extension scheme (see Section 4.3).
Remark 1056. We recall the notation $[\cdot, \cdot)]$ for not specifying open or closed bounds for intervals.

## Definition 1057 (length of interval).

Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. The length of interval from $a$ to $b$ is $\ell(([a, b)]) \stackrel{\text { def. }}{=} b-a$.
Lemma 1058 (length is nonnegative). Let $a, b \in \mathbb{R}$, Assume that $a \leqslant b$. Then, $\ell((a, b)) \geqslant 0$.
Proof. Direct consequence of Definition 1057 (length of interval).
Lemma 1059 (length is homogeneous). $\quad$ We have $\ell(\varnothing)=0$.

Proof. Let $a \in \mathbb{R}$. Then, for instance, we have $\varnothing=(a, a)$. Therefore, from Definition 1057 (length of interval), we have $\ell(\varnothing)=0$.

Lemma 1060 (length of partition). $\quad$ Let $a, b, c \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have

$$
\begin{equation*}
\ell((a, b) \cap(c, \infty))+\ell\left((a, b) \cap(c, \infty)^{c}\right)=\ell((a, b)) . \tag{13.28}
\end{equation*}
$$

Proof. Let $I \stackrel{\text { def. }}{=}(a, b)$, and $E \stackrel{\text { def. }}{=}(c, \infty)$. We have $E^{c}=(-\infty, c]$.
Case $\boldsymbol{c} \leqslant \boldsymbol{a}$. Then, we have $I \cap E=I$ and $I \cap E^{c}=\varnothing$. Hence, from Lemma 1059 (length is homogeneous), we have $\ell(I \cap E)+\ell\left(I \cap E^{c}\right)=\ell(I)$.
Case $\boldsymbol{a}<\boldsymbol{c}<\boldsymbol{b}$. Then, we have $I \cap E=(a, c)$ and $I \cap E^{c}=[c, b)$. Hence, from Definition 1057 (length of interval), and abelian additive group properties of $\mathbb{R}$, we have

$$
\ell(I \cap E)+\ell\left(I \cap E^{c}\right)=\ell((a, c))+\ell([c, b))=(c-a)+(b-c)=b-a=\ell(I)
$$

Case $\boldsymbol{b} \leqslant \boldsymbol{c}$. Then, we have $I \cap E=\varnothing$ and $I \cap E^{c}=I$. Hence, from Lemma 1059 (length is homogeneous , we have $\ell(I \cap E)+\ell\left(I \cap E^{c}\right)=\ell(I)$.

Therefore, we always have $\ell(I \cap E)+\ell\left(I \cap E^{c}\right)=\ell(I)$.

## Definition 1061 (set of countable cover with bounded open intervals).

Let $A \subset \mathbb{R}$. The set of countable covers of $A$ with bounded open intervals is

$$
\begin{equation*}
C_{A} \stackrel{\text { def. }}{=}\left\{\left(\left(a_{n}, b_{n}\right)\right)_{n \in \mathbb{N}} \mid\left(\forall n \in \mathbb{N}, a_{n}, b_{n} \in \mathbb{R} \wedge a_{n} \leqslant b_{n}\right) \quad \wedge \quad A \subset \bigcup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right)\right\} . \tag{13.29}
\end{equation*}
$$

Lemma 1062 (set of countable cover with bounded open intervals is nonempty).
Let $A \subset \mathbb{R}$. Then, we have $C_{A} \neq \varnothing$.
Proof. For all $n \in \mathbb{N}$, let $I_{n} \stackrel{\text { def. }}{=}(-n, n)$. Then, from the Archimedean property of $\mathbb{R}$, we have $\mathbb{R} \subset \bigcup_{n \in \mathbb{N}} I_{n}$. Therefore, from Definition 1061 (set of countable cover with bounded open intervals), and properties of inclusion, we have $\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{A}$.

## Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate).

The Lebesgue measure candidate is the function $\lambda^{\star}: \mathcal{P}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}_{+}$defined by

$$
\begin{equation*}
\forall A \subset \mathbb{R}, \quad \lambda^{\star}(A) \stackrel{\text { def. }}{=} \inf \left\{\sum_{n \in \mathbb{N}} \ell\left(I_{n}\right) \mid\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{A}\right\} . \tag{13.30}
\end{equation*}
$$

Remark 1064. Note that $\lambda^{\star}$ in the previous definition is not $\sigma$-additive on $\mathcal{P}(\mathbb{R})$, e.g. see [34, Ex. $2.28 \mathrm{pp} .86-87]$. Thus, it is not a measure on $\mathcal{P}(\mathbb{R})$. In fact, there is no measure on $\mathcal{P}(\mathbb{R})$ that generalizes the length of interval, e.g. see [34, Ex. 2.29 pp. 87-88].

Lemma 1065 ( $\lambda^{\star}$ is nonnegative). $\quad$ Let $A \subset \mathbb{R}$. Then, we have $\lambda^{\star}(A) \geqslant 0$.
Proof. Direct consequence of Lemma 1058 (length is nonnegative), and Lemma 490 (infimum is monotone (function)).

Lemma 1066 ( $\lambda^{\star}$ is homogeneous).
We have $\lambda^{\star}(\varnothing)=0$.
Proof. From Lemma 1065 ( $\lambda^{\star}$ is nonnegative), we have $\lambda(\varnothing) \geqslant 0$.
For all $n \in \mathbb{N}$, let $I_{n} \stackrel{\text { def. }}{=}(0,0)=\varnothing$. Let $n \in \mathbb{N}$. Then, from Definition 1057 (length of interval), and field properties of $\mathbb{R}$, we have $\ell\left(I_{n}\right)=0-0=0$. Thus, we have $\sum_{n \in \mathbb{N}} \ell\left(I_{n}\right)=0$. Moreover, we have $\varnothing \subset \bigcup_{n \in \mathbb{N}} I_{n}$. Thus, we have $\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{\varnothing}$. Hence, from Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), and Definition 9 (infimum, lower bound), we have $\lambda^{\star}(\varnothing) \leqslant 0$.

Therefore $\lambda^{\star}(\varnothing)=0$.
Lemma 1067 ( $\lambda^{\star}$ is monotone). Let $A, B \subset \mathbb{R}$. Assume that $A \subset B$. Then, $\lambda^{\star}(A) \leqslant \lambda^{\star}(B)$.
Proof. Let $\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{B}$. Then, from Definition 1061 (set of countable cover with bounded open intervals), and transitivity of inclusion, we have $A \subset B \subset \bigcup_{n \in \mathbb{N}} I_{n}$. Thus, from Definition 1061 (set of countable cover with bounded open intervals), we have $\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{A}$. Hence, from Definition $1063\left(\lambda^{\star}\right.$, Lebesgue measure candidate), and Definition 9 (infimum, lower bound, for $A$ ), we have $\lambda^{\star}(A) \leqslant \sum_{n \in \mathbb{N}} \ell\left(I_{n}\right)$.

Therefore, from Definition 9 (infimum, greatest lower bound, for B), and Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), we have $\lambda^{\star}(\bar{A}) \leqslant \lambda^{\star}(B)$.

Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive).
Let $\left(A_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}$. Then,
$\lambda^{\star}\left(\bigcup_{n \in \mathbb{N}} A_{n}\right) \leqslant \sum_{n \in \mathbb{N}} \lambda^{\star}\left(A_{n}\right)$.
Proof. Let $A \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} A_{n}$.
Case $\exists \boldsymbol{n}_{\mathbf{0}} \in \mathbb{N}$ such that $\boldsymbol{\lambda}^{\star}\left(\boldsymbol{A}_{\boldsymbol{n}_{\mathbf{0}}}\right)=\boldsymbol{\infty}$. Then, from Definition 317 (addition in $\overline{\mathbb{R}}$, rule 1 applies), we have $\lambda^{\star}(A) \leqslant \infty=\sum_{n \in \mathbb{N}} \lambda^{\star}\left(A_{n}\right)$.
Case $\forall \boldsymbol{n} \in \mathbb{N}, \boldsymbol{\lambda}^{\star}\left(\boldsymbol{A}_{\boldsymbol{n}}\right)<\infty$. Let $\varepsilon>0$. Let $n \in \mathbb{N}$. Then, from Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), and Lemma 11 (finite infimum), there exists $\left(I_{n, m}\right)_{m \in \mathbb{N}} \in C_{A_{n}}$ such that $\sum_{m \in \mathbb{N}} \ell\left(I_{n, m}\right)<\lambda^{\star}\left(A_{n}\right)+\frac{\varepsilon}{2^{n+1}}$. Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}^{2}$ be a bijection. Then, from Definition 1061 (set of countable cover with bounded open intervals), Lemma 225 (double countable union, with $\underline{I=J} \stackrel{\text { def. }}{=} \mathbb{N}$ ), Lemma 224 (definition of double countable union, with $I=J \stackrel{\text { def. }}{=} \mathbb{N}$ ), and countability of $\mathbb{N}^{2}$, we have

$$
A=\bigcup_{n \in \mathbb{N}} A_{n} \subset \bigcup_{n \in \mathbb{N}}\left(\bigcup_{m \in \mathbb{N}} I_{n, m}\right)=\bigcup_{n, m \in \mathbb{N}} I_{n, m}=\bigcup_{p \in \mathbb{N}} I_{\varphi(p)} .
$$

Then, from Definition 1061 (set of countable cover with bounded open intervals), $\left(I_{\varphi(n)}\right)_{n \in \mathbb{N}}$ belongs to $C_{A}$, and thus, from Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), Definition 9 (infimum, lower bound), Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$), Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$), and additive properties of $\mathbb{R}_{+}$, we have

$$
\lambda^{\star}(A) \leqslant \sum_{p \in \mathbb{N}} \ell\left(I_{\varphi(p)}\right)=\sum_{n, m \in \mathbb{N}} \ell\left(I_{n, m}\right)=\sum_{n \in \mathbb{N}}\left(\sum_{m \in \mathbb{N}} \ell\left(I_{n, m}\right)\right) \leqslant \sum_{n \in \mathbb{N}} \lambda^{\star}\left(A_{n}\right)+\varepsilon .
$$

Hence, from Lemma 381 (limit is monotone), we have $\lambda^{\star}(A) \leqslant \sum_{n \in \mathbb{N}} \lambda^{\star}\left(A_{n}\right)$.
Therefore, we always have $\lambda^{\star}(A) \leqslant \sum_{n \in \mathbb{N}} \lambda^{\star}\left(A_{n}\right)$.

## Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval).

Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have $\lambda^{\star}([(a, b))=b-a$.
Proof. Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$.
(1). Closed interval. Let $\varepsilon>0$. Then, we have $[a, b] \subset\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)$. Thus, from Definition 1061 (set of countable cover with bounded open intervals), Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), Definition 9 (infimum, lower bound), Lemma 1066 ( $\lambda^{\star}$ is homogeneous), Definition 1057 (length of interval), and ordered field properties of $\mathbb{R}$, the sequence $\left(\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right), \varnothing, \ldots\right)$ belongs to $C_{[a, b]}$, and $\lambda^{\star}([a, b]) \leqslant \ell\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)=b-a+\varepsilon$. Hence, from Lemma 381 (limit is monotone $)$, we have $\lambda^{\star}([a, b]) \leqslant b-a$.

Let $\left(\left(a_{n}, b_{n}\right)\right)_{n \in \mathbb{N}} \in C_{[a, b]}$ (with $\forall n \in \mathbb{N}, a_{n} \leqslant b_{n}$ ). Then, from Definition 1061 (set of countable cover with bounded open intervals), and Lemma 299 (finite cover of compact interval), there exist $q \in \mathbb{N}$ and $\left(i_{p}\right)_{p \in[0 . . q]} \in \mathbb{N}$ pairwise distinct such that $[a, b] \subset \bigcup_{p \in[0 . . q]}\left(a_{i_{p}}, b_{i_{p}}\right)$ with $a_{i_{0}}<a, b<b_{i_{q}}$, and for all $p \in[0 . . q-1], a_{i_{p+1}}<b_{i_{p}}$. Then, from ordered field properties of $\mathbb{R}$, Definition 1057 (length of interval), and Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), we have

$$
\begin{aligned}
b-a<b_{i_{q}}-a_{i_{0}} \leqslant b_{i_{q}}+\sum_{p \in[0 . . q-1]}\left(-a_{i_{p+1}}+b_{i_{p}}\right)-a_{i_{0}} & =\sum_{p \in[0 . . q]}\left(b_{i_{p}}-a_{i_{p}}\right) \\
& =\sum_{p \in[0 . . q]} \ell\left(\left(a_{i_{p}}, b_{i_{p}}\right)\right) \leqslant \sum_{n \in \mathbb{N}} \ell\left(\left(a_{n}, b_{n}\right)\right) .
\end{aligned}
$$

Hence, from Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), and Definition 9 (infimum, greatest lower bound $)$, we have $b-a \leqslant \lambda^{\star}([a, b])$.

Therefore, we have $\lambda^{\star}([a, b])=b-a$.
(2). Open interval. Let $\varepsilon>0$. Assume that $\varepsilon<b-a$. Then, we have

$$
\left[a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}\right] \subset(a, b) \subset[a, b] .
$$

Hence, from (1), and Lemma 1067 ( $\lambda^{\star}$ is monotone), we have $b-a-\varepsilon \leqslant \lambda^{\star}((a, b)) \leqslant b-a$. Therefore, from Lemma 381 (limit is monotone), we have $\lambda^{\star}((a, b))=b-a$.
(3). Left-open right-closed interval. Let $\varepsilon>0$. Assume that $\varepsilon<b-a$. Then, we have

$$
[a+\varepsilon, b] \subset(a, b] \subset[a, b] .
$$

Hence, from (1), and Lemma 1067 ( $\lambda^{\star}$ is monotone), we have $b-a-\varepsilon \leqslant \lambda^{\star}((a, b]) \leqslant b-a$. Therefore, from Lemma 381 (limit is monotone), we have $\lambda^{\star}((a, b])=b-a$.
(4). Left-closed right-open interval. Let $\varepsilon>0$. Assume that $\varepsilon<b-a$. Then, we have

$$
(a, b-\varepsilon] \subset[a, b) \subset[a, b] .
$$

Hence, from (1), and Lemma 1067 ( $\lambda^{\star}$ is monotone), we have $b-a-\varepsilon \leqslant \lambda^{\star}([a, b)) \leqslant b-a$. Therefore, from Lemma 381 (limit is monotone), we have $\lambda^{\star}([a, b))=b-a$.

## Remark 1070.

Note that similar proofs using monotonicity provide for all $a \in \mathbb{R}, \lambda^{\star}((-\infty, a))=\lambda^{\star}((a, \infty))=\infty$.
Definition 1071 (L, Lebesgue $\sigma$-algebra). The set $\mathcal{L}$ of subsets of $\mathbb{R}$ defined by

$$
\begin{equation*}
\mathcal{L} \stackrel{\text { def. }}{=}\left\{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^{\star}(A)=\lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E^{c}\right)\right\} \tag{13.31}
\end{equation*}
$$

is called the Lebesgue $\sigma$-algebra.
Remark 1072. The set $\mathcal{L}$ is shown below to be a $\sigma$-algebra; hence its name.

$$
\mathcal{L}=\left\{E \subset \mathbb{R} \mid \forall A \subset \mathbb{R}, \lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E^{c}\right) \leqslant \lambda^{\star}(A)\right\} .
$$

Proof. Direct consequence of Lemma 211 (compatibility of pseudopartition with intersection, with $\operatorname{card}(I)$ equals 2), Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, antisymmetry), and Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra).

## Lemma 1074 ( $\mathcal{L}$ is closed under complement). $\mathcal{L}$ is closed under complement.

Proof. Let $E \in \mathcal{L}$. Let $A \subset \mathbb{R}$. Then, from the double complement law, Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative), and Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra), we have

$$
\lambda^{\star}\left(A \cap E^{c}\right)+\lambda^{\star}\left(A \cap\left(E^{c}\right)^{c}\right)=\lambda^{\star}\left(A \cap E^{c}\right)+\lambda^{\star}(A \cap E)=\lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E^{c}\right)=\lambda^{\star}(A)
$$

Therefore, $E^{c} \in \mathcal{L}$.

## Lemma 1075 ( $\mathcal{L}$ is closed under finite union). $\mathcal{L}$ is closed under finite union.

Proof. For all $n \in[2 . . \infty)$, let $P(n)$ be the property: $\forall\left(E_{i}\right)_{i \in[1 . . n]} \in \mathcal{L}, \bigcup_{i \in[1 . . n]} E_{i} \in \mathcal{L}$.
Induction: $\boldsymbol{P}(\mathbf{2})$. Let $E_{1}, E_{2} \in \mathcal{L}$. Let $E \stackrel{\text { def. }}{=} E_{1} \cup E_{2}$.
Let $A \subset \mathbb{R}$. Then, from properties of intersection, union and complement, we have

$$
A \cap E=A \cap\left(E_{1} \uplus\left(E_{1}^{c} \cap E_{2}\right)\right)=\left(A \cap E_{1}\right) \cup\left(A \cap\left(E_{1}^{c} \cap E_{2}\right)\right) .
$$

Thus, from Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive), associativity of intersection, and Definition 1071 $\left(\mathcal{L}\right.$, Lebesgue $\sigma$-algebra, $\underline{E_{2}}$, then $E_{1}$ belong to $\left.\mathcal{L}\right)$, we have

$$
\begin{aligned}
\lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E^{c}\right) & =\lambda^{\star}\left(\left(A \cap E_{1}\right) \cup\left(A \cap\left(E_{1}^{c} \cap E_{2}\right)\right)\right)+\lambda^{\star}\left(A \cap\left(E_{1}^{c} \cap E_{2}^{c}\right)\right) \\
& \leqslant \lambda^{\star}\left(A \cap E_{1}\right)+\lambda^{\star}\left(\left(A \cap E_{1}^{c}\right) \cap E_{2}\right)+\lambda^{\star}\left(\left(A \cap E_{1}^{c}\right) \cap E_{2}^{c}\right) \\
& =\lambda^{\star}\left(A \cap E_{1}\right)+\lambda^{\star}\left(A \cap E_{1}^{c}\right) \\
& =\lambda^{\star}(A) .
\end{aligned}
$$

Hence, from Lemma 1073 (equivalent definition of $\mathcal{L}$ ), we have $E \in \mathcal{L}$.
Induction; $\boldsymbol{P}(\boldsymbol{n})$ implies $\boldsymbol{P}(\boldsymbol{n}+\mathbf{1})$. Let $n \in[2 . . \infty)$. Assume that $P(n)$ holds.
Let $\left(E_{i}\right)_{i \in[1 . . n+1]} \in \mathcal{L}$. Let $E \stackrel{\text { def. }}{=} \bigcup_{i \in[1 . . n]} E_{i}$. Then, from $P(n)$, we have $E \in \mathcal{L}$, and from associativity of union, and $P(2)$, we have $\bigcup_{i \in[1 . . n+1]} E_{i}=E \cup E_{n+1} \in \mathcal{L}$.

Therefore, $P(n)$ holds for all $n \in[2 . . \infty)$.
Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection). $\mathcal{L}$ is closed under finite intersection.
Proof. Direct consequence of De Morgan's laws, Lemma 1074 ( $\mathcal{L}$ is closed under complement), and Lemma 1075 ( $\mathcal{L}$ is closed under finite union).

Lemma 1077 ( $\mathcal{L}$ is algebra of sets). $\mathcal{L}$ is an algebra of sets on $\mathbb{R}$.
Proof. Let $A \subset \mathbb{R}$. Then, from properties of intersection, and Lemma 1066 ( $\lambda^{\star}$ is homogeneous), we have $\lambda^{\star}(A \cap \mathbb{R})+\lambda^{\star}\left(A \cap \mathbb{R}^{c}\right)=\lambda^{\star}(A)+\lambda^{\star}(\varnothing)=\lambda^{\star}(A)$. Hence, $\mathbb{R} \in \mathcal{L}$.

Therefore, from Lemma 1074 ( $\mathcal{L}$ is closed under complement), Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection), Lemma 545 (closedness under intersection and set difference, $\mathcal{L}$ is closed under set difference), Lemma 637 (other equivalent definition of algebra of sets), $\mathcal{L}$ is an algebra of sets on $\mathbb{R}$.

Lemma 1078 ( $\lambda^{\star}$ is additive on $\mathcal{L}$ ).
Let $n \in[2 . . \infty) . \operatorname{Let}\left(E_{i}\right)_{i \in[1 . . n]} \in \mathcal{L}$.
Assume that the $E_{i}$ 's are pairwise disjoint. Then, $\lambda^{\star}$ is additive on $\mathcal{L}$ :

$$
\begin{equation*}
\forall A \subset \mathbb{R}, \quad \lambda^{\star}\left(A \cap \biguplus_{i \in[1 . . n]} E_{i}\right)=\sum_{i \in[1 . . n]} \lambda^{\star}\left(A \cap E_{i}\right) . \tag{13.33}
\end{equation*}
$$

Proof. For all $n \in[2 . . \infty)$, let $P(n)$ be the property: for all $\left(E_{i}\right)_{i \in[1 . . n]} \in \mathcal{L}$,

$$
\left(\forall i, j \in[1 . . n], i \neq j \Rightarrow E_{i} \cap E_{j}=\varnothing\right) \quad \Longrightarrow \quad \lambda^{\star}\left(A \cap \biguplus_{i \in[1 . . n]} E_{i}\right)=\sum_{i \in[1 . . n]} \lambda^{\star}\left(A \cap E_{i}\right) .
$$

Induction: $\boldsymbol{P}(\mathbf{2})$. Let $E_{1}, E_{2} \in \mathcal{L}$. Assume that $E_{1} \cap E_{2}=\varnothing$. Let $A \subset \mathbb{R}$. Then, from properties of intersection, union, and complement, Lemma 211 (compatibility of pseudopartition with intersection, with $\operatorname{card}(I)=2)$, and Definition $1071\left(\mathcal{L}\right.$, Lebesgue $\sigma$-algebra, $\left.E_{1} \in \mathcal{L}\right)$, we have

$$
\begin{aligned}
\lambda^{\star}\left(A \cap\left(E_{1} \uplus E_{2}\right)\right) & =\lambda^{\star}\left(\left(A \cap E_{1}\right) \uplus\left(A \cap E_{2}\right)\right) \\
& =\lambda^{\star}\left(\left[\left(A \cap E_{1}\right) \uplus\left(A \cap E_{2}\right)\right] \cap E_{1}\right)+\lambda^{\star}\left(\left[\left(A \cap E_{1}\right) \uplus\left(A \cap E_{2}\right)\right] \cap E_{1}^{c}\right) \\
& =\lambda^{\star}\left(A \cap E_{1}\right)+\lambda^{\star}\left(A \cap E_{2}\right) .
\end{aligned}
$$

Induction: $\boldsymbol{P}(\boldsymbol{n})$ implies $\boldsymbol{P}(\boldsymbol{n}+\mathbf{1})$. Let $n \in[2 . . \infty)$. Assume that $P(n)$ holds.
Let $\left(E_{i}\right)_{i \in[1 . . n+1]} \in \mathcal{L}$. Assume that for all $i, j \in[1 . . n+1], i \neq j$ implies $E_{i} \cap E_{j}=\varnothing$. Let $E$ be the disjoint union $\biguplus_{i \in[1 . . n]} E_{i}$. Then, from properties of intersection, union, and complement, $P(2)$, and $P(n)$, we have $E \cap E_{n+1}=\varnothing$, and

$$
\lambda^{\star}\left(A \cap\left(E \uplus E_{n+1}\right)\right)=\lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E_{n+1}\right)=\sum_{i \in[1 . . n]} \lambda^{\star}\left(A \cap E_{i}\right)+\lambda^{\star}\left(A \cap E_{n+1}\right) .
$$

Therefore, $P(n)$ holds for all $n \in[2 . . \infty)$.
Remark 1079. The additive property of $\lambda^{\star}$ per se actually corresponds to the case $A=\mathbb{R}$.

Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ).
$\lambda^{\star}$ is $\sigma$-additive on $(\mathbb{R}, \mathcal{L})$.
Proof. Let $\left(E_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}$. Assume that for all $i, j \in \mathbb{N}, i \neq j$ implies $E_{i} \cap E_{j}=\varnothing$.
Let $E \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} E_{n}$. Let $n \in \mathbb{N}$. Then, from Lemma 1078 ( $\lambda^{\star}$ is additive on $\mathcal{L}$, with $A \stackrel{\text { def. }}{=} \mathbb{R}$ ), and Lemma 1067 ( $\lambda^{\star}$ is monotone), we have

$$
\sum_{i \in[0 . . n]} \lambda^{\star}\left(E_{i}\right)=\lambda^{\star}\left(\bigcup_{i \in[0 . . n]} E_{i}\right) \leqslant \lambda^{\star}(E) .
$$

Thus, from Lemma 381 (limit is monotone), we have

$$
\sum_{n \in \mathbb{N}} \lambda^{\star}\left(E_{n}\right) \leqslant \lambda^{\star}(E)
$$

Hence, from Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive), we have

$$
\lambda^{\star}(E)=\sum_{n \in \mathbb{N}} \lambda^{\star}\left(E_{n}\right)
$$

Therefore, from Definition 874 ( $\sigma$-additivity), $\lambda^{\star}$ is $\sigma$-additive over $(\mathbb{R}, \mathcal{L})$.

Lemma 1081 (disjoint union of union in $\mathcal{L}$ ).
Let $I \subset \mathbb{N}$ be nonempty.
Let $J \stackrel{\text { def. }}{=}[0 . . \operatorname{card}(I))$. Let $\varphi: J \rightarrow I$ be the increasing numbering of $I$. Let $\left(E_{i}\right)_{i \in I} \in \mathcal{L}$.
Then, there exists $\left(F_{i}\right)_{i \in I} \in \mathcal{L}$ pairwise disjoint such that for for all $i \in I, F_{i} \subset E_{i}$, and we have

$$
\begin{equation*}
\forall n \in J, \quad \bigcup_{j \in[0 . . n]} E_{\varphi(j)}=\biguplus_{j \in[0 . . n]} F_{\varphi(j)} \in \mathcal{L} . \tag{13.34}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1077 ( $\mathcal{L}$ is algebra of sets), and Lemma 652 (disjoint union of union in algebra of sets).

Lemma 1082 ( $\mathcal{L}$ is closed under countable union). $\quad \mathcal{L}$ is closed under countable union.
Proof. Let $I \subset \mathbb{N}$ be nonempty. Let $\left(E_{i}\right)_{i \in I} \in \mathcal{L}$. Let $E \stackrel{\text { def. }}{=} \bigcup_{i \in I} E_{i}$.
Case $\operatorname{card}(\boldsymbol{I})<\infty$. Then, from Lemma 1075 ( $\mathcal{L}$ is closed under finite union), we have $E \in \mathcal{L}$.
Case $\operatorname{card}(\boldsymbol{I})=\infty$. Then, from Lemma 1081 (disjoint union of union in $\mathcal{L}$ ), let $\varphi: \mathbb{N} \rightarrow I$ be the increasing numbering of $I$, let $\left(F_{i}\right)_{i \in I} \in \mathcal{L}$ pairwise disjoint such that for all $n \in \mathbb{N}$, we have $\biguplus_{i \in[0 . . n]} F_{\varphi(i)}=\bigcup_{i \in[0 . . n]} E_{\varphi(i)}$.

Let $n \in \mathbb{N}$. Then, from properties of union, and monotonicity of complement, we have $E^{c} \subset F^{c}$ where $F \stackrel{\text { def. }}{=} \biguplus_{i \in[0 . . n]} F_{\varphi(i)}$. Let $A \subset \mathbb{R}$. Then, from Lemma 1067 ( $\lambda^{\star}$ is monotone, with $\left.A \cap E^{c} \subset A \cap F^{c}\right)$, Lemma $1078\left(\lambda^{\star}\right.$ is additive on $\left.\mathcal{L}\right)$, Lemma 1075 ( $\mathcal{L}$ is closed under finite union, $\underline{F \in \mathcal{L}}$ ), and Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra, with $F$ ), we have

$$
\begin{aligned}
\sum_{i \in[0 . . n]} \lambda^{\star}\left(A \cap F_{\varphi(i)}\right)+\lambda^{\star}\left(A \cap E^{c}\right) & \leqslant \sum_{i \in[0 . . n]} \lambda^{\star}\left(A \cap F_{\varphi(i)}\right)+\lambda^{\star}\left(A \cap F^{c}\right) \\
& =\lambda^{\star}(A \cap F)+\lambda^{\star}\left(A \cap F^{c}\right)=\lambda^{\star}(A)
\end{aligned}
$$

Thus, from Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive), distributivity of intersection over union, and Lemma 381 (limit is monotone), we have

$$
\lambda^{\star}\left(A \cap \bigcup_{n \in \mathbb{N}} F_{\varphi(n)}\right)+\lambda^{\star}\left(A \cap E^{c}\right) \leqslant \sum_{n \in \mathbb{N}} \lambda^{\star}\left(A \cap F_{\varphi(n)}\right)+\lambda^{\star}\left(A \cap E^{c}\right) \leqslant \lambda^{\star}(A) .
$$

Hence, since the limit is a function, we have $\bigcup_{n \in \mathbb{N}} F_{\varphi(n)}=\bigcup_{n \in \mathbb{N}} E_{\varphi(n)}=E$, and

$$
\lambda^{\star}(A \cap E)+\lambda^{\star}\left(A \cap E^{c}\right) \leqslant \lambda^{\star}(A)
$$

Therefore, from Lemma 1073 (equivalent definition of $\mathcal{L}$ ), we have $E \in \mathcal{L}$.

## Lemma 1083 (rays are Lebesgue-measurable). Let $a \in \mathbb{R}$. Then, we have

$$
\begin{equation*}
(a, \infty),(-\infty, a],(-\infty, a),[a, \infty), \in \mathcal{L} \tag{13.35}
\end{equation*}
$$

Proof. (1). $(a, \infty) \in \mathcal{L}$.
Let $A \subset \mathbb{R}$. Let $\alpha \stackrel{\text { def. }}{=} \lambda^{\star}(A \cap(a, \infty))+\lambda^{\star}(A \cap(-\infty, a])$. Let us show that $\alpha \leqslant \lambda^{\star}(A)$.
Case $\boldsymbol{\lambda}^{\star}(\boldsymbol{A})=\infty$. Trivial. Case $\boldsymbol{\lambda}^{\star}(\boldsymbol{A})$ finite. Let $\varepsilon>0$. From Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), and Lemma 11 (finite infimum), let $\left(I_{n}\right)_{n \in \mathbb{N}} \in C_{A}$ such that

$$
\lambda^{\star}(A) \leqslant \sum_{n \in \mathbb{N}} \ell\left(I_{n}\right) \leqslant \lambda^{\star}(A)+\varepsilon
$$

For all $n \in \mathbb{N}$, let $I_{n}^{\prime} \stackrel{\text { def. }}{=} I_{n} \cap(a, \infty)$ and $I^{\prime \prime}{ }_{n} \stackrel{\text { def. }}{=} I_{n} \cap(-\infty, a]$. Let $n \in \mathbb{N}$. From Lemma 235 (intervals are closed under finite intersection), $I_{n}^{\prime}$ and $I^{\prime \prime}{ }_{n}$ are intervals (possibly empty). Then, from Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval), and since $I_{n}^{\prime}$ and $I^{\prime \prime}{ }_{n}$ are contiguous, we
have $\lambda^{\star}\left(I_{n}^{\prime}\right)+\lambda^{\star}\left(I^{\prime \prime}{ }_{n}\right)=\ell\left(I_{n}^{\prime}\right)+\ell\left(I^{\prime \prime}{ }_{n}\right)=\ell\left(I_{n}\right)$. Hence, from Lemma 1067 ( $\lambda^{\star}$ is monotone), and Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive), we have

$$
\alpha \leqslant \lambda^{\star}\left(\bigcup_{n \in \mathbb{N}} I_{n}^{\prime}\right)+\lambda^{\star}\left(\bigcup_{n \in \mathbb{N}} I^{\prime \prime}{ }_{n}\right) \leqslant \sum_{n \in \mathbb{N}}\left(\lambda^{\star}\left(I_{n}^{\prime}\right)+\lambda^{\star}\left(I_{n}^{\prime \prime}{ }_{n}\right)\right)=\sum_{n \in \mathbb{N}} \ell\left(I_{n}\right) \leqslant \lambda^{\star}(A)+\varepsilon
$$

Therefore, from Lemma 381 (limit is monotone), we have $\alpha \leqslant \lambda^{\star}(A)$, and from Lemma 1073 (equivalent definition of $\mathcal{L}$ ), we have $(a, \infty) \in \mathcal{L}$.
(2). $(-\infty, a] \in \mathcal{L}$.

Direct consequence of Lemma 1074 ( $\mathcal{L}$ is closed under complement), and (1).
(3). $(-\infty, a) \in \mathcal{L}$.

Direct consequence of Lemma 1082 ( $\mathcal{L}$ is closed under countable union), and (2) with

$$
(-\infty, a)=\bigcup_{n \in \mathbb{N}}\left(-\infty, a-\frac{1}{n+1}\right]
$$

(4). $[a, \infty) \in \mathcal{L}$.

Direct consequence of Lemma 1074 ( $\mathcal{L}$ is closed under complement), and (3).

## Lemma 1084 (intervals are Lebesgue-measurable).

Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have
(13.36)

$$
(a, b),[a, b],[a, b),(a, b] \in \mathcal{L}
$$

Proof. Direct consequence of Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection), and Lemma 1083 (rays are Lebesgue-measurable) with

$$
\begin{array}{ll}
(a, b)=(-\infty, b) \cap(a, \infty), & {[a, b]=(-\infty, b] \cap[a, \infty),} \\
{[a, b)=(-\infty, b) \cap[a, \infty),} & (a, b]=(-\infty, b] \cap(a, \infty) .
\end{array}
$$

## Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra).

$(\mathbb{R}, \mathcal{L})$ is a measurable space.
Proof. Direct consequence of Lemma 1077 ( $\mathcal{L}$ is algebra of sets), Definition 634 (algebra of sets), Lemma 1082 ( $\mathcal{L}$ is closed under countable union), and Lemma 696 ( $\sigma$-algebra).
Lemma $1086\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{L}\right) . \quad\left(\mathbb{R}, \mathcal{L}, \lambda_{\mid \mathcal{L}}^{\star}\right)$ is a measure space.

Proof. Direct consequence of Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra), Lemma 1065 ( $\lambda^{\star}$ is nonnegative), Lemma 1066 ( $\lambda^{\star}$ is homogeneous), Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ), and Lemma 913 (measure).

## Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L}) . \quad$ We have $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$.

Proof. Direct consequence of Lemma 710 (properties of $\sigma$-algebra generation, monotonicity), Lemma 796 (Borel subsets of $\mathbb{R}$ ), and Lemma 1084 (intervals are Lebesgue-measurable).

Lemma $1088\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{B}(\mathbb{R})\right) . \quad\left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda_{\mid \mathcal{B}(\mathbb{R})}^{\star}\right)$ is a measure space.
Proof. Direct consequence of Lemma $1086\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{L}\right)$, and Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub-$\sigma$-algebra of $\mathcal{L}$ ).

## Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).

There exists a unique measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that generalizes the length of bounded open intervals. This measure is denoted $\lambda \stackrel{\text { def. }}{=} \lambda_{\mid \mathcal{B}(\mathbb{R})}^{\star}$; it is called the (Borel-)Lebesgue measure on Borel subsets (of $\mathbb{R}$ ).

Proof. Existence. Direct consequence of Lemma 1088 ( $\lambda^{\star}$ is measure on $\mathcal{B}(\mathbb{R})$ ), and Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval, for open intervals).

## Uniqueness.

Let $G \stackrel{\text { def. }}{=}\{[a, b)\}_{a<b}$. Then, we have $G \subset \mathcal{P}(\mathbb{R})$ and $G \neq \varnothing$. Hence, from Lemma 235 (intervals are closed under finite intersection), and Definition 602 ( $\pi$-system), $G$ is a $\pi$-system.
From Lemma 796 (Borel subsets of $\mathbb{R}$, $\{(a, b)\}_{a<b}$ and $\{[a, b)\}_{a<b}$ generate $\mathcal{B}(\mathbb{R})$ ), Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra, generators belong to $\mathcal{B}(\mathbb{R})$ ), we have

$$
(\forall a, b \in \mathbb{R}, \quad a<b \quad \Longrightarrow \quad(a, b),[a, b) \in \mathcal{B}(\mathbb{R})), \quad \text { and } \quad \Sigma_{\mathbb{R}}(G)=\mathcal{B}(\mathbb{R})
$$

For all $n \in \mathbb{N}$, let $I_{n} \stackrel{\text { def. }}{=}[n, n+1)$. Then, $\left(I_{n}\right)_{n \in \mathbb{N}}$ is obviously pairwise disjoint, and from the Archimedean property of $\mathbb{R}$, we have $\mathbb{R}=\biguplus_{n \in \mathbb{N}} I_{n}$. Let $n \in \mathbb{N}$. Then, $I_{n} \in G$, and from Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval), we have $\lambda^{\star}\left(I_{n}\right)=1<\infty$.

Let $\mu$ be a measure on $\mathcal{B}(\mathbb{R})$ such that, for all $a, b \in \mathbb{R}, a<b$ implies $\mu((a, b))=b-a$.
Let $a, b \in \mathbb{R}$. Assume that $a<b$. For all $p \in \mathbb{N}$, let $A_{p} \stackrel{\text { def. }}{=}\left(a-\frac{1}{p+1}, b\right)$. Then, from the nonincreasing property of $\left(p \mapsto \frac{1}{p+1}\right)$ in $\mathbb{R}_{+}$and its limit 0 when $p \rightarrow \infty$, the definition of inclusion, additive group properties of $\mathbb{R}$, linearity of the limit in $\mathbb{R}$, and Lemma 927 (measure is continuous from above, with $\mu_{1}$ and $\left.\left(A_{p}\right)_{p \in \mathbb{N}}\right)$, the sequence $\left(A_{p}\right)_{p \in \mathbb{N}}$ is nonincreasing, for all $p \in \mathbb{N}$, we have $\mu_{1}\left(A_{p}\right)=b-\overline{a+\frac{1}{p+1}}<\infty$, and

$$
\mu_{1}([a, b))=\mu_{1}\left(\bigcap_{p \in \mathbb{N}} A_{p}\right)=\inf _{p \in \mathbb{N}} \mu_{1}\left(A_{p}\right)=\lim _{p \rightarrow \infty}\left(b-a+\frac{1}{p+1}\right)=b-a .
$$

Hence, from Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval), $\mu$ and $\lambda_{\mid \mathcal{B}(\mathbb{R})}^{\star}$ coincide on $G$.
Therefore, from Lemma 992 (uniqueness of measures extended from a $\pi$-system, with $X \stackrel{\text { def. }}{=} \mathbb{R}$, $\underline{\Sigma} \stackrel{\text { def. }}{=} \mathcal{B}(\mathbb{R}), \mu_{1} \stackrel{\text { def. }}{=} \lambda^{\star}, \mu_{2}=\mu$, and $X_{n} \stackrel{\text { def. }}{=} I_{n}$, we have, for all $A \in \mathcal{B}(\mathbb{R}), \mu(A)=\lambda_{\mid \mathcal{B}(\mathbb{R})}^{\star}(A)$.

Remark 1091. Note that $\mathbf{N}\left(\mathbb{R}, \mathcal{L}, \lambda^{\star}\right)=\mathbf{N}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \subset \mathcal{L}$, e.g. see [34, Ex. 2.33 pp. 92-94].

## Lemma 1092 (Lebesgue measure generalizes length of interval).

Let $a, b \in \mathbb{R}$. Assume that $a \leqslant b$. Then, we have $\lambda(([a, b))=b-a$.
Proof. Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ), and Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval).

Remark 1093. Note that similarly to $\lambda^{\star}$, we can prove for all $a \in \mathbb{R}, \lambda((-\infty, a))=\lambda(((a, \infty))=\infty$.
Lemma 1094 (Lebesgue measure is $\sigma$-finite). $\quad$ The measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is $\sigma$-finite.
Proof. Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ), Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval, with $b=-a \in \mathbb{N}$ ), the Archimedean property of $\mathbb{R}$, and Definition 885 ( $\sigma$-finiteness, with $A_{n} \stackrel{\text { def. }}{=}(-n, n)$ ).

Lemma 1095 (Lebesgue measure is diffuse). $\quad$ The measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is diffuse.
Proof. Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ), Lemma 1069 $\left(\lambda^{\star}\right.$ generalizes length of interval, with $\left.b=a\right)$, and Definition 856 (diffuseness, $\{a\}=[a, a]$ ).

## Chapter 14

## Simple function

## Contents

14.1 Indicator function ..... 236
14.2 Simple function ..... 238
14.3 Nonnegative simple function ..... 245
14.4 Adapted sequence ..... 248

[^1] number of values in a normed vector space over a valued field of characteristic 0 (namely $\mathbb{R}$ or $\mathbb{C}$ ).

### 14.1 Indicator function

Remark 1097. In this section, the measurable number space $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$, or $\overline{\mathbb{R}}$ (see Definition 748).
Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, set of measurable indicator functions). $\quad$ Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. The set of measurable indicator functions on $\mathbb{K}$ is denoted $\mathcal{I} \mathcal{F}_{\mathfrak{K}}(X, \Sigma)$ (or simply $\left.\mathcal{I} \mathcal{F}_{\mathbb{K}}\right)$; it is defined by $\mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma) \stackrel{\text { def. }}{=}\left\{\mathbb{1}_{A} \mid A \in \Sigma\right\}$.

When $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, it is simply denoted $\mathcal{I F}(X, \Sigma)$ (or $\mathcal{I F})$.
Remark 1099. Functions in $\mathcal{I F}_{\mathfrak{K}}$ are shown below to be measurable; hence its name.
Lemma 1100 (indicator and support are inverse). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $A \in \Sigma$, let $f \in \mathcal{I} \mathcal{F}_{\mathfrak{k}}$. Then, we have
(14.1) $\quad \mathbb{1}_{A} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}, \quad\left\{f \neq 0_{\mathbb{K}}\right\} \in \Sigma, \quad\left\{\mathbb{1}_{A} \neq 0_{\mathbb{K}}\right\}=A, \quad$ and $\quad \mathbb{1}_{\left\{f \neq 0_{\mathfrak{K}}\right\}}=f$.

Proof. Direct consequence of Definition 748 (measurability and structure), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions), the definition of the indicator function, and the definition of inverse image.

Lemma 1101 ( $\mathcal{I F}_{\mathfrak{K}}$ is measurable).
Let $(X, \Sigma)$ be a measurable space.
Let $\mathbb{K}$ be a measurable number space. Then, we have $\mathcal{I} \mathcal{F}_{\mathfrak{K}} \subset \mathcal{M}_{\mathbb{K}}$.
Moreover, when $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, we have $\mathcal{I F} \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$.
Proof. Direct consequence of Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions), Lemma 811 (measurability of indicator function), Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}, \underline{\mathcal{M}_{\mathbb{R}} \subset \mathcal{M}}$ ), nonnegativeness of indicator function $\left(\mathcal{I F} \subset \mathcal{M}_{+}\right)$, and Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\left.\mathcal{M}_{+} \subset \mathcal{M}\right)$.

Lemma 1102 ( $\mathcal{I F}_{\mathbb{K}}$ is $\sigma$-additive).
Let $(X, \Sigma)$ be a measurable space.
Let $\mathbb{K}$ be a measurable number space. Let $I \subset \mathbb{N}$. Let $\left(A_{i}\right)_{i \in I} \in \Sigma$ be pairwise disjoint.
Then, we have $\sum_{i \in I} \mathbb{1}_{A_{i}}=\mathbb{1}_{\uplus_{i \in I} A_{i}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}$.
Proof. Direct consequence of Definition 748 (measurability and structure), Lemma 525 (masking is $\sigma$-additive, with $E \stackrel{\text { def. }}{=} \mathbb{K})$, Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions $)$, Definition 741 (measurable space, $\underline{\Sigma}$ is a $\sigma$-algebra) , and Lemma $696\left(\sigma\right.$-algebra, $\underline{P_{\Sigma} \Rightarrow U c}$ ).

Lemma 1103 ( $\mathcal{I} \mathcal{F}_{\mathbb{K}}$ is closed under multiplication). $\quad$ Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $A, B \in \Sigma$. Then, we have $\mathbb{1}_{A} \mathbb{1}_{B}=\mathbb{1}_{A \cap B} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}$.

Proof. Direct consequence of Definition 748 (measurability and structure), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, set of measurable indicator functions), the formula for the indicator of intersection (product), and Lemma 697 (equivalent definition of $\sigma$-algebra, $\underline{\left.P_{\Sigma} \Rightarrow I\right) . ~}$

Remark 1104.
We recall the notation $\mathbb{1}_{U}^{V}$ to denote the indicator function of $U$ defined on $V$ (when $U \subset V$ ).

Lemma 1105 ( $\mathcal{I} \mathcal{F}_{\mathbb{K}}$ is closed under extension by zero). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \mathbb{K}$ and $\hat{f}: X \rightarrow \mathbb{K}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$. Then, we have $f_{\left.\right|_{A}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma)$. If so, there exists $B \in \Sigma$ such that $B \subset A, f_{\left.\right|_{A}}=\mathbb{1}_{B}^{A}$, and $\hat{f} \mathbb{1}_{A}=\mathbb{1}_{B}^{X}$.

Proof. "Left" implies "right". Direct consequence of Definition 748 (measurability and structure), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, set of measurable indicator functions, with $X \stackrel{\text { def. }}{=} A$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma \bar{\cap} A$, thus $f_{\left.\right|_{A}}=\mathbb{1}_{B}^{A}$ with $B \in \Sigma \bar{\cap} A$ ), Lemma 764 (measurability of measurable subspace, with $Y \stackrel{\text { def. } A}{=}$ belongs to $\Sigma$, thus $B \in \Sigma$ and $B \subset A$ ), Lemma 524 (restriction is masking, with $E \stackrel{\text { def. }}{=} \mathbb{K}$ ), the definition of the indicator function $\left(\hat{f} \mathbb{1}_{A}=\mathbb{1}_{B}^{X}\right)$, and Definition $1098\left(\overline{\mathcal{I} \mathcal{F}_{\mathfrak{K}}}\right.$, set of measurable indicator functions, $\left.\hat{f} \mathbb{1}_{A} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}(X, \Sigma)\right)$.
"Right" implies "left". Direct consequence of Definition 748 (measurability and structure), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{k}}\right.$, set of measurable indicator functions, $\hat{f} \mathbb{1}_{A}=\mathbb{1}_{B}^{X}$ with $B \in \Sigma$ such that $\underline{B \subset A}$ ), Lemma 764 (measurability of measurable subspace, with $Y \stackrel{\text { def. }}{=} A \in \Sigma$, thus $B \in \Sigma \bar{\cap} A$ ), Lemma 524 (restriction is masking, with $\left.E \stackrel{\text { def. }}{=} \mathbb{K}, f_{\left.\right|_{A}}=\mathbb{1}_{B}^{A}\right)$, and Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions, with $X \stackrel{\text { def. }}{=} A$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma \bar{\cap} A$, thus $f_{\left.\right|_{A}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A)$ ).
Therefore, we have the equivalence and both representations.
Lemma 1106 ( $\mathcal{I F}_{\mathbb{K}}$ is closed under restriction).
Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space. Let $f: X \rightarrow \mathbb{K}$. Then, we have $f \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}(X, \Sigma)$ iff for all $A \in \Sigma, f_{\left.\right|_{A}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A)$.

Proof. "Left" implies "right". Direct consequence of Definition 748 (measurability and structure), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions, $\underline{\left.f=\mathbb{1}_{B}^{X} \text { with } B \in \Sigma\right) \text {, the rule }}$ for the restriction of an indicator function $\left(f_{\left.\right|_{A}}=\mathbb{1}_{B \cap A}^{A}\right)$, Lemma 764 (measurability of measurable subspace, $\underline{B \cap A \in \Sigma \bar{\cap} A}$ ), and Definition $1098\left(\mathcal{I}_{\mathfrak{K}}\right.$, set of measurable indicator functions, with $X \stackrel{\text { def. }}{=} A$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma \bar{\cap} A$, thus $\left.f_{\left.\right|_{A}} \in \mathcal{I F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A)\right)$.
"Right" implies "left". Direct consequence of Lemma 697 (equivalent definition of $\sigma$-algebra, A def. $X \in \Sigma$ ), Lemma 764 (measurability of measurable subspace, $\Sigma \bar{\cap} X=\Sigma$ ), and the definition of restriction of function $\left(f_{\left.\right|_{X}}=f\right.$, thus $\left.f \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma)\right)$.
Therefore, we have the equivalence.

## Lemma 1107 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$, masking is $\sigma$-additive).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be a measurable number space.
Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \mathbb{K}$.
Then, we have $f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}}$, and $f \mathbb{1}_{A} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}$ iff for all $i \in I$, $f \mathbb{1}_{A_{i}} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}$.
Proof. (1). Identity. Direct consequence of Definition 748 (measurability and structure), and Lemma 525 (masking is $\sigma$-additive).
"Left" implies "right". Direct consequence of Definition 209 (pseudopartition, $A_{i} \subset A$ ), Lemma 1100 (indicator and support are inverse, $f \mathbb{1}_{A}=\mathbb{1}_{B}$ with $B \in \Sigma$ such that $B \subset A$ ), and Lemma $1103\left(\mathcal{I}_{\mathfrak{K}}\right.$ is closed under multiplication, $\left.f \mathbb{1}_{A_{i}}=\mathbb{1}_{B \cap A_{i}} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}\right)$.
"Right" implies "left". Direct consequence of Definition 748 (measurability and structure), Lemma 1100 (indicator and support are inverse, $f \mathbb{1}_{A_{i}}=\mathbb{1}_{B_{i}}$ with $B_{i} \in \Sigma$ and $B_{i} \subset A_{i}$ ), compatibility of inclusion with disjoint union (the $B_{i}$ 's are pairwise disjoint), Lemma 1102 $\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is $\sigma$-additive, $\left.\sum_{i \in I} \mathbb{1}_{B_{i}} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}\right)$, and (1) $\left(f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}\right)$.
Therefore, we have the equivalence.

### 14.2 Simple function

## Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. The vector subspace of finite vector linear combinations of indicator functions of measurable subsets is called vector space of (vector) simple functions (from $X$ ) to $E$; it is denoted $\mathcal{S F}_{E}(X, \Sigma)$ (or simply $\mathcal{S \mathcal { F } _ { E }}$ ); it is defined by

$$
\begin{equation*}
\mathcal{S F}_{E}(X, \Sigma) \stackrel{\text { def. }}{=}\left\{\sum_{i \in[0 . . n]} f_{i} u_{i} \mid n \in \mathbb{N} \wedge\left(f_{i}\right)_{i \in[0 . . n]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}} \wedge\left(u_{i}\right)_{i \in[0 . . n]} \in E\right\} . \tag{14.2}
\end{equation*}
$$

Let $f \in \mathcal{S F}_{E}$. A triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times E$ such that $f=\sum_{i \in[0 . . n]} \mathbb{1}_{A_{i}} u_{i}$ is called simple representation (of $f$ ). If so, for all $x \in X \backslash \bigcup_{i \in[0 . . n]} A_{i}$, we have $f(x)=0$.

Moreover, when $E \stackrel{\text { def. }}{=} \mathbb{K}$, we have $\mathcal{S} \mathcal{F}_{\mathbb{K}}=\operatorname{span}\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right)$, and when $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, it is simply denoted $\mathcal{S F}(X, \Sigma)$ (or $\mathcal{S F}$ ).

Proof. Direct consequence of the definition of the indicator function, Lemma 71 (zero times yields zero, $\underline{0_{k} u=0_{E}}$ ), Definition 61 (vector space, $\left(E,+_{E}\right)$ is an abelian group), Definition 1098 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$, set of measurable indicator functions), and Lemma 1100 (indicator and support are inverse).

Remark 1109. Note that in a simple representation, the values $u_{i}$ 's may not be unique and may not be values of the function, and the supports $A_{i}$ 's may be empty, may not be related to preimages of $u_{i}$ 's, may overlap and thus may not form a partition of $X$.

Remark 1110. We recall that $\{f=u\}$ denotes the subset $f^{-1}(\{u\})$.

## Lemma 1111 ( $\mathcal{S} \mathcal{F}_{E}$ canonical representation).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Let $f: X \rightarrow E$. Then, we have $f \in \mathcal{S F}_{E}$ iff $f(X)$ is finite, and for all $u \in f(X),\{f=u\}$ belongs to $\Sigma$.

If so, we have the following canonical representation:

$$
\begin{equation*}
f=\sum_{u \in f(X)} \mathbb{1}_{\{f=u\}} u \tag{14.3}
\end{equation*}
$$

i.e. there exists a triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times E$ which is unique up to a permutation of $[0 . . n]$ such that

$$
\begin{align*}
\forall i \in[0 . . n], & A_{i}=\left\{f=u_{i}\right\} \neq \varnothing  \tag{14.4}\\
& \left(A_{i}\right)_{i \in[0 . . n]} \text { forms a partition of } X,  \tag{14.5}\\
& f(X)=\left\{u_{i} \mid i \in[0 . . n]\right\}  \tag{14.6}\\
& f=\sum_{i \in[0 . . n]} \mathbb{1}_{A_{i}} u_{i} . \tag{14.7}
\end{align*}
$$

Moreover, when $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, the triple is unique, for instance by specifying

$$
\begin{equation*}
\forall i \in[0 . . n-1], \quad u_{i}<u_{i+1} . \tag{14.8}
\end{equation*}
$$

Proof. "Left" implies "right". Assume that $f \in \mathcal{S} \mathcal{F}_{E}$. Then, from Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation), let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times E$ be a simple representation of $f$. Thus, from the definition of the indicator function, $f$ can only take the values $\sum_{i \in[0 . . n]} \delta_{i} u_{i}$ where $\left(\delta_{i}\right)_{i \in[0 . . n]} \in\{0,1\}$. Hence, the cardinality of $f(X)$ is at most $2^{n+1}$.

Let $\left(\delta_{i}\right)_{i \in[0 . . n]} \in\{0,1\}$. Let $u \stackrel{\text { def. }}{=} \sum_{i \in[0 . . n]} \delta_{i} u_{i} \in f(X)$. Let $I_{u} \subset[0 . . n]$ such that $i \in I_{u}$ iff $\delta_{i}=1$. Then, we have ${ }^{1} u=\sum_{i \in I_{u}} u_{i}$. Unfortunately, several partial sums of $u_{i}$ 's may lead to the same value $u$. Thus, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable union and intersection), we have

$$
\{f=u\}=\bigcup_{\substack{I_{u} \subset \mathcal{P}([0 . . n]) \\ \sum_{i \in I_{u}} u_{i}=u}}\left(\bigcap_{i \in I_{u}} A_{i}\right) \in \Sigma .
$$

From properties of inverse image, the collection $(\{f=u\})_{u \in f(X)}$ makes a finite partition of the whole set $X$. Let $x \in X$. Let $u \stackrel{\text { def. }}{=} f(x) \in E$. Then, from Definition 61 (vector space), we have $f(x)=u=1 u=\mathbb{1}_{\{f=u\}}(x) u$. Hence, property (14.3) holds. And when $E \stackrel{\text { def. }}{=} \mathbb{R}$, property (14.7) is equivalent, up to a nondecreasing reordering of $f(X)$.
"Right" implies "left". Let $n \stackrel{\text { def. }}{=} \operatorname{card}(f(X))-1 \in \mathbb{N}$. Let $\left(u_{i}\right)_{i \in[0 . . n]} \in E$ be the $n+1$ distinct values taken by the function $f$, i.e. such that $f(X)=\left\{u_{i} \mid i \in[0 . . n]\right\}$. For all $i \in[0 . . n]$, let

$$
A_{i} \stackrel{\text { def. }}{=}\left\{f=u_{i}\right\} \in \Sigma .
$$

Then, from the definition and properties of image and inverse image, and the definition of partition, the $n$ subsets $\left(A_{i}\right)_{i \in[0 . . n]}$ constitutes a partition of $X$. Thus, from the definition of the indicator function, we have $f=\sum_{i \in[0 . . n]} \mathbb{1}_{A_{i}} u_{i}$. Hence, from Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation), we have $f \in \mathcal{S} \mathcal{F}_{E}$.

Uniqueness. Direct consequence of (14.4)-(14.8) since $\operatorname{card}(f(X))=n+1$ (i.e. $n$ is unique), $f(X)=\left\{a_{i} \mid i \in[0 . . n]\right\}$ with $a_{i}<a_{i+1}$ (i.e. the $a_{i}$ 's are unique), and $A_{i}=\left\{f=a_{i}\right\}$ (i.e. the $A_{i}$ 's are unique).

Therefore, we have the equivalence, and the representation is unique when $E \stackrel{\text { def. }}{=} \mathbb{R}$.

Remark 1112. To sum up, in a canonical representation, the set of $u_{i}$ values is unique and is the set of values of the function, the supports $A_{i}$ 's are the nonempty preimages of the $u_{i}$ 's and form a partition of $X$. Hence, the canonical representation of the zero function is $\mathbb{1}_{X} 0_{E}$.

Note that properties (14.3) and (14.7) are identical, up to a permutation of $f(X)$.

Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a measurable normed vector space over $\mathbb{K}$. Then, we have $\mathcal{S} \mathcal{F}_{E} \subset \mathcal{M}_{E}$.

Moreover, when $E \stackrel{\text { def. }}{=} \mathbb{K}$, we have $\mathcal{I} \mathcal{F}_{\mathbb{K}} \subset \mathcal{S} \mathcal{F}_{\mathbb{K}} \subset \mathcal{M}_{\mathbb{K}}$, and when $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, we have $\mathcal{I F} \subset \mathcal{S F} \subset \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$.

Proof. Direct consequence of Definition 748 (measurability and structure), Lemma 1111 ( $\mathcal{S F}_{E}$ ca-

 finite is $\left.\mathcal{M}_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}\right)$.

[^2]
## Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Let $f: X \rightarrow E$. Then, we have $f \in \mathcal{S} \mathcal{F}_{E}$ iff there exist $n \in \mathbb{N},\left(A_{i}\right)_{i \in[0 . . n]} \in \Sigma$, and $\left(u_{i}\right)_{i \in[0 . . n]} \in E$ such that

$$
\begin{align*}
\forall i \in[0 . . n], & A_{i} \subset\left\{f=u_{i}\right\},  \tag{14.9}\\
& X=\biguplus_{i \in[0 . . n]} A_{i},  \tag{14.10}\\
& f=\sum_{i \in[0 . . n]} \mathbb{1}_{A_{i}} u_{i} . \tag{14.11}
\end{align*}
$$

If so, the triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times E$ is called disjoint representation (of $f$ ).
Proof. Direct consequence of Lemma 1111 ( $\mathcal{S} \mathcal{F}_{E}$ canonical representation, "left" implies "right"), and Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation, "right" implies "left").

Remark 1115. In a disjoint representation, the values $u_{i}$ 's may not be unique ( $u_{i}=u_{j}$ with $i \neq j$ is possible) and may not be the values of the function ( $A_{i}=\varnothing$ is possible). More precisely, the supports $A_{i}$ 's may be empty but are subsets of the associated preimage of $u_{i}$ (thus the values that are not taken by the function are associated with an empty support). They form a pseudopartition of $X$ (see Definition 209).

Note that the nonempty $A_{i}$ 's form a partition of $(\{f=u\})_{u \in f(X)}$ (see next lemma).

## Lemma $1116\left(\mathcal{S F}_{E}\right.$ disjoint representation is subpartition of canonical representa-

 tion). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Let $f \in \mathcal{S} \mathcal{F}_{E}$. Let $\left(n,\left(A_{i}\right)_{i \in[0 . . m]},\left(u_{i}\right)_{i \in[0 . . m]}\right) \in \mathbb{N} \times \Sigma \times E$ be a disjoint representation of $f$, and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(v_{j}\right)_{j \in[0 . . m]}\right) \in \mathbb{N} \times \Sigma \times E$ be its canonical representation.For all $j \in[0 . . m]$, let $I_{j}^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid \varnothing \neq A_{i} \subset B_{j}\right\}$. Let $I^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid A_{i} \neq \varnothing\right\}$.
Then, we have

$$
\begin{equation*}
\forall j \in[0 . . m], \quad\left(\forall i \in I_{j}^{\prime}, u_{i}=v_{j}\right) \quad \wedge \quad B_{j}=\biguplus_{i \in I_{j}^{\prime}} A_{i} \tag{14.12}
\end{equation*}
$$

$$
\begin{equation*}
\left(\forall p, q \in[0 . . m], p \neq q \Rightarrow I_{p}^{\prime} \cap I_{q}^{\prime}=\varnothing\right) \quad \wedge \quad \biguplus_{j \in[0 . . m]} I_{j}^{\prime}=I^{\prime} \tag{14.13}
\end{equation*}
$$

Proof. Let $i \in[0 . . n]$ and $j \in[0 . . m]$. Assume that $A_{i} \neq \varnothing$.
(0a) $\boldsymbol{A}_{\boldsymbol{i}} \subset \boldsymbol{B}_{\boldsymbol{j}} \Rightarrow \boldsymbol{u}_{\boldsymbol{i}}=\boldsymbol{v}_{\boldsymbol{j}}$. Let $x \in A_{i} \subset B_{j}$. Then, from Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation, $\left.f(x)=u_{i}\right)$, and Lemma $1111\left(\mathcal{S F}_{E}\right.$ canonical representation, $\left.f(x)=v_{j}\right)$, we have $u_{i}=v_{j}$.
(0b) $\boldsymbol{u}_{\boldsymbol{i}}=\boldsymbol{v}_{\boldsymbol{j}} \Rightarrow \boldsymbol{A}_{\boldsymbol{i}} \subset \boldsymbol{B}_{\boldsymbol{j}}$. Then, from Lemma 1114 ( $\mathcal{S} \mathcal{F}_{E}$ disjoint representation), and Lemma $1111\left(\mathcal{S F} \mathcal{E}_{E}\right.$ canonical representation), we have $A_{i} \subset\left\{f=u_{i}\right\}=\left\{f=v_{j}\right\}=B_{j}$.
(1). Let $j \in[0 . . m]$. From Lemma $1114\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation), the $A_{i}$ 's are pairwise disjoint. Let $A_{j}^{\prime} \stackrel{\text { def. }}{=} \biguplus_{i \in I_{j}^{\prime}} A_{i}$. Let $x \in B_{j}$. Then, from Lemma $1111\left(\mathcal{S} \mathcal{F}_{E}\right.$ canonical representation,
 there exists $i \in[0 . . n]$ such that $x \in A_{i}$, i.e. $A_{i} \neq \varnothing$ and $u_{i}=f(x)=v_{j}$. Thus, from (0b), we have $i \in I_{j}^{\prime}$, and $B_{j} \subset A_{j}^{\prime}$. Hence, since the other inclusion is obvious, we have equality, and from (0a), property (14.12) holds.
(2a). Let $p, q \in[0 . . m]$. Assume that $p \neq q$.
Let $i \in I_{p}^{\prime} \cap I_{q}^{\prime}$. Then, from Lemma $1111\left(\mathcal{S \mathcal { F } _ { E }}\right.$ canonical representation, pairwise disjunction), we have $\varnothing \neq A_{i} \subset B_{p} \cap B_{q}=\varnothing$, which is impossible. Hence, we have $I_{p}^{\prime} \cap I_{q}^{\prime}=\varnothing$.
(2b). Let $i \in I^{\prime}$, i.e. $A_{i} \neq \varnothing$. Then, from Lemma $1114\left(\mathcal{S F} \mathcal{F}_{E}\right.$ disjoint representation, $u_{i} \in f(X)$ ), and Lemma $1111\left(\mathcal{S F} \mathcal{F}_{E}\right.$ canonical representation, $\underline{\left.f(X)=\left\{v_{j} \mid j \in[0 . . m]\right\}\right) \text {, there exists } j \in[0 . . m]}$
such that $u_{i}=v_{j}$. Thus, from (0b), we have $i \in I_{j}^{\prime}$, and $I^{\prime} \subset \uplus_{j \in[0 . . m]} I_{j}^{\prime}$. Hence, since the other inclusion is obvious, we have equality, and from (2a), property (14.13) holds.

Therefore, both properties hold.

Lemma 1117 (SF ${ }_{E}$ is vector space).
Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Then, $\mathcal{S \mathcal { F } _ { E }}$ is a subvector space of $E^{X}$. Moreover, when $E \stackrel{\text { def. }}{=} \mathbb{K}, \mathcal{S} \mathcal{F}_{\mathbb{K}}$ is a subalgebra of $\mathbb{K}^{X}$.

Let $f, g \in \mathcal{S F}_{E}$. Let $a \in \mathbb{K}$. Let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right)$ and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(v_{j}\right)_{j \in[0 . . m]}\right)$ in $\mathbb{N} \times \Sigma \times E$ be representations of $f$ and $g$. Let $N \stackrel{\text { def. }}{=} n m+n+m$. Let $\varphi:[0 . . N] \rightarrow[0 . . n] \times[0 . . m]$ be a bijection. For all $(i, j) \in[0 . . n] \times[0 . . m]$, let $C_{i, j} \stackrel{\text { def. }}{=} A_{i} \cap B_{j}$, and $w_{i, j}^{+} \stackrel{\text { def. }}{=} u_{i}+v_{j}$.
Then, if both representations are disjoint, $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . . N]},\left(w_{\varphi(k)}^{+}\right)_{k \in[0 . . N]}\right) \in \mathbb{N} \times \Sigma \times E$ is also a disjoint representation of $f+g$, and $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times E$ is a representation of af, of the same nature as that of $f$ (when $a \neq 0_{\mathbb{K}}$ for canonical representation).

Moreover, when $E \stackrel{\text { def. }}{=} \mathbb{K}$, for all $(i, j) \in[0 . . n] \times[0 . . m]$, let $w_{i, j}^{*} \stackrel{\text { def. }}{=} u_{i} * v_{j}$. Then, $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . . N]},\left(w_{\varphi(k)}^{*}\right)_{k \in[0 . . N]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{K}$ is a representation of $f * g$, which is disjoint when both representations of $f$ and $g$ are.

Proof. Since $N+1=(n+1)(m+1)$, such a bijection $\varphi$ exists.
(1). From Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation), and Lemma 81 (closed under vector operations is subspace), $\mathcal{S} \mathcal{F}_{E}$ is a vector subspace of $E^{X}$. From commutativity of multiplication in $\mathbb{K}$, and Definition 61 (vector space, compatibility of scalar multiplication with field $\underline{\text { multiplication }}),\left(n,\left(A_{i}\right)_{i \in[0 \ldots n]},\left(a u_{i}\right)_{i \in[0 . . n]}\right)$ is a representation of $a f$.

Moreover, when $E \stackrel{\text { def. }}{=} \mathbb{K}$, from distributivity of multiplication over addition and closedness of multiplication in $\mathbb{K}$, Lemma 1103 ( $\mathcal{I} \mathcal{F}_{\mathbb{K}}$ is closed under multiplication), Lemma 1108 $\left(\mathcal{S} \mathcal{F}_{E}\right.$ simple representation, with $\left.E \stackrel{\text { def. }}{=} \mathbb{K}\right)$, and Lemma 286 (vector subspace and closed under
 $\mathcal{S} \mathcal{F}_{\mathbb{K}}$ is a subalgebra of $\mathbb{K}^{X}$.

Assume now that both representations of $f$ and $g$ are disjoint.
(2a). From Lemma $1114\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation), the representation of $a f$ is also disjoint.
(2b). From Lemma $1114\left(\mathcal{S F}_{E}\right.$ disjoint representation), compatibility of intersection with pairwise disjunction, idempotent law for intersection, and left and right distributivity of intersection over union, the $A_{i} \cap B_{j}$ 's are pairwise disjoint, and we have

$$
X=\biguplus_{i \in[0 . . n]} A_{i} \cap \biguplus_{j \in[0 \ldots m]} B_{j}=\biguplus_{k \in[0 . . N]} C_{\varphi(k)}
$$

Hence, from Definition 209 (pseudopartition), $\left(C_{\varphi(k)}\right)_{k \in[0 . . N]}$ forms a pseudopartition of $X$.
(2c). Let $i \in[0 . . n]$. Then, from field properties of $\mathbb{K}$, the definition of the indicator function $\left(\mathbb{1}_{X} \equiv \mathbb{1}_{\mathbb{K}}\right)$, Lemma $1114\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation, $\left.X=\biguplus_{j \in[0 . m]} B_{j}\right)$, Lemma 1102 $\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is $\sigma$-additive, with $\left.I \stackrel{\text { def. }}{=}[0 . . m]\right)$, left distributivity of multiplication over addition in $\mathbb{K}$, and Lemma $11 \overline{03\left(\mathcal{I F}_{\mathfrak{K}} \text { is closed } \text { under multiplication), we have }\right.}$

$$
\mathbb{1}_{A_{i}}=\mathbb{1}_{A_{i}} \mathbb{1}_{\biguplus_{j \in[0 . . m]} B_{j}}=\mathbb{1}_{A_{i}}\left(\sum_{j \in[0 \ldots m]} \mathbb{1}_{B_{j}}\right)=\sum_{j \in[0 \ldots m]}\left(\mathbb{1}_{A_{i}} \mathbb{1}_{B_{j}}\right)=\sum_{j \in[0 . . m]} \mathbb{1}_{C_{i, j}} .
$$

Thus, from distributivity of scalar multiplication over field addition in $E$, the lower bound property of intersection $\left(C_{i, j} \subset \boldsymbol{A}_{i}\right)$, (2b), and Lemma $1114\left(\mathcal{S F} \mathcal{F}_{E}\right.$ disjoint representation), we have

$$
f=\sum_{i \in[0 . . n]}\left(\sum_{j \in[0 . . m]} \mathbb{1}_{C_{i, j}}\right) u_{i}=\sum_{(i, j) \in[0 . . n] \times[0 . . m]} \mathbb{1}_{C_{i, j}} u_{i},
$$

and the representation is disjoint. Similarly, we obtain the disjoint representation

$$
g=\sum_{(i, j) \in[0 . . n] \times[0 \ldots m]} \mathbb{1}_{C_{i, j}} v_{j} .
$$

Hence, from additive group properties of $\boldsymbol{E}$, and distributivity of scalar multiplication over field addition in $\boldsymbol{E},\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . . N]},\left(w_{\varphi(k)}^{+}\right)_{k \in[0 . . N]}\right)$ is a representation of $f+g$.
(2d). Let $u, v \in E$. Let $\diamond$ be a binary operator over $E$ (using infix notation). Then, from the definition of inverse image, we have $(f \diamond g)(\{f=u\} \cap\{g=v\}) \subset\{u \diamond v\}$. Hence, from the identity $A \subset \psi^{-1}(B) \Leftrightarrow \psi(A) \subset B$ (with $\psi \stackrel{\text { def. }}{=} f \diamond g$ ), we have

$$
\{f=u\} \cap\{g=v\} \subset\{f \diamond g=u \diamond v\}
$$

(2e). Let $k \in[0 . . N]$. Let $(i, j) \stackrel{\text { def. }}{=} \varphi(k) \in[0 . . n] \times[0 . . m]$. Then, from Lemma $1114\left(\mathcal{S F}_{E}\right.$ disjoint representation, $A_{i} \subset\left\{f=u_{i}\right\}$ and $B_{j} \subset\left\{g=v_{j}\right\}$ ), monotonicity of intersection ( $\boldsymbol{A}_{i} \cap \boldsymbol{B}_{j}$ is a subset of $\left\{\bar{f}=u_{i}\right\} \cap\left\{g=v_{j}\right\}$ ), and (2d) with the binary operators vector addition and field multiplication (when $E \stackrel{\text { def. }}{=} \mathbb{K}$ ), we have $C_{\varphi(k)} \subset\left\{f+g=c_{\varphi(k)}^{+}\right\}$and $C_{\varphi(k)} \subset\left\{f * g=c_{\varphi(k)}^{*}\right\}$. Hence, from Lemma $1114\left(\mathcal{S F}_{E}\right.$ disjoint representation), and (2b), both representations for $f+g$ and $f * g$ (when $E \stackrel{\text { def. }}{=} \mathbb{K}$ ) are disjoint.
(2f). Assume finally that the representation of $f$ is canonical, and that $a \neq 0_{\mathbb{k}}$.
Then, from Lemma $1111\left(\mathcal{S F}_{E}\right.$ canonical representation), and the bijective property of scalar multiplication by $a$ in $\boldsymbol{E}$, the $a u_{i}$ 's are pairwise distinct, $A_{i}=\left\{a f=a u_{i}\right\} \neq \varnothing$, and the representation of $a f$ is also canonical.

Remark 1118. Note that in the previous lemma, $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 \ldots N]},\left(w_{\varphi(k)}^{+}\right)_{k \in[0 \ldots N]}\right)$ is not a representation of $f+g$ when there is nonempty overlapping in the representation of $f$ or $g$. Indeed, the values in overlapped parts are counted too many times.

Note also that even though the representations of $f$ and $g$ are canonical, the given representations of $f+g$ and $f * g$ are only disjoint. Indeed, the $u_{i}+v_{j}$ 's and $u_{i} * v_{j}$ 's may not be pairwise distinct, and thus the $A_{i} \cap B_{j}$ 's may no longer be their inverse images.

## Lemma 1119 (SF $\mathcal{E}_{E}$ is closed under extension by zero).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$.
Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow E$ and $\hat{f}: X \rightarrow E$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{E}(A, \Sigma \bar{\cap} A)$ iff $\mathbb{1}_{A} \hat{f} \in \mathcal{S} \mathcal{F}_{E}(X, \Sigma)$.
If so, there exists $\left(n,\left(\hat{g}_{i}\right)_{i \in[0 . . n]},\left(u_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma) \times E$ such that

$$
\begin{equation*}
\forall i \in[0 . . n], \mathbb{1}_{A} \hat{g}_{i}=\hat{g}_{i}, \quad f_{\left.\right|_{A}}=\sum_{i \in[0 . . n]}\left(\hat{g}_{i}\right)_{\left.\right|_{A}} u_{i} \quad \text { and } \quad \mathbb{1}_{A} \hat{f}=\sum_{i \in[0 . . n]} \hat{g}_{i} u_{i} . \tag{14.14}
\end{equation*}
$$

Proof. "Left" implies "right". From Lemma $1108\left(\mathcal{S} \mathcal{F}_{E}\right.$ simple representation, with $\left.f \stackrel{\text { def. }}{=} f_{\left.\right|_{A}}\right)$, and Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, set of measurable indicator functions, with $\left.(X, \Sigma) \stackrel{\text { def. }}{=}(A, \Sigma \bar{\cap} A)\right)$, let $n \in \mathbb{N},\left(B_{i}\right)_{i \in[0 . . n]} \in \Sigma \bar{\cap} A$ and $\left(u_{i}\right)_{i \in[0 . . n]} \in E$ such that

$$
\left(f_{i}\right)_{i \in[0 . . n]} \stackrel{\text { def. }}{=}\left(\mathbb{1}_{B_{i}}^{A}\right)_{i \in[0 . . n]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A) \quad \text { and } \quad f_{\mid A}=\sum_{i \in[0 . . n]} f_{i} u_{i} .
$$

Then, from Lemma $1105\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under extension by zero, with $f=g_{i} \stackrel{\text { def. }}{=} \mathbb{1}_{B_{i},}^{Y}$ and $\hat{f}=\hat{g}_{i}$ set to $\mathbb{1}_{B_{i}}^{X}$ ), we have $f_{i}=\left(\hat{g}_{i}\right)_{\mid A}$ and $\mathbb{1}_{A} \hat{g}_{i}=\hat{g}_{i} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma)$. Hence, from Lemma 524 (restriction is masking, with $f \stackrel{\text { def. }}{=} \sum_{i \in[0 . n]} g_{i} u_{i}$ and $\hat{f} \stackrel{\text { def. }}{=} \sum_{i \in[0 \ldots n]} \hat{g}_{i} u_{i}$ ), and Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation), we have $\mathbb{1}_{A} \hat{f}=\sum_{i \in[0 . . n]} \hat{g}_{i} u_{i} \in \mathcal{S} \mathcal{F}_{E}(X, \Sigma)$.
"Right" implies "left". From Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation, with $\left.f \stackrel{\text { def. }}{=} \mathbb{1}_{A} \hat{f}\right)$, and Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions), let $n \in \mathbb{N},\left(B_{i}\right)_{i \in[0 . . n]} \in \Sigma$ and $\left(u_{i}\right)_{i \in[0 . . n]} \in E$ such that

$$
\left(\hat{g}_{i}\right)_{i \in[0 . . n]} \stackrel{\text { def. }}{=}\left(\mathbb{1}_{B_{i}}^{X}\right)_{i \in[0 . . n]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma) \quad \text { and } \quad \mathbb{1}_{A} \hat{f}=\sum_{i \in[0 . . n]} \hat{g}_{i} u_{i} .
$$

Then, from Lemma 524 (restriction is masking), and linearity of restriction of function,

$$
f_{\left.\right|_{A}}=\left(\mathbb{1}_{A} \hat{f}\right)_{\left.\right|_{A}}=\sum_{i \in[0 . . n]}\left(\hat{g}_{i}\right)_{\left.\right|_{A}} u_{i}
$$

Hence, from Lemma $1105\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under extension by zero, with $Y=X, \hat{f} \stackrel{\text { def. }}{=} \hat{g}_{i}$ and $\underline{\mathbb{1}_{A}} \hat{g}_{i}=\hat{g}_{i} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}(X, \Sigma)$ ), and Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation), we have

$$
\left(\hat{g}_{i}\right)_{\left.\right|_{A}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A) \quad \text { and } \quad f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{E}(A, \Sigma \bar{\cap} A)
$$

Therefore, we have the equivalence and both representations.
Lemma 1120 ( $\mathcal{S F}_{E}$ is closed under restriction).
Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Let $f: X \rightarrow E$. Then, we have $f \in \mathcal{S F}_{E}(X, \Sigma)$ iff for all $A \in \Sigma, f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{E}(A, \Sigma \bar{\cap} A)$.

## Proof. "Left" implies "right".

Direct consequence of Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation, $f=\sum_{i \in[0 . n]} f_{i} u_{i}$ with $n \in \mathbb{N}$, $\left(f_{i}\right)_{i \in[0 . . n]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(X, \Sigma)$ and $\left.\left(u_{i}\right)_{i \in[0 . . n]} \in E\right)$, compatibility of vector linear combination and restriction of function $\left(f_{\left.\right|_{A}}=\sum_{i \in[0 . . n]} f_{i_{A}} u_{i}\right)$, Lemma $1106\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under restriction, $f_{\left.i\right|_{A}} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}(A, \Sigma \bar{\cap} A)$ ), and Lemma $1108\left(\mathcal{S \mathcal { F } _ { E }}\right.$ simple representation, $f_{\left.\right|_{A}}$ belongs to $\left.\underline{\mathcal{S} \mathcal{F}_{E}(A, \Sigma \bar{\cap} A)}\right)$.
"Right" implies "left". Lemma 697 (equivalent definition of $\sigma$-algebra, $A \xlongequal{\text { def. }} X \in \Sigma$ ), Lemma 764 (measurability of measurable subspace, $\Sigma \bar{\cap} X=\Sigma$ ), and the definition of restriction of function $\left(f_{\mid x}=f\right.$, thus $\left.f \in \mathcal{S} \mathcal{F}_{E}(X, \Sigma)\right)$.
Therefore, we have the equivalence.

## Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive).

Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{K}$ be $\mathbb{R}$, or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$. Let $I \subset \mathbb{N}$ and $n \in \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow E$.
Then, we have $\mathbb{1}_{A} f=\sum_{i \in I} \mathbb{1}_{A_{i}} f$, and $\mathbb{1}_{A} f \in \mathcal{S \mathcal { F } _ { E }}$ implies for all $i \in I, \mathbb{1}_{A_{i}} f \in \mathcal{S} \mathcal{F}_{E}$.
Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$, we have for all $i \in[0 . . n], \mathbb{1}_{A_{i}} f \in \mathcal{S} \mathcal{F}_{E}$ implies $\mathbb{1}_{A} f \in \mathcal{S} \mathcal{F}_{E}$.
Proof. (1). Identity. Exactly Lemma 525 (masking is $\sigma$-additive).
"Left" implies "right". Assume that $\mathbb{1}_{A} f \in \mathcal{S} \mathcal{F}_{E}$. Then, from Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation), let $p \in \mathbb{N},\left(f_{j}\right)_{j \in[0 . . p]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}$, and $\left(u_{j}\right)_{j \in[0 . . p]} \in E$ such that

$$
\mathbb{1}_{A} f=\sum_{j \in[0 . . p]} f_{j} u_{j} .
$$

Let $i \in I$. Let $j \in[0 . . p]$. Let $g_{j}^{i} \stackrel{\text { def. }}{=} \mathbb{1}_{A_{i}} f_{j}$. Then, from Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, set of measurable indicator functions, $\underline{f}_{j}=\mathbb{1}_{B}$ with $\left.B \in \Sigma\right)$, and Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under multiplication), we have $g_{j}^{i}=\mathbb{1}_{A_{i} \cap B} \in \mathcal{I} \mathcal{F}_{\mathfrak{K}}$. Moreover, since $A_{i} \subset A$ (i.e. $\left.A_{i}=A \cap A_{i}\right)$, from Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under multiplication), and Definition 61 (vector space, compatibility of scalar multiplication with field multiplication, and distributivity of scalar multiplication wrt vector addition), we have

$$
\mathbb{1}_{A_{i}} f=\mathbb{1}_{A \cap A_{i}} f=\mathbb{1}_{A_{i}}\left(\mathbb{1}_{A} f\right)=\sum_{j \in[0 . . p]} \mathbb{1}_{A_{i}} f_{j} u_{j}=\sum_{j \in[0 . . p]} g_{j}^{i} u_{j} .
$$

Hence, from Lemma $1108\left(\mathcal{S \mathcal { F } _ { E }}\right.$ simple representation $)$, we have $\mathbb{1}_{A_{i}} f \in \mathcal{S} \mathcal{F}_{E}$.
"Right" implies "left" when $\boldsymbol{I} \stackrel{\text { def. }}{=}[\mathbf{0 . . n}]$. Assume that for all $i \in[0 . . n], \mathbb{1}_{A_{i}} f \in \mathcal{S} \mathcal{F}_{E}$. Then, from Lemma $1108\left(\mathcal{S \mathcal { F } _ { E }}\right.$ simple representation $)$, for all $i \in[0 . . n]$, let $p^{i} \in \mathbb{N},\left(f_{j}^{i}\right)_{j \in\left[0 . . p^{i}\right]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}}$, and $\left(u_{j}^{i}\right)_{j \in\left[0 . . p^{i}\right]} \in E$ such that $\mathbb{1}_{A_{i}} f=\sum_{j \in\left[0 . . p^{i}\right]} f_{j}^{i} u_{j}^{i}$. Let $p \stackrel{\text { def. }}{=}-1+\sum_{i \in[0 . . n]}\left(1+p^{i}\right) \in \mathbb{N}$,

$$
\left(f_{j}\right)_{j \in[0 . . p]} \stackrel{\text { def. }}{=}\left(\left(\mathbb{1}_{A_{i}} f_{j}^{i}\right)_{j \in\left[0 . . p^{i}\right]}\right)_{i \in[0 . . n]} \in \mathcal{I} \mathcal{F}_{\mathbb{K}} \quad \text { and } \quad\left(u_{j}\right)_{j \in[0 . . p]} \stackrel{\text { def. }}{=}\left(\left(u_{j}^{i}\right)_{j \in\left[0 . . p^{i}\right]}\right)_{i \in[0 . . n]} \in E .
$$

Then, from Lemma $1102\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is $\sigma$-additive, with $\left.I \stackrel{\text { def. }}{=}[0 . . n]\right)$, and Definition 61 (vector space, commutativity and associativity of vector addition in $E$ ), we have

$$
\mathbb{1}_{A} f=\sum_{i \in[0 . . n]} \mathbb{1}_{A_{i}} f=\sum_{i \in[0 . . n]} \sum_{j \in\left[0 . . p^{i}\right]} f_{j}^{i} u_{j}^{i}=\sum_{j \in[0 . . p]} f_{j} u_{j} .
$$

Hence, from Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation $)$, we have $\mathbb{1}_{A} f \in \mathcal{S} \mathcal{F}_{E}$.
Therefore, we have both implications.

### 14.3 Nonnegative simple function

Remark 1122. In this section, we consider the case where $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ is equipped with the usual total order. We recall that the notation $\mathcal{S F}$ denotes the vector space of simple functions to the field of real numbers $\mathbb{R}$, see Lemma 1108.

## Definition 1123 ( $\mathcal{S F}_{+}$, subset of nonnegative simple functions).

Let $(X, \Sigma)$ be a measurable space. The subset of nonnegative simple functions is denoted $\mathcal{S F} \mathcal{F}_{+}(X, \Sigma)$ (or simply $\mathcal{S} \mathcal{F}_{+}$); it is defined by $\mathcal{S} \mathcal{F}_{+} \stackrel{\text { def. }}{=}\left\{f \in \mathcal{S F} \mid f(X) \subset \mathbb{R}_{+}\right\}$.

## Lemma 1124 ( $\mathcal{S} \mathcal{F}_{+}$is measurable).

Let $(X, \Sigma)$ be a measurable space. Then, we have $\mathcal{I F} \subset \mathcal{S F} \mathcal{F}_{+} \subset \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$.
Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions), Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions), nonnegativeness of the indicator function, Lemma $1113\left(\mathcal{S \mathcal { F } _ { E }}\right.$ is measurable, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).

## Lemma 1125 (SF $\mathcal{F}_{+}$disjoint representation).

Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \mathbb{R}$.
Then, $f \in \mathcal{S} \mathcal{F}_{+}$iff there exists a triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$such that

$$
\begin{align*}
\forall i \in[0 . . n], & A_{i} \subset\left\{f=a_{i}\right\},  \tag{14.15}\\
& X=\biguplus_{i \in[0 . . n]} A_{i},  \tag{14.16}\\
& f=\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}} . \tag{14.17}
\end{align*}
$$

Proof. "Left" implies "right". Direct consequence of Definition $1123\left(\mathcal{S F}{ }_{+}\right.$, subset of nonnegative simple functions), Lemma $1114\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation, with $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}, f\left(A_{i}\right)=\left\{a_{i}\right\}$ is included in $\mathbb{R}_{+}$), and commutativity of multiplication in $\mathbb{R}_{+}$.
"Right" implies "left". Direct consequence of commutativity of multiplication in $\mathbb{R}_{+}$, Lemma $1114\left(\mathcal{S F} \mathcal{F}_{E}\right.$ disjoint representation, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}, f \in \mathcal{S F}\right)$, closedness of addition in $\mathbb{R}_{+}(f \geqslant 0)$, and Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions).

Therefore, we have the equivalence.
Lemma 1126 (SF ${ }_{+}$canonical representation).
Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \mathbb{R}$.
Then, $f \in \mathcal{S F} \mathcal{F}_{+}$iff there exists a unique triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$such that

$$
\begin{align*}
\forall i \in[0 . . n-1], & a_{i}<a_{i+1},  \tag{14.18}\\
\forall i \in[0 . . n], & A_{i}=\left\{f=a_{i}\right\} \neq \varnothing \\
& X=\biguplus_{i \in[0 . . n]} A_{i} \\
& f=\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}}
\end{align*}
$$

which can also be written (with $\{f=y\} \in \Sigma$ for all $y \in f(X)$ )

$$
\begin{equation*}
f=\sum_{y \in f(X)} y \mathbb{1}_{\{f=y\}} . \tag{14.22}
\end{equation*}
$$

Proof. "Left" implies "right". Direct consequence of Definition $1123\left(\mathcal{S F} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions), Lemma $1111\left(\mathcal{S} \mathcal{F}_{E}\right.$ canonical representation, with $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, with $\underline{a_{i} \in f(X) \subset \mathbb{R}_{+}}$, and commutativity of multiplication in $\mathbb{R}_{+}$.
"Right" implies "left". Direct consequence of commutativity of multiplication in $\mathbb{R}_{+}$, Lemma $1111\left(\mathcal{S F}_{E}\right.$ canonical representation, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}, f \in \mathcal{S F}\right)$, closedness of addition in $\mathbb{R}_{+}(f \geqslant 0)$, and Definition $1123\left(\mathcal{S \mathcal { F } _ { + }}\right.$, subset of nonnegative simple functions).
Therefore, we have the equivalence.
Lemma 1127 (SF $\mathcal{C}_{+}$disjoint representation is subpartition of canonical representation). Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{S} \mathcal{F}_{+}$.
Let $\left(n,\left(A_{i}\right)_{i \in[0 . . m]},\left(a_{i}\right)_{i \in[0 . . m]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be a disjoint representation of $f$,
and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(b_{j}\right)_{j \in[0 . . m]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be its canonical representation.
For all $j \in[0 . . m]$, let $I_{j}^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid \varnothing \neq A_{i} \subset B_{j}\right\}$. Let $I^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid A_{i} \neq \varnothing\right\}$.
Then, we have

$$
\begin{gather*}
\forall j \in[0 . . m], \quad B_{j}=\biguplus_{i \in I_{j}^{\prime}} A_{i} \wedge \quad\left(\forall i \in I_{j}^{\prime}, a_{i}=b_{j}\right)  \tag{14.23}\\
\left(\forall p, q \in[0 . . m], p \neq q \Rightarrow I_{p}^{\prime} \cap I_{q}^{\prime}=\varnothing\right) \quad \wedge \quad \biguplus_{j \in[0 . . m]} I_{j}^{\prime}=I^{\prime} \tag{14.24}
\end{gather*}
$$

Proof. Direct consequence of Lemma $1116\left(\mathcal{S F}_{E}\right.$ disjoint representation is subpartition of canonical representation, with $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R})$, Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions $)$, Lemma $1125\left(\overline{\mathcal{S F}}+\right.$ disjoint representation, $\left.\underline{a}_{i} \geqslant 0\right)$, and Lemma $1126\left(\mathcal{S F} \mathcal{F}_{+}\right.$canonical representation, $b_{i} \geqslant 0$ ).

## Lemma 1128 (SF ${ }_{+}$simple representation).

Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \mathbb{R}$. Then, we have $f \in \mathcal{S} \mathcal{F}_{+}$iff there exists a triple $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$such that $f=\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}}$.

If so, for all $x \in X \backslash \bigcup_{i \in[0 . . n]} A_{i}$, we have $f(x)=0$.
Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions), Lemma $1108\left(\mathcal{S} \mathcal{F}_{E}\right.$ simple representation, with $\left.E=\mathbb{K} \xlongequal{\text { def. }} \mathbb{R}\right)$, commutativity of multiplication in $\mathbb{R}_{+}$, and closedness of addition in $\mathbb{R}_{+}\left(\sum_{i \in[0 . . n]} a_{i} \mathbb{1}_{A_{i}} \geqslant 0\right)$.

Lemma 1129 (SF ${ }_{+}$is closed under positive algebra operations). $\quad$ Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{S} \mathcal{F}_{+}$. Let $\alpha \in \mathbb{R}_{+}$. Then, we have $f+g, \alpha f, f g \in \mathcal{S} \mathcal{F}_{+}$.

Let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right)$ and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(b_{j}\right)_{j \in[0 . . m]}\right)$ in $\mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be representations of $f$ and $g$. Let $N \stackrel{\text { def. }}{=} n m+n+m$. Let $\varphi:[0 . . N] \rightarrow[0 . . n] \times[0 . . m]$ be a bijection. For all $(i, j) \in[0 . . n] \times[0 . . m]$, let $C_{i, j} \stackrel{\text { def. }}{=} A_{i} \cap B_{j}, c_{i, j}^{+} \stackrel{\text { def. }}{=} a_{i}+b_{j}$, and $c_{i, j}^{*} \stackrel{\text { def. }}{=} a_{i} b_{j}$.
Then, if both representations are disjoint, $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . N]},\left(c_{\varphi(k)}^{+}\right)_{k \in[0 . . N]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$is also a disjoint representation of $f+g,\left(n,\left(A_{i}\right)_{i \in[0 \ldots n]},\left(\alpha a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$is a representation of $\alpha f$, of the same nature as that of $f$ (when $\alpha \neq 0$ for canonical representation), and $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . . N]},\left(c_{\varphi(k)}^{*}\right)_{k \in[0 \ldots N]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$is a representation of $f g$, which is disjoint when both representations of $f$ and $g$ are.

Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions), Lemma $1117\left(\mathcal{S} \mathcal{F}_{E}\right.$ is vector space, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, Definition 277 (algebra over a field), closedness of addition and multiplication in $\mathbb{R}_{+}$. Lemma $1125\left(\mathcal{S F} \mathcal{F}_{+}\right.$disjoint representation), and associativity and commutativity of intersection.

Lemma 1130 ( $\mathcal{S F} \mathcal{F}_{+}$is closed under extension by zero). Let $(X, \Sigma)$ be a measurable space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$.
If so, there exists $\left(n,\left(\hat{g}_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \mathcal{I F}(X, \Sigma) \times \mathbb{R}_{+}$such that

$$
\begin{equation*}
\forall i \in[0 . . n], \hat{g}_{i} \mathbb{1}_{A}=\hat{g}_{i}, \quad f_{\mid A}=\sum_{i \in[0 . . n]} a_{i}\left(\hat{g}_{i}\right)_{\left.\right|_{A}} \quad \text { and } \quad \hat{f} \mathbb{1}_{A}=\sum_{i \in[0 . . n]} a_{i} \hat{g}_{i} . \tag{14.25}
\end{equation*}
$$

Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions, $\boldsymbol{\mathcal { F }}_{+}$ equals $\left.\mathcal{S F} \cap \mathbb{R}_{+}^{X}\right)$, Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, Lemma 524 (restriction is masking, with $E \stackrel{\text { def. }}{=} \mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ ), compatibility of restriction and masking with nonnegativeness, and commutativity of multiplication in $\mathbb{R}_{+}$.

Lemma $1131\left(\mathcal{S F}{ }_{+}\right.$is closed under restriction). Let $(X, \Sigma)$ be a measurable space. Let $f: X \rightarrow \mathbb{R}$. Then, we have $f \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$ iff for all $A \in \Sigma, f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A)$.

Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions, $\boldsymbol{\mathcal { F }}_{+}$ equals $\left.\mathcal{S F} \cap \mathbb{R}_{+}^{X}\right)$, Lemma $1120\left(\mathcal{S} \mathcal{F}_{E}\right.$ is closed under restriction, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and compatibility of restriction with nonnegativeness.

Lemma 1132 ( $\mathcal{S F}_{+}$, masking is $\sigma$-additive). Let $(X, \Sigma)$ be a measurable space.
Let $I \subset \mathbb{N}$ and $n \in \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \mathbb{R}$.
Then, we have $f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}}$, and $f \mathbb{1}_{A} \in \mathcal{S F} \mathcal{F}_{+}$implies for all $i \in I, f \mathbb{1}_{A_{i}} \in \mathcal{S F} \mathcal{F}_{+}$.
Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$, we have for all $i \in[0 . . n], f \mathbb{1}_{A_{i}} \in \mathcal{S} \mathcal{F}_{+}$implies $f \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}$.
Proof. Direct consequence of Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions, $\mathcal{S \mathcal { F }}_{+}$ equals $\left.\mathcal{S F} \cap \mathbb{R}_{+}^{X}\right)$, Lemma $1121\left(\mathcal{S} \mathcal{F}_{E}\right.$, masking is $\sigma$-additive, with $\left.E \stackrel{\text { def. }}{=} \mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and commutativity of multiplication in $\mathbb{R}_{+}$.

## Remark 1133.

The next lemma implements the second step of the Lebesgue scheme (see Section 4.4).
Note that taking $f, g \in \mathcal{I F}$ in property (14.27) would only allow to add two indicator functions, and not a finite number to reach all simple functions.

## Lemma 1134 (Lebesgue scheme step 2).

Let $(X, \Sigma)$ be a measurable space. Let $P$ be a predicate on functions from $X$ to $\overline{\mathbb{R}}_{+}$. Assume that $P$ is compatible with positive linear combination of indicator functions:

$$
\begin{align*}
\forall a \in & \mathbb{R}_{+}, \forall f \in \mathcal{I F}, \quad P(f) \quad \Longrightarrow \quad P(a f),  \tag{14.26}\\
& \forall f, g \in \mathcal{S \mathcal { F } _ { + }}, \quad P(f) \wedge P(g) \quad \Longrightarrow \quad P(f+g) . \tag{14.27}
\end{align*}
$$

Then, $P$ holds on $\mathcal{I F}$ implies $P$ holds on $\mathcal{S F}_{+}$.
Proof. Direct consequence of Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions), Lemma $1128\left(\mathcal{S F}_{+}\right.$simple representation $)$, and induction on the number of operands of the finite sum.

### 14.4 Adapted sequence

## Definition 1135 (adapted sequence).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Let $\left(\varphi_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F} \mathcal{F}_{+}$. The sequence $\left(\varphi_{n}\right)_{n \in \mathbb{N}}$ is called adapted sequence for $f$ iff it is nondecreasing and $f=\lim _{n \in \mathbb{N}} \varphi_{n}=\sup _{n \in \mathbb{N}} \varphi_{n}$.

Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. For all $n \in \mathbb{N}$, let $\varphi_{n}: X \rightarrow \mathbb{R}_{+}$defined by

$$
\forall x \in X, \quad \varphi_{n}(x) \stackrel{\text { def. }}{=} \begin{cases}\frac{\left\lfloor 2^{n} f(x)\right\rfloor}{2^{n}} & \text { when } f(x)<n  \tag{14.28}\\ n & \text { otherwise } .\end{cases}
$$

Then, $\left(\varphi_{n}\right)_{n \in \mathbb{N}}$ is an adapted sequence for $f$.
Proof. Let $n \in \mathbb{N}$. Let $x \in X$. Assume that $f(x)<n$. Let $i \stackrel{\text { def. }}{=}\left\lfloor 2^{n} f(x)\right\rfloor \in \mathbb{N}$. Then, from the definition of the floor function, we have $0 \leqslant i \leqslant 2^{n} f(x)<i+1 \leqslant n 2^{n}$. Thus, for all $i \in \mathbb{N}$ such that $0 \leqslant i<n 2^{n}$, for all $x \in X$ such that $\frac{i}{2^{n}} \leqslant f(x)<\frac{i+1}{2^{n}}$, we have $\varphi_{n}(x)=\frac{i}{2^{n}}$. Hence,

$$
\varphi_{n}=\sum_{0 \leqslant i<n 2^{n}} \frac{i}{2^{n}} \mathbb{1}_{\left\{\frac{i}{2^{n}} \leqslant f<\frac{i+1}{2^{n}}\right\}}+n \mathbb{1}_{\{f \geqslant n\}} .
$$

Moreover, from Lemma 820 (measurability of numeric function), and Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection (with $\operatorname{card}(I)=2)$ ), we have

$$
\left\{\frac{i}{2^{n}} \leqslant f<\frac{i+1}{2^{n}}\right\}=\left\{\frac{i}{2^{n}} \leqslant f\right\} \cap\left\{f<\frac{i+1}{2^{n}}\right\} \in \Sigma, \quad \text { and } \quad\{f \geqslant n\} \in \Sigma
$$

Hence, from Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and Definition 1123 $\left(\mathcal{S F} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions), we have $\varphi_{n} \in \mathcal{S} \mathcal{F}_{+}$.

Let $n \in \mathbb{N}$. Let $x \in X$. Case $\boldsymbol{n}+\mathbf{1} \leqslant \boldsymbol{f}(\boldsymbol{x})$. Then, we have

$$
\varphi_{n}(x)=n<n+1=\varphi_{n+1}(x)
$$

Case $\boldsymbol{f}(\boldsymbol{x})<\boldsymbol{n}+\mathbf{1}$. Let $i \stackrel{\text { def. }}{=}\left\lfloor 2^{n+1} f(x)\right\rfloor$. Then, we have

$$
\varphi_{n+1}(x)=\frac{i}{2^{n+1}} \quad \text { with } 0 \leqslant \frac{i}{2^{n+1}} \leqslant f(x)<\frac{i+1}{2^{n+1}}
$$

Case $\frac{\boldsymbol{i}}{\boldsymbol{2}^{n+1}} \geqslant \boldsymbol{n}$. Then, $f(x) \geqslant n$ and we have

$$
\varphi_{n}(x)=n \leqslant \frac{i}{2^{n+1}}=\varphi_{n+1}(x)
$$

Case $\frac{\boldsymbol{i}}{2^{n+1}}<\boldsymbol{n}$ and $\boldsymbol{i}$ even. Let $j \stackrel{\text { def. }}{=} \frac{i}{2} \in \mathbb{N}$. Then, from ordered field properties of $\mathbb{R}$, we have

$$
\frac{j}{2^{n}} \leqslant f(x)<\frac{j}{2^{n}}+\frac{1}{2^{n+1}}<\frac{j+1}{2^{n}}
$$

Thus, from the definition of the floor function, we have $j=\left\lfloor 2^{n} f(x)\right\rfloor$ and

$$
\varphi_{n}(x)=\frac{j}{2^{n}}=\frac{i}{2^{n+1}}=\varphi_{n+1}(x)
$$

Case $\frac{i}{2^{n+1}}<\boldsymbol{n}$ and $\boldsymbol{i}$ odd. Let $j \stackrel{\text { def. }}{=} \frac{i-1}{2} \in \mathbb{N}$. Then, from ordered field properties of $\mathbb{R}$, we have

$$
\frac{j}{2^{n}} \leqslant \frac{j}{2^{n}}+\frac{1}{2^{n+1}} \leqslant f(x)<\frac{j+1}{2^{n}}
$$

Thus, from the definition of the floor function, we have $j=\left\lfloor 2^{n} f(x)\right\rfloor$ and

$$
\varphi_{n}(x)=\frac{j}{2^{n}}=\frac{i-1}{2^{n+1}}<\frac{i}{2^{n+1}}=\varphi_{n+1}(x)
$$

Thus, we always have $\varphi_{n}(x) \leqslant \varphi_{n+1}(x)$. Hence, the sequence $\left(\varphi_{n}\right)_{n \in \mathbb{N}}$ is nondecreasing.
Let $x \in X$. We have $f(x) \in \overline{\mathbb{R}}_{+}$. Case $\boldsymbol{f}(\boldsymbol{x})=\infty$. Then, for all $n \in \mathbb{N}$, we have $\varphi_{n}(x)=n$. Hence, $\lim _{n \rightarrow \infty} \varphi_{n}(x)=f(x)$. Case $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}_{+}$. Then, from the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $f(x)<N$. Let $n \in \mathbb{N}$ such that $N \leqslant n$. Then, from the definition of the floor function, we have

$$
2^{n} \varphi_{n}(x)=\left\lfloor 2^{n} f(x)\right\rfloor \leqslant 2^{n} f(x)<\left\lfloor 2^{n} f(x)\right\rfloor+1=2^{n} \varphi_{n}(x)+1
$$

Thus, from ordered field properties of $\mathbb{R}\left(\right.$ with $\left.2^{n}>0\right)$, we have $f(x)-\frac{1}{2^{n}}<\varphi_{n}(x) \leqslant f(x)$. Hence, from the squeeze theorem, we have $\lim _{n \rightarrow \infty} \varphi_{n}(x)=f(x)$.

Therefore, from Definition 1135 (adapted sequence), $\left(\varphi_{n}\right)_{n \in \mathbb{N}}$ is an adapted sequence for $f$.
Lemma 1137 ( $\mathcal{M}_{+}$is closed under extension by zero). Let $(X, \Sigma)$ be a measurable space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\mid Y}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$.
If so, there exists $\left(\hat{\varphi}_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}, \hat{\varphi}_{n} \mathbb{1}_{A}=\hat{\varphi}_{n}, \quad f_{\left.\right|_{A}}=\sup _{n \in \mathbb{N}}\left(\hat{\varphi}_{n}\right)_{\left.\right|_{A}} \quad \text { and } \quad \hat{f} \mathbb{1}_{A}=\sup _{n \in \mathbb{N}} \hat{\varphi}_{n} \tag{14.29}
\end{equation*}
$$

Proof. "Left" implies "right". From Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$, with $f \stackrel{\text { def. }}{=} f_{\left.\right|_{A}}$ ), and Lemma $1128\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation, with $(X, \Sigma) \stackrel{\text { def. }}{=}(A, \Sigma \bar{\cap} A)$ ), for all $p \in \mathbb{N}$, let $n^{p} \in \mathbb{N}$, $\left(a_{i}^{p}\right)_{i \in\left[0 . . n^{p}\right]} \in \mathbb{R}_{+}$and $\left(B_{i}^{p}\right)_{i \in\left[0 . . n^{p}\right]} \in \Sigma \bar{\cap} A$ such that

$$
\left(\psi_{p}\right)_{p \in \mathbb{N}} \stackrel{\text { def. }}{=} \sum_{i \in\left[0 . . n^{p}\right]} a_{i}^{p} \mathbb{1}_{B_{i}^{p}}^{A} \in \mathcal{S} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A) \quad \text { and } \quad f_{\left.\right|_{A}}=\sup _{p \in \mathbb{N}} \psi_{p}
$$

Then, from Lemma $1130\left(\mathcal{S F}+\right.$ is closed under extension by zero, with $f=\varphi_{p} \stackrel{\text { def. }}{=} \sum_{i \in\left[0 \ldots n^{p]}\right]} a_{i}^{p} \mathbb{1}_{B_{i}^{p}}^{Y}$ and $\hat{f}=\hat{\varphi}_{p} \stackrel{\text { def. }}{=} \sum_{i \in\left[0 \ldots n^{p}\right]} a_{i}^{p} \mathbb{1}_{B_{i}^{p}}^{X}$, for all $p \in \mathbb{N}$, we have

$$
\psi_{p}=\left(\hat{\varphi}_{p}\right)_{\left.\right|_{A}} \quad \text { and } \quad \hat{\varphi}_{p} \mathbb{1}_{A}=\hat{\varphi}_{p} \in \mathcal{S F} \mathcal{F}_{+}(X, \Sigma)
$$

Hence, from Lemma 524 (restriction is masking, with $f \stackrel{\text { def. }}{=} \sup _{p \in \mathbb{N}} \varphi_{p}$ and $\hat{f} \stackrel{\text { def. }}{=} \sup _{p \in \mathbb{N}} \hat{\varphi}_{p}$ ), Lemma $1124\left(\mathcal{S F} \mathcal{F}_{+}\right.$is measurable, thus $\hat{\varphi}_{p} \in \mathcal{M}_{+}(\overline{X, \Sigma})$ ), and Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum), we have $\hat{f} \mathbb{1}_{A}=\sup _{p \in \mathbb{N}} \hat{\varphi}_{p} \in \mathcal{M}_{+}(X, \Sigma)$.
"Right" implies "left". From Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$, with $f \stackrel{\text { def. }}{=} \hat{f} \mathbb{1}_{A}$ ), and Lemma $1128(\mathcal{S F}+$ simple representation $)$, for all $p \in \mathbb{N}$, let $n^{p} \in \mathbb{N},\left(\overline{a_{i}^{p}}\right)_{i \in\left[0 . . n^{p}\right]} \in \mathbb{R}_{+}$and $\left(B_{i}^{p}\right)_{i \in\left[0 . n^{p}\right]} \in \Sigma$ such that

$$
\left(\hat{\varphi}_{p}\right)_{p \in \mathbb{N}} \stackrel{\text { def. }}{=} \sum_{i \in\left[0 \ldots n^{p}\right]} a_{i}^{p} \mathbb{1}_{B_{i}^{p}}^{A} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma) \quad \text { and } \quad \hat{f} \mathbb{1}_{A}=\sup _{p \in \mathbb{N}} \hat{\varphi}_{p}
$$

Then, from Lemma 524 (restriction is masking), and compatibility of restriction of function with supremum, we have $f_{\left.\right|_{A}}=\left(\hat{f} \mathbb{1}_{A}\right)_{\left.\right|_{A}}=\sup _{p \in \mathbb{N}} \psi_{p_{\left.\right|_{A}}}$. Hence, from Lemma 1130 $\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under extension by zero, with $Y=X, \hat{f} \stackrel{\text { def. }}{=} \hat{\varphi}_{p}$ and $\left.\hat{\varphi}_{p} \mathbb{1}_{A}=\hat{\varphi}_{p} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)\right)$, Lemma $1124(\mathcal{S F})_{+}$is measurable, with $(X, \Sigma) \stackrel{\text { def. }}{=}(A, \Sigma \bar{\cap} A)$ ), and Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$, we have $\left(\hat{\varphi}_{p}\right)_{\left.\right|_{A}} \in \overline{\mathcal{S F}} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A) \subset \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$ and $f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$. Therefore, we always have the equivalence and the representations.

Lemma 1138 ( $\mathcal{M}_{+}$is closed under restriction). Let $(X, \Sigma)$ be a measurable space.
Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f \in \mathcal{M}_{+}(X, \Sigma)$ iff for all $A \in \Sigma, f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$.
Proof. "Left" implies "right". Direct consequence of Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$, $f=\sup _{n \in \mathbb{N}} \varphi_{n}$ with $\left(\varphi_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F}+(X, \Sigma)$ ), compatibility of restriction of function with supremum $\left(f_{\left.\right|_{A}}=\sup _{n \in \mathbb{N}} \varphi_{\left.n\right|_{A}}\right)$, Lemma $1131\left(\mathcal{S F} \mathcal{F}_{+}\right.$is closed under restriction, $\varphi_{\left.n\right|_{A}}$ belongs to $\mathcal{S F}{ }_{+}(A, \Sigma \bar{\cap} A)$ ), Lemma $1124\left(\mathcal{S F}{ }_{+}\right.$is measurable, $\varphi_{\left.n\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$ ), and Lemma 848 $\left(\mathcal{M}_{+}\right.$is closed under supremum, $\left.\underline{f}_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)\right)$.
"Right" implies "left". Lemma 697 (equivalent definition of $\sigma$-algebra, $A \stackrel{\text { def. }}{=} X \in \Sigma$ ), Lemma 764 (measurability of measurable subspace, $\Sigma \bar{\cap} X=\Sigma$ ), and the definition of restriction of function $\left(f_{\mid x}=f\right.$, thus $\left.f \in \mathcal{M}_{+}(X, \Sigma)\right)$.

Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive).
Let $(X, \Sigma)$ be a measurable space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \overline{\mathbb{R}}$.
Then, we have $f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}}$, and $f \mathbb{1}_{A} \in \mathcal{M}_{+}$iff for all $i \in I$, $f \mathbb{1}_{A_{i}} \in \mathcal{M}_{+}$.
Proof. (1). Identity. Exactly Lemma 525 (masking is $\sigma$-additive, with $E \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ ).
"Left" implies "right". Direct consequence of Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under multiplication, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ and $\mathbb{1}_{A_{i}}=\mathbb{1}_{A \cap A_{i}}=\mathbb{1}_{A} \mathbb{1}_{A_{i}}$ ), associativity of multiplication in $\mathbb{R}$, Lemma 1124

"Right" implies "left".
Direct consequence of (1), and Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum).
Therefore, we have the equivalence.

## Remark 1140.

The next lemma implements the third step of the Lebesgue scheme (see Section 4.4).

## Lemma 1141 (Lebesgue scheme step 3).

Let $(X, \Sigma)$ be a measurable space. Let $P$ be a predicate on functions from $X$ to $\overline{\mathbb{R}}_{+}$.
Assume that $P$ is compatible with the limit of nondecreasing nonnegative simple functions:

$$
\begin{equation*}
\forall\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F} \mathcal{F}_{+}, \quad\left(\forall n \in \mathbb{N}, f_{n} \leqslant f_{n+1} \wedge P\left(f_{n}\right)\right) \quad \Longrightarrow \quad P\left(\lim _{n \rightarrow \infty} f_{n}\right) \tag{14.30}
\end{equation*}
$$

Then, $P$ holds on $\mathcal{S F} \mathcal{F}_{+}$implies $P$ holds on $\mathcal{M}_{+}$.
Proof. Direct consequence of Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$).

## Chapter 15

## Lebesgue integration of nonnegative function

## Contents

15.1 Integration of indicator function ..... 252
15.2 Integration of nonnegative simple function ..... 254
15.3 Integration of nonnegative measurable function ..... 260
15.4 Measure and integration over product space ..... 273
15.4.1 Measure over product space ..... 273
15.4.2 Lebesgue measure over product space ..... 280
15.4.3 Integration over product space of nonnegative function ..... 281

### 15.1 Integration of indicator function

Remark 1143. In this section, functions take their values in $\{0,1\}$, and the expressions involving integrals are taken in $\overline{\mathbb{R}}_{+}$. We recall that the notation $\mathcal{I F}$ denotes the set of measurable indicator functions on the field of real numbers $\mathbb{R}$, see Definition 1098.

Definition 1144 (integral in $\mathcal{I F}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{I F}$. The integral of $f$ (for the measure $\mu$ ) is denoted $\int f d \mu$; it is defined by

$$
\begin{equation*}
\int f d \mu \stackrel{\text { def. }}{=} \mu(\{f \neq 0\}) \quad \in \overline{\mathbb{R}}_{+} . \tag{15.1}
\end{equation*}
$$

## Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Then, we have $\int \mathbb{1}_{A} d \mu=\mu(A)$.
Proof. Direct consequence of Lemma 1100 (indicator and support are inverse, with $\mathbb{K}$ set to $\mathbb{R}$, $\mathbb{1}_{A} \in \mathcal{I F}$ and $\left\{\mathbb{1}_{A} \neq 0\right\}=A \in \Sigma$ ), and Definition 1144 (integral in $\mathcal{I F}$ ).

Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive). Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Then, we have

$$
\begin{equation*}
\int \sum_{i \in I} \mathbb{1}_{A_{i}} d \mu=\mu(A)=\sum_{i \in I} \int \mathbb{1}_{A_{i}} d \mu . \tag{15.2}
\end{equation*}
$$

Proof. Direct consequence of Lemma $1102\left(\mathcal{I F}_{\mathbb{K}}\right.$ is $\sigma$-additive, with $\left.\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$, with $A$ and the $A_{i}$ 's), Lemma 913 (measure, $\underline{\sigma \text {-additivity), }}$ Definition 874 ( $\sigma$-additivity).

Lemma 1147 (integral in $\mathcal{I F}$ is monotone). Let $(X, \Sigma, \mu)$ be a measure space.
Let $A, B \in \Sigma$. Assume that $A \subset B$. Then, we have $\int \mathbb{1}_{A} d \mu \leqslant \int \mathbb{1}_{B} d \mu$.
Proof. Direct consequence of Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ), and Lemma 919 (measure is monotone).
Lemma 1148 (integral in $\mathcal{I F}$ over subset). Let $(X, \Sigma, \mu)$ be a measure space.
Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{I F}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{I F}(X, \Sigma)$.
If so, there exists $B \in \Sigma$ such that $B \subset A, f_{\left.\right|_{A}}=\mathbb{1}_{B}^{A}, \hat{f} \mathbb{1}_{A}=\mathbb{1}_{B}^{X}$, and we have

$$
\begin{equation*}
\int f_{\left.\right|_{A}} d \mu_{A}=\mu(B)=\int \hat{f} \mathbb{1}_{A} d \mu \tag{15.3}
\end{equation*}
$$

This integral is denoted $\int_{A} f d \mu$; it is called integral of $f$ over $A$.
Proof. Direct consequence of Lemma $1105\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under extension by zero, with $\left.\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, Definition 1144 (integral in $\mathcal{I F}$ ), and Lemma 932 (trace measure is restricted measure, thus $\mu_{A}(B)$ equals $\mu(B)$ ).

## Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $Y$ be a set such that $A \subset Y \subset X$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$. Then, $\hat{f} \mathbb{1}_{A}=\sum_{i \in I} \hat{f} \mathbb{1}_{A_{i}}$, and $\hat{f} \mathbb{1}_{A} \in \mathcal{I F}$ iff for all $i \in I, \hat{f} \mathbb{1}_{A_{i}} \in \mathcal{I F}$. If so, we have

$$
\begin{equation*}
\int_{A} f d \mu=\sum_{i \in I} \int_{A_{i}} f d \mu \tag{15.4}
\end{equation*}
$$

Proof. Direct consequence of Lemma $1107\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$, masking is $\sigma$-additive, with $\mathbb{K} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$ and $\left.f \stackrel{\text { def. }}{=} \hat{f}\right)$, Lemma 1148 (integral in $\mathcal{I F}$ over subset, with $A$ and the $A_{i}$ 's), and Lemma 1146 (integral in $\mathcal{I} \mathcal{F}$ is $\sigma$-additive).

Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$ and $\mu(A)=0$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$, and either $f_{\left.\right|_{A}} \in \mathcal{I F}(A, \Sigma \bar{\cap} A)$, or $\hat{f} \mathbb{1}_{A} \in \mathcal{I F}(X, \Sigma)$. Then, we have both, and

$$
\begin{equation*}
\int_{A} f d \mu=0 \tag{15.5}
\end{equation*}
$$

Proof.
Direct consequence of Lemma 1148 (integral in $\mathcal{I F}$ over subset, thus exists $B \in \Sigma$ such that $B \subset A$ and $\int_{A} f d \mu=\mu(B)$ ), Lemma 913 (measure, nonnegativeness), Lemma 919 (measure is monotone, thus $\mu(B) \leqslant \mu(A)=0$ ), and the definition of order (antisymmetry, thus $\mu(B)=0$ ).

Remark 1151. In the next lemma, as in the countable case, when $Y$ is uncountable, the sum (of nonnegative values) is understood as the supremum for all finite subsets:

$$
\begin{equation*}
\sum_{y \in Y} f(y) \stackrel{\text { def. }}{=} \sup _{\substack{Z \subset Y \\ \operatorname{card}(Z)<\infty}} \sum_{z \in Z} f(z) \tag{15.6}
\end{equation*}
$$

Lemma 1152 (integral in $\mathcal{I F}$ with counting measure).
Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{I F}$. Then, we have

$$
\begin{equation*}
\int f d \delta_{Y}=\sum_{y \in Y} f(y) \tag{15.7}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1100 (indicator and support are inverse), Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$, with $A=\{f \neq 0\} \in \Sigma$ ), Lemma 935 (counting measure), the definition of the cardinality, and the definition of the indicator function.

### 15.2 Integration of nonnegative simple function

Remark 1153. In this section, functions take their values in $\mathbb{R}_{+}$, and the expressions involving integrals are taken in $\overline{\mathbb{R}}_{+}$. We recall that the notation $\mathcal{S \mathcal { F }}{ }_{+}$denotes the set of nonnegative simple functions to the field of real numbers $\mathbb{R}$, see Definition 1123 .

Lemma 1154 (integral in $\mathcal{S F}_{+}$).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Then, we have $f=\sum_{y \in f(X)} y \mathbb{1}_{\{f=y\}}$, and the sum $\sum_{y \in f(X)} y \mu(\{f=y\})$ is well-defined in $\overline{\mathbb{R}}_{+}$.

The integral of $f$ (for the measure $\mu$ ) is still denoted $\int f d \mu$; it is defined by

$$
\begin{equation*}
\int f d \mu \stackrel{\text { def. }}{=} \sum_{y \in f(X)} y \int \mathbb{1}_{\{f=y\}} d \mu=\sum_{y \in f(X)} y \mu(\{f=y\}) \quad \in \overline{\mathbb{R}}_{+} . \tag{15.8}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1126 ( $\mathcal{S} \mathcal{F}_{+}$canonical representation), Lemma 913 (measure, nonnegativeness), Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory), $a_{i} \in \mathbb{R}_{+} \subset \overline{\mathbb{R}}_{+}$), Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed), and Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).

Remark 1155. Keeping the same notation for the integrals is justified in the next lemma.
Lemma 1156 (integral in $\mathcal{S F}_{+}$generalizes integral in $\mathcal{I F}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{I F}$. Then, the values of $\int f d \mu$ provided by Definition 1144 (integral in $\mathcal{I F}$ ), and Lemma 1154 (integral in $\mathcal{S F}{ }_{+}$) coincide.

In other terms, for all $A \in \Sigma$, we have $\mathbb{1}_{A} \in \mathcal{S F} \mathcal{F}_{+}$and $\int \mathbb{1}_{A} d \mu=\mu(A)$.
Proof. Direct consequence of Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}, A \in \Sigma)$, Lemma $1128\left(\mathcal{S F}+\right.$ simple representation, with $n=0, A_{0}=A$, and $\left.a_{0}=1\right)$, Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ), and Lemma 1154 (integral in $\mathcal{S F}_{+}$, with


Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S} \mathcal{F}_{+} . \operatorname{Let}\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be a disjoint representation of $f$. Then, we have

$$
\begin{equation*}
\int f d \mu=\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i}\right) \tag{15.9}
\end{equation*}
$$

Proof. From Lemma $1126(\mathcal{S F}+$ canonical representation $)$, let $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(b_{j}\right)_{j \in[0 . . m]}\right)$ be the canonical representation of $f$ in $\mathbb{N} \times \Sigma \times \mathbb{R}_{+}$.
For all $j \in[0 . . m]$, let $I_{j}^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid \varnothing \neq A_{i} \subset B_{j}\right\}$. Let $I^{\prime} \stackrel{\text { def. }}{=}\left\{i \in[0 . . n] \mid A_{i} \neq \varnothing\right\}$.
Then, from Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$), Lemma $1127\left(\mathcal{S F} \mathcal{F}_{+}\right.$disjoint representation is subpartition of canonical representation, both properties), Lemma 913 (measure, $\sigma$-additivity and homogeneity), Definition 874 ( $\sigma$-additivity, with $I \stackrel{\text { def. }}{=} I_{j}^{\prime}$ ), and left distributivity of multiplication over addition in $\mathbb{R}$, we have

$$
\begin{aligned}
\int f d \mu & =\sum_{j \in[0 . . m]} b_{j} \mu\left(B_{j}\right)=\sum_{j \in[0 . . m]} b_{j} \mu\left(\biguplus_{i \in I_{j}^{\prime}} A_{i}\right) \\
& =\sum_{j \in[0 . . m]} \sum_{i \in I_{j}^{\prime}} a_{i} \mu\left(A_{i}\right)=\sum_{i \in I^{\prime}} a_{i} \mu\left(A_{i}\right)=\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i}\right) .
\end{aligned}
$$

Lemma 1158 (integral in $\mathcal{S F}_{+}$of zero is zero).
Let $(X, \Sigma, \mu)$ be a measure space. Then, $0 \in \mathcal{S} \mathcal{F}_{+}$, and we have $\int 0 d \mu=0$.
Proof. Direct consequence of the definition of the indicator function ( $\mathbb{1}_{\varnothing} \equiv 0$ ), Lemma 696 ( $\sigma$-algebra, $\varnothing \in \Sigma$ ), Lemma 1156 (integral in $\mathcal{S F} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$, with $A \stackrel{\text { def. }}{=} \varnothing$ ), and Lemma 913 (measure, $\mu(\varnothing)=0$ ).

Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{S F}_{+}$. Let $a \in \mathbb{R}_{+}$. Then, $f+g, a f \in \mathcal{S} \mathcal{F}_{+}$, and we have

$$
\begin{equation*}
\int(f+g) d \mu=\int f d \mu+\int g d \mu, \quad \text { and } \quad \int a f d \mu=a \int f d \mu \tag{15.10}
\end{equation*}
$$

Proof. From Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations $), f+g, a f \in \mathcal{S} \mathcal{F}_{+}$.
Addition. From Lemma $1125\left(\mathcal{S F} \mathcal{F}_{+}\right.$disjoint representation $)$, let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right)$ and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(b_{j}\right)_{j \in[0 . . m]}\right)$ in $\mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be disjoint representations of $f$ and $g$. In particular, we have $X=\biguplus_{i \in[0 . . n]} A_{i}=\biguplus_{j \in[0 . . m]} B_{j}$.

Let $N \stackrel{\text { def. }}{=} n m+n+m$. Let $\varphi:[0 . . N] \rightarrow[0 . . n] \times[0 . . m]$ be a bijection (with $N+1=(n+1)(m+1)$ ). For all $(i, j) \in[0 . . n] \times[0 . . m]$, let $C_{i, j} \stackrel{\text { def. }}{=} A_{i} \cap B_{j}$ and $c_{i, j} \stackrel{\text { def. }}{=} a_{i}+b_{j}$. Then, from Lemma $1129\left(\mathcal{S} \mathcal{F}_{+}\right.$ is closed under positive algebra operations, addition), $\left(N,\left(C_{\varphi(k)}\right)_{k \in[0 . . N]},\left(c_{\varphi(k)}\right)_{k \in[0 . . m]}\right)$ is a disjoint representation of $f+g$. Thus, from Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$ (disjoint)), associativity and commutativity of addition in $\mathbb{R}\left(\sum_{k}=\sum_{i, j}\right)$, Lemma 922 (measure over countable pseudopartition, first with $A_{i}$ and the pseudopartition $\left(B_{j}\right)_{j \in[0 . . m]}$, then with $B_{j}$ and the pseudopartition $\left.\left(A_{i}\right)_{i \in[0 . . n]}\right)$, and left distributivity of multiplication over addition in $\mathbb{R}$, we have

$$
\begin{aligned}
\int(f+g) d \mu & =\sum_{k \in[0 . . N]} c_{\varphi(k)} \mu\left(C_{\varphi(k)}\right)=\sum_{(i, j) \in[0 . . n] \times[0 . . m]} c_{i, j} \mu\left(C_{i, j}\right), \\
\int f d \mu & =\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i}\right)=\sum_{(i, j) \in[0 . . n] \times[0 \ldots m]} a_{i} \mu\left(C_{i, j}\right), \\
\int g d \mu & =\sum_{j \in[0 . . m]} b_{j} \mu\left(B_{j}\right)
\end{aligned}=\sum_{(i, j) \in[0 . . n] \times[0 . . m]} b_{j} \mu\left(C_{i, j}\right) .
$$

Hence, from left distributivity of multiplication over addition in $\mathbb{R}$, we have the equality.
Nonnegative scalar multiplication. Case $a=0$.
Direct consequence of field properties of $\mathbb{R}$ ( $0 f$ is the zero function), Lemma 421 (zeroproduct property in $\overline{\mathbb{R}}$ (measure theory)), and Lemma 1158 (integral in $\mathcal{S} \mathcal{F}_{+}$of zero is zero).
Case $\boldsymbol{a}>\mathbf{0}$. Let $y \in f(X)$. Let $z=a y \in(a f)(X)$. Then, from Definition 326 (multiplication in $\overline{\mathbb{R}}$, for all $a \in \mathbb{R}_{+}^{\star}, \frac{a}{a}=1$ ), we have $\{a f=z\}=\{f=y\}$ Hence, from Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$), we have

$$
\int a f d \mu=\sum_{z \in(a f)(X)} z \mu(\{a f=z\})=\sum_{y \in f(X)} a y \mu(\{f=y\})=a \int f d \mu
$$

Therefore, the integral in $\mathcal{S \mathcal { F } _ { + }}$ is positive linear.
Lemma 1160 (equivalent definition of the integral in $\mathcal{S \mathcal { F } _ { + }}$ (simple)).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be a simple representation of $f$. Then, we have

$$
\begin{equation*}
\int f d \mu=\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i}\right) . \tag{15.11}
\end{equation*}
$$

Proof. Direct consequence of Lemma $1128\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation, thus such representation exists), Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear), and Lemma 1156 (integral in $\mathcal{S F}_{+}$


Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{S} \mathcal{F}_{+}$. Then, we have

$$
\begin{equation*}
f \leqslant g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu \tag{15.12}
\end{equation*}
$$

Proof. Assume that $f \leqslant g$. Then, from Definition $1123\left(\mathcal{S F}{ }_{+}\right.$, subset of nonnegative simple functions), Lemma $1117\left(\mathcal{S F} \mathcal{F}_{E}\right.$ is vector space, with $E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ ), Definition 277 (algebra over a field, $\mathcal{S \mathcal { F }}$ is a vector space), and Definition 61 (vector space, ( $\underline{\mathcal{S F},+ \text { ) is an abelian group), we have }}$ $g=\overline{f+(g-f) \text { with } f, g}-f \in \mathcal{S} \mathcal{F}_{+}$. Therefore, from Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear), and Lemma 1154 (integral in $\mathcal{S \mathcal { F } _ { + }}$, nonnegativeness with $g-f \in \mathcal{S F} \mathcal{F}_{+}$), we have

$$
\int g d \mu=\int f d \mu+\int(g-f) d \mu \geqslant \int f d \mu
$$

## Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \mu=\sup _{\substack{\varphi \in \mathcal{S} \mathcal{F}_{+} \\ \varphi \leqslant f}} \int \varphi d \mu \tag{15.13}
\end{equation*}
$$

Proof. From Definition 2 (supremum, upper bound), and since $f \in \mathcal{S} \mathcal{F}_{+}$, we have

$$
\int f d \mu \leqslant \sup _{\substack{\varphi \in \mathcal{S} \mathcal{F}_{+} \\ \varphi \leqslant f}} \int \varphi d \mu .
$$

Conversely, let $\varphi \in \mathcal{S} \mathcal{F}_{+}$such that $\varphi \leqslant f$. Then, from Lemma 1161 (integral in $\mathcal{S} \mathcal{F}_{+}$is monotone), and Definition 2 (supremum, least upper bound), we have

$$
\sup _{\substack{\varphi \in \mathcal{S \mathcal { F } _ { + }} \\ \varphi \leqslant f}} \int \varphi d \mu \leqslant \int f d \mu
$$

Therefore, we have the equality.
Lemma 1163 (integral in $\mathcal{S F}_{+}$over subset).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{S} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$.
If so, there exists $\left(n,\left(\hat{g}_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \mathcal{I F}(X, \Sigma) \times \mathbb{R}_{+}$such that
(15.14) $\forall i \in[0 . . n], \hat{g}_{i} \mathbb{1}_{A}=\hat{g}_{i}, \quad f_{\mid A}=\sum_{i \in[0 . . n]} a_{i}\left(\hat{g}_{i}\right)_{\left.\right|_{A}}, \quad \hat{f} \mathbb{1}_{A}=\sum_{i \in[0 . . n]} a_{i} \hat{g}_{i}$,
and we have

$$
\begin{equation*}
\int f_{\left.\right|_{A}} d \mu_{A}=\sum_{i \in[0 \ldots . . n]} a_{i} \int_{A} \hat{g}_{i} d \mu=\int f \mathbb{1}_{A} d \mu \tag{15.15}
\end{equation*}
$$

This integral is still denoted $\int_{A} f d \mu$; it is still called integral of $f$ over $A$.

Proof. Direct consequence of Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero $)$, Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear), Lemma 1156 (integral in $\mathcal{S F} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$ ), and Lemma 1148 (integral in $\mathcal{I F}$ over subset).

Lemma 1164 (integral in $\mathcal{S F}_{+}$over subset is $\sigma$-additive). $\quad$ Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ and $n \in \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$.
Let $Y$ be a set such that $A \subset Y \subset X$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, $\hat{f} \mathbb{1}_{A}=\sum_{i \in I} \hat{f} \mathbb{1}_{A_{i}}$, and $\hat{f} \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}$implies for all $i \in I, \hat{f} \mathbb{1}_{A_{i}} \in \mathcal{S} \mathcal{F}_{+}$. If so, we have

$$
\begin{equation*}
\int_{A} f d \mu=\sum_{i \in I} \int_{A_{i}} f d \mu \tag{15.16}
\end{equation*}
$$

Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$, we have for all $i \in[0 . . n], \hat{f} \mathbb{1}_{A_{i}} \in \mathcal{S} \mathcal{F}_{+}$implies $\hat{f} \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}$.

Proof. Direct consequence of Lemma $1132\left(\mathcal{S F}_{+}\right.$, masking is $\sigma$-additive), Lemma 1163 (integral in $\mathcal{S F}{ }_{+}$over subset, with $A$ and the $A_{i}$ 's), Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear, additivity), and induction on the number of operands.

Lemma 1165 (integral in $\mathcal{S F}{ }_{+}$over negligible subset). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$ and $\mu(A)=0$. Let $f: Y \rightarrow \mathbb{R}$ and $\hat{f}: X \rightarrow \mathbb{R}$. Assume that $\hat{f}_{\mid Y}=f$, and either $f_{\left.\right|_{A}} \in \mathcal{S F} \mathcal{F}_{+}(A, \Sigma \bar{\cap} A)$, or $\hat{f} \mathbb{1}_{A} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$. Then, we have both, and

$$
\begin{equation*}
\int_{A} f d \mu=0 \tag{15.17}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1163 (integral in $\mathcal{S F}{ }_{+}$over subset, thus exist $n \in \mathbb{N}$, scalars $\left(a_{i}\right)_{i \in[0 . . n]} \in \mathbb{R}_{+}$and $\left(\hat{g}_{i}\right)_{i \in[0 . . n]} \in \mathcal{I F}(X, \Sigma)$ such that $\hat{g}_{i} \mathbb{1}_{A}=\hat{g}_{i}$ and $\left.\int_{A} f d \mu=\sum_{i \in[0 . n]} a_{i} \int_{A} \hat{g}_{i} d \mu\right)$, Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset, thus $\int_{A} \hat{g}_{i} d \mu=0$ ), and field properties of $\mathbb{R}$ (thus $\left.\sum_{i \in[0 . . n]} a_{i} \int_{A} \hat{g}_{i} d \mu=0\right)$.

Lemma 1166 (density measure in $\mathcal{S F}_{+}$). Let $(X, \Sigma, \mu)$ be a measure space. Let $w \in \mathcal{S} \mathcal{F}_{+}$. Then, the function $\left(B \in \Sigma \mapsto \int w \mathbb{1}_{B} d \mu\right)$ is a measure on $(X, \Sigma)$.

This function is denoted $\mu_{w}$; it is called measure of density $w$, (or simply density measure).

Proof. Let $B \in \Sigma$. Then, from Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations, multiplication), and Lemma 1154 (integral in $\mathcal{S \mathcal { F } _ { + }}$ ), we have $w \mathbb{1}_{B} \in \mathcal{S \mathcal { F } _ { + }}$, and $\int w \mathbb{1}_{B} d \mu \geqslant 0$.

From Lemma $696(\sigma$-algebra, $\varnothing \in \Sigma)$, the definition of the indicator function $\left(\mathbb{1}_{\varnothing} \equiv 0\right)$, field properties of $\mathbb{R}$, and Definition $1123\left(\mathcal{S F}{ }_{+}\right.$, subset of nonnegative simple functions), we have $w \mathbb{1}_{\varnothing}=\mathbb{1}_{\varnothing} \in \mathcal{S F} \mathcal{F}_{+}$. Hence, from Lemma 1156 (integral in $\mathcal{S F} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$ ), and Lemma 913 (measure, homogeneity), we have $\int w \mathbb{1}_{\varnothing} d \mu=\mu(\varnothing)=0$.

From Lemma $1125\left(\mathcal{S F} \mathcal{F}_{+}\right.$disjoint representation $)$, let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$ be a disjoint representation of $w$, i.e. where the $A_{i}$ 's are pairwise disjoint. Let $J \subset \mathbb{N}$. Let $\left(B_{j}\right)_{j \in J}$ be pairwise disjoint in $\Sigma$, and let $B \stackrel{\text { def. }}{=} \biguplus_{j \in J} B_{j}$. Then, from Lemma $696\left(\sigma\right.$-algebra, $\left.P_{\Sigma} \Rightarrow U c\right)$, Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations, multiplication), and Lemma 1125 $\left(\mathcal{S F}{ }_{+}\right.$disjoint representation), we have $w \mathbb{1}_{B} \in \mathcal{S F} \mathcal{F}_{+}$, and $\left(n,\left(\overline{\left.A_{i} \cap B\right)_{i \in[0 . n]}},\left(a_{i}\right)_{i \in[0 . . n]}\right)\right.$ is a disjoint representation of $w \mathbb{1}_{B}$. Hence, from Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$ (disjoint)), distributivity of intersection over union, Lemma 913 (measure, $\sigma$-additivity), Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), Lemma 394 (ad-
dition in $\overline{\mathbb{R}}_{+}$is associative), and Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative), we have

$$
\begin{aligned}
\int w \mathbb{1}_{\uplus_{j \in J} B_{j}} d \mu=\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i} \cap \biguplus_{j \in J} B_{j}\right)=\sum_{i \in[0 . . n]} a_{i} \mu\left(\biguplus_{j \in J}\left(A_{i} \cap B_{j}\right)\right) \\
=\sum_{i \in[0 . . n]} a_{i}\left(\sum_{j \in J} \mu\left(A_{i} \cap B_{j}\right)\right)=\sum_{j \in J}\left(\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i} \cap B_{j}\right)\right)=\sum_{j \in J} \int w \mathbb{1}_{B_{j}} d \mu .
\end{aligned}
$$

Therefore, from Lemma 913 (measure), $\left(A \mapsto \int w \mathbb{1}_{A} d \mu\right)$ is a measure on $(X, \Sigma)$.
Lemma 1167 (integral and density measure in $\mathcal{S F}_{+}$). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right) \in \mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be a disjoint representation of $f$. Then, we have

$$
\begin{align*}
\forall i \in[0 . . n], \quad \mu_{f}\left(A_{i}\right) & =a_{i} \mu\left(A_{i}\right),  \tag{15.18}\\
\int f d \mu & =\sum_{i \in[0 . . n]} \mu_{f}\left(A_{i}\right) . \tag{15.19}
\end{align*}
$$

Proof. (1). Let $i \in[0 . . n]$. Then, from Lemma $1129\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under positive algebra operations, multiplication), Lemma 1125 ( $\mathcal{S F} \mathcal{F}_{+}$disjoint representation, pairwise disjunction), the definition of the indicator function $\left(\mathbb{1}_{\varnothing} \equiv 0\right)$, and field properties of $\mathbb{R}$, we have

$$
f \mathbb{1}_{A_{i}}=\left(\sum_{i^{\prime} \in[0 . . n]} a_{i^{\prime}} \mathbb{1}_{A_{i^{\prime}}}\right) \mathbb{1}_{A_{i}}=\left(\sum_{i^{\prime} \in[0 . . n]} a_{i^{\prime}} \mathbb{1}_{A_{i^{\prime}} \cap A_{i}}\right)=a_{i} \mathbb{1}_{A_{i}} .
$$

Hence, from Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$), and Lemma 1157 (equivalent definition of the integral in $\mathcal{S F} \mathcal{F}_{+}$(disjoint), with $f \stackrel{\text { def. }}{=} a_{i} \mathbb{1}_{A_{i}}$ ), we have

$$
\mu_{f}\left(A_{i}\right)=\int f \mathbb{1}_{A_{i}} d \mu=\int a_{i} \mathbb{1}_{A_{i}} d \mu=a_{i} \mu\left(A_{i}\right)
$$

(2). From Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)), and (1), we have

$$
\int f d \mu=\sum_{i \in[0 . . n]} a_{i} \mu\left(A_{i}\right)=\sum_{i \in[0 . . n]} \mu_{f}\left(A_{i}\right) .
$$

Therefore both properties hold.

## Lemma 1168 (density integral in $\mathcal{S F}{ }_{+}$).

Let $(X, \Sigma, \mu)$ be a measure space. Let $w, f \in \mathcal{S} \mathcal{F}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \mu_{w}=\int f w d \mu \tag{15.20}
\end{equation*}
$$

Proof. From Lemma $1125(\mathcal{S F}+$ disjoint representation $)$, let $\left(n,\left(A_{i}\right)_{i \in[0 . . n]},\left(a_{i}\right)_{i \in[0 . . n]}\right)$ and $\left(m,\left(B_{j}\right)_{j \in[0 . . m]},\left(b_{j}\right)_{j \in[0 . . m]}\right)$ in $\mathbb{N} \times \Sigma \times \mathbb{R}_{+}$be disjoint representations of $f$ and $w$.

Then, from Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$, with $f$ ), Lemma 1166 (density measure in $\mathcal{S F} \mathcal{F}_{+}$), Lemma $1129\left(\mathcal{S F}+\right.$ is closed under positive algebra operations, multiplication with $w$ and $\left.\mathbb{1}_{A_{i}}\right)$, and Lemma 1154 (integral in $\mathcal{S F}{ }_{+}$, with $w \mathbb{1}_{A_{i}}$ ), we have

$$
\int f d \mu_{w}=\sum_{i \in[0 . . n]} a_{i} \mu_{w}\left(A_{i}\right)=\sum_{i \in[0 . . n]} a_{i}\left(\int w \mathbb{1}_{A_{i}} d \mu\right)=\sum_{i \in[0 . . n]} a_{i}\left(\sum_{j \in[0 \ldots m]} b_{j} \mu\left(B_{j} \cap A_{i}\right)\right) .
$$

Moreover, from Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations, multiplication with $f$ and $w$ ), Lemma 1154 (integral in $\mathcal{S F} \mathcal{F}_{+}$), and Lemma 419 (multiplication in $\overline{\mathbb{R}} \overline{\text { is distributive }}$ $\overline{\text { over addition }}$ (measure theory)), we have

$$
\begin{aligned}
\int f w d \mu & =\int\left(\sum_{(i, j) \in[0 . . n] \times[0 . . m]} a_{i} b_{j} \mathbb{1}_{A_{i} \cap B_{j}}\right) d \mu \\
& =\sum_{(i, j) \in[0 . . n] \times[0 \ldots m]} a_{i} b_{j} \mu\left(A_{i} \cap B_{j}\right)=\sum_{i \in[0 . . n]} a_{i}\left(\sum_{j \in[0 . . m]} b_{j} \mu\left(A_{i} \cap B_{j}\right)\right) .
\end{aligned}
$$

Therefore, from commutativity of intersection, we have the equality.

## Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure).

Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Then, we have
(15.21)

$$
\int f d 0=0 .
$$

Proof. Direct consequence of Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$), Lemma 933 (trivial measure), and Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ).

Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure).
Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{S} \mathcal{F}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \delta_{Y}=\sum_{y \in Y} f(y) \tag{15.22}
\end{equation*}
$$

Proof. Direct consequence of Lemma $1128\left(\mathcal{S F}{ }_{+}\right.$simple representation), Lemma 1156 (integral in $\mathcal{S F} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$ ), Lemma 1152 (integral in $\mathcal{I F}$ with counting measure), Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative), and Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative).

### 15.3 Integration of nonnegative measurable function

Remark 1171. In this section, functions take their values in $\overline{\mathbb{R}}_{+}$, and the expressions involving integrals are also taken in $\overline{\mathbb{R}}_{+}$.

## Lemma 1172 (integral in $\mathcal{M}_{+}$). <br> Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$.

 Then, $\left\{\int \varphi d \mu \mid \varphi \in \mathcal{S} \mathcal{F}_{+} \wedge \varphi \leqslant f\right\}$ admits a supremum.The integral of $f$ (for the measure $\mu$ ) is still denoted $\int f d \mu$; it is defined by

$$
\begin{equation*}
\int f d \mu \stackrel{\text { def. }}{=} \sup _{\substack{\varphi \in \mathcal{S} \mathcal{F}_{+} \\ \varphi \leqslant f}} \int \varphi d \mu \quad \in \overline{\mathbb{R}}_{+} . \tag{15.23}
\end{equation*}
$$

A function $f: X \rightarrow \overline{\mathbb{R}}_{+}$is said $\mu$-integrable (in $\mathcal{M}_{+}$) iff $f \in \mathcal{M}_{+}$and $\int f d \mu<\infty$.
Proof. Direct consequence of Lemma 1154 (integral in $\mathcal{S F}_{+}$), and Definition 2 (supremum, with the Lebesgue integral in $\mathcal{S F} \mathcal{F}_{+}$over $\left.\{\varphi \in \mathcal{S F}+\mid \varphi \leqslant f\}\right)$,

Remark 1173. Keeping the same notation for the integrals is justified in the next lemma.
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S F}_{+}$).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{S F}_{+}$. Then, the values of $\int f d \mu$ provided by Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$), and Lemma 1172 (integral in $\mathcal{M}_{+}$) coincide.

Proof. Direct consequence of Lemma 1154 (integral in $\mathcal{S F}_{+}$), Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous), and Lemma 1172 (integral in $\mathcal{M}_{+}$).

Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function).
Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Then, we have $\mathbb{1}_{A} \in \mathcal{M}_{+}$and $\int \mathbb{1}_{A} d \mu=\mu(A)$. Moreover, when $\mu(A)$ is finite, then $\mathbb{1}_{A}$ is $\mu$-integrable.

Proof. Direct consequence of Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable), Lemma 1174 (integral in $\mathcal{M}_{+}$
 Lemma 1172 (integral in $\mathcal{M}_{+}$).

Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Let $a \in \mathbb{R}_{+}$. Then, af $\in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int a f d \mu=a \int f d \mu . \tag{15.24}
\end{equation*}
$$

Proof. From Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication), af $\in \mathcal{M}_{+}$.
Case $\boldsymbol{a}=\mathbf{0}$. Then, from Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions), Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}\right)$, and Definition 61 (vector space, $\left.\underline{0 \in \mathcal{S} \mathcal{F}}\right)$, we have

$$
0 f=0 \in \mathcal{S \mathcal { F } _ { + }} \subset \mathcal{M}_{+}
$$

Hence, from Lemma 1172 (integral in $\mathcal{M}_{+}$), Lemma 1160 (equivalent definition of the integral in $\mathcal{S F} \mathcal{F}_{+}$(simple)), Lemma 1154 (integral in $\mathcal{S \mathcal { F } _ { + }}$ ), and Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), we have

$$
\int 0 f d \mu=\int 0 d \mu=0=0 \times \int f d \mu
$$

Case $\boldsymbol{a}>\mathbf{0}$. Then, from Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)), and Definition 326 (multiplication in $\overline{\mathbb{R}}$, for all $a \in \mathbb{R}_{+}^{\star}, \frac{a}{a}=1$ ), we have $(\star) \forall b \in \overline{\mathbb{R}}_{+}, \frac{1}{a}(a b)=b$.

Let $\varphi \in \mathcal{S} \mathcal{F}_{+}$such that $\varphi \leqslant f$. Then, from Lemma $1129\left(\mathcal{S F}{ }_{+}\right.$is closed under positive algebra operations), we have $a \varphi \in \mathcal{S} \mathcal{F}_{+}$. Moreover, from Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), property $(\star)$, Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear, positive homogeneity), Lemma 1172 (integral in $\mathcal{M}_{+}$), and Definition 2 (supremum, upper bound), we have $a \varphi \leqslant a f$ and

$$
\int \varphi d \mu=\frac{1}{a} \int a \varphi d \mu \leqslant \frac{1}{a} \int a f d \mu
$$

Thus, from Lemma 1172 (integral in $\mathcal{M}_{+}$), Definition 2 (supremum, least upper bound), and property ( $\star$ ), we have

$$
a \int f d \mu=a \sup _{\substack{\varphi \in \mathcal{S \mathcal { F } _ { + }} \\ \varphi \leqslant f}} \int \varphi d \mu \leqslant a \frac{1}{a} \int a f d \mu=\int a f d \mu
$$

Moreover, the same result also holds for function $a f \in \mathcal{M}_{+}$and number $\frac{1}{a}>0$, and with property $(\star)$, we have $\int a f d \mu \leqslant a \int \frac{1}{a}(a f) d \mu=a \int f d \mu$. Hence, we have the equality.

Therefore, we always have the equality.
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero).
Let $(X, \Sigma, \mu)$ be a measure space. Then, $0 \in \mathcal{M}_{+}$, and we have $\int 0 d \mu=0$.
Proof. Direct consequence of Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous, with $a \stackrel{\text { def. }}{=} 0$ ), Definition 326 (multiplication in $\overline{\mathbb{R}}$ ), and Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)).

## Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
f \leqslant g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu \tag{15.25}
\end{equation*}
$$

Proof. Assume that $f \leqslant g$. Then, $\left\{\varphi \in \mathcal{S} \mathcal{F}_{+} \mid \varphi \leqslant f\right\} \subset\left\{\varphi \in \mathcal{S} \mathcal{F}_{+} \mid \varphi \leqslant g\right\}$. Therefore, from Lemma 1172 (integral in $\mathcal{M}_{+}$), and Lemma 489 (supremum is monotone (set)). we have

$$
\int f d \mu \leqslant \int g d \mu
$$

Remark 1179. The next proof follows step 3 of the Lebesgue scheme (see Section 4.4).
See also the sketch of the proof in Section 5.8.

## Theorem 1180 (Beppo Levi, monotone convergence).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$.
Assume that the sequence is pointwise nondecreasing. Then, $\lim _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \lim _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) \tag{15.26}
\end{equation*}
$$

Proof. From Lemma 314 ( $\overline{\mathbb{R}}$ is complete), and Lemma 501 (supremum is limit), the existence of the pointwise limit is guaranteed in $\overline{\mathbb{R}}$, and we have $f \stackrel{\text { def. }}{=} \lim _{n \rightarrow \infty} f_{n}=\sup _{n \in \mathbb{N}} f_{n}$.

From Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent), we have $f \in \mathcal{M}_{+}$. Then, from Lemma 1172 (integral in $\mathcal{M}_{+}$), the integral of $f$ is well-defined. Moreover, from Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and Lemma $314(\overline{\mathbb{R}}$ is complete), the sequence $\left(\int f_{n} d \mu\right)_{n \in \mathbb{N}}$ is nondecreasing, hence convergent in $\overline{\mathbb{R}}$ towards

$$
\lim _{n \rightarrow \infty} \int f_{n} d \mu=\sup _{n \in \mathbb{N}} \int f_{n} d \mu
$$

Thus, from Definition 2 (supremum, upper bound), and Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), for all $n \in \mathbb{N}$, we have

$$
\int f_{n} d \mu \leqslant \int \sup _{n \in \mathbb{N}} f_{n} d \mu=\int f d \mu
$$

Hence, from Definition 2 (supremum, least upper bound), we have

$$
\lim _{n \rightarrow \infty} \int f_{n} d \mu=\sup _{n \in \mathbb{N}} \int f_{n} d \mu \leqslant \int f d \mu
$$

Let $a \in(0,1)$. Let $\varphi \in \mathcal{S} \mathcal{F}_{+}$. Assume that $\varphi \leqslant f$. Let $n \in \mathbb{N}$. Let $A_{n} \stackrel{\text { def. }}{=}\left\{a \varphi \leqslant f_{n}\right\} \subset X$. Then, from Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R},-a \varphi \in \mathcal{S F}\right)$, Lemma 1113 $\left(\mathcal{S \mathcal { F } _ { E }}\right.$ is measurable, with $\left.E=\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R},-a \varphi \in \mathcal{M}\right)$, and Lemma $822(\mathcal{M}$ is closed under addition when defined, $-a \varphi$ takes finite values), we have $f_{n}-a \varphi \in \mathcal{M}$. Thus, from Lemma 820 (measurability of numeric function, with $f_{n}-a \varphi$ ), and Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function), we have $A_{n} \in \Sigma$, and $\mathbb{1}_{A_{n}} \in \mathcal{M}_{+}$. Moreover, from Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity), the sequence $\left(A_{n}\right)_{n \in \mathbb{N}}$ is nondecreasing. Let $x \in X$. Case $\varphi(x)=0$. Then, for all $n \in \mathbb{N}, x \in A_{n}$. Case $\varphi(x)>0$. Then, since $0<a<1$, we have $a \varphi(x)<\varphi(x) \leqslant f(x)$. Thus, from Lemma 381 (limit is monotone), there exists $N \in \mathbb{N}$ such that for all $n \geqslant N, a \varphi(x) \leqslant f_{n}(x)$, i.e. $x \in A_{N}$. Hence, we have $X=\bigcup_{n \in \mathbb{N}} A_{n}$.

Let $B \in \Sigma$. Then, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection (with $\operatorname{card}(I)=2$ ), monotonicity of intersection, and distributivity of intersection over union, the sequence $\left(B \cap A_{n}\right)_{n \in \mathbb{N}}$ belongs to $\Sigma$, is nondecreasing, and $\bigcup_{n \in \mathbb{N}}\left(B \cap A_{n}\right)=B$. Thus, from Lemma 925 (measure is continuous from below), and Definition 879 (continuity from below), we have $\mu(B)=\mu\left(\bigcup_{n \in \mathbb{N}}\left(B \cap A_{n}\right)\right)=\lim _{n \rightarrow \infty} \mu\left(B \cap A_{n}\right)$.

Then, from Lemma 1160 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(simple)), there exists $k \in \mathbb{N},\left(a_{i}\right)_{i \in[0 . . k]} \in \mathbb{R}_{+}$, and $\left(B_{i}\right)_{i \in[0 . k]} \in \Sigma$ such that

$$
\varphi=\sum_{i \in[0 . . k]} a_{i} \mathbb{1}_{B_{i}}, \quad \text { and } \quad \int \varphi d \mu=\sum_{i \in[0 . . k]} a_{i} \mu\left(B_{i}\right) .
$$

Thus, from Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear), we have

$$
\forall n \in \mathbb{N}, \int \varphi \mathbb{1}_{A_{n}} d \mu=\int \sum_{i \in[0 . . k]} a_{i} \mathbb{1}_{B_{i} \cap A_{n}} d \mu=\sum_{i \in[0 . . k]} a_{i} \mu\left(B_{i} \cap A_{n}\right)
$$

Hence, from the previous result on the measure $\mu$, field properties of $\mathbb{R}\left(\frac{a}{a}=1\right)$, Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear, with $a_{i} \in \mathbb{R}_{+}$and $a \in \mathbb{R}_{+}^{\star}$ ), and Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear, with $a \in \mathbb{R}_{+}^{\star}$ ), we have

$$
\int \varphi d \mu=\sum_{i \in[0 . . k]} a_{i} \lim _{n \rightarrow \infty} \mu\left(B_{i} \cap A_{n}\right)=\lim _{n \rightarrow \infty} \int \varphi \mathbb{1}_{A_{n}} d \mu=\frac{1}{a} \lim _{n \rightarrow \infty} \int a \varphi \mathbb{1}_{A_{n}} d \mu
$$

Let $n \in \mathbb{N}$. Then, from the definition of the $A_{n}$ 's, and the definition of the indicator function, we have $a \varphi \mathbb{1}_{A_{n}} \leqslant f_{n} \mathbb{1}_{A_{n}} \leqslant f_{n}$. Thus, from Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), Lemma 381 (limit is monotone), and Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), we have

$$
\int \varphi d \mu \leqslant \frac{1}{a} \lim _{n \rightarrow \infty} \int f_{n} d \mu .
$$

Then, from left continuity of the division at $a=1$, and Lemma 381 (limit is monotone), we have

$$
\forall \varphi \in \mathcal{S F}_{+}, \varphi \leqslant f \Rightarrow \int \varphi d \mu \leqslant \lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

Hence, from Lemma 1172 (integral in $\mathcal{M}_{+}$), and Definition 2 (supremum, least upper bound),

$$
\int f d \mu=\sup _{\substack{\varphi \in \mathcal{S} \mathcal{F}_{+} \\ \varphi \leqslant f}} \int \varphi d \mu \leqslant \lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

Therefore, we have $\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu$.
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Then, $\infty f \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \infty f d \mu=\infty \int f d \mu \tag{15.27}
\end{equation*}
$$

Proof. From Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\left.f \geqslant 0\right)$, Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous), and Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication), $(n f)_{n \in \mathbb{N}}$ is a nondecreasing sequence in $\mathcal{M}_{+}$such that $\lim _{n \rightarrow \infty} n f=\infty f \in \mathcal{M}_{+}$. Therefore, from Theorem 1180 (Beppo Levi, monotone convergence), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), and Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous), we have

$$
\begin{aligned}
\int \infty f d \mu & =\int\left(\lim _{n \rightarrow \infty} n\right) f d \mu=\int\left(\lim _{n \rightarrow \infty} n f\right) d \mu \\
& =\lim _{n \rightarrow \infty}\left(\int n f d \mu\right)=\lim _{n \rightarrow \infty}\left(n \int f d \mu\right)=\left(\lim _{n \rightarrow \infty} n\right) \int f d \mu=\infty \int f d \mu .
\end{aligned}
$$

## Lemma 1182 (usage of adapted sequences).

Let $(X, \Sigma, \mu)$ be a measure space.
Let $f \in \mathcal{M}_{+}$. Let $\left(\varphi_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F}{ }_{+}$be an adapted sequence of $f$. Then, we have
(15.28)

$$
\int f d \mu=\int \lim _{n \rightarrow \infty} \varphi_{n} d \mu=\lim _{n \rightarrow \infty} \int \varphi_{n} d \mu
$$

Proof. Direct consequence of Definition 1135 (adapted sequence), Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$), Theorem 1180 (Beppo Levi, monotone convergence). and Lemma 1174 (integral in $\mathcal{M}_{+}$ generalizes integral in $\mathcal{S \mathcal { F } _ { + }}$ ).

Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Then, $f+g \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int(f+g) d \mu=\int f d \mu+\int g d \mu \tag{15.29}
\end{equation*}
$$

Proof. From Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition), we have $f+g \in \mathcal{M}_{+}$.
From Lemma 1136 (adapted sequence in $\left.\mathcal{M}_{+}\right)$, let $\left(\varphi_{n}\right)_{n \in \mathbb{N}},\left(\psi_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S} \mathcal{F}_{+}$be adapted sequences for $f$ and $g$. Then, from Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear, additivity), Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order), Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous), and Definition 1135 (adapted sequence), $\left(\varphi_{n}+\psi_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F} \mathcal{F}_{+}$is an adapted sequence for $f+g$. Let $n \in \mathbb{N}$. Then, from Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear, additivity), we have

$$
\int\left(\varphi_{n}+\psi_{n}\right) d \mu=\int \varphi_{n} d \mu+\int \psi_{n} d \mu
$$

Therefore, from Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous), and Lemma 1182 (usage of adapted sequences), we have

$$
\int(f+g) d \mu=\int f d \mu+\int g d \mu
$$

Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Let $a \in \overline{\mathbb{R}}_{+}$. Then, $f+g$, af $\in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int(f+g) d \mu=\int f d \mu+\int g d \mu, \quad \text { and } \quad \int a f d \mu=a \int f d \mu \tag{15.30}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), and Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ).

## Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$. Then, $\sum_{n \in \mathbb{N}} f_{n} \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int\left(\sum_{n \in \mathbb{N}} f_{n}\right) d \mu=\sum_{n \in \mathbb{N}}\left(\int f_{n} d \mu\right) . \tag{15.31}
\end{equation*}
$$

Proof. Direct consequence of Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum $)$, and Theorem 1180 (Beppo Levi, monotone convergence, with nondecreasing sequence $\left.\left(\sum_{i \in\left[0 \_n\right]} f_{i}\right)_{n \in \mathbb{N}}\right)$.

Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts). Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{M}$. Then, we have

$$
\begin{equation*}
\int|f| d \mu=\int f^{+} d \mu+\int f^{-} d \mu \tag{15.32}
\end{equation*}
$$

Proof. Direct consequence of Lemma 842 ( $\mathcal{M}$ is closed under absolute value, $|f|$ belongs to $\left.\mathcal{M}_{+}\right)$, Lemma 836 (measurability of nonnegative and nonpositive parts, $f^{+}, f^{-} \in \mathcal{M}_{+}$), Lemma 532 (decomposition into nonnegative and nonpositive parts, $|f|=f^{+}+f^{-}$), and Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).

Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts). Let $(X, \Sigma)$ be a measurable space. Let $f, g \in \mathcal{M}$ such that $f+g$ is well-defined and belongs to $\mathcal{M}$. Then, we have in $\overline{\mathbb{R}}_{+}$

$$
\begin{equation*}
\int(f+g)^{+} d \mu+\int f^{-} d \mu+\int g^{-} d \mu=\int(f+g)^{-} d \mu+\int f^{+} d \mu+\int g^{+} d \mu \tag{15.33}
\end{equation*}
$$

Proof. Direct consequence of Lemma 836 (measurability of nonnegative and nonpositive parts, $\left.f^{ \pm}, g^{ \pm},(f+g)^{ \pm} \in \mathcal{M}_{+}\right)$, Lemma 533 (compatibility of nonpositive and nonnegative parts with addition), Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition), and Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).

Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \mu=0 \quad \Longleftrightarrow \quad f \stackrel{\mu \text { a.e. }}{=} 0 \tag{15.34}
\end{equation*}
$$

Proof. Let $A \stackrel{\text { def. }}{=}\{f>0\}=f^{-1}(0, \infty]$. Then, from Lemma 820 (measurability of numeric function, $\underline{A \in \Sigma}$ ), Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, $\mathbb{1}_{A} \in \mathcal{M}_{+}$and $\int \mathbb{1}_{A} d \mu=\mu(A)$ ), Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication, $\left.\infty f, \infty \mathbb{1}_{A} \in \mathcal{M}_{+}\right)$, Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ), and Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$ (measure theory), $\propto f=\infty \mathbb{1}_{A}$ ), we have

$$
\infty \int f d \mu=\int \infty f d \mu=\int \infty \mathbb{1}_{A} d \mu=\infty \int \mathbb{1}_{A} d \mu=\infty \mu(A)
$$

Therefore, from Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory), multiplication by $\infty$ is definite in $\left.\overline{\mathbb{R}}_{+}\right)$, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $\{f=0\}^{c}=A$ ), Lemma 948 (negligibility of measurable subset), and Definition 953 (property almost satisfied), we have

$$
\begin{aligned}
\int f d \mu=0 & \Longleftrightarrow \infty \int f d \mu=0 \quad \Longleftrightarrow \quad \infty \mu(A)=0 \\
& \Longleftrightarrow \mu(A)=0 \quad \Longleftrightarrow \quad \mu\left(\{f=0\}^{c}\right)=0 \quad \Longleftrightarrow \quad f^{\mu \text { a.e. }}=0
\end{aligned}
$$

Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $I \subset \mathbb{N}$ be nonempty. Let $P$ and $Q$ be predicates on $\overline{\mathbb{R}}^{I}$, $P$ being lifted into a predicate on $\left(\overline{\mathbb{R}}^{I}\right)^{X}$. Assume that $P(0, \ldots, 0)$ holds, and that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I} \in \mathcal{M}_{+}$, we have

$$
\begin{equation*}
\forall x \in X, P(f(x)) \quad \Longrightarrow \quad Q\left(\int f_{i} d \mu\right)_{i \in I} \tag{15.35}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I} \in \mathcal{M}_{+}$, we have

$$
\begin{equation*}
P(g) \mu \text { a.e. } \Longrightarrow Q\left(\int g_{i} d \mu\right)_{i \in I} . \tag{15.36}
\end{equation*}
$$

Proof. Let $g=\left(g_{i}\right)_{i \in I} \in \mathcal{M}_{+}$. Assume that $P(g)$ holds $\mu$-almost everywhere.
From Lemma 955 (equivalent definition of property almost satisfied), let $A \in \Sigma$ such that $\mu\left(A^{c}\right)=0$ and $A \subset\{P(g)\}$, i.e. such that $P\left(g_{\left.\right|_{A}}\right)$ holds. Let $f=\left(f_{i}\right)_{i \in I} \stackrel{\text { def. }}{=}\left(g_{i} \mathbb{1}_{A}\right)_{i \in I}$. Then, from the definition of the indicator function, and since $P(0, \ldots, 0)$, we have $P(f)$ everywhere. Thus, from Lemma 913 (measure), Definition 741 (measurable space, $\Sigma$ is a $\sigma$-algebra), Lemma 696 ( $\sigma$-algebra, $A^{c} \in \Sigma$ ), Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, $\mathbb{1}_{A}, \mathbb{1}_{A^{c}} \in \mathcal{M}_{+}$), and Lemma $844 \overline{\left(\mathcal{M}_{+}\right.}$is closed under multiplication $)$, for all $i \in I$, we have $f_{i}, \overline{g_{i} \mathbb{1}_{A^{c}} \in \mathcal{M}_{+} \text {. Hence, }}$ from assumption, the property $Q\left(\int f_{i} d \mu\right)_{i \in I}$ holds.

Let $i \in I$. Then, from the definition of the indicator function $\left(\left\{\mathbb{1}_{A^{c}}=0\right\}^{c}=A^{c}\right)$, Lemma 948 (negligibility of measurable subset, $\left\{\mathbb{1}_{A^{c}}=0\right\}^{c} \in \mathbf{N}$ ), Definition 953 (property almost satisfied, $\mathbb{1}_{A^{c}} \stackrel{\mu}{=} \stackrel{\text { a.e. }}{=}$ ), Lemma 985 (compatibility of almost equality with operator, with the unary operator left multiplication by $g_{i}$ ), and Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), we have $g_{i} \mathbb{1}_{A^{c}} \stackrel{\mu \text { a.e. }}{=} g_{i} 0=0$. Hence, from properties of the indicator function, Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), Lemma 1183 (integral in $\mathcal{M}_{+}$ is additive), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite), and Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ), we have

$$
\int g_{i} d \mu=\int g_{i}\left(\mathbb{1}_{A}+\mathbb{1}_{A^{c}}\right) d \mu=\int f_{i} d \mu
$$

Therefore, $Q\left(\int g_{i} d \mu\right)_{i \in I}$ also holds.
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
f \stackrel{\mu \text { a.e. }}{=} g \quad \Longrightarrow \quad \int f d \mu=\int g d \mu \text {. } \tag{15.37}
\end{equation*}
$$

Proof. Direct consequence of reflexivity of equality, Lemma 1189 (extended almost modus po-
 in $\mathcal{M}_{+}$, integral is a function).

Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
f \stackrel{\mu \text { a.e. }}{\leqslant} g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu . \tag{15.38}
\end{equation*}
$$

Proof. Direct consequence of the definition of order (reflexivity), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, thus $0 \leqslant 0$ ), Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$, with $\operatorname{card}(I)=2$ and $P=Q \stackrel{\text { def. }}{=} \leqslant$ ), and Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone).

Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{\geqslant} 0$.
Then, we have $f^{-\mu} \stackrel{\text { a.e. }}{=} 0$, and $\int f^{-} d \mu=0$.
Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts, $\left\{f^{-} \neq 0\right\}$ equals $\{f<0\}$ ), Definition 974 (almost binary relation, $\{f<0\} \in \mathbf{N}$, thus $f^{-\mu} \stackrel{\text { a.e. }}{=} 0$ ), and Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),

Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{=} g$. Then, we have $f^{ \pm} \stackrel{\mu \text { a.e. }}{=} g^{ \pm}$in $\mathcal{M}_{+}$, and $\int f^{ \pm} d \mu=\int g^{ \pm} d \mu$.

Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts), Lemma 985 (compatibility of almost equality with operator, with the operators $\max (\cdot, 0)$ and $\max (-\cdot, 0)$ ), Lemma 836 (measurability of nonnegative and nonpositive parts, $f^{ \pm}, g^{ \pm} \in \mathcal{M}_{+}$), and Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality).

Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{M}$. Assume that $f, g \stackrel{\mu \text { a.e. }}{\geqslant} 0$. Then, $f+g$ is well-defined almost everywhere, and we have $f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$.

Moreover, there exist $f^{\prime}, g^{\prime} \in \mathcal{M}$ such that $f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime \mu} \stackrel{\text { a.e. }}{=} g$, and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$, and for all such functions, $f^{\prime}+g^{\prime} \in \mathcal{M}, f^{\prime}, g^{\prime}, f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \stackrel{\mu \text { a.e. }}{\geqslant} 0, f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$, and we have

$$
\begin{equation*}
\int(f \stackrel{\mu \text { a.e. }}{+} g)^{+} d \mu=\int\left(f^{\prime}+g^{\prime}\right)^{+} d \mu=\int\left(f^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{+} d \mu=\int f^{+} d \mu+\int g^{+} d \mu . \tag{15.39}
\end{equation*}
$$

Proof. From Definition 974 (almost binary relation, $\{f<0\},\{g<0\} \in \mathbf{N}$ ), Lemma 951 (subset of negligible is negligible, $\{f=\infty\} \cap\{g=-\infty\} \subset\{g<0\}$ and $\{f=-\infty\} \cap\{g=\infty\} \subset\{f<0\}$ are in $\mathbf{N}$ ), Lemma 952 ( $\mathbf{N}$ is closed under countable union, with card $(I)=2$ ), Definition 997 (summability domain), and Lemma 1000 (negligibility and summability domain), $f+g$ is welldefined almost everywhere. Thus, from Lemma 1002 (almost sum, thus $f^{\prime}$ and $g^{\prime}$ exist), and Lemma 1003 (compatibility of almost sum with almost equality), we have $f^{\prime}+g^{\prime}, f^{\mu \text { a.e. }}+g \in \mathcal{M}$, and $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$.

Hence, from Lemma 836 (measurability of nonnegative and nonpositive parts, $\left(f^{\prime}\right)^{ \pm},\left(g^{\prime}\right)^{ \pm}$and $\left(f^{\prime}+g^{\prime}\right)^{ \pm}$belong to $\left.\mathcal{M}_{+}\right)$, and Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions), we have

$$
\int\left(f^{\prime}\right)^{ \pm} d \mu=\int f^{ \pm} d \mu, \quad \int\left(g^{\prime}\right)^{ \pm} d \mu=\int g^{ \pm} d \mu, \quad \int\left(f^{\prime}+g^{\prime}\right)^{ \pm} d \mu=\int\left(f^{\mu \text { a.e. }} g\right)^{ \pm} d \mu
$$

Then, from Lemma 995 (compatibility of almost nonnegativeness with almost equality, thus ${f^{\prime}}^{\mu \text { a.e. }} 0$ and $g^{\prime} \stackrel{\mu \text { a.e. }}{\geqslant} 0$ ), and Lemma 984 (compatibility of almost binary relation with operator, with the
binary operator addition), we have $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{\geqslant} 0$. Hence, from Lemma 995 (compatibility of almost nonnegativeness with almost equality), we also have $f \stackrel{\mu \text { a.e. }}{+} g \stackrel{\mu \text { a.e. }}{\geqslant} 0$.

Therefore, from Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function, thus $\int\left(f^{\prime}\right)^{-} d \mu=\int\left(g^{\prime}\right)^{-} d \mu=\int\left(f^{\prime}+g^{\prime}\right)^{-} d \mu=0$ ), Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts, with $f \stackrel{\text { def. }}{=} f^{\prime}$ and $\left.g \stackrel{\text { def. }}{=} g^{\prime}\right)$, and Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality), we have $\int\left(f^{\prime}+g^{\prime}\right)^{+} d \mu$ equals $\int\left(f^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{+} d \mu$, and (15.39) holds.

Lemma 1195 (generalized Bienaymé-Chebyshev inequality). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $f \in \mathcal{M}(X)$. Let $g \in \mathcal{M}_{+}(\overline{\mathbb{R}})$. Let $a \in \overline{\mathbb{R}}$. Assume that $g$ is nondecreasing. Then, we have $\{f \geqslant a\} \in \Sigma$, $g \circ f \in \mathcal{M}_{+}$, and

$$
\begin{equation*}
g(a) \mu(\{f \geqslant a\}) \leqslant \int g \circ f d \mu \tag{15.40}
\end{equation*}
$$

Proof. Let $A \stackrel{\text { def. }}{=}\{f \geqslant a\}$. Let $x \in A$. Then, from the definition of nondecreasingness, we have $g(a) \leqslant g \circ f(x)$. Let $x \in A^{c}$. Then, from Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $g \geqslant 0$ ), we have $0 \leqslant g \circ f(x)$. Hence, from the definition of the indicator function, and Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), we have

$$
g(a) \mathbb{1}_{A} \leqslant g \circ f
$$

Therefore, from Lemma 820 (measurability of numeric function), Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ), Lemma 757 (compatibility of measurability with composition, $g \circ f \in \mathcal{M}$ ), Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication), Lemma 1178 (integral in $\mathcal{M}_{+}$ is monotone), and Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), we have $A \in \Sigma$, and

$$
g(a) \mu(A)=g(a) \int \mathbb{1}_{A} d \mu \leqslant \int g \circ f d \mu .
$$

Lemma 1196 (Bienaymé-Chebyshev inequality). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $p \in(0, \infty)$. Let $a \in \overline{\mathbb{R}}_{+}$. Then, we have $\{|f| \geqslant a\} \in \Sigma,|f|^{p} \in \mathcal{M}_{+}$, and

$$
\begin{equation*}
a^{p} \mu(\{|f| \geqslant a\}) \leqslant \int|f|^{p} d \mu . \tag{15.41}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1195 (generalized Bienaymé-Chebyshev inequality, with $|f|$ and $g \stackrel{\text { def. }}{=}\left(x \mapsto|x|^{p}\right)$.

Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite). Let $(X, \Sigma, \mu)$ be a measure space.
Let $f \in \mathcal{M}_{+}$be $\mu$-integrable. Then, we have $\mu(\{f=\infty\})=0$, i.e. $f \stackrel{\mu \text { a.e. }}{<} \infty$.
Proof. Direct consequence of Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, $f \in \mathcal{M}$ ), Lemma 1196 (Bienaymé-Chebyshev inequality, with $p \stackrel{\text { def. }}{=} 1$ and $a \stackrel{\text { def. }}{=} \infty$ ), Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, $|f|=f$ ), Definition 312 (extended real numbers, $\overline{\mathbb{R}}, A \stackrel{\text { def. }}{=}\{|f| \geqslant \infty\}=\{f=\infty\}$ ), Lemma 1172 (integral in $\left.\mathcal{M}_{+}, \int f d \mu<\infty\right)$, Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory), $\mu(A)=0$ ), Lemma 948 (negligibility of measurable subset, $A \in \mathbf{N}$ ), and Definition 953 (property almost satisfied, $\underline{A^{c}=\{f<\infty\} \text { ). }}$

Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$. Then, $f$ is $\mu$-integrable in $\mathcal{M}_{+}$iff $f \in \mathcal{M}_{+}$ and there exists $g: X \rightarrow \overline{\mathbb{R}}_{+}$such that $g$ is $\mu$-integrable in $\mathcal{M}_{+}$and $f \stackrel{\mu \text { a.e. }}{\leqslant} g$.

## Proof. "Left" implies "right".

Direct consequence of Lemma 983 (almost order is order relation, reflexivity, thus $f \stackrel{\mu \text { a.e. }}{\leqslant} g \stackrel{\text { def. }}{=} f$ ).
"Right" implies "left". Direct consequence of Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity), and Lemma 1172 (integral in $\mathcal{M}_{+}, \underline{\mu \text {-integrability). }}$

Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$.
Then, we have $f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$.
If so, there exists $\left(\hat{\varphi}_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}, \hat{\varphi}_{n} \mathbb{1}_{A}=\hat{\varphi}_{n}, \quad f_{\left.\right|_{A}}=\sup _{n \in \mathbb{N}}\left(\hat{\varphi}_{n}\right)_{\left.\right|_{A}}, \quad \hat{f} \mathbb{1}_{A}=\sup _{n \in \mathbb{N}} \hat{\varphi}_{n} \tag{15.42}
\end{equation*}
$$

and we have

$$
\begin{equation*}
\int f_{\left.\right|_{A}} d \mu_{A}=\sup _{n \in \mathbb{N}} \int_{A} \hat{\varphi}_{n} d \mu=\int \hat{f} \mathbb{1}_{A} d \mu \tag{15.43}
\end{equation*}
$$

This integral is still denoted $\int_{A} f d \mu$; it is still called integral of $f$ over $A$.
Proof. Direct consequence of Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero), Theorem 1180 (Beppo Levi, monotone convergence), Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S F}_{+}$), and Lemma 1163 (integral in $\mathcal{S F}+$ over subset).

## Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive).

Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $Y$ be a set such that $A \subset Y \subset X$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$. Then, $\hat{f} \mathbb{1}_{A}=\sum_{i \in I} \hat{f} \mathbb{1}_{A_{i}}$, and $\hat{f} \mathbb{1}_{A} \in \mathcal{M}_{+}$iff for all $i \in I, \hat{f} \mathbb{1}_{A_{i}} \in \mathcal{M}_{+}$. If so, we have

$$
\begin{equation*}
\int_{A} f d \mu=\sum_{i \in I} \int_{A_{i}} f d \mu . \tag{15.44}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive), Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $A$ and the $A_{i}$ 's), and Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive).

Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$ and $\mu(A)=0$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$, and either $f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$, or $\hat{f} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$. Then, we have both, and

$$
\begin{equation*}
\int_{A} f d \mu=0 \tag{15.45}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, thus exists nonnegative simple functions $\left(\hat{\varphi}_{n}\right)_{n \in \mathbb{N}}$ such that $\hat{\varphi}_{n} \mathbb{1}_{A}=\hat{\varphi}_{n}$ and $\left.\int_{A} f d \mu=\sup _{n \in \mathbb{N}} \overline{\int_{A} \hat{\varphi}_{n} d \mu}\right)$, Lemma 1165 (integral in $\mathcal{S F}_{+}$over negligible subset, thus $\int_{A} \hat{\varphi}_{n} d \mu=0$ ), and Lemma 482 (extrema of constant function, thus $\left.\sup _{n \in \mathbb{N}} \int_{A} \hat{\varphi}_{n} d \mu=0\right)$.

Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton).
Let $(X, \Sigma, \mu)$ be a measure space. Let $a \in X$. Assume that $\{a\} \in \Sigma$. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$. Then, we have $f \mathbb{1}_{\{a\}}=f(a) \mathbb{1}_{\{a\}} \in \mathcal{M}_{+}$, and

$$
\begin{equation*}
\int_{\{a\}} f d \mu=f(a) \mu(\{a\}) \tag{15.46}
\end{equation*}
$$

Proof. From the definition of the indicator function, Lemma 811 (measurability of indicator function), and Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication), we have $f \mathbb{1}_{\{a\}}=f(a) \mathbb{1}_{\{a\}} \in \mathcal{M}_{+}$. Then, from Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $A \stackrel{\text { def. }}{=}\{a\}$, $Y^{\text {def. }} X$ and $\hat{f} \stackrel{\text { def. }}{=} f$ ), Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear, homogeneity in $\overline{\mathbb{R}}_{+}$), and Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function), we have

$$
\int_{\{a\}} f d \mu=\int f \mathbb{1}_{\{a\}} d \mu=f(a) \int \mathbb{1}_{\{a\}} d \mu=f(a) \mu(\{a\}) .
$$

## Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$. Assume that either $f \in \mathcal{M}_{+}(X, \Sigma)$, or for all $A \in \Sigma, f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$. Then, we have both, and

$$
\begin{equation*}
\int f d \mu<\infty \quad \Longleftrightarrow \quad \forall A \in \Sigma, \quad \int_{A} f d \mu<\infty \tag{15.47}
\end{equation*}
$$

Proof. From Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$, and Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero, with $Y=X$ and $\hat{f}=f$ ), we have

$$
f \in \mathcal{M}_{+}(X, \Sigma), \quad \text { and } \quad \forall A \in \Sigma, \quad f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \quad \wedge \quad f \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma) .
$$

"Left" implies "right". Let $A \in \Sigma$. Then, from Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $Y=X$ and $\hat{f}=f$ ), the definition of the indicator function (thus $f \mathbb{1}_{A} \leqslant f$ ), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and transitivity of order, we have

$$
\int_{A} f d \mu=\int f \mathbb{1}_{A} d \mu \leqslant \int f d \mu<\infty .
$$

"Right" implies "left". Direct consequence of Lemma 697 (equivalent definition of $\sigma$-algebra, thus $X \in \Sigma$ ), the definition of the indicator function (thus $\mathbb{1}_{X} \equiv 1$ ), and Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $A=Y=X$ and $\hat{f}=f$ ).

Therefore, we have the equivalence.
Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$.
Then, $f$ is $\mu$-integrable in $\mathcal{M}_{+}(X, \Sigma)$ iff for all $A \in \Sigma$, $f_{\left.\right|_{A}}$ is $\mu_{A^{\prime}}$-integrable in $\mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$.
Proof. From Lemma 1172 (integral in $\mathcal{M}_{+}$, integrability with $(X, \Sigma, \mu)$, then $\left(A, \Sigma \bar{\cap} A, \mu_{A}\right)$ ), Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction), Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction), and Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $Y=X$ and $\hat{f}=f$ ), we have

$$
\begin{aligned}
f \mu \text {-integrable in } \mathcal{M}_{+}(X, \Sigma) & \Longleftrightarrow f \in \mathcal{M}_{+}(X, \Sigma) \wedge \int f d \mu<\infty \\
\Longleftrightarrow & \left(\forall A \in \Sigma, \quad f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)\right. \\
& \wedge\left(\forall A \in \Sigma, \quad \int_{A} f d \mu<\infty\right) \\
\Longleftrightarrow & \forall A \in \Sigma, \quad f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \wedge \int f_{\left.\right|_{A}} d \mu_{A}<\infty \\
\Longleftrightarrow & \forall A \in \Sigma, \quad f_{\left.\right|_{A}} \mu_{A} \text {-integrable in } \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) .
\end{aligned}
$$

Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive). Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$.
Assume that either $f \mathbb{1}_{A} \in \mathcal{M}_{+}$, or for all $i \in I$, $f \mathbb{1}_{A_{i}} \in \mathcal{M}_{+}$. Then, we have both, and

$$
\begin{align*}
\int_{A} f d \mu<\infty & \Longrightarrow \forall i \in I, \quad \int_{A_{i}} f d \mu<\infty,  \tag{15.48}\\
\forall i \in[0 . . n], \quad \int_{A_{i}} f d \mu<\infty & \Longrightarrow \int_{A} f d \mu<\infty . \quad(\text { when } I \stackrel{\text { def. }}{=}[0 . . n] \text { with } n \in \mathbb{N}) .
\end{align*}
$$

Proof. From Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive $)$, we have

$$
f \mathbb{1}_{A} \in \mathcal{M}_{+}, \quad \text { and } \quad \forall i \in I, \quad f \mathbb{1}_{A_{i}} \in \mathcal{M}_{+}
$$

(15.48). Let $i \in I$. Then, from Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $Y=X, \hat{f}=f$ and $A \stackrel{\text { def. }}{=} A_{i}$, then $A$ ), Definition 209 (pseudopartition, thus $A_{i} \subset A$ ), the definition of the indicator function (thus $f \mathbb{1}_{\boldsymbol{A}_{i}} \leqslant \boldsymbol{f} \mathbb{1}_{A}$ ), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and transitivity of order, we have

$$
\int_{A_{i}} f d \mu=\int f \mathbb{1}_{A_{i}} d \mu \leqslant \int f \mathbb{1}_{A} d \mu=\int_{A} f d \mu<\infty
$$

(15.49). Direct consequence of Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive), and closedness of finite addition in $\mathbb{R}$.

Therefore, we have both implications.
Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive). $\quad$ Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$.
Then, we have $f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}}$, and $f \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}_{+}$implies for all $i \in I, f \mathbb{1}_{A_{i}}$ is $\mu$-integrable in $\mathcal{M}_{+}$.

Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$ with $n \in \mathbb{N}$, we have for all $i \in[0 . . n], f \mathbb{1}_{A_{i}}$ is $\mu$-integrable in $\mathcal{M}_{+}$ implies $f \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}_{+}$.

Proof. Direct consequence of Lemma 525 (masking is $\sigma$-additive, with $E \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$, thus the identity holds), Lemma 1172 (integral in $\mathcal{M}_{+}$, integrability, with $f \mathbb{1}_{A}$ and $\left.f \mathbb{1}_{A_{i}}\right)$, Lemma $1139\left(\mathcal{M}_{+}\right.$, masking is $\sigma$-additive), and Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive).

## Remark 1207. See the sketch of next proof in Section 5.7.

## Theorem 1208 (Fatou's lemma).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$. Then, $\liminf _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \liminf _{n \rightarrow \infty} f_{n} d \mu \leqslant \liminf _{n \rightarrow \infty} \int f_{n} d \mu \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) . \tag{15.50}
\end{equation*}
$$

Proof. From Lemma 518 (limit inferior bounded from below, with $m \stackrel{\text { def. }}{=} 0$ ), and Lemma $830(\mathcal{M}$ is closed under limit inferior), we have $\liminf _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}$.

Let $n \in \mathbb{N}$. Let $g_{n} \stackrel{\text { def. }}{=} \inf _{p \in \mathbb{N}} f_{n+p}$. Then, from Lemma 488 (infimum is monotone (set)), Lemma 492 (infimum of bounded is bounded, with $X \stackrel{\text { def. }}{=} \mathbb{N}$ and $a \stackrel{\text { def. }}{=} 0$ ), and Lemma $828(\mathcal{M}$ is closed under infimum), $\left(g_{n}\right)_{n \in \mathbb{N}}$ is a nondecreasing sequence in $\mathcal{M}_{+}$. Thus, from Lemma 314 ( $\overline{\mathbb{R}}$ is complete, supremum exists), Lemma 501 (supremum is limit), Definition 516 (pointwise
convergence), Lemma 849 ( $\mathcal{M}_{+}$is closed under limit when pointwise convergent), and Lemma 504 (limit inferior), $\left(g_{n}\right)_{n \in \mathbb{N}}$ is pointwise convergent in $\overline{\mathbb{R}}_{+}$towards

$$
g \stackrel{\text { def. }}{=} \lim _{n \in \mathbb{N}} g_{n}=\sup _{n \in \mathbb{N}} g_{n}=\liminf _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}
$$

Hence, from Theorem 1180 (Beppo Levi, monotone convergence, with $\left.\left(g_{n}\right)_{n \in \mathbb{N}}\right), g$ is measurable and nonnegative, and we have

$$
\int \liminf _{n \rightarrow \infty} f_{n} d \mu=\int g d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu
$$

Let $n, p \in \mathbb{N}$. Let $x \in X$. Then, from Definition 9 (infimum, lower bound), we have

$$
g_{n}(x) \leqslant f_{n+p}(x)
$$

Then, from Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), we have

$$
\int g_{n} d \mu \leqslant \int f_{n+p} d \mu
$$

Hence, from Definition 9 (infimum, greatest lower bound), we have

$$
\int g_{n} d \mu \leqslant \inf _{p \in \mathbb{N}} \int f_{n+p} d \mu
$$

Therefore, from Lemma 381 (limit is monotone), and Lemma 504 (limit inferior), we have

$$
\int \liminf _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu \leqslant \liminf _{n \rightarrow \infty} \int f_{n} d \mu
$$

Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence).
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}_{+}$. Let $f: X \rightarrow \overline{\mathbb{R}}_{+}$.
Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, and for all $n \in \mathbb{N}, f_{n} \leqslant f$. Then, $f \in \mathcal{M}_{+}$, and we have

$$
\begin{equation*}
\int \lim _{n \rightarrow \infty} f_{n} d \mu=\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \tag{15.51}
\end{equation*}
$$

Proof. From Lemma 849 ( $\mathcal{M}_{+}$is closed under limit when pointwise convergent), we have

$$
f=\lim _{n \rightarrow \infty} f_{n} \in \mathcal{M}_{+}
$$

Moreover, from Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), we have

$$
\forall n \in \mathbb{N}, \quad \int f_{n} d \mu \leqslant \int f d \mu
$$

Thus, from Definition 2 (supremum, least upper bound), Lemma 517 (limit inferior and limit superior of pointwise convergent), and Theorem 1208 (Fatou's lemma), we have

$$
\limsup _{n \rightarrow \infty} \int f_{n} d \mu \leqslant \int f d \mu=\int \lim _{n \rightarrow \infty} f_{n} d \mu=\int \liminf _{n \rightarrow \infty} f_{n} d \mu \leqslant \liminf _{n \rightarrow \infty} \int f_{n} d \mu
$$

Therefore, from Lemma 522 (limit inferior, limit superior and pointwise convergence), we have

$$
\liminf _{n \rightarrow \infty} \int f_{n} d \mu=\limsup _{n \rightarrow \infty} \int f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu=\int f d \mu
$$

Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure).
Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{M}_{+}$. Then, we have
(15.52)

$$
\int f d 0=0 .
$$

Proof. Direct consequence of Lemma 1182 (usage of adapted sequences), Lemma 1169 (integral in $\mathcal{S F}+$ with trivial measure), and Lemma 34 (stationary sequence is convergent).

## Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).

Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \delta_{Y}=\sum_{y \in Y} f(y) \tag{15.53}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1182 (usage of adapted sequences), Lemma 1170 (integral in $\mathcal{S F}{ }_{+}$with counting measure), Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous), and Lemma 45 (compatibility of limit with continuous functions).

Lemma 1212 (integral in $\mathcal{M}_{+}$with counting measure on $\mathbb{N}$ ).
Let $f: \mathbb{N} \rightarrow \overline{\mathbb{R}}_{+}$be a nonnegative sequence. Then, we have

$$
\begin{equation*}
\int f d \delta_{\mathbb{N}}=\sum_{n \in \mathbb{N}} f(n) . \tag{15.54}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure, with $Y=X$ set to $\mathbb{N}$ and $\Sigma \stackrel{\text { def. }}{=} \mathcal{P}(\mathbb{N}))$.

Remark 1213. Note that the previous lemma makes the sums of nonnegative series be Lebesgue integrals with counting measure on natural numbers. Thus, the theory of nonnegative series can be derived from Lebesgue integration for nonnegative measurable functions.

## Lemma 1214 (integral in $\mathcal{M}_{+}$with Dirac measure).

Let $(X, \Sigma)$ be a measurable space. Let $\{a\} \in \Sigma$. Let $f \in \mathcal{M}_{+}$. Then, we have

$$
\begin{equation*}
\int f d \delta_{a}=f(a) \tag{15.55}
\end{equation*}
$$

Proof. Direct consequence of Definition 939 (Dirac measure), and Lemma 1211 (integral in $\mathcal{M}_{+}$ with counting measure, with $Y \stackrel{\text { def. }}{=}\{a\})$.

### 15.4 Measure and integration over product space

Remark 1215. The "overline-i" notations from the next definition should help in writing expressions in the general case. But, for the sake of simplicity, we only present this section in the case of the product of two measure spaces. When $i \in\{1,2\}$, the complement $\{1,2\} \backslash\{i\}$ is $\{3-i\}$.

## Definition 1216 (overline-i notation to exclude term in product).

Let $I \subset \mathbb{N}$. For all $i \in \mathbb{N}$, let $\bar{I}^{i} \stackrel{\text { def. }}{=} I \backslash\{i\}$, i.e. $i \in I$ implies $I=\bar{I}^{i} \uplus\{i\}$ (and $\bar{I}^{i}=I$ when $i \notin I$ ).
Let $m \in[2 . . \infty)$. Let $\left(X_{i}, \Sigma_{i}\right)_{i \in[1 . . m]}$ be $m$ measurable spaces. Let

$$
X \stackrel{\text { def. }}{=} \prod_{i \in[1 . . m]} X_{i}, \quad \text { and } \quad \Sigma \stackrel{\text { def. }}{=} \bigotimes_{i \in[1 . . m]} \Sigma_{i} .
$$

Let $i \in[1 . . m]$. We define the following "overline-i" notations:

$$
\begin{equation*}
\bar{X}^{i} \stackrel{\text { def. }}{=} \prod_{j \in \overline{[1 . . m]}^{i}} X_{j}, \quad \text { and } \quad \bar{\Sigma}^{i} \stackrel{\text { def. }}{=} \bigotimes_{j \in \overline{[1 . . m]}^{i}} \Sigma_{j} . \tag{15.56}
\end{equation*}
$$

### 15.4.1 Measure over product space

Remark 1217. The definition of sections and some of their properties are presented in Section 9.4.

Lemma 1218 (measure of section). Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be measure spaces. Let $A \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Then, the function $F_{i}^{A}$ defined by

$$
\begin{equation*}
\forall x_{i} \in X_{i}, \quad F_{i}^{A}\left(x_{i}\right) \stackrel{\text { def. }}{=} \mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) \tag{15.57}
\end{equation*}
$$

is well-defined on $X_{i}$, and takes its values in $\overline{\mathbb{R}}_{+}$. It is called measure of $i$-th section of $A$.
Proof. Direct consequence of Lemma 786 (measurability of section), and Lemma 913 (measure, $\underline{\text { nonnegativeness). }}$

Lemma 1219 (measure of section of product). $\quad$ Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be measure spaces. Let $A_{1} \in \Sigma_{1}$ and $A_{2} \in \Sigma_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Then, we have

$$
\begin{equation*}
F_{i}^{A_{1} \times A_{2}}=\mu_{j}\left(A_{j}\right) \mathbb{1}_{A_{i}} . \tag{15.58}
\end{equation*}
$$

Proof. Direct consequence of Lemma 775 (product of measurable subsets is measurable, $\underline{A_{1} \times A_{2}}$ belongs to $\Sigma$ ), Lemma 1218 (measure of section), Lemma 784 (section of product), Lemma 913 (measure, homogeneity and nonnegativeness with $\mu_{j}$ ), Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)), and the definition of the indicator function.

Remark 1220. The next proof follows the monotone class theorem scheme (see Section 4.2).
See also the sketch of the proof in Section 5.6.
Lemma 1221 (measurability of measure of section (finite)).
Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be measure spaces. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Assume that $\mu_{j}$ is finite. Let $A \in \Sigma_{1} \otimes \Sigma_{2}$. Then, we have $F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.

Proof. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$. From Definition 227 (product of subset systems), Definition 774 (tensor product of $\sigma$-algebras), and Definition 642 (generated algebra of sets), let $\bar{\Sigma} \stackrel{\text { def. }}{=} \Sigma_{1} \overline{\times} \Sigma_{2}$, $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$, and $\mathcal{A} \stackrel{\text { def. }}{=} \mathcal{A}_{X}(\bar{\Sigma})$. Let $\mathcal{S}_{i} \stackrel{\text { def. }}{=}\left\{A \in \Sigma \mid F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)\right\}$.
(1). $\bar{\Sigma} \subset \mathcal{S}_{i}$. Let $A \in \bar{\Sigma}$. Let $A_{1} \in \Sigma_{1}$ and $A_{2} \in \Sigma_{2}$ such that $A=A_{1} \times A_{2}$.

From Lemma 1219 (measure of section of product, $F_{i}^{A}=\mu_{j}\left(A_{j}\right) \mathbb{1}_{A_{i}}$ ), Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, $\mathbb{1}_{A_{i}} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$ ), Lemma 913 (measure, nonnegativeness with $\left.\mu_{j}\right)$, and Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication, with $a \stackrel{\text { def. }}{=} \mu_{j}\left(A_{j}\right)$ ), $\overline{F_{i}^{A}}$ belongs to $\mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$. Thus, from Lemma 775 (product of measurable subsets is measurable, $\underline{A \in \Sigma})$, and the definition of $\mathcal{S}_{i}$, we have $A \in \mathcal{S}_{i}$. Hence, we have $\bar{\Sigma} \subset \mathcal{S}_{i}$.
(2). $\mathcal{A} \subset \mathcal{S}_{i}$. Let $A \in \mathcal{A}$.
 contains algebra of sets, with $G \stackrel{\text { def. }}{=} \bar{\Sigma}$, thus $\mathcal{A} \subset \Sigma$ ), Lemma 730 (algebra of sets generated by product of $\sigma$-algebras), and Lemma 775 (product of measurable subsets is measurable, $\bar{\Sigma} \subset \Sigma$ ), we have $A \in \Sigma$, and there exist $n \in \mathbb{N}$, and $\left(A_{p}\right)_{p \in[0 . . n]}$ in $\bar{\Sigma} \subset \Sigma$ such that, for all $p, q \in[0 . . n]$, $p \neq q \Rightarrow A_{p} \cap A_{q}=\varnothing$, and $A=\biguplus_{p \in[0 . . n]} A_{p}$.

Let $x_{i} \in X_{i}$. Then, from Lemma 1218 (measure of section), Lemma 787 (countable union of sections is measurable, with $I \stackrel{\text { def. }}{=}[0 . . n]$ ), Lemma 785 (compatibility of section with set operations, the $s_{i}\left(x_{i}, A_{p}\right)$ 's are pairwise disjoint), Lemma 913 (measure, $\sigma$-additivity with $\mu_{j}$ ), and Definition 874 ( $\sigma$-additivity, with $I \stackrel{\text { def. }}{=}[0 . . n]$ and $\mu \stackrel{\text { def. }}{=} \mu_{j}$ ), we have

$$
F_{i}^{A}\left(x_{i}\right)=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right)=\mu_{j}\left(\biguplus_{p \in[0 . . n]} s_{i}\left(x_{i}, A_{p}\right)\right)=\sum_{p \in[0 . . n]} \mu_{j}\left(s_{i}\left(x_{i}, A_{p}\right)\right)=\sum_{p \in[0 . . n]} F_{i}^{A_{p}}\left(x_{i}\right) .
$$

Thus, from (1) $\left(A_{p} \in \mathcal{S}_{i}\right)$, the definition of $\mathcal{S}_{i}$, and Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum, with $I \stackrel{\text { def. }}{=}[0 . . n]$, thus $\left.F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)\right)$, we have $A \in \mathcal{S}_{i}$. Hence, we have $\mathcal{A} \subset \mathcal{S}_{i}$.
(3). $\mathcal{S}_{\boldsymbol{i}}$ is monotone class. Let $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S}_{i} \subset \Sigma$.

Let $x_{i} \in X_{i}$. Let $n \in \mathbb{N}$. Then, from Lemma 786 (measurability of section), Lemma 787 (countable union of sections is measurable), and Lemma 788 (countable intersection of sections is measurable), we have

$$
s_{i}\left(x_{i}, A_{n}\right), \quad s_{i}\left(x_{i}, \bigcup_{n \in \mathbb{N}} A_{n}\right)=\bigcup_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right), \quad s_{i}\left(x_{i}, \bigcap_{n \in \mathbb{N}} A_{n}\right)=\bigcap_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right) \quad \in \Sigma_{j}
$$

Assume first that, for all $n \in \mathbb{N}, A_{n} \subset A_{n+1}$. Let $A \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} A_{n}$.
Let $x_{i} \in X_{i}$. Then, from Lemma 1218 (measure of section), the definition of $A$, Lemma 785 (compatibility of section with set operations, $\left(s_{i}\left(x_{i}, A_{n}\right)\right)_{n \in \mathbb{N}}$ is nondecreasing), and Lemma 925 (measure is continuous from below, with $\mu_{j}$ ), we have

$$
\begin{aligned}
F_{i}^{A}\left(x_{i}\right)=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) & =\mu_{j}\left(s_{i}\left(x_{i}, \bigcup_{n \in \mathbb{N}} A_{n}\right)\right) \\
& =\mu_{j}\left(\bigcup_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right)\right)=\sup _{n \in \mathbb{N}} \mu_{j}\left(s_{i}\left(x_{i}, A_{n}\right)\right)=\sup _{n \in \mathbb{N}} F_{i}^{A_{n}}\left(x_{i}\right) .
\end{aligned}
$$

Thus, from the definition of $\mathcal{S}_{i}$, and Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $), F_{i}^{A}$ belongs to $\mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$. Hence, from the definition of $\mathcal{S}_{i}$, we have $A \in \mathcal{S}_{i}$, i.e. $\mathcal{S}_{i}$ is closed under nondecreasing union.

Assume now that, for all $n \in \mathbb{N}, A_{n} \supset A_{n+1}$. Let $A \stackrel{\text { def. }}{=} \bigcap_{n \in \mathbb{N}} A_{n}$.
Let $x_{i} \in X_{i}$. Then, from Lemma 1218 (measure of section), the definition of $A$, Lemma 785 (compatibility of section with set operations, $\left(s_{i}\left(x_{i}, A_{n}\right)\right)_{n \in \mathbb{N}}$ is nonincreasing), Definition 857 (finiteness, with $\mu_{j}$ ), Lemma 858 (finite is bounded, $\underline{\mu_{j}}\left(s_{i}\left(x_{i}, A_{0}\right)\right.$ ) is finite), and Lemma 927 (measure
is continuous from above, with $\mu_{j}$ and $n_{0} \stackrel{\text { def. }}{=} 0$, we have

$$
\begin{aligned}
F_{i}^{A}\left(x_{i}\right)=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) & =\mu_{j}\left(s_{i}\left(x_{i}, \bigcap_{n \in \mathbb{N}} A_{n}\right)\right) \\
& =\mu_{j}\left(\bigcap_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right)\right)=\inf _{n \in \mathbb{N}} \mu_{j}\left(s_{i}\left(x_{i}, A_{n}\right)\right)=\inf _{n \in \mathbb{N}} F_{i}^{A_{n}}\left(x_{i}\right) .
\end{aligned}
$$

Thus, from the definition of $\mathcal{S}_{i}$, and Lemma $847\left(\mathcal{M}_{+}\right.$is closed under infimum $), F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$. Then, from the definition of $\mathcal{S}_{i}$, we have $A \in \mathcal{S}_{i}$, i.e. $\mathcal{S}_{i}$ is closed under nonincreasing intersection.

Hence, from Definition 664 (monotone class), $\mathcal{S}_{i}$ is a monotone class.
Therefore, from Definition 774 (tensor product of $\sigma$-algebras, $\Sigma=\Sigma_{X}(\bar{\Sigma})$ ), (2), (3), and Lemma 738 (usage of monotone class theorem, with $G \stackrel{\text { def. }}{=} \bar{\Sigma}$ ), we have for all $A \in \Sigma, F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.

## Remark 1222. See the sketch of next proof in Section 5.6.

Lemma 1223 (measurability of measure of section). Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $A \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i \in\{1,2\}$. Then, we have $F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.

Proof. Let $j \stackrel{\text { def. }}{=} 3-i$. From Lemma 886 (equivalent definition of $\sigma$-finiteness, with $\mu_{j}$ ), there exists $\left(B_{j, n}\right)_{n \in \mathbb{N}} \in \Sigma_{j}$ such that, for all $n \in \mathbb{N}, B_{j, n} \subset B_{j, n+1}, \mu_{j}\left(B_{j, n}\right)<\infty$, and $\overline{X_{j}}=\bigcup_{n \in \mathbb{N}} B_{j, n}$.

For all $n \in \mathbb{N}$, for all $A \in \Sigma_{1} \otimes \Sigma_{2}$, from Lemma 931 (restricted measure, with $Y \stackrel{\text { def. }}{=} B_{j, n}$ ), and Lemma 1218 (measure of section, with $\mu_{j} \stackrel{\text { def. }}{=} \mu_{j, n}$ ), let $\mu_{j, n}$ be the restricted measure, and $F_{i, n}^{A}$ be the measure of $i$-th section of $A$ defined by

$$
\mu_{j, n} \stackrel{\text { def. }}{=}\left(\mu_{j}\right)_{B_{j, n}}^{\prime}=\left(A_{j} \in \Sigma_{j} \longmapsto \mu_{j}\left(A_{j} \cap B_{j, n}\right)\right), \quad \text { and } \quad F_{i, n}^{A} \stackrel{\text { def. }}{=}\left(x_{i} \in X_{i} \longmapsto \mu_{j, n}\left(s_{i}\left(x_{i}, A\right)\right)\right) .
$$

Let $n \in \mathbb{N}$. Then, from Definition 857 (finiteness, $\mu_{j, n}(X)=\mu_{j}\left(B_{j, n}\right)<\infty$ ), $\mu_{j, n}$ is a finite measure on $\left(X_{j}, \Sigma_{j}\right)$. Moreover, from Lemma 1221 (measurability of measure of section (finite), with $\mu_{j} \stackrel{\text { def. }}{=} \mu_{j, n}$ finite $)$, we have $F_{i, n}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.

Let $x_{i} \in X_{i}$. Then, from Lemma 786 (measurability of section), the definition of $B_{j, n}$, and distributivity of intersection over union, we have

$$
\Sigma_{j} \ni s_{i}\left(x_{i}, A\right)=s_{i}\left(x_{i}, A\right) \cap \bigcup_{n \in \mathbb{N}} B_{j, n}=\bigcup_{n \in \mathbb{N}}\left(s_{i}\left(x_{i}, A\right) \cap B_{j, n}\right)
$$

Thus, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable intersection (with $\operatorname{card}(I)=2$ ), we have $s_{i}\left(x_{i}, A\right) \cap B_{j, n} \in \Sigma_{j}$. Hence, from Lemma 1218 (measure of section), monotonicity of intersection $\left(\left(s_{i}\left(x_{i}, \boldsymbol{A}\right) \cap \boldsymbol{B}_{j, n}\right)_{n \in \mathbb{N}}\right.$ is nondecreasing), Lemma 925 (measure is continuous from below, with $\mu_{j}$ and $A_{n} \stackrel{\text { def. }}{=} s_{i}\left(x_{i}, A\right) \cap B_{j, n}$ ), and the definition of $\mu_{j, n}$,

$$
\begin{aligned}
F_{i}^{A}\left(x_{i}\right)=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) & =\mu_{j}\left(s_{i}\left(x_{i}, A\right) \cap \bigcup_{n \in \mathbb{N}} B_{j, n}\right)=\mu_{j}\left(\bigcup_{n \in \mathbb{N}}\left(s_{i}\left(x_{i}, A\right) \cap B_{j, n}\right)\right) \\
& =\sup _{n \in \mathbb{N}} \mu_{j}\left(s_{i}\left(x_{i}, A\right) \cap B_{j, n}\right)=\sup _{n \in \mathbb{N}} \mu_{j, n}\left(s_{i}\left(x_{i}, A\right)\right)=\sup _{n \in \mathbb{N}} F_{i, n}^{A}\left(x_{i}\right) .
\end{aligned}
$$

Therefore, from Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$, we have

$$
F_{i}^{A}=\sup _{n \in \mathbb{N}} F_{i, n}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)
$$

Definition 1224 (tensor product measure). Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and ( $X_{2}, \Sigma_{2}, \mu_{2}$ ) be measure spaces. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. A measure $\mu$ on the product measurable space $(X, \Sigma)$ is called tensor product measure (on $(X, \Sigma)$ relying on $\mu_{1}$ and $\mu_{2}$ ) iff

$$
\begin{equation*}
\forall A_{1} \in \Sigma_{1}, \forall A_{2} \in \Sigma_{2}, \quad \mu\left(A_{1} \times A_{2}\right)=\mu_{1}\left(A_{1}\right) \mu_{2}\left(A_{2}\right) \tag{15.59}
\end{equation*}
$$

## Definition 1225 (candidate tensor product measure).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be measure spaces. Let $i \in\{1,2\}$. The function defined by

$$
\begin{equation*}
\forall A \in \Sigma_{1} \otimes \Sigma_{2}, \quad\left(\mu_{1} \otimes \mu_{2}\right)^{i}(A) \stackrel{\text { def. }}{=} \int F_{i}^{A} d \mu_{i} \tag{15.60}
\end{equation*}
$$

is called candidate tensor product measure.
Remark 1226. See the sketch of next proof in Section 5.6.

Lemma 1227 (candidate tensor product measure is tensor product measure).
Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $i \in\{1,2\}$. Then, the candidate tensor product measure $\left(\mu_{1} \otimes \mu_{2}\right)^{i}$ is a tensor product measure on $\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$.

Proof. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$.
Measure. Let $j \stackrel{\text { def. }}{=} 3-i$.
Let $A \in \Sigma$. Then, from Lemma 1218 (measure of section), Definition 1225 (candidate tensor product measure), Lemma 1223 (measurability of measure of section, with the $\sigma$-finite measure $\mu_{i}$ ), Lemma 1172 (integral in $\mathcal{M}_{+}$, nonnegativeness, with $X \stackrel{\text { def. }}{=} X_{i}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{i}$ ), and Lemma 1218


From Lemma 785 (compatibility of section with set operations, $\underline{\varnothing}$ ), Lemma 913 (measure, homogeneity with $\mu_{j}$ ), and Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero), we have

$$
\left(\mu_{1} \otimes \mu_{2}\right)^{i}(\varnothing)=\int \mu_{j}\left(s_{i}\left(x_{i}, \varnothing\right)\right) d \mu_{i}=\int \mu_{j}(\varnothing) d \mu_{i}=\int 0 d \mu_{i}=0
$$

Let $\left(A_{n}\right)_{n \in \mathbb{N}} \in \Sigma$. Assume that the $A_{n}$ 's are pairwise disjoint. Let $A \stackrel{\text { def. }}{=} \biguplus_{n \in \mathbb{N}} A_{n}$. Let $x_{i}$ be in $X_{i}$. Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$. Assume that $p \neq q$, Then, from Lemma 786 (measurability of section), Lemma 787 (countable union of sections is measurable, with $I \stackrel{\text { def. }}{=} \mathbb{N}$ ), and Lemma 785 (compatibility of section with set operations, with intersection and $\bar{\varnothing}$ ), $s_{i}\left(x_{i}, A_{n}\right) \in \Sigma_{j}$, and

$$
\begin{gathered}
s_{i}\left(x_{i}, A\right)=s_{i}\left(x_{i}, \bigcup_{n \in \mathbb{N}} A_{n}\right)=\bigcup_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right) \in \Sigma_{j}, \\
s_{i}\left(x_{i}, A_{p}\right) \cap s_{i}\left(x_{i}, A_{q}\right)=s_{i}\left(x_{i}, A_{p} \cap A_{q}\right)=s_{i}\left(x_{i}, \varnothing\right)=\varnothing
\end{gathered}
$$

Thus, from Lemma 1218 (measure of section), Lemma 913 (measure, $\sigma$-additivity with $\mu_{j}$ ), and Definition 874 ( $\sigma$-additivity), we have

$$
F_{i}^{A}\left(x_{i}\right)=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right)=\mu_{j}\left(\biguplus_{n \in \mathbb{N}} s_{i}\left(x_{i}, A_{n}\right)\right)=\sum_{n \in \mathbb{N}} \mu_{j}\left(s_{i}\left(x_{i}, A_{n}\right)\right)=\sum_{n \in \mathbb{N}} F_{i}^{A_{n}}\left(x_{i}\right) .
$$

Then, from Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive), and Definition 1225 (candidate tensor product measure), we have

$$
\left(\mu_{1} \otimes \mu_{2}\right)^{i}(A)=\int \sum_{n \in \mathbb{N}} F_{i}^{A_{n}} d \mu_{i}=\sum_{n \in \mathbb{N}} \int F_{i}^{A_{n}} d \mu_{i}=\sum_{n \in \mathbb{N}}\left(\mu_{1} \otimes \mu_{2}\right)^{i}\left(A_{n}\right)
$$

Hence, from Definition 874 ( $\sigma$-additivity), $\left(\mu_{1} \otimes \mu_{2}\right)^{i}$ is $\sigma$-additive.
Therefore, from Lemma 913 (measure), $\left(\mu_{1} \otimes \mu_{2}\right)^{i}$ is a measure on $(X, \Sigma)$.
Identity. Let $A_{1} \in \Sigma_{1}$. Let $A_{2} \in \Sigma_{2}$. Let $A \stackrel{\text { def. }}{=} A_{1} \times A_{2}$.
Then, from Lemma 775 (product of measurable subsets is measurable, $A \in \Sigma$ ), Definition 1225 (candidate tensor product measure), Lemma 1219 (measure of section of product), Lemma 913 (measure, nonnegativity with $\mu_{j}$ ), Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear, with $a \stackrel{\text { def. }}{=} \mu_{j}\left(A_{j}\right)$ in $\overline{\mathbb{R}}_{+}$), Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, with $\mu \stackrel{\text { def. }}{=} \mu_{i}$ and $A_{i} \in \Sigma_{i}$ ), Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)), and since $\{1,2\}=\{i, j\}$, we have

$$
\left(\mu_{1} \otimes \mu_{2}\right)^{i}(A)=\int \mu_{j}\left(A_{j}\right) \mathbb{1}_{A_{i}} d \mu_{i}=\mu_{j}\left(A_{j}\right) \mu_{i}\left(A_{i}\right)=\mu_{1}\left(A_{1}\right) \mu_{2}\left(A_{2}\right)
$$

Therefore, from Definition 1224 (tensor product measure), $\left(\mu_{1} \otimes \mu_{2}\right)^{i}$ is a tensor product measure.

## Lemma 1228 (tensor product of finite measures).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be finite measure spaces. Let $\mu$ be a tensor product measure on $\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$ relying on $\mu_{1}$ and $\mu_{2}$. Then, $\mu$ is finite.

Proof. Direct consequence of Lemma 913 (measure), Lemma 697 (equivalent definition of $\sigma$-algebra, $X_{1} \in \Sigma_{1}, X_{2} \in \Sigma_{2}$, and $X_{1} \times X_{2} \in \Sigma_{1} \otimes \Sigma_{2}$ ), Definition 1224 (tensor product measure), Definition 857 (finiteness), and closedness of multiplication in $\mathbb{R}_{+}$.

## Lemma 1229 (tensor product of $\sigma$-finite measures).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Let $\mu$ be a tensor product measure on $(X, \Sigma)$ relying on $\mu_{1}$ and $\mu_{2}$. Then, $\mu$ is $\sigma$-finite.

In particular, let $i \in\{1,2\}$. Let $\left(B_{i, n}\right)_{n \in \mathbb{N}} \in \Sigma_{i}$ such that

$$
\begin{equation*}
\left(\forall n \in \mathbb{N}, \quad B_{i, n} \subset B_{i, n+1} \quad \wedge \quad \mu_{i}\left(B_{i, n}\right)<\infty\right) \quad \wedge \quad X_{i}=\bigcup_{n \in \mathbb{N}} B_{i, n} \tag{15.61}
\end{equation*}
$$

For all $n \in \mathbb{N}$, let $B_{n} \stackrel{\text { def. }}{=} B_{1, n} \times B_{2, n}$. Then, we have
(15.62) $\quad\left(\forall n \in \mathbb{N}, \quad B_{n} \in \Sigma \wedge B_{n} \subset B_{n+1} \wedge \mu\left(B_{n}\right)<\infty\right) \wedge X=\bigcup_{n \in \mathbb{N}} B_{n}$.

Proof. Existence of the $B_{i, n}$ 's comes from Lemma 886 (equivalent definition of $\sigma$-finiteness).
Let $n \in \mathbb{N}$. Then, from the definition of the $B_{n}$ 's, Lemma 775 (product of measurable subsets is measurable), monotonicity of Cartesian product, Definition 1224 (tensor product measure), and closedness of multiplication in $\mathbb{R}_{+}$, we have

$$
B_{n} \in \Sigma \quad \wedge \quad B_{n} \subset B_{n+1} \wedge \mu\left(B_{n}\right)=\mu_{1}\left(B_{1, n}\right) \mu_{2}\left(B_{2, n}\right)<\infty
$$

Moreover, from monotonicity of union, we have $\bigcup_{n \in \mathbb{N}} B_{n} \subset X$. Conversely, let $\left(x_{1}, x_{2}\right)$ be in $X$. Let $i \in\{1,2\}$. Then, from Lemma 886 (equivalent definition of $\sigma$-finiteness, with $\mu_{i}$ ), there exists $n_{i} \in \mathbb{N}$ such that $x_{i} \in B_{i, n_{i}}$. Thus, from monotonicity of the $B_{i, n}$ 's, $x_{i}$ belongs to $B_{i, \max \left(n_{1}, n_{2}\right)}$. Then, from the definition of the $B_{n}$ 's, we have $\left(x_{1}, x_{2}\right) \in B_{\max \left(n_{1}, n_{2}\right)}$, i.e. $X$ is included in $\bigcup_{n \in \mathbb{N}} B_{n}$. Hence, we have $X=\bigcup_{n \in \mathbb{N}} B_{n}$.

Therefore, from Definition 885 ( $\sigma$-finiteness), $\mu$ is $\sigma$-finite.
Remark 1230. The uniqueness part of the next proof reproduces the schemes of the proof of Lemma 1221, it follows the monotone class theorem scheme (see Section 4.2).

See also the sketch of the proof in Section 5.6.

Lemma 1231 (uniqueness of tensor product measure (finite)).
Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be finite measure spaces. Then, there exists a unique tensor product measure on the product measurable space $\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$.

Proof. Existence. Direct consequence of Lemma 887 (finite is $\sigma$-finite, thus $\mu_{1}$ and $\mu_{2}$ are $\sigma$-finite), and Lemma 1227 (candidate tensor product measure is tensor product measure).

Uniqueness. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}, \bar{\Sigma} \stackrel{\text { def. }}{=} \Sigma_{1} \overline{\times} \Sigma_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Let $m$ and $\tilde{m}$ be tensor product measures on $(X, \Sigma)$ relying on $\mu_{1}$ and $\mu_{2}$. Let $\mathcal{S} \stackrel{\text { def. }}{=}\{A \in \Sigma \mid m(A)=\tilde{m}(A)\}$. Let $\mathcal{A} \stackrel{\text { def. }}{=} \mathcal{A}_{X}(\bar{\Sigma})$.
(1). $\overline{\mathbf{\Sigma}} \subset \mathcal{S}$. Direct consequence of Lemma 775 (product of measurable subsets is measurable), Definition 1224 (tensor product measure, with $m$ and $\tilde{m}$ ), and the definition of $\mathcal{S}$.

## (2). $\mathcal{A} \subset \mathcal{S}$. Let $A \in \mathcal{A}$.

Then, from Definition 774 (tensor product of $\sigma$-algebras, $\Sigma=\Sigma_{X}(\bar{\Sigma})$ ), Lemma 719 ( $\sigma$-algebra contains algebra of sets, with $G \stackrel{\text { def. }}{=} \bar{\Sigma}$, thus $\mathcal{A} \subset \Sigma$ ), Lemma 730 (algebra of sets generated by product of $\sigma$-algebras), and Lemma 775 (product of measurable subsets is measurable, $\bar{\Sigma} \subset \Sigma$ ), we have $A \in \Sigma$, and there exist $n \in \mathbb{N}$, and $\left(A_{p}\right)_{p \in[0 . . n]} \in \bar{\Sigma} \subset \Sigma$ such that, for all $p, q \in[\overline{[0 . . n], p} \neq q$ implies emptiness of $A_{p} \cap A_{q}$, and $A=\biguplus_{p \in[0 . . n]} A_{p}$. Thus, from Lemma 926 (equivalent definition of measure (from below), additivity for $m$ and $\tilde{m}$ ), Definition 862 (additivity), (1) ( $A_{p} \in \mathcal{S}$ ), the definition of $\mathcal{S}$, and Definition 317 (addition in $\overline{\mathbb{R}}$, addition is a function (same arguments yield the same result)), we have

$$
m(A)=m\left(\biguplus_{p \in[0 . . n]} A_{p}\right)=\sum_{p \in[0 . . n]} m\left(A_{p}\right)=\sum_{p \in[0 . . n]} \tilde{m}\left(A_{p}\right)=\tilde{m}\left(\biguplus_{p \in[0 . . n]} A_{p}\right)=\tilde{m}(A) .
$$

Then, from the definition of $\mathcal{S}$, we have $A \in \mathcal{S}$. Hence, we have $\mathcal{A} \subset \mathcal{S}$.
(3). $\mathcal{S}$ is monotone class. Let $\left(A_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S} \subset \Sigma$.

Then, from Lemma 697 (equivalent definition of $\sigma$-algebra, closedness under countable union and intersection), we have $\bigcup_{n \in \mathbb{N}} A_{n}, \bigcap_{n \in \mathbb{N}} A_{n} \in \Sigma$.

Assume first that, for all $n \in \mathbb{N}, A_{n} \subset A_{n+1}$. Let $A \stackrel{\text { def. }}{=} \bigcup_{n \in \mathbb{N}} A_{n}$.
Then, from Lemma 925 (measure is continuous from below, with $m$ and $\tilde{m}$ ), the definition of $\mathcal{S}$, and Definition 2 (supremum, supremum is a function), we have

$$
m(A)=m\left(\bigcup_{n \in \mathbb{N}} A_{n}\right)=\sup _{n \in \mathbb{N}} m\left(A_{n}\right)=\sup _{n \in \mathbb{N}} \tilde{m}\left(A_{n}\right)=\tilde{m}\left(\bigcup_{n \in \mathbb{N}} A_{n}\right)=\tilde{m}(A)
$$

Hence, from the definition of $\mathcal{S}_{i}$, we have $A \in \mathcal{S}$, i.e. $\mathcal{S}$ is closed under nondecreasing union.
Assume now that, for all $n \in \mathbb{N}, A_{n} \supset A_{n+1}$. Let $A \stackrel{\text { def. }}{=} \bigcap_{n \in \mathbb{N}} A_{n}$.
Then, from Lemma 1228 (tensor product of finite measures, with $m$ and $\tilde{m}$, both relying on finite measures $\mu_{1}$ and $\mu_{2}$ ), Lemma 858 (finite is bounded, $m\left(A_{0}\right), \tilde{m}\left(A_{0}\right)<\infty$ ), Lemma 927 (measure is continuous from above, with $\mu \stackrel{\text { def. }}{=} m, \tilde{m}$ and $n_{0} \stackrel{\text { def. }}{=} 0$, the definition of $\mathcal{S}$, and Definition 9 (infimum, infimum is a function), we have

$$
m(A)=m\left(\bigcap_{n \in \mathbb{N}} A_{n}\right)=\inf _{n \in \mathbb{N}} m\left(A_{n}\right)=\inf _{n \in \mathbb{N}} \tilde{m}\left(A_{n}\right)=\tilde{m}\left(\bigcap_{n \in \mathbb{N}} A_{n}\right)=\tilde{m}(A) .
$$

Thus, from the definition of $\mathcal{S}_{i}$, we have $A \in \mathcal{S}$, i.e. $\mathcal{S}$ is closed under nonincreasing intersection. Hence, from Definition 664 (monotone class), $\mathcal{S}$ is a monotone class.
Therefore, from Definition 774 (tensor product of $\sigma$-algebras, $\Sigma=\Sigma_{X}(\bar{\Sigma})$ ), (2), (3), and Lemma 738 (usage of monotone class theorem, with $G \stackrel{\text { def. }}{=} \bar{\Sigma}$ ), we have $m=\tilde{m}$.

## Remark 1232.

The uniqueness part of the next proof reproduces the schemes of the proof of Lemma 1223.
See also the sketch of the proof in Section 5.6.

## Lemma 1233 (uniqueness of tensor product measure).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$ and $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Then, there exists a unique tensor product measure on $(X, \Sigma)$.

This measure is denoted $\mu_{1} \otimes \mu_{2}$, and for all $i \in\{1,2\}$ with $j \stackrel{\text { def. }}{=} 3-i$, we have

$$
\begin{equation*}
\forall A \in \Sigma, \quad\left(\mu_{1} \otimes \mu_{2}\right)(A)=\int \mu_{j}\left(s_{i}\left(x_{i}, A\right)\right) d \mu_{i} . \tag{15.63}
\end{equation*}
$$

It is called the (tensor) product measure of $\mu_{1}$ and $\mu_{2}$, and $\left(X, \Sigma, \mu_{1} \otimes \mu_{2}\right)$ is called (tensor) product measure space.

## Proof. Existence.

Direct consequence of Lemma 1227 (candidate tensor product measure is tensor product measure).
Uniqueness. Let $m$ and $\tilde{m}$ be tensor product measures on $(X, \Sigma)$ relying on $\mu_{1}$ and $\mu_{2}$.
From Lemma 886 (equivalent definition of $\sigma$-finiteness), for all $i \in\{1,2\}$, there exists $\left(B_{i, n}\right)_{n \in \mathbb{N}}$ in $\Sigma_{i}$ such that, for all $n \in \mathbb{N}, B_{i, n} \subset B_{i, n+1}, \mu_{i}\left(B_{i, n}\right)<\infty$, and $X_{i}=\bigcup_{n \in \mathbb{N}} B_{i, n}$. For all $n \in \mathbb{N}$, let $B_{n} \stackrel{\text { def. }}{=} B_{1, n} \times B_{2, n}$. Then, from Lemma 1229 (tensor product of $\sigma$-finite measures, with $\mu \stackrel{\text { def. }}{=} m, \tilde{m})$, we have for all $n \in \mathbb{N}, B_{n} \in \Sigma, B_{n} \subset B_{n+1}, m\left(B_{n}\right), \tilde{m}\left(B_{n}\right)<\infty$, and $X=\bigcup_{n \in \mathbb{N}} B_{n}$.

For all $i \in\{1,2\}$, for all $n \in \mathbb{N}$, from Lemma 697 (equivalent definition of $\sigma$-algebra, $(A \in \Sigma$ implies $\left.A \cap B_{n} \in \Sigma\right)$ ), and Lemma 931 (restricted measure, with $Y \stackrel{\text { def. }}{=} B_{i, n}$ and $\mu \stackrel{\text { def. }}{=} \mu_{i}$, then $Y \stackrel{\text { def. }}{=} B_{n}$ and $\left.\mu \stackrel{\text { def. }}{=} m, \tilde{m}\right)$, let $\mu_{i, n}, m_{n}$ and $\tilde{m}_{n}$ be the restricted measures defined by

$$
\begin{gathered}
\mu_{i, n} \stackrel{\text { def. }}{=}\left(\mu_{i}\right)_{B_{i, n}}^{\prime}=\left(A_{i} \in \Sigma_{i} \longmapsto \mu_{i}\left(A_{i} \cap B_{i, n}\right)\right), \\
m_{n} \stackrel{\text { def. }}{=} m_{B_{n}}^{\prime}=\left(A \in \Sigma \longmapsto m\left(A \cap B_{n}\right)\right), \quad \text { and } \quad \tilde{m}_{n} \stackrel{\text { def. }}{=}(\tilde{m})_{B_{n}}^{\prime}=\left(A \in \Sigma \longmapsto \tilde{m}\left(A \cap B_{n}\right)\right) .
\end{gathered}
$$

Let $i \in\{1,2\}$. Let $n \in \mathbb{N}$. Then, from Definition 857 (finiteness, with $\mu_{i, n}\left(X_{i}\right)=\mu_{i}\left(B_{i, n}\right)$, $m_{n}(X)=m\left(B_{n}\right)$ and $\tilde{m}_{n}(X)=\tilde{m}\left(B_{n}\right)$ finite $, \mu_{i, n}, m_{n}$ and $\tilde{m}_{n}$ are finite measures.

Let $n \in \mathbb{N}$. Let $A_{1} \in \Sigma_{1}$ and $A_{2} \in \Sigma_{2}$. Then, from Lemma 775 (product of measurable subsets is measurable, $\left.\underline{A_{1} \times A_{2} \in \Sigma}\right)$, the definitions of $m_{n}, B_{n}, \mu_{1, n}, \mu_{2, n}$ and $\tilde{m}_{n}$, compatibility of intersection with Cartesian product, and Definition 1224 (tensor product measure, with $\mu \stackrel{\text { def. }}{=} m, \tilde{m})$, we have

$$
\begin{aligned}
m_{n}\left(A_{1} \times A_{2}\right) & =m\left(\left(A_{1} \times A_{2}\right) \cap\left(B_{1, n} \times B_{2, n}\right)\right)=m\left(\left(A_{1} \cap B_{1, n}\right) \times\left(A_{2} \cap B_{2, n}\right)\right) \\
& =\mu_{1}\left(A_{1} \cap B_{1, n}\right) \mu_{2}\left(A_{2} \cap B_{2, n}\right)=\mu_{1, n}\left(A_{1}\right) \mu_{2, n}\left(A_{2}\right) \\
& =\tilde{m}\left(\left(A_{1} \cap B_{1, n}\right) \times\left(A_{2} \cap B_{2, n}\right)\right)=\tilde{m}\left(\left(A_{1} \times A_{2}\right) \cap\left(B_{1, n} \times B_{2, n}\right)\right)=\tilde{m}_{n}\left(A_{1} \times A_{2}\right)
\end{aligned}
$$

Thus, from Definition 1224 (tensor product measure, with $\mu_{1} \stackrel{\text { def. }}{=} \mu_{1, n}, \mu_{2} \stackrel{\text { def. }}{=} \mu_{2, n}$ and $\mu \stackrel{\text { def. }}{=} m_{n}$ and $\left.\tilde{m}_{n}\right), m_{n}$ and $\tilde{m}_{n}$ are tensor product measures on $(X, \Sigma)$ both relying on finite measures $\mu_{1, n}$ and $\mu_{2, n}$. Hence, from Lemma 1231 (uniqueness of tensor product measure (finite)), we have $m_{n}=\tilde{m}_{n}$.

Let $A \in \Sigma$. Then, from Lemma 697 (equivalent definition of $\sigma$-algebra, $A \cap B_{n} \in \Sigma$, then $\left.\bigcup_{n \in \mathbb{N}}\left(A \cap B_{n}\right) \in \Sigma\right)$, distributivity of intersection over union, monotonicity of intersection $\left(\left(\boldsymbol{A} \cap \boldsymbol{B}_{n}\right)_{n \in \mathbb{N}}\right.$ is nondecreasing), Lemma 925 (measure is continuous from below, with $m$, $\underline{\tilde{m}}$ and $A_{n} \stackrel{\text { def. }}{=} A \cap B_{n}$ ), the definition of $m_{n}$ and $\tilde{m}_{n}$, and Definition 2 (supremum, supremum is a
function), we have

$$
\begin{aligned}
m(A) & =m\left(A \cap \bigcup_{n \in \mathbb{N}} B_{n}\right)=m\left(\bigcup_{n \in \mathbb{N}}\left(A \cap B_{n}\right)\right) \\
& =\sup _{n \in \mathbb{N}} m\left(A \cap B_{n}\right)=\sup _{n \in \mathbb{N}} m_{n}(A)=\sup _{n \in \mathbb{N}} \tilde{m}_{n}(A)=\sup _{n \in \mathbb{N}} \tilde{m}\left(A \cap B_{n}\right) \\
& =\tilde{m}\left(\bigcup_{n \in \mathbb{N}}\left(A \cap B_{n}\right)\right)=\tilde{m}\left(A \cap \bigcup_{n \in \mathbb{N}} B_{n}\right)=\tilde{m}(A) .
\end{aligned}
$$

Therefore, we have $m=\tilde{m}$.
Identity. Direct consequence of the uniqueness result above, Lemma 1227 (candidate tensor product measure is tensor product measure), Definition 1225 (candidate tensor product measure), and Lemma 1218 (measure of section).

## Lemma 1234 (negligibility of measurable section).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $\mu \stackrel{\text { def. }}{=} \mu_{1} \otimes \mu_{2}$. Let $A \in \Sigma_{1} \otimes \Sigma_{2}$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Then, $A$ is $\mu$-negligible iff $\mu(A)=0$ iff for almost all $x_{i} \in X_{i}$, $\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right)=0$ iff for almost all $x_{i} \in X_{i}, s_{i}\left(x_{i}, A\right)$ is $\mu_{j}$-negligible.

Proof. Direct consequence of Lemma 786 (measurability of section, $s_{i}\left(x_{i}, A\right) \in \Sigma_{j}$ ), Lemma 948 (negligibility of measurable subset, with $A$, then $s_{i}\left(x_{i}, A\right)$ ), Lemma 1233 (uniqueness of tensor product measure), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite, with $\mu_{i}$ and $\left(x_{i} \mapsto \mu_{j}\left(s_{i}\left(x_{i}, A\right)\right)\right)$ ).

### 15.4.2 Lebesgue measure over product space

## Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ).

There is a unique measure on $\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right)\right)$ that generalizes the area of bounded open boxes.
It is denoted $\lambda^{\otimes 2} \stackrel{\text { def. }}{=} \lambda \otimes \lambda$, and it is called the Lebesgue measure on (Borel subsets of) $\mathbb{R}^{2}$.
Proof. Direct consequence of Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ), Lemma 1094 (Lebesgue measure is $\sigma$-finite), and Lemma 1233 (uniqueness of tensor product measure).

## Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes).

Let $a_{1}, b_{1}, a_{2}, b_{2} \in \mathbb{R}$. Assume that $a_{1} \leqslant b_{1}$ and $a_{2} \leqslant b_{2}$. Then, we have
(15.64)

$$
\lambda^{\otimes 2}\left(\left[\left(a_{1}, b_{1}\right)\right] \times\left[\left(a_{2}, b_{2}\right\rangle\right)=\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right) .\right.
$$

Proof. Direct consequence of Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ), Definition 1224 (tensor product measure), Lemma 1092 (Lebesgue measure generalizes length of interval), and the definition of the area of boxes of $\mathbb{R}^{2}$.

## Lemma 1237 (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines).

Let $A \subset \mathbb{R}^{2}$ be a box (i.e. the Cartesian product of two intervals).
Then, we have $\lambda^{\otimes 2}(A)=0$ iff $A$ is a line (i.e. at least one of the two intervals is a singleton).
Proof. Direct consequence of Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes, with


Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite).
The measure space $\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right), \lambda^{\otimes 2}\right)$ is $\sigma$-finite.

Proof. Direct consequence of Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ), Lemma 1094 (Lebesgue measure is $\sigma$-finite), and Lemma 1229 (tensor product of $\sigma$-finite measures).

Lemma 1239 (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse).
The measure space $\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right), \lambda^{\otimes 2}\right)$ is diffuse.
Proof. Direct consequence of Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes, with $\underline{a_{1}=b_{1}}$ and $a_{2}=b_{2}$ ), and Definition 856 (diffuseness, $\left\{\left(a_{1}, a_{2}\right)\right\}=\left[a_{1}, a_{1}\right] \times\left[a_{2}, a_{2}\right]$ ).

### 15.4.3 Integration over product space of nonnegative function

## Definition 1240 (partial function of function from product space).

Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be measure spaces. Let $f: X_{1} \times X_{2} \rightarrow \overline{\mathbb{R}}_{+}$.
Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$. Let $\psi \stackrel{\text { def. }}{=}\left(\left(x_{i}, x_{j}\right) \mapsto\left(x_{1}, x_{2}\right)\right)$.
For all $x_{i} \in X_{i}, f_{x_{i}}$ denotes the function $\left(x_{j} \mapsto f \circ \psi\left(x_{i}, x_{j}\right)\right)$ from $X_{j}$ to $\overline{\mathbb{R}}_{+}$.
Moreover, when $f_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), I_{f, i}$ denotes the function $\left(x_{i} \mapsto \int f_{x_{i}} d \mu_{j}\right)$ from $X_{i}$ to $\overline{\mathbb{R}}_{+}$.
Remark 1241. The next proof follows steps 1 to 3 of the Lebesgue scheme (see Section 4.4).
See also the sketch of the proof in Section 5.5.
Theorem 1242 (Tonelli). Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces.
Let $f \in \mathcal{M}_{+}\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$. Let $i \in\{1,2\}$. Let $j \stackrel{\text { def. }}{=} 3-i$.
Then, for all $x_{i} \in X_{i}, f_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), I_{f, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$, and we have

$$
\begin{equation*}
\int f d\left(\mu_{1} \otimes \mu_{2}\right)=\int I_{f, i} d \mu_{i} \quad\left(\text { in } \overline{\mathbb{R}}_{+}\right) \tag{15.65}
\end{equation*}
$$

Proof. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}, \Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$, and $\mu \stackrel{\text { def. }}{=} \mu_{1} \otimes \mu_{2}$. Let $x_{i} \in X_{i}$.
Let $P$ be the predicate defined on functions from $X$ to $\overline{\mathbb{R}}_{+}$by

$$
f_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right) \wedge I_{f, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right) \wedge \int f d \mu=\int I_{f, i} d \mu_{i}
$$

(1). $\boldsymbol{P}$ holds on $\mathcal{I} \mathcal{F}(\boldsymbol{X}, \boldsymbol{\Sigma})$. Let $f \in \mathcal{I F}(X, \Sigma)$.

Let $A \stackrel{\text { def. }}{=}\{f \neq 0\}$. Then, from Lemma 1100 (indicator and support are inverse), we have $A \in \Sigma$ and $f=\mathbb{1}_{A}$. Then, from Definition 1240 (partial function of function from product space), Lemma 789 (indicator of section), Lemma 786 (measurability of section, $s_{i}\left(x_{i}, A\right) \in \Sigma_{j}$ ), and Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, with $A \stackrel{\text { def. }}{=} s_{i}\left(x_{i}, A\right)$ and $\mu_{j}$ ), we have

$$
f_{x_{i}}=\mathbb{1}_{s_{i}\left(x_{i}, A\right)} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), \quad \text { and } \quad I_{f, i}=\int \mathbb{1}_{s_{i}\left(x_{i}, A\right)} d \mu_{j}=\mu_{j}\left(s_{i}\left(x_{i}, A\right)\right)
$$

Hence, from Lemma 1218 (measure of section), and Lemma 1223 (measurability of measure of section), we have $I_{f, i}=F_{i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.

Therefore, from Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, with $\mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$ and $\underline{\mu}$ ), and Lemma 1233 (uniqueness of tensor product measure), we have

$$
\int f d \mu=\mu(A)=\int F_{i}^{A} d \mu_{i}=\int I_{f, i} d \mu_{i}
$$

(2). $P$ holds on $\mathcal{S} \mathcal{F}_{+}(X, \Sigma)$.

Let $f \in \mathcal{I F}(X, \Sigma), g \in \mathcal{S} \mathcal{F}_{+}(X, \Sigma)$ and $a \in \mathbb{R}_{+}$. Assume that $P(f)$ and $P(g)$ hold.
Then, from left linearity of composition, Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear,
with $\mu_{j}$ and $(f+g)_{x_{i}}$, then $\left.(a f)_{x_{i}}\right)$, Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition, with $\left.\mu_{i}\right)$, and Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication, with $\left.\mu_{i}\right)$, we have

$$
\begin{gathered}
(f+g)_{x_{i}}=f_{x_{i}}+g_{x_{i}},(a f)_{x_{i}}=a f_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right) \\
I_{f+g, i}=I_{f, i}+I_{g, i}, I_{a f, i}=a I_{f, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)
\end{gathered}
$$

Hence, from Lemma $1128\left(\mathcal{S F}{ }_{+}\right.$simple representation, $\left.f, a f \in \mathcal{S F}(X, \Sigma)\right)$, Lemma $1129\left(\mathcal{S F}{ }_{+}\right.$is closed under positive algebra operations, $f+g \in \mathcal{S F} \mathcal{F}_{+}(\overline{X, \Sigma})$ ), Lemma $1124\left(\mathcal{S F} \mathcal{F}_{+}\right.$is measurable, $\underline{f, g, f+g, a f \in \mathcal{M}_{+}(X, \Sigma)}$ ), and Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear, with $\mu$ and $\mu_{i}$ ),

$$
\begin{aligned}
\int(f+g) d \mu=\int f d \mu+\int g d \mu & =\int I_{f, i} d \mu_{i}+\int I_{g, i} d \mu_{i}=\int I_{f+g, i} d \mu_{i} \\
\int a f d \mu=a \int f d \mu & =a \int I_{f, i} d \mu_{i}=\int I_{a f, i} d \mu_{i}
\end{aligned}
$$

i.e. $P$ is compatible with positive linear combination of indicator functions.

Therefore, from (1), and Lemma 1134 (Lebesgue scheme step 2), $P$ holds on $\mathcal{S F}_{+}(X, \Sigma)$.
(3). $P$ holds on $\mathcal{M}_{+}(X, \Sigma)$.

Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{S F}+(X, \Sigma)$. Assume that $\left(f_{n}\right)_{n \in \mathbb{N}}$ is nondecreasing and $P\left(f_{n}\right)$ holds for all $n \in \mathbb{N}$. Then, from Lemma 381 (limit is monotone), composition law for the limit, Theorem 1180 (Beppo Levi, monotone convergence, with $\mu_{j}$ and $\left.\left(\lim _{n \rightarrow \infty} f_{n}\right)_{x_{i}}\right)$, and Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent, with $\mu_{i}$ ), we have

$$
\begin{gathered}
\left(\left(f_{n}\right)_{x_{i}}\right)_{n \in \mathbb{N}} \text { is nondecreasing, } \quad\left(\lim _{n \rightarrow \infty} f_{n}\right)_{x_{i}}=\lim _{n \rightarrow \infty}\left(f_{n}\right)_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), \\
I_{\lim _{n \rightarrow \infty} f_{n}, i}=\int\left(\lim _{n \rightarrow \infty} f_{n}\right)_{x_{i}} d \mu_{j}=\lim _{n \rightarrow \infty} \int\left(f_{n}\right)_{x_{i}} d \mu_{j}=\lim _{n \rightarrow \infty} I_{f_{n}, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right) .
\end{gathered}
$$

Hence, from Theorem 1180 (Beppo Levi, monotone convergence, with $\mu$ and $\mu_{j}$ ), we have

$$
\int \lim _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu=\lim _{n \rightarrow \infty} \int I_{f_{n}, i} d \mu_{i}=\int I_{\lim _{n \rightarrow \infty} f_{n}, i} d \mu_{i}
$$

i.e. $P$ is compatible with the limit of nondecreasing nonnegative simple functions.

Therefore, from (2), and Lemma 1141 (Lebesgue scheme step 3), $P$ holds on $\mathcal{M}_{+}(X, \Sigma)$.
Lemma 1243 (Tonelli over subset). Let $\left(X_{1}, \Sigma_{1}, \mu_{1}\right)$ and $\left(X_{2}, \Sigma_{2}, \mu_{2}\right)$ be $\sigma$-finite measure spaces. Let $\Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}$. Let $A \subset Y \subset X_{1} \times X_{2}$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$.
Assume that $f_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$. Let $i \in\{1,2\}$ with $j \stackrel{\text { def. }}{=} 3-i$. Let $\psi \stackrel{\text { def. }}{=}\left(\left(x_{i}, x_{j}\right) \mapsto\left(x_{1}, x_{2}\right)\right)$.
For all $x_{i} \in X_{i}$, let $A_{x_{i}} \stackrel{\text { def. }}{=} s_{i}\left(x_{i}, A\right)$, and $f_{x_{i}}^{A} \stackrel{\text { def. }}{=}\left(x_{j} \in A_{x_{i}} \mapsto f \circ \psi\left(x_{i}, x_{j}\right)\right)$.
Let $I_{f, i}^{A} \stackrel{\text { def. }}{=}\left(x_{i} \in X_{i} \mapsto \int_{A_{x_{i}}} f_{x_{i}}^{A} d \mu_{j}\right)$.
Then, for all $x_{i} \in X_{i}, f_{x_{i}}^{A} \in \mathcal{M}_{+}\left(A_{x_{i}}, \Sigma_{j} \bar{\cap} A_{x_{i}}\right), I_{f, i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$, and we have

$$
\begin{equation*}
\int_{A} f d\left(\mu_{1} \otimes \mu_{2}\right)=\int I_{f, i}^{A} d \mu_{i} . \tag{15.66}
\end{equation*}
$$

Proof. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}$. Let $\mu \stackrel{\text { def. }}{=} \mu_{1} \otimes \mu_{2}$. Let $\hat{f}: X \rightarrow \overline{\mathbb{R}}_{+}$such that $\hat{f}_{\left.\right|_{Y}}=f$. Then, from Lemma 1199 (integral in $\mathcal{M}_{+}$over subset), we have $\hat{f} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$ and $\int_{A} f d \mu=\int \hat{f} \mathbb{1}_{A} d \mu$. Hence, from Theorem 1242 (Tonelli, with $f \stackrel{\text { def. }}{=} \hat{f} \mathbb{1}_{A}$ ), and Definition 1240 (partial function of function from product space), for all $x_{i} \in X_{i},\left(\hat{f} \mathbb{1}_{A}\right)_{x_{i}} \in \mathcal{M}_{+}\left(X_{j}, \Sigma_{j}\right), I_{\hat{f} \mathbb{1}_{A}, i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$, and we have $\int \hat{f} \mathbb{1}_{A} d \mu=\int I_{\hat{f} \mathbb{1}_{A}, i} d \mu_{i}$.

Let $x_{i} \in X_{i}$. Then, by construction, and from Lemma 789 (indicator of section), we have $f_{x_{i}}^{A}=\left(\hat{f} \mathbb{1}_{A}\right)_{x_{i}}=\hat{f}_{x_{i}} \mathbb{1}_{A_{x_{i}}}$. Therefore, from Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $\mu_{j}$ ), we have $f_{x_{i}}^{A} \in \mathcal{M}_{+}\left(A_{x_{i}}, \Sigma_{j} \bar{\cap} A_{x_{i}}\right), I_{f, i}^{A}=\int\left(\hat{f} \mathbb{1}_{A}\right)_{x_{i}} d \mu_{j}=I_{\hat{f} \mathbb{1}_{A}, i}$, i.e. $I_{f, i}^{A} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$ and $\int_{A} f d \mu=\int I_{f, i}^{A} d \mu_{i}$.

## Lemma 1244 (Tonelli for tensor product).

For all $i \in\{1,2\}$, let $\left(X_{i}, \Sigma_{i}, \mu_{i}\right)$ be a $\sigma$-finite measure space, and $f_{i} \in \mathcal{M}_{+}\left(X_{i}, \Sigma_{i}\right)$.
Then, $f_{1} \otimes f_{2} \in \mathcal{M}_{+}\left(X_{1} \times X_{2}, \Sigma_{1} \otimes \Sigma_{2}\right)$, and we have

$$
\begin{equation*}
\int\left(f_{1} \otimes f_{2}\right) d\left(\mu_{1} \otimes \mu_{2}\right)=\left(\int f_{1} d \mu_{1}\right)\left(\int f_{2} d \mu_{2}\right) . \tag{15.67}
\end{equation*}
$$

Proof. Let $X \stackrel{\text { def. }}{=} X_{1} \times X_{2}, \Sigma \stackrel{\text { def. }}{=} \Sigma_{1} \otimes \Sigma_{2}, \mu \stackrel{\text { def. }}{=} \mu_{1} \otimes \mu_{2}$, and $f \stackrel{\text { def. }}{=} f_{1} \otimes f_{2}$.
Then, from Definition 851 (tensor product of numeric functions), Lemma 853 (measurability of tensor product of numeric functions, with $\mathbb{K} \stackrel{\text { def. }}{=} \overline{\mathbb{R}}_{+}$), Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), and Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), $f$ belongs to $\mathcal{M}_{+}(X, \Sigma)$.

Then, from Theorem 1242 (Tonelli, with $i \stackrel{\text { def. }}{=}$ ), Definition 1240 (partial function of function from product space), Definition 851 (tensor product of numeric functions), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous, with $f \stackrel{\text { def. }}{=} f_{2}$ and $a \stackrel{\text { def. }}{=} f_{1}\left(x_{1}\right) \in \overline{\mathbb{R}}_{+}$), and Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)), we have

$$
\int f d \mu=\int I_{f, 1} d \mu_{1} \quad \text { where } \forall x_{1} \in X_{1}, I_{f, 1}\left(x_{1}\right) \stackrel{\text { def. }}{=} \int f_{1}\left(x_{1}\right) f_{2} d \mu_{2}=\left(\int f_{2} d \mu_{2}\right) f_{1}\left(x_{1}\right) .
$$

Therefore, from Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous, with $a \stackrel{\text { def. }}{=} \int f_{2} d \mu_{2}$ belongs to $\overline{\mathbb{R}}_{+}$and $f \stackrel{\text { def. }}{=} f_{1}$ ), and Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)), we have

$$
\int f d \mu=\int\left(\int f_{2} d \mu_{2}\right) f_{1} d \mu_{1}=\left(\int f_{1} d \mu_{1}\right)\left(\int f_{2} d \mu_{2}\right) .
$$

## Chapter 16

## Lebesgue integration of real function

## Contents

16.1 Definition of the integral ..... 286
16.2 Notations for specific cases ..... 293
16.3 The seminormed vector space $\mathcal{L}^{p}$ ..... 297
16.3.1 The $p$-seminorm ..... 297
16.3.2 Hölder inequality ..... 302
16.3.3 Minkowski inequality ..... 305
16.3.4 The seminormed vector space $\mathcal{L}^{p}$ ..... 309
16.3.5 Lebesgue's dominated convergence theorem ..... 318
16.3.6 The Riesz-Fischer theorem ..... 323

[^3]
### 16.1 Definition of the integral

Remark 1246. This section follows step 4 of the Lebesgue scheme (see Section 4.4).
Definition 1247 (integrability).
Let $(X, \Sigma, \mu)$ be a measure space.
A function $f: X \rightarrow \overline{\mathbb{R}}$ is said $\mu$-integrable (in $\mathcal{M}$ ) iff $f^{+}$and $f^{-}$are $\mu$-integrable in $\mathcal{M}_{+}$.
Moreover, when $f$ is finite and $\mu$-integrable, we may say that it is $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$.
Lemma 1248 (integrable is measurable). Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Assume that $f$ is $\mu$-integrable in $\mathcal{M}$. Then, we have $f \in \mathcal{M}$.

Proof. Direct consequence of Definition 1247 (integrability), Lemma 1172 (integral in $\mathcal{M}_{+}$, definition of $\mu$-integrability with $f^{+}$and $f^{-}$), and Lemma 836 (measurability of nonnegative an $\bar{d}$ nonpositive parts).

## Lemma 1249 (equivalent definition of integrability).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$.
Then, $f$ is $\mu$-integrable in $\mathcal{M}$ iff $f \in \mathcal{M}$, and $|f|$ is $\mu$-integrable in $\mathcal{M}_{+}$, i.e. $\int|f| d \mu<\infty$.
Proof. "Left" implies "right". Direct consequence of Definition 1247 (integrability), Lemma 1248 (integrable is measurable), Lemma 842 ( $\mathcal{M}$ is closed under absolute value), Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts), Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed), and Lemma 1172 (integral in $\mathcal{M}_{+},|f|$ is $\mu$-integrable in $\mathcal{M}_{+}$).
"Right" implies "left". Direct consequence of Lemma 836 (measurability of nonnegative and nonpositive parts), Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts), Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$, contrapositive), Lemma 1172 (integral in $\mathcal{M}_{+}$, $f^{+}$and $f^{-}$are $\mu$-integrable in $\mathcal{M}_{+}$), and Definition 1247 (integrability).

Therefore, we have the equivalence.

## Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Assume that $f$ is nonnegative.
Then, $f$ is $\mu$-integrable in $\mathcal{M}$ iff $f$ is $\mu$-integrable in $\mathcal{M}_{+}$.
Proof. Direct consequence of Lemma 1249 (equivalent definition of integrability), and Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, $\lfloor f \mid=f$ ).

Lemma 1251 (integrable is almost finite).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $f: X \rightarrow \overline{\mathbb{R}}$ be $\mu$-integrable in $\mathcal{M}$. Then, we have $\mu(\{f= \pm \infty\})=0$, i.e. $|f| \stackrel{\mu \text { a.e. }}{<} \infty$.
Proof. Direct consequence of Lemma 1249 (equivalent definition of integrability, $|f|$ is integrable in $\mathcal{M}_{+}$), Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even, $\{f= \pm \infty\}=\{|f|=\infty\}$ ), and Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite, with $|f|$ ).

## Lemma 1252 (almost bounded by integrable is integrable).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, $f$ is $\mu$-integrable in $\mathcal{M}$ iff $f \in \mathcal{M}$ and there exists $g: X \rightarrow \overline{\mathbb{R}}_{+}$such that $g$ is $\mu$-integrable in $\mathcal{M}_{+}$and $|f| \stackrel{\mu \text { a.e. }}{\leqslant} g$.

Proof. Direct consequence of Lemma 1249 (equivalent definition of integrability), Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable, with $\left.f \stackrel{\text { def. }}{=}|f|\right)$, the tautology $(P \Rightarrow Q)$ implies $(\boldsymbol{P} \wedge \boldsymbol{Q} \Leftrightarrow \boldsymbol{P})$, and Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value, thus $f \in \mathcal{M} \wedge|f| \in \mathcal{M}_{+}$ is equivalent to $f \in \mathcal{M})$.

Lemma 1253 (integrability is closed under extension by zero). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\mid Y}=f$. Then, $f_{\mid A}$ is $\mu_{A}$-integrable in $\mathcal{M}(A, \Sigma \bar{\cap} A)$ iff $\hat{f} \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}(X, \Sigma)$.

If so, the function $f$ is said $\mu$-integrable over $A$. Moreover, $\left(\hat{f}^{ \pm}\right)_{\left.\right|_{A}}$ are $\mu$-integrable in $\mathcal{M}_{+}(A, \Sigma \bar{\cap} A), \hat{f}^{ \pm} \mathbb{1}_{A}$ are $\mu$-integrable in $\mathcal{M}_{+}(X, \Sigma)$, and we have

$$
\begin{gather*}
\mathcal{D}^{+}\left(\left(\hat{f}^{+}\right)_{\left.\right|_{A}},-\left(\hat{f}^{-}\right)_{\left.\right|_{A}}\right)=\mathcal{D}^{+}\left(\hat{f}^{+} \mathbb{1}_{A},-\hat{f}^{-} \mathbb{1}_{A}\right)=X,  \tag{16.1}\\
f_{\left.\right|_{A}}=\left(\hat{f}^{+}\right)_{\left.\right|_{A}}-\left(\hat{f}^{-}\right)_{\left.\right|_{A}} \quad \text { and } \quad \hat{f} \mathbb{1}_{A}=\hat{f}^{+} \mathbb{1}_{A}-\hat{f}^{-} \mathbb{1}_{A} . \tag{16.2}
\end{gather*}
$$

Proof. From Definition 1247 (integrability, with $f_{\left.\right|_{A}}$, then $\hat{f} \mathbb{1}_{A}$ ), Lemma 1172 (integral in $\mathcal{M}_{+}$, integrability, with $f_{\left.\right|_{A}}$, then $\hat{f} \mathbb{1}_{A}$ ), Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), Lemma 1199 (integral in $\mathcal{M}_{+}$over subset, with $f \stackrel{\text { def. }}{=} f^{ \pm}$and $\hat{f} \stackrel{\text { def. }}{=} \hat{f}^{ \pm}$), and Lemma 534 (compatibility of nonpositive and nonnegative parts with mask), we have

$$
\begin{aligned}
f_{\left.\right|_{A}} \mu_{A} \text {-integrable in } \mathcal{M}(A, \Sigma \bar{\cap} A) & \Longleftrightarrow\left(f_{\left.\right|_{A}}\right)^{ \pm} \mu_{A} \text {-integrable in } \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \\
& \Longleftrightarrow\left(f_{\left.\right|_{A}}\right)^{ \pm} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \wedge \int\left(f_{\left.\right|_{A}}\right)^{ \pm} d \mu<\infty \\
& \Longleftrightarrow\left(f^{ \pm}\right)_{\left.\right|_{A}} \in \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \wedge \int\left(f^{ \pm}\right)_{\left.\right|_{A}} d \mu<\infty \\
& \Longleftrightarrow \hat{f}^{ \pm} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma) \wedge \int \hat{f}^{ \pm} \mathbb{1}_{A} d \mu<\infty \\
& \Longleftrightarrow\left(\hat{f} \mathbb{1}_{A}\right)^{ \pm} \in \mathcal{M}_{+}(X, \Sigma) \wedge \int\left(\hat{f} \mathbb{1}_{A}\right)^{ \pm} d \mu<\infty \\
& \Longleftrightarrow\left(\hat{f} \mathbb{1}_{A}\right)^{ \pm} \mu \text {-integrable in } \mathcal{M}_{+}(X, \Sigma) \\
& \Longleftrightarrow \hat{f} \mathbb{1}_{A} \mu \text {-integrable in } \mathcal{M}(X, \Sigma) .
\end{aligned}
$$

Moreover, from Definition 1247 (integrability), the definition of the indicator function (thus masking is idempotent), Lemma 531 (nonnegative and nonpositive parts are orthogonal, thus $\left(\hat{f}^{+}\right)_{\left.\right|_{1}}$ and $\left(\hat{f}^{-}\right)_{\left.\right|_{A}}$ on the one hand, and $\hat{f}^{+} \mathbb{1}_{A}$ and $\hat{f}^{-} \mathbb{1}_{A}$ on the other, do not take infinite values at the same point), Lemma 998 (summability on summability domain, thus $\mathcal{D}^{+}\left(\left(\hat{f}^{+}\right)_{\left.\right|_{A}},-\left(\hat{f}^{-}\right)_{\left.\right|_{A}}\right)=\mathcal{D}^{+}\left(\hat{f}^{+} \mathbb{1}_{A},-\hat{f}^{-} \mathbb{1}_{A}\right)=X$ ), Lemma 532 (decomposition into nonnegative and nonpositive parts), Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), and Lemma 534 (compatibility of nonpositive and nonnegative parts with mask), $\left(\hat{f}^{ \pm}\right)_{\left.\right|_{A}}$ are $\mu$ integrable in $\mathcal{M}_{+}(A, \Sigma \bar{\cap} A), \hat{f}^{ \pm} \mathbb{1}_{A}$ are $\mu$-integrable in $\mathcal{M}_{+}(X, \Sigma)$ and all satisfy (16.1) and (16.2).

## Lemma 1254 (integrability is closed under restriction).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$.
Then, $f$ is $\mu$-integrable in $\mathcal{M}(X, \Sigma)$ iff for all $A \in \Sigma$, $f_{\left.\right|_{A}}$ is $\mu_{A}$-integrable in $\mathcal{M}(A, \Sigma \bar{\cap} A)$.
Proof. From Definition 1247 (integrability, with $(X, \Sigma, \mu)$, then $\left(A, \Sigma \bar{\cap} A, \mu_{A}\right)$ ), Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction), and Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction), we have

$$
\begin{aligned}
f \mu \text {-integrable in } \mathcal{M}(X, \Sigma) & \Longleftrightarrow f^{ \pm} \mu \text {-integrable in } \mathcal{M}_{+}(X, \Sigma) \\
& \Longleftrightarrow \forall A \in \Sigma, \quad f^{ \pm}{ }_{\left.\right|_{A}} \mu_{A} \text {-integrable in } \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \\
& \Longleftrightarrow \forall A \in \Sigma, \quad f_{\left.\right|_{A}}^{ \pm} \mu_{A} \text {-integrable in } \mathcal{M}_{+}(A, \Sigma \bar{\cap} A) \\
& \Longleftrightarrow \forall A \in \Sigma, \quad f_{\left.\right|_{A}} \mu_{A} \text {-integrable in } \mathcal{M}(A, \Sigma \bar{\cap} A) .
\end{aligned}
$$

Lemma 1255 (integrability, masking is $\sigma$-additive).
Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, we have $f \mathbb{1}_{A}=\sum_{i \in I} f \mathbb{1}_{A_{i}}$, and $f \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}$ implies for all $i \in I, f \mathbb{1}_{A_{i}}$ is $\mu$-integrable in $\mathcal{M}$.

Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$ with $n \in \mathbb{N}$, we have for all $i \in[0 . . n], f \mathbb{1}_{A_{i}}$ is $\mu$-integrable in $\mathcal{M}$ implies $f \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}$.

Proof. Direct consequence of Lemma 525 (masking is $\sigma$-additive, with $E \stackrel{\text { def. }}{=} \overline{\mathbb{R}}$, thus the identity holds), Definition 1247 (integrability, with $f \mathbb{1}_{A}$ and $f \mathbb{1}_{A_{i}}$ ), Lemma 1206 (integrable in $\mathcal{M}_{+}$,
 mask).

Lemma 1256 (integral in $\mathcal{M}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Then, $\int f^{+} d \mu-\int f^{-} d \mu$ is finite.

The Lebesgue integral of $f$ (for the measure $\mu$ ) is still denoted $\int f d \mu$; it is defined by
(16.3)

$$
\int f d \mu \stackrel{\text { def. }}{=} \int f^{+} d \mu-\int f^{-} d \mu \quad \in \mathbb{R}
$$

Proof. Direct consequence of Definition 1247 (integrability), Lemma 1172 (integral in $\mathcal{M}_{+}, \underline{f^{+}}$and $\underline{f^{-}}$are $\mu$-integrable in $\mathcal{M}_{+}$, thus their integrals are finite).

Remark 1257. Keeping the same notation for the integrals is justified in the next lemma.

Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$). Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Assume that $f$ is nonnegative. Then, both Lemma 1172 (integral in $\mathcal{M}_{+}$), and Lemma 1256 (integral in $\mathcal{M}$ ) provide the same value for the integral of $f$.

Proof. Direct consequence of Definition 528 (nonnegative and nonpositive parts, $f^{+}=f$ and $\underline{f^{-}=0}$ ), Definition 1247 (integrability), Lemma 1172 (integral in $\mathcal{M}_{+}, \underline{f \text { is } \mu \text {-integrable in } \mathcal{M}_{+} \text {), }}$ Lemma 1256 (integral in $\mathcal{M}$ ), and Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero).

## Lemma 1259 (integral in $\mathcal{M}$ of zero is zero).

Let $(X, \Sigma, \mu)$ be a measure space. Then, 0 is $\mu$-integrable in $\mathcal{M}$, and $\int 0 d \mu=0$.
Proof. Direct consequence of Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero), Lemma 1172 (integral in $\mathcal{M}_{+}$, thus 0 is $\mu$-integrable in $\mathcal{M}_{+}$), Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$), and Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$).

Lemma 1260 (integral in $\mathcal{M}$ of indicator function). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Assume that $\mu(A)<\infty$. Then, $\mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}$, and we have

$$
\begin{equation*}
\int \mathbb{1}_{A} d \mu=\mu(A) \quad \in \mathbb{R} \tag{16.4}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function), Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$), and Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$).

Definition 1261 (merge integral in $\mathcal{M}$ and $\mathcal{M}_{+}$).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. The integral of $f$ (for the measure $\mu$ ) exists iff $f$ is nonnegative (and $\int f d \mu \in \overline{\mathbb{R}}_{+}$), or $f$ is $\mu$-integrable (and $\int f d \mu \in \mathbb{R}$ ).

## Lemma 1262 (compatibility of integral with almost equality).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Let $g \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{=} g$. Then, $g$ is $\mu$-integrable, and we have

$$
\begin{equation*}
\int f d \mu=\int g d \mu \tag{16.5}
\end{equation*}
$$

Proof. Direct consequence of Lemma 985 (compatibility of almost equality with operator, with the binary operator $\max$ and the unary operator additive inverse, $f^{+} \stackrel{\mu \text { a.e. }}{=} g^{+}$and $\left.f^{-\mu} \stackrel{\text { a.e. }}{=} \overline{g^{-}}\right)$, Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality), Definition 1247 (integrability), and Lemma 1256 (integral in $\mathcal{M}$ ).

Lemma 1263 (finite representative of integrable in $\mathcal{M})$. Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Then, there exists $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ such that $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$ and $\int f d \mu=\int \tilde{f} d \mu$.

Moreover, if $f$ is nonnegative (i.e. if $f$ is $\mu$-integrable in $\mathcal{M}_{+}$), then $\tilde{f} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.
Proof. Direct consequence of Lemma 1251 (integrable is almost finite, $\{f= \pm \infty\}$ is $\mu$-negligible), Lemma $821\left(\mathcal{M}\right.$ is closed under finite part, $\tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{f-1}(\mathbb{R})$, thus $\left.\{f \neq \tilde{f}\}=\{f= \pm \infty\}\right)$, Lemma 1262 (compatibility of integral with almost equality), Lemma 811 (measurability of indicator function), and Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$.

Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$ be nonempty. Let $P$ be a predicate on $\overline{\mathbb{R}}^{I}$, lifted into a predicate on $\left(\overline{\mathbb{R}}^{I}\right)^{X}$. Let $Q$ be a predicate on $\mathbb{R}^{I}$. Assume that $P(0, \ldots, 0)$ holds, and that for all $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I} \in \mathcal{M}_{\mathbb{R}}$ be $\mu$-integrable, we have

$$
\begin{equation*}
\forall x \in X, P(f(x)) \quad \Longrightarrow \quad Q\left(\int f_{i} d \mu\right)_{i \in I} \tag{16.6}
\end{equation*}
$$

Then, for all $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I} \mu$-integrable in $\mathcal{M}$, we have

$$
\begin{equation*}
P(g) \mu \text { a.e. } \Longrightarrow Q\left(\int g_{i} d \mu\right)_{i \in I} . \tag{16.7}
\end{equation*}
$$

Proof. Let $g \stackrel{\text { def. }}{=}\left(g_{i}\right)_{i \in I}$ be $\mu$-integrable functions in $\mathcal{M}$. Assume that $P(g) \mu$ a.e. holds.
Let $B \stackrel{\text { def. }}{=}\{P(g)\}$, and $C \stackrel{\text { def. }}{=} \bigcap_{i \in I} g_{i}^{-1}(\mathbb{R})$. Then, from Lemma 1251 (integrable is almost finite, with $g_{i}$ ), Lemma 960 (countable almost properties is almost property, with $P_{0} \stackrel{\text { def. }}{=}(P(g))$ and for all $i \in I, P_{i+1} \stackrel{\text { def. }}{=}\left(g_{i}\right.$ finite $)$ ), there exists $A \subset B \cap C$ such that $A, A^{c} \in \Sigma$, and $\mu\left(A^{c}\right)=0$.

For all $i \in I$, let $f_{i} \stackrel{\text { def. }}{=} g_{i} \mathbb{1}_{A}$, and let $f \stackrel{\text { def. }}{=}\left(f_{i}\right)_{i \in I}$. Then, from Lemma 1008 (masking almost nowhere, with $g_{i}$ ), Lemma 833 (measurability and masking), the definition of the indicator function, and Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$, for all $i \in I, f_{i} \stackrel{\mu \text { a.e. }}{=} g_{i}, f_{i} \in \mathcal{M}_{\mathbb{R}}$, we have $P(f)$ everywhere, and thus $Q\left(\int f_{i} d \mu\right)_{i \in I}$ holds. Therefore, from Lemma 1262 (compatibility of integral with almost equality, with $f \stackrel{\text { def. }}{=} f_{i}$ and $\left.g \stackrel{\text { def. }}{=} g_{i}\right)$, we also have $Q\left(\int g_{i} d \mu\right)_{i \in I}$.

Remark 1265. The previous lemma allows to extend a property on the integrals of real-valued $\mu$ integrable functions to the same property on extended real-valued $\mu$-integrable functions whereas the pointwise premises may be satisfied only $\mu$-almost everywhere.

Lemma 1266 (integral in $\mathcal{M}$ is homogeneous).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Let $a \in \mathbb{R}$. Then, af is $\mu$-integrable in $\mathcal{M}$, and we have

$$
\begin{equation*}
\int a f d \mu=a \int f d \mu \tag{16.8}
\end{equation*}
$$

Proof. From Lemma 1249 (equivalent definition of integrability, thus $f \in \mathcal{M}$ and $\int|f| d \mu<\infty$ ), Lemma $826(\mathcal{M}$ is closed under scalar multiplication, thus af $\in \mathcal{M})$, the definition of the absolute value in $\mathbb{R}$ (thus $\left.|a| \in \mathbb{R}_{+}\right)$, Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value, thus $\left.|f| \in \mathcal{M}_{+}\right)$, Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative, thus $|a f|=|a||f|$ ), Lemma $1176 \overline{\left(\text { integral in } \mathcal{M}_{+}\right.}$ is positive homogeneous), and closedness of multiplication in $\mathbb{R}_{+}, \int|a f| d \mu=|a| \int|f| d \mu<\infty$. Hence, from Lemma 1249 (equivalent definition of integrability), af is $\mu$-integrable in $\mathcal{M}$.

From Lemma 836 (measurability of nonnegative and nonpositive parts), Definition 528 (nonnegative and nonpositive parts), and Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), we have $f^{ \pm} \in \mathcal{M}_{+},(a f)^{ \pm}=a f^{ \pm} \in \mathcal{M}_{+}$when $a \geqslant 0$, and $(a f)^{ \pm}=-a f^{\mp}$ belong to $\mathcal{M}_{+}$when $a<0$. Therefore, from Lemma 1256 (integral in $\mathcal{M}$, with af, then $f$ ), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous, with $a f^{ \pm}$(when $a \geqslant 0$ ) or $-a f^{\mp}$ (when $\underline{a<0)}$ ), and field properties of $\mathbb{R}$ (all integrals below are finite), we have in both cases

$$
\int a f d \mu=\int(a f)^{+} d \mu-\int(a f)^{-} d \mu=a \int f^{+} d \mu-a \int f^{-} d \mu=a \int f d \mu
$$

Lemma 1267 (integral in $\mathcal{M}$ of constant function). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Let $a \in \mathbb{R}$. Assume that $\mu(A)<\infty$. Then, $a \mathbb{1}_{A}$ is $\mu$-integrable in $\mathcal{M}$, and we have

$$
\begin{equation*}
\int a \mathbb{1}_{A} d \mu=a \mu(A) \quad \in \mathbb{R} . \tag{16.9}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1260 (integral in $\mathcal{M}$ of indicator function), and Lemma 1266 (integral in $\mathcal{M}$ is homogeneous, with $f \stackrel{\text { def. }}{=} \mathbb{1}_{A}$ ).

Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g, f^{\prime}, g^{\prime} \in \mathcal{M}$. Assume that $f+g$ is well-defined almost everywhere, $f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime} \stackrel{\mu \text { a.e. }}{=} g$, and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$. Then, $f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and we have

$$
\begin{equation*}
\int|f \stackrel{\mu \text { a.e. }}{+} g| d \mu=\int\left|f^{\prime}+g^{\prime}\right| d \mu \leqslant \int\left|f^{\prime}\right| d \mu+\int\left|g^{\prime}\right| d \mu=\int|f| d \mu+\int|g| d \mu . \tag{16.10}
\end{equation*}
$$

Proof. From Lemma 1002 (almost sum, such $f^{\prime}$ and $g^{\prime}$ exist), and Lemma 1003 (compatibility of almost sum with almost equality), we have $f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$. Hence, from Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value), Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$ with almost equality), and Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$, we have

$$
\begin{gathered}
|f|,|g|,\left|f^{\prime}\right|,\left|g^{\prime}\right|,\left|f^{\prime}+g^{\prime}\right|,\left|f^{\prime}\right|+\left|g^{\prime}\right|,|f \stackrel{\mu \text { a.e. }}{+} g| \in \mathcal{M}_{+} \\
\int\left|f^{\prime}\right| d \mu=\int|f| d \mu, \quad \int\left|g^{\prime}\right| d \mu=\int|g| d \mu, \quad \text { and } \quad \int\left|f^{\prime}+g^{\prime}\right| d \mu=\int|f \stackrel{\mu \text { a.e. }}{+} g| d \mu
\end{gathered}
$$

Therefore, from Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), we have

$$
\int\left|f^{\prime}+g^{\prime}\right| d \mu \leqslant \int\left(\left|f^{\prime}\right|+\left|g^{\prime}\right|\right) d \mu=\int\left|f^{\prime}\right| d \mu+\int\left|g^{\prime}\right| d \mu
$$

Lemma 1269 (integral in $\mathcal{M}$ is additive). Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g$ be $\mu$-integrable in $\mathcal{M}$. Let $f^{\prime}, g^{\prime} \in \mathcal{M}$. Assume that $f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime \mu} \stackrel{\text { a.e. }}{=} g$, and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$. Then, $f+g$ is well-defined almost everywhere, $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$ are $\mu$-integrable in $\mathcal{M}$, and we have

$$
\begin{equation*}
\int\left(f^{\mu \text { a.e. }}+g\right) d \mu=\int\left(f^{\prime}+g^{\prime}\right) d \mu=\int f^{\prime} d \mu+\int g^{\prime} d \mu=\int f d \mu+\int g d \mu \tag{16.11}
\end{equation*}
$$

Moreover, when $\mathcal{D}^{+}(f, g)=X$, e.g. when $f, g$ are $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}, f+g$ is $\mu$-integrable in $\mathcal{M}$, and we have

$$
\begin{equation*}
\int(f+g) d \mu=\int f d \mu+\int g d \mu \tag{16.12}
\end{equation*}
$$

Proof. From Lemma 1249 (equivalent definition of integrability, thus $f, g \in \mathcal{M}$ and $\int|f| d \mu, \int|g| d \mu$ are finite), Lemma 1002 (almost sum, such $f^{\prime}$ and $g^{\prime}$ exist), and Lemma 1003 (compatibility of almost sum with almost equality), $f+g$ is well-defined $\mu$-almost everywhere. Then, from Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ), and closedness of addition in $\mathbb{R}_{+}, f^{\prime}+$ $g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}, f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$, and we have $\int|f \stackrel{\mu \text { a.e. }}{+} g| d \mu=\int\left|f^{\prime}+g^{\prime}\right| d \mu<\infty$. Hence, from Lemma 1249 (equivalent definition of integrability), $f^{\prime}+g^{\prime}$ and $f \stackrel{\mu \text { a.e. }}{+} g$ are $\mu$-integrable in $\mathcal{M}$.

From Lemma 1262 (compatibility of integral with almost equality, with $f^{\prime}$ and $g^{\prime}$ ), $f^{\prime}$ and $g^{\prime}$ are also $\mu$-integrable in $\mathcal{M}$, and we have $\int f^{\prime} d \mu=\int f d \mu$ and $\int g^{\prime} d \mu=\int g d \mu$. Then, from Lemma 836 (measurability of nonnegative and nonpositive parts), $f^{\prime \pm}, g^{\prime \pm}$, and $\left(f^{\prime}+g^{\prime}\right)^{ \pm}$belong to $\mathcal{M}_{+}$. Therefore, from Lemma 1256 (integral in $\mathcal{M}$, with $f^{\prime}+g^{\prime}$, then $f^{\prime}$ and $g^{\prime}$ ), Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts, with $f^{\prime}$ and $\underline{g^{\prime}}$ ), and field properties of $\mathbb{R}$ (all integrals below are finite), we have

$$
\begin{aligned}
\int\left(f^{\prime}+g^{\prime}\right) d \mu & =\int\left(f^{\prime}+g^{\prime}\right)^{+} d \mu-\int\left(f^{\prime}+g^{\prime}\right)^{-} d \mu \\
& =\int\left(f^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{+} d \mu-\left(\int\left(f^{\prime}\right)^{-} d \mu+\int\left(g^{\prime}\right)^{-} d \mu\right) \\
& =\left(\int\left(f^{\prime}\right)^{+} d \mu-\int\left(f^{\prime}\right)^{-} d \mu\right)+\left(\int\left(g^{\prime}\right)^{+} d \mu-\int\left(g^{\prime}\right)^{-} d \mu\right) \\
& =\int f^{\prime} d \mu+\int g^{\prime} d \mu
\end{aligned}
$$

The specific case $\mathcal{D}^{+}(f, g)=X$ is a direct consequence of Lemma 1004 (almost sum is sum).

## Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear).

Let $(X, \Sigma, \mu)$ be a measure space. Then, the integral is linear on $\mu$-integrable functions in $\mathcal{M}_{\mathbb{R}}$ : let $a \in \mathbb{R}$, and $f, g$ be $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$, then af, $f+g$ are $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$, and we have

$$
\begin{equation*}
\int a f d \mu=a \int f d \mu, \quad \text { and } \quad \int(f+g) d \mu=\int f d \mu+\int g d \mu . \tag{16.13}
\end{equation*}
$$

Proof. Direct consequence of Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space, thus af,f+g $\left.\in \mathcal{M}_{\mathbb{R}}\right)$, Lemma 819 $\left(\mathcal{M}\right.$ and finite is $\mathcal{M}_{\mathbb{R}}$, thus $f \in \mathcal{M}$ ), Lemma 1266 (integral in $\mathcal{M}$ is homogeneous), and Lemma 1269 (integral in $\mathcal{M}$ is additive).

## Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g$ be $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$. Then, we have

$$
\begin{equation*}
f \leqslant g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu \quad(\text { in } \mathbb{R}) . \tag{16.14}
\end{equation*}
$$

Proof. From Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$, Definition 61 (vector space), additive group properties of $\mathcal{M}_{\mathbb{R}}$ (thus $g-f \in \mathcal{M}_{\mathbb{R}}$ ), and Definition 835 $\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), we have $g-f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset$ $\mathcal{M}$. Hence, from Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$, with $(f, g) \stackrel{\text { def. }}{=}(g,-f)$ ), Lemma 1004 (almost sum is sum, thus $g \stackrel{\mu \text { a.e. }}{+}(-f)=g-f$ ), Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even, thus $|-f|=|f|$ ), Lemma 1249 (equivalent definition of integrability, with $f$ and $g$, then $\underline{g-f})$, and additive group properties of $\mathbb{R}$ (closedness under addition), we have

$$
\int|g-f| d \mu \leqslant \int|g| d \mu+\int|f| d \mu<\infty, \quad \text { and } \quad g-f \text { is } \mu \text {-integrable. }
$$

Therefore, from Lemma 1172 (integral in $\mathcal{M}_{+}$, nonnegativeness), Lemma 1269 (integral in $\mathcal{M}$ is additive, with $f, g-f \in \mathcal{M}_{\mathbb{R}}$ ), and ordered field properties of $\mathbb{R}$, we have

$$
\int f d \mu=\int f d \mu \leqslant \int f d \mu+\int(g-f) d \mu=\int g d \mu
$$

## Lemma 1272 (integral in $\mathcal{M}$ is almost monotone).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g$ be $\mu$-integrable in $\mathcal{M}$. Then, we have

$$
\begin{equation*}
f \stackrel{\mu \text { a.e. }}{\leqslant} g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu \quad(\text { in } \mathbb{R}) . \tag{16.15}
\end{equation*}
$$

Proof. Direct consequence of Lemma 313 ( order in $\overline{\mathbb{R}}$ is total, reflexivity (thus $0 \leqslant 0$ ) , Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone), and Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$,


## Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f$ be $\mu$-integrable in $\mathcal{M}$. Then, we have

$$
\begin{equation*}
\left|\int f d \mu\right| \leqslant \int|f| d \mu \tag{16.16}
\end{equation*}
$$

Proof. From Definition 1247 (integrability), and Lemma 1249 (equivalent definition of integrability), $f^{+}, f^{-}$and $|f|$ are $\mu$-integrable in $\mathcal{M}_{+}$, i.e. all integrals below are finite. Therefore, from Lemma 1256 (integral in $\mathcal{M}$, with $f$, then $|f|$ ), the triangle inequality for the absolute value in $\mathbb{R}$, Lemma 530 (nonnegative and nonpositive parts are nonnegative), Lemma 1172 (integral in $\mathcal{M}_{+}$, nonnegativeness), valued field properties of $\mathbb{R}(a \geqslant 0 \Rightarrow|a|=a)$, and Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts), we have

$$
\left|\int f d \mu\right|=\left|\int f^{+} d \mu-\int f^{-} d \mu\right| \leqslant \int f^{+} d \mu+\int f^{-} d \mu=\int|f| d \mu
$$

### 16.2 Notations for specific cases

Lemma 1274 (integral over subset).
Let $(X, \Sigma, \mu)$ be a measure space. Let $A \subset Y \subset X$. Assume that $A \in \Sigma$. Let $f: Y \rightarrow \overline{\mathbb{R}}$, and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$, and $f$ is $\mu$-integrable over $A$. Then, $\left(\hat{f}^{ \pm}\right)_{\left.\right|_{A}}$ are $\mu$ integrable in $\mathcal{M}_{+}(A, \Sigma \bar{\cap} A), \hat{f}^{ \pm} \mathbb{1}_{A}$ are $\mu$-integrable in $\mathcal{M}_{+}(X, \Sigma)$, and we have

$$
\begin{gather*}
\mathcal{D}^{+}\left(\left(\hat{f}^{+}\right)_{\left.\right|_{A}},-\left(\hat{f}^{-}\right)_{\left.\right|_{A}}\right)=\mathcal{D}^{+}\left(\hat{f}^{+} \mathbb{1}_{A},-\hat{f}^{-} \mathbb{1}_{A}\right)=X,  \tag{16.17}\\
f_{\left.\right|_{A}}=\left(\hat{f}^{+}\right)_{\left.\right|_{A}}-\left(\hat{f}^{-}\right)_{\left.\right|_{A}}, \quad \hat{f} \mathbb{1}_{A}=\hat{f}^{+} \mathbb{1}_{A}-\hat{f}^{-} \mathbb{1}_{A},  \tag{16.18}\\
\int f_{\left.\right|_{A}} d \mu_{A}=\int_{A} \hat{f}^{+} d \mu-\int_{A} \hat{f}^{-} d \mu=\int \hat{f} \mathbb{1}_{A} d \mu . \tag{16.19}
\end{gather*}
$$

This integral is still denoted $\int_{A} f d \mu$; it is still called integral of $f$ over $A$.
Proof. Direct consequence of Lemma 1253 (integrability is closed under extension by zero), Lemma 1266 (integral in $\mathcal{M}$ is homogeneous, with $a \stackrel{\text { def. }}{=}-1$ ), Lemma 1269 (integral in $\mathcal{M}$ is additive, with $\mathcal{D}^{+}\left(\hat{g}^{+},-\hat{g}^{-}\right)=\mathcal{D}^{+}\left(\left(\hat{g}^{+}\right)_{\left.\right|_{\Lambda}},-\left(\hat{g}^{-}\right)_{\dagger_{\Lambda}}\right)=X$ ), Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$), and Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).

Lemma 1275 (integral over subset is $\sigma$-additive). Let $(X, \Sigma, \mu)$ be a measure space. Let $I \subset \mathbb{N}$. Let $A,\left(A_{i}\right)_{i \in I} \in \Sigma$. Assume that $A=\biguplus_{i \in I} A_{i}$. Let $Y$ be a set such that $A \subset Y \subset X$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\left.\right|_{Y}}=f$. Then, $\hat{f} \mathbb{1}_{A}=\sum_{i \in I} \hat{f} \mathbb{1}_{A_{i}}$, and $f$ is $\mu$-integrable over $A$ implies for all $i \in I, f$ is $\mu$-integrable over $A_{i}$. If so, we have

$$
\begin{equation*}
\int_{A} f d \mu=\sum_{i \in I} \int_{A_{i}} f d \mu \tag{16.20}
\end{equation*}
$$

Moreover, when $I \stackrel{\text { def. }}{=}[0 . . n]$ with $n \in \mathbb{N}$, we have for all $i \in[0 . . n]$, $f$ is $\mu$-integrable over $A_{i}$ implies $f$ is $\mu$-integrable over $A$.

Proof. From Lemma 1255 (integrability, masking is $\sigma$-additive), we have the first identity and both implications.
Identity (16.20). Assume that $f$ is $\mu$-integrable over $A$, and over all the $A_{i}$ 's. Then, from Lemma 1274 (integral over subset, with $A$ and all the $A_{i}$ 's), Lemma 1172 (integral in $\mathcal{M}_{+}$, integrability), Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive, with $A$ and all the $A_{i}$ 's), and commutativity and associativity of the sum of absolutely convergent series, $\hat{f}^{ \pm} \mathbb{1}_{A}$ and for all $i \in I, \hat{f}^{ \pm} \mathbb{1}_{A_{i}}$ are $\mu$-integrable in $\mathcal{M}_{+}$, i.e. $\hat{f}^{ \pm} \mathbb{1}_{A}, \hat{f}^{ \pm} \mathbb{1}_{A_{i}} \in \mathcal{M}_{+}$and $\int_{A} \hat{f}^{ \pm} d \mu, \int_{A_{i}} \hat{f}^{ \pm} d \mu$ are finite, and we have

$$
\begin{aligned}
\int_{A} f d \mu & =\int_{A} \hat{f}^{+} d \mu-\int_{A} \hat{f}^{-} d \mu=\sum_{i \in I} \int_{A_{i}} \hat{f}^{+} d \mu-\sum_{i \in I} \int_{A_{i}} \hat{f}^{-} d \mu \\
& =\sum_{i \in I}\left(\int_{A_{i}} \hat{f}^{+} d \mu-\int_{A_{i}} \hat{f}^{+} d \mu\right)=\sum_{i \in I} \int_{A_{i}} f d \mu
\end{aligned}
$$

## Lemma 1276 (integral over negligible subset). Let $(X, \Sigma, \mu)$ be a measure space.

 Let $A \subset Y \subset X$. Assume that $A \in \Sigma$ and $\mu(A)=0$. Let $f: Y \rightarrow \overline{\mathbb{R}}$ and $\hat{f}: X \rightarrow \overline{\mathbb{R}}$. Assume that $\hat{f}_{\mid Y}=f$, and $f$ is $\mu$-integrable over $A$. Then, we have$$
\begin{equation*}
\int_{A} f d \mu=0 . \tag{16.21}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1274 (integral over subset, thus $\hat{f}^{ \pm} \mathbb{1}_{A} \in \mathcal{M}_{+}(X, \Sigma)$ and $\int_{A} f d \mu=\int_{A} \hat{f}^{+} d \mu-\int_{A} \hat{f}^{-} d \mu$ ), Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset, thus $\int_{A} \hat{f}^{ \pm} d \mu$ are equal to 0 ), and additive group properties of $\mathbb{R}$ (thus $0-0=0$ ).

## Lemma 1277 (integral over singleton). Let $(X, \Sigma, \mu)$ be a measure space.

Let $a \in X$. Assume that $\{a\} \in \Sigma$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Then, $f$ is $\mu$-integrable over $\{a\}$ iff

$$
\begin{equation*}
(f(a) \text { and } \mu(\{a\}) \text { are finite }) \quad \vee \quad f(a)=0 \quad \vee \quad \mu(\{a\})=0 \tag{16.22}
\end{equation*}
$$

If so, we have

$$
\begin{equation*}
\int_{\{a\}} f d \mu=f(a) \mu(\{a\}) . \tag{16.23}
\end{equation*}
$$

Proof. From Lemma 1253 (integrability is closed under extension by zero, with $A \stackrel{\text { def. }}{=}\{a\}, Y \stackrel{\text { def. }}{=} X$ and $\hat{f} \stackrel{\text { def. }}{=} f$ ), Lemma 1249 (equivalent definition of integrability), Lemma 1172 (integral in $\mathcal{M}_{+}$, integrability), Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton, with $|f|$, thus $\left|f \mathbb{1}_{\{a\}}\right|=|f(a)| \mathbb{1}_{\{a\}}$ belong to $\mathcal{M}_{+}$and $\int\left|f \mathbb{1}_{\{a\}}\right| d \mu=|f(a)| \mu(\{a\})$ ), Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory)), closedness of the absolute value in $\mathbb{R}$, Definition 339 (absolute value in $\overline{\mathbb{R}}$, absolute value is closed in $\{ \pm \infty\}$ ), and Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite), we have
$f$ is $\mu$-integrable over $\{a\} \Longleftrightarrow f \mathbb{1}_{\{a\}}$ is $\mu$-integrable in $\mathcal{M}$

$$
\begin{aligned}
& \Longleftrightarrow\left|f \mathbb{1}_{\{a\}}\right| \text { is } \mu \text {-integrable in } \mathcal{M}_{+} \\
& \Longleftrightarrow\left|f \mathbb{1}_{\{a\}}\right| \in \mathcal{M}_{+} \wedge \quad \int\left|f \mathbb{1}_{\{a\}}\right| d \mu<\infty \\
& \Longleftrightarrow|f(a)| \mu(\{a\})<\infty \\
& \Longleftrightarrow\left(f(a) \in \mathbb{R} \wedge \mu(\{a\}) \in \mathbb{R}_{+}\right) \quad \vee \quad f(a)=0 \quad \vee \quad \mu(\{a\})=0
\end{aligned}
$$

Assume that $f$ is $\mu$-integrable over $\{a\}$. Then, from Lemma 1274 (integral over subset, with $A \stackrel{\text { def. }}{=}\{a\}, Y \stackrel{\text { def. }}{=} X$ and $\hat{f} \stackrel{\text { def. }}{=} f$ ), Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton), right distributivity of multiplication over addition in $\mathbb{R}$, and Lemma 532 (decomposition into nonnegative and nonpositive parts, for $f$ ), we have

$$
\int_{\{a\}} f d \mu=\int_{\{a\}} f^{+} d \mu-\int_{\{a\}} f^{-} d \mu=f^{+}(a) \mu(\{a\})-f^{-}(a) \mu(\{a\})=f(a) \mu(\{a\}) .
$$

## Lemma 1278 (integral over interval).

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a measure space. Assume that $\mu$ is diffuse. Let $a, b \in \overline{\mathbb{R}}$ such that $a \leqslant b$. Let $f$ be $\mu$-integrable over $(a, b)$. Then, the integral over the interval remains the same when the interval is closed at one or both of its finite extremities.

The integral of $f$ over an interval with extremities $a$ and $b$ is called integral of $f$ from $a$ to $b$; it is denoted $\int_{a}^{b} f(x) d \mu(x)$. When $a<b$, we assume the convention $\int_{b}^{a} f(x) d \mu(x) \stackrel{\text { def. }}{=}-\int_{a}^{b} f(x) d \mu(x)$.

Proof. Direct consequence of Definition 856 (diffuseness), Lemma 1275 (integral over subset is $\sigma$-additive, on $[a, b]=\{a\} \uplus(a, b) \uplus\{b\})$, Lemma 744 (some Borel subsets, singletons of $\mathbb{R}$ are measurable), and Lemma 1277 (integral over singleton).

## Lemma 1279 (Chasles relation, integral over split intervals).

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a measure space. Assume that $\mu$ is diffuse. Let $a, b, c \in \overline{\mathbb{R}}$. Let $f$ be $\mu$-integrable over $(\min (a, b, c), \max (a, b, c))$. Then, we have

$$
\begin{equation*}
\int_{a}^{b} f(x) d \mu(x)=\int_{a}^{c} f(x) d \mu(x)+\int_{c}^{b} f(x) d \mu(x) . \tag{16.24}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1275 (integral over subset is $\sigma$-additive). Lemma 1278 (integral over interval, convention for reverse bounds), and field properties of $\mathbb{R}$.

It is trivial when $a \leqslant c \leqslant b$. For instance, when $c<b<a$, we have

$$
\int_{c}^{a} f(x) d \mu(x)=\int_{c}^{b} f(x) d \mu(x)+\int_{b}^{a} f(x) d \mu(x)
$$

Hence, we have

$$
\begin{aligned}
\int_{a}^{b} f(x) d \mu(x) & =-\int_{b}^{a} f(x) d \mu(x) \\
& =-\int_{c}^{a} f(x) d \mu(x)+\int_{c}^{b} f(x) d \mu(x) \\
& =\int_{a}^{c} f(x) d \mu(x)+\int_{c}^{b} f(x) d \mu(x)
\end{aligned}
$$

## Lemma 1280 (integral with trivial measure).

Let $(X, \Sigma)$ be a measurable space. Let $f \in \mathcal{M}$. Then, $f$ is 0 -integrable, and we have

$$
\begin{equation*}
\int f d 0=0 \tag{16.25}
\end{equation*}
$$

Proof. From Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure), Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), and Lemma 1249 (equivalent definition of integrability), $f$ is 0 -integrable in $\mathcal{M}$. Then, from Lemma 1256 (integral in $\mathcal{M}$ ), Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure), and additive group properties of $\mathbb{R}$, we have

$$
\int f d 0=\int f^{+} d 0-\int f^{-} d 0=0
$$

## Lemma 1281 (integral with counting measure).

Let $(X, \Sigma)$ be a measurable space. Let $Y \subset X$. Let $f \in \mathcal{M}$. Assume that $\sum_{y \in Y}|f(y)|$ is finite. Then, $f$ is $\delta_{Y}$-integrable, and we have

$$
\begin{equation*}
\int f d \delta_{Y}=\sum_{y \in Y} f(y) \tag{16.26}
\end{equation*}
$$

Proof. From Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure, $\int|f| d \delta_{Y}$ is finite), and Lemma 1249 (equivalent definition of integrability), $f$ is $\delta_{Y}$-integrable in $\mathcal{M}$. Then, from Lemma 1256 (integral in $\mathcal{M}$ ), Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure), and associativity and commutativity of (possibly uncountable) addition for absolutely convergent sums,

$$
\int f d \delta_{Y}=\int f^{+} d \delta_{Y}-\int f^{-} d \delta_{Y}=\sum_{y \in Y} f^{+}(y)-\sum_{y \in Y} f^{-}(y)=\sum_{y \in Y} f(y)
$$

Lemma 1282 (integral with counting measure on $\mathbb{N}$ ).
Let $f: \mathbb{N} \rightarrow \overline{\mathbb{R}}$ be a sequence. Assume that $\sum_{n \in \mathbb{N}}|f(n)|$ is finite. Then, $f$ is $\delta_{\mathbb{N}}$-integrable in $\mathcal{M}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and we have

$$
\begin{equation*}
\int f d \delta_{\mathbb{N}}=\sum_{n \in \mathbb{N}} f(n) \tag{16.27}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1281 (integral with counting measure, with $Y=X \stackrel{\text { def. }}{=} \mathbb{N}$ and $\Sigma \stackrel{\text { def. }}{=} \mathcal{P}(\mathbb{N}))$.

Remark 1283. Note that the previous lemma makes the sums of absolutely convergent series be Lebesgue integrals with counting measure on natural numbers. Thus, the theory of absolutely converging series can be derived from the theory of Lebesgue integration.

Lemma 1284 (integral with Dirac measure).
Let $(X, \Sigma)$ be a measurable space. Let $\{a\} \in \Sigma$. Let $f \in \mathcal{M}$. Assume that $f(a)$ is finite. Then, $f$ is $\delta_{a}$-integrable, and we have

$$
\begin{equation*}
\int f d \delta_{a}=f(a) \tag{16.28}
\end{equation*}
$$

Proof. Direct consequence of Definition 939 (Dirac measure), and Lemma 1281 (integral with counting measure, with $Y \stackrel{\text { def. }}{=}\{a\})$.

## Definition 1285 (integral for Lebesgue measure on $\mathbb{R}$ ).

Let $a, b \in \overline{\mathbb{R}}$. Let $Y \stackrel{\text { def. }}{=}[(\min (a, b), \max (a, b)]$. Let $f: Y \rightarrow \mathbb{R}$. Assume that $f$ is $\lambda$-integrable over $Y$ (for the Lebesgue measure $\lambda$ ). The integral of $f$ from $a$ to $b$ is denoted $\int_{a}^{b} f(x) d x$.

### 16.3 The seminormed vector space $\mathcal{L}^{p}$

### 16.3.1 The $p$-seminorm

Lemma 1286 (p-seminorm). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $p \in(0, \infty)$. Then, $|f|^{p},|f| \in \mathcal{M}_{+}$, and $\int|f|^{p} d \mu$ and esssup ${ }^{+}|f|$ are well-defined (in $\overline{\mathbb{R}}_{+}$).

Let $\widetilde{N}_{p} \stackrel{\text { def. }}{=}\left(f \in \mathcal{M} \mapsto \int|f|^{p} d \mu \in \overline{\mathbb{R}}_{+}\right)$. The function $N_{p} \stackrel{\text { def. }}{=} \widetilde{N}_{p}^{\frac{1}{p}}$ is called $p$-seminorm, or $\mathcal{L}^{p}$-seminorm. In particular, for $p=1$, we have $N_{1}=\widetilde{N}_{1}=\left(f \in \mathcal{M} \longmapsto \int|f| d \mu \in \overline{\mathbb{R}}_{+}\right)$.

The function $N_{\infty} \stackrel{\text { def. }}{=}\left(f \in \mathcal{M} \mapsto \operatorname{ess} \sup ^{+}|f| \in \overline{\mathbb{R}}_{+}\right)$is called $\infty$-seminorm, or $\mathcal{L}^{\infty}$-seminorm.
Proof. Direct consequence of Lemma $841(\mathcal{M}$ is closed under power function, with $p$ and 1, thus $|f|^{p},|f| \in \mathcal{M}_{+}$), Lemma 1172 (integral in $\mathcal{M}_{+}$), Lemma 361 (power function in $\overline{\mathbb{R}}$, with $\frac{1}{p} \in \mathbb{R}_{+}^{\star}$ ), and Lemma 1036 (equivalent definition of nonnegative essential supremum).

Remark 1287. When $p \in[1, \infty)$, the function $N_{p}$ is shown below to be a seminorm on the vector space $\mathcal{L}^{p}$, hence its name. When $p \in(0,1)$, the function $N_{p}$ is only a quasinorm, for which the triangle inequality is relaxed, and its specific study is out of the scope of this document.

Lemma 1288 ( $N_{p}$ with trivial measure).
Let $(X, \Sigma, \mu)$ be a trivial measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{M}$. Then, we have $N_{p}(f)=0$.
Proof. Case $p \in(0, \infty)$. Direct consequence of ordered field properties of $\mathbb{R}\left(\frac{1}{p} \in(0, \infty)\right)$, Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure), Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory) , and Lemma 1286 ( $p$-seminorm, thus $N_{p}(f)=\widetilde{N}_{p}(f)=0$ ).
Case $\boldsymbol{p}=\boldsymbol{\infty}$. Direct consequence of Lemma 1286 ( p-seminorm) , and Lemma 1037 (nonnegative essential supremum with trivial measure).
Therefore, $N_{p}$ is always zero.
Lemma 1289 ( $N_{p}$ of indicator function).
Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Let $p \in(0, \infty)$. Then, $\mathbb{1}_{A} \in \mathcal{M}$, and we have

$$
N_{p}\left(\mathbb{1}_{A}\right)=(\mu(A))^{\frac{1}{p}}, \quad \text { and } \quad N_{\infty}\left(\mathbb{1}_{A}\right)= \begin{cases}1 & \text { if } \mu(A)>0  \tag{16.29}\\ 0 & \text { otherwise } .\end{cases}
$$

Proof. From Lemma 811 (measurability of indicator function), we have $\mathbb{1}_{A} \in \mathcal{M}$.
Case $p \in(0, \infty)$. Direct consequence of properties of the power function in $\mathbb{R}$ (thus $\left.\left|\mathbb{1}_{A}\right|^{p}=\mathbb{1}_{A} \geqslant 0\right)$, Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function, thus $\int \mathbb{1}_{A} d \mu=\mu(A)$ ), and Lemma 1286 ( $p$-seminorm).

Case $\mu$ trivial, and $p=\infty$.
Direct consequence of Lemma 933 (trivial measure, thus $\mu(A)>0$ cannot occur), Lemma 1037 (nonnegative essential supremum with trivial measure), and Lemma 1286 (p-seminorm).
Case $\mu$ nontrivial, and $p=\infty$.
Direct consequence of Lemma 1041 (nonnegative essential supremum for extended simple functions, with $n \stackrel{\text { def. }}{=} 1, A_{0} \stackrel{\text { def. }}{=} A, A_{1} \stackrel{\text { def. }}{=} A^{c}, a_{0} \stackrel{\text { def. }}{=} 1$ and $a_{1} \stackrel{\text { def. }}{=} 0$ ), and Lemma 1286 ( $p$-seminorm).

Therefore, we always have the equalities.
Lemma 1290 ( $N_{p}$ is nonnegative).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{M}$. Then, we have $N_{p}(f) \geqslant 0$.

Proof. Case $\boldsymbol{p} \in(\mathbf{0}, \infty)$. Direct consequence of Lemma 1286 ( $p$-seminorm, with $p \in(0, \infty)$, thus $\tilde{N}_{p}(f) \geqslant 0$ ), and Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative, with $\frac{1}{p} \in(0, \infty)$ ).
Case $\boldsymbol{p}=\infty$. Direct consequence of Lemma 1286 ( p-seminorm, $N_{\infty}(f) \in \overline{\mathbb{R}}_{+}$).
Therefore, $N_{p}$ is always nonnegative.
Lemma 1291 (link between $N_{1}$ and $N_{p}$. Let $(X, \Sigma, \mu)$ be a measure space.
Let $p \in(0, \infty)$. Let $f \in \mathcal{M}$, and $g \in \mathcal{M}_{+}$. Then, $|f|^{p}, g^{\frac{1}{p}} \in \mathcal{M}_{+} \subset \mathcal{M}$, and we have in $\overline{\mathbb{R}}_{+}$
(16.30) $\quad N_{p}^{p}(f)=\widetilde{N}_{p}(f)=N_{1}\left(|f|^{p}\right), \quad N_{p}(f)=N_{1}^{\frac{1}{p}}\left(|f|^{p}\right)$,
(16.31) $\quad N_{1}(g)=\int g d \mu=\int\left(g^{\frac{1}{p}}\right)^{p} d \mu=\widetilde{N}_{p}\left(g^{\frac{1}{p}}\right)=N_{p}^{p}\left(g^{\frac{1}{p}}\right), \quad$ and $\quad N_{1}^{\frac{1}{p}}(g)=N_{p}\left(g^{\frac{1}{p}}\right)$.

Proof. Direct consequence of Lemma $841\left(\mathcal{M}\right.$ is closed under power function, thus $|f|^{p}, g^{\frac{1}{p}} \in \mathcal{M}_{+}$, a subset of $\mathcal{M}$ ), Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)), Lemma 1286 ( $p$-seminorm, with $(p, f)$ and $\left(1,|f|^{p}\right)$, then $(1, g)$ and $\left(p, g^{\frac{1}{p}}\right)$ ), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication, with $\frac{1}{p}$ and $p$ ), and field properties of $\mathbb{R}\left(\right.$ thus $\left.\frac{1}{p} p=1\right)$.

Lemma 1292 (integrable is finite $N_{1}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in X \rightarrow \overline{\mathbb{R}}$. Then, $f$ is $\mu$-integrable in $\mathcal{M}$ iff $f \in \mathcal{M}$ and $N_{1}(f)<\infty$.

Proof. Direct consequence of Lemma 1249 (equivalent definition of integrability), and Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=} 1$ ).

Lemma 1293 ( $N_{\infty}$ bounded is almost bounded).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $a \in \overline{\mathbb{R}}_{+}$. Then, we have

$$
\begin{equation*}
N_{\infty}(f) \leqslant a \quad \Longleftrightarrow \quad|f|^{\mu \text { a.e. }} \leqslant a \tag{16.32}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1286 (p-seminorm), Lemma 1039 (nonnegative essential supremum is almost upper bound, $a \in C_{|f|}$ ), and Lemma 1046 (nonnegative essential supremum bounded by constant).

## Lemma 1294 (compatibility of $N_{p}$ with power function).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $\alpha \in[0, \infty)$. Let $f \in \mathcal{M}$.
Then, $|f|^{\alpha} \in \mathcal{M}_{+} \subset \mathcal{M}$ and $|f|^{\alpha} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$ when $f \in \mathcal{M}_{\mathbb{R}}$, and we have

$$
\begin{equation*}
\mu \text { nontrivial } \vee \alpha \neq 0 \quad \Longrightarrow \quad N_{p}^{\alpha}(f)=N_{\frac{p}{\alpha}}\left(|f|^{\alpha}\right) \tag{16.33}
\end{equation*}
$$

In particular, we have $N_{\infty}^{\alpha}(f)=N_{\infty}\left(|f|^{\alpha}\right)$ when $\alpha \neq 0$, and $N_{p}^{0}(f)=N_{\infty}\left(|f|^{0}\right)=1$ when $\mu$ is nontrivial.

Proof. From Lemma 841 ( $\mathcal{M}$ is closed under power function, with $p \stackrel{\text { def. }}{=} \alpha$ finite $),|f|^{\alpha}$ belongs to $\mathcal{M}_{+} \subset \mathcal{M}$ when $f \in \mathcal{M}$, and to $\mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$ when $f \in \overline{\mathcal{M}_{\mathbb{R}}}$.
Case $p, \alpha \in(0, \infty)$. From ordered field properties of $\mathbb{R}$ (thus $\frac{1}{p}, \frac{p}{\alpha} \in(0, \infty)$ ), Lemma 1286 ( $p$-seminorm, with $(p, f)$, then $\left(\frac{p}{\alpha},|f|^{\alpha}\right)$ ), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication, with $(p, q) \stackrel{\text { def. }}{=}\left(\frac{1}{p}, \alpha\right)$, then $\left(\alpha, \frac{p}{\alpha}\right)$ ), and Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory), with $p \stackrel{\text { def. } \alpha}{=}$ ), we have

$$
N_{p}^{\alpha}(f)=\left(\left(\int|f|^{p} d \mu\right)^{\frac{1}{p}}\right)^{\alpha}=\left(\int|f|^{p} d \mu\right)^{\frac{\alpha}{p}}=\left(\left.\left.\int| | f\right|^{\alpha}\right|^{\frac{p}{\alpha}} d \mu\right)^{\frac{\alpha}{p}}=N_{\frac{p}{\alpha}}\left(|f|^{\alpha}\right)
$$

Case $\boldsymbol{p}=\infty$ or $\boldsymbol{\alpha}=\mathbf{0}$. Then, from Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$, thus $\frac{1}{\alpha}=\infty$ ), Definition 326 (multiplication in $\overline{\mathbb{R}}$, with $p>0$, thus $\frac{p}{\alpha}=\infty$ ), and Lemma 1286 ( $p$-seminorm, with $\underline{p}$ def. $\infty)$, we have $N_{\frac{p}{\alpha}}=N_{\infty}$. Case $\boldsymbol{p}=\infty$ and $\boldsymbol{\alpha} \in(\mathbf{0}, \infty)$. Direct consequence of Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)), and Lemma 1054 (nonnegative essential supremum is compatible with power function, with any measure $\mu$, thus $N_{\infty}^{\alpha}(f)=N_{\infty}\left(|f|^{\alpha}\right)$ ). Case $\boldsymbol{\mu}$ nontrivial, and $\boldsymbol{\alpha}=\mathbf{0}$. Direct consequence of Lemma 431 $\overline{(\text { one-exponentiation }}$ property in $\overline{\mathbb{R}}$ (measure theory), thus $N_{p}^{0}(f)=|f|^{0}=1$ ), and Lemma 1040 (nonnegative essential supremum of constant function, thus $\left.N_{\infty}\left(|f|^{0}\right)=1=N_{p}^{0}(f)\right)$.
Therefore, the equality always holds.
Lemma 1295 (compatibility of $N_{p}$ with absolute value). Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{M}$. Then, $|f| \in \mathcal{M}_{+} \subset \mathcal{M}$ and we have $N_{p}(|f|)=N_{p}(f)$.

Proof. Direct consequence of Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$, thus $\frac{p}{1}=p$ ), Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 , thus $|f|^{1}=|f|$ and $N_{p}^{1}(f)=N_{p}(f)$ ), , and Lemma 1294 (compatibility of $N_{p}$ with power function, with $\alpha \stackrel{\text { def. }}{=}$ ).

## Lemma 1296 ( $N_{p}$ is almost definite).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Then, $N_{p}$ is almost definite:

$$
\begin{equation*}
\forall f \in \mathcal{M}, \quad N_{p}(f)=0 \quad \Longleftrightarrow \quad f \stackrel{\mu \text { a.e. }}{=} 0 \tag{16.34}
\end{equation*}
$$

Proof. Case $\boldsymbol{p} \in(\mathbf{0}, \boldsymbol{\infty})$. Direct consequence of Lemma 1286 ( $p$-seminorm), ordered field properties of $\mathbb{R}$ (thus $\frac{1}{p} \in(0, \infty)$ ), Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite, with $p \stackrel{\text { def. }}{=} \frac{1}{p}$ ), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite, with $f \stackrel{\text { def. }}{=}|f|^{p}$ ), and Lemma 1007 (power function is almost definite, with $\frac{1}{p}$ and $p$ ).

## Case $p=\infty$.

Direct consequence of Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=} \infty$ ), Lemma 1050 (nonnegative essential supremum is almost definite), and Lemma $10 \overline{05 \text { (absolute value is almost definite). }}$
Therefore, $N_{p}$ is always almost definite.
Lemma 1297 (compatibility of $N_{p}$ with almost equality). $\quad$ Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f, g \in \mathcal{M}$. Assume that $f \stackrel{\mu \text { a.e. }}{=} g$. Then, we have $N_{p}(f)=N_{p}(g)$.

Proof. From Lemma 985 (compatibility of almost equality with operator, with the absolute value and the power function of degree $p$ ), we have $|f| \stackrel{\mu \text { a.e. }}{=}|g|$, and $|f|^{p} \stackrel{\mu \text { a.e. }}{=}|g|^{p}$.
Case $\boldsymbol{p} \in(\mathbf{0}, \boldsymbol{\infty})$. Direct consequence of Lemma 1262 (compatibility of integral with almost equality), Lemma 361 (power function in $\overline{\mathbb{R}}$, with $\frac{1}{p}>0$ ), and Lemma 1286 (p-seminorm).
Case $\boldsymbol{p}=\infty$. Direct consequence of Lemma 842 ( $\mathcal{M}$ is closed under absolute value, $\left.|f|,|g| \in \mathcal{M}_{+}\right)$, Lemma 1049 (compatibility of nonnegative essential supremum with almost equality), and Lemma 1286 ( $p$-seminorm).
Therefore, the equality always holds.
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f, g \in \mathcal{M}$. Then, we have

$$
\begin{align*}
0 \leqslant f \leqslant g \quad & \Longrightarrow \quad N_{p}(f) \leqslant N_{p}(g)  \tag{16.35}\\
|f| \leqslant g & \Longrightarrow \quad N_{p}(f) \leqslant N_{p}(g)
\end{align*}
$$

Moreover, in both cases when $p$ is finite, $|g|^{p}$ is $\mu$-integrable implies $|f|^{p}$ is $\mu$-integrable.

Proof. Property (16.35). From Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), we have $f, g \in \mathcal{M}_{+}$. Case $\boldsymbol{p} \in(\mathbf{0}, \boldsymbol{\infty})$. Direct consequence of Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, with $f, g$ and $p$, then with $\int|f|^{p} d \mu, \int|g|^{p} d \mu$ and $\frac{1}{p}$ ), Lemma $841(\mathcal{M}$ is closed under power function), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and Lemma 1286 ( $p$-seminorm). Case $\boldsymbol{p}=\infty$. Direct consequence of Lemma 958 (everywhere is almost everywhere, thus $f \stackrel{\mu}{\leqslant}$ ), Lemma 1045 (nonnegative essential supremum is almost monotone), and Lemma 1286 ( $p$-seminorm).

Property (16.36). Direct consequence of Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative, thus $|f| \geqslant 0$ ), Lemma 1295 (compatibility of $N_{p}$ with absolute value, thus $|f| \in \mathcal{M}$, and $\left.N_{p}(|f|)=N_{p} \overline{(f)}\right)$, and Property (16.35) (with $f \stackrel{\text { def. }}{=}|f|$ ).

Integrability. Direct consequence of Properties (16.35) and (16.36) (thus $N_{p}(f) \leqslant N_{p}(g)$ ), Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, thus $N_{p}^{p}\left(|f|^{p}\right) \leqslant N_{p}^{p}\left(|g|^{p}\right)$ ), Lemma 1291 (link between $N_{1}$ and $N_{p}$, thus $N_{1}\left(|f|^{p}\right)=N_{p}^{p}(f)$ and $N_{p}^{p}(g)=N_{1}\left(|g|^{p}\right)$ ), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity), and Lemma 1292 (integrable is finite $N_{1}$, with $g$ and $f$ ).

Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f, g \in \mathcal{M}$. Then, we have

$$
\begin{align*}
0 \stackrel{\mu \text { a.e. }}{\leqslant} f \stackrel{\mu \text { a.e. }}{\leqslant} g & \Longrightarrow N_{p}(f) \leqslant N_{p}(g),  \tag{16.37}\\
|f| \stackrel{\mu \text { a.e. }}{\leqslant} g & \Longrightarrow N_{p}(f) \leqslant N_{p}(g) . \tag{16.38}
\end{align*}
$$

Moreover, in both cases when $p$ is finite, $|g|^{p}$ is $\mu$-integrable implies $|f|^{p}$ is $\mu$-integrable.
Proof. Direct consequence of Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions, the integrability property is unchanged), Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=}\{0,1\}, P_{0} \stackrel{\text { def. }}{=}(0 \leqslant f)$ and $P_{1} \stackrel{\text { def. }}{=}(f \leqslant g)$ ), Lemma 1297 (compatibility of $N_{p}$ with almost equality, thus $P_{2}(f, g) \stackrel{\text { def. }}{=}\left(N_{p}(f) \leqslant N_{p}(g)\right)$ is compatible with almost equality), and Lemma 965 (extended almost modus ponens (simplified 0), with card $(I)=2, P(f, g) \stackrel{\text { def. }}{=} P_{0} \wedge P_{1}$ and $S \stackrel{\text { def. }}{=} P_{2}$, then $P(f, g) \stackrel{\text { def. }}{=}(|f| \leqslant g)$ and the same $S)$.

Lemma 1300 ( $N_{p}$ is absolutely homogeneous). Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Then, $N_{p}$ is absolutely homogeneous of degree 1 :

$$
\begin{equation*}
\forall a \in \overline{\mathbb{R}}, \forall f \in \mathcal{M}, \quad a f \in \mathcal{M} \quad \wedge \quad N_{p}(a f)=|a| N_{p}(f) . \tag{16.39}
\end{equation*}
$$

Proof. From Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), we have af $\in \mathcal{M}$.
Case $\boldsymbol{p} \in(\mathbf{0}, \infty)$. Direct consequence of Lemma 1286 ( $p$-seminorm, thus $|a f|^{p},|f|^{p} \in \mathcal{M}_{+}$), Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory), thus $|a|^{p} \geqslant 0$ ), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous, for $a$ finite, thus $\tilde{N}_{p}$ is absolutely homogeneous of degree p), Lemma 361 (power function in $\overline{\mathbb{R}}$, thus $| \pm \infty|^{p}=\infty$ ), Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication, with $q \stackrel{\text { def. }}{=} \frac{1}{p}$ ), and field properties of $\mathbb{R}$.
Case $\boldsymbol{p}=\infty$. Direct consequence of Lemma 1286 ( $p$-seminorm, thus $|a f|,|f| \in \mathcal{M}_{+}$), Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative, thus $|a| \geqslant 0)$, Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication, $|a||f| \in \mathcal{M}_{+}$), and Lemma 1044 (nonnegative essential supremum is positive homogeneous).

Therefore, $N_{p}$ is always absolutely homogeneous.

Lemma 1301 ( $N_{p}$ of constant function).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $A \in \Sigma$. Let $a \in \overline{\mathbb{R}}$. Let $p \in(0, \infty)$. Then, $a \mathbb{1}_{A} \in \mathcal{M}$, and we have
(16.40) $\quad N_{p}\left(a \mathbb{1}_{A}\right)=|a|(\mu(A))^{\frac{1}{p}}, \quad$ and $\quad N_{\infty}\left(a \mathbb{1}_{A}\right)= \begin{cases}|a| & \text { if } \mu(A)>0, \\ 0 & \text { otherwise. }\end{cases}$

Proof. Direct consequence of Lemma 1289 ( $N_{p}$ of indicator function), Lemma 1300 ( $N_{p}$ is absolutely homogeneous), and Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ).

### 16.3.2 Hölder inequality

Lemma 1302 (generalized Hölder inequality (with $\infty$ )).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f, g \in \mathcal{M}$. Then, $f g \in \mathcal{M}$, and we have

$$
\begin{equation*}
N_{p}(f g) \leqslant N_{\infty}(f) N_{p}(g) \quad \wedge \quad N_{p}(f g) \leqslant N_{p}(f) N_{\infty}(g) \tag{16.41}
\end{equation*}
$$

Proof. From Lemma 824 ( $\mathcal{M}$ is closed under multiplication), Lemma 1286 ( $p$-seminorm, with $p$ and $\infty$ ), and Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), we have $f g \in \overline{\mathcal{M}}$, and $N_{p}(f g), N_{p}(f), N_{\infty}(f), N_{p}(g), N_{\infty}(g), N_{\infty}(f) N_{p}(g)$ and $N_{p}(f) N_{\infty}(g)$ exist and belong to $\overline{\mathbb{R}}_{+}$.

Case $\boldsymbol{p} \in(\mathbf{0}, \infty)$. From Lemma 1286 ( $\mathbf{p}$-seminorm), Lemma 1039 (nonnegative essential supremum is almost upper bound, with $f \in \mathcal{M})$, we have $|f| \stackrel{\mu \text { a.e. }}{\leqslant} N_{\infty}(f)$. Then, from Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)), Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), Lemma 984 (compatibility of almost binary relation with operator, with unary operator power function, then the unary operator right multiplication by $\left.|g(x)|^{p}\right)$, and Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), we have $|f g|^{p} \stackrel{\mu \text { a.e. }}{\leqslant} N_{\infty}^{p}(f)|g|^{p}$. Thus, from Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone), Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory), thus $N_{\infty}^{p}(f) \geqslant 0$ ), and Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear), we have

$$
N_{p}^{p}(f g)=\int|f g|^{p} d \mu \leqslant N_{\infty}^{p}(f) \int|g|^{p} d \mu=N_{\infty}^{p}(f) N_{p}^{p}(g)
$$

Hence, from Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, with $\frac{1}{p}$ ), Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication, with $q \stackrel{\text { def. }}{=} \frac{1}{p}$, and field properties of $\mathbb{R}$, we have $N_{p}(f g) \leqslant N_{\infty}(f) N_{p}(g)$. Then, by exchanging the roles of $f$ and $g$, and from Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)), we also have $N_{p}(f g) \leqslant N_{p}(f) N_{\infty}(g)$.

Case $\boldsymbol{p}=\infty$. Direct consequence of Lemma 1286 ( p-seminorm) , and Lemma 1053 (nonnegative essential supremum is submultiplicative).

Therefore, the inequalities always hold.
Remark 1303. Note that from now on, we only consider $p$-seminorms with $p \geqslant 1$.
Lemma 1304 (Hölder inequality). Let $(X, \Sigma, \mu)$ be a measure space. Let $p, q \in[1, \infty]$. Assume that $p$ and $q$ are Hölder conjugates. Let $f, g \in \mathcal{M}$. Then, $f g \in \mathcal{M}$, and we have

$$
\begin{equation*}
N_{1}(f g) \leqslant N_{p}(f) N_{q}(g) . \tag{16.42}
\end{equation*}
$$

Proof. From Lemma $824(\mathcal{M}$ is closed under multiplication), we have $f g \in \mathcal{M}$. Thus, from Definition 451 (Hölder conjugates), and Lemma 1286 ( $p$-seminorm), $N_{1}(f g), N_{p}(f)$, and $N_{q}(g)$ exist, and belong to $\overline{\mathbb{R}}_{+}$. Moreover, from Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory) ), we have $N_{p}(f) N_{q}(g) \in \overline{\mathbb{R}}_{+}$.
Case $\boldsymbol{N}_{\boldsymbol{p}}(\boldsymbol{f}) \boldsymbol{N}_{\boldsymbol{q}}(\boldsymbol{g})=\mathbf{0}$. Direct consequence of Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory), thus $N_{p}(f)=0$ or $\left.N_{q}(g)=0\right)$, Lemma $1296\left(N_{p}\right.$ is almost definite, thus $f \stackrel{\mu \text { a.e. }}{=} 0$ or $\left.g^{\mu \text { a.e. }}=0\right)$, Lemma 985 (compatibility of almost equality with operator, thus $f g \stackrel{\mu \text { a.e. }}{=} 0$ ), Lemma 1296 ( $N_{p}$ is almost definite, thus $N_{1}(f g)=0$ ), and the definition of order (reflexivity, thus $N_{1}(f g)=0 \leqslant 0=N_{p} \overline{\left.(f) N_{q}(g)\right)}$.
Case $N_{p}(f) N_{q}(g)=\infty$.


Case $(p, q)=(1, \infty)$, or $(p, q)=(\infty, 1)$.
Direct consequence of Lemma 1302 (generalized Hölder inequality (with $\infty$ ), with $p=1$ ).
Case $N_{p}(f) N_{q}(g) \in(0, \infty)$, and $p, q \in(1, \infty)$.
Then, from Lemma 1290 ( $N_{p}$ is nonnegative, with $p$ and $q$ ), Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory), contrapositive), and Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$, contrapositive), we have $N_{p}(f), N_{q}(g) \in(0, \infty)$. Thus, from ordered field properties of $\mathbb{R}$ $\left(1 / N_{p}(f), 1 / N_{q}(f) \in(0, \infty)\right.$ ), and Definition 326 (multiplication in $\left.\overline{\mathbb{R}}\right)$, let $\tilde{f} \stackrel{\text { def. }}{=} f / N_{p}(f)$ and $\tilde{g} \stackrel{\text { def. }}{=} g / N_{q}(g)$. Hence, from Lemma $1300\left(N_{p}\right.$ is absolutely homogeneous, with $p, q$ and 1$)$, Lemma $1290\left(N_{p}\right.$ is nonnegative, with $p$ and $\left.q\right)$, and valued field properties of $\mathbb{R}$, we have

$$
N_{p}(\tilde{f})=N_{q}(\tilde{g})=1, \quad \text { and } \quad N_{1}(\tilde{f} \tilde{g})=\frac{N_{1}(f g)}{N_{p}(f) N_{q}(g)}
$$

Let $x \in X$. Then, from Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative), and Lemma 453 (Young's inequality for products (measure theory)), we have in $\overline{\mathbb{R}}_{+}$

$$
|\tilde{f}(x) \tilde{g}(x)|=|\tilde{f}(x)||\tilde{g}(x)| \leqslant \frac{|\tilde{f}(x)|^{p}}{p}+\frac{|\tilde{g}(x)|^{q}}{q}
$$

Moreover, from Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), Lemma 824 ( $\mathcal{M}$ is closed under multiplication), Lemma 842 ( $\mathcal{M}$ is closed under absolute value), Lemma 841 ( $\mathcal{M}$ is closed under power function), ordered field properties of $\mathbb{R}$ ( thus $\frac{1}{p}, \frac{1}{q} \in(0,1) \subset \overline{\mathbb{R}}_{+}$), Lemma 845 $\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$, and Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition), we have

$$
\tilde{f}, \tilde{g}, \tilde{f} \tilde{g} \in \mathcal{M}, \quad \text { and } \quad|\tilde{f}|,|\tilde{g}|,|\tilde{f} \tilde{g}|,|\tilde{f}|^{p},|\tilde{g}|^{q}, \frac{|\tilde{f}|^{p}}{p}, \frac{|\tilde{g}|^{q}}{q}, \frac{|\tilde{f}|^{p}}{p}+\frac{|\tilde{g}|^{q}}{q} \in \mathcal{M}_{+} .
$$

Thus, from Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=} 1$ ), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), and Definition 451 (Hölder conjugates), we have

$$
\begin{aligned}
\frac{N_{1}(f g)}{N_{p}(f) N_{q}(g)}=N_{1}(\tilde{f} \tilde{g})=\int|\tilde{f} \tilde{g}| d \mu & \leqslant \int\left(\frac{|\tilde{f}|^{p}}{p}+\frac{|\tilde{g}|^{q}}{q}\right) d \mu \\
& =\frac{1}{p} \int|\tilde{f}|^{p} d \mu+\frac{1}{q} \int|\tilde{g}|^{q} d \mu=\frac{N_{p}(\tilde{f})}{p}+\frac{N_{q}(\tilde{g})}{q}=\frac{1}{p}+\frac{1}{q}=1
\end{aligned}
$$

Hence, from ordered field properties of $\mathbb{R}$, we have $N_{1}(f g) \leqslant N_{p}(f) N_{q}(g)$.
Therefore, the inequality always holds.
Lemma 1305 (generalized Hölder inequality).
Let $(X, \Sigma, \mu)$ be a measure space.
Let $p, p_{1}, p_{2} \in[1, \infty]$ such that $\frac{1}{p} \stackrel{\text { def. }}{=} \frac{1}{p_{1}}+\frac{1}{p_{2}} \leqslant 1$. Let $f_{1}, f_{2} \in \mathcal{M}$. Then, $f_{1} f_{2} \in \mathcal{M}$, and
(16.43)

$$
N_{p}\left(f_{1} f_{2}\right) \leqslant N_{p_{1}}\left(f_{1}\right) N_{p_{2}}\left(f_{2}\right)
$$

Proof. From Lemma $824\left(\mathcal{M}\right.$ is closed under multiplication), we have $f_{1} f_{2} \in \mathcal{M}$.
Case $\boldsymbol{p}_{\mathbf{1}}, \boldsymbol{p}_{\mathbf{2}} \in[\mathbf{1}, \boldsymbol{\infty})$. Then, $p$ is finite. Let $q_{1} \stackrel{\text { def. }}{=} \frac{p_{1}+p_{2}}{p_{2}}$, and $q_{2} \stackrel{\text { def. }}{=} \frac{p_{1}+p_{2}}{p_{1}}$.
Then, from ordered field properties of $\mathbb{R}$, and Definition 451 (Hölder conjugates), $q_{1}$ and $q_{2}$ are Hölder conjugates in $(1, \infty)$, and $\frac{p_{1}}{q_{1}}=\frac{p_{2}}{q_{2}}=\frac{p_{1} p_{2}}{p_{1}+p_{2}}=p$. Let $g_{1} \stackrel{\text { def. }}{=}\left|f_{1}\right|^{p}$, and $g_{2} \stackrel{\text { def. }}{=}\left|f_{2}\right|^{p}$.

Let $i \in\{1,2\}$. Then, from Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication), we have $\left|g_{i}\right|^{q_{i}}=\left|f_{i}\right|^{p_{i}}$.

Then, from Lemma $841\left(\mathcal{M}\right.$ is closed under power function, thus $\left.g_{1}, g_{2} \in \mathcal{M}_{+} \subset \mathcal{M}\right)$, Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication, thus $\left.g_{1} g_{2} \in \mathcal{M}_{+}\right)$, Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory), thus $g_{1} g_{2}=\left|f_{1} f_{2}\right|^{p}$ ), Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory), $\left\lfloor g_{1} g_{2} \mid=g_{1} g_{2}\right.$ ), Lemma 1291 (link between $N_{1}$ and $N_{p}$, with $\left(p, f_{1} f_{2}\right),\left(q_{i}, g_{i}\right)$ and $\left(p_{i}, f_{i}\right)$, with $i \in\{1,2\}$ ), Lemma 1304 (Hölder inequality, with $q_{1}, q_{2}$ and $\underline{g}_{1}, g_{2}$ ), field properties of $\mathbb{R}$, and Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, with $\frac{1}{p}$ ),

$$
\begin{aligned}
N_{p}^{p}\left(f_{1} f_{2}\right)=N_{1}\left(g_{1} g_{2}\right) & \leqslant N_{q_{1}}\left(g_{1}\right) N_{q_{2}}\left(g_{2}\right) \\
& =N_{1}^{\frac{1}{q_{1}}}\left(\left|g_{1}\right|^{q_{1}}\right) N_{1}^{\frac{1}{q_{2}}}\left(\left|g_{2}\right|^{q_{2}}\right)=N_{1}^{\frac{p_{2}}{p_{1}+p_{2}}}\left(\left|f_{1}\right|^{p_{1}}\right) N_{1}^{\frac{p_{1}}{p_{1}+p_{2}}}\left(\left|f_{2}\right|^{p_{2}}\right) \\
& =N_{p_{1}}^{\frac{p_{1} p_{2}}{p_{1}+p_{2}}}\left(f_{1}\right) N_{p_{2}}^{\frac{p_{1} p_{2}}{p_{2}+p_{2}}}\left(f_{2}\right)=N_{p_{1}}^{p}\left(f_{1}\right) N_{p_{2}}^{p}\left(f_{2}\right),
\end{aligned}
$$

and $N_{p}\left(f_{1} f_{2}\right) \leqslant N_{p_{1}}\left(f_{1}\right) N_{p_{2}}\left(f_{2}\right)$.
Case $\boldsymbol{p}_{\boldsymbol{i}}=\infty$ with $\boldsymbol{i} \in\{\mathbf{1}, \mathbf{2}\}$. Then, we have $p_{j}=p$ with $j=3-i$.
Direct consequence of Lemma 1302 (generalized Hölder inequality (with $\infty$ ), with $p_{j}, f_{1}$ and $f_{2}$, left inequality when $i=1$ and right inequality when $i=2$ ).
Therefore, the inequality always holds.

## Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty)$. Let $q \in(1, \infty]$. Assume that $p$ and $q$ are Hölder conjugates. Let $f \in \mathcal{M}$. Then, $|f|^{p-1},|f|^{0} \in \mathcal{M}$, and we have

$$
\begin{align*}
p>1 & \Longrightarrow \quad N_{q}\left(|f|^{p-1}\right)=N_{p}^{p-1}(f)  \tag{16.44}\\
\mu \text { nontrivial } & \Longrightarrow \quad N_{\infty}\left(|f|^{0}\right)=N_{1}^{0}(f)=1 \tag{16.45}
\end{align*}
$$

Proof. Case $p, q \in(1, \infty)$.
Then, from Definition 451 (Hölder conjugates), and field properties of $\mathbb{R}$ (with $q \neq 0$ ), we have $p q=p+q,(p-1) q=p$, and $\frac{p}{q}=p-1$. Hence, from Lemma 1291 (link between $N_{1}$ and $N_{p}, \underline{\text { with }}$ $\left(q,|f|^{p-1}\right)$, then $(p, f)$ ), Lemma 1286 ( $p$-seminorm, with $\left(q,|f|^{p-1}\right)$, then $(p, f)$ ), and Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication), we have

$$
\begin{gathered}
\widetilde{N}_{q}\left(|f|^{p-1}\right)=N_{1}\left(|f|^{(p-1) q}\right)=N_{1}\left(|f|^{p}\right)=\widetilde{N}_{p}(f), \quad \text { and } \\
N_{q}\left(|f|^{p-1}\right)=\widetilde{N}_{p}^{\frac{1}{q}}(f)=N_{p}^{\frac{p}{q}}(f)=N_{p}^{p-1}(f)
\end{gathered}
$$

Case $\mu$ nontrivial, and $p=1$.
Direct consequence of Definition 451 (Hölder conjugates, thus $q=\infty$ ), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative, thus $|f| \geqslant 0$ ), Lemma $1290\left(N_{p}\right.$ is nonnegative, thus $\left.N_{1}(f) \geqslant 0\right)$, Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory), thus $|f|^{0}=N_{1}^{0}(f)=1$ ), and Lemma 1040 (nonnegative essential supremum of constant function, with $\mu$ nontrivial, thus $\left.N_{\infty}\left(|f|^{0}\right)=1=N_{1}^{0}(f)\right)$.
Therefore, the equality always holds.
Remark 1307. Note that the previous lemma does not hold when $p=\infty$ (and $q=1$ ). Indeed, let $a, b \in \overline{\mathbb{R}}$ such that $a<b$, and let $f=\mathbb{1}_{[a, b)}$. Then, we have

$$
\begin{gathered}
N_{q}\left(f^{p-1}\right)=N_{1}\left(\mathbb{1}_{[a, b)}^{\infty}\right)=N_{1}\left(\mathbb{1}_{[a, b)}\right)=\mu([a, b)), \\
N_{p}^{p-1}(f)=N_{\infty}^{\infty}\left(\mathbb{1}_{[a, b)}\right)=1^{\infty}=1
\end{gathered}
$$

The equality of both previous expressions is not compatible with the $\sigma$-additivity property of the measure $\mu$; e.g. we would have

$$
\mu([0,2))=\mu([0,1))=\mu([1,2))=1, \quad \text { and } \quad \mu([0,2))=\mu([0,1) \uplus[1,2))=1+1=2
$$

### 16.3.3 Minkowski inequality

Remark 1308. In the next lemma, the summability domain $\mathcal{D}^{+}(f, g)$ and the almost sum $f+g$ are respectively defined in Definition 997, and Lemma 1002.

## Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ). <br> Let $(X, \Sigma, \mu)$ be a measure

 space. Let $p \in[1, \infty]$. Let $f, g, f^{\prime}, g^{\prime} \in \mathcal{M}$. Assume that $f+g$ is well-defined almost everywhere, $f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime} \stackrel{\mu \text { a.e. }}{=} g$, and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$. Then, $f^{\prime}+g^{\prime}, f^{\mu \text { a.e. }}+g \in \mathcal{M}$, and we have$$
\begin{equation*}
N_{p}\left(f^{\mu \text { a.e. }}+g\right)=N_{p}\left(f^{\prime}+g^{\prime}\right) \leqslant N_{p}\left(f^{\prime}\right)+N_{p}\left(g^{\prime}\right)=N_{p}(f)+N_{p}(g) \tag{16.46}
\end{equation*}
$$

Moreover, when $\mathcal{D}^{+}(f, g)=X$, e.g. when $f, g \in \mathcal{M}_{\mathbb{R}}, f+g \in \mathcal{M}$, and we have
(16.47)

$$
N_{p}(f+g) \leqslant N_{p}(f)+N_{p}(g) .
$$

Proof. From Lemma 1002 (almost sum, such $f^{\prime}$ and $g^{\prime}$ exist), and Lemma 1003 (compatibility of almost sum with almost equality), we have $f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$. Hence, from Lemma 1286 ( $p$-seminorm), and Lemma 1297 (compatibility of $N_{p}$ with almost equality), $N_{p}(f), N_{p}(g), N_{p}\left(f^{\prime}\right), N_{p}\left(g^{\prime}\right), N_{p}\left(f^{\prime}+g^{\prime}\right), N_{p}\left(f_{+}^{\mu \text { a.e. }} g\right)$ are well-defined in $\overline{\mathbb{R}}_{+}$, and we have

$$
\begin{gathered}
N_{p}\left(f^{\prime}\right)=N_{p}(f), \quad N_{p}\left(g^{\prime}\right)=N_{p}(g) \quad\left(\text { thus } N_{p}\left(f^{\prime}\right)+N_{p}\left(g^{\prime}\right)=N_{p}(f)+N_{p}(g)\right), \\
\text { and } \quad N_{p}\left(f^{\prime}+g^{\prime}\right)=N_{p}\left(f_{\stackrel{\mu \text { a.e. }}{+} g)}\right.
\end{gathered}
$$

Moreover, from Lemma $841(\mathcal{M}$ is closed under power function $)$, and Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition), for all $p \in[1, \infty)$, we have

$$
|f|^{p},|g|^{p},\left|f^{\prime}\right|^{p},\left|g^{\prime}\right|^{p},\left|f^{\prime}+g^{\prime}\right|^{p},\left|f^{\prime}\right|^{p}+\left|g^{\prime}\right|^{p},|f \stackrel{\mu \text { a.e. }}{+} g|^{p} \in \mathcal{M}_{+} \subset \mathcal{M}
$$

Case $\boldsymbol{N}_{\boldsymbol{p}}(\boldsymbol{f})=\infty$ or $\boldsymbol{N}_{\boldsymbol{p}}(\boldsymbol{g})=\infty$. Direct consequence of Lemma 1290 ( $N_{p}$ is nonnegative),

Case $\boldsymbol{N}_{\boldsymbol{p}}\left(\boldsymbol{f}^{\prime}+\boldsymbol{g}^{\prime}\right)=\mathbf{0}$. Direct consequence of Lemma 1290 ( $N_{p}$ is nonnegative), and Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed, $\left.N_{p}(f)+N_{p}(g) \geqslant 0\right)$.
Case $\boldsymbol{p}=1$. Direct consequence of Lemma 1286 ( $p$-seminorm, with $f^{\prime}+g^{\prime}$, then $f^{\prime}$ and $g^{\prime}$ ), and Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ).
Case $\boldsymbol{p}=\boldsymbol{\infty}$. Direct consequence of Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality, with $f^{\prime}+g^{\prime}$ well-defined everywhere), Lemma 958 (everywhere is almost everywhere), Lemma 1045 (nonnegative essential supremum is almost monotone, with $f \stackrel{\text { def. }}{=}\left|f^{\prime}+g^{\prime}\right|, g \stackrel{\text { def. }}{=}\left|f^{\prime}\right|+\left|g^{\prime}\right| \in \mathcal{M}_{+}$), Lemma 1051 (nonnegative essential supremum is subadditive, with $f \stackrel{\text { def. }}{=}\left|f^{\prime}\right|, g \stackrel{\text { def. }}{=}\left|g^{\prime}\right| \in \mathcal{M}_{+}$), and Lemma 1286 ( $p$-seminorm).
Case $\boldsymbol{p} \in(1, \infty), \boldsymbol{N}_{\boldsymbol{p}}(\boldsymbol{f}), \boldsymbol{N}_{\boldsymbol{p}}(\boldsymbol{g})<\infty$, and $\boldsymbol{N}_{\boldsymbol{p}}\left(\boldsymbol{f}^{\prime}+\boldsymbol{g}^{\prime}\right)>\mathbf{0}$. From Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), and Lemma 371 (power function satisfies extended relaxed triangle inequality, with $f^{\prime}+g^{\prime}$ well-defined everywhere), we have

$$
0 \leqslant\left|f^{\prime}+g^{\prime}\right|^{p} \leqslant 2^{p-1}\left(\left|f^{\prime}\right|^{p}+\left|g^{\prime}\right|^{p}\right)
$$

Then, from Lemma 1286 ( p-seminorm), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear, additivity and positive homogeneity), Lemma 367 (power func-
 patibility of the power function in $\mathbb{R}$ with multiplication $\left(\left(2^{p-1}\right)^{\frac{1}{p}}=2^{1-\frac{1}{p}}\right)$, Lemma 1291
(link between $N_{1}$ and $N_{p}$ ), closedness of the power function in $\mathbb{R}_{+}$(with powers $1-\frac{1}{p}$, $p$ and $\frac{1}{p}$, all in $(0, \infty)$ ), closedness of addition in $\mathbb{R}_{+}$, and closedness of multiplication in $\mathbb{R}_{+}$(in short, all terms in the right-hand side are finite), we have

$$
\begin{gathered}
\widetilde{N}_{p}\left(f^{\prime}+g^{\prime}\right)=\int\left|f^{\prime}+g^{\prime}\right|^{p} d \mu \leqslant 2^{p-1}\left(\widetilde{N}_{p}\left(f^{\prime}\right)+\widetilde{N}_{p}\left(g^{\prime}\right)\right), \quad \text { and } \\
N_{p}\left(f^{\prime}+g^{\prime}\right)=\widetilde{N}_{p}^{\frac{1}{p}}\left(f^{\prime}+g^{\prime}\right) \leqslant 2^{1-\frac{1}{p}}\left(N_{p}^{p}\left(f^{\prime}\right)+N_{p}^{p}\left(g^{\prime}\right)\right)^{\frac{1}{p}}<\infty .
\end{gathered}
$$

Moreover, from Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition, with $p \stackrel{\text { def. }}{=} 1$ and $q \stackrel{\text { def. }}{=} p-1>0$ ), Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality, with $f^{\prime}+g^{\prime}$ well-defined everywhere), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative), Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory), with $p-1>0$ ), Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)), Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), and Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),

$$
\left|f^{\prime}+g^{\prime}\right|^{p}=\left|f^{\prime}+g^{\prime}\right|\left|f^{\prime}+g^{\prime}\right|^{p-1} \leqslant\left(\left|f^{\prime}\right|+\left|g^{\prime}\right|\right)\left|f^{\prime}+g^{\prime}\right|^{p-1}=\left|f^{\prime}\right| f^{\prime}+\left.g^{\prime}\right|^{p-1}\left|+\left|g^{\prime}\right| f^{\prime}+g^{\prime}\right|^{p-1} \mid .
$$

Hence, from Lemma 1286 ( $p$-seminorm, with $p$, then 1 ), Lemma 841 ( $\mathcal{M}$ is closed under power function, with $p-1 \in(0, \infty))$, Lemma $824\left(\mathcal{M}\right.$ is closed under multiplication, $f^{\prime}\left|f^{\prime}+g^{\prime}\right|^{p-1}$ and $g^{\prime}\left|f^{\prime}+g^{\prime}\right|^{p-1}$ belong to $\mathcal{M}$ ), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), and Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), we have

$$
\begin{aligned}
\tilde{N}_{p}\left(f^{\prime}+g^{\prime}\right)=\int\left|f^{\prime}+g^{\prime}\right|^{p} d \mu & \leqslant \int\left|f^{\prime}\right| f^{\prime}+\left.g^{\prime}\right|^{p-1}\left|d \mu+\int\right| g^{\prime}\left|f^{\prime}+g^{\prime}\right|^{p-1} \mid d \mu \\
& =N_{1}\left(f^{\prime}\left|f^{\prime}+g^{\prime}\right|^{p-1}\right)+N_{1}\left(g^{\prime}\left|f^{\prime}+g^{\prime}\right|^{p-1}\right)
\end{aligned}
$$

Let $q$ be the Hölder conjugate of $p$. Then, from Lemma 1291 (link between $N_{1}$ and $N_{p}$ ), Lemma 1304 (Hölder inequality, with $\left(f,\left|f^{\prime}+g^{\prime}\right|^{p-1}\right)$ and $\left(g,\left|f^{\prime}+g^{\prime}\right|^{p-1}\right)$ ), Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), and Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates, with $p \in(1, \infty)$ ), we have

$$
\begin{aligned}
N_{p}^{p}\left(f^{\prime}+g^{\prime}\right)=\widetilde{N}_{p}\left(f^{\prime}+g^{\prime}\right) & \leqslant\left(N_{p}\left(f^{\prime}\right)+N_{p}\left(g^{\prime}\right)\right) N_{q}\left(\left|f^{\prime}+g^{\prime}\right|^{p-1}\right) \\
& =\left(N_{p}\left(f^{\prime}\right)+N_{p}\left(g^{\prime}\right)\right) N_{p}^{p-1}\left(f^{\prime}+g^{\prime}\right)
\end{aligned}
$$

Hence, from ordered field properties of $\mathbb{R}\left(\right.$ with $\left.N_{p}^{p-1}\left(f^{\prime}+g^{\prime}\right) \in(0, \infty)\right)$, we have

$$
N_{p}\left(f^{\prime}+g^{\prime}\right) \leqslant N_{p}\left(f^{\prime}\right)+N_{p}\left(g^{\prime}\right)
$$

Therefore, the inequality always holds.
The specific case $\mathcal{D}^{+}(f, g)=X$ is a direct consequence of Lemma 1004 (almost sum is sum).
Lemma 1310 (Minkowski equality (nonnegative functions)). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $f, g, f^{\prime}, g^{\prime} \in \mathcal{M}$. Assume that $f, g \stackrel{\mu \text { a.e. }}{\geqslant} 0, f^{\prime} \stackrel{\mu \text { a.e. }}{=} f, g^{\prime \mu} \stackrel{\mu \text { a.e. }}{=} g$ and $\mathcal{D}^{+}\left(f^{\prime}, g^{\prime}\right)=X$. Then, $f+g$ is well-defined almost everywhere, $f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and we have

$$
\begin{equation*}
N_{1}\left(f^{\mu \text { a.e. }}+g\right)=N_{1}\left(f^{\prime}+g^{\prime}\right)=N_{1}\left(f^{\prime}\right)+N_{1}\left(g^{\prime}\right)=N_{1}(f)+N_{1}(g) \tag{16.48}
\end{equation*}
$$

Moreover, when $\mathcal{D}^{+}(f, g)=X$, e.g. when $f, g \in \mathcal{M}_{+}, f+g \in \mathcal{M}$, and we have

$$
\begin{equation*}
N_{1}(f+g)=N_{1}(f)+N_{1}(g) . \tag{16.49}
\end{equation*}
$$

Proof. From Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions), and Lemma 1003 (compatibility of almost sum with almost equality), $f+g$ is well-defined almost everywhere, $f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}, f^{\prime}, g^{\prime}, f^{\prime}+g^{\prime}, f \stackrel{\mu \text { a.e. }}{+} g \stackrel{\mu \text { a.e. }}{\geqslant} 0$, $f^{\prime}+g^{\prime} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g$, and we have

$$
\int\left(f^{\prime}+g^{\prime}\right)^{+} d \mu=\int\left(f^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{+} d \mu
$$

Then, from Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),

$$
\int\left(f^{\prime}\right)^{-} d \mu=\int\left(g^{\prime}\right)^{-} d \mu=\int\left(f^{\prime}+g^{\prime}\right)^{-} d \mu=0
$$

Hence, from Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=}$ ), and Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts), we have

$$
\begin{aligned}
N_{1}\left(f^{\prime}+g^{\prime}\right) & =\int\left|f^{\prime}+g^{\prime}\right| d \mu=\int\left(f^{\prime}+g^{\prime}\right)^{+} d \mu+\int\left(f^{\prime}+g^{\prime}\right)^{-} d \mu \\
& =\int\left(f^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{+} d \mu=\int\left(f^{\prime}\right)^{+} d \mu+\int\left(f^{\prime}\right)^{-} d \mu+\int\left(g^{\prime}\right)^{+} d \mu+\int\left(g^{\prime}\right)^{-} d \mu \\
& =\int\left|f^{\prime}\right| d \mu+\int\left|g^{\prime}\right| d \mu=N_{1}\left(f^{\prime}\right)+N_{1}\left(g^{\prime}\right) .
\end{aligned}
$$

Therefore, from Lemma 1297 (compatibility of $N_{p}$ with almost equality, with $\left(f^{\mu \text { a.e. }} \quad g, f^{\prime}+g^{\prime}\right)$, then $\left(f^{\prime}, f\right)$, and $\left.\left(g^{\prime}, g\right)\right)$, we have the equalities.

## Lemma 1311 (Minkowski equality (linearly dependent functions)).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $f, g \in \mathcal{M}$. Let $a \in \overline{\mathbb{R}}_{+}$. Assume that $f=a g$. Then, $f+g$ is well-defined, belongs to $\mathcal{M}$, and we have

$$
\begin{equation*}
N_{p}(f+g)=N_{p}(f)+N_{p}(g) . \tag{16.50}
\end{equation*}
$$

Proof. From Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication, ag $\in \mathcal{M}$ ), Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)), Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory), thus $g(x) \geqslant 0 \Rightarrow a g(x) \geqslant 0$, and $g(x) \leqslant 0 \Longrightarrow a g(x) \leqslant 0)$, and Lemma 822 ( $\mathcal{M}$ is closed under addition when defined, thus $g$ and $a g$ never take opposite infinite values at the same point), $f+g$ is well-defined, and belongs to $\mathcal{M}$.

Therefore, from Definition 317 (addition in $\overline{\mathbb{R}}$, rule 1), Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
 in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), we have $|a+1|=a+1=|a|+1$, and

$$
N_{p}(f+g)=(|a|+1) N_{p}(g)=N_{p}(f)+N_{p}(g)
$$

Lemma 1312 (Minkowski equality (almost linearly dependent functions)). Let ( $X, \Sigma, \mu$ ) be a measure space. Let $p \in[1, \infty]$. Let $f, g \in \mathcal{M}$. Let $a \in \overline{\mathbb{R}}_{+}$. Assume that $f \stackrel{\mu \text { a.e. }}{=}$ ag. Then, $f+g$ is well-defined almost everywhere, $f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, and we have

$$
\begin{equation*}
N_{p}(f \stackrel{\mu \text { a.e. }}{+} g)=N_{p}(f)+N_{p}(g) \tag{16.51}
\end{equation*}
$$

Proof. From Lemma 955 (equivalent definition of property almost satisfied), let $B \in \Sigma$ such that $B \subset\{f=a g\}$ and $\mu\left(B^{c}\right)=0$. Let $\tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{B}$, and $\tilde{g} \stackrel{\text { def. }}{=} g \mathbb{1}_{B}$. Then, from Lemma 833
(measurability and masking, with $f$ and $g$ ), and Lemma 1008 (masking almost nowhere, with $f$ and $g$ ), we have $\tilde{f}, \tilde{g} \in \mathcal{M}, \tilde{f} \stackrel{\tilde{\mu} \text { a.e. }}{=} f$ and $\tilde{g}{ }^{\mu} \xlongequal{\mu} \xlongequal{\text { a.e. }} g$.

Let $x \in X$. Case $\boldsymbol{x} \in \boldsymbol{B}$. Then, from the definition of the indicator function, we have $\tilde{f}(x)=f(x)=a g(x)=a \tilde{g}(x)$. Case $x \notin B$. Then, from the definition of the indicator function, and Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)), we have $\tilde{f}(x)=0=a \tilde{g}(x)$. Hence, we have $\tilde{f}=a \tilde{g}$, and from Lemma 1311 (Minkowski equality (linearly dependent functions)), $\tilde{f}+\tilde{g}$ is well-defined, belongs to $\mathcal{M}$, and we have $N_{p}(\tilde{f}+\tilde{g})=N_{p}(\tilde{f})+N_{p}(\tilde{g})$.

Therefore, from Lemma 1003 (compatibility of almost sum with almost equality), and Lemma 1297 (compatibility of $N_{p}$ with almost equality), we have $\tilde{f}+\tilde{g} \stackrel{\mu \text { a.e. }}{=} f \stackrel{\mu \text { a.e. }}{+} g, f \stackrel{\mu \text { a.e. }}{+} g \in \mathcal{M}$, $f+g$ is well-defined almost everywhere, $N_{p}\left(f^{\mu \text { a.e. }} g\right)=N_{p}(\tilde{f}+\tilde{g}), N_{p}(\tilde{f})=N_{p}(f), N_{p}(\tilde{g})=N_{p}(g)$, and the equality is satisfied.

### 16.3.4 The seminormed vector space $\mathcal{L}^{p}$

## Remark 1313.

Note that in the beginning of this section, we consider again $p$-seminorms with $p>0$.
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. The vector space of $p$-powered integrable functions (for $p<\infty$ ), or the vector space of essentially bounded measurable functions (for $p=\infty$ ), is denoted $\mathcal{L}^{p}(X, \Sigma, \mu)$ (or simply $\mathcal{L}^{p}$ ); it is defined by
(16.52)

$$
\mathcal{L}^{p}(X, \Sigma, \mu) \stackrel{\text { def. }}{=}\left\{f \in \mathcal{M}_{\mathbb{R}} \mid N_{p}(f)<\infty\right\} .
$$

## Remark 1315.

The set $\mathcal{L}^{p}$ is shown below to be a (seminormed) vector space when $p \geqslant 1$, hence its name.

## Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $p \in(0, \infty)$.
Then, $f \in \mathcal{L}^{p}$ iff $f \in \mathcal{M}$ is finite, and $|f|^{p}$ is $\mu$-integrable in $\mathcal{M}_{+}$, i.e. $\int|f|^{p} d \mu<\infty$.
In particular, $f \in \mathcal{L}^{1}$ iff $f$ is $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$ iff $f \in \mathcal{M}$ is finite and $\int|f| d \mu<\infty$.
Proof. Direct consequence of Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ), Lemma 1286 (p-seminorm), Lemma 363 (finite power function in $\overline{\mathbb{R}})$, and Lemma 841 ( $\mathcal{M}$ is closed under power function).

## Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure).

Let $(X, \Sigma)$ be a measurable space. Let $p \in(0, \infty]$. Then, we have $\mathcal{L}^{p}(X, \Sigma, 0)=\mathcal{M}_{\mathbb{R}}$.
Proof. Direct consequence of Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma 1288 ( $N_{p}$ with trivial measure), and Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).

Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$). Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{L}^{p}$. Assume that $f \geqslant 0$. Then, we have $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+} \subset \mathcal{M}$.

Proof. Direct consequence of Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $\left.f \in \mathcal{M}_{\mathbb{R}}\right)$, Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}\right)$, and Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).

Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $\mathcal{I}: \mathcal{L}^{1} \rightarrow \mathbb{R}$ be the function defined by: for all $f \in \mathcal{L}^{1}, \mathcal{I}(f) \stackrel{\text { def. }}{=} \int f d \mu$. Then, $\mathcal{I}$ is a nondecreasing positive linear form on $\mathcal{L}^{1}$.

Moreover, for all $f \in \mathcal{L}^{1}$, we have $|\mathcal{I}(f)| \leqslant \mathcal{I}(|f|)=N_{1}(f)$.
Proof. (0). Existence. Direct consequence of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=} 1$ ), and Lemma 1256 (integral in $\mathcal{M}$, thus $\mathcal{I}(f)$ exists and belongs to $\mathbb{R}$ ).
(1). Monotonicity. Direct consequence of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $f, g \in \mathcal{L}^{1}$ are $\mu$-integrable in $\mathcal{M}_{\mathbb{R}}$ ), and Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone).
(2). Nonnegativeness. Direct consequence of Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$, thus

(3). Linearity. Direct consequence of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ), and Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear).
Therefore, from Definition 64 (linear map), and Definition 66 (linear form), $\mathcal{I}$ is a positive linear form on $\mathcal{L}^{1}$.
(4). Inequality. Direct consequence of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ), Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value), and Lemma 1286 (p-seminorm).

Lemma 1320 (integral in $\mathcal{L}^{1}$ is almost monotone).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f, g \in \mathcal{L}^{1}$. Then, we have

$$
\begin{equation*}
f \stackrel{\mu \text { a.e. }}{\leqslant} g \quad \Longrightarrow \quad \int f d \mu \leqslant \int g d \mu \quad(\text { in } \mathbb{R}) \tag{16.53}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=} 1$, thus $f, g$ are $\mu$-integrable), and Lemma 1272 (integral in $\mathcal{M}$ is almost monotone).

Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty)$. Let $f \in \mathcal{M}$, and $g \in \mathcal{M}_{+}$. Then, $|f|^{p}, g^{\frac{1}{p}} \in \mathcal{M}_{+} \subset \mathcal{M}$, we have in $\overline{\mathbb{R}}_{+}$

$$
\begin{align*}
& N_{p}^{p}(f)=\tilde{N}_{p}(f)=N_{1}\left(|f|^{p}\right), \quad N_{p}(f)=N_{1}^{\frac{1}{p}}\left(|f|^{p}\right),  \tag{16.54}\\
& N_{1}(g)=\int g d \mu=\int\left(g^{\frac{1}{p}}\right)^{p} d \mu=\widetilde{N}_{p}\left(g^{\frac{1}{p}}\right)=N_{p}^{p}\left(g^{\frac{1}{p}}\right), \quad N_{1}^{\frac{1}{p}}(g)=N_{p}\left(g^{\frac{1}{p}}\right), \tag{16.55}
\end{align*}
$$

and the equivalences

$$
\begin{equation*}
f \in \mathcal{L}^{p} \Longleftrightarrow|f|^{p} \in \mathcal{L}^{1} \Longleftrightarrow|f|^{p} \text { is finite and } \mu \text {-integrable in } \mathcal{M}_{+}, \tag{16.56}
\end{equation*}
$$

$$
\text { (16.57) } \quad g \in \mathcal{L}^{1} \Longleftrightarrow g^{\frac{1}{p}} \in \mathcal{L}^{p} \Longleftrightarrow g^{\frac{1}{p}} \text { is finite, and } g \text { is } \mu \text {-integrable in } \mathcal{M}_{+} \text {. }
$$

Proof. Direct consequence of Lemma 1291 (link between $N_{1}$ and $N_{p}$ ), Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ), and Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication).

Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{L}^{p}$. Then, we have $|f| \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$.
Proof. Direct consequence of Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value, thus $\left.|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}\right)$, Lemma 1295 (compatibility of $N_{p}$ with absolute value, thus $\left.N_{p}(|f|)<\infty\right)$.

## Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in(0, \infty]$. Let $f \in \mathcal{M}$. Then, we have the equivalence

$$
f \in \mathcal{L}^{p} \quad \Longleftrightarrow \quad \exists g \in \mathcal{L}^{p},|f| \leqslant g
$$

Proof. "Left" implies "right". Direct consequence of Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value, $g \stackrel{\text { def. }}{=}|f| \in \mathcal{L}^{p}$ ), and reflexivity of order in $\mathbb{R}$.
"Right" implies "left". Let $g \in \mathcal{L}^{p}$. Assume that $|f| \leqslant g$. Then, from Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $g \in \mathcal{M}$ is finite), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative), Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity, with $0 \leqslant|f| \leqslant g=|g|<\infty)$, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, with $|f|$ and $g$ ), Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$, thus $f$ is finite), and Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$, with $|f|$ and $|f|$ ), we have $|f| \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}, g=|g| \in \mathcal{M}_{+}$, and $f \in \mathcal{M}_{\mathbb{R}}$.
Case $\boldsymbol{p} \in(\mathbf{0}, \infty)$. From Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone, with $p$, then $\frac{1}{p}$, both in $(0, \infty)$ ), Lemma $841\left(\mathcal{M}\right.$ is closed under power function, $|f|^{p},|g|^{p} \in \mathcal{M}_{+}$), and Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), we have $|f|^{p} \leqslant g^{p}=|g|^{p}$, and $\left(\int|f|^{p} d \mu\right)^{\frac{1}{p}} \leqslant\left(\int|g|^{p} d \mu\right)^{\frac{1}{p}}$.
Case $\boldsymbol{p}=\infty$. From Lemma 1045 (nonnegative essential supremum is almost monotone), and Lemma 958 (everywhere is almost everywhere), we have ess sup ${ }^{+}|f| \leqslant \operatorname{ess}^{2} \sup ^{+} g=\operatorname{ess}^{\sup }{ }^{+}|g|$.

Hence, in both cases, from Lemma 1286 ( $p$-seminorm), Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, with $g$, then $f$ ), and we have $N_{p}(f) \leqslant N_{p}(g)<\infty$, and $f \in \mathcal{L}^{p}$.
Therefore, we have the equivalence.

Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}$. Let $p \in(0, \infty]$. Assume that $N_{p}(f)$ is finite, i.e. that $|f|^{p}$ is $\mu$-integrable in $\mathcal{M}_{+}$when $p$ is finite, or that $f$ is essentially bounded when $p \stackrel{\text { def. }}{=} \infty$. Then, there exists $\tilde{f} \in \mathcal{L}^{p}$ such that $f \stackrel{\mu}{ } \stackrel{\text { a.e. }}{=} \tilde{f}$ and $N_{p}(f)=N_{p}(\tilde{f})<\infty$.

Proof. Let $A \stackrel{\text { def. }}{=} f^{-1}(\mathbb{R})$, and $\tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A}$. Then, from Lemma $821(\mathcal{M}$ is closed under finite part), we have $A \in \Sigma$ and $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$.
Case $\boldsymbol{p} \in(\mathbf{0}, \boldsymbol{\infty})$. Then, from Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory), thus $|f|^{p}= \pm \infty \Leftrightarrow|f|^{p}=\infty$ ), Lemma 1251 (integrable is almost finite, thus $\mu\left(\left\{|f|^{p}=\infty\right\}\right)=0$ ), and Lemma 363 (finite power function in $\overline{\mathbb{R}}$, thus $A^{c}=\left\{|f|^{p}=\infty\right\}$ ), we have $\mu\left(A^{c}\right)=0$.
Case $\boldsymbol{p}=\infty$. Then, from Lemma 1048 (essentially bounded is almost finite), we have $\mu\left(A^{c}\right)=0$. Therefore, in both cases, from Lemma 1008 (masking almost nowhere, thus $\tilde{f}^{\mu} \stackrel{\text { a.e. }}{=} f$ ), Lemma 1297 (compatibility of $N_{p}$ with almost equality), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have $N_{p}(\tilde{f})=N_{p}(f)<\infty$, and $\tilde{f} \in \mathcal{L}^{p}$.

Remark 1325. From the previous lemma, we can restrict the study of integrable functions to the sole case of real-valued functions, and rely on the finite part, see Definition 526.

Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $f \in \mathcal{M}_{+}$. Assume that $\int f d \mu$ is finite
Then, there exists $\tilde{f} \in \mathcal{L}^{1} \cap \mathcal{M}_{+}$such that $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$ and $N_{1}(f)=N_{1}(\tilde{f})<\infty$.
Proof. From Lemma 1172 (integral in $\mathcal{M}_{+}$, thus $f$ is $\mu$-integrable in $\mathcal{M}_{+}$), Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ), Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=} 1$ ), properties of the absolute value (thus $|f|=f$ ), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), there exists $\tilde{f} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$such that

$$
f \stackrel{\mu \text { a.e. }}{=} \tilde{f}, \quad N_{1}(\tilde{f})=\int \tilde{f} d \mu=\int f d \mu<\infty, \quad \text { and } \quad \tilde{f} \in \mathcal{L}^{1}
$$

## Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p$ in $(0, \infty]$.
Then, $\mathcal{L}^{p}$ is closed under scalar multiplication, and $N_{p}$ is absolutely homogeneous of degree 1.
Proof. Let $a \in \mathbb{R}$. Let $f \in \mathcal{L}^{p} \subset \mathcal{M}_{\mathbb{R}}$. Then, from Definition $1314\left(\mathcal{L}^{p}\right.$, vector space of $p$-powered integrable functions), we have $N_{p}(f)<\infty$. Hence, from Lemma $814\left(\mathcal{M}_{\mathfrak{K}}\right.$ is vector space, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, thus af $\in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1300 ( $N_{p}$ is absolutely homogeneous), and closedness of the absolute value and multiplication in $\mathbb{R}$, we have $N_{p}(a f)=|a| N_{p}(f)<\infty$.

Therefore, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions), af $\in \mathcal{L}^{p}$, and $N_{p}$ is absolutely homogeneous of degree 1.

Lemma 1328 (constant function is $\mathcal{L}^{p}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $A \in \Sigma$. Let $a \in \mathbb{R}$. Let $p \in(0, \infty)$. Then, $a \mathbb{1}_{A} \in \mathcal{L}^{\infty}$, and when $\mu(A)<\infty, a \mathbb{1}_{A} \in \mathcal{L}^{p}$, and we have

$$
\begin{equation*}
\int a \mathbb{1}_{A} d \mu=a \mu(A) \quad \in \mathbb{R} \tag{16.59}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1301 ( $N_{p}$ of constant function, thus $a \mathbb{1}_{A} \in \mathcal{M}$ and equalities on $p$-seminorms), closedness of multiplication in $\mathbb{R}$, Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$, thus $\left.\overline{a \mathbb{1}_{A} \in \mathcal{M}_{\mathbb{R}}}\right)$, closedness of the power function in $\mathbb{R}$ (thus $\left.N_{p}\left(a \mathbb{1}_{A}\right), N_{\infty}\left(a \mathbb{1}_{A}\right)<\infty\right)$, Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions, with $p \in(0, \infty)$ and $\left.p \stackrel{\text { def. }}{=} \infty\right)$, and Lemma 1267 (integral in $\mathcal{M}$ of constant function, thus (16.59) holds).

## Lemma 1329 (first mean value inequality).

Let $(X, \Sigma, \mu)$ be a measure space.
Let $A \in \Sigma$. Let $f \in \mathcal{M}$. Let $m, M \in \mathbb{R}$. Let $p \in(0, \infty)$. Assume that $m \leqslant f_{\left.\right|_{A}} \leqslant M$.
Then, $f \mathbb{1}_{A} \in \mathcal{L}^{\infty}$, and when $\mu(A)<\infty, f \mathbb{1}_{A} \in \mathcal{L}^{p}$, and we have

$$
\begin{equation*}
m \mu(A) \leqslant \int_{A} f d \mu \leqslant M \mu(A) \tag{16.60}
\end{equation*}
$$

For instance, this holds with $m \stackrel{\text { def. }}{=} \inf (f(A))$ and $M \stackrel{\text { def. }}{=} \sup (f(A))$ when they are both finite, and then, inequalities are strict iff $f_{\left.\right|_{A}}$ is not equal to one of its bounds $\mu_{A}$-almost everywhere.

Proof. From Lemma 913 (measure, nonnegativeness), we have $\mu(A) \geqslant 0$.
Let $a \in \mathbb{R}$. Then, from Lemma 1328 (constant function is $\mathcal{L}^{p}$ ), we have $a \mathbb{1}_{A} \in \mathcal{L}^{\infty}$, and when $\mu(A)<\infty, a \mathbb{1}_{A} \in \mathcal{L}^{p}$, and $\int a \mathbb{1}_{A} d \mu=a \mu(A) \in \mathbb{R}$.

Thus, from ordered field properties of $\mathbb{R}$ (thus $a \stackrel{\text { def. }}{=} \max (-m, M) \in \mathbb{R})$, and the previous result, we have $g \stackrel{\text { def. }}{=} \max (-m, M) \mathbb{1}_{A} \in \mathcal{L}^{\infty}, g \in \mathcal{L}^{p}$ when $\mu(A)<\infty$, and $\int g d \mu$ equals $\max (-m, M) \mu(A) \in \mathbb{R}$.
Case $\mathbf{0} \leqslant \boldsymbol{m} \leqslant \boldsymbol{M}$. Then, we have $-m \leqslant M$ and $\left|f \mathbb{1}_{A}\right| \leqslant M=\max (-m, M)$.
Case $\boldsymbol{m}<\mathbf{0} \leqslant \boldsymbol{M}$. Then, we have $\left|f \mathbb{1}_{A}\right| \leqslant \max (-m, M)$.
Case $\boldsymbol{m} \leqslant \boldsymbol{M}<\mathbf{0}$. Then, we have $M \leqslant-m$ and $\left|f \mathbb{1}_{A}\right| \leqslant-m=\max (-m, M)$.
Hence, in all cases, from Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$, with $\left|f \mathbb{1}_{A}\right| \leqslant g$ ), we have $f \mathbb{1}_{A} \in \mathcal{L}^{\infty}$, and when $\mu(A)<\infty, f \in \mathcal{L}^{p}$.

Therefore, from the definition of restriction of function and masking (thus we have $m \mathbb{1}_{\boldsymbol{A}} \leqslant \boldsymbol{f} \mathbb{1}_{\boldsymbol{A}} \leqslant M \mathbb{1}_{\boldsymbol{A}}$ ), Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}, \mathcal{I}$ is nondecreasing), the first result (with $a \stackrel{\text { def. }}{=} m, M$ and $\mu(A)<\infty$ ), and Lemma 1274 (integral over subset, with $Y^{\text {def. }} X$ and $\left.\hat{f} \stackrel{\text { def. }}{=} f\right)$, we have

$$
m \mu(A) \leqslant \int f \mathbb{1}_{A} d \mu=\int_{A} f d \mu=\int f \mathbb{1}_{A} d \mu \leqslant M \mu(A)
$$

Then, from Definition 9 (infimum, lower bound, thus $m \leqslant f_{\left.\right|_{A}}$ ), and Definition 2 (supremum, upper bound, thus $f_{\left.\right|_{A}} \leqslant M$ ), this holds when $m \stackrel{\text { def. }}{=} \inf (f(A))$ and $M \stackrel{\text { def. }}{=} \sup (f(A))$. Moreover, from Lemma 1274 (integral over subset), the first result (with $a \stackrel{\text { def. }}{=} m$ and $\mu(A)$ finite), Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ), additive group properties of $\mathbb{R}$, Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$, thus $f_{l_{A}}-m$ belongs to $\mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$ ), Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite), and Lemma 985 (compatibility of almost equality with operator, with the unary operator translation by $m$ ), we have

$$
\int_{A} f d \mu=m \mu(A) \quad \Longleftrightarrow \quad \int\left(f_{\left.\right|_{A}}-m\right) d \mu_{A}=0 \quad \Longleftrightarrow \quad f_{\left.\right|_{A}} \stackrel{\mu_{A} \text { a.e. }}{=} m
$$

Similarly, with $M-f_{\left.\right|_{A}} \in \mathcal{L}^{1}(A, \Sigma \bar{\cap} A) \cap \mathcal{M}_{+}(A, \Sigma \bar{\cap} A)$, we have $\int_{A} f d \mu=M \mu(A) \Leftrightarrow f_{\left.\right|_{A}} \stackrel{\mu_{A} \text { a.e. }}{=} M$. Hence, from the tautology $(P \Leftrightarrow Q) \Leftrightarrow(\neg P \Leftrightarrow \neg Q)$, we have strict inequalities in (16.61) iff $f_{\left.\right|_{A}}$ is not equal to $m$ nor to $M \mu_{A}$-almost everywhere.

## Remark 1330.

Note that in the previous lemma, when the measure is trivial, strict inequalities and the restriction of the function being not equal almost everywhere to one of its bounds are both wrong!

## Lemma 1331 (first mean value theorem).

Let $(X, \Sigma, \mu)$ be a finite measure space. Let $A \in \Sigma$ be nonempty. Let $f \in \mathcal{M}$. Let $m \stackrel{\text { def. }}{=} \inf (f(A))$ and $M \stackrel{\text { def. }}{=} \sup (f(A))$. Let $p \in(0, \infty]$. Assume that $m, M \in \mathbb{R},(m, M) \subset f(A)$, and $f_{\left.\right|_{A}}$ is not equal to $m$ nor $M \mu_{A}$-almost everywhere. Then, $f \mathbb{1}_{A} \in \mathcal{L}^{\infty}$, and when $\mu(A)<\infty, f \mathbb{1}_{A} \in \mathcal{L}^{p}$, and we have

$$
\begin{equation*}
\exists x \in A, \quad \int_{A} f d \mu=f(x) \mu(A) . \tag{16.61}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1329 (first mean value inequality, with strict inequalities, thus $\mu(A)=0$ is impossible), Lemma 913 (measure, nonnegativeness, thus $\mu(A) \in \mathbb{R}_{+}^{\star}$ ), ordered field properties of $\mathbb{R}$, and Definition 229 (interval, with $X \stackrel{\text { def. }}{=} \mathbb{R}$, there exists $y \in(m, M) \subset f(A)$ and $x \in A$ such that $\left.\int_{A} f d \mu=y \mu(A)=f(x) \mu(A)\right)$.

Remark 1332. Note that the previous lemma holds for instance when $f$ is continuous and bounded.

## Remark 1333.

Note that from now on, we consider again only $p$-seminorms and $\mathcal{L}^{p}$ spaces with $p \geqslant 1$.
Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $k \in[1 . . \infty)$. Let $p,\left(p_{i}\right)_{i \in[1 . . k]} \in[1, \infty]$ such that $\frac{1}{p} \stackrel{\text { def. }}{=} \sum_{i \in[1 . . k]} \frac{1}{p_{i}} \leqslant 1$. For all $i \in[1 . . k]$, let $f_{i} \in \mathcal{L}^{p_{i}}$. Then, we have $f \stackrel{\text { def. }}{=} \prod_{i \in[1 . . k]} f_{i} \in \mathcal{L}^{p}$, and

$$
\begin{equation*}
N_{p}(f) \leqslant \prod_{i \in[1 . . k]} N_{p_{i}}\left(f_{i}\right) \tag{16.62}
\end{equation*}
$$

Proof. From Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma $819(\mathcal{M}$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$, and Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra), for all $i \in[1 . . k]$, we have $f_{i} \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}$ and $N_{p_{i}}\left(f_{i}\right)<\infty$, and thus $f \in \mathcal{M}_{\mathbb{R}}$.

Therefore, from Lemma 1305 (generalized Hölder inequality), induction on $\boldsymbol{k}$, and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have

$$
N_{p}(f) \leqslant \prod_{i \in[1 . . k]} N_{p_{i}}\left(f_{i}\right)<\infty, \quad \text { and } \quad f \in \mathcal{L}^{p} .
$$

## Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p, q \in[1, \infty]$. Assume that $p<q$. Let $r \in[p, q]$. Let $\alpha \stackrel{\text { def. }}{=} \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}$. Let $f \in \mathcal{L}^{p} \cap \mathcal{L}^{q}$.
Then, we have $f \in \mathcal{L}^{r}$, and

$$
\begin{equation*}
N_{r}(f) \leqslant N_{p}^{\alpha}(f) N_{q}^{1-\alpha}(f) . \tag{16.63}
\end{equation*}
$$

Proof. From Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ ), we have

$$
\alpha \in[0,1], \quad \text { and } \quad \frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q} .
$$

Case $\boldsymbol{\mu}$ trivial. Direct consequence of Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure, thus $\mathcal{L}^{p}=\mathcal{L}^{q}=\mathcal{L}^{r}$ equal $\mathcal{M}_{\mathbb{R}}$ ), Lemma 1288 ( $N_{p}$ with trivial measure, thus $\left.N_{p}(f)=N_{q}(f)=N_{r}(f)=0\right)$, Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory), thus $0^{0}=0^{1-1}=1$ ), Lemma 432 (zero-expo-
 (zero is identity element for addition in $\overline{\mathbb{R}}$ ), and Lemma 313 (order in $\overline{\mathbb{R}}$ is total, reflexivity).

Case $\boldsymbol{\mu}$ nontrivial. From Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, with $p$ and $q$ ), closedness of the absolute value in $\mathbb{R}$, and compatibility of the power function in $\mathbb{R}$ with addition, we have $f \in \mathcal{M}_{\mathbb{R}}, N_{p}(f), N_{q}(f)<\infty$, and $|f|=|f|^{\alpha}|f|^{1-\alpha}$. Then, from ordered field properties of $\mathbb{R}\left(\alpha, 1-\alpha \in[0,1] \subset \mathbb{R}_{+}\right)$, Lemma 841 ( $\mathcal{M}$ is closed under power function, with $p \stackrel{\text { def. }}{=} \alpha$ or $1-\alpha$ ), Lemma 1294 (compatibility of $N_{p}$ with power function), Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory), with $a \stackrel{\text { def. }}{=} N_{p}(f)$ or $N_{q}(f)$
and $b \stackrel{\text { def. }}{=} \alpha$ or $1-\alpha$, all finite), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have

$$
\begin{array}{lll}
|f|^{\alpha} \in \mathcal{M}_{\mathbb{R}}, & N_{\frac{p}{\alpha}}\left(|f|^{\alpha}\right)=N_{p}^{\alpha}(f)<\infty, & |f|^{\alpha} \in \mathcal{L}^{\frac{p}{\alpha}}, \quad \text { and } \\
|f|^{1-\alpha} \in \mathcal{M}_{\mathbb{R}}, & N_{\frac{q}{1-\alpha}}\left(|f|^{1-\alpha}\right)=N_{q}^{1-\alpha}(f)<\infty, & |f|^{1-\alpha} \in \mathcal{L}^{\frac{q}{1-\alpha}} .
\end{array}
$$

Hence, from Lemma 1295 (compatibility of $N_{p}$ with absolute value), Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$, with $\left(p, p_{1}, p_{2}\right) \stackrel{\text { def. }}{=}\left(r, \frac{p}{\alpha}, \frac{q}{1-\alpha}\right) \in[1, \infty], f_{1} \stackrel{\text { def. }}{=}|f|^{\alpha}$ and $\left.f_{2} \stackrel{\text { def. }}{=}|f|^{1-\alpha}\right)$, and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have

$$
N_{r}(f)=N_{r}(|f|) \leqslant N_{\frac{p}{\alpha}}\left(|f|^{\alpha}\right) N_{\frac{q}{1-\alpha}}\left(|f|^{1-\alpha}\right)=N_{p}^{\alpha}(f) N_{q}^{1-\alpha}(f)<\infty, \quad \text { and } \quad f \in \mathcal{L}^{r}
$$

Therefore, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions), we always have $f \in \mathcal{L}^{r}$ and the inequality.

Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $f, g \in \mathcal{L}^{p}$. Then, we have $f+g \in \mathcal{L}^{p}$, and $N_{p}(f+g) \leqslant N_{p}(f)+N_{p}(g)$.

Moreover, let $n \in \mathbb{N}$, and $\left(f_{i}\right)_{i \in[0 . . n]} \in \mathcal{L}^{p}$. Then, $\sum_{i \in[0 . . n]} f_{i} \in \mathcal{L}^{p}$, and we have

$$
\begin{equation*}
N_{p}\left(\sum_{i \in[0 . . n]} f_{i}\right) \leqslant \sum_{i \in[0 . . n]} N_{p}\left(f_{i}\right) \tag{16.64}
\end{equation*}
$$

Proof. From Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma $819(\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ), and Lemma $814\left(\mathcal{M}_{\mathbb{k}}\right.$ is vector space), we have $f, g \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}, N_{p}(f)$ and $N_{p}(g)$ are finite, and $f+g \in \mathcal{M}_{\mathbb{R}}$ is well-defined everywhere.

Therefore, from Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have

$$
N_{p}(f+g) \leqslant N_{p}(f)+N_{p}(g)<\infty, \quad \text { and } \quad f+g \in \mathcal{L}^{p} .
$$

Then, the finite-sum version follows from induction on $n$.

## Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p$ in $[1, \infty]$. Then, $\left(\mathcal{L}^{p}, N_{p}\right)$ is a seminormed vector space. In particular, $\mathcal{L}^{p}$ is closed under scalar multiplication and addition.

Proof. From Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite, thus $|0|=0$ ), Lemma 1300 ( $N_{p}$ is absolutely homogeneous, with $a \stackrel{\text { def. }}{=} 0$, thus $N_{p}(0)=0$ is finite $)$, Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $\mathcal{L}^{p} \subset \mathcal{M}_{\mathbb{R}}$ and $\left.0 \in \mathcal{L}^{p}\right)$, and Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$ and $0: X \rightarrow \mathbb{R}$ as zero), $\mathcal{L}^{p}$ is a subset of the vector space $\mathcal{M}_{\mathbb{R}}$ containing the zero, and we have $N_{p}\left(\mathcal{L}^{p}\right) \subset \mathbb{R}$.

Let $f, g \in \mathcal{L}^{p} \subset \mathcal{M}_{\mathbb{R}}$. Then, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions), we have $N_{p}(f), N_{p}(g)<\infty$. Thus, from Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space, with $\mathbb{K} \stackrel{\text { def. }}{=} \mathbb{R}$, thus $f+g \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ), and closedness of addition in $\mathbb{R}$, we have $N_{p}(f+g) \leqslant N_{p}(f)+N_{p}(g)<\infty$. Hence, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), $f+g \in \mathcal{L}^{p}$, and $N_{p}$ satisfies the triangle inequality.

Therefore, from Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication), Lemma 81 (closed under vector operations is subspace, $\mathcal{L}^{p}$ is a vector subspace of $\mathcal{M}_{\mathbb{R}}$ ), Definition 77 (subspace, $\mathcal{L}^{p}$ is a vector space), and Definition 288 (seminorm, $N_{p}$ is a seminorm over $\left.\mathcal{L}^{p}\right),\left(\mathcal{L}^{p}, N_{p}\right)$ is a seminormed vector space.

Remark 1338. In the next lemmas, the belonging (of $f_{n}$ 's, $f$ and $\tilde{f}$ ) to $\mathcal{L}^{p}$ could be relaxed into a belonging only from some rank, and/or into equality almost everywhere to functions in $\mathcal{L}^{p}$.

Lemma 1339 (convergence in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{p}$. Then, for all $n \in \mathbb{N}, N_{p}\left(f_{n}-f\right)$ is well-defined. The sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ is said convergent towards $f$ in $\mathcal{L}^{p}$ iff $\lim _{n \rightarrow \infty} N_{p}\left(f_{n}-f\right)=0$; it is denoted $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$.

Proof. Direct consequence of Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, thus $f_{n}-f \in \mathcal{L}^{p}$ ), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $f_{n}-f \in \mathcal{M}$ ), and Lemma 1286 ( $p$-seminorm).

Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty)$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{p}$. Then, we have

$$
\begin{equation*}
f_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{p}} f \Longleftrightarrow \lim _{n \rightarrow \infty} \tilde{N}_{p}\left(f_{n}-f\right)=0 \Longleftrightarrow \lim _{n \rightarrow \infty} \int\left|f_{n}-f\right|^{p} d \mu=0 \tag{16.65}
\end{equation*}
$$

Proof. Direct consequence of Lemma 1339 (convergence in $\mathcal{L}^{p}$ ), Lemma 1286 ( $p$-seminorm, defini$\underline{\text { tion of } \tilde{N}_{p} \text { and } N_{p} \text { for } p \in[1, \infty) \text { ), Lemma } 369 \text { (power function in } \overline{\mathbb{R}} \text { is definite, with } \frac{1}{p} \in \overline{(0, \infty)} \text {, }}$ thus $0=0^{\frac{1}{p}}$ ), Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous), and Lemma 45 (compatibility of limit with continuous functions, with the power function and $p \in(0, \infty)$ ).

Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal). Let $(X, \Sigma, \mu)$ be a measure space. Let $p$ be in $[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f, \tilde{f} \in \mathcal{L}^{p}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{p}} f$ and $f_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{p}} \tilde{f}$. Then, we have $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$.

Proof. Let $\varepsilon>0$. Then, from Lemma 1339 (convergence in $\mathcal{L}^{p}$, with $f$ and $\tilde{f}$ ), Definition 27 (convergent sequence), let $M_{1}, M_{2} \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$
n \geqslant M_{1} \Rightarrow N_{p}\left(f_{n}-f\right) \leqslant \frac{\varepsilon}{2}, \quad \text { and } \quad n \geqslant M_{2} \Rightarrow N_{p}\left(f_{n}-\tilde{f}\right) \leqslant \frac{\varepsilon}{2}
$$

Let $M \stackrel{\text { def. }}{=} \max \left(M_{1}, M_{2}\right)$. Let $n \in \mathbb{N}$. Assume that $n \geqslant M$. Then, from Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication, with the scalar -1), Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ), and ordered field properties of $\mathbb{R}$, we have $N_{p}(f-\tilde{f}) \leqslant N_{p}\left(f-f_{n}\right)+N_{p}\left(f_{n}-\tilde{f}\right) \leqslant \varepsilon$.

Therefore, from Lemma 264 (pseudo-point separation), Lemma 1296 ( $N_{p}$ is almost definite), and Lemma 985 (compatibility of almost equality with operator, with the unary operator translation by $\tilde{f})$, we have $N_{p}(f-\tilde{f})=0$ and $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$.

## Remark 1342.

The next theorem is different from Theorem 1180 in two aspects. First, the sequence is not supposed to be nonnegative. But, as it is monotone and in $\mathcal{L}^{1}$ (and thus finite), one can subtract the first item to recover a nonnegative sequence. Second, it is assumed that $\left(\int f_{n} d \mu\right)_{n \in \mathbb{N}}$ is bounded. Hence, the pointwise limit is finite almost everywhere, and we obtain the pointwise convergence almost everywhere towards a finite function, which thus belongs to $\mathcal{L}^{1}$.

Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{1}$. Assume that the sequence is pointwise nondecreasing, and that $\sup _{n \in \mathbb{N}} \int f_{n} d \mu<\infty$. Then, there exists $f \in \mathcal{L}^{1}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, $f_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{1}} f, f_{0} \leqslant f$, for all $n \in \mathbb{N}, f_{n+1} \stackrel{\mu \text { a.e. }}{\leqslant} f$, and we have
(16.66)

$$
\int f d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad(\text { in } \mathbb{R})
$$

Proof. 1. Swap integral and limit. For all $n \in \mathbb{N}$, let $g_{n} \stackrel{\text { def. }}{=} f_{n}-f_{0}$.
Then, from Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, with $p \stackrel{\text { def. }}{=} 1$, thus $g_{n} \in \mathcal{L}^{1}$ ), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $f_{n}$ is finite), ordered field properties of $\mathbb{R}$ (thus
$0 \leqslant \boldsymbol{g}_{n} \leqslant \boldsymbol{g}_{n+1}$ ), and Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$), the sequence $\left(g_{n}\right)_{n \in \mathbb{N}}$ is pointwise nondecreasing in $\mathcal{M}_{+}$. Moreover, from Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$, monotonicity), and Lemma 501 (supremum is limit), the sequence ( $\left.\int f_{n} d \mu\right)_{n \in \mathbb{N}}$ is nondecreasing in $\mathbb{R}$, and for all $n \in \mathbb{N}$, we have

$$
\int f_{n} d \mu \leqslant \sup _{n \in \mathbb{N}} \int f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu<\infty
$$

Thus, from Theorem 1180 (Beppo Levi, monotone convergence), Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$, linearity), linearity of the limit in $\mathbb{R}$, and Lemma 34 (stationary sequence is convergent), $g \stackrel{\text { def. }}{=} \lim _{n \rightarrow \infty} g_{n} \in \mathcal{M}_{+}$, and we have

$$
\int g d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu-\int f_{0} d \mu<\infty
$$

Hence, from Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ), there exists $\tilde{g} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$such that we have

$$
\tilde{g} \stackrel{\mu \text { a.e. }}{=} g, \quad N_{1}(\tilde{g})=\int \tilde{g} d \mu=\int g d \mu<\infty, \quad \text { and } \quad \tilde{g} \in \mathcal{L}^{1} .
$$

Let $f \stackrel{\text { def. }}{=} \tilde{g}+f_{0}$. Then, from Lemma $1337\left(\mathcal{L}^{p}\right.$ is seminormed vector space, thus $\left.f \in \mathcal{L}^{1}\right)$, Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $f_{0}, \tilde{g}$ and $f$ are finite), Definition $8 \overline{35\left(\mathcal{M}_{+}, \text {sub- }\right.}$ set of nonnegative measurable numeric functions, thus $\tilde{g} \geqslant 0$ ), ordered field properties of $\mathbb{R}$, Lemma 985 (compatibility of almost equality with operator, with the unary operator translation by $\underline{f_{0}}$ ), and Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear), we have

$$
f \geqslant f_{0}, \quad \text { and } \quad f \stackrel{\mu \text { a.e. }}{=} g+f_{0}=\lim _{n \rightarrow \infty} g_{n}+f_{0}=\lim _{n \rightarrow \infty} f_{n}
$$

Hence, from Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$, linearity), and field properties of $\mathbb{R}$ (all integrals below are finite), we have

$$
\int f d \mu=\int \tilde{g} d \mu+\int f_{0} d \mu=\int g d \mu+\int f_{0} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu
$$

Let $n \in \mathbb{N}$. Then, from Definition 2 (supremum, upper bound), Lemma 501 (supremum is limit, with $\left(f_{n}\right)_{n \in \mathbb{N}}$ nondecreasing), and Lemma 994 (transitivity of almost inequality and equality), we have $f_{n} \leqslant \sup _{n \rightarrow \infty} f_{n}=\lim _{n \rightarrow \infty} f_{n} \stackrel{\mu \text { a.e. }}{=} f$, and thus $f_{n} \stackrel{\mu \text { a.e. }}{\leqslant} f$.
2. Convergence in $\mathcal{L}^{1}$. Let $n \in \mathbb{N}$.

Then, from Lemma 984 (compatibility of almost binary relation with operator, with the unary operator translation by $f_{n}$, thus $f-f_{n} \stackrel{\mu \text { a.e. }}{\geqslant} 0$ ), Lemma 1006 (absolute value is almost equal to almost nonnegative function, thus $\left|f_{n}-f\right| \stackrel{\mu \text { a.e. }}{=} f-f_{n}$ ), Lemma 1286 ( $p$-seminorm, with $p \stackrel{\text { def. }}{=} 1$ ), Lemma 1262 (compatibility of integral with almost equality), and Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$, linearity), we have $N_{1}\left(f_{n}-f\right)=\int\left(f-f_{n}\right) d \mu=\int f d \mu-\int f_{n} d \mu$. Hence, from linearity of the limit in $\mathbb{R}$, and Lemma 34 (stationary sequence is convergent), we have

$$
\lim _{n \rightarrow \infty} N_{1}\left(f_{n}-f\right)=\int f d \mu-\lim _{n \rightarrow \infty} \int f_{n} d \mu=0
$$

Therefore, from Lemma 1339 (convergence in $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=}$ ), we have $f_{n} \underset{n \rightarrow \infty}{\stackrel{\mathcal{L}^{1}}{\longrightarrow}} f$.
Remark 1344. The next theorem is different from Theorem 1208 in two aspects. First, the sequence is only supposed to be almost nonnegative. Second, it is assumed that the $f_{n}$ 's are in $\mathcal{L}^{1}$ and that $\left(\int f_{n} d \mu\right)_{n \in \mathbb{N}}$ is bounded. Thus, the integrals remain finite and the liminf remains almost finite. Hence, it is almost equal to a function in $\mathcal{L}^{1}$.

Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ). Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{1}$. Assume that the sequence is almost nonnegative, and that $\sup _{n \in \mathbb{N}} \int f_{n} d \mu<\infty$.
Then, there exists $f \in \mathcal{L}^{1}$ such that $f \stackrel{\mu \text { a.e. }}{=} \liminf _{n \rightarrow \infty} f_{n}$, and we have

$$
\begin{equation*}
\int f d \mu=\int \liminf _{n \rightarrow \infty} f_{n} d \mu \leqslant \liminf _{n \rightarrow \infty} \int f_{n} d \mu \quad\left(\text { in } \mathbb{R}_{+}\right) . \tag{16.67}
\end{equation*}
$$

Proof. From Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=} \mathbb{N}$ and $P_{n}$ set to $\left(f_{n} \geqslant 0\right)$ ), there exists $A \in \Sigma$ such that $A \subset \bigcap_{n \in \mathbb{N}}\left\{f_{n} \geqslant 0\right\}$ and $\mu\left(A^{c}\right)=0$.

For all $n \in \mathbb{N}$, let $\tilde{f}_{n} \stackrel{\text { def. }}{=} f_{n} \mathbb{1}_{A}$. Let $n \in \mathbb{N}$. Then, from Lemma 1008 (masking almost nowhere, with $f_{n}$ ), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=} 1$, thus $f_{n} \in \mathcal{M}$ is finite), Lemma 833 (measurability and masking), the definition of the indicator function, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), and Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ), we have $\tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n}$, and $\tilde{f}_{n} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$.

Let $\tilde{f} \stackrel{\text { def. }}{=} \liminf _{n \in \mathbb{N}} \tilde{f}_{n}$. Then, from Lemma 985 (compatibility of almost equality with operator, with the infinitary operator $\liminf$, thus $\tilde{f} \stackrel{\mu \text { a.e. }}{=} \liminf _{n \in \mathbb{N}} f_{n}$ ), Lemma 1262 (compatibility of integral with almost equality, with $\tilde{f}$, then $\tilde{f}_{n}$ ), Theorem 1208 (Fatou's lemma, with $\tilde{f}_{n}$ ), Definition 2 (supremum, upper bound), and Lemma 519 (limit inferior bounded from above, with $\left.M \stackrel{\text { def. }}{=} \sup _{n \in \mathbb{N}} \int f_{n} d \mu\right)$, we have $\tilde{f} \in \mathcal{M}_{+}$, and

$$
\begin{aligned}
\int \liminf _{n \rightarrow \infty} f_{n} d \mu=\int \tilde{f} d \mu=\int \liminf _{n \rightarrow \infty} & \tilde{f}_{n} d \mu \\
& \leqslant \liminf _{n \rightarrow \infty} \int \tilde{f}_{n} d \mu=\liminf _{n \rightarrow \infty} \int f_{n} d \mu \leqslant \sup _{n \in \mathbb{N}} \int f_{n} d \mu<\infty
\end{aligned}
$$

Therefore, from Lemma 1290 ( $N_{p}$ is nonnegative, with $p \stackrel{\text { def. }}{=} 1$ ), and Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ), there exists $f \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}$such that we have

$$
f \stackrel{\mu \text { a.e. }}{=} \tilde{f}=\liminf _{n \in \mathbb{N}} f_{n}, \quad 0 \leqslant N_{1}(f)=\int f d \mu=\int \tilde{f} d \mu<\infty, \quad \text { and } \quad f \in \mathcal{L}^{1}
$$

### 16.3.5 Lebesgue's dominated convergence theorem

## Remark 1346. See the sketch of next proof in Section 5.4.

## Theorem 1347 (Lebesgue, dominated convergence).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{M}$. Let $f: X \rightarrow \overline{\mathbb{R}}$. Let $g \in \mathcal{L}^{1}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\xrightarrow{\mu \text { a.e. }}} f$, and for all $n \in \mathbb{N},\left|f_{n}\right| \leqslant g$. Then, $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{1}, f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{1}}{\longrightarrow}} f$, and we have

$$
\begin{equation*}
\int f d \mu=\int \lim _{n \rightarrow \infty} f_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad(\text { in } \mathbb{R}) \tag{16.68}
\end{equation*}
$$

Proof. From Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent), Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous), Lemma 381 (limit is monotone), and Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=} 1$ ), we have $f \in \mathcal{M},|f| \leqslant g$, and $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{1}$. Thus, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions), nonnegativeness of the absolute value in $\mathbb{R}$, transitivity of order in $\mathbb{R}$, properties of the absolute value in $\mathbb{R}$ (thus $|g|=g$ ), and Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=} 1$ ), $\left(f_{n}\right)_{n \in \mathbb{N}}, f$ and $g$ are finite, $g$ is nonnegative, and $\left(\int f_{n} d \mu\right)_{n \in \mathbb{N}}, \int f d \mu$ and $\int g d \bar{\mu}$ are finite.

Let $n \in \mathbb{N}$. Let $g_{n} \stackrel{\text { def. }}{=}\left|f_{n}-f\right|$. Then, from Lemma $1337\left(\mathcal{L}^{p}\right.$ is seminormed vector space, with $p \stackrel{\text { def. }}{=} 1$ ), Lemma $1322\left(\mathcal{L}^{p}\right.$ is closed under absolute value), Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is
 inition of the absolute value in $\mathbb{R}$ (thus $|0|=0$ ), and Definition 516 (pointwise convergence), we have $g_{n} \in \mathcal{L}^{1} \cap \mathcal{M}_{+}$, and $\lim _{n \rightarrow \infty} g_{n}=\left|\lim _{n \rightarrow \infty} f_{n}-f\right|=0$. Moreover, from Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality, thus $g_{n} \leqslant\left|f_{n}\right|+|f| \leqslant 2 g$ ), Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, with $p \stackrel{\text { def. }}{=} 1)$, Lemma $1318\left(\mathcal{L}^{p}\right.$ and nonnegative is $\mathcal{M}_{+}$, with $\left.2 g-g_{n} \geqslant 0\right)$, Lemma 517 (limit inferior and limit superior of pointwise convergent), and linearity of the limit in $\mathbb{R}$, we have $2 g-g_{n} \in \mathcal{L}^{1} \cap \mathcal{M}_{+}$, and $\liminf _{n \rightarrow \infty}\left(2 g-g_{n}\right)=2 g-\lim _{n \rightarrow \infty} g_{n}=2 g$.

Then, from Theorem 1208 (Fatou's lemma, with $f_{n} \stackrel{\text { def. }}{=} 2 g-g_{n}$ ), Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ), and Lemma 510 (duality limit inferior-limit superior), we have

$$
\begin{aligned}
2 \int g d \mu=\int \liminf _{n \rightarrow \infty}\left(2 g-g_{n}\right) d \mu & \leqslant \liminf _{n \rightarrow \infty} \int\left(2 g-g_{n}\right) d \mu \\
& =\liminf _{n \rightarrow \infty}\left(2 \int g d \mu-\int g_{n} d \mu\right)=2 \int g d \mu-\limsup _{n \rightarrow \infty} \int g_{n} d \mu
\end{aligned}
$$

Hence, from ordered field properties of $\mathbb{R}$ (with $\int g d \mu$ finite), we have $\lim \sup _{n \rightarrow \infty} g_{n} \leqslant 0$. Therefore, from Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}, \int g_{n} d \mu \geqslant 0$ ), Lemma 518 (limit inferior bounded from below, $\liminf _{n \rightarrow \infty} \int g_{n} d \mu \geqslant 0$ ), Lemma 522 (limit inferior, limit superior and pointwise convergence, with limsup $\leqslant 0 \leqslant \liminf$ ), Lemma 1286 ( $p$-seminorm), and Lemma 1339 (convergence in $\mathcal{L}^{p}$ ), we have

$$
0=\liminf _{n \rightarrow \infty} \int g_{n} d \mu=\limsup _{n \rightarrow \infty} \int g_{n} d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu=\lim _{n \rightarrow \infty} N_{1}\left(f_{n}-f\right)
$$

i.e. the sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ is convergent towards $f$ in $\mathcal{L}^{1}$.

Moreover, from nonnegativeness of the absolute value in $\mathbb{R}$, and Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ), we have

$$
0 \leqslant\left|\int f_{n} d \mu-\int f d \mu\right|=\left|\int\left(f_{n}-f\right) d \mu\right| \leqslant \int\left|f_{n}-f\right| d \mu=\int g_{n} d \mu
$$

Hence, from the squeeze theorem, and linearity of the limit in $\mathbb{R}$, we have in $\mathbb{R}$

$$
\lim _{n \rightarrow \infty} \int f_{n} d \mu=\int f d \mu=\int \lim _{n \rightarrow \infty} f_{n} d \mu
$$

Remark 1348. See the sketch of next proof in Section 5.3.

## Theorem 1349 (Lebesgue, extended dominated convergence).

Let $(X, \Sigma, \mu)$ be a measure space. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f, g \in \mathcal{M}$. Assume that $f_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} f, g$ is $\mu$ integrable, and for all $n \in \mathbb{N},\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, $\left(f_{n}\right)_{n \in \mathbb{N}}$ and $f$ are $\mu$-integrable, and there exist $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{1}$ such that, for all $n \in \mathbb{N}, \tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n}, \tilde{f}^{\mu} \stackrel{\text { a.e. }}{=} f, \tilde{f}_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{1}} \tilde{f}$, and we have

$$
\begin{equation*}
\int f d \mu=\int \tilde{f} d \mu=\int \lim _{n \rightarrow \infty} \tilde{f}_{n} d \mu=\lim _{n \rightarrow \infty} \int \tilde{f}_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n} d \mu \quad(\text { in } \mathbb{R}) \tag{16.69}
\end{equation*}
$$

Proof. Let $B \stackrel{\text { def. }}{=}\left\{f=\liminf _{n \rightarrow \infty} f_{n}\right\} \cap\left\{f=\lim \sup _{n \rightarrow \infty} f_{n}\right\}$, and $\forall n \in \mathbb{N}, C_{n} \stackrel{\text { def. }}{=}\left\{\left|f_{n}\right| \leqslant g\right\}$. Then, from Definition 974 (almost binary relation, with inequality), Lemma 517 (limit inferior and limit superior of pointwise convergent), Definition 953 (property almost satisfied, thus
 $P_{0} \stackrel{\text { def. }}{=}\left(f=\liminf _{n \rightarrow \infty} f_{n}\right), P_{1} \stackrel{\text { def. }}{=}\left(f=\limsup _{n \rightarrow \infty} f_{n}\right)$ and for all $\left.n \in \mathbb{N}, P_{n+2} \stackrel{\text { def. }}{=}\left(\left|f_{n}\right| \leqslant g\right)\right)$, and Lemma 948 (negligibility of measurable subset), there exists $D \subset B \cap \bigcap_{n \in \mathbb{N}} C_{n}$ such that $D$ and $D^{c}$ are measurable, and $\mu\left(D^{c}\right)=0$, i.e. $D^{c} \in \mathbf{N}$.

Let $A \stackrel{\text { def. }}{=} D \cap g^{-1}\left(\mathbb{R}_{+}\right)$, for all $n \in \mathbb{N}, \tilde{f}_{n} \stackrel{\text { def. }}{=} f_{n} \mathbb{1}_{A}, \tilde{f} \stackrel{\text { def. }}{=} f \mathbb{1}_{A}$ and $\tilde{g} \stackrel{\text { def. }}{=} g \mathbb{1}_{A}$. Then, from Lemma 1251 (integrable is almost finite, with g), Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative, thus $\left|f_{n}\right| \geqslant 0$ ), Lemma 313 (order in $\overline{\mathbb{R}}$ is total, transitivity, thus $D \subset\{g \geqslant 0\}$ ), Lemma 951 (subset of negligible is negligible, thus $\{g \geqslant 0\}^{c} \in \mathbf{N}$ ), and Definition 953 (property almost satisfied, with $P \stackrel{\text { def. }}{=}(g \geqslant 0)), g$ is $\mu$-almost everywhere finite and nonnegative. Then, from Lemma 1009 (finite nonnegative part, with $(f, A) \stackrel{\text { def. }}{=}(g, D)$, we have $A \in \Sigma, \tilde{g} \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}, \mu\left(A^{c}\right)=0$, and $g \stackrel{\mu \text { a.e. }}{=} \tilde{g}$. Hence, from Lemma 1297 (compatibility of $N_{p}$ with almost equality), Lemma 1286 ( $p$-seminorm), Lemma 1249 (equivalent definition of integrability), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$ powered integrable functions, with $p \stackrel{\text { def. }}{=} 1$, we have $N_{1}(\tilde{g})=N_{1}(g)=\int|g| d \mu<\infty$, i.e. $\tilde{g} \in \mathcal{L}^{1}$. Moreover, from Lemma 833 (measurability and masking, with $f_{n}$ and f), and Lemma 1008 (masking almost nowhere, with $f_{n}$ and $f$ ), we have $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, \tilde{f} \in \overline{\mathcal{M}}$, for all $n \in \mathbb{N}, f_{n} \stackrel{\mu \text { a.e. }}{=} \tilde{f}_{n}$, and $f \stackrel{\mu \text { a.e. }}{=} \tilde{f}$.

Let $x \in X$. Let $n \in \mathbb{N}$. Case $\boldsymbol{x} \in A$. Then, from the definition of the indicator function, and since $A \subset B \cap C_{n}$, we have

$$
\lim _{n \rightarrow \infty} \tilde{f}_{n}(x)=\lim _{n \rightarrow \infty} f_{n}(x)=f(x)=\tilde{f}(x), \quad \text { and } \quad\left|\tilde{f}_{n}(x)\right|=\left|f_{n}(x)\right| \leqslant g(x)=\tilde{g}(x)
$$

Case $x \notin A$. Then, from the definition of the indicator function (thus $\tilde{f}_{n}(x), \tilde{f}(x)$ and $\tilde{\boldsymbol{g}}(x)$ are all equal to 0 ), Lemma 34 (stationary sequence is convergent), and Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite), we have $\lim _{n \rightarrow \infty} \tilde{f}_{n}(x)=0=\tilde{f}(x)$, and $\left|\tilde{f}_{n}(x)\right|=0 \leqslant 0=\tilde{g}(x)$. Hence, in all cases, from Theorem 1347 (Lebesgue, dominated convergence, with $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, \tilde{f}$ and $\underline{\tilde{g}}$ ), and Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, with $p \stackrel{\text { def. }}{=}$ ), we have $\tilde{f}_{n}, \tilde{f} \in \mathcal{L}^{1}, \tilde{f}_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{1}} \tilde{f}$, $\int \tilde{f} d \mu=\lim _{n \rightarrow \infty} \int \tilde{f}_{n} d \mu$, and $\tilde{f}_{n}-\tilde{f}$ is well-defined.

Therefore, from Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $p \stackrel{\text { def. }}{=}$ ), Lemma 1249 (equivalent definition of integrability, thus $\tilde{f}$ and $\tilde{f}_{n}$ are $\mu$-integrable), and Lemma 1262 (compatibility of integral with almost equality, with $(f, g) \stackrel{\text { def. }}{=}(\tilde{f}, f)$, then $\left(\tilde{f}_{n}, f_{n}\right)$ ), for all $n \in \mathbb{N}, f$ and $f_{n}$ are $\mu$-integrable, and we have $\int f d \mu=\int \tilde{f} d \mu$, and $\int \tilde{f}_{n} d \mu=\int f_{n} d \mu$.

## Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty)$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{M}$. Let $g \in \mathcal{L}^{p}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, and for all $n \in \mathbb{N},\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} g$. Then, there exists $\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}, \tilde{f} \in \mathcal{L}^{p}$ such that we have

$$
\begin{equation*}
\forall n \in \mathbb{N}, \quad \tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n} \quad \wedge \quad N_{p}\left(\tilde{f}_{n}\right)=N_{p}\left(f_{n}\right) \leqslant N_{p}(g)<\infty \tag{16.70}
\end{equation*}
$$

$$
\begin{equation*}
\tilde{f}^{\mu} \stackrel{\text { a.e. }}{=} f \wedge \quad N_{p}(\tilde{f})=N_{p}(f) \leqslant N_{p}(g)<\infty \tag{16.71}
\end{equation*}
$$

$$
\begin{equation*}
\tilde{f}_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} \tilde{f} \tag{16.72}
\end{equation*}
$$

$$
\begin{equation*}
\int|f|^{p} d \mu=\int|\tilde{f}|^{p} d \mu=\int \lim _{n \rightarrow \infty}\left|\tilde{f}_{n}\right|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|\tilde{f}_{n}\right|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|f_{n}\right|^{p} d \mu \quad(\text { in } \mathbb{R}) \tag{16.73}
\end{equation*}
$$

Proof. 1. Swap integral and limit. For all $n \in \mathbb{N}$, let $h_{n} \stackrel{\text { def. }}{=}\left|f_{n}\right|^{p}$, and $h \stackrel{\text { def. }}{=}|f|^{p}$.
Then, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $g \in \mathcal{M}_{\mathbb{R}}$ and $\left.N_{p}(g)<\infty\right)$, Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$, Lemma $841(\mathcal{M}$ is closed under power function, with $f_{n}, f$ and $g$ ), and Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $g$ ), $\left(h_{n}\right)_{n \in \mathbb{N}}, h$ and $|g|^{p}$ belong to $\mathcal{M}_{+} \subset \mathcal{M}, N_{p}(g)<\infty$, and $|g|^{p}$ is $\mu$-integrable.

Let $n \in \mathbb{N}$. Then, from Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions), we have $N_{p}\left(f_{n}\right) \leqslant N_{p}(g)<\infty$. Moreover, from Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative, thus $\left|h_{n}\right|=h_{n}$ ), Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone), and Lemma 984 (compatibility of almost binary relation with operator, with the unary operator power function), we have $\left|h_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant}|g|^{p}$.

Then, from Lemma 1012 (compatibility of almost limit with power function, with $g \stackrel{\text { def. }}{=} 0$, thus $\left.h_{n} \xrightarrow[h \rightarrow \infty]{\mu \text { a.e. }} h\right)$, and Theorem 1349 (Lebesgue, extended dominated convergence, with $f_{n} \stackrel{\text { def. }}{=} h_{n}, f \stackrel{\text { def. }}{=} h$ and $\left.g \stackrel{\text { def. }}{=}|g|^{p}\right),\left(h_{n}\right)_{n \in \mathbb{N}}$ and $h$ are $\mu$-integrable, and we have

$$
\int|f|^{p} d \mu=\int h d \mu=\lim _{n \rightarrow \infty} \int h_{n} d \mu=\lim _{n \rightarrow \infty} \int\left|f_{n}\right|^{p} d \mu<\infty
$$

Moreover, from Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$, with $f_{n}$ and $f$ ), there exists $\tilde{f},\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{p}$ such that $\tilde{f} \stackrel{\mu \text { a.e. }}{=} f, N_{p}(\tilde{f})=N_{p}(f)<\infty$, and for all $n \in \mathbb{N}, \tilde{f}_{n} \stackrel{\mu \text { a.e. }}{=} f_{n}$ and $N_{p}\left(\tilde{f}_{n}\right)=N_{p}\left(f_{n}\right) \leqslant N_{p}(g)<\infty$. Hence, from Lemma 1014 (compatibility of almost limit with almost equality, with $g_{n} \stackrel{\text { def. }}{=} \tilde{f}_{n}$ and $g=\tilde{f}$ ), Lemma 962 (almost everywhere for almost the same, with $\operatorname{card}(I)=1, f \stackrel{\text { def. }}{=} f_{n}$ and $P(f) \stackrel{\text { def. }}{=}\left(\left|f_{n}\right| \leqslant g\right)$ ), Lemma 1015 (compatibility of almost limit with almost inequality, with $f_{n} \stackrel{\text { def. }}{=} \tilde{f}_{n}$ and $\left.f \stackrel{\text { def. }}{=} \tilde{f}\right)$, and Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions, with $f \stackrel{\text { def. }}{=} \tilde{f})$, we have

$$
\tilde{f}_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} \tilde{f}, \quad\left(\left|\tilde{f}_{n}\right|\right)_{n \in \mathbb{N}},|\tilde{f}| \stackrel{\mu \text { a.e. }}{\leqslant} g, \quad \text { and } \quad N_{p}(\tilde{f})=N_{p}(f) \leqslant N_{p}(g)<\infty .
$$

2. Convergence in $\mathcal{L}^{p}$. For all $n \in \mathbb{N}$, let $k_{n} \stackrel{\text { def. }}{=}\left|\tilde{f}_{n}-\tilde{f}\right|^{p}$.

Then, from Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, thus $\tilde{f}_{n}-\tilde{f}, 2 g \in \mathcal{L}^{p}$ ), Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $\tilde{f}_{n}-\tilde{f}, 2 g \in \mathcal{M}_{\mathbb{R}}$ and $\left.N_{p}(2 g)<\infty\right)$, Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$, Lemma $841\left(\mathcal{M}\right.$ is closed under power function, with $\tilde{f}_{n}-\tilde{f}$ and $\left.2 g\right)$, Lemma 754 (constant function is measurable, with constant 0), Lemma 429 ( absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory), with $a \stackrel{\text { def. }}{=} 2=|2|$ ), and Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $2 g$ ), $\left(k_{n}\right)_{n \in \mathbb{N}}, 0,2^{p}|g|^{p} \in \mathcal{M} \overline{\mathcal{M}}, N_{p}(2 g)<\infty$, and $2^{p}|g|^{p}$ is $\mu$-integrable. Moreover, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $\tilde{f} \in \mathcal{M}_{\mathbb{R}}$ ), Lemma 1012 (compatibility of almost limit with power function, with $\left(f_{n}\right)_{n \in \mathbb{N}} \stackrel{\text { def. }}{=}\left(\tilde{f}_{n}\right)_{n \in \mathbb{N}}$ and $\left.f, g \stackrel{\text { def. }}{=} \tilde{f}\right)$, and

Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite), for all $n \in \mathbb{N}, \tilde{f}_{n}-\tilde{f}$ is well-defined, and we have $k_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\text { a.e. }}{\rightarrow \infty}}|\tilde{f}-\tilde{f}|^{p}=0$.

Let $n \in \mathbb{N}$. Then, from Lemma 371 (power function satisfies extended relaxed triangle inequality, with $\tilde{f}_{n}-\tilde{f}$ well-defined), and Lemma 984 (compatibility of almost binary relation with operator, with the unary operator power function, then the binary operator addition), we have $k_{n} \leqslant 2^{p-1}\left(\left|\tilde{f}_{n}\right|^{p}+|\tilde{f}|^{p}\right) \stackrel{\mu \text { a.e. }}{\leqslant} 2^{p}|g|^{p}$.

Therefore, from Theorem 1349 (Lebesgue, extended dominated convergence, with $f_{n} \stackrel{\text { def. }}{=} k_{n}$, $\underline{f \stackrel{\text { def. }}{=} k \text { and } g \stackrel{\text { def. }}{=} 2^{p}|g|^{p} \text { ), Lemma } 1259 \text { (integral in } \mathcal{M} \text { of zero is zero), and Lemma } 1340 \text { (equivalent }}$ definition of convergence in $\left.\mathcal{L}^{p}\right),\left(k_{n}\right)_{n \in \mathbb{N}}$ are $\mu$-integrable, and we have

$$
0=\int 0 d \mu=\lim _{n \rightarrow \infty} \int k_{n} d \mu=\lim _{n \rightarrow \infty} \int\left|\tilde{f}_{n}-\tilde{f}\right|^{p} d \mu, \quad \text { and } \quad \tilde{f}_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{p}} \tilde{f}
$$

## Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty)$. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$.
Assume that the sequence is pointwise nondecreasing, and that $\sup _{n \in \mathbb{N}} N_{p}\left(f_{n}\right)<\infty$.
Then, there exists $f \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$such that $f_{n} \underset{n \rightarrow \infty}{\stackrel{\mu \text { a.e. }}{\rightarrow}} f, f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$, for all $n \in \mathbb{N}, f_{n} \stackrel{\mu \text { a.e. }}{\leqslant} f$, and

$$
\begin{equation*}
\int f^{p} d \mu=\lim _{n \rightarrow \infty} \int f_{n}^{p} d \mu \quad\left(\text { in } \mathbb{R}_{+}\right) \tag{16.74}
\end{equation*}
$$

Proof. 1. Swap integral and limit. For all $n \in \mathbb{N}$, let $g_{n} \stackrel{\text { def. }}{=}\left|f_{n}\right|^{p}$. Let $n \in \mathbb{N}$. Then, from Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, thus $f_{n}$ is finite), Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions, thus $f_{n} \geqslant 0$ ), properties of the power function in $\mathbb{R}$, and Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$, with $f \stackrel{\text { def. }}{=} f_{n}$ ), $g_{n}=f_{n}^{p}$ is finite and nonnegative, $\left(g_{n}\right)_{n \in \mathbb{N}}$ is nondecreasing, $g_{n} \in \mathcal{L}^{1}$, and we have $\sup _{n \in \mathbb{N}} \int g_{n} d \mu<\infty$. Hence, from Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$, with $f_{n} \stackrel{\text { def. }}{=} g_{n}$ ), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$, with $\left.p \stackrel{\text { def. }}{=} 1\right)$, Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions $)$, and Lemma 1172 (integral in $\mathcal{M}_{+}, \underset{\mu \text { a.e. }}{\text { nonnegativeness }}$ ), there exists $g \in \mathcal{L}^{1} \cap \mathcal{M}_{+}$such that $g_{n} \underset{n \rightarrow \infty}{\mu \text { a.e. }} g$, $0 \leqslant g_{0} \leqslant g$, for all $n \in \mathbb{N}, 0 \leqslant g_{n+1} \leqslant g$, and we have in $\mathbb{R}_{+}$

$$
\int g d \mu=\lim _{n \rightarrow \infty} \int g_{n} d \mu=\lim _{n \rightarrow \infty} \int f_{n}^{p} d \mu
$$

Let $f \stackrel{\text { def. }}{=} g^{\frac{1}{p}}$. Then, from Definition $1314\left(\mathcal{L}^{p}\right.$, vector space of $p$-powered integrable functions, thus $\left.g \in \mathcal{M}_{\mathbb{R}} \cap \mathcal{M}_{+}\right)$, Lemma $841\left(\mathcal{M}\right.$ is closed under power function, with $\frac{1}{p}$ and $\left.g\right)$, and Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$, with $g$ ), we have

$$
f \in \mathcal{L}^{p} \cap \mathcal{M}_{+}, \quad \int f^{p} d \mu=\widetilde{N}_{p}(f)=N_{1}(g)=\int g d \mu, \quad \text { and } \quad(16.74) \text { holds }
$$

Moreover, from monotonicity of the power function in $\mathbb{R}_{+}$, and Lemma 984 (compatibility of almost binary relation with operator, with the unary operator power function), Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous, with $\frac{1}{p}$ ), Lemma 1011 (compatibility of almost limit with continuous function, with $\left(g_{n}\right)_{n \in \mathbb{N}}$ and $g_{x} \stackrel{\text { def. }}{=}\left(y \mapsto|y|^{\frac{1}{p}}\right)$, we have for all $n \in \mathbb{N}, f_{n} \stackrel{\mu \text { a.e. }}{\leqslant} f$, and $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} f$.
2. Convergence in $\mathcal{L}^{p}$. For all $n \in \mathbb{N}$, let $h_{n} \stackrel{\text { def. }}{=}\left|f_{n}-f\right|$.

Then, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $f_{n}, f \in \mathcal{M}_{\mathbb{R}}$ ), Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\mathbb{R}$, thus $\left.f_{n}-f, 0 \in \mathcal{M}_{\mathbb{R}} \subset \mathcal{M}\right)$,

Lemma $842\left(\mathcal{M}\right.$ is closed under absolute value), Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, closedness under scalar multiplication), Lemma 1012 (compatibility of almost limit with power function, with $p \stackrel{\text { def. }}{=} 1$ and $g \stackrel{\text { def. }}{=} f$ ), properties of the absolute value in $\mathbb{R}$ (definiteness, identity function on nonnegative values, and the triangle inequality), Lemma 984 (compatibility of almost binary relation with operator, with the unary operator translation by $f$ ), we have $h_{n}, 0 \in \mathcal{M}, 2 f \in \mathcal{L}^{p}, h_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }}|f-f|=0$, and $\left|h_{n}\right| \leqslant\left|f_{n}\right|+|f| \mid \stackrel{\mu \text { a.e. }}{\leqslant} 2 f$.

Therefore, from Lemma 1259 (integral in $\mathcal{M}$ of zero is zero), definiteness of the power function in $\mathbb{R}$, Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$, with $f_{n} \stackrel{\text { def. }}{=} h_{n}$, $\underline{f} \stackrel{\text { def. }}{=} 0$ and $g \stackrel{\text { def. }}{=} 2 f$ ), and Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ), we have

$$
0=\int|0|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|f_{n}-f\right|^{p} d \mu, \quad \text { and } \quad f_{n} \underset{n \rightarrow \infty}{\mathcal{L}^{p}} f .
$$

### 16.3.6 The Riesz-Fischer theorem

Remark 1353. In the next theorem, the notion of Cauchy sequence is meant for the pseudometric induced by the seminorm $N_{p}$.

See the sketch of the proof in Section 5.1.

## Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}} \in \mathcal{L}^{p}$ be a Cauchy sequence.
Then, there exists $f \in \mathcal{L}^{p}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$.
Moreover, there exists $\left(k_{n}\right)_{n \in \mathbb{N}} \in \mathbb{N}$ increasing and $h \in \mathcal{L}^{p}$ such that for all $n \in \mathbb{N},\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$, and $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} f$. When $p=\infty$, one can take $k_{n} \stackrel{\text { def. }}{=} n$, for all $n \in \mathbb{N}$.

Proof. Let $l, m \in \mathbb{N}$. Then, from Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space), Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have $f_{l}, f_{l}-f_{m} \in \mathcal{L}^{p} \subset \mathcal{M}_{\mathbb{R}}$, and $\left|f_{l}\right|,\left|f_{l}-f_{m}\right| \in \mathcal{L}^{p} \cap \mathcal{M}_{+} \subset \mathcal{M}_{\mathbb{R}}$.

1. Case $p \in[\mathbf{1}, \infty)$. Let $i \in \mathbb{N}$. Then, from Definition 35 (Cauchy sequence, with $\varepsilon \stackrel{\text { def. }}{=} \frac{1}{2^{i}}$ ), let $n_{i} \in \mathbb{N}$ such that for all $l, m \in \mathbb{N}, l, m \geqslant n_{i}$ implies $N_{p}\left(f_{l}-f_{m}\right) \leqslant \frac{1}{2^{i}}$.

Let $j \in \mathbb{N}$. Let $k_{0} \stackrel{\text { def. }}{=} n_{0}$. For all $i \in \mathbb{N}$, let $k_{i+1} \stackrel{\text { def. }}{=} \max \left(n_{i+1}, k_{i}+1\right)$. Thus, from ordered field properties of $\mathbb{R},\left(k_{i}\right)_{i \in \mathbb{N}}$ is an increasing sequence, and we have $n_{i} \leqslant k_{i} \leqslant k_{i}+j$, and $N_{p}\left(f_{k_{i}+j}-f_{k_{i}}\right) \leqslant \frac{1}{2^{i}}$. For instance, this is valid for $j \stackrel{\text { def. }}{=} k_{i+1}-k_{i}$.
1.1. Use monotone convergence in $\mathcal{L}^{p}$. For all $n \in \mathbb{N}$, let $S_{n} \stackrel{\text { def. }}{=} \sum_{i \in[0 . . n)}\left|f_{k_{i+1}}-f_{k_{i}}\right|$.

Then, from nonnegativeness of the absolute value in $\mathbb{R},\left(S_{n}\right)_{n \in \mathbb{N}}$ is nonnegative and nondecreasing. Let $n \in \mathbb{N}$. Then, from Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$, finite version), Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$), Lemma 1295 (compatibility of $N_{p}$ with absolute value), and ordered field properties of $\mathbb{R}, S_{n} \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$, and we have

$$
N_{p}\left(S_{n}\right) \leqslant \sum_{i \in[0 . . n)} N_{p}\left(f_{k_{i+1}}-f_{k_{i}}\right) \leqslant \sum_{i \in[0 . . n)} \frac{1}{2^{i}} \leqslant \sum_{i \in \mathbb{N}} \frac{1}{2^{i}}=2<\infty
$$

Hence, from Lemma 493 (supremum of bounded is bounded, thus $\sup _{n \in \mathbb{N}} N_{p}\left(S_{n}\right)<\infty$ ), and Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$, with $f_{n} \stackrel{\text { def. }}{=} S_{n}$ ), there exists $S \in \mathcal{L}^{p} \cap \mathcal{M}_{+}$ such that $S_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} S$, and for all $n \in \mathbb{N}, S_{n} \stackrel{\mu \text { a.e. }}{\leqslant} S$.

For all $n \in \mathbb{N}$, let $B \stackrel{\text { def. }}{=}\left\{S=\liminf _{n \rightarrow \infty} S_{n}\right\} \cap\left\{S=\limsup _{n \rightarrow \infty} S_{n}\right\}$ and $C_{n} \stackrel{\text { def. }}{=}\left\{S_{n} \leqslant S\right\}$. Then, from Lemma 960 (countable almost properties is almost property, with $I \stackrel{\text { def. }}{=} \mathbb{N}, P_{0}$ set to $\left(S=\liminf _{n \rightarrow \infty} S_{n}\right), P_{1} \stackrel{\text { def. }}{=}\left(S=\limsup _{n \rightarrow \infty} S_{n}\right)$ and $\left.P_{n+2} \stackrel{\text { def. }}{=}\left(S_{n} \leqslant S\right)\right)$, there exists $A \in \Sigma$ such that $A \subset B \cap \bigcap_{n \in \mathbb{N}} C_{n}$ and $\mu\left(A^{c}\right)=0$.
1.2. Use extended dominated convergence in $\mathcal{L}^{p}$. Let $n, q \in \mathbb{N}$. Let $x \in A$. Let $\varepsilon>0$. Then, from additive group properties of $\mathbb{R}$, and the triangle inequality for the absolute value in $\mathbb{R}$, we have

$$
\left|f_{k_{n+q}}(x)-f_{k_{n}}(x)\right| \leqslant \sum_{i \in[0 . . q)}\left|f_{k_{n+i+1}}(x)-f_{k_{n+i}}(x)\right|=S_{n+q}(x)-S_{n}(x) \leqslant S(x)-S_{n}(x) \leqslant S(x) .
$$

Thus, from Definition 27 (convergent sequence), Definition 36 (equivalent definition of Cauchy sequence), and the completeness of $\mathbb{R}$, there exists $M \in \mathbb{N}$ such that $\left|S(x)-S_{n}(x)\right| \leqslant \varepsilon$ for all $n \geqslant M,\left(f_{k_{n}}(x)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and thus there exists $\tilde{f}(x)$ in $\mathbb{R}$ such that $f_{k_{n}}(x) \underset{n \rightarrow \infty}{\longrightarrow} \tilde{f}(x)$. Hence, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $f_{k_{n}} \in \mathcal{M}_{\mathbb{R}}$ ), and Lemma 1016 (compatibility of almost limit with measurability), there exists $f \in \mathcal{M}_{\mathbb{R}}$ such that $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$.

For all $n, q \in \mathbb{N}$, let $h_{n} \stackrel{\text { def. }}{=}\left|f_{k_{n}}-f\right|=\left|f-f_{k_{n}}\right|$, and $g_{q}^{n} \stackrel{\text { def. }}{=}\left|f_{k_{n+q}}-f_{k_{n}}\right| \in \mathcal{L}^{p}$.
Then, from properties of the absolute value in $\mathbb{R}$ (thus $\left|h_{n}\right|=h_{n}$ ), Lemma 1012 (compatibility of almost limit with power function, first with $f_{n} \stackrel{\text { def. }}{=} f_{k_{n}}$ and $g \stackrel{\text { def. }}{=} f$, thus $h_{n} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} 0$, then $\underline{\text { with } f_{q} \stackrel{\text { def. }}{=} f_{k_{n+9}} \text { and } g \stackrel{\text { def. }}{=} f_{k_{n}} \text {, thus } g_{q}^{n} \xrightarrow[q \rightarrow \infty]{\mu \text { a.e. }} h_{n}}$ ), Lemma 1015 (compatibility of almost limit with almost inequality, with $f_{q} \stackrel{\text { def. }}{=} g_{q}^{n}, f \stackrel{\text { def. }}{=} h_{n}$ and $g \stackrel{\text { def. }}{=} S$, thus $h_{n} \stackrel{\mu \text { a.e. }}{\leqslant} S$ ), Lemma 1001 (absolute value and almost inequality, with $g \stackrel{\text { def. }}{=} f_{k_{n}}$ finite and $h \stackrel{\text { def. }}{=} S$, thus $|f| \stackrel{\mu \text { a.e. }}{\leqslant} S+\left|f_{k_{n}}\right|$, Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, thus $S+\left|f_{k_{n}}\right| \in \mathcal{L}^{p}$ ), and Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ), we have $f \in \mathcal{L}^{p}$. Let $n \in \mathbb{N}$. Then, from Lemma 1001 (absolute value and almost inequality, with $f \stackrel{\text { def. }}{=} f_{k_{n}}, g \stackrel{\text { def. }}{=} f$ finite and $\left.h \stackrel{\text { def. }}{=} S\right)$, Lemma $1322\left(\mathcal{L}^{p}\right.$ is closed under absolute value $)$, and Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space), we have $\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$ with $h \stackrel{\text { def. }}{=} S+|f| \in \mathcal{L}^{p}$.

Therefore, from Lemma 1259 (integral in $\mathcal{M}$ of zero is zero), definiteness of the power function in $\mathbb{R}$, and Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$, with $f_{n}$ set to $h_{n}, f \stackrel{\text { def. }}{=} 0$ and $g \stackrel{\text { def. }}{=} S$, we have for all $n \in \mathbb{N}, N_{p}\left(h_{n}\right) \leqslant N_{p}(S)<\infty$ (i.e. $\left|f_{k_{n}}-f\right|^{p}$ is $\mu$-integrable), and

$$
0=\int|0|^{p} d \mu=\lim _{n \rightarrow \infty} \int\left|f_{k_{n}}-f\right|^{p} d \mu
$$

Hence, from Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$, thus $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$ ), and Lemma 269 (convergent subsequence of Cauchy sequence, with the pseudometric induced by the $\xrightarrow[\text { seminorm } N_{p}]{ }$ ), we have $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$.
2. Case $\boldsymbol{p}=\infty$. From Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)), and Lemma 1293 ( $N_{\infty}$ bounded is almost bounded), we have

$$
\forall k \in \mathbb{N}, \exists K_{k} \in \mathbb{N}, \forall l, m \in \mathbb{N}, \quad l, m \geqslant K_{k} \quad \Longrightarrow \quad\left|f_{l}-f_{m}\right| \stackrel{\mu \text { a.e. }}{\leqslant} \frac{1}{k+1} \leqslant 1
$$

where the rank $K_{k}$ is independent of the variable of the $f_{n}$ 's. Then, from countability of $\mathbb{N}^{2}$ (used twice), Lemma 960 (countable almost properties is almost property, with $P_{l, m}^{k}$ set to $\left(\left|f_{l}-f_{m}\right| \leqslant \frac{1}{k+1}\right)$ for $\left.l, m \geqslant K_{k}\right)$, there exists $A \in \Sigma$ such that

$$
A \subset \bigcap_{k \in \mathbb{N} l, m \geqslant K_{k}}\left\{\left|f_{l}-f_{m}\right| \leqslant \frac{1}{k+1}\right\}, \quad \text { and } \quad \mu\left(A^{c}\right)=0 .
$$

Let $x \in A$. Then, from Definition 268 (equivalent definition of Cauchy sequence (discrete criterion)), completeness of $\mathbb{R}$, and Definition 953 (property almost satisfied), $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence, and there exists $\tilde{f}(x) \in \mathbb{R}$ such that $f_{n}(x) \underset{n \rightarrow \infty}{\longrightarrow} \tilde{f}(x)$, i.e. $f_{n} \underset{n \rightarrow \infty}{\stackrel{\mu \text { a.e. }}{\rightarrow}} \tilde{f}$. Hence, from Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions, thus $f_{n} \in \mathcal{M}_{\mathbb{R}}$ ), and Lemma 1016 (compatibility of almost limit with measurability), there exists $f \in \mathcal{M}_{\mathbb{R}}$ such that $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} f$.

For all $n, q \in \mathbb{N}$, let $h_{n} \stackrel{\text { def. }}{=}\left|f_{n}-f\right|=\left|f-f_{n}\right|$, and $g_{q}^{n} \stackrel{\text { def. }}{=}\left|f_{K_{0}+q}-f_{K_{0}+n}\right| \in \mathcal{L}^{\infty}$.
Then, from properties of the absolute value in $\mathbb{R}$ (thus $\left|h_{n}\right|=h_{n}$ ), Lemma 1012 (compatibility of almost limit with power function, first with $g \stackrel{\text { def. }}{=} f$, thus $h_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} 0$, then with $f_{q} \stackrel{\text { def. }}{=} f_{K_{0}+q}$ and $g \stackrel{\text { def. }}{=} f_{K_{0}+n}$, thus $g_{q}^{n} \xrightarrow[q \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} h_{K_{0}+n}$ ), Lemma 1015 (compatibility of almost limit with almost inequality, with $f_{q} \stackrel{\text { def. }}{=} g_{q}^{n}, f \stackrel{\text { def. }}{=} h_{K_{0}+n}$ and $g \stackrel{\text { def. }}{=} 1$, thus $h_{K_{0}+n} \stackrel{\mu \text { a.e. }}{\leqslant} 1$ ), Lemma 1001 (absolute value and almost inequality, with $g \stackrel{\text { def. }}{=} f_{K_{0}+n}$ finite and $h \stackrel{\text { def. }}{=} 1$, thus $|f| \stackrel{\mu \text { a.e. }}{\leqslant} 1+\left|f_{K_{0}+n}\right|$ ), Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space, thus $1+\left|f_{K_{0}+n}\right| \in \mathcal{L}^{\infty}$ ), and Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ), we have $f \in \mathcal{L}^{\infty}$.

Therefore, from Lemma 1339 (convergence in $\mathcal{L}^{p}$ ), Definition 27 (convergent sequence), and Lemma $1293\left(N_{\infty}\right.$ bounded is almost bounded), we have $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{\infty}}{\rightarrow}} f$.

Let $n \in \mathbb{N}$. Then, from Lemma 1001 (absolute value and almost inequality, with $f \stackrel{\text { def. }}{=} f_{K_{0}+n}$, $g \stackrel{\text { def. }}{=} f$ finite and $h \stackrel{\text { def. }}{=} 1$, thus $\left.\left|f_{K_{0}+n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} 1+|f|\right)$, Lemma $1322\left(\mathcal{L}^{p}\right.$ is closed under absolute value), Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space), and Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), we have $\left|f_{n}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$ with $h \stackrel{\text { def. }}{=} \max _{k<K_{0}}\left\{\left|f_{k}\right|, 1+|f|\right\} \in \mathcal{L}^{\infty}$.

Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{p}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$. Then, there exist $\left(k_{n}\right)_{n \in \mathbb{N}} \in \mathbb{N}$ increasing and $h \in \mathcal{L}^{p}$ such that $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f$, and for all $n \in \mathbb{N}$, $\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$. When $p=\infty$, one can take $k_{n} \stackrel{\text { def. }}{=} n$, for all $n \in \mathbb{N}$.

Proof. Direct consequence of Lemma 37 (convergent sequence is Cauchy), Lemma 1354 (RieszFischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$, thus there exists $\tilde{\mu}, h \in \mathcal{L}^{p}$ and $\left(k_{n}\right)_{n \in \mathbb{N}} \in \mathbb{N}$ increasing such that $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mathcal{L}^{p}}{\longrightarrow}} \tilde{f}$, for all $n \in \mathbb{N},\left|f_{k_{n}}\right| \stackrel{\mu \text { a.e. }}{\leqslant} h$ and $\left.f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\xrightarrow{\mu \text { a.e. }} \tilde{f}}\right)$, Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal, thus $\tilde{f} \stackrel{\mu \text { a.e. }}{=} f$ ), and Lemma 1014 (compatibility of almost limit with almost equality, with $f_{n}, g_{n} \stackrel{\text { def. }}{=} f_{k_{n}}, f \stackrel{\text { def. }}{=} \tilde{f}$ and $g \stackrel{\text { def. }}{=} f$, thus $\left.f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} f\right)$.

Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ).
Let $(X, \Sigma, \mu)$ be a measure space. Let $p \in[1, \infty]$. Let $\left(f_{n}\right)_{n \in \mathbb{N}}, f \in \mathcal{L}^{p}$. Let $g \in \mathcal{M}$. Assume that $f_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{p}} f$, and $f_{n} \xrightarrow[n \rightarrow \infty]{\stackrel{\mu \text { a.e. }}{\rightarrow}} g$. Then, we have $f \stackrel{\mu \text { a.e. }}{=} g$.

Proof. Direct consequence of Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$, thus there
 with subsequence, thus $f_{k_{n}} \xrightarrow[n \rightarrow \infty]{\mu \text { a.e. }} g$ ), and Lemma 1013 (almost limits are almost equal, thus $f \stackrel{\mu \text { a.e. }}{=} g$ ).

## Chapter 17

## Conclusions, perspectives

We have presented very detailed proofs of the main basic results in measure theory and Lebesgue integration such as the Beppo Levi (monotone convergence) theorem, Fatou's lemma, the Tonelli theorem, Lebesgue's dominated convergence theorem, and the Riesz-Fischer theorem.

The short-term purpose of this work was to help the formalization in a formal proof assistant such as Coq of the basic concepts in measure theory and Lebesgue integration and of the proofs of their main properties. First milestones towards this are dedicated to the integral of nonnegative measurable functions [10] where special attention was paid to the formalization of $\sigma$-algebras and of simple functions, and to the Tonelli theorem [11].

Our mean-term purpose is now to continue up to the formalization of $L^{p}$ Lebesgue spaces and of $W^{m, p}$ Sobolev spaces as Banach spaces, and in particular $L^{2}$ and $H^{m} \stackrel{\text { def. }}{=} W^{m, 2}$ as Hilbert spaces. This will include parts of the distribution theory.

The long-term purpose of these studies is the formal proof of programs implementing the Finite Element Method. As a consequence, after having addressed the formalization of the Lax-Milgram theorem $[20,9]$, we will also have to write very detailed pen-and-paper proofs for the concepts and results of the interpolation and approximation theory to define the Finite Element Method itself.

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## Appendix A

## Lists of statements

This appendix collects the references (name and number) for all statements present in Part II. These are split into definitions, lemmas, and theorems.

## List of Definitions

209 Definition (pseudopartition) ..... 44
226 Definition (trace of subset system) ..... 48
227 Definition (product of subset systems) ..... 48
229 Definition (interval) ..... 49
240 Definition (topological space) ..... 51
242 Definition (generated topology) ..... 51
246 Definition (topological basis) ..... 51
256 Definition (order topology) ..... 54
260 Definition (second-countability) ..... 54
262 Definition (pseudometric) ..... 55
267 Definition (cluster point) ..... 55
270 Definition (relation compatible with vector operations) ..... 57
277 Definition (algebra over a field) ..... 58
280 Definition (inherited algebra operations) ..... 58
284 Definition (subalgebra) ..... 59
288 Definition (seminorm) ..... 60
295 Definition (Banach space) ..... 60
296 Definition (semi-inner product) ..... 61
302 Definition (Hölder conjugates in $\mathbb{R}$ ) ..... 63
312 Definition (extended real numbers, $\overline{\mathbb{R}}$ ) ..... 66
317 Definition (addition in $\overline{\mathbb{R}}$ ) ..... 66
326 Definition (multiplication in $\overline{\mathbb{R}}$ ) ..... 67
339 Definition (absolute value in $\overline{\mathbb{R}}$ ) ..... 69
352 Definition (exponential and logarithm in $\overline{\mathbb{R}}$ ) ..... 71
354 Definition (exponentiation in $\overline{\mathbb{R}}$ ) ..... 71
374 Definition (tangent and arctangent in $\overline{\mathbb{R}}$ ) ..... 74
405 Definition (multiplication in $\overline{\mathbb{R}}_{+}$) ..... 80
415 Definition (multiplication in $\overline{\mathbb{R}}$ (measure theory)) ..... 81
451 Definition (Hölder conjugates) ..... 86
456 Definition (connected component in $\mathbb{R}$ ) ..... 88
468 Definition (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ) ..... 90
516 Definition (pointwise convergence) ..... 102
526 Definition (finite part) ..... 104
528 Definition (nonnegative and nonpositive parts) ..... 104
578 Definition (subset system) ..... 118
580 Definition (subset system class) ..... 118
584 Definition (generated subset system) ..... 118
602 Definition ( $\pi$-system) ..... 122
606 Definition (generated $\pi$-system) ..... 122
611 Definition (ring of sets) ..... 124
617 Definition (generated ring of sets) ..... 125
626 Definition (semiring of sets) ..... 126
634 Definition (algebra of sets) ..... 129
642 Definition (generated algebra of sets) ..... 130
654 Definition (semi-algebra of sets) ..... 132
664 Definition (monotone class) ..... 134
666 Definition (generated monotone class) ..... 134
670 Definition (monotone class and symmetric set difference) ..... 134
677 Definition ( $\lambda$-system) ..... 137
682 Definition (generated $\lambda$-system) ..... 138
689 Definition ( $\lambda$-system and intersection) ..... 139
708 Definition (generated $\sigma$-algebra) ..... 142
739 Definition (premeasurable space) ..... 150
741 Definition (measurable space) ..... 150
743 Definition (Borel $\sigma$-algebra) ..... 150
748 Definition (measurability and structure) ..... 151
749 Definition (measurable function) ..... 152
758 Definition ( $\mathcal{M}_{Y}$, set of measurable functions to $Y$ ) ..... 153
774 Definition (tensor product of $\sigma$-algebras) ..... 158
783 Definition (section in Cartesian product) ..... 160
817 Definition ( $\mathcal{M}$, set of measurable numeric functions) ..... 168
835 Definition ( $\mathcal{M}_{+}$, subset of nonnegative measurable numeric functions) ..... 173
851 Definition (tensor product of numeric functions) ..... 175
855 Definition (homogeneity) ..... 178
856 Definition (diffuseness) ..... 178
857 Definition (finiteness) ..... 178
859 Definition (subadditivity) ..... 178
861 Definition (superadditivity) ..... 178
862 Definition (additivity) ..... 179
865 Definition (subtractivity) ..... 179
870 Definition ( $\sigma$-subadditivity) ..... 180
872 Definition ( $\sigma$-superadditivity) ..... 181
874 Definition ( $\sigma$-additivity) ..... 181
879 Definition (continuity from below) ..... 182
882 Definition (continuity from above) ..... 183
885 Definition ( $\sigma$-finiteness) ..... 185
888 Definition (content) ..... 186
939 Definition (Dirac measure) ..... 199
942 Definition (negligible subset) ..... 200
943 Definition (complete measure) ..... 200
945 Definition (considerable subset) ..... 200
953 Definition (property almost satisfied) ..... 201
969 Definition (almost definition) ..... 203
974 Definition (almost binary relation) ..... 204
997 Definition (summability domain) ..... 212
1034 Definition (nonnegative essential supremum) ..... 221
1047 Definition (essentially bounded) ..... 224
1057 Definition (length of interval) ..... 227
1061 Definition (set of countable cover with bounded open intervals) ..... 227
1063 Definition ( $\lambda^{\star}$, Lebesgue measure candidate) ..... 227
1071 Definition ( $\mathcal{L}$, Lebesgue $\sigma$-algebra) ..... 229
1098 Definition $\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions) ..... 236
1123 Definition $\left(\mathcal{S \mathcal { F } _ { + }}\right.$, subset of nonnegative simple functions) ..... 245
1135 Definition (adapted sequence) ..... 248
1144 Definition (integral in $\mathcal{I F}$ ) ..... 252
1216 Definition (overline- $i$ notation to exclude term in product) ..... 273
1224 Definition (tensor product measure) ..... 276
1225 Definition (candidate tensor product measure) ..... 276
1240 Definition (partial function of function from product space) ..... 281
1247 Definition (integrability) ..... 286
1261 Definition (merge integral in $\mathcal{M}$ and $\mathcal{M}_{+}$) ..... 288
1285 Definition (integral for Lebesgue measure on $\mathbb{R}$ ) ..... 296
1314 Definition ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions) ..... 309

## List of Lemmas

211 Lemma (compatibility of pseudopartition with intersection) ..... 44
212 Lemma (countable union is monotone) ..... 44
213 Lemma (equality of countable union) ..... 44
215 Lemma (disjoint union of union) ..... 44
217 Lemma (monotone union of union) ..... 45
219 Lemma (empty monotone intersection of union) ..... 46
221 Lemma (monotone union of monotone intersection) ..... 47
222 Lemma (technical inclusion for countable union) ..... 47
223 Lemma (order is meaningless in countable union) ..... 47
224 Lemma (definition of double countable union) ..... 47
225 Lemma (double countable union) ..... 47
231 Lemma (equivalent definition of interval) ..... 49
232 Lemma (empty open interval) ..... 49
235 Lemma (intervals are closed under finite intersection) ..... 50
236 Lemma (empty intersection of open intervals) ..... 50
238 Lemma (trace interval on a subset) ..... 50
241 Lemma (intersection of topologies) ..... 51
243 Lemma (generated topology is minimum) ..... 51
244 Lemma (equivalent definition of generated topology) ..... 51
245 Lemma (topology generation is monotone) ..... 51
247 Lemma (equivalent definition of topological basis) ..... 52
248 Lemma (augmented topological basis) ..... 52
249 Lemma (topological basis is generator) ..... 52
250 Lemma (topological subbasis) ..... 52
252 Lemma (trace topology on subset) ..... 53
254 Lemma (box topology on Cartesian product) ..... 53
258 Lemma (topological basis of order topology) ..... 54
259 Lemma (trace of order topology) ..... 54
261 Lemma (compatibility of second-countability with Cartesian product) ..... 54
264 Lemma (pseudo-point separation) ..... 55
265 Lemma (equivalent definition of convergent sequence) ..... 55
266 Lemma (compatibility of subsequence with convergence) ..... 55
268 Lemma (equivalent definition of Cauchy sequence (discrete criterion)) ..... 56
269 Lemma (convergent subsequence of Cauchy sequence) ..... 56
271 Lemma (quotient vector operations) ..... 57
272 Lemma (quotient vector space, equivalence relation) ..... 57
273 Lemma (quotient vector space) ..... 57
274 Lemma (linear map on quotient vector space) ..... 58
279 Lemma ( $\mathbb{K}$ is $\mathbb{K}$-algebra) ..... 58
282 Lemma (algebra of functions to algebra) ..... 58
283 Lemma ( $\mathbb{K}^{X}$ is algebra) ..... 59
286 Lemma (vector subspace and closed under multiplication is subalgebra) ..... 59
287 Lemma (closed under algebra operations is subalgebra) ..... 59
290 Lemma (seminorm is homogeneous) ..... 60
291 Lemma (seminorm is nonnegative) ..... 60
292 Lemma (seminormed vector space is pseudometric space) ..... 60
293 Lemma (definite seminorm is norm) ..... 60
298 Lemma (definite semi-inner product is inner product) ..... 61
299 Lemma (finite cover of compact interval) ..... 62
300 Lemma (power function satisfies relaxed triangle inequality) ..... 63
303 Lemma (2 is self-Hölder conjugate in $\mathbb{R}$ ) ..... 63
304 Lemma (Young's inequality for products in $\mathbb{R}$ ) ..... 63
305 Lemma (Young's inequality for products in $\mathbb{R}$, case $p=2$ ) ..... 64
307 Lemma (topology on $\mathbb{R}$ ) ..... 64
308 Lemma (trace topology on subset of $\mathbb{R}$ ) ..... 64
310 Lemma (topology on $\mathbb{R}^{m}$ ) ..... 64
311 Lemma (trace topology on subset of $\mathbb{R}^{m}$ ) ..... 65
313 Lemma (order in $\overline{\mathbb{R}}$ is total) ..... 66
314 Lemma ( $\overline{\mathbb{R}}$ is complete) ..... 66
318 Lemma (zero is identity element for addition in $\overline{\mathbb{R}}$ ) ..... 66
319 Lemma (addition in $\mathbb{R}$ is associative when defined) ..... 66
320 Lemma (addition in $\overline{\mathbb{R}}$ is commutative when defined) ..... 66
321 Lemma (addition in $\mathbb{R}$ is compatible with order when defined) ..... 67
322 Lemma (infinity-sum property in $\overline{\mathbb{R}}$ ) ..... 67
323 Lemma (finite-sum property in $\overline{\mathbb{R}}$ ) ..... 67
324 Lemma (additive inverse in $\overline{\mathbb{R}}$ is involution) ..... 67
325 Lemma (additive inverse in $\overline{\mathbb{R}}$ is monotone) ..... 67
328 Lemma (one is identity element for multiplication in $\overline{\mathbb{R}}$ ) ..... 67
329 Lemma (multiplication in $\overline{\mathbb{R}}$ is associative when defined) ..... 68
330 Lemma (multiplication in $\overline{\mathbb{R}}$ is commutative when defined) ..... 68
331 Lemma (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined) ..... 68
332 Lemma (multiplication in $\mathbb{R}$ is right distributive over addition when defined) ..... 68
333 Lemma (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined) ..... 68
334 Lemma (zero-product property in $\mathbb{R}$ ) ..... 68
335 Lemma (one-product property in $\overline{\mathbb{R}}$ ) ..... 68
336 Lemma (infinity-product property in $\overline{\mathbb{R}}$ ) ..... 68
337 Lemma (finite-product property in $\overline{\mathbb{R}}$ ) ..... 69
338 Lemma (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined) ..... 69
340 Lemma (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ) ..... 69
341 Lemma (finite absolute value in $\mathbb{R}$ ) ..... 69
342 Lemma (absolute value in $\overline{\mathbb{R}}$ is nonnegative) ..... 69
343 Lemma (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative) ..... 69
344 Lemma (inverse of absolute value in $\overline{\mathbb{R}}$ ) ..... 70
345 Lemma (absolute value in $\mathbb{R}$ is idempotent) ..... 70
346 Lemma (absolute value in $\overline{\mathbb{R}}$ is even) ..... 70
347 Lemma (absolute value in $\overline{\mathbb{R}}$ is definite) ..... 70
348 Lemma (absolute value in $\overline{\mathbb{R}}$ is multiplicative) ..... 70
349 Lemma (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality) ..... 70
350 Lemma (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality) ..... 70
351 Lemma (absolute value in $\overline{\mathbb{R}}$ satisfies countable triangle inequality) ..... 71
353 Lemma (exponential and logarithm in $\overline{\mathbb{R}}$ are inverse) ..... 71
355 Lemma (exponentiation in $\overline{\mathbb{R}}$ ) ..... 71
357 Lemma (exponentiation in $\overline{\mathbb{R}}$ of power 1) ..... 71
358 Lemma (zero-exponentiation property in $\mathbb{R}$ ) ..... 71
359 Lemma (one-exponentiation property in $\overline{\mathbb{R}}$ ) ..... 72
360 Lemma (infinity-exponentiation property in $\overline{\mathbb{R}}$ ) ..... 72
361 Lemma (power function in $\overline{\mathbb{R}}$ ) ..... 72
362 Lemma (power function in $\overline{\mathbb{R}}$ is absolute value) ..... 72
363 Lemma (finite power function in $\overline{\mathbb{R}}$ ) ..... 72
364 Lemma (power function in $\overline{\mathbb{R}}$ is nonnegative) ..... 72
365 Lemma (compatibility of power function in $\overline{\mathbb{R}}$ with addition) ..... 72
366 Lemma (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication) ..... 72
367 Lemma (power function in $\overline{\mathbb{R}}$ is monotone) ..... 73
368 Lemma (inverse of power function in $\overline{\mathbb{R}}$ ) ..... 73
369 Lemma (power function in $\overline{\mathbb{R}}$ is definite) ..... 73
370 Lemma (power function in $\overline{\mathbb{R}}$ is multiplicative) ..... 73
371 Lemma (power function satisfies extended relaxed triangle inequality) ..... 73
372 Lemma (topology on $\overline{\mathbb{R}}$ ) ..... 73
373 Lemma (trace topology on subset of $\overline{\mathbb{R}}$ ) ..... 74
375 Lemma (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms) ..... 74
377 Lemma (metric on $\overline{\mathbb{R}}$ ) ..... 74
378 Lemma (equivalent topologies on $\overline{\mathbb{R}}$ ) ..... 75
379 Lemma (convergence towards $\pm \infty$ ) ..... 75
380 Lemma (convergence towards $\pm \infty$ (discrete)) ..... 75
381 Lemma (limit is monotone) ..... 75
382 Lemma (cluster point at $\pm \infty$ (discrete)) ..... 75
383 Lemma (limit is invariant by translation) ..... 76
385 Lemma (topology on $\overline{\mathbb{R}}^{m}$ ) ..... 76
386 Lemma (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ) ..... 76
387 Lemma (addition in $\overline{\mathbb{R}}$ is continuous) ..... 76
388 Lemma (multiplication in $\overline{\mathbb{R}}$ is continuous) ..... 77
389 Lemma (absolute value in $\overline{\mathbb{R}}$ is continuous) ..... 77
390 Lemma (power function in $\overline{\mathbb{R}}$ is continuous) ..... 77
391 Lemma (limit in $\overline{\mathbb{R}}$ is linear) ..... 77
392 Lemma ( $\overline{\mathbb{R}}_{+}$is complete) ..... 77
393 Lemma (addition in $\overline{\mathbb{R}}_{+}$is closed) ..... 77
394 Lemma (addition in $\overline{\mathbb{R}}_{+}$is associative) ..... 77
395 Lemma (addition in $\overline{\mathbb{R}}_{+}$is commutative) ..... 77
396 Lemma (addition in $\overline{\mathbb{R}}_{+}$is compatible with order) ..... 77
397 Lemma (infinity-sum property in $\overline{\mathbb{R}}_{+}$) ..... 78
398 Lemma (definition of series in $\overline{\mathbb{R}}_{+}$) ..... 78
399 Lemma (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition) ..... 78
400 Lemma (series in $\overline{\mathbb{R}}_{+}$is monotone) ..... 78
401 Lemma (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds) ..... 79
402 Lemma (order is meaningless in series in $\overline{\mathbb{R}}_{+}$) ..... 79
403 Lemma (definition of double series in $\overline{\mathbb{R}}_{+}$) ..... 79
404 Lemma (double series in $\overline{\mathbb{R}}_{+}$) ..... 79
407 Lemma (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined) ..... 80
408 Lemma (zero-product property in $\overline{\mathbb{R}}_{+}$) ..... 80
409 Lemma (one-product property in $\overline{\mathbb{R}}_{+}$) ..... 80
410 Lemma (infinity-product property in $\overline{\mathbb{R}}_{+}$) ..... 80
411 Lemma (finite-product property in $\overline{\mathbb{R}}_{+}$) ..... 80
412 Lemma (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone) ..... 80
413 Lemma (addition in $\overline{\mathbb{R}}_{+}$is continuous) ..... 80
414 Lemma (limit in $\overline{\mathbb{R}}_{+}$is positive linear) ..... 80
417 Lemma (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)) ..... 81
418 Lemma (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)) ..... 81
419 Lemma (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)) ..... 81
420 Lemma (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)) ..... 81
421 Lemma (zero-product property in $\overline{\mathbb{R}}$ (measure theory)) ..... 81
422 Lemma (one-product property in $\overline{\mathbb{R}}$ (measure theory)) ..... 81
423 Lemma (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)) ..... 82
424 Lemma (finite-product property in $\overline{\mathbb{R}}$ (measure theory)) ..... 82
425 Lemma (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)) ..... 82
427 Lemma (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 82
428 Lemma (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 82
429 Lemma (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) ..... 82
430 Lemma (exponentiation in $\overline{\mathbb{R}}$ (measure theory)) ..... 83
431 Lemma (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)) ..... 83
432 Lemma (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 83
433 Lemma (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 83
434 Lemma (finite-positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 83
435 Lemma (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 83
436 Lemma (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) ..... 84
437 Lemma (less-than-one-exponentiation property in $\mathbb{R}_{+}$(measure theory)) ..... 84
438 Lemma (great-than-one-exponentiation property in $\mathbb{R}_{+}$(measure theory)) ..... 84
439 Lemma (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)) ..... 84
441 Lemma (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)) ..... 84
442 Lemma (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)) ..... 85
443 Lemma (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)) ..... 85
444 Lemma (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)) ..... 85
445 Lemma (power function in $\overline{\mathbb{R}}$ is even (measure theory)) ..... 85
446 Lemma (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) ..... 85
448 Lemma (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)) ..... 85
449 Lemma (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)) ..... 86
453 Lemma (Young's inequality for products (measure theory)) ..... 86
454 Lemma (Young's inequality for products, case $p=2$ (measure theory)) ..... 86
455 Lemma (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ ) ..... 87
457 Lemma (empty connected component) ..... 88
458 Lemma (connected component of open subset of $\mathbb{R}$ is open interval) ..... 88
459 Lemma (connected component of open subset of $\mathbb{R}$ is maximal) ..... 88
460 Lemma (connected components of open subset of $\mathbb{R}$ are equal or disjoint) ..... 88
462 Lemma (rational approximation of lower bound of open interval) ..... 89
463 Lemma (rational approximation of upper bound of open interval) ..... 89
464 Lemma (open intervals with rational bounds cover open interval) ..... 89
466 Lemma ( $\mathbb{R}^{m}$ is second-countable) ..... 89
469 Lemma (compatibility between projection and injections in $\overline{\mathbb{R}}$ ) ..... 90
471 Lemma (compatibility of projection and injections in $\overline{\mathbb{R}}$ with subset operations) ..... 90
474 Lemma (open down) ..... 91
475 Lemma (open up) ..... 91
476 Lemma (open subsets of $\overline{\mathbb{R}}$ ) ..... 91
477 Lemma (countable connected components of open subsets of $\overline{\mathbb{R}}$ ) ..... 91
478 Lemma (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ) ..... 92
479 Lemma ( $\overline{\mathbb{R}}$ is second-countable) ..... 92
480 Lemma ( $\overline{\mathbb{R}}^{m}$ is second-countable) ..... 92
482 Lemma (extrema of constant function) ..... 93
483 Lemma (compatibility of extrema with affine transformation) ..... 93
484 Lemma (equivalent definition of finite infimum) ..... 93
485 Lemma (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ) ..... 94
486 Lemma (equivalent definition of infimum) ..... 94
487 Lemma (infimum is smaller than supremum) ..... 95
488 Lemma (infimum is monotone (set)) ..... 95
489 Lemma (supremum is monotone (set)) ..... 95
490 Lemma (infimum is monotone (function)) ..... 95
491 Lemma (supremum is monotone (function)) ..... 96
492 Lemma (infimum of bounded is bounded) ..... 96
493 Lemma (supremum of bounded is bounded) ..... 96
494 Lemma (compatibility of infimum with absolute value) ..... 96
495 Lemma (compatibility of supremum with absolute value) ..... 96
496 Lemma (compatibility of translation with infimum of sequence) ..... 96
497 Lemma (compatibility of translation with supremum of sequence) ..... 96
498 Lemma (infimum of nonincreasing sequence) ..... 96
499 Lemma (supremum of nondecreasing sequence) ..... 97
500 Lemma (infimum is limit) ..... 97
501 Lemma (supremum is limit) ..... 97
502 Lemma (series in $\overline{\mathbb{R}}_{+}$is supremum) ..... 98
503 Lemma (constant sum of monotone nonnegative functions) ..... 98
504 Lemma (limit inferior) ..... 99
505 Lemma (limit inferior is $\infty$ ) ..... 99
506 Lemma (equivalent definition of the limit inferior) ..... 99
507 Lemma (limit inferior is invariant by translation) ..... 100
508 Lemma (limit inferior is monotone) ..... 100
509 Lemma (limit superior) ..... 100
510 Lemma (duality limit inferior-limit superior) ..... 101
511 Lemma (equivalent definition of limit superior) ..... 101
512 Lemma (limit inferior is smaller than limit superior) ..... 101
513 Lemma (limit superior is monotone) ..... 101
514 Lemma (compatibility of limit inferior with absolute value) ..... 101
515 Lemma (compatibility of limit superior with absolute value) ..... 102
517 Lemma (limit inferior and limit superior of pointwise convergent) ..... 102
518 Lemma (limit inferior bounded from below) ..... 102
519 Lemma (limit inferior bounded from above) ..... 102
520 Lemma (limit superior bounded from below) ..... 103
521 Lemma (limit superior bounded from above) ..... 103
522 Lemma (limit inferior, limit superior and pointwise convergence) ..... 103
524 Lemma (restriction is masking) ..... 104
525 Lemma (masking is $\sigma$-additive) ..... 104
527 Lemma (finite part is finite) ..... 104
529 Lemma (equivalent definition of nonnegative and nonpositive parts) ..... 104
530 Lemma (nonnegative and nonpositive parts are nonnegative) ..... 104
531 Lemma (nonnegative and nonpositive parts are orthogonal) ..... 104
532 Lemma (decomposition into nonnegative and nonpositive parts) ..... 105
533 Lemma (compatibility of nonpositive and nonnegative parts with addition) ..... 105
534 Lemma (compatibility of nonpositive and nonnegative parts with mask) ..... 105
535 Lemma (compatibility of nonpositive and nonnegative parts with restriction) ..... 105
538 Lemma (nonempty and with empty or full) ..... 108
539 Lemma (with empty and full) ..... 108
541 Lemma (closedness under local complement and complement) ..... 108
542 Lemma (closedness under disjoint union and local complement) ..... 109
543 Lemma (closedness under set difference and local complement) ..... 109
544 Lemma (closedness under symmetric difference, local complement, and disjoint union) ..... 109
545 Lemma (closedness under intersection and set difference) ..... 109
546 Lemma (closedness under union and disjoint union) ..... 109
547 Lemma (closedness under union and intersection) ..... 109
548 Lemma (closedness under union and set difference) ..... 110
550 Lemma (closedness under symmetric difference and set difference) ..... 110
551 Lemma (closedness under symmetric difference and intersection) ..... 110
552 Lemma (nonempty is with empty) ..... 110
553 Lemma (with full is with empty) ..... 110
554 Lemma (disjoint decomposition) ..... 110
555 Lemma (disjoint decomposition of complement and set difference) ..... 111
556 Lemma (disjoint decomposition of local complement and complement) ..... 111
557 Lemma (disjoint decomposition of set difference and local complement) ..... 111
558 Lemma (closedness under finite operations) ..... 111
559 Lemma (closedness under finite union and intersection) ..... 111
561 Lemma (closedness under intersection and finite disjoint union closure) ..... 111
562 Lemma (finite disjoint union closure is closed under finite disjoint union) ..... 112
563 Lemma (closedness under set difference and finite disjoint union closure) ..... 112
566 Lemma (closedness under countable disjoint union and local complement) ..... 113
567 Lemma (closedness under countable union and intersection) ..... 113
568 Lemma (closedness under countable disjoint and monotone union) ..... 113
569 Lemma (closedness under countable monotone and disjoint union) ..... 113
570 Lemma (closedness under countable disjoint union and countable union) ..... 113
571 Lemma (closedness under countable monotone union and countable union) ..... 113
572 Lemma (disjoint union of union in subset system) ..... 114
574 Lemma (disjoint union of union in subset system (variant)) ..... 114
575 Lemma (monotone union of union in subset system) ..... 116
576 Lemma (empty monotone intersection of union in subset system) ..... 116
577 Lemma (monotone union of monotone intersection in subset system) ..... 117
582 Lemma (intersection of subset system classes) ..... 118
583 Lemma (subset system classes are monotone) ..... 118
585 Lemma (subset system generation preserves class) ..... 118
586 Lemma (properties of subset system generation) ..... 118
587 Lemma (other properties of subset system generation) ..... 119
588 Lemma (generated subset systems are monotone) ..... 119
590 Lemma (upgrade generated subset system) ..... 119
591 Lemma (cascading generated subset systems) ..... 119
592 Lemma (double cascading generated subset systems) ..... 120
594 Lemma (class of nonempty subset systems) ..... 120
595 Lemma (generated nonempty subset systems) ..... 120
596 Lemma (subset system generation preserves class (with nonemptiness)) ..... 120
597 Lemma (properties of generated subset system (with nonemptiness)) ..... 120
598 Lemma (other properties of subset system generation (with nonemptiness)) ..... 121
599 Lemma (generated subset systems are monotone (with nonemptiness)) ..... 121
600 Lemma (upgrade generated subset system (with nonemptiness)) ..... 121
601 Lemma (cascading generated subset systems (with nonemptiness)) ..... 121
604 Lemma (equivalent definition of $\pi$-system) ..... 122
605 Lemma (intersection of $\pi$-systems) ..... 122
607 Lemma (generated $\pi$-system is $\pi$-system) ..... 122
608 Lemma (properties of $\pi$-system generation) ..... 122
609 Lemma (other properties of $\pi$-system generation) ..... 122
612 Lemma (equivalent definition of ring of sets) ..... 124
614 Lemma (ring of sets is $\pi$-system) ..... 124
615 Lemma (other properties of ring of sets) ..... 124
616 Lemma (intersection of rings of sets) ..... 125
618 Lemma (generated ring of sets is ring of sets) ..... 125
619 Lemma (properties of ring of sets generation) ..... 125
620 Lemma (other properties of ring of sets generation) ..... 125
621 Lemma (ring of sets contains $\pi$-system) ..... 125
622 Lemma (upgrade $\pi$-system to ring of sets) ..... 125
623 Lemma (ring of sets generated by $\pi$-system) ..... 126
624 Lemma (disjoint union of union in ring of sets) ..... 126
627 Lemma (equivalent definition of semiring of sets) ..... 126
629 Lemma (semiring of sets is $\pi$-system) ..... 127
630 Lemma (other properties of semiring of sets) ..... 127
631 Lemma (ring of sets is semiring of sets) ..... 127
632 Lemma (explicit ring of sets) ..... 127
635 Lemma (equivalent definition of algebra of sets) ..... 129
637 Lemma (other equivalent definition of algebra of sets) ..... 129
638 Lemma (algebra of sets is ring of sets) ..... 129
639 Lemma (other properties of algebra of sets) ..... 130
640 Lemma (algebra of sets is $\pi$-system) ..... 130
641 Lemma (intersection of algebras of sets) ..... 130
643 Lemma (generated algebra of sets is algebra of sets) ..... 130
644 Lemma (properties of algebra of sets generation) ..... 130
645 Lemma (other properties of algebra of sets generation) ..... 130
646 Lemma (algebra of sets contains ring of sets) ..... 131
647 Lemma (upgrade ring of sets to algebra of sets) ..... 131
648 Lemma (algebra of sets generated by ring of sets) ..... 131
649 Lemma (algebra of sets contains $\pi$-system) ..... 131
650 Lemma (upgrade $\pi$-system to algebra of sets) ..... 131
651 Lemma (algebra of sets generated by $\pi$-system) ..... 131
652 Lemma (disjoint union of union in algebra of sets) ..... 132
655 Lemma (equivalent definition of semi-algebra of sets) ..... 132
657 Lemma (semi-algebra of sets is semiring of sets) ..... 133
658 Lemma (other properties of semi-algebra of sets) ..... 133
659 Lemma (semi-algebra of sets is $\pi$-system) ..... 133
660 Lemma (algebra of sets is semi-algebra of sets) ..... 133
661 Lemma (algebra of sets is semiring of sets) ..... 133
662 Lemma (explicit algebra of sets) ..... 133
665 Lemma (intersection of monotone classes) ..... 134
667 Lemma (generated monotone class is monotone class) ..... 134
668 Lemma (properties of monotone class generation) ..... 134
669 Lemma (other properties of monotone class generation) ..... 134
671 Lemma ( $\mathcal{C} \backslash$ is symmetric) ..... 135
672 Lemma ( $\mathcal{C}$ is monotone class) ..... 135
674 Lemma (monotone class is closed under set difference) ..... 135
675 Lemma (monotone class generated by algebra of sets) ..... 136
678 Lemma (equivalent definition of $\lambda$-system) ..... 137
679 Lemma ( $\lambda$-system is monotone class) ..... 137
680 Lemma (other properties of $\lambda$-system) ..... 137
681 Lemma (intersection of $\lambda$-systems) ..... 138
683 Lemma (generated $\lambda$-system is $\lambda$-system) ..... 138
684 Lemma (properties of $\lambda$-system generation) ..... 138
685 Lemma (other properties of $\lambda$-system generation) ..... 138
686 Lemma ( $\lambda$-system contains monotone class) ..... 138
687 Lemma (upgrade monotone class to $\lambda$-system) ..... 138
688 Lemma ( $\lambda$-system generated by monotone class) ..... 139
690 Lemma ( $\Lambda^{\cap}$ is symmetric) ..... 139
691 Lemma ( $\Lambda^{\cap}$ is $\lambda$-system) ..... 139
692 Lemma ( $\lambda$-system with intersection) ..... 139
693 Lemma ( $\lambda$-system is closed under intersection) ..... 140
694 Lemma ( $\lambda$-system generated by $\pi$-system) ..... 140
696 Lemma ( $\sigma$-algebra) ..... 141
697 Lemma (equivalent definition of $\sigma$-algebra) ..... 141
699 Lemma ( $\sigma$-algebra is algebra of sets) ..... 141
700 Lemma (other properties of $\sigma$-algebra) ..... 141
701 Lemma ( $\sigma$-algebra is $\lambda$-system) ..... 142
702 Lemma ( $\sigma$-algebra is monotone class) ..... 142
703 Lemma ( $\sigma$-algebra is ring of sets) ..... 142
704 Lemma ( $\sigma$-algebra is $\pi$-system) ..... 142
705 Lemma ( $\sigma$-algebra is semi-algebra of sets) ..... 142
706 Lemma ( $\sigma$-algebra is semiring of sets) ..... 142
707 Lemma (intersection of $\sigma$-algebras) ..... 142
709 Lemma (generated $\sigma$-algebra is $\sigma$-algebra) ..... 142
710 Lemma (properties of $\sigma$-algebra generation) ..... 143
711 Lemma (other properties of $\sigma$-algebra generation) ..... 143
712 Lemma (countable union $\sigma$-algebra generator) ..... 143
713 Lemma ( $\sigma$-algebra contains $\lambda$-system) ..... 143
714 Lemma (upgrade $\lambda$-system to $\sigma$-algebra) ..... 143
715 Lemma ( $\sigma$-algebra generated by $\lambda$-system) ..... 143
716 Lemma ( $\sigma$-algebra contains monotone class) ..... 144
717 Lemma (upgrade monotone class to $\sigma$-algebra) ..... 144
718 Lemma ( $\sigma$-algebra generated by monotone class) ..... 144
719 Lemma ( $\sigma$-algebra contains algebra of sets) ..... 144
720 Lemma (upgrade algebra of sets to $\sigma$-algebra) ..... 144
721 Lemma ( $\sigma$-algebra generated by algebra of sets) ..... 144
722 Lemma ( $\sigma$-algebra contains ring of sets) ..... 145
723 Lemma (upgrade ring of sets to $\sigma$-algebra) ..... 145
724 Lemma ( $\sigma$-algebra generated by ring of sets) ..... 145
725 Lemma ( $\sigma$-algebra contains $\pi$-system) ..... 145
726 Lemma (upgrade $\pi$-system to $\sigma$-algebra) ..... 145
727 Lemma ( $\sigma$-algebra generated by $\pi$-system) ..... 145
728 Lemma (disjoint union of union in $\sigma$-algebra) ..... 146
730 Lemma (algebra of sets generated by product of $\sigma$-algebras) ..... 146
734 Lemma (usage of Dynkin $\pi-\lambda$ theorem) ..... 147
738 Lemma (usage of monotone class theorem) ..... 148
740 Lemma (properties of premeasurable space) ..... 150
742 Lemma (properties of measurable space) ..... 150
744 Lemma (some Borel subsets) ..... 150
745 Lemma (countable union Borel $\sigma$-algebra generator) ..... 151
750 Lemma (inverse $\sigma$-algebra) ..... 152
751 Lemma (image $\sigma$-algebra) ..... 152
752 Lemma (identity function is measurable) ..... 152
753 Lemma (countable range is measurable) ..... 152
754 Lemma (constant function is measurable) ..... 152
755 Lemma (inverse image of generating family) ..... 152
756 Lemma (equivalent definition of measurable function) ..... 153
757 Lemma (compatibility of measurability with composition) ..... 153
759 Lemma ( $\mathcal{M}_{Y}$ inverse image of countable is measurable) ..... 153
760 Lemma ( $\mathcal{M}_{Y}$ countable range) ..... 154
761 Lemma (continuous is measurable) ..... 154
763 Lemma (trace $\sigma$-algebra) ..... 155
764 Lemma (measurability of measurable subspace) ..... 155
765 Lemma (generating measurable subspace) ..... 155
766 Lemma (generating on a partition) ..... 156
768 Lemma (Borel sub- $\sigma$-algebra) ..... 156
769 Lemma (Borel subsets of partition) ..... 156
770 Lemma (source restriction of measurable function) ..... 156
771 Lemma (destination restriction of measurable function) ..... 156
772 Lemma (measurability of function defined on a pseudopartition) ..... 157
775 Lemma (product of measurable subsets is measurable) ..... 158
776 Lemma (measurability of function to product space) ..... 158
777 Lemma (canonical projection is measurable) ..... 158
778 Lemma (permutation is measurable) ..... 159
779 Lemma (generating product measurable space) ..... 159
780 Lemma (Borel subsets in product measurable subspace) ..... 159
784 Lemma (section of product) ..... 160
785 Lemma (compatibility of section with set operations) ..... 160
786 Lemma (measurability of section) ..... 160
787 Lemma (countable union of sections is measurable) ..... 160
788 Lemma (countable intersection of sections is measurable) ..... 161
789 Lemma (indicator of section) ..... 161
790 Lemma (measurability of function from product space) ..... 161
794 Lemma (some Borel subsets of numbers) ..... 164
796 Lemma (Borel subsets of $\mathbb{R}$ ) ..... 164
797 Lemma (countable generator of Borel subsets of $\mathbb{R}$ ) ..... 165
798 Lemma (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ) ..... 165
800 Lemma (generators of Borel subsets of $\overline{\mathbb{R}}$ ) ..... 165
802 Lemma (Borel subsets of $\overline{\mathbb{R}}$ ) ..... 165
803 Lemma (Borel subsets of $\mathbb{R}_{+}$) ..... 165
804 Lemma (Borel subsets of $\mathbb{R}_{+}$) ..... 166
806 Lemma (Borel subsets of $\mathbb{C}$ ) ..... 166
808 Lemma (inverse image of countable is measurable) ..... 167
809 Lemma (inverse image is measurable) ..... 167
810 Lemma ( $\mathcal{M}_{\mathfrak{K}}$ countable range) ..... 167
811 Lemma (measurability of indicator function) ..... 167
813 Lemma ( $\mathcal{M}_{\mathbb{K}}$ is algebra) ..... 167
814 Lemma ( $\mathcal{M}_{\mathbb{K}}$ is vector space) ..... 168
815 Lemma ( $\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\mathbb{R}$ ) ..... 168
816 Lemma (measurability of numeric function to $\mathbb{R}$ ) ..... 168
819 Lemma ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ) ..... 168
820 Lemma (measurability of numeric function) ..... 169
821 Lemma ( $\mathcal{M}$ is closed under finite part) ..... 169
822 Lemma ( $\mathcal{M}$ is closed under addition when defined) ..... 169
823 Lemma ( $\mathcal{M}$ is closed under finite sum when defined) ..... 170
824 Lemma ( $\mathcal{M}$ is closed under multiplication) ..... 170
825 Lemma ( $\mathcal{M}$ is closed under finite product) ..... 171
826 Lemma ( $\mathcal{M}$ is closed under scalar multiplication) ..... 171
828 Lemma ( $\mathcal{M}$ is closed under infimum) ..... 171
829 Lemma ( $\mathcal{M}$ is closed under supremum) ..... 171
830 Lemma ( $\mathcal{M}$ is closed under limit inferior) ..... 172
831 Lemma ( $\mathcal{M}$ is closed under limit superior) ..... 172
832 Lemma ( $\mathcal{M}$ is closed under limit when pointwise convergent) ..... 172
833 Lemma (measurability and masking) ..... 172
834 Lemma (measurability of restriction) ..... 172
836 Lemma (measurability of nonnegative and nonpositive parts) ..... 173
837 Lemma ( $\mathcal{M}_{+}$is closed under masking) ..... 173
838 Lemma ( $\mathcal{M}_{+}$is closed under finite part) ..... 173
839 Lemma ( $\mathcal{M}$ is closed under nonnegative finite part) ..... 173
840 Lemma (power function in $\overline{\mathbb{R}}$ is measurable) ..... 174
841 Lemma ( $\mathcal{M}$ is closed under power function) ..... 174
842 Lemma ( $\mathcal{M}$ is closed under absolute value) ..... 174
843 Lemma ( $\mathcal{M}_{+}$is closed under addition) ..... 174
844 Lemma $\left(\mathcal{M}_{+}\right.$is closed under multiplication) ..... 174
845 Lemma ( $\mathcal{M}_{+}$is closed under nonnegative scalar multiplication) ..... 174
847 Lemma ( $\mathcal{M}_{+}$is closed under infimum) ..... 175
848 Lemma $\left(\mathcal{M}_{+}\right.$is closed under supremum) ..... 175
849 Lemma ( $\mathcal{M}_{+}$is closed under limit when pointwise convergent) ..... 175
850 Lemma ( $\mathcal{M}_{+}$is closed under countable sum) ..... 175
853 Lemma (measurability of tensor product of numeric functions) ..... 175
858 Lemma (finite is bounded) ..... 178
863 Lemma (decomposition over finite pseudopartition) ..... 179
864 Lemma (binary inclusion-exclusion principle) ..... 179
866 Lemma (subtractivity) ..... 179
867 Lemma (monotonicity) ..... 180
869 Lemma (subadditivity) ..... 180
871 Lemma ( $\sigma$-subadditive is subadditive) ..... 180
873 Lemma ( $\sigma$-superadditive is superadditive) ..... 181
875 Lemma ( $\sigma$-additive is additive) ..... 181
876 Lemma (decomposition over countable pseudopartition) ..... 181
878 Lemma ( $\sigma$-subadditivity) ..... 181
880 Lemma (continuity from below) ..... 182
881 Lemma ( $\sigma$-additivity (from below)) ..... 183
883 Lemma (continuity from above) ..... 183
884 Lemma ( $\sigma$-additivity (from above)) ..... 184
886 Lemma (equivalent definition of $\sigma$-finiteness) ..... 185
887 Lemma (finite is $\sigma$-finite) ..... 185
890 Lemma (content over finite pseudopartition) ..... 186
891 Lemma (content satisfies the binary inclusion-exclusion principle) ..... 186
892 Lemma (content is subtractive) ..... 186
893 Lemma (content is monotone) ..... 186
894 Lemma (content satisfies the finite Boole inequality) ..... 187
895 Lemma (finite content is bounded) ..... 187
896 Lemma ( $\sigma$-finite content) ..... 187
897 Lemma (premeasure) ..... 188
899 Lemma (premeasure is additive) ..... 188
900 Lemma (premeasure satisfies the binary inclusion-exclusion principle) ..... 188
901 Lemma (premeasure is subtractive) ..... 188
902 Lemma (premeasure is monotone) ..... 188
903 Lemma (finite premeasure is bounded) ..... 189
904 Lemma ( $\sigma$-finite premeasure) ..... 189
905 Lemma (premeasure over countable pseudopartition) ..... 189
906 Lemma (premeasure satisfies the Boole inequality) ..... 189
907 Lemma (compatibility of null premeasure with countable union) ..... 189
908 Lemma (premeasure is continuous from below) ..... 190
909 Lemma (equivalent definition of premeasure (from below)) ..... 190
910 Lemma (premeasure is continuous from above) ..... 190
911 Lemma (equivalent definition of premeasure (from above)) ..... 190
912 Lemma (properties of premeasure space) ..... 191
913 Lemma (measure) ..... 194
915 Lemma (measure space is premeasure space) ..... 194
916 Lemma (measure is additive) ..... 194
917 Lemma (measure satisfies the binary inclusion-exclusion principle) ..... 194
918 Lemma (measure is subtractive) ..... 194
919 Lemma (measure is monotone) ..... 194
920 Lemma (finite measure is bounded) ..... 195
921 Lemma ( $\sigma$-finite measure) ..... 195
922 Lemma (measure over countable pseudopartition) ..... 195
923 Lemma (measure satisfies the Boole inequality) ..... 195
924 Lemma (compatibility of null measure with countable union) ..... 195
925 Lemma (measure is continuous from below) ..... 195
926 Lemma (equivalent definition of measure (from below)) ..... 196
927 Lemma (measure is continuous from above) ..... 196
928 Lemma (equivalent definition of measure (from above)) ..... 196
929 Lemma (properties of measure space) ..... 196
930 Lemma (trace measure) ..... 197
931 Lemma (restricted measure) ..... 197
932 Lemma (trace measure is restricted measure) ..... 197
933 Lemma (trivial measure) ..... 198
934 Lemma (equivalent definition of trivial measure) ..... 198
935 Lemma (counting measure) ..... 198
936 Lemma (finiteness of counting measure) ..... 198
937 Lemma ( $\sigma$-finite counting measure) ..... 198
940 Lemma (equivalent definition of Dirac measure) ..... 199
941 Lemma (Dirac measure is finite) ..... 199
946 Lemma (equivalent definition of considerable subset) ..... 200
947 Lemma (negligibility with trivial measure) ..... 200
948 Lemma (negligibility of measurable subset) ..... 200
949 Lemma (empty set is negligible) ..... 200
951 Lemma (subset of negligible is negligible) ..... 200
952 Lemma ( $\mathbf{N}$ is closed under countable union) ..... 200
955 Lemma (equivalent definition of property almost satisfied) ..... 201
956 Lemma (property almost satisfied with trivial measure) ..... 201
957 Lemma (measurable property almost satisfied) ..... 201
958 Lemma (everywhere is almost everywhere) ..... 201
959 Lemma (almost modus ponens) ..... 201
960 Lemma (countable almost properties is almost property) ..... 201
962 Lemma (almost everywhere for almost the same) ..... 202
963 Lemma (extended almost modus ponens) ..... 202
965 Lemma (extended almost modus ponens (simplified 0)) ..... 203
966 Lemma (extended almost modus ponens (simplified 1)) ..... 203
967 Lemma (extended almost modus ponens (simplified 2)) ..... 203
970 Lemma (extension to almost defined) ..... 203
972 Lemma (generalized almost modus ponens) ..... 204
975 Lemma (compatibility of almost binary relation with reflexivity) ..... 204
977 Lemma (compatibility of almost binary relation with extended reflexivity) ..... 205
978 Lemma (compatibility of almost binary relation with symmetry) ..... 205
979 Lemma (compatibility of almost binary relation with extended antisymmetry) ..... 205
980 Lemma (compatibility of almost binary relation with transitivity) ..... 205
981 Lemma (almost equivalence is equivalence relation) ..... 205
982 Lemma (almost equality is equivalence relation) ..... 206
983 Lemma (almost order is order relation) ..... 206
984 Lemma (compatibility of almost binary relation with operator) ..... 206
985 Lemma (compatibility of almost equality with operator) ..... 206
988 Lemma (definiteness is almost definiteness) ..... 206
992 Lemma (uniqueness of measures extended from a $\pi$-system) ..... 208
993 Lemma (transitivity of almost nonnegativeness) ..... 212
994 Lemma (transitivity of almost inequality and equality) ..... 212
995 Lemma (compatibility of almost nonnegativeness with almost equality) ..... 212
996 Lemma (almost finite) ..... 212
998 Lemma (summability on summability domain) ..... 212
999 Lemma (measurability of summability domain) ..... 212
1000 Lemma (negligibility and summability domain) ..... 213
1001 Lemma (absolute value and almost inequality) ..... 213
1002 Lemma (almost sum) ..... 213
1003 Lemma (compatibility of almost sum with almost equality) ..... 214
1004 Lemma (almost sum is sum) ..... 214
1005 Lemma (absolute value is almost definite) ..... 214
1006 Lemma (absolute value is almost equal to almost nonnegative function) ..... 214
1007 Lemma (power function is almost definite) ..... 214
1008 Lemma (masking almost nowhere) ..... 215
1009 Lemma (finite nonnegative part) ..... 215
1010 Lemma (compatibility of almost limit with subsequence) ..... 215
1011 Lemma (compatibility of almost limit with continuous function) ..... 215
1012 Lemma (compatibility of almost limit with power function) ..... 216
1013 Lemma (almost limits are almost equal) ..... 216
1014 Lemma (compatibility of almost limit with almost equality) ..... 216
1015 Lemma (compatibility of almost limit with almost inequality) ..... 216
1016 Lemma (compatibility of almost limit with measurability) ..... 216
1017 Lemma (essential supremum) ..... 218
1019 Lemma (essential supremum with trivial measure) ..... 218
1020 Lemma (essential supremum is bounded by supremum) ..... 218
1021 Lemma (essential supremum is minimum) ..... 218
1022 Lemma (essential supremum is almost upper bound) ..... 219
1023 Lemma (essential supremum of constant function) ..... 219
1024 Lemma (essential supremum of extended simple functions) ..... 219
1026 Lemma (essential supremum of infinite times function) ..... 220
1027 Lemma (essential supremum is positive homogeneous) ..... 220
1029 Lemma (essential supremum is almost monotone) ..... 220
1030 Lemma (essential supremum bounded by constant) ..... 221
1031 Lemma (essential supremum of absolute value is nonnegative) ..... 221
1032 Lemma (compatibility of essential supremum with almost equality) ..... 221
1036 Lemma (equivalent definition of nonnegative essential supremum) ..... 221
1037 Lemma (nonnegative essential supremum with trivial measure) ..... 222
1038 Lemma (nonnegative essential supremum is minimum) ..... 222
1039 Lemma (nonnegative essential supremum is almost upper bound) ..... 222
1040 Lemma (nonnegative essential supremum of constant function) ..... 222
1041 Lemma (nonnegative essential supremum for extended simple functions) ..... 223
1043 Lemma (nonnegative essential supremum of infinite times function) ..... 223
1044 Lemma (nonnegative essential supremum is positive homogeneous) ..... 223
1045 Lemma (nonnegative essential supremum is almost monotone) ..... 224
1046 Lemma (nonnegative essential supremum bounded by constant) ..... 224
1048 Lemma (essentially bounded is almost finite) ..... 224
1049 Lemma (compatibility of nonnegative essential supremum with almost equality) ..... 224
1050 Lemma (nonnegative essential supremum is almost definite) ..... 225
1051 Lemma (nonnegative essential supremum is subadditive) ..... 225
1053 Lemma (nonnegative essential supremum is submultiplicative) ..... 226
1054 Lemma (nonnegative essential supremum is compatible with power function) ..... 226
1058 Lemma (length is nonnegative) ..... 227
1059 Lemma (length is homogeneous) ..... 227
1060 Lemma (length of partition) ..... 227
1062 Lemma (set of countable cover with bounded open intervals is nonempty) ..... 227
1065 Lemma ( $\lambda^{\star}$ is nonnegative) ..... 228
1066 Lemma ( $\lambda^{\star}$ is homogeneous) ..... 228
1067 Lemma ( $\lambda^{\star}$ is monotone) ..... 228
1068 Lemma ( $\lambda^{\star}$ is $\sigma$-subadditive) ..... 228
1069 Lemma ( $\lambda^{\star}$ generalizes length of interval) ..... 229
1073 Lemma (equivalent definition of $\mathcal{L}$ ) ..... 230
1074 Lemma ( $\mathcal{L}$ is closed under complement) ..... 230
1075 Lemma ( $\mathcal{L}$ is closed under finite union) ..... 230
1076 Lemma ( $\mathcal{L}$ is closed under finite intersection) ..... 230
1077 Lemma ( $\mathcal{L}$ is algebra of sets) ..... 230
1078 Lemma ( $\lambda^{\star}$ is additive on $\mathcal{L}$ ) ..... 231
1080 Lemma ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ) ..... 231
1081 Lemma (disjoint union of union in $\mathcal{L}$ ) ..... 232
1082 Lemma ( $\mathcal{L}$ is closed under countable union) ..... 232
1083 Lemma (rays are Lebesgue-measurable) ..... 232
1084 Lemma (intervals are Lebesgue-measurable) ..... 233
1085 Lemma ( $\mathcal{L}$ is $\sigma$-algebra) ..... 233
$1086 \operatorname{Lemma}\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{L}\right)$ ..... 233
1087 Lemma $(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$ ..... 233
$1088 \operatorname{Lemma}\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{B}(\mathbb{R})\right)$ ..... 233
1092 Lemma (Lebesgue measure generalizes length of interval) ..... 234
1094 Lemma (Lebesgue measure is $\sigma$-finite) ..... 234
1095 Lemma (Lebesgue measure is diffuse) ..... 234
1100 Lemma (indicator and support are inverse) ..... 236
1101 Lemma ( $\mathcal{I} \mathcal{F}_{\mathfrak{k}}$ is measurable) ..... 236
1102 Lemma ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is $\sigma$-additive) ..... 236
1103 Lemma ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is closed under multiplication) ..... 236
1105 Lemma ( $\mathcal{I} \mathcal{F}_{\mathbb{k}}$ is closed under extension by zero) ..... 236
1106 Lemma ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is closed under restriction) ..... 237
1107 Lemma $\left(\mathcal{I F}_{\mathfrak{K}}\right.$, masking is $\sigma$-additive) ..... 237
1108 Lemma ( $\mathcal{S} \mathcal{F}_{E}$ simple representation) ..... 238
1111 Lemma ( $\mathcal{S} \mathcal{F}_{E}$ canonical representation) ..... 238
1113 Lemma $\left(\mathcal{S \mathcal { F } _ { E }}\right.$ is measurable) ..... 239
1114 Lemma ( $\mathcal{S \mathcal { F } _ { E }}$ disjoint representation) ..... 240
1116 Lemma ( $\mathcal{S} \mathcal{F}_{E}$ disjoint representation is subpartition of canonical representation) ..... 240
1117 Lemma ( $\mathcal{S \mathcal { F } _ { E }}$ is vector space) ..... 241
1119 Lemma ( $\mathcal{S \mathcal { F } _ { E }}$ is closed under extension by zero) ..... 242
1120 Lemma $\left(\mathcal{S F} \mathcal{F}_{E}\right.$ is closed under restriction) ..... 243
1121 Lemma ( $\mathcal{S \mathcal { F } _ { E }}$, masking is $\sigma$-additive) ..... 243
1124 Lemma ( $\mathcal{S F} \mathcal{F}_{+}$is measurable) ..... 245
1125 Lemma $\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation) ..... 245
1126 Lemma $\left(\mathcal{S} \mathcal{F}_{+}\right.$canonical representation) ..... 245
1127 Lemma ( $\mathcal{S F}+$ disjoint representation is subpartition of canonical representation) ..... 246
1128 Lemma $\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation) ..... 246
1129 Lemma $\left(\mathcal{S F} \mathcal{F}_{+}\right.$is closed under positive algebra operations) ..... 246
1130 Lemma $\left(\mathcal{S F} \mathcal{F}_{+}\right.$is closed under extension by zero) ..... 247
1131 Lemma $\left(\mathcal{S F} \mathcal{F}_{+}\right.$is closed under restriction) ..... 247
1132 Lemma ( $\mathcal{S F} \mathcal{F}_{+}$, masking is $\sigma$-additive) ..... 247
1134 Lemma (Lebesgue scheme step 2) ..... 247
1136 Lemma (adapted sequence in $\mathcal{M}_{+}$) ..... 248
1137 Lemma ( $\mathcal{M}_{+}$is closed under extension by zero) ..... 249
1138 Lemma $\left(\mathcal{M}_{+}\right.$is closed under restriction) ..... 250
1139 Lemma ( $\mathcal{M}_{+}$, masking is $\sigma$-additive) ..... 250
1141 Lemma (Lebesgue scheme step 3) ..... 250
1145 Lemma (equivalent definition of integral in $\mathcal{I F}$ ) ..... 252
1146 Lemma (integral in $\mathcal{I F}$ is $\sigma$-additive) ..... 252
1147 Lemma (integral in $\mathcal{I F}$ is monotone) ..... 252
1148 Lemma (integral in $\mathcal{I F}$ over subset) ..... 252
1149 Lemma (integral in $\mathcal{I F}$ over subset is $\sigma$-additive) ..... 252
1150 Lemma (integral in $\mathcal{I F}$ over negligible subset) ..... 253
1152 Lemma (integral in $\mathcal{I F}$ with counting measure) ..... 253
1154 Lemma (integral in $\mathcal{S \mathcal { F } _ { + }}$ ) ..... 254
1156 Lemma (integral in $\mathcal{S F}_{+}$generalizes integral in $\mathcal{I F}$ ) ..... 254
1157 Lemma (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)) ..... 254
1158 Lemma (integral in $\mathcal{S F}+$ of zero is zero) ..... 255
1159 Lemma (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear) ..... 255
1160 Lemma (equivalent definition of the integral in $\mathcal{S \mathcal { F } _ { + }}$ (simple)) ..... 255
1161 Lemma (integral in $\mathcal{S \mathcal { F } _ { + }}$ is monotone) ..... 256
1162 Lemma (integral in $\mathcal{S \mathcal { F } _ { + }}$ is continuous) ..... 256
1163 Lemma (integral in $\mathcal{S F}+$ over subset) ..... 256
1164 Lemma (integral in $\mathcal{S \mathcal { F }}+$ over subset is $\sigma$-additive) ..... 257
1165 Lemma (integral in $\mathcal{S F} \mathcal{F}_{+}$over negligible subset) ..... 257
1166 Lemma (density measure in $\mathcal{S F}{ }_{+}$) ..... 257
1167 Lemma (integral and density measure in $\mathcal{S F}_{+}$) ..... 258
1168 Lemma (density integral in $\mathcal{S F}_{+}$) ..... 258
1169 Lemma (integral in $\mathcal{S F} \mathcal{F}_{+}$with trivial measure) ..... 259
1170 Lemma (integral in $\mathcal{S F}+$ with counting measure) ..... 259
1172 Lemma (integral in $\mathcal{M}_{+}$) ..... 260
1174 Lemma (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S \mathcal { F } _ { + } \text { ) }}$ ..... 260
1175 Lemma (integral in $\mathcal{M}_{+}$of indicator function) ..... 260
1176 Lemma (integral in $\mathcal{M}_{+}$is positive homogeneous) ..... 260
1177 Lemma (integral in $\mathcal{M}_{+}$of zero is zero) ..... 261
1178 Lemma (integral in $\mathcal{M}_{+}$is monotone) ..... 261
1181 Lemma (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ) ..... 263
1182 Lemma (usage of adapted sequences) ..... 263
1183 Lemma (integral in $\mathcal{M}_{+}$is additive) ..... 263
1184 Lemma (integral in $\mathcal{M}_{+}$is positive linear) ..... 264
1185 Lemma (integral in $\mathcal{M}_{+}$is $\sigma$-additive) ..... 264
1186 Lemma (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts) ..... 264
1187 Lemma (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonneg- ative parts) ..... 264
1188 Lemma (integral in $\mathcal{M}_{+}$is almost definite) ..... 264
1189 Lemma (extended almost modus ponens for integral in $\mathcal{M}_{+}$) ..... 265
1190 Lemma (compatibility of integral in $\mathcal{M}_{+}$with almost equality) ..... 265
1191 Lemma (integral in $\mathcal{M}_{+}$is almost monotone) ..... 266
1192 Lemma (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function) ..... 266
1193 Lemma (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal func- tions) ..... 266
1194 Lemma (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions) ..... 266
1195 Lemma (generalized Bienaymé-Chebyshev inequality) ..... 267
1196 Lemma (Bienaymé-Chebyshev inequality) ..... 267
1197 Lemma (integrable in $\mathcal{M}_{+}$is almost finite) ..... 267
1198 Lemma (almost bounded by integrable in $\mathcal{M}_{+}$is integrable) ..... 267
1199 Lemma (integral in $\mathcal{M}_{+}$over subset) ..... 268
1200 Lemma (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive) ..... 268
1201 Lemma (integral in $\mathcal{M}_{+}$over negligible subset) ..... 268
1202 Lemma (integral in $\mathcal{M}_{+}$over singleton) ..... 268
1203 Lemma (finite integral in $\mathcal{M}_{+}$is closed under restriction) ..... 269
1204 Lemma (integrable in $\mathcal{M}_{+}$is closed under restriction) ..... 269
1205 Lemma (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive) ..... 270
1206 Lemma (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive) ..... 270
1209 Lemma (integral in $\mathcal{M}_{+}$of pointwise convergent sequence) ..... 271
1210 Lemma (integral in $\mathcal{M}_{+}$with trivial measure) ..... 272
1211 Lemma (integral in $\mathcal{M}_{+}$with counting measure) ..... 272
1212 Lemma (integral in $\mathcal{M}_{+}$with counting measure on $\mathbb{N}$ ) ..... 272
1214 Lemma (integral in $\mathcal{M}_{+}$with Dirac measure) ..... 272
1218 Lemma (measure of section) ..... 273
1219 Lemma (measure of section of product) ..... 273
1221 Lemma (measurability of measure of section (finite)) ..... 273
1223 Lemma (measurability of measure of section) ..... 275
1227 Lemma (candidate tensor product measure is tensor product measure) ..... 276
1228 Lemma (tensor product of finite measures) ..... 277
1229 Lemma (tensor product of $\sigma$-finite measures) ..... 277
1231 Lemma (uniqueness of tensor product measure (finite)) ..... 278
1233 Lemma (uniqueness of tensor product measure) ..... 279
1234 Lemma (negligibility of measurable section) ..... 280
1235 Lemma (Lebesgue measure on $\mathbb{R}^{2}$ ) ..... 280
1236 Lemma (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes) ..... 280
1237 Lemma (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines) ..... 280
1238 Lemma (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite) ..... 280
1239 Lemma (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse) ..... 281
1243 Lemma (Tonelli over subset) ..... 282
1244 Lemma (Tonelli for tensor product) ..... 283
1248 Lemma (integrable is measurable) ..... 286
1249 Lemma (equivalent definition of integrability) ..... 286
1250 Lemma (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$) ..... 286
1251 Lemma (integrable is almost finite) ..... 286
1252 Lemma (almost bounded by integrable is integrable) ..... 286
1253 Lemma (integrability is closed under extension by zero) ..... 287
1254 Lemma (integrability is closed under restriction) ..... 287
1255 Lemma (integrability, masking is $\sigma$-additive) ..... 288
1256 Lemma (integral in $\mathcal{M}$ ) ..... 288
1258 Lemma (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$) ..... 288
1259 Lemma (integral in $\mathcal{M}$ of zero is zero) ..... 288
1260 Lemma (integral in $\mathcal{M}$ of indicator function) ..... 288
1262 Lemma (compatibility of integral with almost equality) ..... 289
1263 Lemma (finite representative of integrable in $\mathcal{M}$ ) ..... 289
1264 Lemma (extended almost modus ponens for integral in $\mathcal{M}$ ) ..... 289
1266 Lemma (integral in $\mathcal{M}$ is homogeneous) ..... 289
1267 Lemma (integral in $\mathcal{M}$ of constant function) ..... 290
1268 Lemma (Minkowski inequality in $\mathcal{M}(p=1)$ ) ..... 290
1269 Lemma (integral in $\mathcal{M}$ is additive) ..... 291
1270 Lemma (integral in $\mathcal{M}_{\mathbb{R}}$ is linear) ..... 291
1271 Lemma (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone) ..... 291
1272 Lemma (integral in $\mathcal{M}$ is almost monotone) ..... 292
1273 Lemma (compatibility of integral in $\mathcal{M}$ with absolute value) ..... 292
1274 Lemma (integral over subset) ..... 293
1275 Lemma (integral over subset is $\sigma$-additive) ..... 293
1276 Lemma (integral over negligible subset) ..... 293
1277 Lemma (integral over singleton) ..... 294
1278 Lemma (integral over interval) ..... 294
1279 Lemma (Chasles relation, integral over split intervals) ..... 294
1280 Lemma (integral with trivial measure) ..... 295
1281 Lemma (integral with counting measure) ..... 295
1282 Lemma (integral with counting measure on $\mathbb{N}$ ) ..... 295
1284 Lemma (integral with Dirac measure) ..... 296
1286 Lemma ( $p$-seminorm) ..... 297
1288 Lemma ( $N_{p}$ with trivial measure) ..... 297
1289 Lemma ( $N_{p}$ of indicator function) ..... 297
1290 Lemma ( $N_{p}$ is nonnegative) ..... 297
1291 Lemma (link between $N_{1}$ and $N_{p}$ ) ..... 298
1292 Lemma (integrable is finite $N_{1}$ ) ..... 298
1293 Lemma ( $N_{\infty}$ bounded is almost bounded) ..... 298
1294 Lemma (compatibility of $N_{p}$ with power function) ..... 298
1295 Lemma (compatibility of $N_{p}$ with absolute value) ..... 299
1296 Lemma ( $N_{p}$ is almost definite) ..... 299
1297 Lemma (compatibility of $N_{p}$ with almost equality) ..... 299
1298 Lemma ( $N_{p}$ is monotone for nonnegative functions) ..... 299
1299 Lemma ( $N_{p}$ is almost monotone for almost nonnegative functions) ..... 300
1300 Lemma ( $N_{p}$ is absolutely homogeneous) ..... 300
1301 Lemma ( $N_{p}$ of constant function) ..... 301
1302 Lemma (generalized Hölder inequality (with $\infty$ )) ..... 302
1304 Lemma (Hölder inequality) ..... 302
1305 Lemma (generalized Hölder inequality) ..... 303
1306 Lemma (compatibility of $N_{p}$ with Hölder conjugates) ..... 304
1309 Lemma (Minkowski inequality in $\mathcal{M}$ ) ..... 305
1310 Lemma (Minkowski equality (nonnegative functions)) ..... 306
1311 Lemma (Minkowski equality (linearly dependent functions)) ..... 307
1312 Lemma (Minkowski equality (almost linearly dependent functions)) ..... 307
1316 Lemma (equivalent definition of $\mathcal{L}^{p}$ ) ..... 309
1317 Lemma ( $\mathcal{L}^{p}$ with trivial measure) ..... 309
1318 Lemma ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$) ..... 309
1319 Lemma (integral is positive linear form on $\mathcal{L}^{1}$ ) ..... 309
1320 Lemma (integral in $\mathcal{L}^{1}$ is almost monotone) ..... 310
1321 Lemma (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ) ..... 310
1322 Lemma ( $\mathcal{L}^{p}$ is closed under absolute value) ..... 310
1323 Lemma (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ) ..... 310
1324 Lemma (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ) ..... 311
1326 Lemma (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ) ..... 311
1327 Lemma ( $\mathcal{L}^{p}$ is closed under scalar multiplication) ..... 311
1328 Lemma (constant function is $\mathcal{L}^{p}$ ) ..... 311
1329 Lemma (first mean value inequality) ..... 312
1331 Lemma (first mean value theorem) ..... 312
1334 Lemma (generalized Hölder inequality in $\mathcal{L}^{p}$ ) ..... 313
1335 Lemma (interpolation inequality in $\mathcal{L}^{p}$ ) ..... 313
1336 Lemma (Minkowski inequality in $\mathcal{L}^{p}$ ) ..... 314
1337 Lemma ( $\mathcal{L}^{p}$ is seminormed vector space) ..... 314
1339 Lemma (convergence in $\mathcal{L}^{p}$ ) ..... 315
1340 Lemma (equivalent definition of convergence in $\mathcal{L}^{p}$ ) ..... 315
1341 Lemma (limits in $\mathcal{L}^{p}$ are almost equal) ..... 315
1355 Lemma (subsequence of convergent sequence in $\mathcal{L}^{p}$ ) ..... 325
1356 Lemma (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ) ..... 325

## List of Theorems

461 Theorem (countable connected components of open subsets of $\mathbb{R}$ ) ..... 88
465 Theorem ( $\mathbb{R}$ is second-countable) ..... 89
732 Theorem (Dynkin $\pi-\lambda$ theorem) ..... 147
736 Theorem (monotone class) ..... 148
1090 Theorem (Carathéodory, Lebesgue measure on $\mathbb{R}$ ) ..... 234
1180 Theorem (Beppo Levi, monotone convergence) ..... 261
1208 Theorem (Fatou's lemma) ..... 270
1242 Theorem (Tonelli) ..... 281
1343 Theorem (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ) ..... 315
1345 Theorem (Fatou's lemma in $\mathcal{L}^{1}$ ) ..... 317
1347 Theorem (Lebesgue, dominated convergence) ..... 318
1349 Theorem (Lebesgue, extended dominated convergence) ..... 319
1351 Theorem (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ) ..... 320
1352 Theorem (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ) ..... 321
1354 Theorem (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ) ..... 323

## Appendix B

## The proof cites explicitly...

This appendix gathers the explicit citations of the statements listed in Appendix A that appear in the proof of each result (lemmas and theorems). Statements from [20] are anonymized.

Printing is not advised!

The proof of Lemma 211 (compatibility of pseudopartition with intersection)
cites explicitly:
Definition 209 (pseudopartition).
The proof of Lemma 212 (countable union is monotone)
has no explicit citation.
The proof of Lemma 213 (equality of countable union)
cites explicitly:
Lemma 212 (countable union is monotone).
The proof of Lemma 215 (disjoint union of union)
cites explicitly:
Lemma 213 (equality of countable union).
The proof of Lemma 217 (monotone union of union)
cites explicitly:
Lemma 213 (equality of countable union).
The proof of Lemma 219 (empty monotone intersection of union)
has no explicit citation.
The proof of Lemma 221 (monotone union of monotone intersection)
has no explicit citation.
The proof of Lemma 222 (technical inclusion for countable union)
has no explicit citation.
The proof of Lemma 223 (order is meaningless in countable union)
cites explicitly:
Lemma 222 (technical inclusion for countable union).
The proof of Lemma 224 (definition of double countable union)
cites explicitly:
Lemma 223 (order is meaningless in countable union).
The proof of Lemma 225 (double countable union)
cites explicitly:
Lemma 224 (definition of double countable union).

The proof of Lemma 231 (equivalent definition of interval)
cites explicitly:
Statement(s) from [20],
Definition 229 (interval).
The proof of Lemma 232 (empty open interval)
cites explicitly:
Definition 229 (interval).
The proof of Lemma 235 (intervals are closed under finite intersection)
cites explicitly:
Definition 229 (interval).
The proof of Lemma 236 (empty intersection of open intervals)
cites explicitly:
Lemma 232 (empty open interval),
Lemma 235 (intervals are closed under finite intersection).
The proof of Lemma 238 (trace interval on a subset)
cites explicitly:
Definition 226 (trace of subset system),
Definition 229 (interval).
The proof of Lemma 241 (intersection of topologies)
cites explicitly:
Definition 240 (topological space).
The proof of Lemma 243 (generated topology is minimum)
cites explicitly:
Lemma 241 (intersection of topologies),
Definition 242 (generated topology).
The proof of Lemma 244 (equivalent definition of generated topology)
cites explicitly:
Definition 240 (topological space),
Lemma 243 (generated topology is minimum).
The proof of Lemma 245 (topology generation is monotone)
cites explicitly:
Lemma 243 (generated topology is minimum).
The proof of Lemma 247 (equivalent definition of topological basis)
cites explicitly:
Definition 240 (topological space),
Lemma 243 (generated topology is minimum),
Lemma 244 (equivalent definition of generated topology),
Definition 246 (topological basis).
The proof of Lemma 248 (augmented topological basis)
cites explicitly:
Definition 246 (topological basis).
The proof of Lemma 249 (topological basis is generator)
cites explicitly:
Lemma 244 (equivalent definition of generated topology),
Definition 246 (topological basis).

The proof of Lemma 250 (topological subbasis)
cites explicitly:
Definition 246 (topological basis),
Lemma 249 (topological basis is generator).
The proof of Lemma 252 (trace topology on subset)
cites explicitly:
Definition 226 (trace of subset system),
Definition 240 (topological space),
Definition 246 (topological basis).
The proof of Lemma 254 (box topology on Cartesian product)
cites explicitly:
Definition 227 (product of subset systems),
Definition 240 (topological space),
Definition 246 (topological basis),
Lemma 247 (equivalent definition of topological basis).
The proof of Lemma 258 (topological basis of order topology)
cites explicitly:
Lemma 235 (intervals are closed under finite intersection),
Lemma 244 (equivalent definition of generated topology),
Definition 246 (topological basis),
Definition 256 (order topology).
The proof of Lemma 259 (trace of order topology)
cites explicitly:
Lemma 238 (trace interval on a subset),
Lemma 252 (trace topology on subset),
Lemma 258 (topological basis of order topology).
The proof of Lemma 261 (compatibility of second-countability with Cartesian product)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 254 (box topology on Cartesian product),
Definition 260 (second-countability).
The proof of Lemma 264 (pseudo-point separation)
cites explicitly:
Definition 262 (pseudometric).
The proof of Lemma 265 (equivalent definition of convergent sequence)
cites explicitly:
Statement(s) from [20].
The proof of Lemma 266 (compatibility of subsequence with convergence)
cites explicitly:
Statement(s) from [20].
The proof of Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion))
cites explicitly:
Statement(s) from [20],
Lemma 265 (equivalent definition of convergent sequence).

The proof of Lemma 269 (convergent subsequence of Cauchy sequence)
cites explicitly:
Statement(s) from [20],
Definition 262 (pseudometric).
The proof of Lemma 271 (quotient vector operations)
cites explicitly:
Definition 270 (relation compatible with vector operations).
The proof of Lemma 272 (quotient vector space, equivalence relation)
cites explicitly:
Statement(s) from [20],
Lemma 271 (quotient vector operations).
The proof of Lemma 273 (quotient vector space)
cites explicitly:
Statement(s) from [20],
Definition 270 (relation compatible with vector operations),
Lemma 272 (quotient vector space, equivalence relation).
The proof of Lemma 274 (linear map on quotient vector space)
cites explicitly:
Statement(s) from [20],
Lemma 271 (quotient vector operations),
Lemma 273 (quotient vector space).
The proof of Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra)
cites explicitly:
Definition 277 (algebra over a field).
The proof of Lemma 282 (algebra of functions to algebra)
cites explicitly:
Statement(s) from [20],
Definition 277 (algebra over a field),
Definition 280 (inherited algebra operations).
The proof of Lemma $283\left(\mathbb{K}^{X}\right.$ is algebra)
cites explicitly:
Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra),
Lemma 282 (algebra of functions to algebra).
The proof of Lemma 286 (vector subspace and closed under multiplication is subalge-
bra)
cites explicitly:
Statement(s) from [20],
Definition 277 (algebra over a field),
Definition 284 (subalgebra).
The proof of Lemma 287 (closed under algebra operations is subalgebra)
cites explicitly:
Statement(s) from [20],
Lemma 286 (vector subspace and closed under multiplication is subalgebra).
The proof of Lemma 290 (seminorm is homogeneous)
cites explicitly:
Statement(s) from [20],
Definition 288 (seminorm).

The proof of Lemma 291 (seminorm is nonnegative)
cites explicitly:
Statement(s) from [20],
Definition 288 (seminorm),
Lemma 290 (seminorm is homogeneous).
The proof of Lemma 292 (seminormed vector space is pseudometric space)
cites explicitly:
Statement(s) from [20],
Definition 262 (pseudometric),
Definition 288 (seminorm),
Lemma 290 (seminorm is homogeneous),
Lemma 291 (seminorm is nonnegative).
The proof of Lemma 293 (definite seminorm is norm)
cites explicitly:
Statement(s) from [20],
Definition 288 (seminorm).
The proof of Lemma 298 (definite semi-inner product is inner product)
cites explicitly:
Statement(s) from [20],
Definition 296 (semi-inner product).
The proof of Lemma 299 (finite cover of compact interval)
has no explicit citation.
The proof of Lemma 300 (power function satisfies relaxed triangle inequality) has no explicit citation.

The proof of Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ )
cites explicitly:
Definition 302 (Hölder conjugates in $\mathbb{R}$ ).
The proof of Lemma 304 (Young's inequality for products in $\mathbb{R}$ )
cites explicitly:
Definition 302 (Hölder conjugates in $\mathbb{R}$ ).
The proof of Lemma 305 (Young's inequality for products in $\mathbb{R}$, case $p=2$ )
cites explicitly:
Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ),
Lemma 304 (Young's inequality for products in $\mathbb{R}$ ).
The proof of Lemma 307 (topology on $\mathbb{R}$ )
cites explicitly:
Lemma 249 (topological basis is generator),
Lemma 258 (topological basis of order topology).
The proof of Lemma 308 (trace topology on subset of $\mathbb{R}$ )
cites explicitly:
Lemma 252 (trace topology on subset),
Lemma 307 (topology on $\mathbb{R}$ ).
The proof of Lemma 310 (topology on $\mathbb{R}^{m}$ )
cites explicitly:
Definition 227 (product of subset systems),
Lemma 250 (topological subbasis),
Lemma 254 (box topology on Cartesian product),
Lemma 307 (topology on $\mathbb{R}$ ).

The proof of Lemma 311 (trace topology on subset of $\mathbb{R}^{m}$ )
cites explicitly:
Lemma 252 (trace topology on subset),
Lemma 310 (topology on $\mathbb{R}^{m}$ ).
The proof of Lemma 313 (order in $\overline{\mathbb{R}}$ is total)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
The proof of Lemma 314 ( $\overline{\mathbb{R}}$ is complete)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
The proof of Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 323 (finite-sum property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).

The proof of Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined)
cites explicitly:
Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined),
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined).
The proof of Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 335 (one-product property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution),
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ).

The proof of Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent)
cites explicitly:
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative).
The proof of Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite).
The proof of Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality) cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).
The proof of Lemma 351 (absolute value in $\overline{\mathbb{R}}$ satisfies countable triangle inequality) cites explicitly:
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality).
The proof of Lemma 353 (exponential and logarithm in $\overline{\mathbb{R}}$ are inverse)
cites explicitly:
Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ).
The proof of Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ),
Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ).
The proof of Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 )
cites explicitly:
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ).
The proof of Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ),
Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ),
Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ).

The proof of Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ),
Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ),
Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ).
The proof of Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ),
Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ ),
Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ).
The proof of Lemma 361 (power function in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ).
The proof of Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value)
cites explicitly:
Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 ),
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 363 (finite power function in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ),
Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone).
The proof of Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 361 (power function in $\overline{\mathbb{R}}$ ),
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite).

The proof of Lemma 371 (power function satisfies extended relaxed triangle inequality)
cites explicitly:
Lemma 300 (power function satisfies relaxed triangle inequality),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 372 (topology on $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 256 (order topology),
Lemma 258 (topological basis of order topology),
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
The proof of Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 226 (trace of subset system),
Lemma 252 (trace topology on subset),
Lemma 259 (trace of order topology),
Lemma 307 (topology on $\mathbb{R}$ ).
The proof of Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms)
cites explicitly:
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ),
Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$ ).
The proof of Lemma 377 (metric on $\overline{\mathbb{R}}$ )
cites explicitly:
Statement(s) from [20],
Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$ ),
Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms).
The proof of Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 249 (topological basis is generator),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ),
Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms).
The proof of Lemma 379 (convergence towards $\pm \infty$ )
cites explicitly:
Lemma 372 (topology on $\overline{\mathbb{R}}$ ).
The proof of Lemma 380 (convergence towards $\pm \infty$ (discrete))
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 379 (convergence towards $\pm \infty$ ).
The proof of Lemma 381 (limit is monotone)
cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 380 (convergence towards $\pm \infty$ (discrete)).
The proof of Lemma 382 (cluster point at $\pm \infty$ (discrete))
cites explicitly:
Definition 267 (cluster point),
Lemma 380 (convergence towards $\pm \infty$ (discrete)).

The proof of Lemma 383 (limit is invariant by translation)
has no explicit citation.
The proof of Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ )
cites explicitly:
Lemma 254 (box topology on Cartesian product),
Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ).
The proof of Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ )
cites explicitly:
Definition 226 (trace of subset system),
Lemma 252 (trace topology on subset),
Lemma 310 (topology on $\mathbb{R}^{m}$ ),
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ).
The proof of Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ).
The proof of Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ).
The proof of Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ).
The proof of Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear)
cites explicitly:
Statement(s) from [20],
Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous),
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous).
The proof of Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete $)$
cites explicitly:
Lemma 314 ( $\overline{\mathbb{R}}$ is complete),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ).
The proof of Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ).
The proof of Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined).
The proof of Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined).

The proof of Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined).
The proof of Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ ).
The proof of Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 392 ( $\overline{\mathbb{R}}_{+}$is complete),
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed).
The proof of Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition)
cites explicitly:
Statement(s) from [20],
Definition 209 (pseudopartition),
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone)
cites explicitly:
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition).
The proof of Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds)
cites explicitly:
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone).
The proof of Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds).
The proof of Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Statement(s) from [20],
Lemma 381 (limit is monotone),
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 407 (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 408 (zero-product property in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).

The proof of Lemma 409 (one-product property in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 411 (finite-product property in $\overline{\mathbb{R}}_{+}$)
cites explicitly:
Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).
The proof of Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone)
cites explicitly:
Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined), Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$).

The proof of Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous).
The proof of Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear)
cites explicitly:
Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear).
The proof of Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)) cites explicitly:
Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory))
cites explicitly:
Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 419 ( multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)) cites explicitly:
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined),
Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined), Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).

The proof of Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory))
cites explicitly:
Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)) cites explicitly:
Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).

The proof of Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory))
cites explicitly:
Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory))
cites explicitly:
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory))
cites explicitly:
Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory))
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) cites explicitly:
Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory))
cites explicitly:
Definition 354 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory))
cites explicitly:
Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ ).
The proof of Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)) cites explicitly:
Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 434 (finite-positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).

The proof of Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) cites explicitly:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) cites explicitly:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory))
cites explicitly:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)) cites explicitly:
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 361 ( power function in $\overline{\mathbb{R}}$ ),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory))
cites explicitly:
Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ),
Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ),
Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition),
Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory))
cites explicitly:
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory))
cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)).
The proof of Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)) cites explicitly:

Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 445 (power function in $\overline{\mathbb{R}}$ is even (measure theory))
cites explicitly:
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even).
The proof of Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) cites explicitly:
Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ),
Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 448 (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)) cites explicitly:
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory))
cites explicitly:
Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)).
The proof of Lemma 453 (Young's inequality for products (measure theory)) cites explicitly:
Lemma 304 (Young's inequality for products in $\mathbb{R}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)).
The proof of Lemma 454 (Young's inequality for products, case $p=2$ (measure theory))
cites explicitly:
Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ),
Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)),
Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 453 (Young's inequality for products (measure theory)).
The proof of Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ ) cites explicitly:
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$),
Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone).
The proof of Lemma 457 (empty connected component)
cites explicitly:
Definition 456 (connected component in $\mathbb{R}$ ).
The proof of Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval)
cites explicitly:
Statement(s) from [20],

Definition 229 (interval),
Definition 240 (topological space),
Definition 456 (connected component in $\mathbb{R}$ ).
The proof of Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal)
cites explicitly:
Definition 456 (connected component in $\mathbb{R}$ ),
Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval).
The proof of Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint)
cites explicitly:
Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal).
The proof of Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ )
cites explicitly:
Statement(s) from [20],
Definition 229 (interval),
Definition 456 (connected component in $\mathbb{R}$ ),
Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint).
The proof of Lemma 462 (rational approximation of lower bound of open interval) has no explicit citation.

The proof of Lemma 463 (rational approximation of upper bound of open interval) cites explicitly:
Lemma 462 (rational approximation of lower bound of open interval).
The proof of Lemma 464 (open intervals with rational bounds cover open interval) cites explicitly:
Statement(s) from [20],
Definition 240 (topological space),
Lemma 462 (rational approximation of lower bound of open interval),
Lemma 463 (rational approximation of upper bound of open interval).
The proof of Theorem 465 ( $\mathbb{R}$ is second-countable)
cites explicitly:
Lemma 250 (topological subbasis),
Definition 260 (second-countability),
Lemma 307 (topology on $\mathbb{R}$ ),
Lemma 464 (open intervals with rational bounds cover open interval).
The proof of Lemma $466\left(\mathbb{R}^{m}\right.$ is second-countable)
cites explicitly:
Lemma 261 (compatibility of second-countability with Cartesian product),
Lemma 310 (topology on $\mathbb{R}^{m}$ ),
Theorem $465(\mathbb{R}$ is second-countable).
The proof of Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ).
The proof of Lemma 471 (compatibility of projection and injections in $\overline{\mathbb{R}}$ with subset operations)
cites explicitly:
Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ).

## The proof of Lemma 474 (open down)

cites explicitly:
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ).
The proof of Lemma 475 (open up)
cites explicitly:
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ),
Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}$ ),
Lemma 474 (open down).
The proof of Lemma 476 (open subsets of $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 246 (topological basis),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ),
Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ ),
Lemma 474 (open down),
Lemma 475 (open up).
The proof of Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ )
cites explicitly:
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ),
Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).
The proof of Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ )
cites explicitly:
Lemma 462 (rational approximation of lower bound of open interval),
Lemma 463 (rational approximation of upper bound of open interval),
Lemma 464 (open intervals with rational bounds cover open interval).
The proof of Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable)
cites explicitly:
Lemma 250 (topological subbasis),
Definition 260 (second-countability),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ),
Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ),
Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ).
The proof of Lemma 480 ( $\overline{\mathbb{R}}^{m}$ is second-countable)
cites explicitly:
Lemma 261 (compatibility of second-countability with Cartesian product),
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ),
Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable).
The proof of Lemma 482 (extrema of constant function)
cites explicitly:
Statement(s) from [20].
The proof of Lemma 483 (compatibility of extrema with affine transformation) cites explicitly:
Statement(s) from [20],
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined).
The proof of Lemma 484 (equivalent definition of finite infimum)
cites explicitly:
Statement(s) from [20],
Lemma 265 (equivalent definition of convergent sequence).

The proof of Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ) cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 482 (extrema of constant function),
Lemma 484 (equivalent definition of finite infimum).
The proof of Lemma 486 (equivalent definition of infimum)
cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 380 (convergence towards $\pm \infty$ (discrete)),
Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ).
The proof of Lemma 487 (infimum is smaller than supremum)
cites explicitly:
Statement(s) from [20],
Lemma 313 (order in $\overline{\mathbb{R}}$ is total).
The proof of Lemma 488 (infimum is monotone (set))
cites explicitly:
Statement(s) from [20],
Lemma 313 (order in $\overline{\mathbb{R}}$ is total).
The proof of Lemma 489 (supremum is monotone (set))
cites explicitly:
Statement(s) from [20],
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 488 (infimum is monotone (set)).
The proof of Lemma 490 (infimum is monotone (function))
cites explicitly:
Statement(s) from [20],
Lemma 313 (order in $\overline{\mathbb{R}}$ is total).
The proof of Lemma 491 (supremum is monotone (function))
cites explicitly:
Statement(s) from [20],
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 490 (infimum is monotone (function)).
The proof of Lemma 492 (infimum of bounded is bounded)
cites explicitly:
Lemma 482 (extrema of constant function),
Lemma 490 (infimum is monotone (function)).
The proof of Lemma 493 (supremum of bounded is bounded)
cites explicitly:
Lemma 482 (extrema of constant function),
Lemma 491 (supremum is monotone (function)).
The proof of Lemma 494 (compatibility of infimum with absolute value) cites explicitly:
Statement(s) from [20],
Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ),

Lemma 487 (infimum is smaller than supremum),
Lemma 490 (infimum is monotone (function)),
Lemma 491 (supremum is monotone (function)).
The proof of Lemma 495 (compatibility of supremum with absolute value)
cites explicitly:
Statement(s) from [20],
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even),
Lemma 494 (compatibility of infimum with absolute value).
The proof of Lemma 496 (compatibility of translation with infimum of sequence)
cites explicitly:
Lemma 488 (infimum is monotone (set)).
The proof of Lemma 497 (compatibility of translation with supremum of sequence)
cites explicitly:
Lemma 489 (supremum is monotone (set)).
The proof of Lemma 498 (infimum of nonincreasing sequence)
cites explicitly:
Statement(s) from [20],
Lemma 496 (compatibility of translation with infimum of sequence).
The proof of Lemma 499 (supremum of nondecreasing sequence)
cites explicitly:
Statement(s) from [20],
Lemma 498 (infimum of nonincreasing sequence).
The proof of Lemma 500 (infimum is limit)
cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 379 (convergence towards $\pm \infty$ ),
Lemma 482 (extrema of constant function).
The proof of Lemma 501 (supremum is limit)
cites explicitly:
Statement(s) from [20],
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous),
Lemma 500 (infimum is limit).
The proof of Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum)
cites explicitly:
Statement(s) from [20],
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma 488 (infimum is monotone (set)).
The proof of Lemma 503 (constant sum of monotone nonnegative functions)
cites explicitly:
Statement(s) from [20],
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous),
Lemma 500 (infimum is limit),
Lemma 501 (supremum is limit).
The proof of Lemma 504 (limit inferior)
cites explicitly:
Lemma 314 ( $\overline{\mathbb{R}}$ is complete),
Lemma 488 (infimum is monotone (set)).

The proof of Lemma 505 (limit inferior is $\infty$ )
cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 504 (limit inferior).
The proof of Lemma 506 (equivalent definition of the limit inferior)
cites explicitly:
Statement(s) from [20],
Definition 267 (cluster point),
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 381 (limit is monotone),
Lemma 382 (cluster point at $\pm \infty$ (discrete)),
Lemma 504 (limit inferior),
Lemma 505 (limit inferior is $\infty$ ).
The proof of Lemma 507 (limit inferior is invariant by translation) cites explicitly:
Definition 267 (cluster point),
Lemma 383 (limit is invariant by translation),
Lemma 506 (equivalent definition of the limit inferior).
The proof of Lemma 508 (limit inferior is monotone)
cites explicitly:
Lemma 490 (infimum is monotone (function)),
Lemma 491 (supremum is monotone (function)),
Lemma 504 (limit inferior),
Lemma 507 (limit inferior is invariant by translation).
The proof of Lemma 509 (limit superior)
cites explicitly:
Lemma 314 ( $\overline{\mathbb{R}}$ is complete).
The proof of Lemma 510 (duality limit inferior-limit superior)
cites explicitly:
Statement(s) from [20],
Lemma 504 (limit inferior),
Lemma 509 (limit superior).
The proof of Lemma 511 (equivalent definition of limit superior)
cites explicitly:
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 510 (duality limit inferior-limit superior).
The proof of Lemma 512 (limit inferior is smaller than limit superior) cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 511 (equivalent definition of limit superior).
The proof of Lemma 513 (limit superior is monotone)
cites explicitly:
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 508 (limit inferior is monotone),
Lemma 510 (duality limit inferior-limit superior).

The proof of Lemma 514 (compatibility of limit inferior with absolute value)
cites explicitly:
Lemma 381 (limit is monotone),
Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous),
Lemma 494 (compatibility of infimum with absolute value),
Lemma 504 (limit inferior),
Lemma 509 (limit superior).
The proof of Lemma 515 (compatibility of limit superior with absolute value)
cites explicitly:
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even),
Lemma 510 (duality limit inferior-limit superior),
Lemma 514 (compatibility of limit inferior with absolute value).
The proof of Lemma 517 (limit inferior and limit superior of pointwise convergent)
cites explicitly:
Lemma 506 (equivalent definition of the limit inferior),
Lemma 511 (equivalent definition of limit superior),
Definition 516 (pointwise convergence).
The proof of Lemma 518 (limit inferior bounded from below)
cites explicitly:
Statement(s) from [20],
Lemma 508 (limit inferior is monotone),
Lemma 517 (limit inferior and limit superior of pointwise convergent).
The proof of Lemma 519 (limit inferior bounded from above)
cites explicitly:
Statement(s) from [20],
Lemma 508 (limit inferior is monotone),
Lemma 517 (limit inferior and limit superior of pointwise convergent).
The proof of Lemma 520 (limit superior bounded from below)
cites explicitly:
Lemma 510 (duality limit inferior-limit superior),
Lemma 519 (limit inferior bounded from above).
The proof of Lemma 521 (limit superior bounded from above)
cites explicitly:
Lemma 510 (duality limit inferior-limit superior),
Lemma 518 (limit inferior bounded from below).
The proof of Lemma 522 (limit inferior, limit superior and pointwise convergence)
cites explicitly:
Statement(s) from [20],
Lemma 505 (limit inferior is $\infty$ ),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 510 (duality limit inferior-limit superior),
Lemma 511 (equivalent definition of limit superior),
Lemma 512 (limit inferior is smaller than limit superior).
The proof of Lemma 524 (restriction is masking) has no explicit citation.

The proof of Lemma 525 (masking is $\sigma$-additive)
cites explicitly:
Definition 209 (pseudopartition).

The proof of Lemma 527 (finite part is finite)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 526 (finite part).
The proof of Lemma 529 (equivalent definition of nonnegative and nonpositive parts)
cites explicitly:
Statement(s) from [20],
Definition 528 (nonnegative and nonpositive parts).
The proof of Lemma 530 (nonnegative and nonpositive parts are nonnegative)
cites explicitly:
Statement(s) from [20],
Definition 528 (nonnegative and nonpositive parts).
The proof of Lemma 531 (nonnegative and nonpositive parts are orthogonal)
cites explicitly:
Statement(s) from [20],
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ).
The proof of Lemma 532 (decomposition into nonnegative and nonpositive parts)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Lemma 531 (nonnegative and nonpositive parts are orthogonal).
The proof of Lemma 533 (compatibility of nonpositive and nonnegative parts with addition)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$),
Definition 528 (nonnegative and nonpositive parts),
Lemma 530 (nonnegative and nonpositive parts are nonnegative),
Lemma 531 (nonnegative and nonpositive parts are orthogonal),
Lemma 532 (decomposition into nonnegative and nonpositive parts).
The proof of Lemma 534 (compatibility of nonpositive and nonnegative parts with mask)
cites explicitly:
Definition 528 (nonnegative and nonpositive parts).
The proof of Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction)
cites explicitly:
Definition 528 (nonnegative and nonpositive parts).
The proof of Lemma 538 (nonempty and with empty or full)
has no explicit citation.
The proof of Lemma 539 (with empty and full)
has no explicit citation.
The proof of Lemma 541 (closedness under local complement and complement)
has no explicit citation.

The proof of Lemma 542 (closedness under disjoint union and local complement) has no explicit citation.

The proof of Lemma 543 (closedness under set difference and local complement) has no explicit citation.

The proof of Lemma 544 (closedness under symmetric difference, local complement, and disjoint union)
has no explicit citation.
The proof of Lemma 545 (closedness under intersection and set difference) has no explicit citation.

The proof of Lemma 546 (closedness under union and disjoint union) has no explicit citation.

The proof of Lemma 547 (closedness under union and intersection) has no explicit citation.

The proof of Lemma 548 (closedness under union and set difference) cites explicitly: Lemma 545 (closedness under intersection and set difference), Lemma 547 (closedness under union and intersection).

The proof of Lemma 550 (closedness under symmetric difference and set difference) has no explicit citation.

The proof of Lemma 551 (closedness under symmetric difference and intersection) has no explicit citation.

The proof of Lemma 552 (nonempty is with empty)
has no explicit citation.
The proof of Lemma 553 (with full is with empty)
has no explicit citation.
The proof of Lemma 554 (disjoint decomposition)
has no explicit citation.
The proof of Lemma 555 (disjoint decomposition of complement and set difference) has no explicit citation.

The proof of Lemma 556 (disjoint decomposition of local complement and complement)
has no explicit citation.
The proof of Lemma 557 (disjoint decomposition of set difference and local complement)
has no explicit citation.
The proof of Lemma 558 (closedness under finite operations)
has no explicit citation.
The proof of Lemma 559 (closedness under finite union and intersection)
cites explicitly:
Lemma 547 (closedness under union and intersection), Lemma 558 (closedness under finite operations).

The proof of Lemma 561 (closedness under intersection and finite disjoint union closure)
has no explicit citation.
The proof of Lemma 562 (finite disjoint union closure is closed under finite disjoint union)
has no explicit citation.
The proof of Lemma 563 (closedness under set difference and finite disjoint union closure)
cites explicitly:
Lemma 561 (closedness under intersection and finite disjoint union closure),
Lemma 562 (finite disjoint union closure is closed under finite disjoint union).
The proof of Lemma 566 (closedness under countable disjoint union and local complement)
cites explicitly:
Lemma 542 (closedness under disjoint union and local complement).
The proof of Lemma 567 (closedness under countable union and intersection)
has no explicit citation.
The proof of Lemma 568 (closedness under countable disjoint and monotone union)
cites explicitly:
Lemma 215 (disjoint union of union).
The proof of Lemma 569 (closedness under countable monotone and disjoint union) cites explicitly:
Lemma 217 (monotone union of union).
The proof of Lemma 570 (closedness under countable disjoint union and countable union)
cites explicitly:
Lemma 215 (disjoint union of union).
The proof of Lemma 571 (closedness under countable monotone union and countable union)
cites explicitly:
Lemma 569 (closedness under countable monotone and disjoint union),
Lemma 570 (closedness under countable disjoint union and countable union).
The proof of Lemma 572 (disjoint union of union in subset system)
cites explicitly:
Lemma 215 (disjoint union of union),
Lemma 558 (closedness under finite operations).
The proof of Lemma 574 (disjoint union of union in subset system (variant))
cites explicitly:
Lemma 215 (disjoint union of union),
Lemma 552 (nonempty is with empty).
The proof of Lemma 575 (monotone union of union in subset system)
cites explicitly:
Lemma 217 (monotone union of union),
Lemma 558 (closedness under finite operations).
The proof of Lemma 576 (empty monotone intersection of union in subset system) cites explicitly:
Lemma 219 (empty monotone intersection of union).

The proof of Lemma 577 (monotone union of monotone intersection in subset system)
cites explicitly:
Lemma 221 (monotone union of monotone intersection).
The proof of Lemma 582 (intersection of subset system classes)
cites explicitly:
Definition 580 (subset system class).
The proof of Lemma 583 (subset system classes are monotone)
cites explicitly:
Definition 580 (subset system class).
The proof of Lemma 585 (subset system generation preserves class)
cites explicitly:
Definition 578 (subset system),
Definition 580 (subset system class),
Definition 584 (generated subset system).
The proof of Lemma 586 (properties of subset system generation)
cites explicitly:
Definition 578 (subset system),
Definition 580 (subset system class),
Definition 584 (generated subset system),
Lemma 585 (subset system generation preserves class).
The proof of Lemma 587 (other properties of subset system generation)
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation).
The proof of Lemma 588 (generated subset systems are monotone)
cites explicitly:
Lemma 583 (subset system classes are monotone), Definition 584 (generated subset system).

The proof of Lemma 590 (upgrade generated subset system)
cites explicitly:
Definition 580 (subset system class),
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation).
The proof of Lemma 591 (cascading generated subset systems)
cites explicitly:
Lemma 583 (subset system classes are monotone),
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation).
The proof of Lemma 592 (double cascading generated subset systems) has no explicit citation.

The proof of Lemma 594 (class of nonempty subset systems)
cites explicitly:
Definition 580 (subset system class),
Lemma 582 (intersection of subset system classes).

The proof of Lemma 595 (generated nonempty subset systems)
cites explicitly:
Definition 584 (generated subset system),
Lemma 594 (class of nonempty subset systems).
The proof of Lemma 596 (subset system generation preserves class (with nonemptiness))
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 594 (class of nonempty subset systems),
Lemma 595 (generated nonempty subset systems).
The proof of Lemma 597 (properties of generated subset system (with nonemptiness))
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 594 (class of nonempty subset systems),
Lemma 595 (generated nonempty subset systems).
The proof of Lemma 598 (other properties of subset system generation (with nonemptiness))
cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 595 (generated nonempty subset systems).
The proof of Lemma 599 (generated subset systems are monotone (with nonemptiness))
cites explicitly:
Lemma 588 (generated subset systems are monotone),
Lemma 595 (generated nonempty subset systems).
The proof of Lemma 600 (upgrade generated subset system (with nonemptiness))
cites explicitly:
Lemma 590 (upgrade generated subset system),
Lemma 595 (generated nonempty subset systems),
Lemma 596 (subset system generation preserves class (with nonemptiness)).
The proof of Lemma 601 (cascading generated subset systems (with nonemptiness)) cites explicitly:
Lemma 591 (cascading generated subset systems),
Lemma 595 (generated nonempty subset systems).
The proof of Lemma 604 (equivalent definition of $\pi$-system)
cites explicitly:
Lemma 558 (closedness under finite operations),
Definition 602 ( $\pi$-system).
The proof of Lemma 605 (intersection of $\pi$-systems)
cites explicitly:
Lemma 604 (equivalent definition of $\pi$-system).
The proof of Lemma 607 (generated $\pi$-system is $\pi$-system)
cites explicitly:
Lemma 596 (subset system generation preserves class (with nonemptiness)),
Lemma 605 (intersection of $\pi$-systems),
Definition 606 (generated $\pi$-system).

The proof of Lemma 608 (properties of $\pi$-system generation)
cites explicitly:
Lemma 597 (properties of generated subset system (with nonemptiness)),
Lemma 605 (intersection of $\pi$-systems),
Definition 606 (generated $\pi$-system).
The proof of Lemma 609 (other properties of $\pi$-system generation)
cites explicitly:
Lemma 598 (other properties of subset system generation (with nonemptiness)),
Lemma 605 (intersection of $\pi$-systems),
Definition 606 (generated $\pi$-system).
The proof of Lemma 612 (equivalent definition of ring of sets)
cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 544 (closedness under symmetric difference, local complement, and disjoint union),
Lemma 545 (closedness under intersection and set difference),
Lemma 546 (closedness under union and disjoint union),
Lemma 550 (closedness under symmetric difference and set difference),
Lemma 551 (closedness under symmetric difference and intersection),
Lemma 558 (closedness under finite operations),
Definition 611 (ring of sets).
The proof of Lemma 614 (ring of sets is $\pi$-system)
cites explicitly:
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 612 (equivalent definition of ring of sets).
The proof of Lemma 615 (other properties of ring of sets)
cites explicitly:
Lemma 558 (closedness under finite operations),
Definition 602 ( $\pi$-system),
Lemma 612 (equivalent definition of ring of sets),
Lemma 614 (ring of sets is $\pi$-system).
The proof of Lemma 616 (intersection of rings of sets)
cites explicitly:
Lemma 612 (equivalent definition of ring of sets).
The proof of Lemma 618 (generated ring of sets is ring of sets)
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets).
The proof of Lemma 619 (properties of ring of sets generation)
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 612 (equivalent definition of ring of sets),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets).
The proof of Lemma 620 (other properties of ring of sets generation)
cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets).

## The proof of Lemma 621 (ring of sets contains $\pi$-system)

cites explicitly:
Lemma 599 (generated subset systems are monotone (with nonemptiness)),
Definition 606 (generated $\pi$-system),
Lemma 614 (ring of sets is $\pi$-system),
Definition 617 (generated ring of sets).
The proof of Lemma 622 (upgrade $\pi$-system to ring of sets)
cites explicitly:
Lemma 600 (upgrade generated subset system (with nonemptiness)),
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 605 (intersection of $\pi$-systems),
Definition 606 (generated $\pi$-system),
Lemma 612 (equivalent definition of ring of sets),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets),
Lemma 621 (ring of sets contains $\pi$-system).
The proof of Lemma 623 (ring of sets generated by $\pi$-system)
cites explicitly:
Lemma 601 (cascading generated subset systems (with nonemptiness)), Lemma 605 (intersection of $\pi$-systems),
Definition 606 (generated $\pi$-system),
Lemma 614 (ring of sets is $\pi$-system),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets).
The proof of Lemma 624 (disjoint union of union in ring of sets)
cites explicitly:
Lemma 572 (disjoint union of union in subset system),
Lemma 612 (equivalent definition of ring of sets).
The proof of Lemma 627 (equivalent definition of semiring of sets)
cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 552 (nonempty is with empty),
Definition 626 (semiring of sets).
The proof of Lemma 629 (semiring of sets is $\pi$-system)
cites explicitly:
Lemma 604 (equivalent definition of $\pi$-system),
Definition 626 (semiring of sets).
The proof of Lemma 630 (other properties of semiring of sets)
cites explicitly:
Lemma 557 (disjoint decomposition of set difference and local complement),
Definition 602 ( $\pi$-system),
Definition 626 (semiring of sets),
Lemma 629 (semiring of sets is $\pi$-system).
The proof of Lemma 631 (ring of sets is semiring of sets)
cites explicitly:
Lemma 554 (disjoint decomposition),
Lemma 612 (equivalent definition of ring of sets),
Definition 626 (semiring of sets).

The proof of Lemma 632 (explicit ring of sets)
cites explicitly:
Lemma 562 (finite disjoint union closure is closed under finite disjoint union),
Lemma 563 (closedness under set difference and finite disjoint union closure),
Definition 611 (ring of sets),
Lemma 612 (equivalent definition of ring of sets),
Lemma 618 (generated ring of sets is ring of sets),
Lemma 619 (properties of ring of sets generation),
Definition 626 (semiring of sets),
Lemma 627 (equivalent definition of semiring of sets).
The proof of Lemma 635 (equivalent definition of algebra of sets)
cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 539 (with empty and full),
Lemma 547 (closedness under union and intersection),
Lemma 558 (closedness under finite operations), Definition 634 (algebra of sets).

The proof of Lemma 637 (other equivalent definition of algebra of sets)
cites explicitly:
Lemma 541 (closedness under local complement and complement),
Lemma 543 (closedness under set difference and local complement),
Lemma 545 (closedness under intersection and set difference),
Lemma 635 (equivalent definition of algebra of sets).
The proof of Lemma 638 (algebra of sets is ring of sets)
cites explicitly:
Lemma 612 (equivalent definition of ring of sets),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 637 (other equivalent definition of algebra of sets).
The proof of Lemma 639 (other properties of algebra of sets)
cites explicitly:
Lemma 546 (closedness under union and disjoint union),
Lemma 558 (closedness under finite operations),
Lemma 612 (equivalent definition of ring of sets),
Lemma 615 (other properties of ring of sets),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 638 (algebra of sets is ring of sets).
The proof of Lemma 640 (algebra of sets is $\pi$-system)
cites explicitly:
Lemma 614 (ring of sets is $\pi$-system),
Lemma 638 (algebra of sets is ring of sets).
The proof of Lemma 641 (intersection of algebras of sets)
cites explicitly:
Definition 634 (algebra of sets).
The proof of Lemma 643 (generated algebra of sets is algebra of sets)
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets).

The proof of Lemma 644 (properties of algebra of sets generation)
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets).
The proof of Lemma 645 (other properties of algebra of sets generation)
cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets).
The proof of Lemma 646 (algebra of sets contains ring of sets)
cites explicitly:
Lemma 588 (generated subset systems are monotone),
Definition 617 (generated ring of sets),
Lemma 638 (algebra of sets is ring of sets),
Definition 642 (generated algebra of sets).
The proof of Lemma 647 (upgrade ring of sets to algebra of sets)
cites explicitly:
Lemma 590 (upgrade generated subset system),
Lemma 612 (equivalent definition of ring of sets),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 638 (algebra of sets is ring of sets),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets),
Lemma 646 (algebra of sets contains ring of sets).
The proof of Lemma 648 (algebra of sets generated by ring of sets)
cites explicitly:
Lemma 591 (cascading generated subset systems),
Lemma 616 (intersection of rings of sets),
Definition 617 (generated ring of sets),
Lemma 638 (algebra of sets is ring of sets),
Lemma 641 (intersection of algebras of sets), Definition 642 (generated algebra of sets).

The proof of Lemma 649 (algebra of sets contains $\pi$-system) cites explicitly:
Lemma 621 (ring of sets contains $\pi$-system),
Lemma 646 (algebra of sets contains ring of sets).
The proof of Lemma 650 (upgrade $\pi$-system to algebra of sets) cites explicitly:
Lemma 545 (closedness under intersection and set difference),
Lemma 547 (closedness under union and intersection),
Lemma 550 (closedness under symmetric difference and set difference),
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 614 (ring of sets is $\pi$-system),
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 647 (upgrade ring of sets to algebra of sets).

The proof of Lemma 651 (algebra of sets generated by $\pi$-system)
cites explicitly:
Lemma 592 (double cascading generated subset systems),
Definition 606 (generated $\pi$-system),
Definition 617 (generated ring of sets),
Lemma 623 (ring of sets generated by $\pi$-system),
Definition 642 (generated algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets).
The proof of Lemma 652 (disjoint union of union in algebra of sets)
cites explicitly:
Lemma 624 (disjoint union of union in ring of sets),
Lemma 638 (algebra of sets is ring of sets).
The proof of Lemma 655 (equivalent definition of semi-algebra of sets) cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 555 (disjoint decomposition of complement and set difference),
Lemma 556 (disjoint decomposition of local complement and complement),
Lemma 557 (disjoint decomposition of set difference and local complement),
Definition 626 (semiring of sets),
Definition 654 (semi-algebra of sets).
The proof of Lemma 657 (semi-algebra of sets is semiring of sets)
cites explicitly:
Definition 654 (semi-algebra of sets).
The proof of Lemma 658 (other properties of semi-algebra of sets)
cites explicitly:
Lemma 630 (other properties of semiring of sets),
Lemma 657 (semi-algebra of sets is semiring of sets).
The proof of Lemma 659 (semi-algebra of sets is $\pi$-system)
cites explicitly:
Lemma 629 (semiring of sets is $\pi$-system),
Lemma 657 (semi-algebra of sets is semiring of sets).
The proof of Lemma 660 (algebra of sets is semi-algebra of sets)
cites explicitly:
Lemma 554 (disjoint decomposition),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 655 (equivalent definition of semi-algebra of sets).
The proof of Lemma 661 (algebra of sets is semiring of sets)
cites explicitly:
Lemma 657 (semi-algebra of sets is semiring of sets),
Lemma 660 (algebra of sets is semi-algebra of sets).
The proof of Lemma 662 (explicit algebra of sets) cites explicitly:
Lemma 618 (generated ring of sets is ring of sets),
Lemma 632 (explicit ring of sets),
Lemma 647 (upgrade ring of sets to algebra of sets),
Definition 654 (semi-algebra of sets).
The proof of Lemma 665 (intersection of monotone classes)
cites explicitly:
Definition 664 (monotone class).

The proof of Lemma 667 (generated monotone class is monotone class) cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 665 (intersection of monotone classes),
Definition 666 (generated monotone class).
The proof of Lemma 668 (properties of monotone class generation)
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 665 (intersection of monotone classes),
Definition 666 (generated monotone class).
The proof of Lemma 669 (other properties of monotone class generation) cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 665 (intersection of monotone classes),
Definition 666 (generated monotone class).
The proof of Lemma 671 ( $\mathcal{C} \backslash$ is symmetric)
cites explicitly:
Definition 670 (monotone class and symmetric set difference).
The proof of Lemma 672 ( $\mathcal{C}$ is monotone class)
cites explicitly:
Definition 664 (monotone class),
Definition 670 (monotone class and symmetric set difference).
The proof of Lemma 674 (monotone class is closed under set difference) cites explicitly:
Lemma 667 (generated monotone class is monotone class),
Lemma 668 (properties of monotone class generation),
Definition 670 (monotone class and symmetric set difference),
Lemma 671 ( $\mathcal{C}$ \is symmetric),
Lemma 672 ( $\mathcal{C}$ \is monotone class).
The proof of Lemma 675 (monotone class generated by algebra of sets) cites explicitly:
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 667 (generated monotone class is monotone class),
Lemma 674 (monotone class is closed under set difference).
The proof of Lemma 678 (equivalent definition of $\lambda$-system)
cites explicitly:
Lemma 541 (closedness under local complement and complement),
Lemma 542 (closedness under disjoint union and local complement),
Lemma 566 (closedness under countable disjoint union and local complement),
Lemma 568 (closedness under countable disjoint and monotone union),
Lemma 569 (closedness under countable monotone and disjoint union),
Definition 677 ( $\lambda$-system).
The proof of Lemma 679 ( $\lambda$-system is monotone class)
cites explicitly:
Lemma 567 (closedness under countable union and intersection),
Definition 664 (monotone class),
Definition 677 ( $\lambda$-system),
Lemma 678 (equivalent definition of $\lambda$-system).

The proof of Lemma 680 (other properties of $\lambda$-system)
cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 539 (with empty and full),
Lemma 558 (closedness under finite operations),
Definition 664 (monotone class),
Definition 677 ( $\lambda$-system),
Lemma 679 ( $\lambda$-system is monotone class).

## The proof of Lemma 681 (intersection of $\lambda$-systems)

cites explicitly:
Definition 677 ( $\lambda$-system).
The proof of Lemma 683 (generated $\lambda$-system is $\lambda$-system)
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 681 (intersection of $\lambda$-systems),
Definition 682 (generated $\lambda$-system).
The proof of Lemma 684 (properties of $\lambda$-system generation)
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 681 (intersection of $\lambda$-systems),
Definition 682 (generated $\lambda$-system).
The proof of Lemma 685 (other properties of $\lambda$-system generation)
cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 681 (intersection of $\lambda$-systems), Definition 682 (generated $\lambda$-system).

The proof of Lemma 686 ( $\lambda$-system contains monotone class) cites explicitly:
Lemma 599 (generated subset systems are monotone (with nonemptiness)),
Definition 666 (generated monotone class),
Lemma 679 ( $\lambda$-system is monotone class),
Definition 682 (generated $\lambda$-system).
The proof of Lemma 687 (upgrade monotone class to $\lambda$-system)
cites explicitly:
Lemma 541 (closedness under local complement and complement),
Lemma 567 (closedness under countable union and intersection),
Lemma 590 (upgrade generated subset system),
Definition 664 (monotone class),
Lemma 665 (intersection of monotone classes),
Definition 666 (generated monotone class),
Lemma 678 (equivalent definition of $\lambda$-system),
Lemma 681 (intersection of $\lambda$-systems),
Definition 682 (generated $\lambda$-system),
Lemma 686 ( $\lambda$-system contains monotone class).
The proof of Lemma 688 ( $\lambda$-system generated by monotone class)
cites explicitly:
Lemma 591 (cascading generated subset systems),
Lemma 665 (intersection of monotone classes),
Definition 666 (generated monotone class),

Lemma 679 ( $\lambda$-system is monotone class),
Lemma 681 (intersection of $\lambda$-systems), Definition 682 (generated $\lambda$-system).

The proof of Lemma $690\left(\Lambda^{\cap}\right.$ is symmetric)
cites explicitly:
Definition 689 ( $\lambda$-system and intersection).
The proof of Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system)
cites explicitly:
Definition 677 ( $\lambda$-system),
Lemma 678 (equivalent definition of $\lambda$-system).
The proof of Lemma 692 ( $\lambda$-system with intersection)
cites explicitly:
Lemma 683 (generated $\lambda$-system is $\lambda$-system),
Lemma 684 (properties of $\lambda$-system generation),
Definition 689 ( $\lambda$-system and intersection),
Lemma 690 ( $\Lambda^{\cap}$ is symmetric),
Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system).
The proof of Lemma 693 ( $\lambda$-system is closed under intersection)
cites explicitly:
Definition 689 ( $\lambda$-system and intersection),
Lemma 692 ( $\lambda$-system with intersection).
The proof of Lemma 694 ( $\lambda$-system generated by $\pi$-system)
cites explicitly:
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 683 (generated $\lambda$-system is $\lambda$-system),
Lemma 693 ( $\lambda$-system is closed under intersection).
The proof of Lemma 696 ( $\sigma$-algebra)
cites explicitly:
Definition 634 (algebra of sets).
The proof of Lemma 697 (equivalent definition of $\sigma$-algebra)
cites explicitly:
Lemma 538 (nonempty and with empty or full),
Lemma 539 (with empty and full),
Lemma 567 (closedness under countable union and intersection),
Lemma 696 ( $\sigma$-algebra).
The proof of Lemma 699 ( $\sigma$-algebra is algebra of sets)
cites explicitly:
Lemma 696 ( $\sigma$-algebra).
The proof of Lemma 700 (other properties of $\sigma$-algebra)
cites explicitly:
Definition 634 (algebra of sets),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 639 (other properties of algebra of sets),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 699 ( $\sigma$-algebra is algebra of sets).

The proof of Lemma 701 ( $\sigma$-algebra is $\lambda$-system)
cites explicitly:
Definition 677 ( $\lambda$-system),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra).
The proof of Lemma 702 ( $\sigma$-algebra is monotone class)
cites explicitly:
Lemma 679 ( $\lambda$-system is monotone class),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system).
The proof of Lemma 703 ( $\sigma$-algebra is ring of sets)
cites explicitly:
Lemma 638 (algebra of sets is ring of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets).
The proof of Lemma 704 ( $\sigma$-algebra is $\pi$-system)
cites explicitly:
Lemma 640 (algebra of sets is $\pi$-system),
Lemma 699 ( $\sigma$-algebra is algebra of sets).
The proof of Lemma 705 ( $\sigma$-algebra is semi-algebra of sets)
cites explicitly:
Lemma 660 (algebra of sets is semi-algebra of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets).
The proof of Lemma 706 ( $\sigma$-algebra is semiring of sets)
cites explicitly:
Lemma 661 (algebra of sets is semiring of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets).
The proof of Lemma 707 (intersection of $\sigma$-algebras)
cites explicitly:
Lemma 696 ( $\sigma$-algebra).
The proof of Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra)
cites explicitly:
Lemma 585 (subset system generation preserves class),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra).
The proof of Lemma 710 (properties of $\sigma$-algebra generation)
cites explicitly:
Lemma 586 (properties of subset system generation),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra).
The proof of Lemma 711 (other properties of $\sigma$-algebra generation)
cites explicitly:
Lemma 587 (other properties of subset system generation),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra).
The proof of Lemma 712 (countable union $\sigma$-algebra generator)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 711 (other properties of $\sigma$-algebra generation).

## The proof of Lemma 713 ( $\sigma$-algebra contains $\lambda$-system)

cites explicitly:
Lemma 588 (generated subset systems are monotone),
Definition 682 (generated $\lambda$-system),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Definition 708 (generated $\sigma$-algebra).

## The proof of Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra)

cites explicitly:
Lemma 545 (closedness under intersection and set difference),
Lemma 570 (closedness under countable disjoint union and countable union),
Lemma 590 (upgrade generated subset system),
Definition 677 ( $\lambda$-system),
Lemma 681 (intersection of $\lambda$-systems),
Definition 682 (generated $\lambda$-system),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra),
Lemma 713 ( $\sigma$-algebra contains $\lambda$-system).
The proof of Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system)
cites explicitly:
Lemma 591 (cascading generated subset systems),
Lemma 681 (intersection of $\lambda$-systems), Definition 682 (generated $\lambda$-system), Lemma 701 ( $\sigma$-algebra is $\lambda$-system), Lemma 707 (intersection of $\sigma$-algebras), Definition 708 (generated $\sigma$-algebra).

The proof of Lemma 716 ( $\sigma$-algebra contains monotone class)
cites explicitly:
Lemma 686 ( $\lambda$-system contains monotone class),
Lemma 713 ( $\sigma$-algebra contains $\lambda$-system).
The proof of Lemma 717 (upgrade monotone class to $\sigma$-algebra)
cites explicitly:
Lemma 539 (with empty and full),
Lemma 542 (closedness under disjoint union and local complement),
Lemma 546 (closedness under union and disjoint union),
Lemma 547 (closedness under union and intersection),
Definition 677 ( $\lambda$-system),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra).
The proof of Lemma 718 ( $\sigma$-algebra generated by monotone class)
cites explicitly:
Lemma 592 (double cascading generated subset systems),
Definition 666 (generated monotone class),
Definition 682 (generated $\lambda$-system),
Lemma 688 ( $\lambda$-system generated by monotone class),
Definition 708 (generated $\sigma$-algebra),
Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system).

The proof of Lemma 719 ( $\sigma$-algebra contains algebra of sets)
cites explicitly:
Lemma 588 (generated subset systems are monotone),
Definition 642 (generated algebra of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Definition 708 (generated $\sigma$-algebra).
The proof of Lemma 720 (upgrade algebra of sets to $\sigma$-algebra)
cites explicitly:
Lemma 570 (closedness under countable disjoint union and countable union),
Lemma 571 (closedness under countable monotone union and countable union),
Lemma 590 (upgrade generated subset system),
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 639 (other properties of algebra of sets),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets),
Lemma 696 ( $\sigma$-algebra),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra),
Lemma 719 ( $\sigma$-algebra contains algebra of sets).
The proof of Lemma 721 ( $\sigma$-algebra generated by algebra of sets)
cites explicitly:
Lemma 591 (cascading generated subset systems),
Lemma 641 (intersection of algebras of sets),
Definition 642 (generated algebra of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Lemma 707 (intersection of $\sigma$-algebras),
Definition 708 (generated $\sigma$-algebra).
The proof of Lemma 722 ( $\sigma$-algebra contains ring of sets)
cites explicitly:
Lemma 646 (algebra of sets contains ring of sets),
Lemma 719 ( $\sigma$-algebra contains algebra of sets).
The proof of Lemma 723 (upgrade ring of sets to $\sigma$-algebra)
cites explicitly:
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).

## The proof of Lemma 724 ( $\sigma$-algebra generated by ring of sets)

 cites explicitly:Lemma 592 (double cascading generated subset systems),
Definition 617 (generated ring of sets),
Definition 642 (generated algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets),
Definition 708 (generated $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets).
The proof of Lemma 725 ( $\sigma$-algebra contains $\pi$-system)
cites explicitly:
Lemma 621 (ring of sets contains $\pi$-system),
Lemma 722 ( $\sigma$-algebra contains ring of sets).

## The proof of Lemma 726 (upgrade $\pi$-system to $\sigma$-algebra)

cites explicitly:
Lemma 650 (upgrade $\pi$-system to algebra of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).
The proof of Lemma 727 ( $\sigma$-algebra generated by $\pi$-system)
cites explicitly:
Lemma 592 (double cascading generated subset systems),
Definition 606 (generated $\pi$-system),
Definition 617 (generated ring of sets),
Lemma 623 (ring of sets generated by $\pi$-system),
Definition 708 (generated $\sigma$-algebra),
Lemma 724 ( $\sigma$-algebra generated by ring of sets).
The proof of Lemma 728 (disjoint union of union in $\sigma$-algebra)
cites explicitly:
Lemma 572 (disjoint union of union in subset system),
Lemma 700 (other properties of $\sigma$-algebra).
The proof of Lemma 730 (algebra of sets generated by product of $\sigma$-algebras)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 655 (equivalent definition of semi-algebra of sets),
Lemma 662 (explicit algebra of sets),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra).
The proof of Theorem 732 (Dynkin $\pi-\lambda$ theorem)
cites explicitly:
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 694 ( $\lambda$-system generated by $\pi$-system),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 704 ( $\sigma$-algebra is $\pi$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra).
The proof of Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem)
cites explicitly:
Lemma 607 (generated $\pi$-system is $\pi$-system),
Lemma 684 (properties of $\lambda$-system generation),
Definition 708 (generated $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation), Theorem 732 (Dynkin $\pi-\lambda$ theorem).

The proof of Theorem 736 (monotone class)
cites explicitly:
Lemma 635 (equivalent definition of algebra of sets),
Lemma 675 (monotone class generated by algebra of sets),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Lemma 702 ( $\sigma$-algebra is monotone class),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).
The proof of Lemma 738 (usage of monotone class theorem)
cites explicitly:
Lemma 643 (generated algebra of sets is algebra of sets),
Lemma 668 (properties of monotone class generation),
Definition 708 (generated $\sigma$-algebra),

Lemma 710 (properties of $\sigma$-algebra generation),
Theorem 736 (monotone class).
The proof of Lemma 740 (properties of premeasurable space)
cites explicitly:
Lemma 554 (disjoint decomposition),
Definition 611 (ring of sets),
Lemma 612 (equivalent definition of ring of sets),
Lemma 614 (ring of sets is $\pi$-system),
Lemma 615 (other properties of ring of sets),
Lemma 631 (ring of sets is semiring of sets),
Definition 739 (premeasurable space).
The proof of Lemma 742 (properties of measurable space)
cites explicitly:
Lemma 554 (disjoint decomposition),
Lemma 631 (ring of sets is semiring of sets),
Lemma 660 (algebra of sets is semi-algebra of sets),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 702 ( $\sigma$-algebra is monotone class),
Lemma 703 ( $\sigma$-algebra is ring of sets),
Lemma 704 ( $\sigma$-algebra is $\pi$-system),
Definition 739 (premeasurable space),
Definition 741 (measurable space).
The proof of Lemma 744 (some Borel subsets)
cites explicitly:
Definition 240 (topological space),
Lemma 696 ( $\sigma$-algebra),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Definition 743 (Borel $\sigma$-algebra).
The proof of Lemma 745 (countable union Borel $\sigma$-algebra generator)
cites explicitly:
Lemma 712 (countable union $\sigma$-algebra generator),
Definition 743 (Borel $\sigma$-algebra).
The proof of Lemma 750 (inverse $\sigma$-algebra)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 749 (measurable function).
The proof of Lemma 751 (image $\sigma$-algebra)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 749 (measurable function).
The proof of Lemma 752 (identity function is measurable)
cites explicitly:
Definition 749 (measurable function).

The proof of Lemma 753 (countable range is measurable)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra),
Definition 741 (measurable space),
Definition 749 (measurable function).
The proof of Lemma 754 (constant function is measurable)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 753 (countable range is measurable).
The proof of Lemma 755 (inverse image of generating family)
cites explicitly:
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation),
Lemma 750 (inverse $\sigma$-algebra),
Lemma 751 (image $\sigma$-algebra).
The proof of Lemma 756 (equivalent definition of measurable function)
cites explicitly:
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation),
Definition 749 (measurable function),
Lemma 755 (inverse image of generating family).
The proof of Lemma 757 (compatibility of measurability with composition)
cites explicitly:
Definition 749 (measurable function).
The proof of Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable)
cites explicitly:
Lemma 744 (some Borel subsets),
Definition 749 (measurable function),
Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$.
The proof of Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range $)$
cites explicitly:
Lemma 753 (countable range is measurable),
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable).
The proof of Lemma 761 (continuous is measurable)
cites explicitly:
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Definition 743 (Borel $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 756 (equivalent definition of measurable function).
The proof of Lemma 763 (trace $\sigma$-algebra)
cites explicitly:
Definition 226 (trace of subset system),
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 749 (measurable function).

The proof of Lemma 764 (measurability of measurable subspace)
cites explicitly:
Definition 226 (trace of subset system),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 763 (trace $\sigma$-algebra).
The proof of Lemma 765 (generating measurable subspace)
cites explicitly:
Definition 226 (trace of subset system),
Lemma 755 (inverse image of generating family),
Lemma 764 (measurability of measurable subspace).
The proof of Lemma 766 (generating on a partition)
cites explicitly:
Lemma 211 (compatibility of pseudopartition with intersection),
Definition 226 (trace of subset system),
Lemma 696 ( $\sigma$-algebra),
Lemma 765 (generating measurable subspace).
The proof of Lemma 768 (Borel sub- $\sigma$-algebra)
cites explicitly:
Lemma 252 (trace topology on subset),
Definition 743 (Borel $\sigma$-algebra),
Lemma 765 (generating measurable subspace).
The proof of Lemma 769 (Borel subsets of partition)
cites explicitly:
Lemma 766 (generating on a partition).
The proof of Lemma 770 (source restriction of measurable function)
cites explicitly:
Definition 226 (trace of subset system),
Definition 749 (measurable function),
Lemma 757 (compatibility of measurability with composition),
Lemma 763 (trace $\sigma$-algebra).
The proof of Lemma 771 (destination restriction of measurable function)
cites explicitly:
Definition 749 (measurable function).
The proof of Lemma 772 (measurability of function defined on a pseudopartition)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 749 (measurable function).
The proof of Lemma 775 (product of measurable subsets is measurable)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Definition 774 (tensor product of $\sigma$-algebras).
The proof of Lemma 776 (measurability of function to product space)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 756 (equivalent definition of measurable function),

Definition 774 (tensor product of $\sigma$-algebras),
Lemma 775 (product of measurable subsets is measurable).
The proof of Lemma 777 (canonical projection is measurable)
cites explicitly:
Definition 749 (measurable function),
Lemma 776 (measurability of function to product space).
The proof of Lemma 778 (permutation is measurable)
cites explicitly:
Lemma 776 (measurability of function to product space),
Lemma 777 (canonical projection is measurable).
The proof of Lemma 779 (generating product measurable space)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation),
Definition 749 (measurable function),
Lemma 755 (inverse image of generating family),
Definition 774 (tensor product of $\sigma$-algebras),
Lemma 777 (canonical projection is measurable).
The proof of Lemma 780 (Borel subsets in product measurable subspace)
cites explicitly:
Lemma 252 (trace topology on subset),
Lemma 254 (box topology on Cartesian product),
Definition 743 (Borel $\sigma$-algebra),
Lemma 765 (generating measurable subspace),
Lemma 779 (generating product measurable space).

## The proof of Lemma 784 (section of product)

cites explicitly:
Definition 783 (section in Cartesian product).

## The proof of Lemma 785 (compatibility of section with set operations)

cites explicitly:
Definition 783 (section in Cartesian product).
The proof of Lemma 786 (measurability of section)
cites explicitly:
Definition 227 (product of subset systems),
Lemma 696 ( $\sigma$-algebra),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Definition 741 (measurable space),
Definition 774 (tensor product of $\sigma$-algebras),
Lemma 784 (section of product),
Lemma 785 (compatibility of section with set operations).
The proof of Lemma 787 (countable union of sections is measurable)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 785 (compatibility of section with set operations),
Lemma 786 (measurability of section).

The proof of Lemma 788 (countable intersection of sections is measurable)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 785 (compatibility of section with set operations),
Lemma 786 (measurability of section).
The proof of Lemma 789 (indicator of section)
cites explicitly:
Definition 783 (section in Cartesian product).
The proof of Lemma 790 (measurability of function from product space)
cites explicitly:
Definition 749 (measurable function),
Lemma 786 (measurability of section).
The proof of Lemma 794 (some Borel subsets of numbers)
cites explicitly:
Lemma 744 (some Borel subsets).
The proof of Lemma 796 (Borel subsets of $\mathbb{R}$ )
cites explicitly:
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ),
Lemma 696 ( $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra),
Definition 708 (generated $\sigma$-algebra),
Lemma 711 (other properties of $\sigma$-algebra generation),
Lemma 745 (countable union Borel $\sigma$-algebra generator),
Lemma 794 (some Borel subsets of numbers).
The proof of Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ )
cites explicitly:
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ),
Theorem $465(\mathbb{R}$ is second-countable),
Lemma 796 (Borel subsets of $\mathbb{R}$ ).
The proof of Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ )
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 743 (Borel $\sigma$-algebra),
Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 769 (Borel subsets of partition).
The proof of Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 708 (generated $\sigma$-algebra),
Lemma 766 (generating on a partition),
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ).
The proof of Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ )
cites explicitly:
Definition 226 (trace of subset system),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ ).
The proof of Lemma 803 (Borel subsets of $\mathbb{R}_{+}$)
cites explicitly:
Lemma 765 (generating measurable subspace),

Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 796 (Borel subsets of $\mathbb{R}$ ).

## The proof of Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$)

cites explicitly:
Lemma 765 (generating measurable subspace),
Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ).
The proof of Lemma 806 (Borel subsets of $\mathbb{C}$ )
cites explicitly:
Lemma 780 (Borel subsets in product measurable subspace),
Lemma 796 (Borel subsets of $\mathbb{R}$ ).
The proof of Lemma 808 (inverse image of countable is measurable)
cites explicitly:
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable $)$.
The proof of Lemma 809 (inverse image is measurable)
cites explicitly:
Lemma 808 (inverse image of countable is measurable).
The proof of Lemma $810\left(\mathcal{M}_{\mathbb{K}}\right.$ countable range $)$
cites explicitly:
Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range $)$.
The proof of Lemma 811 (measurability of indicator function)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 741 (measurable space),
Definition 748 (measurability and structure),
Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$ countable range).
The proof of Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra $)$
cites explicitly:
Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra),
Lemma 282 (algebra of functions to algebra),
Lemma 287 (closed under algebra operations is subalgebra),
Lemma 754 (constant function is measurable),
Lemma 757 (compatibility of measurability with composition),
Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$,
Lemma 761 (continuous is measurable),
Lemma 776 (measurability of function to product space),
Lemma 780 (Borel subsets in product measurable subspace).
The proof of Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$ is vector space)
cites explicitly:
Statement(s) from [20],
Definition 277 (algebra over a field),
Definition 284 (subalgebra),
Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra).
The proof of Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$ cites explicitly:
Lemma $814\left(\mathcal{M}_{\mathfrak{K}}\right.$ is vector space $)$.

The proof of Lemma 816 (measurability of numeric function to $\mathbb{R}$ )
cites explicitly:
Definition 743 (Borel $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 756 (equivalent definition of measurable function),
Lemma 794 (some Borel subsets of numbers),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$.
The proof of Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ )
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 749 (measurable function),
Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$,
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ),
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$,
Definition 817 ( $\mathcal{M}$, set of measurable numeric functions).
The proof of Lemma 820 (measurability of numeric function)
cites explicitly:
Lemma 372 (topology on $\overline{\mathbb{R}}$ ),
Definition 743 (Borel $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 756 (equivalent definition of measurable function),
Lemma 794 (some Borel subsets of numbers),
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ),
Definition 817 ( $\mathcal{M}$, set of measurable numeric functions).
The proof of Lemma 821 ( $\mathcal{M}$ is closed under finite part)
cites explicitly:
Definition 209 (pseudopartition),
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 526 (finite part),
Lemma 527 (finite part is finite),
Lemma 696 ( $\sigma$-algebra),
Lemma 754 (constant function is measurable),
Lemma 772 (measurability of function defined on a pseudopartition),
Lemma 808 (inverse image of countable is measurable),
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ).
The proof of Lemma 822 ( $\mathcal{M}$ is closed under addition when defined)
cites explicitly:
Definition 209 (pseudopartition),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 526 (finite part),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 743 (Borel $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 754 (constant function is measurable),
Lemma 772 (measurability of function defined on a pseudopartition),
Lemma 809 (inverse image is measurable),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra),
Definition 817 ( $\mathcal{M}$, set of measurable numeric functions),
Lemma 821 ( $\mathcal{M}$ is closed under finite part).

The proof of Lemma 823 ( $\mathcal{M}$ is closed under finite sum when defined)
cites explicitly:
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined $)$.
The proof of Lemma 824 ( $\mathcal{M}$ is closed under multiplication)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Definition 526 (finite part),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 743 (Borel $\sigma$-algebra),
Definition 749 (measurable function),
Lemma 754 (constant function is measurable),
Lemma 772 (measurability of function defined on a pseudopartition),
Lemma 809 (inverse image is measurable),
Lemma 813 ( $\mathcal{M}_{\mathfrak{K}}$ is algebra),
Definition 817 ( $\mathcal{M}$, set of measurable numeric functions),
Lemma 820 (measurability of numeric function),
Lemma 821 ( $\mathcal{M}$ is closed under finite part).
The proof of Lemma 825 ( $\mathcal{M}$ is closed under finite product)
cites explicitly:
Lemma 824 ( $\mathcal{M}$ is closed under multiplication).
The proof of Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication)
cites explicitly:
Lemma 754 (constant function is measurable),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication).
The proof of Lemma 828 ( $\mathcal{M}$ is closed under infimum)
cites explicitly:
Statement(s) from [20],
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 820 (measurability of numeric function).
The proof of Lemma 829 ( $\mathcal{M}$ is closed under supremum)
cites explicitly:
Statement(s) from [20],
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 820 (measurability of numeric function).
The proof of Lemma 830 ( $\mathcal{M}$ is closed under limit inferior $)$
cites explicitly:
Lemma 504 (limit inferior),
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma 829 ( $\mathcal{M}$ is closed under supremum).
The proof of Lemma 831 ( $\mathcal{M}$ is closed under limit superior $)$
cites explicitly:
Lemma 509 (limit superior),
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma 829 ( $\mathcal{M}$ is closed under supremum).
The proof of Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent) cites explicitly:

Lemma 522 (limit inferior, limit superior and pointwise convergence),
Lemma 830 ( $\mathcal{M}$ is closed under limit inferior),
Lemma $831(\mathcal{M}$ is closed under limit superior).
The proof of Lemma 833 (measurability and masking)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 811 (measurability of indicator function),
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ),
Lemma $824(\mathcal{M}$ is closed under multiplication).
The proof of Lemma 834 (measurability of restriction)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 741 (measurable space),
Definition 749 (measurable function),
Definition 817 ( $\mathcal{M}$, set of measurable numeric functions).
The proof of Lemma 836 (measurability of nonnegative and nonpositive parts)
cites explicitly:
Definition 528 (nonnegative and nonpositive parts),
Lemma 530 (nonnegative and nonpositive parts are nonnegative),
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 754 (constant function is measurable),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 829 ( $\mathcal{M}$ is closed under supremum),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma $837\left(\mathcal{M}_{+}\right.$is closed under masking $)$
cites explicitly:
Lemma 833 (measurability and masking),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part)
cites explicitly:
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part)
cites explicitly:
Definition 526 (finite part),
Lemma 529 (equivalent definition of nonnegative and nonpositive parts),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 820 (measurability of numeric function),
Lemma 833 (measurability and masking),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part).
The proof of Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable)
cites explicitly:
Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ),

Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)),
Lemma 753 (countable range is measurable),
Lemma 761 (continuous is measurable),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma 841 ( $\mathcal{M}$ is closed under power function)
cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)), Lemma 757 (compatibility of measurability with composition),
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$, Definition $817(\mathcal{M}$, set of measurable numeric functions),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).
The proof of Lemma 842 ( $\mathcal{M}$ is closed under absolute value)
cites explicitly:
Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value),
Lemma 841 ( $\mathcal{M}$ is closed under power function).
The proof of Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$
cites explicitly:
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$
cites explicitly:
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$ cites explicitly:
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).

The proof of Lemma $847\left(\mathcal{M}_{+}\right.$is closed under infimum)
cites explicitly:
Lemma 492 (infimum of bounded is bounded),
Lemma 828 ( $\mathcal{M}$ is closed under infimum).
The proof of Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum) cites explicitly: Lemma 493 (supremum of bounded is bounded), Lemma 829 ( $\mathcal{M}$ is closed under supremum).

The proof of Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent) cites explicitly:
Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete $)$,

Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions).
The proof of Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum)
cites explicitly:
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent).
The proof of Lemma 853 (measurability of tensor product of numeric functions)
cites explicitly:
Definition 748 (measurability and structure),
Lemma 757 (compatibility of measurability with composition),
Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$,
Lemma 777 (canonical projection is measurable),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra),
Lemma 825 ( $\mathcal{M}$ is closed under finite product),
Definition 851 (tensor product of numeric functions).
The proof of Lemma 858 (finite is bounded)
cites explicitly:
Definition 857 (finiteness).
The proof of Lemma 863 (decomposition over finite pseudopartition)
cites explicitly:
Lemma 211 (compatibility of pseudopartition with intersection),
Definition 862 (additivity).
The proof of Lemma 864 (binary inclusion-exclusion principle)
cites explicitly:
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Definition 862 (additivity).
The proof of Lemma 866 (subtractivity)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 862 (additivity),
Definition 865 (subtractivity).
The proof of Lemma 867 (monotonicity)
cites explicitly:
Definition 862 (additivity).
The proof of Lemma 869 (subadditivity)
cites explicitly:
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 557 (disjoint decomposition of set difference and local complement),
Lemma 574 (disjoint union of union in subset system (variant)),
Definition 859 (subadditivity),
Definition 862 (additivity),
Lemma 867 (monotonicity).
The proof of Lemma 871 ( $\sigma$-subadditive is subadditive)
cites explicitly:
Definition 859 (subadditivity),
Definition 870 ( $\sigma$-subadditivity).

## The proof of Lemma 873 ( $\sigma$-superadditive is superadditive)

cites explicitly:
Definition 861 (superadditivity),
Definition 872 ( $\sigma$-superadditivity).
The proof of Lemma 875 ( $\sigma$-additive is additive)
cites explicitly:
Definition 862 (additivity),
Definition 874 ( $\sigma$-additivity).
The proof of Lemma 876 (decomposition over countable pseudopartition)
cites explicitly:
Lemma 211 (compatibility of pseudopartition with intersection),
Definition 874 ( $\sigma$-additivity).
The proof of Lemma 878 ( $\sigma$-subadditivity)
cites explicitly:
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 557 (disjoint decomposition of set difference and local complement),
Lemma 574 (disjoint union of union in subset system (variant)),
Lemma 867 (monotonicity),
Definition 870 ( $\sigma$-subadditivity),
Definition 874 ( $\sigma$-additivity),
Lemma 875 ( $\sigma$-additive is additive).
The proof of Lemma 880 (continuity from below)
cites explicitly:
Lemma 501 (supremum is limit),
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma 554 (disjoint decomposition),
Lemma 572 (disjoint union of union in subset system),
Definition 862 (additivity),
Lemma 867 (monotonicity),
Definition 874 ( $\sigma$-additivity),
Lemma 875 ( $\sigma$-additive is additive),
Definition 879 (continuity from below).
The proof of Lemma 881 ( $\sigma$-additivity (from below))
cites explicitly:
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma 575 (monotone union of union in subset system),
Definition 862 (additivity),
Definition 874 ( $\sigma$-additivity),
Lemma 875 ( $\sigma$-additive is additive),
Definition 879 (continuity from below),
Lemma 880 (continuity from below).
The proof of Lemma 883 (continuity from above)
cites explicitly:
Statement(s) from [20],
Lemma 483 (compatibility of extrema with affine transformation),
Lemma 498 (infimum of nonincreasing sequence),
Lemma 500 (infimum is limit),
Lemma 554 (disjoint decomposition),
Lemma 577 (monotone union of monotone intersection in subset system),
Definition 865 (subtractivity),

Lemma 866 (subtractivity),
Lemma 867 (monotonicity),
Lemma 875 ( $\sigma$-additive is additive),
Definition 879 (continuity from below),
Lemma 880 (continuity from below),
Definition 882 (continuity from above).
The proof of Lemma 884 ( $\sigma$-additivity (from above))
cites explicitly:
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma 503 (constant sum of monotone nonnegative functions),
Lemma 576 (empty monotone intersection of union in subset system),
Definition 862 (additivity),
Definition 874 ( $\sigma$-additivity),
Lemma 875 ( $\sigma$-additive is additive),
Definition 882 (continuity from above),
Lemma 883 (continuity from above).
The proof of Lemma 886 (equivalent definition of $\sigma$-finiteness)
cites explicitly:
Lemma 575 (monotone union of union in subset system),
Definition 859 (subadditivity),
Definition 885 ( $\sigma$-finiteness).
The proof of Lemma 887 (finite is $\sigma$-finite)
cites explicitly:
Definition 857 (finiteness),
Definition 885 ( $\sigma$-finiteness).
The proof of Lemma 890 (content over finite pseudopartition)
cites explicitly:
Definition 602 ( $\pi$-system),
Lemma 863 (decomposition over finite pseudopartition),
Definition 888 (content).
The proof of Lemma 891 (content satisfies the binary inclusion-exclusion principle)
cites explicitly:
Definition 626 (semiring of sets),
Lemma 864 (binary inclusion-exclusion principle),
Definition 888 (content).
The proof of Lemma 892 (content is subtractive)
cites explicitly:
Lemma 612 (equivalent definition of ring of sets),
Definition 865 (subtractivity),
Lemma 866 (subtractivity),
Definition 888 (content).
The proof of Lemma 893 (content is monotone)
cites explicitly:
Lemma 630 (other properties of semiring of sets),
Lemma 867 (monotonicity),
Definition 888 (content).
The proof of Lemma 894 (content satisfies the finite Boole inequality)
cites explicitly:
Definition 626 (semiring of sets),

Lemma 869 (subadditivity),
Definition 888 (content).
The proof of Lemma 895 (finite content is bounded)
cites explicitly:
Definition 654 (semi-algebra of sets),
Lemma 658 (other properties of semi-algebra of sets),
Lemma 858 (finite is bounded),
Definition 888 (content),
Lemma 893 (content is monotone).
The proof of Lemma 896 ( $\sigma$-finite content)
cites explicitly:
Lemma 612 (equivalent definition of ring of sets),
Lemma 631 (ring of sets is semiring of sets),
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Lemma 894 (content satisfies the finite Boole inequality).
The proof of Lemma 897 (premeasure)
cites explicitly:
Definition 739 (premeasurable space),
Lemma 875 ( $\sigma$-additive is additive),
Definition 888 (content).
The proof of Lemma 899 (premeasure is additive)
cites explicitly:
Definition 888 (content),
Lemma 897 (premeasure).
The proof of Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 891 (content satisfies the binary inclusion-exclusion principle),
Lemma 897 (premeasure).
The proof of Lemma 901 (premeasure is subtractive)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 892 (content is subtractive),
Lemma 897 (premeasure).
The proof of Lemma 902 (premeasure is monotone)
cites explicitly:
Lemma 554 (disjoint decomposition),
Lemma 740 (properties of premeasurable space),
Lemma 893 (content is monotone),
Lemma 897 (premeasure).
The proof of Lemma 903 (finite premeasure is bounded)
cites explicitly:
Lemma 660 (algebra of sets is semi-algebra of sets),
Lemma 742 (properties of measurable space),
Lemma 895 (finite content is bounded),
Lemma 897 (premeasure).

The proof of Lemma 904 ( $\sigma$-finite premeasure)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 896 ( $\sigma$-finite content),
Lemma 897 (premeasure).
The proof of Lemma 905 (premeasure over countable pseudopartition)
cites explicitly:
Definition 602 ( $\pi$-system),
Lemma 740 (properties of premeasurable space),
Lemma 876 (decomposition over countable pseudopartition),
Lemma 897 (premeasure).
The proof of Lemma 906 (premeasure satisfies the Boole inequality)
cites explicitly:
Definition 626 (semiring of sets),
Lemma 740 (properties of premeasurable space),
Lemma 871 ( $\sigma$-subadditive is subadditive),
Lemma 878 ( $\sigma$-subadditivity),
Lemma 897 (premeasure).
The proof of Lemma 907 (compatibility of null premeasure with countable union)
cites explicitly:
Statement(s) from [20],
Lemma 740 (properties of premeasurable space),
Lemma 897 (premeasure),
Lemma 906 (premeasure satisfies the Boole inequality).
The proof of Lemma 908 (premeasure is continuous from below)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 880 (continuity from below),
Lemma 897 (premeasure).
The proof of Lemma 909 (equivalent definition of premeasure (from below))
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 897 (premeasure).
The proof of Lemma 910 (premeasure is continuous from above)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 883 (continuity from above),
Lemma 897 (premeasure).
The proof of Lemma 911 (equivalent definition of premeasure (from above)) cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 884 ( $\sigma$-additivity (from above)),
Lemma 897 (premeasure).
The proof of Lemma 912 (properties of premeasure space)
cites explicitly:
Lemma 740 (properties of premeasurable space),
Lemma 897 (premeasure),
Lemma 899 (premeasure is additive),

Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle),
Lemma 901 (premeasure is subtractive),
Lemma 902 (premeasure is monotone),
Lemma 904 ( $\sigma$-finite premeasure),
Lemma 905 (premeasure over countable pseudopartition),
Lemma 906 (premeasure satisfies the Boole inequality),
Lemma 908 (premeasure is continuous from below),
Lemma 910 (premeasure is continuous from above).
The proof of Lemma 913 (measure)
cites explicitly:
Lemma 700 (other properties of $\sigma$-algebra),
Definition 874 ( $\sigma$-additivity),
Lemma 897 (premeasure).
The proof of Lemma 915 (measure space is premeasure space)
cites explicitly:
Lemma 703 ( $\sigma$-algebra is ring of sets),
Definition 741 (measurable space),
Lemma 897 (premeasure),
Lemma 913 (measure).
The proof of Lemma 916 (measure is additive)
cites explicitly:
Lemma 700 (other properties of $\sigma$-algebra),
Definition 862 (additivity),
Lemma 899 (premeasure is additive),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 917 (measure satisfies the binary inclusion-exclusion principle)
cites explicitly:
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 864 (binary inclusion-exclusion principle),
Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 918 (measure is subtractive)
cites explicitly:
Lemma 700 (other properties of $\sigma$-algebra), Definition 865 (subtractivity), Lemma 901 (premeasure is subtractive), Lemma 915 (measure space is premeasure space).

The proof of Lemma 919 (measure is monotone)
cites explicitly:
Lemma 902 (premeasure is monotone), Lemma 915 (measure space is premeasure space).

The proof of Lemma 920 (finite measure is bounded) cites explicitly: Lemma 903 (finite premeasure is bounded), Lemma 915 (measure space is premeasure space).

The proof of Lemma 921 ( $\sigma$-finite measure) cites explicitly:

Lemma 904 ( $\sigma$-finite premeasure),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 922 (measure over countable pseudopartition)
cites explicitly:
Lemma 905 (premeasure over countable pseudopartition),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 923 (measure satisfies the Boole inequality)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 870 ( $\sigma$-subadditivity),
Lemma 906 (premeasure satisfies the Boole inequality),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 924 (compatibility of null measure with countable union)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 907 (compatibility of null premeasure with countable union),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 925 (measure is continuous from below)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 908 (premeasure is continuous from below),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 926 (equivalent definition of measure (from below))
cites explicitly:
Lemma 909 (equivalent definition of premeasure (from below)),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 927 (measure is continuous from above)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 910 (premeasure is continuous from above),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 928 (equivalent definition of measure (from above))
cites explicitly:
Lemma 911 (equivalent definition of premeasure (from above)),
Lemma 915 (measure space is premeasure space).
The proof of Lemma 929 (properties of measure space)
cites explicitly:
Lemma 742 (properties of measurable space),
Lemma 912 (properties of premeasure space),
Lemma 913 (measure).
The proof of Lemma 930 (trace measure)
cites explicitly:
Lemma 763 (trace $\sigma$-algebra),
Lemma 764 (measurability of measurable subspace),
Lemma 913 (measure).
The proof of Lemma 931 (restricted measure) cites explicitly:

Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 913 (measure).
The proof of Lemma 932 (trace measure is restricted measure)
cites explicitly:
Lemma 764 (measurability of measurable subspace),
Lemma 930 (trace measure),
Lemma 931 (restricted measure).
The proof of Lemma 933 (trivial measure)
cites explicitly:
Lemma 913 (measure).
The proof of Lemma 934 (equivalent definition of trivial measure)
cites explicitly:
Lemma 913 (measure),
Lemma 919 (measure is monotone),
Lemma 933 (trivial measure).
The proof of Lemma 935 (counting measure)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 874 ( $\sigma$-additivity),
Lemma 913 (measure).
The proof of Lemma 936 (finiteness of counting measure)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 857 (finiteness),
Lemma 935 (counting measure).
The proof of Lemma 937 ( $\sigma$-finite counting measure)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 885 ( $\sigma$-finiteness),
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Lemma 913 (measure),
Lemma 935 (counting measure).
The proof of Lemma 940 (equivalent definition of Dirac measure)
cites explicitly:
Lemma 935 (counting measure),
Definition 939 (Dirac measure).
The proof of Lemma 941 (Dirac measure is finite)
cites explicitly:
Lemma 935 (counting measure),
Lemma 936 (finiteness of counting measure),
Definition 939 (Dirac measure).
The proof of Lemma 946 (equivalent definition of considerable subset)
cites explicitly:
Lemma 913 (measure),
Definition 942 (negligible subset),
Definition 945 (considerable subset).

The proof of Lemma 947 (negligibility with trivial measure)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 933 (trivial measure),
Definition 942 (negligible subset).
The proof of Lemma 948 (negligibility of measurable subset)
cites explicitly:
Lemma 913 (measure),
Lemma 919 (measure is monotone),
Definition 942 (negligible subset).
The proof of Lemma 949 (empty set is negligible)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 913 (measure),
Lemma 948 (negligibility of measurable subset).
The proof of Lemma 951 (subset of negligible is negligible)
cites explicitly:
Definition 942 (negligible subset).
The proof of Lemma 952 ( N is closed under countable union) cites explicitly:
Lemma 924 (compatibility of null measure with countable union), Definition 942 (negligible subset).

The proof of Lemma 955 (equivalent definition of property almost satisfied)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 913 (measure),
Definition 942 (negligible subset),
Definition 953 (property almost satisfied).
The proof of Lemma 956 (property almost satisfied with trivial measure)
cites explicitly:
Lemma 947 (negligibility with trivial measure), Definition 953 (property almost satisfied).

The proof of Lemma 957 (measurable property almost satisfied)
cites explicitly:
Lemma 948 (negligibility of measurable subset), Definition 953 (property almost satisfied).

The proof of Lemma 958 (everywhere is almost everywhere)
cites explicitly:
Lemma 949 (empty set is negligible),
Definition 953 (property almost satisfied).
The proof of Lemma 959 (almost modus ponens)
cites explicitly:
Lemma 951 (subset of negligible is negligible),
Definition 953 (property almost satisfied).

The proof of Lemma 960 (countable almost properties is almost property)
cites explicitly:
Lemma 951 (subset of negligible is negligible),
Lemma 952 ( $\mathbf{N}$ is closed under countable union),
Definition 953 (property almost satisfied),
Lemma 955 (equivalent definition of property almost satisfied).
The proof of Lemma 962 (almost everywhere for almost the same)
cites explicitly:
Lemma 958 (everywhere is almost everywhere),
Lemma 959 (almost modus ponens),
Lemma 960 (countable almost properties is almost property).
The proof of Lemma 963 (extended almost modus ponens)
cites explicitly:
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 962 (almost everywhere for almost the same).
The proof of Lemma 965 (extended almost modus ponens (simplified 0))
cites explicitly:
Lemma 963 (extended almost modus ponens).
The proof of Lemma 966 (extended almost modus ponens (simplified 1)) cites explicitly:
Lemma 963 (extended almost modus ponens).
The proof of Lemma 967 (extended almost modus ponens (simplified 2)) cites explicitly:
Lemma 958 (everywhere is almost everywhere),
Lemma 966 (extended almost modus ponens (simplified 1)).
The proof of Lemma 970 (extension to almost defined)
cites explicitly:
Definition 953 (property almost satisfied),
Lemma 959 (almost modus ponens),
Lemma 960 (countable almost properties is almost property).
The proof of Lemma 972 (generalized almost modus ponens)
cites explicitly:
Lemma 959 (almost modus ponens),
Lemma 960 (countable almost properties is almost property),
Lemma 970 (extension to almost defined).
The proof of Lemma 975 (compatibility of almost binary relation with reflexivity) cites explicitly:
Lemma 970 (extension to almost defined),
Definition 974 (almost binary relation).
The proof of Lemma 977 (compatibility of almost binary relation with extended reflexivity)
cites explicitly:
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens),
Definition 974 (almost binary relation).
The proof of Lemma 978 (compatibility of almost binary relation with symmetry) cites explicitly:

Lemma 972 (generalized almost modus ponens), Definition 974 (almost binary relation).

The proof of Lemma 979 (compatibility of almost binary relation with extended antisymmetry)
cites explicitly:
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens),
Definition 974 (almost binary relation).
The proof of Lemma 980 (compatibility of almost binary relation with transitivity) cites explicitly:
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens),
Definition 974 (almost binary relation).
The proof of Lemma 981 (almost equivalence is equivalence relation)
cites explicitly:
Lemma 975 (compatibility of almost binary relation with reflexivity),
Lemma 978 (compatibility of almost binary relation with symmetry),
Lemma 980 (compatibility of almost binary relation with transitivity).
The proof of Lemma 982 (almost equality is equivalence relation)
cites explicitly:
Lemma 981 (almost equivalence is equivalence relation).
The proof of Lemma 983 (almost order is order relation)
cites explicitly:
Lemma 975 (compatibility of almost binary relation with reflexivity),
Lemma 979 (compatibility of almost binary relation with extended antisymmetry),
Lemma 980 (compatibility of almost binary relation with transitivity).
The proof of Lemma 984 (compatibility of almost binary relation with operator)
cites explicitly:
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens),
Definition 974 (almost binary relation).
The proof of Lemma 985 (compatibility of almost equality with operator) cites explicitly:
Lemma 984 (compatibility of almost binary relation with operator).
The proof of Lemma 988 (definiteness is almost definiteness)
cites explicitly:
Lemma 972 (generalized almost modus ponens).
The proof of Lemma 992 (uniqueness of measures extended from a $\pi$-system)
cites explicitly:
Definition 209 (pseudopartition),
Definition 602 ( $\pi$-system),
Lemma 608 (properties of $\pi$-system generation),
Lemma 678 (equivalent definition of $\lambda$-system),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem),
Definition 874 ( $\sigma$-additivity),
Definition 879 (continuity from below),

Lemma 913 (measure),
Lemma 919 (measure is monotone),
Lemma 925 (measure is continuous from below).
The proof of Lemma 993 (transitivity of almost nonnegativeness)
cites explicitly:
Lemma 980 (compatibility of almost binary relation with transitivity).
The proof of Lemma 994 (transitivity of almost inequality and equality)
cites explicitly:
Lemma 958 (everywhere is almost everywhere),
Lemma 977 (compatibility of almost binary relation with extended reflexivity),
Lemma 980 (compatibility of almost binary relation with transitivity).
The proof of Lemma 995 (compatibility of almost nonnegativeness with almost equal-
ity)
cites explicitly:
Lemma 977 (compatibility of almost binary relation with extended reflexivity),
Lemma 993 (transitivity of almost nonnegativeness).
The proof of Lemma 996 (almost finite)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ ),
Lemma 808 (inverse image of countable is measurable),
Lemma 957 (measurable property almost satisfied).
The proof of Lemma 998 (summability on summability domain)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 997 (summability domain).
The proof of Lemma 999 (measurability of summability domain)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 809 (inverse image is measurable).
The proof of Lemma 1000 (negligibility and summability domain)
cites explicitly:
Definition 953 (property almost satisfied),
Lemma 998 (summability on summability domain).
The proof of Lemma 1001 (absolute value and almost inequality)
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ),
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality),
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens),
Lemma 983 (almost order is order relation),
Lemma 1000 (negligibility and summability domain).
The proof of Lemma 1002 (almost sum)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),

Lemma 811 (measurability of indicator function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined), Lemma 824 ( $\mathcal{M}$ is closed under multiplication), Definition 953 (property almost satisfied), Lemma 999 (measurability of summability domain), Lemma 1000 (negligibility and summability domain).

The proof of Lemma 1003 (compatibility of almost sum with almost equality)
cites explicitly:
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 962 (almost everywhere for almost the same),
Lemma 982 (almost equality is equivalence relation),
Lemma 985 (compatibility of almost equality with operator),
Lemma 998 (summability on summability domain),
Lemma 1002 (almost sum).
The proof of Lemma 1004 (almost sum is sum)
cites explicitly:
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$,
Definition 997 (summability domain),
Lemma 998 (summability on summability domain),
Lemma 1002 (almost sum).
The proof of Lemma 1005 (absolute value is almost definite)
cites explicitly:
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite),
Lemma 985 (compatibility of almost equality with operator),
Lemma 988 (definiteness is almost definiteness).
The proof of Lemma 1006 (absolute value is almost equal to almost nonnegative function)
cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 972 (generalized almost modus ponens).
The proof of Lemma 1007 (power function is almost definite)
cites explicitly:
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite),
Lemma 985 (compatibility of almost equality with operator),
Lemma 988 (definiteness is almost definiteness).
The proof of Lemma 1008 (masking almost nowhere)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma 913 (measure),
Definition 942 (negligible subset),
Definition 953 (property almost satisfied).
The proof of Lemma 1009 (finite nonnegative part)
cites explicitly:
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 960 (countable almost properties is almost property),
Lemma 966 (extended almost modus ponens (simplified 1)).

The proof of Lemma 1010 (compatibility of almost limit with subsequence) cites explicitly:
Lemma 266 (compatibility of subsequence with convergence),
Lemma 972 (generalized almost modus ponens).
The proof of Lemma 1011 (compatibility of almost limit with continuous function) cites explicitly:
Statement(s) from [20],
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 958 (everywhere is almost everywhere),
Lemma 960 (countable almost properties is almost property),
Lemma 972 (generalized almost modus ponens).
The proof of Lemma 1012 (compatibility of almost limit with power function)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)),
Lemma 1011 (compatibility of almost limit with continuous function).
The proof of Lemma 1013 (almost limits are almost equal) cites explicitly:
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 960 (countable almost properties is almost property),
Lemma 967 (extended almost modus ponens (simplified 2)).
The proof of Lemma 1014 (compatibility of almost limit with almost equality)
cites explicitly:
Lemma 958 (everywhere is almost everywhere),
Lemma 960 (countable almost properties is almost property),
Lemma 962 (almost everywhere for almost the same).
The proof of Lemma 1015 (compatibility of almost limit with almost inequality) cites explicitly:
Statement(s) from [20],
Lemma 381 (limit is monotone),
Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous),
Lemma 960 (countable almost properties is almost property),
Lemma 967 (extended almost modus ponens (simplified 2)).
The proof of Lemma 1016 (compatibility of almost limit with measurability) cites explicitly:
Statement(s) from [20],
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma $832(\mathcal{M}$ is closed under limit when pointwise convergent),
Lemma 833 (measurability and masking),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Definition 953 (property almost satisfied),
Lemma 960 (countable almost properties is almost property),
Lemma 1008 (masking almost nowhere),
Lemma 1014 (compatibility of almost limit with almost equality).
The proof of Lemma 1017 (essential supremum)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ), Lemma 314 ( $\overline{\mathbb{R}}$ is complete),

Lemma 820 (measurability of numeric function),
Lemma 913 (measure),
Lemma 919 (measure is monotone).
The proof of Lemma 1019 (essential supremum with trivial measure)
cites explicitly:
Lemma 933 (trivial measure),
Lemma 1017 (essential supremum).
The proof of Lemma 1020 (essential supremum is bounded by supremum)
cites explicitly:
Statement(s) from [20],
Lemma 913 (measure),
Lemma 1017 (essential supremum).
The proof of Lemma 1021 (essential supremum is minimum)
cites explicitly:
Statement(s) from [20],
Lemma 486 (equivalent definition of infimum),
Lemma 924 (compatibility of null measure with countable union),
Lemma 1017 (essential supremum).
The proof of Lemma 1022 (essential supremum is almost upper bound)
cites explicitly:
Lemma 948 (negligibility of measurable subset),
Definition 974 (almost binary relation),
Lemma 1017 (essential supremum),
Lemma 1021 (essential supremum is minimum).
The proof of Lemma 1023 (essential supremum of constant function)
cites explicitly:
Lemma 754 (constant function is measurable),
Lemma 913 (measure),
Lemma 934 (equivalent definition of trivial measure),
Lemma 1017 (essential supremum).
The proof of Lemma 1024 (essential supremum of extended simple functions)
cites explicitly:
Definition 209 (pseudopartition),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 811 (measurability of indicator function),
Lemma 823 ( $\mathcal{M}$ is closed under finite sum when defined),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Definition 862 (additivity),
Lemma 913 (measure),
Lemma 916 (measure is additive),
Lemma 919 (measure is monotone),
Lemma 934 (equivalent definition of trivial measure),
Lemma 1017 (essential supremum).
The proof of Lemma 1026 (essential supremum of infinite times function)
cites explicitly:
Definition 209 (pseudopartition),
Lemma 809 (inverse image is measurable),
Lemma 820 (measurability of numeric function),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),

Lemma 933 (trivial measure),
Lemma 1017 (essential supremum),
Lemma 1019 (essential supremum with trivial measure),
Lemma 1024 (essential supremum of extended simple functions).
The proof of Lemma 1027 (essential supremum is positive homogeneous)
cites explicitly:
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 1017 (essential supremum),
Lemma 1021 (essential supremum is minimum),
Lemma 1023 (essential supremum of constant function).
The proof of Lemma 1029 (essential supremum is almost monotone)
cites explicitly:
Lemma 820 (measurability of numeric function),
Lemma 948 (negligibility of measurable subset),
Lemma 951 (subset of negligible is negligible),
Lemma 952 ( $\mathbf{N}$ is closed under countable union),
Definition 974 (almost binary relation),
Lemma 1017 (essential supremum),
Lemma 1021 (essential supremum is minimum).
The proof of Lemma 1030 (essential supremum bounded by constant)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 983 (almost order is order relation),
Lemma 1019 (essential supremum with trivial measure),
Lemma 1022 (essential supremum is almost upper bound),
Lemma 1023 (essential supremum of constant function),
Lemma 1029 (essential supremum is almost monotone).
The proof of Lemma 1031 (essential supremum of absolute value is nonnegative) cites explicitly:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 958 (everywhere is almost everywhere),
Lemma 1030 (essential supremum bounded by constant).
The proof of Lemma 1032 (compatibility of essential supremum with almost equality)
cites explicitly:
Lemma 977 (compatibility of almost binary relation with extended reflexivity),
Lemma 1029 (essential supremum is almost monotone).
The proof of Lemma 1036 (equivalent definition of nonnegative essential supremum)
cites explicitly:
Statement(s) from [20],
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 913 (measure),
Lemma 933 (trivial measure),
Lemma 934 (equivalent definition of trivial measure),
Lemma 1017 (essential supremum).
The proof of Lemma 1037 (nonnegative essential supremum with trivial measure) cites explicitly:

Lemma 933 (trivial measure),
Definition 1034 (nonnegative essential supremum),
Lemma 1036 (equivalent definition of nonnegative essential supremum).
The proof of Lemma 1038 (nonnegative essential supremum is minimum)
cites explicitly:
Lemma 1021 (essential supremum is minimum),
Definition 1034 (nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure).
The proof of Lemma 1039 (nonnegative essential supremum is almost upper bound)
cites explicitly:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 956 (property almost satisfied with trivial measure),
Lemma 1022 (essential supremum is almost upper bound),
Definition 1034 (nonnegative essential supremum).
The proof of Lemma 1040 (nonnegative essential supremum of constant function)
cites explicitly:
Lemma 754 (constant function is measurable),
Lemma 1023 (essential supremum of constant function),
Definition 1034 (nonnegative essential supremum).
The proof of Lemma 1041 (nonnegative essential supremum for extended simple functions)
cites explicitly:
Lemma 1024 (essential supremum of extended simple functions),
Definition 1034 (nonnegative essential supremum).
The proof of Lemma 1043 (nonnegative essential supremum of infinite times function)
cites explicitly:
Lemma 845 ( $\mathcal{M}_{+}$is closed under nonnegative scalar multiplication),
Lemma 933 (trivial measure),
Lemma 934 (equivalent definition of trivial measure),
Lemma 1026 (essential supremum of infinite times function),
Definition 1034 (nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure).
The proof of Lemma 1044 (nonnegative essential supremum is positive homogeneous)
cites explicitly:
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication), Lemma 1027 (essential supremum is positive homogeneous),
Definition 1034 (nonnegative essential supremum),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1038 (nonnegative essential supremum is minimum),
Lemma 1043 (nonnegative essential supremum of infinite times function).
The proof of Lemma 1045 (nonnegative essential supremum is almost monotone) cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 1029 (essential supremum is almost monotone),

Definition 1034 (nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure).
The proof of Lemma 1046 (nonnegative essential supremum bounded by constant) cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 956 (property almost satisfied with trivial measure),
Lemma 1030 (essential supremum bounded by constant),
Definition 1034 (nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure).
The proof of Lemma 1048 (essentially bounded is almost finite)
cites explicitly:
Lemma 958 (everywhere is almost everywhere),
Lemma 983 (almost order is order relation),
Lemma 1046 (nonnegative essential supremum bounded by constant),
Definition 1047 (essentially bounded).
The proof of Lemma 1049 (compatibility of nonnegative essential supremum with almost equality)
cites explicitly:
Lemma 1032 (compatibility of essential supremum with almost equality),
Definition 1034 (nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure).
The proof of Lemma 1050 (nonnegative essential supremum is almost definite) cites explicitly:
Definition 209 (pseudopartition),
Lemma 809 (inverse image is measurable),
Lemma 820 (measurability of numeric function),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 948 (negligibility of measurable subset),
Definition 953 (property almost satisfied),
Lemma 956 (property almost satisfied with trivial measure),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1038 (nonnegative essential supremum is minimum),
Lemma 1040 (nonnegative essential supremum of constant function),
Lemma 1049 (compatibility of nonnegative essential supremum with almost equality).
The proof of Lemma 1051 (nonnegative essential supremum is subadditive)
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1046 (nonnegative essential supremum bounded by constant).
The proof of Lemma 1053 (nonnegative essential supremum is submultiplicative)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication),
Lemma 984 (compatibility of almost binary relation with operator),

Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1046 (nonnegative essential supremum bounded by constant).
The proof of Lemma 1054 (nonnegative essential supremum is compatible with power function)
cites explicitly:
Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)),
Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1040 (nonnegative essential supremum of constant function),
Lemma 1046 (nonnegative essential supremum bounded by constant).

## The proof of Lemma 1058 (length is nonnegative)

cites explicitly:
Definition 1057 (length of interval).
The proof of Lemma 1059 (length is homogeneous)
cites explicitly:
Definition 1057 (length of interval).

## The proof of Lemma 1060 (length of partition)

cites explicitly:
Definition 1057 (length of interval),
Lemma 1059 (length is homogeneous).
The proof of Lemma 1062 (set of countable cover with bounded open intervals is nonempty)
cites explicitly:
Definition 1061 (set of countable cover with bounded open intervals).
The proof of Lemma 1065 ( $\lambda^{\star}$ is nonnegative)
cites explicitly:
Lemma 490 (infimum is monotone (function)),
Lemma 1058 (length is nonnegative).
The proof of Lemma 1066 ( $\lambda^{\star}$ is homogeneous)
cites explicitly:
Statement(s) from [20],
Definition 1057 (length of interval),
Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate), Lemma 1065 ( $\lambda^{\star}$ is nonnegative).

The proof of Lemma 1067 ( $\lambda^{\star}$ is monotone)
cites explicitly:
Statement(s) from [20],
Definition 1061 (set of countable cover with bounded open intervals), Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate).

The proof of Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive)
cites explicitly:
Statement(s) from [20],
Lemma 224 (definition of double countable union),
Lemma 225 (double countable union),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 381 (limit is monotone),
Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$),
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$),
Definition 1061 (set of countable cover with bounded open intervals), Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate).

The proof of Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval)
cites explicitly:
Statement(s) from [20],
Lemma 299 (finite cover of compact interval),
Lemma 381 (limit is monotone),
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Definition 1057 (length of interval),
Definition 1061 (set of countable cover with bounded open intervals),
Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate),
Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma 1067 ( $\lambda^{\star}$ is monotone).
The proof of Lemma 1073 (equivalent definition of $\mathcal{L}$ )
cites explicitly:
Lemma 211 (compatibility of pseudopartition with intersection),
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra).
The proof of Lemma 1074 ( $\mathcal{L}$ is closed under complement)
cites explicitly:
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra) .
The proof of Lemma 1075 ( $\mathcal{L}$ is closed under finite union)
cites explicitly:
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra) ,
Lemma 1073 (equivalent definition of $\mathcal{L}$ ).
The proof of Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection)
cites explicitly:
Lemma 1074 ( $\mathcal{L}$ is closed under complement),
Lemma 1075 ( $\mathcal{L}$ is closed under finite union).
The proof of Lemma 1077 ( $\mathcal{L}$ is algebra of sets)
cites explicitly:
Lemma 545 (closedness under intersection and set difference),
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma 1074 ( $\mathcal{L}$ is closed under complement),
Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection).
The proof of Lemma 1078 ( $\lambda^{\star}$ is additive on $\mathcal{L}$ )
cites explicitly:

Lemma 211 (compatibility of pseudopartition with intersection),
Definition 1071 (L, Lebesgue $\sigma$-algebra).
The proof of Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ )
cites explicitly:
Lemma 381 (limit is monotone),
Definition 874 ( $\sigma$-additivity),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1078 ( $\lambda^{\star}$ is additive on $\left.\mathcal{L}\right)$.
The proof of Lemma 1081 (disjoint union of union in $\mathcal{L}$ )
cites explicitly:
Lemma 652 (disjoint union of union in algebra of sets),
Lemma 1077 ( $\mathcal{L}$ is algebra of sets).
The proof of Lemma 1082 ( $\mathcal{L}$ is closed under countable union) cites explicitly:
Lemma 381 (limit is monotone),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra) ,
Lemma 1073 (equivalent definition of $\mathcal{L}$ ),
Lemma 1075 ( $\mathcal{L}$ is closed under finite union),
Lemma 1078 ( $\lambda^{\star}$ is additive on $\mathcal{L}$ ),
Lemma 1081 (disjoint union of union in $\mathcal{L}$ ).
The proof of Lemma 1083 (rays are Lebesgue-measurable)
cites explicitly:
Statement(s) from [20],
Lemma 235 (intervals are closed under finite intersection),
Lemma 381 (limit is monotone),
Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma 1073 (equivalent definition of $\mathcal{L}$ ),
Lemma 1074 ( $\mathcal{L}$ is closed under complement),
Lemma 1082 ( $\mathcal{L}$ is closed under countable union).
The proof of Lemma 1084 (intervals are Lebesgue-measurable)
cites explicitly:
Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection),
Lemma 1083 (rays are Lebesgue-measurable).
The proof of Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra)
cites explicitly:
Definition 634 (algebra of sets),
Lemma 696 ( $\sigma$-algebra),
Lemma 1077 ( $\mathcal{L}$ is algebra of sets),
Lemma 1082 ( $\mathcal{L}$ is closed under countable union).
The proof of Lemma 1086 ( $\lambda^{\star}$ is measure on $\mathcal{L}$ )
cites explicitly:
Lemma 913 (measure),
Lemma 1065 ( $\lambda^{\star}$ is nonnegative),

Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma $1080\left(\lambda^{\star}\right.$ is $\sigma$-additive on $\left.\mathcal{L}\right)$,
Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra).
The proof of Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$
cites explicitly:
Lemma 710 (properties of $\sigma$-algebra generation),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 1084 (intervals are Lebesgue-measurable).
The proof of Lemma 1088 ( $\lambda^{\star}$ is measure on $\mathcal{B}(\mathbb{R})$ )
cites explicitly:
Lemma 1086 ( $\lambda^{\star}$ is measure on $\left.\mathcal{L}\right)$,
Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$.
The proof of Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ )
cites explicitly:
Lemma 235 (intervals are closed under finite intersection),
Definition 602 ( $\pi$-system),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 927 (measure is continuous from above),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma $1088\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{B}(\mathbb{R})\right)$.
The proof of Lemma 1092 (Lebesgue measure generalizes length of interval) cites explicitly:
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).
The proof of Lemma 1094 (Lebesgue measure is $\sigma$-finite)
cites explicitly:
Definition 885 ( $\sigma$-finiteness),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).
The proof of Lemma 1095 (Lebesgue measure is diffuse)
cites explicitly:
Definition 856 (diffuseness),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).
The proof of Lemma 1100 (indicator and support are inverse)
cites explicitly:
Definition 748 (measurability and structure),
Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions).
The proof of Lemma $1101\left(\mathcal{I F}_{\mathbb{K}}\right.$ is measurable)
cites explicitly:
Lemma 811 (measurability of indicator function),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions).
The proof of Lemma $1102\left(\mathcal{I F} \mathcal{F}_{\mathfrak{K}}\right.$ is $\sigma$-additive $)$
cites explicitly:

Lemma 525 (masking is $\sigma$-additive),
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Definition 748 (measurability and structure),
Definition 1098 ( $\mathcal{I F}_{\mathbb{K}}$, set of measurable indicator functions).
The proof of Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under multiplication)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 748 (measurability and structure),
Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions).
The proof of Lemma $1105\left(\mathcal{I F}_{k}\right.$ is closed under extension by zero)
cites explicitly:
Lemma 524 (restriction is masking),
Definition 748 (measurability and structure),
Lemma 764 (measurability of measurable subspace),
Definition $1098\left(\mathcal{I}_{\mathfrak{K}}\right.$, set of measurable indicator functions).
The proof of Lemma $1106\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under restriction)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 748 (measurability and structure),
Lemma 764 (measurability of measurable subspace),
Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions).
The proof of Lemma $1107\left(\mathcal{I F}_{\mathfrak{K}}\right.$, masking is $\sigma$-additive)
cites explicitly:
Definition 209 (pseudopartition),
Lemma 525 (masking is $\sigma$-additive),
Definition 748 (measurability and structure),
Lemma 1100 (indicator and support are inverse),
Lemma $1102{\left.\text { ( } \mathcal{F}_{\mathfrak{K}} \text { is } \sigma \text {-additive }\right) \text {, }}_{\text {, }}$
Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under multiplication).
The proof of Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation $)$
cites explicitly:
Statement(s) from [20],
Definition $1098\left(\mathcal{I}_{\mathfrak{K}}\right.$, set of measurable indicator functions),
Lemma 1100 (indicator and support are inverse).
The proof of Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation)
cites explicitly:
Statement(s) from [20],
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation).
The proof of Lemma $1113\left(\mathcal{S \mathcal { F } _ { E }}\right.$ is measurable)
cites explicitly:
Definition 748 (measurability and structure),
Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$ countable range),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation),
Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation).
The proof of Lemma $1114\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation)
cites explicitly:

Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation),
Lemma $1111\left(\mathcal{S F}_{E}\right.$ canonical representation).
The proof of Lemma $1116\left(\mathcal{S \mathcal { F } _ { E }}\right.$ disjoint representation is subpartition of canonical representation)
cites explicitly:
Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation),
Lemma $1114\left(\mathcal{S F}_{E}\right.$ disjoint representation).
The proof of Lemma 1117 ( $\mathcal{S \mathcal { F } _ { E }}$ is vector space)
cites explicitly:
Statement(s) from [20],
Definition 209 (pseudopartition),
Lemma 286 (vector subspace and closed under multiplication is subalgebra),
Lemma 1102 ( $\mathcal{I F}_{\mathbb{K}}$ is $\sigma$-additive),
Lemma $1103\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under multiplication),
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation),
Lemma $1111\left(\mathcal{S F}_{E}\right.$ canonical representation),
Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation).
The proof of Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero)
cites explicitly:
Lemma 524 (restriction is masking),
Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions),
Lemma $1105\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under extension by zero),
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation).
The proof of Lemma $1120\left(\mathcal{S F}_{E}\right.$ is closed under restriction)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 764 (measurability of measurable subspace),
Lemma $1106\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under restriction),
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation).
The proof of Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive)
cites explicitly:
Statement(s) from [20],
Lemma 525 (masking is $\sigma$-additive),
Definition $1098\left(\mathcal{I F}_{\mathfrak{K}}\right.$, set of measurable indicator functions),
Lemma $1102\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is $\sigma$-additive),
Lemma $1103\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under multiplication),
Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation).
The proof of Lemma $1124(\mathcal{S F}+$ is measurable $)$
cites explicitly:
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Definition $1098\left(\mathcal{I}_{\mathfrak{K}}\right.$, set of measurable indicator functions),
Lemma 1113 ( $\mathcal{S} \mathcal{F}_{E}$ is measurable),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma $1125(\mathcal{S F}+$ disjoint representation)
cites explicitly:
Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation),
Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma $1126(\mathcal{S F}+$ canonical representation) cites explicitly:

Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma $1127(\mathcal{S F}+$ disjoint representation is subpartition of canonical representation)
cites explicitly:
Lemma $1116\left(\mathcal{S F}_{E}\right.$ disjoint representation is subpartition of canonical representation),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions $)$,
Lemma $1125\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation $)$,
Lemma $1126\left(\mathcal{S F}_{+}\right.$canonical representation $)$.
The proof of Lemma $1128\left(\mathcal{S F}_{+}\right.$simple representation $)$
cites explicitly:
Lemma $1108\left(\mathcal{S F}_{E}\right.$ simple representation),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma 1129 ( $\mathcal{S \mathcal { F } _ { + }}$ is closed under positive algebra operations)
cites explicitly:
Definition 277 (algebra over a field),
Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space),
Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions),
Lemma 1125 ( $\mathcal{S F}+$ disjoint representation).
The proof of Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero)
cites explicitly:
Lemma 524 (restriction is masking),
Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero),
Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma $1131(\mathcal{S F}+$ is closed under restriction $)$
cites explicitly:
Lemma $1120\left(\mathcal{S} \mathcal{F}_{E}\right.$ is closed under restriction),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma $1132\left(\mathcal{S F}_{+}\right.$, masking is $\sigma$-additive $)$
cites explicitly:
Lemma 1121 ( $\mathcal{S}_{E}$, masking is $\sigma$-additive),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions).
The proof of Lemma 1134 (Lebesgue scheme step 2)
cites explicitly:
Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions),
Lemma $1128\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation $)$.
The proof of Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 820 (measurability of numeric function),
Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions),
Definition 1135 (adapted sequence).
The proof of Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero)
cites explicitly:
Lemma 524 (restriction is masking),
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$,
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable),

Lemma $1128\left(\mathcal{S F}_{+}\right.$simple representation),
Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero $)$,
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$).
The proof of Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 764 (measurability of measurable subspace),
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$,
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable),
Lemma $1131\left(\mathcal{S F}_{+}\right.$is closed under restriction),
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$).
The proof of Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive $)$
cites explicitly:
Lemma 525 (masking is $\sigma$-additive),
Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$,
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum),
Lemma $1103\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under multiplication),
Lemma 1124 ( $\mathcal{S F}_{+}$is measurable).
The proof of Lemma 1141 (Lebesgue scheme step 3)
cites explicitly:
Lemma 1136 ( adapted sequence in $\mathcal{M}_{+}$).
The proof of Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ )
cites explicitly:
Lemma 1100 (indicator and support are inverse),
Definition 1144 (integral in $\mathcal{I F}$ ).
The proof of Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive)
cites explicitly:
Definition 874 ( $\sigma$-additivity),
Lemma 913 (measure),
Lemma 1102 ( $\mathcal{I F}_{\mathfrak{K}}$ is $\sigma$-additive),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).
The proof of Lemma 1147 (integral in $\mathcal{I F}$ is monotone)
cites explicitly:
Lemma 919 (measure is monotone),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).
The proof of Lemma 1148 (integral in $\mathcal{I F}$ over subset)
cites explicitly:
Lemma 932 (trace measure is restricted measure),
Lemma $1105\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under extension by zero),
Definition 1144 (integral in $\mathcal{I F}$ ).
The proof of Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive)
cites explicitly:
Lemma 1107 ( $\mathcal{F}_{\mathfrak{K}}$, masking is $\sigma$-additive),
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive),
Lemma 1148 (integral in $\mathcal{I F}$ over subset).
The proof of Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset)
cites explicitly:
Lemma 913 (measure),

Lemma 919 (measure is monotone),
Lemma 1148 (integral in $\mathcal{I F}$ over subset).
The proof of Lemma 1152 (integral in IF with counting measure)
cites explicitly:
Lemma 935 (counting measure),
Lemma 1100 (indicator and support are inverse),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).
The proof of Lemma 1154 (integral in $\mathcal{S F}{ }_{+}$)
cites explicitly:
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 913 (measure),
Lemma 1126 ( $\mathcal{S F}_{+}$canonical representation),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ).
The proof of Lemma 1156 (integral in $\mathcal{S F}+$ generalizes integral in $\mathcal{I F}$ )
cites explicitly:
Definition $1098\left(\mathcal{I F}_{\mathbb{K}}\right.$, set of measurable indicator functions),
Lemma $1128(\mathcal{S F}+$ simple representation),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ),
Lemma 1154 (integral in $\mathcal{S F}_{+}$).
The proof of Lemma 1157 (equivalent definition of the integral in $\mathcal{S \mathcal { F } _ { + }}$ (disjoint)) cites explicitly:
Definition 874 ( $\sigma$-additivity),
Lemma 913 (measure),
Lemma 1126 ( $\mathcal{S F}_{+}$canonical representation),
Lemma $1127\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation is subpartition of canonical representation $)$,
Lemma 1154 (integral in $\mathcal{S F}_{+}$).
The proof of Lemma 1158 (integral in $\mathcal{S F}{ }_{+}$of zero is zero)
cites explicitly:
Lemma 696 ( $\sigma$-algebra),
Lemma 913 (measure),
Lemma 1156 (integral in $\mathcal{S F}{ }_{+}$generalizes integral in $\mathcal{I F}$ ).
The proof of Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 922 (measure over countable pseudopartition),
Lemma $1125\left(\mathcal{S F}_{+}\right.$disjoint representation),
Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations),
Lemma 1154 (integral in $\mathcal{S \mathcal { F }}{ }_{+}$),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(disjoint)),
Lemma 1158 (integral in $\mathcal{S F}_{+}$of zero is zero).
The proof of Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}_{+}$(simple))
cites explicitly:
Lemma $1128(\mathcal{S F}+$ simple representation $)$,
Lemma 1156 (integral in $\mathcal{S F}+$ generalizes integral in $\mathcal{I F}$ ),
Lemma 1159 (integral in $\mathcal{S F}+$ is positive linear).
The proof of Lemma 1161 (integral in $\mathcal{S F _ { + }}$ is monotone) cites explicitly:

Statement(s) from [20],
Definition 277 (algebra over a field),
Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions),
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear).
The proof of Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous)
cites explicitly:
Statement(s) from [20],
Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone).
The proof of Lemma 1163 (integral in $\mathcal{S F}{ }_{+}$over subset)
cites explicitly:
Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero $)$,
Lemma 1148 (integral in $\mathcal{I F}$ over subset),
Lemma 1156 (integral in $\mathcal{S F} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$ ),
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear).
The proof of Lemma 1164 (integral in $\mathcal{S F}{ }_{+}$over subset is $\sigma$-additive)
cites explicitly:
Lemma $1132\left(\mathcal{S F}_{+}\right.$, masking is $\sigma$-additive $)$,
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear),
Lemma 1163 (integral in $\mathcal{S F} \mathcal{F}_{+}$over subset).
The proof of Lemma 1165 (integral in $\mathcal{S F}_{+}$over negligible subset)
cites explicitly:
Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset),
Lemma 1163 (integral in $\mathcal{S F} \mathcal{F}_{+}$over subset).
The proof of Lemma 1166 (density measure in $\mathcal{S \mathcal { F } _ { + } \text { ) }}$
cites explicitly:
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),
Lemma 696 ( $\sigma$-algebra),
Lemma 913 (measure),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions),
Lemma $1125\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation $)$,
Lemma 1129 ( $\mathcal{S F}_{+}$is closed under positive algebra operations),
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1156 (integral in $\mathcal{S F}+$ generalizes integral in $\mathcal{I F}$ ),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)).
The proof of Lemma 1167 (integral and density measure in $\mathcal{S \mathcal { F } _ { + } \text { ) }}$
cites explicitly:
Lemma $1125\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation),
Lemma $1129\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under positive algebra operations),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(disjoint)),
Lemma 1166 (density measure in $\mathcal{S F}{ }_{+}$).

cites explicitly:
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),
Lemma $1125\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation),
Lemma 1129 ( $\mathcal{S F} \mathcal{F}_{+}$is closed under positive algebra operations),

Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1166 (density measure in $\mathcal{S F}_{+}$).
The proof of Lemma 1169 (integral in $\mathcal{S \mathcal { F }}{ }_{+}$with trivial measure)
cites explicitly:
Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ),
Lemma 933 (trivial measure),
Lemma 1154 (integral in $\mathcal{S F}_{+}$).
The proof of Lemma 1170 (integral in $\mathcal{S F}{ }_{+}$with counting measure)
cites explicitly:
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Lemma $1128(\mathcal{S F}+$ simple representation $)$,
Lemma 1152 (integral in $\mathcal{I F}$ with counting measure),
Lemma 1156 (integral in $\mathcal{S F}_{+}$generalizes integral in $\mathcal{I F}$ ).
The proof of Lemma 1172 (integral in $\mathcal{M}_{+}$)
cites explicitly:
Statement(s) from [20],
Lemma 1154 (integral in $\mathcal{S F}{ }_{+}$).
The proof of Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S F}{ }_{+}$)
cites explicitly:
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous),
Lemma 1172 (integral in $\mathcal{M}_{+}$).
The proof of Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function)
cites explicitly:
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable),
Lemma 1156 (integral in $\mathcal{S F}+$ generalizes integral in $\mathcal{I F}$ ),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S} \mathcal{F}_{+}$).
The proof of Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous)
cites explicitly:
Statement(s) from [20],
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication),
Lemma $1108\left(\mathcal{S F} \mathcal{F}_{E}\right.$ simple representation),
Definition $1123\left(\mathcal{S F}_{+}\right.$, subset of nonnegative simple functions),
Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations $)$,
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear),
Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}_{+}$(simple)),
Lemma 1172 (integral in $\mathcal{M}_{+}$).
The proof of Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero)
cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous).

The proof of Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone)
cites explicitly:
Lemma 489 (supremum is monotone (set)), Lemma 1172 (integral in $\mathcal{M}_{+}$).

The proof of Theorem 1180 (Beppo Levi, monotone convergence)
cites explicitly:
Statement(s) from [20],
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 314 ( $\overline{\mathbb{R}}$ is complete),
Lemma 381 (limit is monotone),
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 501 (supremum is limit),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 820 (measurability of numeric function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent $)$,
Definition 879 (continuity from below),
Lemma 925 (measure is continuous from below),
Lemma $1108\left(\mathcal{S} \mathcal{F}_{E}\right.$ simple representation $)$,
Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable),
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear),
Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}{ }_{+}$(simple)),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone).
The proof of Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ) cites explicitly:
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Theorem 1180 (Beppo Levi, monotone convergence).
The proof of Lemma 1182 (usage of adapted sequences)
cites explicitly:
Definition 1135 (adapted sequence),
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$),
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S} \mathcal{F}_{+}$),
Theorem 1180 (Beppo Levi, monotone convergence).
The proof of Lemma 1183 (integral in $\mathcal{M}_{+}$is additive) cites explicitly:
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Definition 1135 (adapted sequence),
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$),
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear),
Lemma 1182 (usage of adapted sequences).

The proof of Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear)
cites explicitly:
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
The proof of Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive)
cites explicitly:
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum $)$,
Theorem 1180 (Beppo Levi, monotone convergence).
The proof of Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts)
cites explicitly:
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
The proof of Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts)
cites explicitly:
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
The proof of Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite)
cites explicitly:
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 820 (measurability of numeric function),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma 948 (negligibility of measurable subset),
Definition 953 (property almost satisfied),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ).
The proof of Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$)
cites explicitly:
Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ),
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 696 ( $\sigma$-algebra),
Definition 741 (measurable space),
Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication),
Lemma 913 (measure),
Lemma 948 (negligibility of measurable subset),
Definition 953 (property almost satisfied),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite).

The proof of Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality)
cites explicitly:
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$).
The proof of Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone)
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$).
The proof of Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function)
cites explicitly:
Definition 528 (nonnegative and nonpositive parts),
Definition 974 (almost binary relation),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite).
The proof of Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions)
cites explicitly:
Definition 528 (nonnegative and nonpositive parts),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality).
The proof of Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions)
cites explicitly:
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 951 (subset of negligible is negligible),
Lemma 952 ( $\mathbf{N}$ is closed under countable union),
Definition 974 (almost binary relation),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 995 (compatibility of almost nonnegativeness with almost equality),
Definition 997 (summability domain),
Lemma 1000 (negligibility and summability domain),
Lemma 1002 (almost sum),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts),
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality),
Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions).
The proof of Lemma 1195 (generalized Bienaymé-Chebyshev inequality) cites explicitly:
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 757 (compatibility of measurability with composition),
Lemma 820 (measurability of numeric function),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone).

The proof of Lemma 1196 (Bienaymé-Chebyshev inequality)
cites explicitly:
Lemma 1195 (generalized Bienaymé-Chebyshev inequality).
The proof of Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 948 (negligibility of measurable subset),
Definition 953 (property almost satisfied),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1196 (Bienaymé-Chebyshev inequality).
The proof of Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable)
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 983 (almost order is order relation),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone).
The proof of Lemma 1199 (integral in $\mathcal{M}_{+}$over subset)
cites explicitly:
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero),
Lemma 1163 (integral in $\mathcal{S F}+$ over subset),
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S} \mathcal{F}_{+}$),
Theorem 1180 (Beppo Levi, monotone convergence).
The proof of Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive)
cites explicitly:
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive $)$,
Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
The proof of Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset)
cites explicitly:
Lemma 482 (extrema of constant function),
Lemma 1165 (integral in $\mathcal{S F}_{+}$over negligible subset),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
The proof of Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton)
cites explicitly:
Lemma 811 (measurability of indicator function),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
The proof of Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero $)$,
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$,
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).

The proof of Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction) cites explicitly:
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$,
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction).
The proof of Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive)
cites explicitly:
Definition 209 (pseudopartition),
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive).
The proof of Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive)
cites explicitly:
Lemma 525 (masking is $\sigma$-additive),
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive).
The proof of Theorem 1208 (Fatou's lemma)
cites explicitly:
Statement(s) from [20],
Lemma 314 ( $\overline{\mathbb{R}}$ is complete),
Lemma 381 (limit is monotone),
Lemma 488 (infimum is monotone (set)),
Lemma 492 (infimum of bounded is bounded),
Lemma 501 (supremum is limit),
Lemma 504 (limit inferior),
Definition 516 (pointwise convergence),
Lemma 518 (limit inferior bounded from below),
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma $830(\mathcal{M}$ is closed under limit inferior),
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Theorem 1180 (Beppo Levi, monotone convergence).
The proof of Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence)
cites explicitly:
Statement(s) from [20],
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 522 (limit inferior, limit superior and pointwise convergence),
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Theorem 1208 (Fatou's lemma).
The proof of Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure)
cites explicitly:
Statement(s) from [20],
Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure),
Lemma 1182 (usage of adapted sequences).
The proof of Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure) cites explicitly:

Statement(s) from [20],
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous),
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure),
Lemma 1182 (usage of adapted sequences).
The proof of Lemma 1212 (integral in $\mathcal{M}_{+}$with counting measure on $\mathbb{N}$ )
cites explicitly:
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).
The proof of Lemma 1214 (integral in $\mathcal{M}_{+}$with Dirac measure)
cites explicitly:
Definition 939 (Dirac measure),
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).
The proof of Lemma 1218 (measure of section)
cites explicitly:
Lemma 786 (measurability of section),
Lemma 913 (measure).
The proof of Lemma 1219 (measure of section of product)
cites explicitly:
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 775 (product of measurable subsets is measurable),
Lemma 784 (section of product),
Lemma 913 (measure),
Lemma 1218 (measure of section).
The proof of Lemma 1221 (measurability of measure of section (finite))
cites explicitly:
Definition 227 (product of subset systems),
Definition 642 (generated algebra of sets),
Definition 664 (monotone class),
Lemma 719 ( $\sigma$-algebra contains algebra of sets),
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras),
Lemma 738 (usage of monotone class theorem),
Definition 774 (tensor product of $\sigma$-algebras),
Lemma 775 (product of measurable subsets is measurable),
Lemma 785 (compatibility of section with set operations),
Lemma 786 (measurability of section),
Lemma 787 (countable union of sections is measurable),
Lemma 788 (countable intersection of sections is measurable),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma 847 ( $\mathcal{M}_{+}$is closed under infimum),
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$,
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum),
Definition 857 (finiteness),
Lemma 858 (finite is bounded),
Definition 874 ( $\sigma$-additivity),
Lemma 913 (measure),
Lemma 925 (measure is continuous from below),
Lemma 927 (measure is continuous from above),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1218 (measure of section),
Lemma 1219 (measure of section of product).

The proof of Lemma 1223 (measurability of measure of section)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 786 (measurability of section),
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$,
Definition 857 (finiteness),
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Lemma 925 (measure is continuous from below),
Lemma 931 (restricted measure),
Lemma 1218 (measure of section),
Lemma 1221 (measurability of measure of section (finite)).
The proof of Lemma 1227 (candidate tensor product measure is tensor product measure)
cites explicitly:
Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)),
Lemma 775 (product of measurable subsets is measurable),
Lemma 785 (compatibility of section with set operations),
Lemma 786 (measurability of section),
Lemma 787 (countable union of sections is measurable),
Definition 874 ( $\sigma$-additivity),
Lemma 913 (measure),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive),
Lemma 1218 (measure of section),
Lemma 1219 (measure of section of product),
Lemma 1223 (measurability of measure of section),
Definition 1224 (tensor product measure),
Definition 1225 (candidate tensor product measure).
The proof of Lemma 1228 (tensor product of finite measures)
cites explicitly:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Definition 857 (finiteness),
Lemma 913 (measure),
Definition 1224 (tensor product measure).
The proof of Lemma 1229 (tensor product of $\sigma$-finite measures)
cites explicitly:
Lemma 775 (product of measurable subsets is measurable),
Definition 885 ( $\sigma$-finiteness),
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Definition 1224 (tensor product measure).
The proof of Lemma 1231 (uniqueness of tensor product measure (finite))
cites explicitly:
Statement(s) from [20],
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 664 (monotone class),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 719 ( $\sigma$-algebra contains algebra of sets),
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras),

Lemma 738 (usage of monotone class theorem),
Definition 774 (tensor product of $\sigma$-algebras),
Lemma 775 (product of measurable subsets is measurable),
Lemma 858 (finite is bounded),
Definition 862 (additivity),
Lemma 887 (finite is $\sigma$-finite),
Lemma 925 (measure is continuous from below),
Lemma 926 (equivalent definition of measure (from below)),
Lemma 927 (measure is continuous from above),
Definition 1224 (tensor product measure),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1228 (tensor product of finite measures).
The proof of Lemma 1233 (uniqueness of tensor product measure)
cites explicitly:
Statement(s) from [20],
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 775 (product of measurable subsets is measurable),
Definition 857 (finiteness),
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Lemma 925 (measure is continuous from below),
Lemma 931 (restricted measure),
Lemma 1218 (measure of section),
Definition 1224 (tensor product measure),
Definition 1225 (candidate tensor product measure),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1229 (tensor product of $\sigma$-finite measures),
Lemma 1231 (uniqueness of tensor product measure (finite)).
The proof of Lemma 1234 (negligibility of measurable section)
cites explicitly:
Lemma 786 (measurability of section),
Lemma 948 (negligibility of measurable subset),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1233 (uniqueness of tensor product measure).
The proof of Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ )
cites explicitly:
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ),
Lemma 1094 (Lebesgue measure is $\sigma$-finite),
Lemma 1233 (uniqueness of tensor product measure).
The proof of Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes)
cites explicitly:
Lemma 1092 (Lebesgue measure generalizes length of interval),
Definition 1224 (tensor product measure),
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ).
The proof of Lemma 1237 (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines)
cites explicitly:
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes).
The proof of Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite)
cites explicitly:
Lemma 1094 (Lebesgue measure is $\sigma$-finite),

Lemma 1229 (tensor product of $\sigma$-finite measures),
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ).
The proof of Lemma 1239 (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse)
cites explicitly:
Definition 856 (diffuseness),
Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes).
The proof of Theorem 1242 (Tonelli)
cites explicitly:
Lemma 381 (limit is monotone),
Lemma 786 (measurability of section),
Lemma 789 (indicator of section),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent),
Lemma 1100 (indicator and support are inverse),
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable),
Lemma $1128\left(\mathcal{S F}_{+}\right.$simple representation),
Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations),
Lemma 1134 (Lebesgue scheme step 2),
Lemma 1141 (Lebesgue scheme step 3),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1218 (measure of section),
Lemma 1223 (measurability of measure of section),
Lemma 1233 (uniqueness of tensor product measure),
Definition 1240 (partial function of function from product space).
The proof of Lemma 1243 (Tonelli over subset)
cites explicitly:
Lemma 789 (indicator of section),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Definition 1240 (partial function of function from product space), Theorem 1242 (Tonelli).

## The proof of Lemma 1244 (Tonelli for tensor product)

 cites explicitly: Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)), Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)), Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions), Definition 851 (tensor product of numeric functions), Lemma 853 (measurability of tensor product of numeric functions), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),Definition 1240 (partial function of function from product space), Theorem 1242 (Tonelli).

The proof of Lemma 1248 (integrable is measurable)
cites explicitly:
Lemma 836 (measurability of nonnegative and nonpositive parts), Lemma 1172 (integral in $\mathcal{M}_{+}$),
Definition 1247 (integrability).
The proof of Lemma 1249 (equivalent definition of integrability) cites explicitly:

Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed),
Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Definition 1247 (integrability),
Lemma 1248 (integrable is measurable).
The proof of Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$) cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 1249 (equivalent definition of integrability).
The proof of Lemma 1251 (integrable is almost finite)
cites explicitly:
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1249 (equivalent definition of integrability).
The proof of Lemma 1252 (almost bounded by integrable is integrable)
cites explicitly:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable),
Lemma 1249 (equivalent definition of integrability).
The proof of Lemma 1253 (integrability is closed under extension by zero)
cites explicitly:
Lemma 531 (nonnegative and nonpositive parts are orthogonal),
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 534 (compatibility of nonpositive and nonnegative parts with mask),
Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction),
Lemma 998 (summability on summability domain),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Definition 1247 (integrability).
The proof of Lemma 1254 (integrability is closed under restriction)
cites explicitly:
Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction),
Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction),
Definition 1247 (integrability).
The proof of Lemma 1255 (integrability, masking is $\sigma$-additive)
cites explicitly:
Lemma 525 (masking is $\sigma$-additive),
Lemma 534 (compatibility of nonpositive and nonnegative parts with mask),
Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Definition 1247 (integrability).
The proof of Lemma 1256 (integral in $\mathcal{M}$ )
cites explicitly:
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Definition 1247 (integrability).
The proof of Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$) cites explicitly:

Definition 528 (nonnegative and nonpositive parts),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Definition 1247 (integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).
The proof of Lemma 1259 (integral in $\mathcal{M}$ of zero is zero)
cites explicitly:
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$).
The proof of Lemma 1260 (integral in $\mathcal{M}$ of indicator function)
cites explicitly:
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$).
The proof of Lemma 1262 (compatibility of integral with almost equality)
cites explicitly:
Lemma 985 (compatibility of almost equality with operator),
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality),
Definition 1247 (integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).
The proof of Lemma 1263 (finite representative of integrable in $\mathcal{M}$ )
cites explicitly:
Lemma 811 (measurability of indicator function),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$,
Lemma 1251 (integrable is almost finite),
Lemma 1262 (compatibility of integral with almost equality).
The proof of Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ) cites explicitly:
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ),
Lemma 833 (measurability and masking),
Lemma 960 (countable almost properties is almost property),
Lemma 1008 (masking almost nowhere),
Lemma 1251 (integrable is almost finite),
Lemma 1262 (compatibility of integral with almost equality).
The proof of Lemma 1266 (integral in $\mathcal{M}$ is homogeneous)
cites explicitly:
Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Definition 528 (nonnegative and nonpositive parts),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1249 (equivalent definition of integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).
The proof of Lemma 1267 (integral in $\mathcal{M}$ of constant function)
cites explicitly:

Lemma 1260 (integral in $\mathcal{M}$ of indicator function),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous).
The proof of Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ )
cites explicitly:
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$,
Lemma 1002 (almost sum),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive),
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality).
The proof of Lemma 1269 (integral in $\mathcal{M}$ is additive)
cites explicitly:
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 1002 (almost sum),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1004 (almost sum is sum),
Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts),
Lemma 1249 (equivalent definition of integrability),
Lemma 1256 (integral in $\mathcal{M}$ ),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ).
The proof of Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear)
cites explicitly:
Lemma $814\left(\mathcal{M}_{\mathfrak{K}}\right.$ is vector space $)$,
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1269 (integral in $\mathcal{M}$ is additive).
The proof of Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone)
cites explicitly:
Statement(s) from [20],
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even),
Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$,
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 1004 (almost sum is sum),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1249 (equivalent definition of integrability),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1269 (integral in $\mathcal{M}$ is additive).
The proof of Lemma 1272 (integral in $\mathcal{M}$ is almost monotone)
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone).
The proof of Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value)
cites explicitly:
Lemma 530 (nonnegative and nonpositive parts are nonnegative),
Lemma 1172 (integral in $\mathcal{M}_{+}$),

Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Definition 1247 (integrability),
Lemma 1249 (equivalent definition of integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).

## The proof of Lemma 1274 (integral over subset)

cites explicitly:
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1269 (integral in $\mathcal{M}$ is additive).
The proof of Lemma 1275 (integral over subset is $\sigma$-additive)
cites explicitly:
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive),
Lemma 1255 (integrability, masking is $\sigma$-additive),
Lemma 1274 (integral over subset).
The proof of Lemma 1276 (integral over negligible subset)
cites explicitly:
Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset),
Lemma 1274 (integral over subset).
The proof of Lemma 1277 (integral over singleton)
cites explicitly:
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite),
Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1249 (equivalent definition of integrability),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1274 (integral over subset).
The proof of Lemma 1278 (integral over interval)
cites explicitly:
Lemma 744 (some Borel subsets),
Definition 856 (diffuseness),
Lemma 1275 (integral over subset is $\sigma$-additive),
Lemma 1277 (integral over singleton).
The proof of Lemma 1279 (Chasles relation, integral over split intervals)
cites explicitly:
Lemma 1275 (integral over subset is $\sigma$-additive),
Lemma 1278 (integral over interval).

## The proof of Lemma 1280 (integral with trivial measure)

cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure),
Lemma 1249 (equivalent definition of integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).

The proof of Lemma 1281 (integral with counting measure)
cites explicitly:
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure),
Lemma 1249 (equivalent definition of integrability),
Lemma 1256 (integral in $\mathcal{M}$ ).
The proof of Lemma 1282 (integral with counting measure on $\mathbb{N}$ )
cites explicitly:
Lemma 1281 (integral with counting measure).
The proof of Lemma 1284 (integral with Dirac measure)
cites explicitly:
Definition 939 (Dirac measure),
Lemma 1281 (integral with counting measure).
The proof of Lemma 1286 ( $p$-seminorm)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1172 (integral in $\mathcal{M}_{+}$).
The proof of Lemma 1288 ( $N_{p}$ with trivial measure)
cites explicitly:
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1289 ( $N_{p}$ of indicator function)
cites explicitly:
Lemma 811 (measurability of indicator function),
Lemma 933 (trivial measure),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1041 (nonnegative essential supremum for extended simple functions),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1290 ( $N_{p}$ is nonnegative)
cites explicitly:
Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1291 (link between $N_{1}$ and $N_{p}$ )
cites explicitly:
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1292 (integrable is finite $N_{1}$ )
cites explicitly:
Lemma 1249 (equivalent definition of integrability), Lemma 1286 ( $p$-seminorm).

The proof of Lemma 1293 ( $N_{\infty}$ bounded is almost bounded) cites explicitly:

Lemma 1039 (nonnegative essential supremum is almost upper bound), Lemma 1046 (nonnegative essential supremum bounded by constant), Lemma 1286 ( $p$-seminorm).

## The proof of Lemma 1294 (compatibility of $N_{p}$ with power function)

cites explicitly:
Definition 326 (multiplication in $\overline{\mathbb{R}}$ ),
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1040 (nonnegative essential supremum of constant function),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1295 (compatibility of $N_{p}$ with absolute value)
cites explicitly:
Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ),
Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 ),
Lemma 1294 (compatibility of $N_{p}$ with power function).
The proof of Lemma 1296 ( $N_{p}$ is almost definite)
cites explicitly:
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite),
Lemma 1005 (absolute value is almost definite),
Lemma 1007 (power function is almost definite),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1297 (compatibility of $N_{p}$ with almost equality)
cites explicitly:
Lemma 361 (power function in $\overline{\mathbb{R}}$ ),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1049 (compatibility of nonnegative essential supremum with almost equality),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions)
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 958 (everywhere is almost everywhere),
Lemma 1045 (nonnegative essential supremum is almost monotone),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1286 ( $p$-seminorm),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Lemma 1292 (integrable is finite $N_{1}$ ),
Lemma 1295 (compatibility of $N_{p}$ with absolute value).

The proof of Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions)
cites explicitly:
Lemma 960 (countable almost properties is almost property),
Lemma 965 (extended almost modus ponens (simplified 0)),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions).
The proof of Lemma 1300 ( $N_{p}$ is absolutely homogeneous)
cites explicitly:
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 361 ( power function in $\overline{\mathbb{R}}$ ),
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma 1044 (nonnegative essential supremum is positive homogeneous),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1301 ( $N_{p}$ of constant function)
cites explicitly:
Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ),
Lemma 1289 ( $N_{p}$ of indicator function),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous).
The proof of Lemma 1302 (generalized Hölder inequality (with $\infty$ ))
cites explicitly:
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)),
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1053 (nonnegative essential supremum is submultiplicative),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone),
Lemma 1286 ( $p$-seminorm).
The proof of Lemma 1304 (Hölder inequality)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 326 ( multiplication in $\overline{\mathbb{R}}$ ),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),

Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Definition 451 (Hölder conjugates), Lemma 453 (Young's inequality for products (measure theory)), Lemma 824 ( $\mathcal{M}$ is closed under multiplication), Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication), Lemma 841 ( $\mathcal{M}$ is closed under power function), Lemma 842 ( $\mathcal{M}$ is closed under absolute value), Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$, Lemma 845 ( $\mathcal{M}_{+}$is closed under nonnegative scalar multiplication), Lemma 985 (compatibility of almost equality with operator), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone), Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), Lemma 1286 ( $p$-seminorm), Lemma 1290 ( $N_{p}$ is nonnegative), Lemma 1296 ( $N_{p}$ is almost definite), Lemma 1300 ( $N_{p}$ is absolutely homogeneous), Lemma 1302 (generalized Hölder inequality (with $\infty$ )).

The proof of Lemma 1305 (generalized Hölder inequality) cites explicitly: Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication), Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)), Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)), Definition 451 (Hölder conjugates), Lemma $824(\mathcal{M}$ is closed under multiplication), Lemma 841 ( $\mathcal{M}$ is closed under power function), Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$, Lemma 1291 (link between $N_{1}$ and $N_{p}$ ), Lemma 1302 (generalized Hölder inequality (with $\infty$ )), Lemma 1304 (Hölder inequality).

The proof of Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates) cites explicitly: Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative), Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication), Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)), Definition 451 (Hölder conjugates), Lemma 1040 (nonnegative essential supremum of constant function), Lemma 1286 ( $p$-seminorm), Lemma 1290 ( $N_{p}$ is nonnegative), Lemma 1291 (link between $N_{1}$ and $N_{p}$ ).

## The proof of Lemma 1309 (Minkowski inequality in M)

cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality), Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition), Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone), Lemma 371 (power function satisfies extended relaxed triangle inequality), Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed), Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),

Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$,
Lemma 958 (everywhere is almost everywhere),
Lemma 1002 (almost sum),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1004 (almost sum is sum),
Lemma 1045 (nonnegative essential supremum is almost monotone),
Lemma 1051 (nonnegative essential supremum is subadditive),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1286 ( $p$-seminorm),
Lemma 1290 ( $N_{p}$ is nonnegative),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1304 (Hölder inequality),
Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates).
The proof of Lemma 1310 (Minkowski equality (nonnegative functions))
cites explicitly:
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1286 ( $p$-seminorm),
Lemma 1297 (compatibility of $N_{p}$ with almost equality).
The proof of Lemma 1311 (Minkowski equality (linearly dependent functions))
cites explicitly:
Definition 317 (addition in $\overline{\mathbb{R}}$ ),
Definition 339 (absolute value in $\overline{\mathbb{R}}$ ),
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous).
The proof of Lemma 1312 (Minkowski equality (almost linearly dependent functions))
cites explicitly:
Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 833 (measurability and masking),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1008 (masking almost nowhere),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1311 (Minkowski equality (linearly dependent functions)).

The proof of Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 363 (finite power function in $\overline{\mathbb{R}}$ ),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1286 ( $p$-seminorm),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure)
cites explicitly:
Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ ),
Lemma 1288 ( $N_{p}$ with trivial measure),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$)
cites explicitly:
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ )
cites explicitly:
Statement(s) from [20],
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1256 (integral in $\mathcal{M}$ ),
Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value),
Lemma 1286 ( $p$-seminorm),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$).
The proof of Lemma 1320 (integral in $\mathcal{L}^{1}$ is almost monotone)
cites explicitly:
Lemma 1272 (integral in $\mathcal{M}$ is almost monotone),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ).
The proof of Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ).
The proof of Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value)
cites explicitly:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1295 (compatibility of $N_{p}$ with absolute value),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ ),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),

Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 958 (everywhere is almost everywhere),
Lemma 1045 (nonnegative essential supremum is almost monotone),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Lemma 1286 ( $p$-seminorm),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value).
The proof of Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 363 (finite power function in $\overline{\mathbb{R}}$ ),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 1008 (masking almost nowhere),
Lemma 1048 (essentially bounded is almost finite),
Lemma 1251 (integrable is almost finite),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ )
cites explicitly:
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1286 ( $p$-seminorm),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of $p$-powered integrable functions).
The proof of Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication)
cites explicitly:
Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$ is vector space),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1328 (constant function is $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ),
Lemma 1267 (integral in $\mathcal{M}$ of constant function),
Lemma 1301 ( $N_{p}$ of constant function),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1329 (first mean value inequality)
cites explicitly:
Statement(s) from [20],
Lemma 913 (measure),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1274 (integral over subset),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1328 (constant function is $\mathcal{L}^{p}$ ).

## The proof of Lemma 1331 (first mean value theorem)

cites explicitly:
Definition 229 (interval),
Lemma 913 (measure),
Lemma 1329 (first mean value inequality).
The proof of Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 813 ( $\mathcal{M}_{\mathfrak{K}}$ is algebra),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 1305 (generalized Hölder inequality),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ),
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 455 (interpolation of multiplicative inverse in segment of $\mathbb{R}$ ),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1288 ( $N_{p}$ with trivial measure),
Lemma 1294 (compatibility of $N_{p}$ with power function),
Lemma 1295 (compatibility of $N_{p}$ with absolute value),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure),
Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ).
The proof of Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space $)$,
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 1309 (Minkowski inequality in M),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions).
The proof of Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space)
cites explicitly:
Statement(s) from [20],
Definition 288 (seminorm),
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite),
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space $)$,
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions), Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication),
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ).
The proof of Lemma 1339 (convergence in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 1286 ( $p$-seminorm),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space).
The proof of Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ) cites explicitly:
Statement(s) from [20],

Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite),
Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous),
Lemma 1286 ( $p$-seminorm),
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ).
The proof of Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal)
cites explicitly:
Statement(s) from [20],
Lemma 264 (pseudo-point separation),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1296 ( $N_{p}$ is almost definite),
Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication),
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ),
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ).
The proof of Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ )
cites explicitly:
Statement(s) from [20],
Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear),
Lemma 501 (supremum is limit),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 985 (compatibility of almost equality with operator),
Lemma 994 (transitivity of almost inequality and equality),
Lemma 1006 (absolute value is almost equal to almost nonnegative function),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1286 ( p-seminorm),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ).
The proof of Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ )
cites explicitly:
Statement(s) from [20],
Lemma 519 (limit inferior bounded from above),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 833 (measurability and masking),
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 960 (countable almost properties is almost property),
Lemma 985 (compatibility of almost equality with operator),
Lemma 1008 (masking almost nowhere),
Theorem 1208 (Fatou's lemma),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1290 ( $N_{p}$ is nonnegative),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ).

## The proof of Theorem 1347 (Lebesgue, dominated convergence)

cites explicitly:
Statement(s) from [20],
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),

Lemma 381 (limit is monotone),
Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous),
Lemma 510 (duality limit inferior-limit superior),
Definition 516 (pointwise convergence),
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 518 (limit inferior bounded from below),
Lemma 522 (limit inferior, limit superior and pointwise convergence),
Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent),
Theorem 1208 (Fatou's lemma),
Lemma 1286 ( $p$-seminorm),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ).
The proof of Theorem 1349 (Lebesgue, extended dominated convergence)
cites explicitly:
Statement(s) from [20],
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite),
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 833 (measurability and masking),
Lemma 948 (negligibility of measurable subset),
Lemma 951 (subset of negligible is negligible),
Definition 953 (property almost satisfied),
Lemma 960 (countable almost properties is almost property),
Definition 974 (almost binary relation),
Lemma 1008 (masking almost nowhere),
Lemma 1009 (finite nonnegative part),
Lemma 1249 (equivalent definition of integrability),
Lemma 1251 (integrable is almost finite),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1286 ( $p$-seminorm),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Theorem 1347 (Lebesgue, dominated convergence).
The proof of Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite),
Lemma 371 (power function satisfies extended relaxed triangle inequality),
Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 754 (constant function is measurable),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 841 ( $\mathcal{M}$ is closed under power function),

Lemma 962 (almost everywhere for almost the same),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1012 (compatibility of almost limit with power function),
Lemma 1014 (compatibility of almost limit with almost equality),
Lemma 1015 (compatibility of almost limit with almost inequality),
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
The proof of Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ )
cites explicitly:
Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous),
Lemma 815 ( $\mathcal{M}_{\mathbb{R}}$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$,
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1011 (compatibility of almost limit with continuous function),
Lemma 1012 (compatibility of almost limit with power function),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).
The proof of Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ) cites explicitly:
Statement(s) from [20],
Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)),
Lemma 269 (convergent subsequence of Cauchy sequence),
Lemma 493 (supremum of bounded is bounded),
Definition 953 (property almost satisfied),
Lemma 960 (countable almost properties is almost property),
Lemma 1001 (absolute value and almost inequality),
Lemma 1012 (compatibility of almost limit with power function),
Lemma 1015 (compatibility of almost limit with almost inequality),
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Lemma 1293 ( $N_{\infty}$ bounded is almost bounded),
Lemma 1295 (compatibility of $N_{p}$ with absolute value),
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),

Lemma 1339 (convergence in $\mathcal{L}^{p}$ ),
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
The proof of Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ )
cites explicitly:
Statement(s) from [20],
Lemma 1014 (compatibility of almost limit with almost equality),
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
The proof of Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ) cites explicitly:
Lemma 1010 (compatibility of almost limit with subsequence),
Lemma 1013 (almost limits are almost equal),
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).

## Appendix C

## Is explicitly cited in the proof of. . .

This appendix gathers the explicit citations that appear in the proof of results (lemmas and theorems) for each statement listed in Appendix A. Statements from [20] are anonymized.

Printing is not advised!

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Statement(s) from [20]
    are explicitly cited in the proof of:
    Lemma }231\mathrm{ (equivalent definition of interval),
    Lemma 265 (equivalent definition of convergent sequence),
    Lemma 266 (compatibility of subsequence with convergence),
    Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)),
    Lemma 269 (convergent subsequence of Cauchy sequence),
    Lemma 272 (quotient vector space, equivalence relation),
    Lemma 273 (quotient vector space),
    Lemma 274 (linear map on quotient vector space),
    Lemma 282 (algebra of functions to algebra),
    Lemma 286 (vector subspace and closed under multiplication is subalgebra),
    Lemma 287 (closed under algebra operations is subalgebra),
    Lemma 290 (seminorm is homogeneous),
    Lemma }291\mathrm{ (seminorm is nonnegative),
    Lemma 292 (seminormed vector space is pseudometric space),
    Lemma }293\mathrm{ (definite seminorm is norm),
    Lemma 298 (definite semi-inner product is inner product),
    Lemma }377\mathrm{ (metric on }\overline{\mathbb{R}}\mathrm{ ),
    Lemma 381 (limit is monotone),
    Lemma 391 (limit in }\overline{\mathbb{R}}\mathrm{ is linear),
    Lemma 399 (series in }\mp@subsup{\overline{\mathbb{R}}}{+}{}\mathrm{ are compatible with pseudopartition),
    Lemma 404 (double series in }\mp@subsup{\overline{\mathbb{R}}}{+}{}\mathrm{ ),
    Lemma 458 (connected component of open subset of \mathbb{R}}\mathrm{ is open interval),
    Theorem 461 (countable connected components of open subsets of \mathbb{R}),
    Lemma 464 (open intervals with rational bounds cover open interval),
    Lemma 482 (extrema of constant function),
    Lemma 483 (compatibility of extrema with affine transformation),
    Lemma 484 (equivalent definition of finite infimum),
    Lemma 485 (equivalent definition of finite infimum in }\overline{\mathbb{R}}\mathrm{ ),
    Lemma 486 (equivalent definition of infimum),
    Lemma }487\mathrm{ (infimum is smaller than supremum),
    Lemma }488\mathrm{ (infimum is monotone (set)),
    Lemma 489 (supremum is monotone (set)),
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Lemma 490 (infimum is monotone (function)),
Lemma 491 (supremum is monotone (function)),
Lemma 494 (compatibility of infimum with absolute value),
Lemma 495 (compatibility of supremum with absolute value),
Lemma 498 (infimum of nonincreasing sequence),
Lemma 499 (supremum of nondecreasing sequence),
Lemma 500 (infimum is limit),
Lemma 501 (supremum is limit),
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma 503 (constant sum of monotone nonnegative functions),
Lemma 505 (limit inferior is $\infty$ ),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 510 (duality limit inferior-limit superior),
Lemma 518 (limit inferior bounded from below),
Lemma 519 (limit inferior bounded from above),
Lemma 522 (limit inferior, limit superior and pointwise convergence),
Lemma 529 (equivalent definition of nonnegative and nonpositive parts),
Lemma 530 (nonnegative and nonpositive parts are nonnegative),
Lemma 531 (nonnegative and nonpositive parts are orthogonal),
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space $)$,
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma 829 ( $\mathcal{M}$ is closed under supremum),
Lemma 883 (continuity from above),
Lemma 907 (compatibility of null premeasure with countable union),
Lemma 1011 (compatibility of almost limit with continuous function),
Lemma 1015 (compatibility of almost limit with almost inequality),
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1020 (essential supremum is bounded by supremum),
Lemma 1021 (essential supremum is minimum),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma 1083 (rays are Lebesgue-measurable),
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation),
Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation),
Lemma $1117\left(\mathcal{S F}_{E}\right.$ is vector space $)$,
Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive $)$,
Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone),
Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Theorem 1180 (Beppo Levi, monotone convergence),
Theorem 1208 (Fatou's lemma),
Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence),
Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure),
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1233 (uniqueness of tensor product measure),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1329 (first mean value inequality),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),

Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1349 (Lebesgue, extended dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ),
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).

## Definition 209 (pseudopartition)

is explicitly cited in the proof of:
Lemma 211 (compatibility of pseudopartition with intersection),
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition),
Lemma 525 (masking is $\sigma$-additive),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1107 ( $\mathcal{I F}_{\mathfrak{K}}$, masking is $\sigma$-additive),
Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space),
Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive).
Lemma 211 (compatibility of pseudopartition with intersection)
is explicitly cited in the proof of:
Lemma 766 (generating on a partition),
Lemma 863 (decomposition over finite pseudopartition),
Lemma 876 (decomposition over countable pseudopartition),
Lemma 1073 (equivalent definition of $\mathcal{L}$ ),
Lemma $1078\left(\lambda^{\star}\right.$ is additive on $\left.\mathcal{L}\right)$.
Lemma 212 (countable union is monotone)
is explicitly cited in the proof of:
Lemma 213 (equality of countable union).
Lemma 213 (equality of countable union) is explicitly cited in the proof of:
Lemma 215 (disjoint union of union),
Lemma 217 (monotone union of union).

## Lemma 215 (disjoint union of union)

is explicitly cited in the proof of:
Lemma 568 (closedness under countable disjoint and monotone union), Lemma 570 (closedness under countable disjoint union and countable union), Lemma 572 (disjoint union of union in subset system), Lemma 574 (disjoint union of union in subset system (variant)).

## Lemma 217 (monotone union of union)

is explicitly cited in the proof of:
Lemma 569 (closedness under countable monotone and disjoint union), Lemma 575 (monotone union of union in subset system).

## Lemma 219 (empty monotone intersection of union)

 is explicitly cited in the proof of: Lemma 576 (empty monotone intersection of union in subset system).
## Lemma 221 (monotone union of monotone intersection)

is explicitly cited in the proof of:
Lemma 577 (monotone union of monotone intersection in subset system).
Lemma 222 (technical inclusion for countable union)
is explicitly cited in the proof of:
Lemma 223 (order is meaningless in countable union).

## Lemma 223 (order is meaningless in countable union)

is explicitly cited in the proof of:
Lemma 224 (definition of double countable union).

## Lemma 224 (definition of double countable union)

is explicitly cited in the proof of:
Lemma 225 (double countable union),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive).
Lemma 225 (double countable union)
is explicitly cited in the proof of:
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive).
Definition 226 (trace of subset system)
is explicitly cited in the proof of:
Lemma 238 (trace interval on a subset),
Lemma 252 (trace topology on subset),
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ),
Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ),
Lemma 763 (trace $\sigma$-algebra),
Lemma 764 (measurability of measurable subspace),
Lemma 765 (generating measurable subspace),
Lemma 766 (generating on a partition),
Lemma 770 (source restriction of measurable function),
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ).

## Definition 227 (product of subset systems)

is explicitly cited in the proof of:
Lemma 254 (box topology on Cartesian product),
Lemma 261 (compatibility of second-countability with Cartesian product),
Lemma 310 (topology on $\mathbb{R}^{m}$ ),
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras),
Lemma 775 (product of measurable subsets is measurable),
Lemma 776 (measurability of function to product space),
Lemma 779 (generating product measurable space),
Lemma 786 (measurability of section),
Lemma 1221 (measurability of measure of section (finite)).
Definition 229 (interval)
is explicitly cited in the proof of:
Lemma 231 (equivalent definition of interval),
Lemma 232 (empty open interval),
Lemma 235 (intervals are closed under finite intersection),
Lemma 238 (trace interval on a subset),
Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval),
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ),
Lemma 1331 (first mean value theorem).

## Lemma 231 (equivalent definition of interval)

 is not yet used.Lemma 232 (empty open interval)
is explicitly cited in the proof of: Lemma 236 (empty intersection of open intervals).

Lemma 235 (intervals are closed under finite intersection) is explicitly cited in the proof of: Lemma 236 (empty intersection of open intervals), Lemma 258 (topological basis of order topology), Lemma 1083 (rays are Lebesgue-measurable), Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).

Lemma 236 (empty intersection of open intervals) is not yet used.

Lemma 238 (trace interval on a subset)
is explicitly cited in the proof of:
Lemma 259 (trace of order topology).
Definition 240 (topological space)
is explicitly cited in the proof of: Lemma 241 (intersection of topologies), Lemma 244 (equivalent definition of generated topology), Lemma 247 (equivalent definition of topological basis), Lemma 252 (trace topology on subset), Lemma 254 (box topology on Cartesian product), Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval), Lemma 464 (open intervals with rational bounds cover open interval), Lemma 744 (some Borel subsets).

## Lemma 241 (intersection of topologies)

is explicitly cited in the proof of: Lemma 243 (generated topology is minimum).
Definition 242 (generated topology) is explicitly cited in the proof of: Lemma 243 (generated topology is minimum).

Lemma 243 (generated topology is minimum) is explicitly cited in the proof of: Lemma 244 (equivalent definition of generated topology), Lemma 245 (topology generation is monotone), Lemma 247 (equivalent definition of topological basis).

## Lemma 244 (equivalent definition of generated topology)

 is explicitly cited in the proof of: Lemma 247 (equivalent definition of topological basis), Lemma 249 (topological basis is generator), Lemma 258 (topological basis of order topology).Lemma 245 (topology generation is monotone) is not yet used.
Definition 246 (topological basis) is explicitly cited in the proof of: Lemma 247 (equivalent definition of topological basis),

Lemma 248 (augmented topological basis),
Lemma 249 (topological basis is generator), Lemma 250 (topological subbasis),
Lemma 252 (trace topology on subset),
Lemma 254 (box topology on Cartesian product),
Lemma 258 (topological basis of order topology),
Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).

## Lemma 247 (equivalent definition of topological basis)

is explicitly cited in the proof of:
Lemma 254 (box topology on Cartesian product).

## Lemma 248 (augmented topological basis)

is not yet used.

## Lemma 249 (topological basis is generator)

is explicitly cited in the proof of:
Lemma 250 (topological subbasis),
Lemma 307 (topology on $\mathbb{R}$ ),
Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ).
Lemma 250 (topological subbasis)
is explicitly cited in the proof of:
Lemma 310 (topology on $\mathbb{R}^{m}$ ),
Theorem $465(\mathbb{R}$ is second-countable),
Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable).
Lemma 252 (trace topology on subset)
is explicitly cited in the proof of:
Lemma 259 (trace of order topology),
Lemma 308 (trace topology on subset of $\mathbb{R}$ ),
Lemma 311 (trace topology on subset of $\mathbb{R}^{m}$ ),
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ),
Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ),
Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 780 (Borel subsets in product measurable subspace).

## Lemma 254 (box topology on Cartesian product)

is explicitly cited in the proof of:
Lemma 261 (compatibility of second-countability with Cartesian product),
Lemma 310 (topology on $\mathbb{R}^{m}$ ),
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ),
Lemma 780 (Borel subsets in product measurable subspace).

## Definition 256 (order topology)

is explicitly cited in the proof of:
Lemma 258 (topological basis of order topology),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ).
Lemma 258 (topological basis of order topology)
is explicitly cited in the proof of:
Lemma 259 (trace of order topology),
Lemma 307 (topology on $\mathbb{R}$ ),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ).
Lemma 259 (trace of order topology)
is explicitly cited in the proof of:
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ).

Definition 260 (second-countability)
is explicitly cited in the proof of:
Lemma 261 (compatibility of second-countability with Cartesian product),
Theorem $465(\mathbb{R}$ is second-countable),
Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable).
Lemma 261 (compatibility of second-countability with Cartesian product) is explicitly cited in the proof of: Lemma $466\left(\mathbb{R}^{m}\right.$ is second-countable), Lemma $480\left(\overline{\mathbb{R}}^{m}\right.$ is second-countable).

Definition 262 (pseudometric)
is explicitly cited in the proof of:
Lemma 264 (pseudo-point separation),
Lemma 269 (convergent subsequence of Cauchy sequence),
Lemma 292 (seminormed vector space is pseudometric space).
Lemma 264 (pseudo-point separation)
is explicitly cited in the proof of:
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal).

## Lemma 265 (equivalent definition of convergent sequence)

is explicitly cited in the proof of:
Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)), Lemma 484 (equivalent definition of finite infimum).

Lemma 266 (compatibility of subsequence with convergence)
is explicitly cited in the proof of:
Lemma 1010 (compatibility of almost limit with subsequence).

## Definition 267 (cluster point)

is explicitly cited in the proof of:
Lemma 382 (cluster point at $\pm \infty$ (discrete)),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 507 (limit inferior is invariant by translation).
Lemma 268 (equivalent definition of Cauchy sequence (discrete criterion)) is explicitly cited in the proof of: Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 269 (convergent subsequence of Cauchy sequence)

is explicitly cited in the proof of:
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Definition 270 (relation compatible with vector operations)
is explicitly cited in the proof of:
Lemma 271 (quotient vector operations),
Lemma 273 (quotient vector space).
Lemma 271 (quotient vector operations)
is explicitly cited in the proof of:
Lemma 272 (quotient vector space, equivalence relation),
Lemma 274 (linear map on quotient vector space).

## Lemma 272 (quotient vector space, equivalence relation)

is explicitly cited in the proof of:
Lemma 273 (quotient vector space).

Lemma 273 (quotient vector space)
is explicitly cited in the proof of:
Lemma 274 (linear map on quotient vector space).
Lemma 274 (linear map on quotient vector space)
is not yet used.
Definition 277 (algebra over a field)
is explicitly cited in the proof of:
Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra) ,
Lemma 282 (algebra of functions to algebra),
Lemma 286 (vector subspace and closed under multiplication is subalgebra),
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space),
Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations),
Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone).

## Lemma 279 ( $\mathbb{K}$ is $\mathbb{K}$-algebra)

is explicitly cited in the proof of:
Lemma 283 ( $\mathbb{K}^{X}$ is algebra),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra).
Definition 280 (inherited algebra operations)
is explicitly cited in the proof of:
Lemma 282 (algebra of functions to algebra).
Lemma 282 (algebra of functions to algebra)
is explicitly cited in the proof of:
Lemma 283 ( $\mathbb{K}^{X}$ is algebra),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra).
Lemma 283 ( $\mathbb{K}^{X}$ is algebra)
is not yet used.
Definition 284 (subalgebra)
is explicitly cited in the proof of:
Lemma 286 (vector subspace and closed under multiplication is subalgebra), Lemma 814 ( $\mathcal{M}_{\mathbb{K}}$ is vector space).

Lemma 286 (vector subspace and closed under multiplication is subalgebra) is explicitly cited in the proof of:
Lemma 287 (closed under algebra operations is subalgebra),
Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space).

## Lemma 287 (closed under algebra operations is subalgebra)

is explicitly cited in the proof of:
Lemma 813 ( $\mathcal{M}_{\mathfrak{k}}$ is algebra).
Definition 288 (seminorm)
is explicitly cited in the proof of:
Lemma 290 (seminorm is homogeneous),
Lemma 291 (seminorm is nonnegative),
Lemma 292 (seminormed vector space is pseudometric space),
Lemma 293 (definite seminorm is norm),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space).
Lemma 290 (seminorm is homogeneous)
is explicitly cited in the proof of:
Lemma 291 (seminorm is nonnegative),
Lemma 292 (seminormed vector space is pseudometric space).

## Lemma 291 (seminorm is nonnegative)

is explicitly cited in the proof of:
Lemma 292 (seminormed vector space is pseudometric space).
Lemma 292 (seminormed vector space is pseudometric space)
is not yet used.
Lemma 293 (definite seminorm is norm)
is not yet used.

## Definition 295 (Banach space)

is not yet used.
Definition 296 (semi-inner product)
is explicitly cited in the proof of:
Lemma 298 (definite semi-inner product is inner product).
Lemma 298 (definite semi-inner product is inner product)
is not yet used.
Lemma 299 (finite cover of compact interval)
is explicitly cited in the proof of:
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval).
Lemma 300 (power function satisfies relaxed triangle inequality)
is explicitly cited in the proof of:
Lemma 371 (power function satisfies extended relaxed triangle inequality).
Definition 302 (Hölder conjugates in $\mathbb{R}$ )
is explicitly cited in the proof of:
Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ ),
Lemma 304 (Young's inequality for products in $\mathbb{R}$ ).
Lemma 303 (2 is self-Hölder conjugate in $\mathbb{R}$ )
is explicitly cited in the proof of:
Lemma 305 (Young's inequality for products in $\mathbb{R}$, case $p=2$ ),
Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)).
Lemma 304 (Young's inequality for products in $\mathbb{R}$ )
is explicitly cited in the proof of:
Lemma 305 (Young's inequality for products in $\mathbb{R}$, case $p=2$ ),
Lemma 453 (Young's inequality for products (measure theory)).
Lemma 305 (Young's inequality for products in $\mathbb{R}$, case $p=2$ )
is not yet used.
Lemma 307 (topology on $\mathbb{R}$ )
is explicitly cited in the proof of:
Lemma 308 (trace topology on subset of $\mathbb{R}$ ), Lemma 310 (topology on $\mathbb{R}^{m}$ ), Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ ), Theorem 465 ( $\mathbb{R}$ is second-countable).

Lemma 308 (trace topology on subset of $\mathbb{R}$ ) is not yet used.

## Lemma 310 (topology on $\mathbb{R}^{m}$ )

is explicitly cited in the proof of:
Lemma 311 (trace topology on subset of $\mathbb{R}^{m}$ ),
Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ),
Lemma $466\left(\mathbb{R}^{m}\right.$ is second-countable $)$.

## Lemma 311 (trace topology on subset of $\mathbb{R}^{m}$ )

is not yet used.

## Definition 312 (extended real numbers, $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 313 (order in $\overline{\mathbb{R}}$ is total),
Lemma 314 ( $\overline{\mathbb{R}}$ is complete),
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution),
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality),
Lemma 372 (topology on $\overline{\mathbb{R}}$ ),
Lemma 381 (limit is monotone),
Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ),
Lemma 486 (equivalent definition of infimum),
Lemma 500 (infimum is limit),
Lemma 505 (limit inferior is $\infty$ ),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 527 (finite part is finite),
Lemma 531 (nonnegative and nonpositive parts are orthogonal),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 935 (counting measure),
Lemma 996 (almost finite),
Lemma 1017 (essential supremum),
Lemma 1030 (essential supremum bounded by constant),
Lemma 1046 (nonnegative essential supremum bounded by constant),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1280 (integral with trivial measure),
Lemma 1304 (Hölder inequality),
Lemma 1309 (Minkowski inequality in M),
Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure).
Lemma 313 (order in $\overline{\mathbb{R}}$ is total)
is explicitly cited in the proof of:
Lemma 380 (convergence towards $\pm \infty$ (discrete)),
Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ),
Lemma 486 (equivalent definition of infimum),
Lemma 487 (infimum is smaller than supremum),
Lemma 488 (infimum is monotone (set)),
Lemma 490 (infimum is monotone (function)),
Lemma 512 (limit inferior is smaller than limit superior),
Lemma 1001 (absolute value and almost inequality),
Lemma 1045 (nonnegative essential supremum is almost monotone),
Lemma 1073 (equivalent definition of $\mathcal{L}$ ),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone),

Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable),
Lemma 1272 (integral in $\mathcal{M}$ is almost monotone),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 314 ( $\mathbb{R}$ is complete)

is explicitly cited in the proof of:
Lemma $392\left(\overline{\mathbb{R}}_{+}\right.$is complete $)$,
Lemma 504 (limit inferior),
Lemma 509 (limit superior),
Lemma 1017 (essential supremum),
Theorem 1180 (Beppo Levi, monotone convergence),
Theorem 1208 (Fatou's lemma).

## Definition 317 (addition in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ ),
Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined),
Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined),
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ ),
Lemma 323 (finite-sum property in $\overline{\mathbb{R}}$ ),
Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution),
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone),
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined),
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality),
Lemma 371 (power function satisfies extended relaxed triangle inequality),
Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous),
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed),
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative),
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative),
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous),
Lemma 453 (Young's inequality for products (measure theory)),
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 866 (subtractivity),
Lemma 998 (summability on summability domain),
Lemma 1002 (almost sum),
Lemma 1012 (compatibility of almost limit with power function),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1051 (nonnegative essential supremum is subadditive),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1309 (Minkowski inequality in M),
Lemma 1311 (Minkowski equality (linearly dependent functions)).

## Lemma 318 (zero is identity element for addition in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure),

Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Lemma 319 (addition in $\overline{\mathbb{R}}$ is associative when defined)
is explicitly cited in the proof of:
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative).
Lemma 320 (addition in $\overline{\mathbb{R}}$ is commutative when defined)
is explicitly cited in the proof of:
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative).
Lemma 321 (addition in $\overline{\mathbb{R}}$ is compatible with order when defined)
is explicitly cited in the proof of:
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order),
Lemma 483 (compatibility of extrema with affine transformation),
Lemma 1001 (absolute value and almost inequality).

## Lemma 322 (infinity-sum property in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$).

## Lemma 323 (finite-sum property in $\overline{\mathbb{R}}$ )

is not yet used.
Lemma 324 (additive inverse in $\overline{\mathbb{R}}$ is involution)
is explicitly cited in the proof of:
Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ).
Lemma 325 (additive inverse in $\overline{\mathbb{R}}$ is monotone)
is explicitly cited in the proof of:
Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ),
Lemma 489 (supremum is monotone (set)),
Lemma 491 (supremum is monotone (function)),
Lemma 511 (equivalent definition of limit superior),
Lemma 513 (limit superior is monotone).
Definition 326 (multiplication in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ ),
Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined),
Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined),
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined),
Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined),
Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ ),
Lemma 335 (one-product property in $\overline{\mathbb{R}}$ ),
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ ),
Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ ),
Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined),
Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 371 (power function satisfies extended relaxed triangle inequality),
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous),
Lemma 407 (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 453 (Young's inequality for products (measure theory)),
Lemma $824(\mathcal{M}$ is closed under multiplication),

Lemma 833 (measurability and masking),
Lemma 1053 (nonnegative essential supremum is submultiplicative), Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Lemma 1294 (compatibility of $N_{p}$ with power function),
Lemma 1304 (Hölder inequality).

## Lemma 328 (one is identity element for multiplication in $\overline{\mathbb{R}}$ )

 is explicitly cited in the proof of:Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 1295 (compatibility of $N_{p}$ with absolute value),
Lemma 1301 ( $N_{p}$ of constant function).
Lemma 329 (multiplication in $\overline{\mathbb{R}}$ is associative when defined)
is explicitly cited in the proof of:
Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)).
Lemma 330 (multiplication in $\overline{\mathbb{R}}$ is commutative when defined)
is explicitly cited in the proof of:
Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined),
Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)).
Lemma 331 (multiplication in $\overline{\mathbb{R}}$ is left distributive over addition when defined) is explicitly cited in the proof of:
Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined), Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)), Lemma 483 (compatibility of extrema with affine transformation).

Lemma 332 (multiplication in $\overline{\mathbb{R}}$ is right distributive over addition when defined) is explicitly cited in the proof of: Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)).

Lemma 333 (multiplication in $\overline{\mathbb{R}}$ is compatible with order when defined) is explicitly cited in the proof of: Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)).

## Lemma 334 (zero-product property in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 408 (zero-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)).

## Lemma 335 (one-product property in $\overline{\mathbb{R}}$ )

 is explicitly cited in the proof of: Lemma 409 (one-product property in $\overline{\mathbb{R}}_{+}$),Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory)).
Lemma 336 (infinity-product property in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)).

## Lemma 337 (finite-product property in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 411 (finite-product property in $\overline{\mathbb{R}}_{+}$).
Lemma 338 (multiplicative inverse in $\overline{\mathbb{R}}$ is monotone when defined)
is explicitly cited in the proof of:
Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone).
Definition 339 (absolute value in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ ),
Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ ),
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative),
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even),
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite),
Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality),
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality),
Lemma 361 (power function in $\overline{\mathbb{R}}$ ),
Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous),
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 1277 (integral over singleton),
Lemma 1311 (Minkowski equality (linearly dependent functions)).
Lemma 340 (equivalent definition of absolute value in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ ),
Lemma 494 (compatibility of infimum with absolute value),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).
Lemma 341 (finite absolute value in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 996 (almost finite),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ).
Lemma 342 (absolute value in $\overline{\mathbb{R}}$ is nonnegative)
is explicitly cited in the proof of:
Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Lemma 1304 (Hölder inequality),
Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates),
Lemma 1309 (Minkowski inequality in M),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 343 (absolute value in $\overline{\mathbb{R}}$ is equal to nonnegative)
is explicitly cited in the proof of:
Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent),
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ),
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1006 (absolute value is almost equal to almost nonnegative function),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$),

Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 344 (inverse of absolute value in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ),
Lemma 1001 (absolute value and almost inequality).

## Lemma 345 (absolute value in $\overline{\mathbb{R}}$ is idempotent)

is explicitly cited in the proof of:
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)).
Lemma 346 (absolute value in $\overline{\mathbb{R}}$ is even)
is explicitly cited in the proof of:
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality),
Lemma 445 (power function in $\overline{\mathbb{R}}$ is even (measure theory)),
Lemma 495 (compatibility of supremum with absolute value),
Lemma 515 (compatibility of limit superior with absolute value),
Lemma 1251 (integrable is almost finite),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone).
Lemma 347 (absolute value in $\overline{\mathbb{R}}$ is definite)
is explicitly cited in the proof of:
Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 1005 (absolute value is almost definite),
Lemma 1277 (integral over singleton),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 348 (absolute value in $\overline{\mathbb{R}}$ is multiplicative)

is explicitly cited in the proof of:
Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous).
Lemma 349 (absolute value in $\overline{\mathbb{R}}$ satisfies triangle inequality) is explicitly cited in the proof of:
Lemma 351 (absolute value in $\overline{\mathbb{R}}$ satisfies countable triangle inequality),
Lemma 1001 (absolute value and almost inequality),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1309 (Minkowski inequality in M),
Theorem 1347 (Lebesgue, dominated convergence).
Lemma 350 (absolute value in $\overline{\mathbb{R}}$ satisfies reverse triangle inequality)
is explicitly cited in the proof of:
Lemma 1001 (absolute value and almost inequality).
Lemma 351 (absolute value in $\overline{\mathbb{R}}$ satisfies countable triangle inequality) is not yet used.

Definition 352 (exponential and logarithm in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 353 (exponential and logarithm in $\overline{\mathbb{R}}$ are inverse),
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ).

## Lemma 353 (exponential and logarithm in $\overline{\mathbb{R}}$ are inverse)

is not yet used.

## Definition 354 (exponentiation in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ ),
Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ ),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)).

## Lemma 355 (exponentiation in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 ),
Lemma 361 (power function in $\overline{\mathbb{R}}$ ),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 453 (Young's inequality for products (measure theory)).

## Lemma 357 (exponentiation in $\overline{\mathbb{R}}$ of power 1 )

is explicitly cited in the proof of:
Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value),
Lemma 1295 (compatibility of $N_{p}$ with absolute value).

## Lemma 358 (zero-exponentiation property in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)).
Lemma 359 (one-exponentiation property in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)).
Lemma 360 (infinity-exponentiation property in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)).
Lemma 361 (power function in $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value),
Lemma 363 (finite power function in $\overline{\mathbb{R}}$ ),
Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative),
Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition),
Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication),
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone),
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite),
Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 371 (power function satisfies extended relaxed triangle inequality),
Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous),
Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),
Lemma 1286 ( $p$-seminorm),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous).
Lemma 362 (power function in $\overline{\mathbb{R}}$ is absolute value)
is explicitly cited in the proof of:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value).

## Lemma 363 (finite power function in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ).
Lemma 364 (power function in $\overline{\mathbb{R}}$ is nonnegative)
is explicitly cited in the proof of:
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ),
Lemma 1290 ( $N_{p}$ is nonnegative).
Lemma 365 (compatibility of power function in $\overline{\mathbb{R}}$ with addition)
is explicitly cited in the proof of:
Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).

## Lemma 366 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication)

is explicitly cited in the proof of:
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ),
Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Lemma 1294 (compatibility of $N_{p}$ with power function),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Lemma 1305 (generalized Hölder inequality),
Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates),
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ).
Lemma 367 (power function in $\overline{\mathbb{R}}$ is monotone)
is explicitly cited in the proof of:
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ ),
Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Lemma 1305 (generalized Hölder inequality),
Lemma 1309 (Minkowski inequality in M),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).
Lemma 368 (inverse of power function in $\overline{\mathbb{R}}$ )
is not yet used.
Lemma 369 (power function in $\overline{\mathbb{R}}$ is definite)
is explicitly cited in the proof of:
Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative),
Lemma 1007 (power function is almost definite),
Lemma 1296 ( $N_{p}$ is almost definite),
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).
Lemma 370 (power function in $\overline{\mathbb{R}}$ is multiplicative)
is explicitly cited in the proof of:
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)).
Lemma 371 (power function satisfies extended relaxed triangle inequality)
is explicitly cited in the proof of:
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 372 (topology on $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ),
Lemma 379 (convergence towards $\pm \infty$ ),
Lemma 392 ( $\overline{\mathbb{R}}_{+}$is complete),
Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ),
Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable),
Lemma 820 (measurability of numeric function).
Lemma 373 (trace topology on subset of $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms),
Lemma 474 (open down),
Lemma 475 (open up).

## Definition 374 (tangent and arctangent in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms),
Lemma 377 (metric on $\overline{\mathbb{R}}$ ).
Lemma 375 (tangent and arctangent in $\overline{\mathbb{R}}$ are homeomorphisms)
is explicitly cited in the proof of:
Lemma 377 (metric on $\overline{\mathbb{R}}$ ),
Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ ).
Lemma 377 (metric on $\overline{\mathbb{R}}$ )
is not yet used.
Lemma 378 (equivalent topologies on $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ ).
Lemma 379 (convergence towards $\pm \infty$ )
is explicitly cited in the proof of:
Lemma 380 (convergence towards $\pm \infty$ (discrete)),
Lemma 500 (infimum is limit).
Lemma 380 (convergence towards $\pm \infty$ (discrete))
is explicitly cited in the proof of:
Lemma 381 (limit is monotone),
Lemma 382 (cluster point at $\pm \infty$ (discrete)),
Lemma 486 (equivalent definition of infimum).
Lemma 381 (limit is monotone)
is explicitly cited in the proof of:
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 514 (compatibility of limit inferior with absolute value),
Lemma 1015 (compatibility of almost limit with almost inequality),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma $1080\left(\lambda^{\star}\right.$ is $\sigma$-additive on $\left.\mathcal{L}\right)$,
Lemma 1082 ( $\mathcal{L}$ is closed under countable union),
Lemma 1083 (rays are Lebesgue-measurable),
Theorem 1180 (Beppo Levi, monotone convergence),
Theorem 1208 (Fatou's lemma),

Theorem 1242 (Tonelli),
Theorem 1347 (Lebesgue, dominated convergence).
Lemma 382 (cluster point at $\pm \infty$ (discrete))
is explicitly cited in the proof of:
Lemma 506 (equivalent definition of the limit inferior).
Lemma 383 (limit is invariant by translation)
is explicitly cited in the proof of:
Lemma 507 (limit inferior is invariant by translation).
Lemma 385 (topology on $\overline{\mathbb{R}}^{m}$ )
is explicitly cited in the proof of:
Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ ),
Lemma $480\left(\overline{\mathbb{R}}^{m}\right.$ is second-countable).
Lemma 386 (trace topology on subset of $\overline{\mathbb{R}}^{m}$ )
is not yet used.
Lemma 387 (addition in $\overline{\mathbb{R}}$ is continuous)
is explicitly cited in the proof of:
Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear),
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous).
Lemma 388 (multiplication in $\overline{\mathbb{R}}$ is continuous)
is explicitly cited in the proof of:
Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear),
Lemma 448 (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)),
Lemma 501 (supremum is limit),
Lemma 511 (equivalent definition of limit superior),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ).
Lemma 389 (absolute value in $\overline{\mathbb{R}}$ is continuous)
is explicitly cited in the proof of:
Lemma 514 (compatibility of limit inferior with absolute value),
Lemma 1015 (compatibility of almost limit with almost inequality),
Theorem 1347 (Lebesgue, dominated convergence).
Lemma 390 (power function in $\overline{\mathbb{R}}$ is continuous)
is explicitly cited in the proof of:
Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)),
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma 391 (limit in $\overline{\mathbb{R}}$ is linear)
is explicitly cited in the proof of:
Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).
Lemma 392 ( $\overline{\mathbb{R}}_{+}$is complete)
is explicitly cited in the proof of:
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent $)$.
Lemma 393 (addition in $\overline{\mathbb{R}}_{+}$is closed)
is explicitly cited in the proof of:
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$),
Lemma 453 (Young's inequality for products (measure theory)),

Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition $)$,
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1249 (equivalent definition of integrability),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).
Lemma 394 (addition in $\overline{\mathbb{R}}_{+}$is associative)
is explicitly cited in the proof of:
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 864 (binary inclusion-exclusion principle),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure).
Lemma 395 (addition in $\overline{\mathbb{R}}_{+}$is commutative)
is explicitly cited in the proof of:
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 1074 ( $\mathcal{L}$ is closed under complement),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure).
Lemma 396 (addition in $\overline{\mathbb{R}}_{+}$is compatible with order)
is explicitly cited in the proof of:
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$),
Lemma 869 (subadditivity),
Lemma 878 ( $\sigma$-subadditivity),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).

## Lemma 397 (infinity-sum property in $\overline{\mathbb{R}}_{+}$)

is explicitly cited in the proof of:
Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 1249 (equivalent definition of integrability).
Lemma 398 (definition of series in $\overline{\mathbb{R}}_{+}$)
is explicitly cited in the proof of:
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition),
Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone),
Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds),
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$),
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum).
Lemma 399 (series in $\overline{\mathbb{R}}_{+}$are compatible with pseudopartition)
is explicitly cited in the proof of:
Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone).
Lemma 400 (series in $\overline{\mathbb{R}}_{+}$is monotone)
is explicitly cited in the proof of:
Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds).
Lemma 401 (subseries in $\overline{\mathbb{R}}_{+}$are lower bounds)
is explicitly cited in the proof of:
Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$).

## Lemma 402 (order is meaningless in series in $\overline{\mathbb{R}}_{+}$)

is explicitly cited in the proof of:
Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$).
Lemma 403 (definition of double series in $\overline{\mathbb{R}}_{+}$)
is explicitly cited in the proof of:
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive).
Lemma 404 (double series in $\overline{\mathbb{R}}_{+}$)
is explicitly cited in the proof of:
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive).
Definition 405 (multiplication in $\overline{\mathbb{R}}_{+}$)
is explicitly cited in the proof of:
Lemma 407 (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined),
Lemma 408 (zero-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 409 (one-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 411 (finite-product property in $\overline{\mathbb{R}}_{+}$),
Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ ),
Lemma 1294 (compatibility of $N_{p}$ with power function).
Lemma 407 (multiplication in $\overline{\mathbb{R}}_{+}$is closed when defined)
is not yet used.
Lemma 408 (zero-product property in $\overline{\mathbb{R}}_{+}$)
is not yet used.
Lemma 409 (one-product property in $\overline{\mathbb{R}}_{+}$)
is not yet used.
Lemma 410 (infinity-product property in $\overline{\mathbb{R}}_{+}$)
is explicitly cited in the proof of:
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 453 (Young's inequality for products (measure theory)),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 1304 (Hölder inequality).
Lemma 411 (finite-product property in $\overline{\mathbb{R}}_{+}$)
is not yet used.
Lemma 412 (multiplicative inverse in $\overline{\mathbb{R}}_{+}$is monotone)
is explicitly cited in the proof of:
Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ ).
Lemma 413 (addition in $\overline{\mathbb{R}}_{+}$is continuous)
is explicitly cited in the proof of:
Lemma 503 (constant sum of monotone nonnegative functions),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive),
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).
Lemma 414 (limit in $\overline{\mathbb{R}}_{+}$is positive linear)
is explicitly cited in the proof of:
Theorem 1180 (Beppo Levi, monotone convergence).

Definition 415 (multiplication in $\overline{\mathbb{R}}$ (measure theory))
is explicitly cited in the proof of:
Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)),
Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)),
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)),
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)),
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)),
Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory)),
Lemma 448 (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 1044 (nonnegative essential supremum is positive homogeneous),
Lemma 1311 (Minkowski equality (linearly dependent functions)),
Lemma 1312 (Minkowski equality (almost linearly dependent functions)).
Lemma 417 (multiplication in $\overline{\mathbb{R}}$ is associative (measure theory)) is explicitly cited in the proof of: Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous).

Lemma 418 (multiplication in $\overline{\mathbb{R}}$ is commutative (measure theory)) is explicitly cited in the proof of:
Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)), Lemma 1227 (candidate tensor product measure is tensor product measure), Lemma 1244 (Tonelli for tensor product),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )).
Lemma 419 (multiplication in $\overline{\mathbb{R}}$ is distributive over addition (measure theory)) is explicitly cited in the proof of: Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$), Lemma 1168 (density integral in $\mathcal{S F}_{+}$), Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ),
Lemma 1311 (Minkowski equality (linearly dependent functions)).
Lemma 420 (multiplication in $\overline{\mathbb{R}}$ is compatible with order (measure theory)) is explicitly cited in the proof of:
Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ),
Lemma 1311 (Minkowski equality (linearly dependent functions)).
Lemma 421 (zero-product property in $\overline{\mathbb{R}}$ (measure theory))
is explicitly cited in the proof of:
Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)),
Lemma 453 (Young's inequality for products (measure theory)),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),

Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality),
Lemma 1219 (measure of section of product),
Lemma 1237 (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines),
Lemma 1304 (Hölder inequality).
Lemma 422 (one-product property in $\overline{\mathbb{R}}$ (measure theory)) is explicitly cited in the proof of:
Lemma 1054 (nonnegative essential supremum is compatible with power function).
Lemma 423 (infinity-product property in $\overline{\mathbb{R}}$ (measure theory)) is explicitly cited in the proof of: Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)).

Lemma 424 (finite-product property in $\overline{\mathbb{R}}$ (measure theory)) is explicitly cited in the proof of: Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory)), Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite).

Lemma 425 (multiplication in $\overline{\mathbb{R}}_{+}$is closed (measure theory)) is explicitly cited in the proof of: Lemma 453 (Young's inequality for products (measure theory)), Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)), Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$, Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$, Lemma 1154 (integral in $\mathcal{S F}_{+}$), Lemma 1244 (Tonelli for tensor product), Lemma 1302 (generalized Hölder inequality (with $\infty$ )), Lemma 1304 (Hölder inequality).

Lemma 427 (infinity-product property in $\overline{\mathbb{R}}_{+}$(measure theory)) is explicitly cited in the proof of: Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)), Lemma 453 (Young's inequality for products (measure theory)), Lemma 1044 (nonnegative essential supremum is positive homogeneous), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite).

## Lemma 428 (finite-product property in $\overline{\mathbb{R}}_{+}$(measure theory))

 is explicitly cited in the proof of: Lemma 1277 (integral over singleton).Lemma 429 (absolute value in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) is explicitly cited in the proof of: Lemma 1300 ( $N_{p}$ is absolutely homogeneous), Lemma 1304 (Hölder inequality), Lemma 1309 (Minkowski inequality in M), Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 430 (exponentiation in $\overline{\mathbb{R}}$ (measure theory))

 is explicitly cited in the proof of: Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory)), Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)),Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)),
Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)), Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)), Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates).

Lemma 431 (one-exponentiation property in $\overline{\mathbb{R}}$ (measure theory))
is explicitly cited in the proof of:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)),
Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1294 (compatibility of $N_{p}$ with power function),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Lemma 432 (zero-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory))
is explicitly cited in the proof of:
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1288 ( $N_{p}$ with trivial measure),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Lemma 433 (values of the exponentiation in $\overline{\mathbb{R}}_{+}$(measure theory))
is explicitly cited in the proof of:
Lemma 434 (finite-positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)), Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)),
Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)), Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).

Lemma 434 (finite-positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) is not yet used.

Lemma 435 (finite-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) is explicitly cited in the proof of: Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).

Lemma 436 (positive-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) is explicitly cited in the proof of: Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)).
Lemma 437 (less-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) is explicitly cited in the proof of: Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)).

Lemma 438 (great-than-one-exponentiation property in $\overline{\mathbb{R}}_{+}$(measure theory)) is explicitly cited in the proof of: Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory)).

Lemma 439 (power function in $\overline{\mathbb{R}}$ is nonnegative (measure theory)) is explicitly cited in the proof of:
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory)), Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable),
Lemma 841 ( $\mathcal{M}$ is closed under power function), Lemma 1300 ( $N_{p}$ is absolutely homogeneous),

Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Lemma 1309 (Minkowski inequality in M),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ), Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

Lemma 441 (compatibility of power function in $\overline{\mathbb{R}}$ with addition (measure theory)) is not yet used.

Lemma 442 (compatibility of power function in $\overline{\mathbb{R}}$ with multiplication (measure theory))
is explicitly cited in the proof of:
Lemma 1054 (nonnegative essential supremum is compatible with power function).
Lemma 443 (compatibility of power function in $\overline{\mathbb{R}}$ with absolute value (measure theory))
is explicitly cited in the proof of:
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ), Lemma 1294 (compatibility of $N_{p}$ with power function), Lemma 1305 (generalized Hölder inequality).

Lemma 444 (power function in $\overline{\mathbb{R}}$ is monotone (measure theory)) is explicitly cited in the proof of: Lemma 1054 (nonnegative essential supremum is compatible with power function), Lemma 1302 (generalized Hölder inequality (with $\infty$ )).

Lemma 445 (power function in $\overline{\mathbb{R}}$ is even (measure theory)) is not yet used.

Lemma 446 (power function in $\overline{\mathbb{R}}$ is multiplicative (measure theory)) is explicitly cited in the proof of: Lemma 1300 ( $N_{p}$ is absolutely homogeneous), Lemma 1302 (generalized Hölder inequality (with $\infty$ )), Lemma 1305 (generalized Hölder inequality), Lemma 1309 (Minkowski inequality in M).

Lemma 448 (multiplication in $\overline{\mathbb{R}}_{+}$is continuous (measure theory)) is not yet used.

Lemma 449 (power function in $\overline{\mathbb{R}}$ is continuous (measure theory)) is explicitly cited in the proof of:
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable), Lemma 1012 (compatibility of almost limit with power function).

Definition 451 (Hölder conjugates)
is explicitly cited in the proof of: Lemma 1304 (Hölder inequality), Lemma 1305 (generalized Hölder inequality), Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates).

Lemma 453 (Young's inequality for products (measure theory)) is explicitly cited in the proof of: Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)), Lemma 1304 (Hölder inequality).

Lemma 454 (Young's inequality for products, case $p=2$ (measure theory)) is not yet used.

Lemma 455 (interpolation of multiplicative inverse in segment of $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Definition 456 (connected component in $\mathbb{R}$ )
is explicitly cited in the proof of:
Lemma 457 (empty connected component),
Lemma 458 ( connected component of open subset of $\mathbb{R}$ is open interval),
Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal),
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ).
Lemma 457 (empty connected component)
is not yet used.
Lemma 458 (connected component of open subset of $\mathbb{R}$ is open interval)
is explicitly cited in the proof of:
Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal).
Lemma 459 (connected component of open subset of $\mathbb{R}$ is maximal)
is explicitly cited in the proof of:
Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint).
Lemma 460 (connected components of open subset of $\mathbb{R}$ are equal or disjoint)
is explicitly cited in the proof of:
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ).
Theorem 461 (countable connected components of open subsets of $\mathbb{R}$ ) is explicitly cited in the proof of:
Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ ).
Lemma 462 (rational approximation of lower bound of open interval)
is explicitly cited in the proof of:
Lemma 463 (rational approximation of upper bound of open interval),
Lemma 464 (open intervals with rational bounds cover open interval),
Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ).

## Lemma 463 (rational approximation of upper bound of open interval)

 is explicitly cited in the proof of: Lemma 464 (open intervals with rational bounds cover open interval), Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ).Lemma 464 (open intervals with rational bounds cover open interval) is explicitly cited in the proof of: Theorem $465(\mathbb{R}$ is second-countable), Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ).

Theorem $465(\mathbb{R}$ is second-countable) is explicitly cited in the proof of: Lemma $466\left(\mathbb{R}^{m}\right.$ is second-countable), Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ ).

Lemma $466\left(\mathbb{R}^{m}\right.$ is second-countable) is not yet used.

Definition 468 (projection and injections between $\mathbb{R}$ and $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}$ ),
Lemma 471 (compatibility of projection and injections in $\overline{\mathbb{R}}$ with subset operations),
Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).
Lemma 469 (compatibility between projection and injections in $\overline{\mathbb{R}}$ ) is explicitly cited in the proof of: Lemma 475 (open up).

Lemma 471 (compatibility of projection and injections in $\overline{\mathbb{R}}$ with subset operations) is not yet used.

## Lemma 474 (open down)

is explicitly cited in the proof of: Lemma 475 (open up), Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).

Lemma 475 (open up) is explicitly cited in the proof of: Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ).

Lemma 476 (open subsets of $\overline{\mathbb{R}}$ ) is explicitly cited in the proof of: Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ).

Lemma 477 (countable connected components of open subsets of $\overline{\mathbb{R}}$ ) is explicitly cited in the proof of: Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable).

Lemma 478 (open intervals with rational bounds cover open interval of $\overline{\mathbb{R}}$ ) is explicitly cited in the proof of: Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable).

Lemma 479 ( $\overline{\mathbb{R}}$ is second-countable) is explicitly cited in the proof of:
Lemma $480\left(\overline{\mathbb{R}}^{m}\right.$ is second-countable).
Lemma $480\left(\overline{\mathbb{R}}^{m}\right.$ is second-countable) is not yet used.

Lemma 482 (extrema of constant function)
is explicitly cited in the proof of:
Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ),
Lemma 492 (infimum of bounded is bounded),
Lemma 493 (supremum of bounded is bounded),
Lemma 500 (infimum is limit),
Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset).

## Lemma 483 (compatibility of extrema with affine transformation)

is explicitly cited in the proof of: Lemma 883 (continuity from above).

## Lemma 484 (equivalent definition of finite infimum)

is explicitly cited in the proof of: Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ ).

## Lemma 485 (equivalent definition of finite infimum in $\overline{\mathbb{R}}$ )

is explicitly cited in the proof of:
Lemma 486 (equivalent definition of infimum).

## Lemma 486 (equivalent definition of infimum)

is explicitly cited in the proof of:
Lemma 1021 (essential supremum is minimum).
Lemma 487 (infimum is smaller than supremum)
is explicitly cited in the proof of:
Lemma 494 (compatibility of infimum with absolute value).

## Lemma 488 (infimum is monotone (set))

is explicitly cited in the proof of:
Lemma 489 (supremum is monotone (set)),
Lemma 496 (compatibility of translation with infimum of sequence),
Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum),
Lemma 504 (limit inferior),
Theorem 1208 (Fatou's lemma).
Lemma 489 (supremum is monotone (set))
is explicitly cited in the proof of:
Lemma 497 (compatibility of translation with supremum of sequence),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone).

## Lemma 490 (infimum is monotone (function))

is explicitly cited in the proof of:
Lemma 491 (supremum is monotone (function)),
Lemma 492 (infimum of bounded is bounded),
Lemma 494 (compatibility of infimum with absolute value),
Lemma 508 (limit inferior is monotone),
Lemma 1065 ( $\lambda^{\star}$ is nonnegative).
Lemma 491 (supremum is monotone (function))
is explicitly cited in the proof of:
Lemma 493 (supremum of bounded is bounded),
Lemma 494 (compatibility of infimum with absolute value),
Lemma 508 (limit inferior is monotone).

## Lemma 492 (infimum of bounded is bounded)

 is explicitly cited in the proof of:Lemma 847 ( $\mathcal{M}_{+}$is closed under infimum), Theorem 1208 (Fatou's lemma).

## Lemma 493 (supremum of bounded is bounded)

is explicitly cited in the proof of:
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum $)$,
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 494 (compatibility of infimum with absolute value)
is explicitly cited in the proof of:
Lemma 495 (compatibility of supremum with absolute value),
Lemma 514 (compatibility of limit inferior with absolute value).
Lemma 495 (compatibility of supremum with absolute value) is not yet used.

## Lemma 496 (compatibility of translation with infimum of sequence)

is explicitly cited in the proof of:
Lemma 498 (infimum of nonincreasing sequence).

## Lemma 497 (compatibility of translation with supremum of sequence)

is not yet used.

## Lemma 498 (infimum of nonincreasing sequence)

is explicitly cited in the proof of:
Lemma 499 (supremum of nondecreasing sequence),
Lemma 883 (continuity from above).

## Lemma 499 (supremum of nondecreasing sequence)

is not yet used.

## Lemma 500 (infimum is limit)

is explicitly cited in the proof of:
Lemma 501 (supremum is limit),
Lemma 503 (constant sum of monotone nonnegative functions), Lemma 883 (continuity from above).

Lemma 501 (supremum is limit) is explicitly cited in the proof of: Lemma 503 (constant sum of monotone nonnegative functions), Lemma 880 (continuity from below), Theorem 1180 (Beppo Levi, monotone convergence), Theorem 1208 (Fatou's lemma), Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).

Lemma 502 (series in $\overline{\mathbb{R}}_{+}$is supremum)
is explicitly cited in the proof of:
Lemma 880 (continuity from below),
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 884 ( $\sigma$-additivity (from above)).

## Lemma 503 (constant sum of monotone nonnegative functions)

is explicitly cited in the proof of:
Lemma 884 ( $\sigma$-additivity (from above)).

## Lemma 504 (limit inferior)

is explicitly cited in the proof of:
Lemma 505 (limit inferior is $\infty$ ),
Lemma 506 (equivalent definition of the limit inferior),
Lemma 508 (limit inferior is monotone),
Lemma 510 (duality limit inferior-limit superior),
Lemma 514 (compatibility of limit inferior with absolute value),
Lemma $830(\mathcal{M}$ is closed under limit inferior $)$,
Theorem 1208 (Fatou's lemma).
Lemma 505 (limit inferior is $\infty$ )
is explicitly cited in the proof of:
Lemma 506 (equivalent definition of the limit inferior),
Lemma 522 (limit inferior, limit superior and pointwise convergence).

## Lemma 506 (equivalent definition of the limit inferior)

is explicitly cited in the proof of:
Lemma 507 (limit inferior is invariant by translation),

Lemma 511 (equivalent definition of limit superior),
Lemma 512 (limit inferior is smaller than limit superior),
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 522 (limit inferior, limit superior and pointwise convergence).

## Lemma 507 (limit inferior is invariant by translation)

is explicitly cited in the proof of:
Lemma 508 (limit inferior is monotone).

## Lemma 508 (limit inferior is monotone)

is explicitly cited in the proof of:
Lemma 513 (limit superior is monotone),
Lemma 518 (limit inferior bounded from below),
Lemma 519 (limit inferior bounded from above).

## Lemma 509 (limit superior)

is explicitly cited in the proof of:
Lemma 510 (duality limit inferior-limit superior),
Lemma 514 (compatibility of limit inferior with absolute value),
Lemma $831(\mathcal{M}$ is closed under limit superior).

## Lemma 510 (duality limit inferior-limit superior)

is explicitly cited in the proof of:
Lemma 511 (equivalent definition of limit superior),
Lemma 513 (limit superior is monotone),
Lemma 515 (compatibility of limit superior with absolute value),
Lemma 520 (limit superior bounded from below),
Lemma 521 (limit superior bounded from above),
Lemma 522 (limit inferior, limit superior and pointwise convergence),
Theorem 1347 (Lebesgue, dominated convergence).

## Lemma 511 (equivalent definition of limit superior)

is explicitly cited in the proof of:
Lemma 512 (limit inferior is smaller than limit superior),
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Lemma 522 (limit inferior, limit superior and pointwise convergence).
Lemma 512 (limit inferior is smaller than limit superior)
is explicitly cited in the proof of:
Lemma 522 (limit inferior, limit superior and pointwise convergence).
Lemma 513 (limit superior is monotone)
is not yet used.

## Lemma 514 (compatibility of limit inferior with absolute value)

is explicitly cited in the proof of:
Lemma 515 (compatibility of limit superior with absolute value).
Lemma 515 (compatibility of limit superior with absolute value)
is not yet used.
Definition 516 (pointwise convergence)
is explicitly cited in the proof of:
Lemma 517 (limit inferior and limit superior of pointwise convergent),
Theorem 1208 (Fatou's lemma),
Theorem 1347 (Lebesgue, dominated convergence).

## Lemma 517 (limit inferior and limit superior of pointwise convergent)

is explicitly cited in the proof of:
Lemma 518 (limit inferior bounded from below),
Lemma 519 (limit inferior bounded from above),
Lemma 1011 (compatibility of almost limit with continuous function),
Lemma 1013 (almost limits are almost equal),
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 518 (limit inferior bounded from below)

is explicitly cited in the proof of:
Lemma 521 (limit superior bounded from above),
Theorem 1208 (Fatou's lemma),
Theorem 1347 (Lebesgue, dominated convergence).

## Lemma 519 (limit inferior bounded from above)

is explicitly cited in the proof of:
Lemma 520 (limit superior bounded from below),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ).
Lemma 520 (limit superior bounded from below)
is not yet used.

## Lemma 521 (limit superior bounded from above)

is not yet used.
Lemma 522 (limit inferior, limit superior and pointwise convergence) is explicitly cited in the proof of:
Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent), Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence), Theorem 1347 (Lebesgue, dominated convergence).

## Lemma 524 (restriction is masking)

is explicitly cited in the proof of:
Lemma $1105\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under extension by zero),
Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero $)$,
Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero $)$,
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero $)$.
Lemma 525 (masking is $\sigma$-additive)
is explicitly cited in the proof of:
Lemma 1102 ( $\mathcal{I F}_{\mathfrak{K}}$ is $\sigma$-additive),
Lemma 1107 ( $\mathcal{F}_{\mathfrak{K}}$, masking is $\sigma$-additive),
Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive),
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1255 (integrability, masking is $\sigma$-additive).

## Definition 526 (finite part)

is explicitly cited in the proof of:
Lemma 527 (finite part is finite),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part).

## Lemma 527 (finite part is finite)

is explicitly cited in the proof of:
Lemma 821 ( $\mathcal{M}$ is closed under finite part).
Definition 528 (nonnegative and nonpositive parts)
is explicitly cited in the proof of:
Lemma 529 (equivalent definition of nonnegative and nonpositive parts),
Lemma 530 (nonnegative and nonpositive parts are nonnegative),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 534 (compatibility of nonpositive and nonnegative parts with mask),
Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous).
Lemma 529 (equivalent definition of nonnegative and nonpositive parts)
is explicitly cited in the proof of:
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part).
Lemma 530 (nonnegative and nonpositive parts are nonnegative)
is explicitly cited in the proof of:
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value).
Lemma 531 (nonnegative and nonpositive parts are orthogonal)
is explicitly cited in the proof of:
Lemma 532 (decomposition into nonnegative and nonpositive parts),
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 1253 (integrability is closed under extension by zero).
Lemma 532 (decomposition into nonnegative and nonpositive parts)
is explicitly cited in the proof of:
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1277 (integral over singleton).
Lemma 533 (compatibility of nonpositive and nonnegative parts with addition)
is explicitly cited in the proof of:
Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts).

Lemma 534 (compatibility of nonpositive and nonnegative parts with mask)
is explicitly cited in the proof of:
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1255 (integrability, masking is $\sigma$-additive).
Lemma 535 (compatibility of nonpositive and nonnegative parts with restriction) is explicitly cited in the proof of:
Lemma 1253 (integrability is closed under extension by zero), Lemma 1254 (integrability is closed under restriction).

Lemma 538 (nonempty and with empty or full)
is explicitly cited in the proof of:
Lemma 612 (equivalent definition of ring of sets),
Lemma 627 (equivalent definition of semiring of sets),
Lemma 635 (equivalent definition of algebra of sets),
Lemma 655 (equivalent definition of semi-algebra of sets),
Lemma 680 (other properties of $\lambda$-system),
Lemma 697 (equivalent definition of $\sigma$-algebra).
Lemma 539 (with empty and full)
is explicitly cited in the proof of:
Lemma 635 (equivalent definition of algebra of sets),
Lemma 680 (other properties of $\lambda$-system),
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).
Lemma 541 (closedness under local complement and complement)
is explicitly cited in the proof of:
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 678 (equivalent definition of $\lambda$-system),
Lemma 687 (upgrade monotone class to $\lambda$-system).
Lemma 542 (closedness under disjoint union and local complement)
is explicitly cited in the proof of:
Lemma 566 (closedness under countable disjoint union and local complement),
Lemma 678 (equivalent definition of $\lambda$-system),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).
Lemma 543 (closedness under set difference and local complement)
is explicitly cited in the proof of:
Lemma 637 (other equivalent definition of algebra of sets).
Lemma 544 (closedness under symmetric difference, local complement, and disjoint union)
is explicitly cited in the proof of:
Lemma 612 (equivalent definition of ring of sets).
Lemma 545 (closedness under intersection and set difference)
is explicitly cited in the proof of:
Lemma 548 (closedness under union and set difference),
Lemma 612 (equivalent definition of ring of sets),
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 650 (upgrade $\pi$-system to algebra of sets),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 1077 ( $\mathcal{L}$ is algebra of sets).
Lemma 546 (closedness under union and disjoint union)
is explicitly cited in the proof of:
Lemma 612 (equivalent definition of ring of sets),
Lemma 639 (other properties of algebra of sets),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).
Lemma 547 (closedness under union and intersection)
is explicitly cited in the proof of:
Lemma 548 (closedness under union and set difference),
Lemma 559 (closedness under finite union and intersection),
Lemma 635 (equivalent definition of algebra of sets),

Lemma 650 (upgrade $\pi$-system to algebra of sets),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).
Lemma 548 (closedness under union and set difference)
is not yet used.
Lemma 550 (closedness under symmetric difference and set difference)
is explicitly cited in the proof of:
Lemma 612 (equivalent definition of ring of sets),
Lemma 650 (upgrade $\pi$-system to algebra of sets).

## Lemma 551 (closedness under symmetric difference and intersection)

 is explicitly cited in the proof of:Lemma 612 (equivalent definition of ring of sets).
Lemma 552 (nonempty is with empty)
is explicitly cited in the proof of:
Lemma 574 (disjoint union of union in subset system (variant)),
Lemma 627 (equivalent definition of semiring of sets).
Lemma 553 (with full is with empty)
is not yet used.
Lemma 554 (disjoint decomposition)
is explicitly cited in the proof of:
Lemma 631 (ring of sets is semiring of sets),
Lemma 660 (algebra of sets is semi-algebra of sets),
Lemma 740 (properties of premeasurable space),
Lemma 742 (properties of measurable space),
Lemma 880 (continuity from below),
Lemma 883 (continuity from above),
Lemma 902 (premeasure is monotone).
Lemma 555 (disjoint decomposition of complement and set difference)
is explicitly cited in the proof of:
Lemma 655 (equivalent definition of semi-algebra of sets).
Lemma 556 (disjoint decomposition of local complement and complement)
is explicitly cited in the proof of:
Lemma 655 (equivalent definition of semi-algebra of sets).
Lemma 557 (disjoint decomposition of set difference and local complement) is explicitly cited in the proof of:
Lemma 630 (other properties of semiring of sets),
Lemma 655 (equivalent definition of semi-algebra of sets),
Lemma 869 (subadditivity),
Lemma 878 ( $\sigma$-subadditivity).
Lemma 558 (closedness under finite operations)
is explicitly cited in the proof of:
Lemma 559 (closedness under finite union and intersection),
Lemma 572 (disjoint union of union in subset system),
Lemma 575 (monotone union of union in subset system),
Lemma 604 (equivalent definition of $\pi$-system),
Lemma 612 (equivalent definition of ring of sets),
Lemma 615 (other properties of ring of sets),
Lemma 635 (equivalent definition of algebra of sets),

Lemma 639 (other properties of algebra of sets), Lemma 680 (other properties of $\lambda$-system).

Lemma 559 (closedness under finite union and intersection) is not yet used.

Lemma 561 (closedness under intersection and finite disjoint union closure) is explicitly cited in the proof of: Lemma 563 (closedness under set difference and finite disjoint union closure).

Lemma 562 (finite disjoint union closure is closed under finite disjoint union) is explicitly cited in the proof of: Lemma 563 (closedness under set difference and finite disjoint union closure), Lemma 632 (explicit ring of sets).

Lemma 563 (closedness under set difference and finite disjoint union closure) is explicitly cited in the proof of: Lemma 632 (explicit ring of sets).

Lemma 566 (closedness under countable disjoint union and local complement) is explicitly cited in the proof of: Lemma 678 (equivalent definition of $\lambda$-system).

Lemma 567 (closedness under countable union and intersection) is explicitly cited in the proof of: Lemma 679 ( $\lambda$-system is monotone class), Lemma 687 (upgrade monotone class to $\lambda$-system), Lemma 697 (equivalent definition of $\sigma$-algebra).

Lemma 568 (closedness under countable disjoint and monotone union) is explicitly cited in the proof of: Lemma 678 (equivalent definition of $\lambda$-system).

Lemma 569 (closedness under countable monotone and disjoint union) is explicitly cited in the proof of: Lemma 571 (closedness under countable monotone union and countable union), Lemma 678 (equivalent definition of $\lambda$-system).

Lemma 570 (closedness under countable disjoint union and countable union) is explicitly cited in the proof of: Lemma 571 (closedness under countable monotone union and countable union), Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra), Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).

Lemma 571 (closedness under countable monotone union and countable union) is explicitly cited in the proof of: Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).

Lemma 572 (disjoint union of union in subset system) is explicitly cited in the proof of: Lemma 624 (disjoint union of union in ring of sets), Lemma 728 (disjoint union of union in $\sigma$-algebra), Lemma 880 (continuity from below).

Lemma 574 (disjoint union of union in subset system (variant)) is explicitly cited in the proof of:
Lemma 869 (subadditivity), Lemma 878 ( $\sigma$-subadditivity).

## Lemma 575 (monotone union of union in subset system)

is explicitly cited in the proof of:
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 886 (equivalent definition of $\sigma$-finiteness).
Lemma 576 (empty monotone intersection of union in subset system)
is explicitly cited in the proof of:
Lemma 884 ( $\sigma$-additivity (from above)).
Lemma 577 (monotone union of monotone intersection in subset system)
is explicitly cited in the proof of:
Lemma 883 (continuity from above).
Definition 578 (subset system)
is explicitly cited in the proof of:
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation).
Definition 580 (subset system class)
is explicitly cited in the proof of:
Lemma 582 (intersection of subset system classes),
Lemma 583 (subset system classes are monotone),
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation),
Lemma 590 (upgrade generated subset system),
Lemma 594 (class of nonempty subset systems).
Lemma 582 (intersection of subset system classes)
is explicitly cited in the proof of:
Lemma 594 (class of nonempty subset systems).

## Lemma 583 (subset system classes are monotone)

is explicitly cited in the proof of:
Lemma 588 (generated subset systems are monotone),
Lemma 591 (cascading generated subset systems).
Definition 584 (generated subset system)
is explicitly cited in the proof of:
Lemma 585 (subset system generation preserves class),
Lemma 586 (properties of subset system generation),
Lemma 588 (generated subset systems are monotone),
Lemma 595 (generated nonempty subset systems).
Lemma 585 (subset system generation preserves class) is explicitly cited in the proof of:
Lemma 586 (properties of subset system generation),
Lemma 587 (other properties of subset system generation),
Lemma 590 (upgrade generated subset system),
Lemma 591 (cascading generated subset systems),
Lemma 596 (subset system generation preserves class (with nonemptiness)),
Lemma 618 (generated ring of sets is ring of sets),
Lemma 643 (generated algebra of sets is algebra of sets),
Lemma 667 (generated monotone class is monotone class),
Lemma 683 (generated $\lambda$-system is $\lambda$-system),
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra).

## Lemma 586 (properties of subset system generation)

is explicitly cited in the proof of:
Lemma 587 (other properties of subset system generation),
Lemma 590 (upgrade generated subset system),
Lemma 591 (cascading generated subset systems),
Lemma 597 (properties of generated subset system (with nonemptiness)),
Lemma 619 (properties of ring of sets generation),
Lemma 644 (properties of algebra of sets generation),
Lemma 668 (properties of monotone class generation),
Lemma 684 (properties of $\lambda$-system generation),
Lemma 710 (properties of $\sigma$-algebra generation).

## Lemma 587 (other properties of subset system generation)

is explicitly cited in the proof of:
Lemma 598 (other properties of subset system generation (with nonemptiness)),
Lemma 620 (other properties of ring of sets generation),
Lemma 645 (other properties of algebra of sets generation),
Lemma 669 (other properties of monotone class generation),
Lemma 685 (other properties of $\lambda$-system generation),
Lemma 711 (other properties of $\sigma$-algebra generation).
Lemma 588 (generated subset systems are monotone)
is explicitly cited in the proof of:
Lemma 599 (generated subset systems are monotone (with nonemptiness)),
Lemma 646 (algebra of sets contains ring of sets),
Lemma 713 ( $\sigma$-algebra contains $\lambda$-system),
Lemma 719 ( $\sigma$-algebra contains algebra of sets).
Lemma 590 (upgrade generated subset system)
is explicitly cited in the proof of:
Lemma 600 (upgrade generated subset system (with nonemptiness)),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).
Lemma 591 (cascading generated subset systems) is explicitly cited in the proof of:
Lemma 601 (cascading generated subset systems (with nonemptiness)),
Lemma 648 (algebra of sets generated by ring of sets),
Lemma 688 ( $\lambda$-system generated by monotone class),
Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets).
Lemma 592 (double cascading generated subset systems)
is explicitly cited in the proof of:
Lemma 651 (algebra of sets generated by $\pi$-system),
Lemma 718 ( $\sigma$-algebra generated by monotone class),
Lemma 724 ( $\sigma$-algebra generated by ring of sets),
Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).

## Lemma 594 (class of nonempty subset systems)

is explicitly cited in the proof of:
Lemma 595 (generated nonempty subset systems),
Lemma 596 (subset system generation preserves class (with nonemptiness)),
Lemma 597 (properties of generated subset system (with nonemptiness)).

## Lemma 595 (generated nonempty subset systems)

is explicitly cited in the proof of:
Lemma 596 (subset system generation preserves class (with nonemptiness)),
Lemma 597 (properties of generated subset system (with nonemptiness)),
Lemma 598 (other properties of subset system generation (with nonemptiness)),
Lemma 599 (generated subset systems are monotone (with nonemptiness)),
Lemma 600 (upgrade generated subset system (with nonemptiness)),
Lemma 601 (cascading generated subset systems (with nonemptiness)).
Lemma 596 (subset system generation preserves class (with nonemptiness)) is explicitly cited in the proof of:
Lemma 600 (upgrade generated subset system (with nonemptiness)),
Lemma 607 (generated $\pi$-system is $\pi$-system).
Lemma 597 (properties of generated subset system (with nonemptiness)) is explicitly cited in the proof of:
Lemma 608 (properties of $\pi$-system generation).
Lemma 598 (other properties of subset system generation (with nonemptiness)) is explicitly cited in the proof of: Lemma 609 (other properties of $\pi$-system generation).
Lemma 599 (generated subset systems are monotone (with nonemptiness)) is explicitly cited in the proof of: Lemma 621 (ring of sets contains $\pi$-system), Lemma 686 ( $\lambda$-system contains monotone class).

Lemma 600 (upgrade generated subset system (with nonemptiness))
is explicitly cited in the proof of:
Lemma 622 (upgrade $\pi$-system to ring of sets).
Lemma 601 (cascading generated subset systems (with nonemptiness)) is explicitly cited in the proof of: Lemma 623 (ring of sets generated by $\pi$-system).

Definition 602 ( $\pi$-system) is explicitly cited in the proof of: Lemma 604 (equivalent definition of $\pi$-system), Lemma 615 (other properties of ring of sets), Lemma 630 (other properties of semiring of sets), Lemma 890 (content over finite pseudopartition), Lemma 905 (premeasure over countable pseudopartition),
Lemma 992 (uniqueness of measures extended from a $\pi$-system), Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).
Lemma 604 (equivalent definition of $\pi$-system)
is explicitly cited in the proof of:
Lemma 605 (intersection of $\pi$-systems),
Lemma 614 (ring of sets is $\pi$-system),
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 629 (semiring of sets is $\pi$-system),
Lemma 650 (upgrade $\pi$-system to algebra of sets),
Lemma 694 ( $\lambda$-system generated by $\pi$-system),
Theorem 732 (Dynkin $\pi-\lambda$ theorem).
Lemma 605 (intersection of $\pi$-systems)
is explicitly cited in the proof of:

Lemma 607 (generated $\pi$-system is $\pi$-system), Lemma 608 (properties of $\pi$-system generation), Lemma 609 (other properties of $\pi$-system generation), Lemma 622 (upgrade $\pi$-system to ring of sets), Lemma 623 (ring of sets generated by $\pi$-system).

## Definition 606 (generated $\pi$-system)

is explicitly cited in the proof of:
Lemma 607 (generated $\pi$-system is $\pi$-system),
Lemma 608 (properties of $\pi$-system generation),
Lemma 609 (other properties of $\pi$-system generation),
Lemma 621 (ring of sets contains $\pi$-system),
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 623 (ring of sets generated by $\pi$-system),
Lemma 651 (algebra of sets generated by $\pi$-system),
Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).
Lemma 607 (generated $\pi$-system is $\pi$-system) is explicitly cited in the proof of: Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem).

## Lemma 608 (properties of $\pi$-system generation)

is explicitly cited in the proof of:
Lemma 992 (uniqueness of measures extended from a $\pi$-system).

## Lemma 609 (other properties of $\pi$-system generation)

 is not yet used.
## Definition 611 (ring of sets)

 is explicitly cited in the proof of: Lemma 612 (equivalent definition of ring of sets), Lemma 632 (explicit ring of sets), Lemma 740 (properties of premeasurable space).
## Lemma 612 (equivalent definition of ring of sets)

 is explicitly cited in the proof of: Lemma 614 (ring of sets is $\pi$-system), Lemma 615 (other properties of ring of sets), Lemma 616 (intersection of rings of sets), Lemma 619 (properties of ring of sets generation), Lemma 622 (upgrade $\pi$-system to ring of sets), Lemma 624 (disjoint union of union in ring of sets), Lemma 631 (ring of sets is semiring of sets), Lemma 632 (explicit ring of sets), Lemma 638 (algebra of sets is ring of sets), Lemma 639 (other properties of algebra of sets), Lemma 647 (upgrade ring of sets to algebra of sets), Lemma 740 (properties of premeasurable space), Lemma 892 (content is subtractive), Lemma 896 ( $\sigma$-finite content).
## Lemma 614 (ring of sets is $\pi$-system)

 is explicitly cited in the proof of: Lemma 615 (other properties of ring of sets), Lemma 621 (ring of sets contains $\pi$-system), Lemma 623 (ring of sets generated by $\pi$-system),Lemma 640 (algebra of sets is $\pi$-system),
Lemma 650 (upgrade $\pi$-system to algebra of sets),
Lemma 740 (properties of premeasurable space).

## Lemma 615 (other properties of ring of sets)

is explicitly cited in the proof of:
Lemma 639 (other properties of algebra of sets),
Lemma 740 (properties of premeasurable space).

## Lemma 616 (intersection of rings of sets)

is explicitly cited in the proof of:
Lemma 618 (generated ring of sets is ring of sets),
Lemma 619 (properties of ring of sets generation),
Lemma 620 (other properties of ring of sets generation),
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 623 (ring of sets generated by $\pi$-system),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets).

## Definition 617 (generated ring of sets)

is explicitly cited in the proof of:
Lemma 618 (generated ring of sets is ring of sets),
Lemma 619 (properties of ring of sets generation),
Lemma 620 (other properties of ring of sets generation),
Lemma 621 (ring of sets contains $\pi$-system),
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 623 (ring of sets generated by $\pi$-system),
Lemma 646 (algebra of sets contains ring of sets),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets),
Lemma 651 (algebra of sets generated by $\pi$-system),
Lemma 724 ( $\sigma$-algebra generated by ring of sets),
Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).
Lemma 618 (generated ring of sets is ring of sets)
is explicitly cited in the proof of:
Lemma 632 (explicit ring of sets),
Lemma 662 (explicit algebra of sets).

## Lemma 619 (properties of ring of sets generation)

is explicitly cited in the proof of:
Lemma 632 (explicit ring of sets).

## Lemma 620 (other properties of ring of sets generation)

 is not yet used.
## Lemma 621 (ring of sets contains $\pi$-system)

is explicitly cited in the proof of:
Lemma 622 (upgrade $\pi$-system to ring of sets),
Lemma 649 (algebra of sets contains $\pi$-system),
Lemma 725 ( $\sigma$-algebra contains $\pi$-system).
Lemma 622 (upgrade $\pi$-system to ring of sets)
is explicitly cited in the proof of:
Lemma 650 (upgrade $\pi$-system to algebra of sets).

## Lemma 623 (ring of sets generated by $\pi$-system)

is explicitly cited in the proof of:
Lemma 651 (algebra of sets generated by $\pi$-system),
Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).
Lemma 624 (disjoint union of union in ring of sets)
is explicitly cited in the proof of:
Lemma 652 (disjoint union of union in algebra of sets).
Definition 626 (semiring of sets)
is explicitly cited in the proof of:
Lemma 627 (equivalent definition of semiring of sets),
Lemma 629 (semiring of sets is $\pi$-system),
Lemma 630 (other properties of semiring of sets),
Lemma 631 (ring of sets is semiring of sets),
Lemma 632 (explicit ring of sets),
Lemma 655 (equivalent definition of semi-algebra of sets),
Lemma 891 (content satisfies the binary inclusion-exclusion principle),
Lemma 894 (content satisfies the finite Boole inequality),
Lemma 906 (premeasure satisfies the Boole inequality).
Lemma 627 (equivalent definition of semiring of sets)
is explicitly cited in the proof of:
Lemma 632 (explicit ring of sets).
Lemma 629 (semiring of sets is $\pi$-system)
is explicitly cited in the proof of:
Lemma 630 (other properties of semiring of sets),
Lemma 659 (semi-algebra of sets is $\pi$-system).
Lemma 630 (other properties of semiring of sets)
is explicitly cited in the proof of:
Lemma 658 (other properties of semi-algebra of sets),
Lemma 893 (content is monotone).
Lemma 631 (ring of sets is semiring of sets)
is explicitly cited in the proof of:
Lemma 740 (properties of premeasurable space),
Lemma 742 (properties of measurable space),
Lemma 896 ( $\sigma$-finite content).
Lemma 632 (explicit ring of sets)
is explicitly cited in the proof of:
Lemma 662 (explicit algebra of sets).
Definition 634 (algebra of sets)
is explicitly cited in the proof of:
Lemma 635 (equivalent definition of algebra of sets),
Lemma 641 (intersection of algebras of sets),
Lemma 696 ( $\sigma$-algebra),
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra).

## Lemma 635 (equivalent definition of algebra of sets)

is explicitly cited in the proof of:
Lemma 637 (other equivalent definition of algebra of sets),
Lemma 638 (algebra of sets is ring of sets),

Lemma 639 (other properties of algebra of sets), Lemma 647 (upgrade ring of sets to algebra of sets), Lemma 660 (algebra of sets is semi-algebra of sets), Lemma 700 (other properties of $\sigma$-algebra), Theorem 736 (monotone class).

## Lemma 637 (other equivalent definition of algebra of sets)

is explicitly cited in the proof of:
Lemma 638 (algebra of sets is ring of sets),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 675 (monotone class generated by algebra of sets),
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 1077 ( $\mathcal{L}$ is algebra of sets).
Lemma 638 (algebra of sets is ring of sets)
is explicitly cited in the proof of:
Lemma 639 (other properties of algebra of sets),
Lemma 640 (algebra of sets is $\pi$-system),
Lemma 646 (algebra of sets contains ring of sets), Lemma 647 (upgrade ring of sets to algebra of sets), Lemma 648 (algebra of sets generated by ring of sets), Lemma 652 (disjoint union of union in algebra of sets), Lemma 703 ( $\sigma$-algebra is ring of sets).

## Lemma 639 (other properties of algebra of sets)

 is explicitly cited in the proof of: Lemma 700 (other properties of $\sigma$-algebra), Lemma 720 (upgrade algebra of sets to $\sigma$-algebra).
## Lemma 640 (algebra of sets is $\pi$-system)

is explicitly cited in the proof of:
Lemma 704 ( $\sigma$-algebra is $\pi$-system).

## Lemma 641 (intersection of algebras of sets)

is explicitly cited in the proof of:
Lemma 643 (generated algebra of sets is algebra of sets),
Lemma 644 (properties of algebra of sets generation),
Lemma 645 (other properties of algebra of sets generation),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets).
Definition 642 (generated algebra of sets)
is explicitly cited in the proof of:
Lemma 643 (generated algebra of sets is algebra of sets),
Lemma 644 (properties of algebra of sets generation),
Lemma 645 (other properties of algebra of sets generation),
Lemma 646 (algebra of sets contains ring of sets),
Lemma 647 (upgrade ring of sets to algebra of sets),
Lemma 648 (algebra of sets generated by ring of sets),
Lemma 651 (algebra of sets generated by $\pi$-system),
Lemma 719 ( $\sigma$-algebra contains algebra of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets),

Lemma 724 ( $\sigma$-algebra generated by ring of sets),
Lemma 1221 (measurability of measure of section (finite)).
Lemma 643 (generated algebra of sets is algebra of sets) is explicitly cited in the proof of:
Lemma 738 (usage of monotone class theorem).
Lemma 644 (properties of algebra of sets generation) is not yet used.

## Lemma 645 (other properties of algebra of sets generation)

 is not yet used.
## Lemma 646 (algebra of sets contains ring of sets)

 is explicitly cited in the proof of: Lemma 647 (upgrade ring of sets to algebra of sets), Lemma 649 (algebra of sets contains $\pi$-system), Lemma 722 ( $\sigma$-algebra contains ring of sets)
## Lemma 647 (upgrade ring of sets to algebra of sets)

is explicitly cited in the proof of:
Lemma 650 (upgrade $\pi$-system to algebra of sets), Lemma 662 (explicit algebra of sets), Lemma 723 (upgrade ring of sets to $\sigma$-algebra).

Lemma 648 (algebra of sets generated by ring of sets) is explicitly cited in the proof of: Lemma 651 (algebra of sets generated by $\pi$-system), Lemma 724 ( $\sigma$-algebra generated by ring of sets).

Lemma 649 (algebra of sets contains $\pi$-system) is not yet used.

Lemma 650 (upgrade $\pi$-system to algebra of sets) is explicitly cited in the proof of: Lemma 726 (upgrade $\pi$-system to $\sigma$-algebra)

Lemma 651 (algebra of sets generated by $\pi$-system) is not yet used.

Lemma 652 (disjoint union of union in algebra of sets) is explicitly cited in the proof of: Lemma 1081 (disjoint union of union in $\mathcal{L}$ ).

## Definition 654 (semi-algebra of sets)

 is explicitly cited in the proof of: Lemma 655 (equivalent definition of semi-algebra of sets), Lemma 657 (semi-algebra of sets is semiring of sets), Lemma 662 (explicit algebra of sets), Lemma 895 (finite content is bounded).
## Lemma 655 (equivalent definition of semi-algebra of sets)

 is explicitly cited in the proof of: Lemma 660 (algebra of sets is semi-algebra of sets), Lemma 730 (algebra of sets generated by product of $\sigma$-algebras).
## Lemma 657 (semi-algebra of sets is semiring of sets)

is explicitly cited in the proof of:
Lemma 658 (other properties of semi-algebra of sets),
Lemma 659 (semi-algebra of sets is $\pi$-system),
Lemma 661 (algebra of sets is semiring of sets).

## Lemma 658 (other properties of semi-algebra of sets)

is explicitly cited in the proof of:
Lemma 895 (finite content is bounded).
Lemma 659 (semi-algebra of sets is $\pi$-system) is not yet used.

## Lemma 660 (algebra of sets is semi-algebra of sets)

is explicitly cited in the proof of: Lemma 661 (algebra of sets is semiring of sets), Lemma 705 ( $\sigma$-algebra is semi-algebra of sets), Lemma 742 (properties of measurable space), Lemma 903 (finite premeasure is bounded).

Lemma 661 (algebra of sets is semiring of sets) is explicitly cited in the proof of: Lemma 706 ( $\sigma$-algebra is semiring of sets).

## Lemma 662 (explicit algebra of sets)

 is explicitly cited in the proof of: Lemma 730 (algebra of sets generated by product of $\sigma$-algebras).
## Definition 664 (monotone class)

is explicitly cited in the proof of:
Lemma 665 (intersection of monotone classes),
Lemma 672 ( $\mathcal{C} \backslash$ is monotone class),
Lemma 679 ( $\lambda$-system is monotone class),
Lemma 680 (other properties of $\lambda$-system),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).
Lemma 665 (intersection of monotone classes) is explicitly cited in the proof of:
Lemma 667 (generated monotone class is monotone class),
Lemma 668 (properties of monotone class generation),
Lemma 669 (other properties of monotone class generation),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 688 ( $\lambda$-system generated by monotone class).
Definition 666 (generated monotone class) is explicitly cited in the proof of:
Lemma 667 (generated monotone class is monotone class),
Lemma 668 (properties of monotone class generation),
Lemma 669 (other properties of monotone class generation),
Lemma 686 ( $\lambda$-system contains monotone class),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 688 ( $\lambda$-system generated by monotone class),
Lemma 718 ( $\sigma$-algebra generated by monotone class).

Lemma 667 (generated monotone class is monotone class)
is explicitly cited in the proof of:
Lemma 674 (monotone class is closed under set difference),
Lemma 675 (monotone class generated by algebra of sets).
Lemma 668 (properties of monotone class generation)
is explicitly cited in the proof of:
Lemma 674 (monotone class is closed under set difference),
Lemma 738 (usage of monotone class theorem).
Lemma 669 (other properties of monotone class generation)
is not yet used.
Definition 670 (monotone class and symmetric set difference)
is explicitly cited in the proof of:
Lemma 671 ( $\mathcal{C} \backslash$ is symmetric),
Lemma 672 ( $\mathcal{C}$ \is monotone class),
Lemma 674 (monotone class is closed under set difference).
Lemma 671 ( $\mathcal{C} \backslash$ is symmetric)
is explicitly cited in the proof of:
Lemma 674 (monotone class is closed under set difference).
Lemma 672 ( $\mathcal{C} \backslash$ is monotone class)
is explicitly cited in the proof of:
Lemma 674 (monotone class is closed under set difference).
Lemma 674 (monotone class is closed under set difference)
is explicitly cited in the proof of:
Lemma 675 (monotone class generated by algebra of sets).
Lemma 675 (monotone class generated by algebra of sets)
is explicitly cited in the proof of:
Theorem 736 (monotone class).
Definition 677 ( $\lambda$-system)
is explicitly cited in the proof of:
Lemma 678 (equivalent definition of $\lambda$-system),
Lemma 679 ( $\lambda$-system is monotone class),
Lemma 680 (other properties of $\lambda$-system),
Lemma 681 (intersection of $\lambda$-systems),
Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 717 (upgrade monotone class to $\sigma$-algebra).

## Lemma 678 (equivalent definition of $\lambda$-system)

is explicitly cited in the proof of:
Lemma 679 ( $\lambda$-system is monotone class),
Lemma 687 (upgrade monotone class to $\lambda$-system),
Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system),
Lemma 992 (uniqueness of measures extended from a $\pi$-system).
Lemma 679 ( $\lambda$-system is monotone class)
is explicitly cited in the proof of:
Lemma 680 (other properties of $\lambda$-system),
Lemma 686 ( $\lambda$-system contains monotone class),

Lemma 688 ( $\lambda$-system generated by monotone class), Lemma 702 ( $\sigma$-algebra is monotone class).

Lemma 680 (other properties of $\lambda$-system) is not yet used.

## Lemma 681 (intersection of $\lambda$-systems)

is explicitly cited in the proof of:
Lemma 683 (generated $\lambda$-system is $\lambda$-system), Lemma 684 (properties of $\lambda$-system generation), Lemma 685 (other properties of $\lambda$-system generation), Lemma 687 (upgrade monotone class to $\lambda$-system), Lemma 688 ( $\lambda$-system generated by monotone class), Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra), Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system).

Definition 682 (generated $\lambda$-system) is explicitly cited in the proof of: Lemma 683 (generated $\lambda$-system is $\lambda$-system), Lemma 684 (properties of $\lambda$-system generation), Lemma 685 (other properties of $\lambda$-system generation), Lemma 686 ( $\lambda$-system contains monotone class), Lemma 687 (upgrade monotone class to $\lambda$-system), Lemma 688 ( $\lambda$-system generated by monotone class), Lemma 713 ( $\sigma$-algebra contains $\lambda$-system), Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra), Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system), Lemma 718 ( $\sigma$-algebra generated by monotone class).

Lemma 683 (generated $\lambda$-system is $\lambda$-system) is explicitly cited in the proof of: Lemma 692 ( $\lambda$-system with intersection), Lemma 694 ( $\lambda$-system generated by $\pi$-system).

## Lemma 684 (properties of $\lambda$-system generation)

 is explicitly cited in the proof of: Lemma 692 ( $\lambda$-system with intersection), Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem).Lemma 685 (other properties of $\lambda$-system generation) is not yet used.

Lemma 686 ( $\lambda$-system contains monotone class) is explicitly cited in the proof of: Lemma 687 (upgrade monotone class to $\lambda$-system), Lemma 716 ( $\sigma$-algebra contains monotone class).

Lemma 687 (upgrade monotone class to $\lambda$-system) is explicitly cited in the proof of: Lemma 717 (upgrade monotone class to $\sigma$-algebra).

Lemma 688 ( $\lambda$-system generated by monotone class) is explicitly cited in the proof of: Lemma 718 ( $\sigma$-algebra generated by monotone class).

Definition 689 ( $\lambda$-system and intersection) is explicitly cited in the proof of:

Lemma 690 ( $\Lambda^{\wedge}$ is symmetric),
Lemma 692 ( $\lambda$-system with intersection),
Lemma 693 ( $\lambda$-system is closed under intersection).

## Lemma 690 ( $\Lambda^{\cap}$ is symmetric)

is explicitly cited in the proof of:
Lemma 692 ( $\lambda$-system with intersection).
Lemma 691 ( $\Lambda^{\cap}$ is $\lambda$-system)
is explicitly cited in the proof of:
Lemma 692 ( $\lambda$-system with intersection).
Lemma 692 ( $\lambda$-system with intersection)
is explicitly cited in the proof of:
Lemma 693 ( $\lambda$-system is closed under intersection).

## Lemma 693 ( $\lambda$-system is closed under intersection)

is explicitly cited in the proof of:
Lemma 694 ( $\lambda$-system generated by $\pi$-system).

## Lemma 694 ( $\lambda$-system generated by $\pi$-system)

is explicitly cited in the proof of:
Theorem 732 (Dynkin $\pi-\lambda$ theorem).

## Lemma 696 ( $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 697 (equivalent definition of $\sigma$-algebra),
Lemma 699 ( $\sigma$-algebra is algebra of sets),
Lemma 707 (intersection of $\sigma$-algebras),
Lemma 712 (countable union $\sigma$-algebra generator),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 744 (some Borel subsets),
Lemma 750 (inverse $\sigma$-algebra),
Lemma 751 (image $\sigma$-algebra),
Lemma 753 (countable range is measurable),
Lemma 763 (trace $\sigma$-algebra),
Lemma 766 (generating on a partition),
Lemma 786 (measurability of section),
Lemma 787 (countable union of sections is measurable),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 833 (measurability and masking),
Lemma 834 (measurability of restriction),
Lemma 923 (measure satisfies the Boole inequality),
Lemma 924 (compatibility of null measure with countable union),
Lemma 925 (measure is continuous from below),
Lemma 927 (measure is continuous from above),
Lemma 935 (counting measure),
Lemma 937 ( $\sigma$-finite counting measure),
Lemma 947 (negligibility with trivial measure),
Lemma 949 (empty set is negligible),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 1008 (masking almost nowhere),
Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra),
Lemma $1102\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is $\sigma$-additive),

Lemma 1158 (integral in $\mathcal{S F}_{+}$of zero is zero),
Lemma 1166 (density measure in $\mathcal{S F}_{+}$),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$).

## Lemma 697 (equivalent definition of $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras),
Lemma 742 (properties of measurable space),
Lemma 754 (constant function is measurable),
Lemma 764 (measurability of measurable subspace),
Lemma 772 (measurability of function defined on a pseudopartition),
Lemma 776 (measurability of function to product space),
Lemma 788 (countable intersection of sections is measurable),
Lemma 811 (measurability of indicator function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma 829 ( $\mathcal{M}$ is closed under supremum),
Lemma 834 (measurability of restriction),
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part),
Lemma 931 (restricted measure),
Lemma 936 (finiteness of counting measure),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 999 (measurability of summability domain),
Lemma $1103\left(\mathcal{I F}_{\mathfrak{k}}\right.$ is closed under multiplication),
Lemma $1106\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under restriction),
Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation),
Lemma $1120\left(\mathcal{S F}_{E}\right.$ is closed under restriction),
Lemma 1136 ( adapted sequence in $\mathcal{M}_{+}$),
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$,
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1223 (measurability of measure of section),
Lemma 1228 (tensor product of finite measures),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1233 (uniqueness of tensor product measure).
Lemma 699 ( $\sigma$-algebra is algebra of sets)
is explicitly cited in the proof of:
Lemma 700 (other properties of $\sigma$-algebra),
Lemma 703 ( $\sigma$-algebra is ring of sets),
Lemma 704 ( $\sigma$-algebra is $\pi$-system),
Lemma 705 ( $\sigma$-algebra is semi-algebra of sets),
Lemma 706 ( $\sigma$-algebra is semiring of sets),
Lemma 719 ( $\sigma$-algebra contains algebra of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets),
Theorem 736 (monotone class),
Lemma 742 (properties of measurable space).
Lemma 700 (other properties of $\sigma$-algebra)
is explicitly cited in the proof of:

Lemma 701 ( $\sigma$-algebra is $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 728 (disjoint union of union in $\sigma$-algebra),
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras),
Lemma 742 (properties of measurable space),
Lemma 753 (countable range is measurable),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 913 (measure),
Lemma 916 (measure is additive),
Lemma 917 (measure satisfies the binary inclusion-exclusion principle),
Lemma 918 (measure is subtractive).

## Lemma 701 ( $\sigma$-algebra is $\lambda$-system)

is explicitly cited in the proof of:
Lemma 702 ( $\sigma$-algebra is monotone class),
Lemma 713 ( $\sigma$-algebra contains $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system),
Theorem 732 (Dynkin $\pi-\lambda$ theorem),
Lemma 742 (properties of measurable space).
Lemma 702 ( $\sigma$-algebra is monotone class)
is explicitly cited in the proof of:
Theorem 736 (monotone class),
Lemma 742 (properties of measurable space).
Lemma 703 ( $\sigma$-algebra is ring of sets)
is explicitly cited in the proof of:
Lemma 742 (properties of measurable space),
Lemma 915 (measure space is premeasure space).
Lemma 704 ( $\sigma$-algebra is $\pi$-system)
is explicitly cited in the proof of:
Theorem 732 (Dynkin $\pi-\lambda$ theorem),
Lemma 742 (properties of measurable space).

## Lemma 705 ( $\sigma$-algebra is semi-algebra of sets)

is not yet used.
Lemma 706 ( $\sigma$-algebra is semiring of sets)
is not yet used.
Lemma 707 (intersection of $\sigma$-algebras)
is explicitly cited in the proof of:
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation),
Lemma 711 (other properties of $\sigma$-algebra generation),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets).
Definition 708 (generated $\sigma$-algebra)
is explicitly cited in the proof of:
Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra),
Lemma 710 (properties of $\sigma$-algebra generation),

Lemma 711 (other properties of $\sigma$-algebra generation),
Lemma 713 ( $\sigma$-algebra contains $\lambda$-system),
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system),
Lemma 718 ( $\sigma$-algebra generated by monotone class), Lemma 719 ( $\sigma$-algebra contains algebra of sets),
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra),
Lemma 721 ( $\sigma$-algebra generated by algebra of sets),
Lemma 724 ( $\sigma$-algebra generated by ring of sets),
Lemma 727 ( $\sigma$-algebra generated by $\pi$-system),
Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem),
Lemma 738 (usage of monotone class theorem),
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ ).

## Lemma 709 (generated $\sigma$-algebra is $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 712 (countable union $\sigma$-algebra generator),
Lemma 744 (some Borel subsets),
Lemma 755 (inverse image of generating family),
Lemma 756 (equivalent definition of measurable function),
Lemma 761 (continuous is measurable),
Lemma 775 (product of measurable subsets is measurable),
Lemma 779 (generating product measurable space),
Lemma 786 (measurability of section),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).

## Lemma 710 (properties of $\sigma$-algebra generation)

is explicitly cited in the proof of:
Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem),
Lemma 738 (usage of monotone class theorem),
Lemma 755 (inverse image of generating family),
Lemma 756 (equivalent definition of measurable function),
Lemma 779 (generating product measurable space),
Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$.

## Lemma 711 (other properties of $\sigma$-algebra generation)

is explicitly cited in the proof of:
Lemma 712 (countable union $\sigma$-algebra generator),
Lemma 796 (Borel subsets of $\mathbb{R}$ ).

## Lemma 712 (countable union $\sigma$-algebra generator)

is explicitly cited in the proof of:
Lemma 745 (countable union Borel $\sigma$-algebra generator).

## Lemma 713 ( $\sigma$-algebra contains $\lambda$-system)

is explicitly cited in the proof of:
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra),
Lemma 716 ( $\sigma$-algebra contains monotone class).
Lemma 714 (upgrade $\lambda$-system to $\sigma$-algebra)
is explicitly cited in the proof of:
Lemma 717 (upgrade monotone class to $\sigma$-algebra), Theorem 732 (Dynkin $\pi-\lambda$ theorem).

## Lemma 715 ( $\sigma$-algebra generated by $\lambda$-system)

is explicitly cited in the proof of:
Lemma 718 ( $\sigma$-algebra generated by monotone class).
Lemma 716 ( $\sigma$-algebra contains monotone class)
is not yet used.
Lemma 717 (upgrade monotone class to $\sigma$-algebra)
is explicitly cited in the proof of:
Theorem 736 (monotone class).
Lemma 718 ( $\sigma$-algebra generated by monotone class)
is not yet used.
Lemma 719 ( $\sigma$-algebra contains algebra of sets)
is explicitly cited in the proof of:
Lemma 720 (upgrade algebra of sets to $\sigma$-algebra), Lemma 722 ( $\sigma$-algebra contains ring of sets),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Lemma 720 (upgrade algebra of sets to $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 723 (upgrade ring of sets to $\sigma$-algebra),
Lemma 726 (upgrade $\pi$-system to $\sigma$-algebra).
Lemma 721 ( $\sigma$-algebra generated by algebra of sets)
is explicitly cited in the proof of:
Lemma 724 ( $\sigma$-algebra generated by ring of sets).
Lemma 722 ( $\sigma$-algebra contains ring of sets)
is explicitly cited in the proof of:
Lemma 725 ( $\sigma$-algebra contains $\pi$-system).
Lemma 723 (upgrade ring of sets to $\sigma$-algebra)
is not yet used.
Lemma 724 ( $\sigma$-algebra generated by ring of sets)
is explicitly cited in the proof of: Lemma 727 ( $\sigma$-algebra generated by $\pi$-system).

Lemma 725 ( $\sigma$-algebra contains $\pi$-system)
is not yet used.
Lemma 726 (upgrade $\pi$-system to $\sigma$-algebra)
is not yet used.

## Lemma 727 ( $\sigma$-algebra generated by $\pi$-system)

is not yet used.
Lemma 728 (disjoint union of union in $\sigma$-algebra)
is not yet used.
Lemma 730 (algebra of sets generated by product of $\sigma$-algebras)
is explicitly cited in the proof of:
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Theorem 732 (Dynkin $\pi-\lambda$ theorem)

is explicitly cited in the proof of:
Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem).

## Lemma 734 (usage of Dynkin $\pi-\lambda$ theorem)

is explicitly cited in the proof of:
Lemma 992 (uniqueness of measures extended from a $\pi$-system).

## Theorem 736 (monotone class)

is explicitly cited in the proof of:
Lemma 738 (usage of monotone class theorem).
Lemma 738 (usage of monotone class theorem)
is explicitly cited in the proof of:
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).
Definition 739 (premeasurable space)
is explicitly cited in the proof of:
Lemma 740 (properties of premeasurable space),
Lemma 742 (properties of measurable space),
Lemma 897 (premeasure).

## Lemma 740 (properties of premeasurable space)

is explicitly cited in the proof of:
Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle),
Lemma 901 (premeasure is subtractive),
Lemma 902 (premeasure is monotone),
Lemma 904 ( $\sigma$-finite premeasure),
Lemma 905 (premeasure over countable pseudopartition),
Lemma 906 (premeasure satisfies the Boole inequality),
Lemma 907 (compatibility of null premeasure with countable union),
Lemma 908 (premeasure is continuous from below),
Lemma 909 (equivalent definition of premeasure (from below)),
Lemma 910 (premeasure is continuous from above),
Lemma 911 (equivalent definition of premeasure (from above)),
Lemma 912 (properties of premeasure space).
Definition 741 (measurable space)
is explicitly cited in the proof of:
Lemma 742 (properties of measurable space),
Lemma 750 (inverse $\sigma$-algebra),
Lemma 751 (image $\sigma$-algebra),
Lemma 753 (countable range is measurable),
Lemma 754 (constant function is measurable),
Lemma 763 (trace $\sigma$-algebra),
Lemma 786 (measurability of section),
Lemma 787 (countable union of sections is measurable),
Lemma 811 (measurability of indicator function),
Lemma 833 (measurability and masking),
Lemma 834 (measurability of restriction),
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part),
Lemma 915 (measure space is premeasure space),
Lemma 935 (counting measure),
Lemma 937 ( $\sigma$-finite counting measure),
Lemma 949 (empty set is negligible),

Lemma 955 (equivalent definition of property almost satisfied),
Lemma 1008 (masking almost nowhere), Lemma 1102 ( $\mathcal{I F}_{\mathbb{K}}$ is $\sigma$-additive), Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$).

Lemma 742 (properties of measurable space)
is explicitly cited in the proof of:
Lemma 903 (finite premeasure is bounded),
Lemma 929 (properties of measure space).

## Definition 743 (Borel $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 744 (some Borel subsets),
Lemma 745 (countable union Borel $\sigma$-algebra generator),
Lemma 761 (continuous is measurable),
Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 780 (Borel subsets in product measurable subspace),
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ),
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma 820 (measurability of numeric function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication).

## Lemma 744 (some Borel subsets)

is explicitly cited in the proof of:
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable),
Lemma 794 (some Borel subsets of numbers),
Lemma 1278 (integral over interval).

## Lemma 745 (countable union Borel $\sigma$-algebra generator)

 is explicitly cited in the proof of: Lemma 796 (Borel subsets of $\mathbb{R}$ ).Definition 748 (measurability and structure)
is explicitly cited in the proof of: Lemma 811 (measurability of indicator function), Lemma 853 (measurability of tensor product of numeric functions), Lemma 1100 (indicator and support are inverse), Lemma $1102\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is $\sigma$-additive $)$, Lemma $1103\left(\mathcal{I} \mathcal{F}_{\mathbb{K}}\right.$ is closed under multiplication), Lemma $1105\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under extension by zero), Lemma $1106\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under restriction), Lemma $1107\left(\mathcal{I F}_{\mathfrak{K}}\right.$, masking is $\sigma$-additive), Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable).

Definition 749 (measurable function)
is explicitly cited in the proof of:
Lemma 750 (inverse $\sigma$-algebra),
Lemma 751 (image $\sigma$-algebra),
Lemma 752 (identity function is measurable),
Lemma 753 (countable range is measurable),
Lemma 756 (equivalent definition of measurable function),
Lemma 757 (compatibility of measurability with composition),
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable),
Lemma 761 (continuous is measurable),
Lemma 763 (trace $\sigma$-algebra),

Lemma 770 (source restriction of measurable function),
Lemma 771 (destination restriction of measurable function),
Lemma 772 (measurability of function defined on a pseudopartition),
Lemma 776 (measurability of function to product space),
Lemma 777 (canonical projection is measurable),
Lemma 779 (generating product measurable space),
Lemma 790 (measurability of function from product space),
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 820 (measurability of numeric function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 834 (measurability of restriction).

## Lemma 750 (inverse $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 755 (inverse image of generating family).

## Lemma 751 (image $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 755 (inverse image of generating family).

## Lemma 752 (identity function is measurable)

is not yet used.

## Lemma 753 (countable range is measurable)

is explicitly cited in the proof of:
Lemma 754 (constant function is measurable),
Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).

## Lemma 754 (constant function is measurable)

is explicitly cited in the proof of:
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma 1023 (essential supremum of constant function),
Lemma 1040 (nonnegative essential supremum of constant function),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 755 (inverse image of generating family)

is explicitly cited in the proof of:
Lemma 756 (equivalent definition of measurable function),
Lemma 765 (generating measurable subspace),
Lemma 779 (generating product measurable space).

## Lemma 756 (equivalent definition of measurable function)

is explicitly cited in the proof of:
Lemma 761 (continuous is measurable),
Lemma 776 (measurability of function to product space),
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma 820 (measurability of numeric function).

## Lemma 757 (compatibility of measurability with composition)

is explicitly cited in the proof of:
Lemma 770 (source restriction of measurable function),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 853 (measurability of tensor product of numeric functions),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality).
Definition $758\left(\mathcal{M}_{Y}\right.$, set of measurable functions to $\left.Y\right)$
is explicitly cited in the proof of:
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 853 (measurability of tensor product of numeric functions).
Lemma $759\left(\mathcal{M}_{Y}\right.$ inverse image of countable is measurable)
is explicitly cited in the proof of:
Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range),
Lemma 808 (inverse image of countable is measurable).
Lemma $760\left(\mathcal{M}_{Y}\right.$ countable range)
is explicitly cited in the proof of:
Lemma 810 ( $\mathcal{M}_{\mathbb{K}}$ countable range).

## Lemma 761 (continuous is measurable)

is explicitly cited in the proof of:
Lemma 813 ( $\mathcal{M}_{\mathfrak{K}}$ is algebra),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).

## Lemma 763 (trace $\sigma$-algebra)

 is explicitly cited in the proof of: Lemma 764 (measurability of measurable subspace), Lemma 770 (source restriction of measurable function), Lemma 930 (trace measure).Lemma 764 (measurability of measurable subspace) is explicitly cited in the proof of: Lemma 765 (generating measurable subspace), Lemma 930 (trace measure), Lemma 932 (trace measure is restricted measure), Lemma $1105\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under extension by zero), Lemma $1106\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under restriction), Lemma $1120\left(\mathcal{S F}_{E}\right.$ is closed under restriction), Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$.

Lemma 765 (generating measurable subspace)
is explicitly cited in the proof of:
Lemma 766 (generating on a partition),
Lemma 768 (Borel sub- $\sigma$-algebra),
Lemma 780 (Borel subsets in product measurable subspace),
Lemma 803 (Borel subsets of $\mathbb{R}_{+}$),
Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$).

## Lemma 766 (generating on a partition)

is explicitly cited in the proof of:
Lemma 769 (Borel subsets of partition),
Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ ).

## Lemma 768 (Borel sub- $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ),
Lemma 803 (Borel subsets of $\mathbb{R}_{+}$),
Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$).

## Lemma 769 (Borel subsets of partition)

is explicitly cited in the proof of:
Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ ).

## Lemma 770 (source restriction of measurable function)

is not yet used.

## Lemma 771 (destination restriction of measurable function)

is not yet used.

## Lemma 772 (measurability of function defined on a pseudopartition)

is explicitly cited in the proof of:
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication).

## Definition 774 (tensor product of $\sigma$-algebras)

is explicitly cited in the proof of:
Lemma 775 (product of measurable subsets is measurable),
Lemma 776 (measurability of function to product space),
Lemma 779 (generating product measurable space),
Lemma 786 (measurability of section),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Lemma 775 (product of measurable subsets is measurable)

is explicitly cited in the proof of:
Lemma 776 (measurability of function to product space),
Lemma 1219 (measure of section of product),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1229 (tensor product of $\sigma$-finite measures),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1233 (uniqueness of tensor product measure).

## Lemma 776 (measurability of function to product space)

is explicitly cited in the proof of:
Lemma 777 (canonical projection is measurable),
Lemma 778 (permutation is measurable),
Lemma 813 ( $\mathcal{M}_{\mathbb{K}}$ is algebra).

## Lemma 777 (canonical projection is measurable)

is explicitly cited in the proof of:
Lemma 778 (permutation is measurable),
Lemma 779 (generating product measurable space),
Lemma 853 (measurability of tensor product of numeric functions).
Lemma 778 (permutation is measurable)
is not yet used.

## Lemma 779 (generating product measurable space)

is explicitly cited in the proof of:
Lemma 780 (Borel subsets in product measurable subspace).
Lemma 780 (Borel subsets in product measurable subspace)
is explicitly cited in the proof of:
Lemma 806 (Borel subsets of $\mathbb{C}$ ),
Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra).

## Definition 783 (section in Cartesian product)

is explicitly cited in the proof of:
Lemma 784 (section of product),
Lemma 785 (compatibility of section with set operations),
Lemma 789 (indicator of section).

## Lemma 784 (section of product)

is explicitly cited in the proof of:
Lemma 786 (measurability of section),
Lemma 1219 (measure of section of product).

## Lemma 785 (compatibility of section with set operations)

is explicitly cited in the proof of:
Lemma 786 (measurability of section),
Lemma 787 (countable union of sections is measurable),
Lemma 788 (countable intersection of sections is measurable),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure).

## Lemma 786 (measurability of section)

is explicitly cited in the proof of:
Lemma 787 (countable union of sections is measurable),
Lemma 788 (countable intersection of sections is measurable),
Lemma 790 (measurability of function from product space),
Lemma 1218 (measure of section),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1223 (measurability of measure of section),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1234 (negligibility of measurable section),
Theorem 1242 (Tonelli).
Lemma 787 (countable union of sections is measurable)
is explicitly cited in the proof of:
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure).
Lemma 788 (countable intersection of sections is measurable)
is explicitly cited in the proof of:
Lemma 1221 (measurability of measure of section (finite)).

## Lemma 789 (indicator of section)

is explicitly cited in the proof of:
Theorem 1242 (Tonelli),
Lemma 1243 (Tonelli over subset).
Lemma 790 (measurability of function from product space) is not yet used.

## Lemma 794 (some Borel subsets of numbers)

is explicitly cited in the proof of:
Lemma 796 (Borel subsets of $\mathbb{R}$ ),
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma 820 (measurability of numeric function).

## Lemma 796 (Borel subsets of $\mathbb{R}$ )

is explicitly cited in the proof of:
Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ ),
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ),
Lemma 803 (Borel subsets of $\mathbb{R}_{+}$),
Lemma 806 (Borel subsets of $\mathbb{C}$ ),
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable),
Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$,
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).

## Lemma 797 (countable generator of Borel subsets of $\mathbb{R}$ )

is not yet used.

## Lemma 798 (Borel subsets of $\overline{\mathbb{R}}$ and $\mathbb{R}$ )

is explicitly cited in the proof of:
Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ ),
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ),
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable).
Lemma 800 (generators of Borel subsets of $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ ).
Lemma 802 (Borel subsets of $\overline{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$),
Lemma 820 (measurability of numeric function).
Lemma 803 (Borel subsets of $\mathbb{R}_{+}$)
is not yet used.
Lemma 804 (Borel subsets of $\overline{\mathbb{R}}_{+}$)
is not yet used.
Lemma 806 (Borel subsets of $\mathbb{C}$ )
is not yet used.

## Lemma 808 (inverse image of countable is measurable)

is explicitly cited in the proof of:
Lemma 809 (inverse image is measurable),
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 996 (almost finite).
Lemma 809 (inverse image is measurable)
is explicitly cited in the proof of:
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma 999 (measurability of summability domain),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1050 (nonnegative essential supremum is almost definite).

Lemma $810\left(\mathcal{M}_{\mathbb{K}}\right.$ countable range)
is explicitly cited in the proof of:
Lemma 811 (measurability of indicator function),
Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable).
Lemma 811 (measurability of indicator function)
is explicitly cited in the proof of:
Lemma 833 (measurability and masking),
Lemma 1002 (almost sum),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1101 ( $\mathcal{I} \mathcal{F}_{\mathfrak{k}}$ is measurable),
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1289 ( $N_{p}$ of indicator function).
Lemma $813\left(\mathcal{M}_{\mathbb{K}}\right.$ is algebra)
is explicitly cited in the proof of:
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space $)$,
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 853 (measurability of tensor product of numeric functions),
Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ).
Lemma $814\left(\mathcal{M}_{\mathbb{K}}\right.$ is vector space $)$
is explicitly cited in the proof of:
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$,
Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear),
Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication),
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space).
Lemma $815\left(\mathcal{M}_{\mathbb{R}}\right.$, vector space of measurable numeric functions to $\left.\mathbb{R}\right)$ is explicitly cited in the proof of:
Lemma 816 (measurability of numeric function to $\mathbb{R}$ ),
Lemma $819\left(\mathcal{M}\right.$ and finite is $\left.\mathcal{M}_{\mathbb{R}}\right)$,
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma 1004 (almost sum is sum),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma 816 (measurability of numeric function to $\mathbb{R}$ )
is not yet used.
Definition 817 (M, set of measurable numeric functions)
is explicitly cited in the proof of:
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ ),
Lemma 820 (measurability of numeric function),
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 834 (measurability of restriction),
Lemma 841 ( $\mathcal{M}$ is closed under power function).
Lemma 819 ( $\mathcal{M}$ and finite is $\mathcal{M}_{\mathbb{R}}$ )
is explicitly cited in the proof of:
Lemma 821 ( $\mathcal{M}$ is closed under finite part),
Lemma 833 (measurability and masking),

Lemma 1016 (compatibility of almost limit with measurability),
Lemma $1101\left(\mathcal{I F}_{\mathbb{K}}\right.$ is measurable), Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable), Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ), Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ), Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$), Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ), Lemma 1328 (constant function is $\mathcal{L}^{p}$ ), Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ), Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ), Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ), Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 820 (measurability of numeric function)

is explicitly cited in the proof of:
Lemma $824(\mathcal{M}$ is closed under multiplication),
Lemma 828 ( $\mathcal{M}$ is closed under infimum),
Lemma 829 ( $\mathcal{M}$ is closed under supremum),
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part), Lemma 1017 (essential supremum),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality).
Lemma 821 ( $\mathcal{M}$ is closed under finite part)
is explicitly cited in the proof of:
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined),
Lemma 824 ( $\mathcal{M}$ is closed under multiplication),
Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part),
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ).
Lemma 822 ( $\mathcal{M}$ is closed under addition when defined)
is explicitly cited in the proof of:
Lemma 823 ( $\mathcal{M}$ is closed under finite sum when defined),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma 1002 (almost sum),
Lemma 1003 (compatibility of almost sum with almost equality),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1311 (Minkowski equality (linearly dependent functions)).
Lemma 823 ( $\mathcal{M}$ is closed under finite sum when defined)
is explicitly cited in the proof of:
Lemma 1024 (essential supremum of extended simple functions).
Lemma 824 ( $\mathcal{M}$ is closed under multiplication)
is explicitly cited in the proof of:
Lemma $825(\mathcal{M}$ is closed under finite product),
Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication),
Lemma 833 (measurability and masking),

Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$,
Lemma 1002 (almost sum),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Lemma 1304 (Hölder inequality),
Lemma 1305 (generalized Hölder inequality),
Lemma 1309 (Minkowski inequality in M).
Lemma 825 ( $\mathcal{M}$ is closed under finite product)
is explicitly cited in the proof of:
Lemma 853 (measurability of tensor product of numeric functions).

## Lemma 826 ( $\mathcal{M}$ is closed under scalar multiplication)

is explicitly cited in the proof of:
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Lemma 1304 (Hölder inequality),
Lemma 1311 (Minkowski equality (linearly dependent functions)).

## Lemma 828 ( $\mathcal{M}$ is closed under infimum)

is explicitly cited in the proof of:
Lemma $830(\mathcal{M}$ is closed under limit inferior),
Lemma 831 ( $\mathcal{M}$ is closed under limit superior),
Lemma $847\left(\mathcal{M}_{+}\right.$is closed under infimum),
Theorem 1208 (Fatou's lemma).
Lemma 829 ( $\mathcal{M}$ is closed under supremum)
is explicitly cited in the proof of:
Lemma $830(\mathcal{M}$ is closed under limit inferior $)$,
Lemma 831 ( $\mathcal{M}$ is closed under limit superior),
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum).
Lemma 830 ( $\mathcal{M}$ is closed under limit inferior $)$
is explicitly cited in the proof of:
Lemma $832(\mathcal{M}$ is closed under limit when pointwise convergent $)$, Theorem 1208 (Fatou's lemma).

Lemma 831 ( $\mathcal{M}$ is closed under limit superior) is explicitly cited in the proof of:
Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent).
Lemma 832 ( $\mathcal{M}$ is closed under limit when pointwise convergent) is explicitly cited in the proof of: Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent), Lemma 1016 (compatibility of almost limit with measurability), Theorem 1347 (Lebesgue, dominated convergence).

## Lemma 833 (measurability and masking)

is explicitly cited in the proof of:
Lemma $837\left(\mathcal{M}_{+}\right.$is closed under masking),
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part),
Lemma 1016 (compatibility of almost limit with measurability),

Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ),
Lemma 1312 (Minkowski equality (almost linearly dependent functions)),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 834 (measurability of restriction)

is not yet used.
Definition $835\left(\mathcal{M}_{+}\right.$, subset of nonnegative measurable numeric functions)
is explicitly cited in the proof of:
Lemma 836 (measurability of nonnegative and nonpositive parts),
Lemma $837\left(\mathcal{M}_{+}\right.$is closed under masking),
Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part $)$,
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable),
Lemma 841 ( $\mathcal{M}$ is closed under power function),
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition),
Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$,
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication $)$,
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent),
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1053 (nonnegative essential supremum is submultiplicative),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1101 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is measurable),
Lemma 1124 ( $\mathcal{S F}_{+}$is measurable),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1244 (Tonelli for tensor product),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).

## Lemma 836 (measurability of nonnegative and nonpositive parts)

is explicitly cited in the proof of:
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part),
Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts),
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost
nonnegative functions),
Lemma 1248 (integrable is measurable),
Lemma 1249 (equivalent definition of integrability),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1269 (integral in $\mathcal{M}$ is additive).
Lemma $837\left(\mathcal{M}_{+}\right.$is closed under masking) is not yet used.

## Lemma $838\left(\mathcal{M}_{+}\right.$is closed under finite part)

is explicitly cited in the proof of:
Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part).

## Lemma 839 ( $\mathcal{M}$ is closed under nonnegative finite part)

is explicitly cited in the proof of:
Lemma 1009 (finite nonnegative part).
Lemma 840 (power function in $\overline{\mathbb{R}}$ is measurable)
is explicitly cited in the proof of:
Lemma 841 ( $\mathcal{M}$ is closed under power function).
Lemma 841 ( $\mathcal{M}$ is closed under power function)
is explicitly cited in the proof of:
Lemma 842 ( $\mathcal{M}$ is closed under absolute value),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1286 ( $p$-seminorm),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Lemma 1294 (compatibility of $N_{p}$ with power function),
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1304 (Hölder inequality),
Lemma 1305 (generalized Hölder inequality),
Lemma 1309 (Minkowski inequality in M),
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).

## Lemma 842 ( $\mathcal{M}$ is closed under absolute value)

is explicitly cited in the proof of:
Lemma 1031 (essential supremum of absolute value is nonnegative),
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1046 (nonnegative essential supremum bounded by constant),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts),
Lemma 1249 (equivalent definition of integrability),
Lemma 1252 (almost bounded by integrable is integrable),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1304 (Hölder inequality),
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma $843\left(\mathcal{M}_{+}\right.$is closed under addition)
is explicitly cited in the proof of:
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum $)$,
Lemma 1051 (nonnegative essential supremum is subadditive),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive),
Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts),
Theorem 1242 (Tonelli),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1304 (Hölder inequality),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).

## Lemma $844\left(\mathcal{M}_{+}\right.$is closed under multiplication $)$

is explicitly cited in the proof of:
Lemma 1053 (nonnegative essential supremum is submultiplicative),
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1305 (generalized Hölder inequality).
Lemma $845\left(\mathcal{M}_{+}\right.$is closed under nonnegative scalar multiplication) is explicitly cited in the proof of:
Lemma 1043 (nonnegative essential supremum of infinite times function), Lemma 1044 (nonnegative essential supremum is positive homogeneous),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality),
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1221 (measurability of measure of section (finite)),
Theorem 1242 (Tonelli),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Lemma 1304 (Hölder inequality).
Lemma $847\left(\mathcal{M}_{+}\right.$is closed under infimum)
is explicitly cited in the proof of:
Lemma 1221 (measurability of measure of section (finite)).
Lemma $848\left(\mathcal{M}_{+}\right.$is closed under supremum)
is explicitly cited in the proof of:
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero $)$,
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$,
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1223 (measurability of measure of section).
Lemma $849\left(\mathcal{M}_{+}\right.$is closed under limit when pointwise convergent)
is explicitly cited in the proof of:
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum),
Theorem 1180 (Beppo Levi, monotone convergence),
Theorem 1208 (Fatou's lemma),
Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence),
Theorem 1242 (Tonelli).
Lemma $850\left(\mathcal{M}_{+}\right.$is closed under countable sum)
is explicitly cited in the proof of:
Lemma $1139\left(\mathcal{M}_{+}\right.$, masking is $\sigma$-additive $)$,
Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive),
Lemma 1221 (measurability of measure of section (finite)).
Definition 851 (tensor product of numeric functions)
is explicitly cited in the proof of:
Lemma 853 (measurability of tensor product of numeric functions),
Lemma 1244 (Tonelli for tensor product).
Lemma 853 (measurability of tensor product of numeric functions)
is explicitly cited in the proof of:
Lemma 1244 (Tonelli for tensor product).

## Definition 855 (homogeneity)

is not yet used.

## Definition 856 (diffuseness)

is explicitly cited in the proof of:
Lemma 1095 (Lebesgue measure is diffuse),
Lemma 1239 (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse),
Lemma 1278 (integral over interval).

## Definition 857 (finiteness)

is explicitly cited in the proof of:
Lemma 858 (finite is bounded),
Lemma 887 (finite is $\sigma$-finite),
Lemma 936 (finiteness of counting measure),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1223 (measurability of measure of section),
Lemma 1228 (tensor product of finite measures),
Lemma 1233 (uniqueness of tensor product measure).

## Lemma 858 (finite is bounded)

is explicitly cited in the proof of:
Lemma 895 (finite content is bounded),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Definition 859 (subadditivity)

is explicitly cited in the proof of:
Lemma 869 (subadditivity),
Lemma 871 ( $\sigma$-subadditive is subadditive),
Lemma 886 (equivalent definition of $\sigma$-finiteness).
Definition 861 (superadditivity)
is explicitly cited in the proof of:
Lemma 873 ( $\sigma$-superadditive is superadditive).

## Definition 862 (additivity)

is explicitly cited in the proof of:
Lemma 863 (decomposition over finite pseudopartition),
Lemma 864 (binary inclusion-exclusion principle),
Lemma 866 (subtractivity),
Lemma 867 (monotonicity),
Lemma 869 (subadditivity),
Lemma 875 ( $\sigma$-additive is additive),
Lemma 880 (continuity from below),
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 884 ( $\sigma$-additivity (from above)),
Lemma 916 (measure is additive),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Lemma 863 (decomposition over finite pseudopartition)

is explicitly cited in the proof of:
Lemma 890 (content over finite pseudopartition).
Lemma 864 (binary inclusion-exclusion principle)
is explicitly cited in the proof of:

Lemma 891 (content satisfies the binary inclusion-exclusion principle),
Lemma 917 (measure satisfies the binary inclusion-exclusion principle).

## Definition 865 (subtractivity)

is explicitly cited in the proof of:
Lemma 866 (subtractivity),
Lemma 883 (continuity from above),
Lemma 892 (content is subtractive),
Lemma 918 (measure is subtractive).

## Lemma 866 (subtractivity)

is explicitly cited in the proof of:
Lemma 883 (continuity from above),
Lemma 892 (content is subtractive).

## Lemma 867 (monotonicity)

is explicitly cited in the proof of:
Lemma 869 (subadditivity),
Lemma 878 ( $\sigma$-subadditivity),
Lemma 880 (continuity from below),
Lemma 883 (continuity from above),
Lemma 893 (content is monotone).

## Lemma 869 (subadditivity)

is explicitly cited in the proof of:
Lemma 894 (content satisfies the finite Boole inequality).
Definition 870 ( $\sigma$-subadditivity)
is explicitly cited in the proof of:
Lemma 871 ( $\sigma$-subadditive is subadditive),
Lemma 878 ( $\sigma$-subadditivity),
Lemma 923 (measure satisfies the Boole inequality).

## Lemma 871 ( $\sigma$-subadditive is subadditive)

is explicitly cited in the proof of:
Lemma 906 (premeasure satisfies the Boole inequality).
Definition 872 ( $\sigma$-superadditivity)
is explicitly cited in the proof of:
Lemma 873 ( $\sigma$-superadditive is superadditive).

## Lemma 873 ( $\sigma$-superadditive is superadditive)

is not yet used.
Definition 874 ( $\sigma$-additivity)
is explicitly cited in the proof of:
Lemma 875 ( $\sigma$-additive is additive),
Lemma 876 (decomposition over countable pseudopartition),
Lemma 878 ( $\sigma$-subadditivity),
Lemma 880 (continuity from below),
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 884 ( $\sigma$-additivity (from above)),
Lemma 913 (measure),
Lemma 935 (counting measure),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ),
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive),

Lemma 1157 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(disjoint)),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure).
Lemma 875 ( $\sigma$-additive is additive) is explicitly cited in the proof of: Lemma 878 ( $\sigma$-subadditivity), Lemma 880 (continuity from below), Lemma 881 ( $\sigma$-additivity (from below)), Lemma 883 (continuity from above), Lemma 884 ( $\sigma$-additivity (from above)), Lemma 897 (premeasure).

## Lemma 876 (decomposition over countable pseudopartition)

 is explicitly cited in the proof of: Lemma 905 (premeasure over countable pseudopartition).
## Lemma 878 ( $\sigma$-subadditivity)

is explicitly cited in the proof of:
Lemma 906 (premeasure satisfies the Boole inequality).
Definition 879 (continuity from below)
is explicitly cited in the proof of: Lemma 880 (continuity from below), Lemma 881 ( $\sigma$-additivity (from below)), Lemma 883 (continuity from above), Lemma 992 (uniqueness of measures extended from a $\pi$-system), Theorem 1180 (Beppo Levi, monotone convergence).

## Lemma 880 (continuity from below)

is explicitly cited in the proof of:
Lemma 881 ( $\sigma$-additivity (from below)),
Lemma 883 (continuity from above),
Lemma 908 (premeasure is continuous from below).

## Lemma 881 ( $\sigma$-additivity (from below))

 is explicitly cited in the proof of:Lemma 909 (equivalent definition of premeasure (from below)).
Definition 882 (continuity from above)
is explicitly cited in the proof of: Lemma 883 (continuity from above), Lemma 884 ( $\sigma$-additivity (from above)).

## Lemma 883 (continuity from above)

is explicitly cited in the proof of:
Lemma 884 ( $\sigma$-additivity (from above)),
Lemma 910 (premeasure is continuous from above).
Lemma 884 ( $\sigma$-additivity (from above))
is explicitly cited in the proof of:
Lemma 911 (equivalent definition of premeasure (from above)).
Definition 885 ( $\sigma$-finiteness)
is explicitly cited in the proof of:
Lemma 886 (equivalent definition of $\sigma$-finiteness),
Lemma 887 (finite is $\sigma$-finite),

Lemma 937 ( $\sigma$-finite counting measure),
Lemma 1094 (Lebesgue measure is $\sigma$-finite),
Lemma 1229 (tensor product of $\sigma$-finite measures).
Lemma 886 (equivalent definition of $\sigma$-finiteness)
is explicitly cited in the proof of:
Lemma 896 ( $\sigma$-finite content),
Lemma 937 ( $\sigma$-finite counting measure),
Lemma 1223 (measurability of measure of section),
Lemma 1229 (tensor product of $\sigma$-finite measures),
Lemma 1233 (uniqueness of tensor product measure).
Lemma 887 (finite is $\sigma$-finite)
is explicitly cited in the proof of:
Lemma 1231 (uniqueness of tensor product measure (finite)).
Definition 888 (content)
is explicitly cited in the proof of:
Lemma 890 (content over finite pseudopartition),
Lemma 891 (content satisfies the binary inclusion-exclusion principle),
Lemma 892 (content is subtractive),
Lemma 893 (content is monotone),
Lemma 894 (content satisfies the finite Boole inequality),
Lemma 895 (finite content is bounded),
Lemma 897 (premeasure),
Lemma 899 (premeasure is additive).
Lemma 890 (content over finite pseudopartition)
is not yet used.

## Lemma 891 (content satisfies the binary inclusion-exclusion principle)

is explicitly cited in the proof of:
Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle).

## Lemma 892 (content is subtractive)

is explicitly cited in the proof of:
Lemma 901 (premeasure is subtractive).
Lemma 893 (content is monotone) is explicitly cited in the proof of:
Lemma 895 (finite content is bounded),
Lemma 902 (premeasure is monotone).

## Lemma 894 (content satisfies the finite Boole inequality)

is explicitly cited in the proof of:
Lemma 896 ( $\sigma$-finite content).
Lemma 895 (finite content is bounded)
is explicitly cited in the proof of:
Lemma 903 (finite premeasure is bounded).
Lemma 896 ( $\sigma$-finite content)
is explicitly cited in the proof of:
Lemma 904 ( $\sigma$-finite premeasure).

## Lemma 897 (premeasure)

is explicitly cited in the proof of: Lemma 899 (premeasure is additive),

Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle),
Lemma 901 (premeasure is subtractive),
Lemma 902 (premeasure is monotone),
Lemma 903 (finite premeasure is bounded),
Lemma 904 ( $\sigma$-finite premeasure),
Lemma 905 (premeasure over countable pseudopartition),
Lemma 906 (premeasure satisfies the Boole inequality),
Lemma 907 (compatibility of null premeasure with countable union),
Lemma 908 (premeasure is continuous from below),
Lemma 909 (equivalent definition of premeasure (from below)),
Lemma 910 (premeasure is continuous from above),
Lemma 911 (equivalent definition of premeasure (from above)),
Lemma 912 (properties of premeasure space),
Lemma 913 (measure),
Lemma 915 (measure space is premeasure space).

## Lemma 899 (premeasure is additive)

is explicitly cited in the proof of:
Lemma 912 (properties of premeasure space),
Lemma 916 (measure is additive).

## Lemma 900 (premeasure satisfies the binary inclusion-exclusion principle)

 is explicitly cited in the proof of: Lemma 912 (properties of premeasure space), Lemma 917 (measure satisfies the binary inclusion-exclusion principle).
## Lemma 901 (premeasure is subtractive)

 is explicitly cited in the proof of: Lemma 912 (properties of premeasure space), Lemma 918 (measure is subtractive).Lemma 902 (premeasure is monotone) is explicitly cited in the proof of: Lemma 912 (properties of premeasure space), Lemma 919 (measure is monotone).

## Lemma 903 (finite premeasure is bounded)

 is explicitly cited in the proof of: Lemma 920 (finite measure is bounded).
## Lemma 904 ( $\sigma$-finite premeasure)

 is explicitly cited in the proof of: Lemma 912 (properties of premeasure space), Lemma 921 ( $\sigma$-finite measure).
## Lemma 905 (premeasure over countable pseudopartition)

is explicitly cited in the proof of: Lemma 912 (properties of premeasure space), Lemma 922 (measure over countable pseudopartition)

Lemma 906 (premeasure satisfies the Boole inequality) is explicitly cited in the proof of: Lemma 907 (compatibility of null premeasure with countable union), Lemma 912 (properties of premeasure space), Lemma 923 (measure satisfies the Boole inequality).

## Lemma 907 (compatibility of null premeasure with countable union)

is explicitly cited in the proof of:
Lemma 924 (compatibility of null measure with countable union).

## Lemma 908 (premeasure is continuous from below)

is explicitly cited in the proof of:
Lemma 912 (properties of premeasure space),
Lemma 925 (measure is continuous from below).
Lemma 909 (equivalent definition of premeasure (from below))
is explicitly cited in the proof of:
Lemma 926 (equivalent definition of measure (from below)).

## Lemma 910 (premeasure is continuous from above)

is explicitly cited in the proof of:
Lemma 912 (properties of premeasure space),
Lemma 927 (measure is continuous from above).

## Lemma 911 (equivalent definition of premeasure (from above))

 is explicitly cited in the proof of:Lemma 928 (equivalent definition of measure (from above)).

## Lemma 912 (properties of premeasure space)

is explicitly cited in the proof of:
Lemma 929 (properties of measure space).

## Lemma 913 (measure)

is explicitly cited in the proof of:
Lemma 915 (measure space is premeasure space),
Lemma 929 (properties of measure space),
Lemma 930 (trace measure),
Lemma 931 (restricted measure),
Lemma 933 (trivial measure),
Lemma 934 (equivalent definition of trivial measure),
Lemma 935 (counting measure),
Lemma 937 ( $\sigma$-finite counting measure),
Lemma 946 (equivalent definition of considerable subset),
Lemma 948 (negligibility of measurable subset),
Lemma 949 (empty set is negligible),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 1008 (masking almost nowhere),
Lemma 1017 (essential supremum),
Lemma 1020 (essential supremum is bounded by supremum),
Lemma 1023 (essential supremum of constant function),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1086 ( $\lambda^{\star}$ is measure on $\mathcal{L}$ ),
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive),
Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset),
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)),
Lemma 1158 (integral in $\mathcal{S \mathcal { F }}+{ }_{+}$of zero is zero),
Lemma 1166 (density measure in $\mathcal{S F}{ }_{+}$),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1218 (measure of section),

Lemma 1219 (measure of section of product),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1228 (tensor product of finite measures),
Lemma 1329 (first mean value inequality),
Lemma 1331 (first mean value theorem).
Lemma 915 (measure space is premeasure space)
is explicitly cited in the proof of:
Lemma 916 (measure is additive),
Lemma 917 (measure satisfies the binary inclusion-exclusion principle),
Lemma 918 (measure is subtractive),
Lemma 919 (measure is monotone),
Lemma 920 (finite measure is bounded),
Lemma 921 ( $\sigma$-finite measure),
Lemma 922 (measure over countable pseudopartition),
Lemma 923 (measure satisfies the Boole inequality),
Lemma 924 (compatibility of null measure with countable union),
Lemma 925 (measure is continuous from below),
Lemma 926 (equivalent definition of measure (from below)),
Lemma 927 (measure is continuous from above),
Lemma 928 (equivalent definition of measure (from above)).

## Lemma 916 (measure is additive)

is explicitly cited in the proof of:
Lemma 1024 (essential supremum of extended simple functions).
Lemma 917 (measure satisfies the binary inclusion-exclusion principle) is not yet used.

## Lemma 918 (measure is subtractive)

is not yet used.

## Lemma 919 (measure is monotone)

is explicitly cited in the proof of:
Lemma 934 (equivalent definition of trivial measure),
Lemma 948 (negligibility of measurable subset),
Lemma 992 (uniqueness of measures extended from a $\pi$-system),
Lemma 1017 (essential supremum),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1147 (integral in $\mathcal{I F}$ is monotone),
Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset).
Lemma 920 (finite measure is bounded)
is not yet used.

## Lemma 921 ( $\sigma$-finite measure)

is not yet used.

## Lemma 922 (measure over countable pseudopartition)

is explicitly cited in the proof of:
Lemma 1159 (integral in $\mathcal{S F} \mathcal{F}_{+}$is positive linear).
Lemma 923 (measure satisfies the Boole inequality) is not yet used.

Lemma 924 (compatibility of null measure with countable union)
is explicitly cited in the proof of:
Lemma 952 ( $\mathbf{N}$ is closed under countable union),
Lemma 1021 (essential supremum is minimum).

## Lemma 925 (measure is continuous from below)

is explicitly cited in the proof of:
Lemma 992 (uniqueness of measures extended from $a \pi$-system), Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1223 (measurability of measure of section),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1233 (uniqueness of tensor product measure).

## Lemma 926 (equivalent definition of measure (from below))

is explicitly cited in the proof of:
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Lemma 927 (measure is continuous from above)

is explicitly cited in the proof of:
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1231 (uniqueness of tensor product measure (finite)).

## Lemma 928 (equivalent definition of measure (from above))

is not yet used.

## Lemma 929 (properties of measure space)

is not yet used.

## Lemma 930 (trace measure)

is explicitly cited in the proof of:
Lemma 932 (trace measure is restricted measure).

## Lemma 931 (restricted measure)

is explicitly cited in the proof of:
Lemma 932 (trace measure is restricted measure),
Lemma 1223 (measurability of measure of section),
Lemma 1233 (uniqueness of tensor product measure).

## Lemma 932 (trace measure is restricted measure)

is explicitly cited in the proof of:
Lemma 1148 (integral in $\mathcal{I F}$ over subset).

## Lemma 933 (trivial measure)

is explicitly cited in the proof of:
Lemma 934 (equivalent definition of trivial measure),
Lemma 947 (negligibility with trivial measure),
Lemma 1019 (essential supremum with trivial measure),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1043 (nonnegative essential supremum of infinite times function),
Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure),
Lemma 1289 ( $N_{p}$ of indicator function).

## Lemma 934 (equivalent definition of trivial measure)

is explicitly cited in the proof of:
Lemma 1023 (essential supremum of constant function),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1036 (equivalent definition of nonnegative essential supremum),
Lemma 1043 (nonnegative essential supremum of infinite times function).

## Lemma 935 (counting measure)

is explicitly cited in the proof of:
Lemma 936 (finiteness of counting measure),
Lemma 937 ( $\sigma$-finite counting measure),
Lemma 940 (equivalent definition of Dirac measure),
Lemma 941 (Dirac measure is finite),
Lemma 1152 (integral in $\mathcal{I F}$ with counting measure).
Lemma 936 (finiteness of counting measure)
is explicitly cited in the proof of:
Lemma 941 (Dirac measure is finite).
Lemma 937 ( $\sigma$-finite counting measure)
is not yet used.
Definition 939 (Dirac measure)
is explicitly cited in the proof of:
Lemma 940 (equivalent definition of Dirac measure),
Lemma 941 (Dirac measure is finite),
Lemma 1214 (integral in $\mathcal{M}_{+}$with Dirac measure),
Lemma 1284 (integral with Dirac measure).
Lemma 940 (equivalent definition of Dirac measure) is not yet used.

## Lemma 941 (Dirac measure is finite)

is not yet used.
Definition 942 (negligible subset)
is explicitly cited in the proof of:
Lemma 946 (equivalent definition of considerable subset),
Lemma 947 (negligibility with trivial measure),
Lemma 948 (negligibility of measurable subset),
Lemma 951 (subset of negligible is negligible),
Lemma 952 ( $\mathbf{N}$ is closed under countable union),
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 1008 (masking almost nowhere).
Definition 943 (complete measure)
is not yet used.
Definition 945 (considerable subset)
is explicitly cited in the proof of:
Lemma 946 (equivalent definition of considerable subset).
Lemma 946 (equivalent definition of considerable subset)
is not yet used.
Lemma 947 (negligibility with trivial measure)
is explicitly cited in the proof of:
Lemma 956 (property almost satisfied with trivial measure).

## Lemma 948 (negligibility of measurable subset)

is explicitly cited in the proof of:
Lemma 949 (empty set is negligible),
Lemma 957 (measurable property almost satisfied),
Lemma 1022 (essential supremum is almost upper bound),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1234 (negligibility of measurable section),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 949 (empty set is negligible)

is explicitly cited in the proof of:
Lemma 958 (everywhere is almost everywhere).

## Lemma 951 (subset of negligible is negligible)

is explicitly cited in the proof of:
Lemma 959 (almost modus ponens),
Lemma 960 (countable almost properties is almost property),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 952 ( N is closed under countable union)

is explicitly cited in the proof of:
Lemma 960 (countable almost properties is almost property),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions).

## Definition 953 (property almost satisfied)

is explicitly cited in the proof of:
Lemma 955 (equivalent definition of property almost satisfied),
Lemma 956 (property almost satisfied with trivial measure),
Lemma 957 (measurable property almost satisfied),
Lemma 958 (everywhere is almost everywhere),
Lemma 959 (almost modus ponens),
Lemma 960 (countable almost properties is almost property),
Lemma 970 (extension to almost defined),
Lemma 1000 (negligibility and summability domain),
Lemma 1002 (almost sum),
Lemma 1008 (masking almost nowhere),
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1050 (nonnegative essential supremum is almost definite),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Theorem 1349 (Lebesgue, extended dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 955 (equivalent definition of property almost satisfied)

is explicitly cited in the proof of:
Lemma 960 (countable almost properties is almost property),

Lemma 963 (extended almost modus ponens),
Lemma 1009 (finite nonnegative part), Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$), Lemma 1312 (Minkowski equality (almost linearly dependent functions)).

## Lemma 956 (property almost satisfied with trivial measure)

is explicitly cited in the proof of:
Lemma 1039 (nonnegative essential supremum is almost upper bound),
Lemma 1046 (nonnegative essential supremum bounded by constant),
Lemma 1050 (nonnegative essential supremum is almost definite).

## Lemma 957 (measurable property almost satisfied)

is explicitly cited in the proof of:
Lemma 996 (almost finite).

## Lemma 958 (everywhere is almost everywhere)

 is explicitly cited in the proof of: Lemma 962 (almost everywhere for almost the same), Lemma 967 (extended almost modus ponens (simplified 2)), Lemma 994 (transitivity of almost inequality and equality), Lemma 1011 (compatibility of almost limit with continuous function), Lemma 1014 (compatibility of almost limit with almost equality), Lemma 1031 (essential supremum of absolute value is nonnegative), Lemma 1048 (essentially bounded is almost finite), Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions), Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ), Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ).
## Lemma 959 (almost modus ponens)

 is explicitly cited in the proof of: Lemma 962 (almost everywhere for almost the same), Lemma 970 (extension to almost defined), Lemma 972 (generalized almost modus ponens).
## Lemma 960 (countable almost properties is almost property)

 is explicitly cited in the proof of: Lemma 962 (almost everywhere for almost the same), Lemma 970 (extension to almost defined), Lemma 972 (generalized almost modus ponens), Lemma 977 (compatibility of almost binary relation with extended reflexivity), Lemma 979 (compatibility of almost binary relation with extended antisymmetry), Lemma 980 (compatibility of almost binary relation with transitivity), Lemma 984 (compatibility of almost binary relation with operator), Lemma 1001 (absolute value and almost inequality), Lemma 1009 (finite nonnegative part), Lemma 1011 (compatibility of almost limit with continuous function), Lemma 1013 (almost limits are almost equal), Lemma 1014 (compatibility of almost limit with almost equality), Lemma 1015 (compatibility of almost limit with almost inequality), Lemma 1016 (compatibility of almost limit with measurability), Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ), Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions), Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),Theorem 1349 (Lebesgue, extended dominated convergence), Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 962 (almost everywhere for almost the same)

is explicitly cited in the proof of:
Lemma 963 (extended almost modus ponens),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1014 (compatibility of almost limit with almost equality),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 963 (extended almost modus ponens)

is explicitly cited in the proof of:
Lemma 965 (extended almost modus ponens (simplified 0)),
Lemma 966 (extended almost modus ponens (simplified 1)).

## Lemma 965 (extended almost modus ponens (simplified 0))

is explicitly cited in the proof of:
Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions).

## Lemma 966 (extended almost modus ponens (simplified 1))

is explicitly cited in the proof of:
Lemma 967 (extended almost modus ponens (simplified 2)),
Lemma 1009 (finite nonnegative part).

## Lemma 967 (extended almost modus ponens (simplified 2))

is explicitly cited in the proof of:
Lemma 1013 (almost limits are almost equal),
Lemma 1015 (compatibility of almost limit with almost inequality).

## Definition 969 (almost definition)

is not yet used.

## Lemma 970 (extension to almost defined)

 is explicitly cited in the proof of: Lemma 972 (generalized almost modus ponens),Lemma 975 (compatibility of almost binary relation with reflexivity).

## Lemma 972 (generalized almost modus ponens)

is explicitly cited in the proof of:
Lemma 977 (compatibility of almost binary relation with extended reflexivity),
Lemma 978 (compatibility of almost binary relation with symmetry),
Lemma 979 (compatibility of almost binary relation with extended antisymmetry),
Lemma 980 (compatibility of almost binary relation with transitivity),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 988 (definiteness is almost definiteness),
Lemma 1001 (absolute value and almost inequality),
Lemma 1006 (absolute value is almost equal to almost nonnegative function),
Lemma 1010 (compatibility of almost limit with subsequence),
Lemma 1011 (compatibility of almost limit with continuous function).

## Definition 974 (almost binary relation)

is explicitly cited in the proof of:
Lemma 975 (compatibility of almost binary relation with reflexivity),
Lemma 977 (compatibility of almost binary relation with extended reflexivity),
Lemma 978 (compatibility of almost binary relation with symmetry),
Lemma 979 (compatibility of almost binary relation with extended antisymmetry),
Lemma 980 (compatibility of almost binary relation with transitivity),
Lemma 984 (compatibility of almost binary relation with operator),
Lemma 1022 (essential supremum is almost upper bound),
Lemma 1029 (essential supremum is almost monotone),

Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 975 (compatibility of almost binary relation with reflexivity)

 is explicitly cited in the proof of:Lemma 981 (almost equivalence is equivalence relation), Lemma 983 (almost order is order relation).

Lemma 977 (compatibility of almost binary relation with extended reflexivity) is explicitly cited in the proof of: Lemma 994 (transitivity of almost inequality and equality), Lemma 995 (compatibility of almost nonnegativeness with almost equality),
Lemma 1032 (compatibility of essential supremum with almost equality).

## Lemma 978 (compatibility of almost binary relation with symmetry)

is explicitly cited in the proof of:
Lemma 981 (almost equivalence is equivalence relation).
Lemma 979 (compatibility of almost binary relation with extended antisymmetry) is explicitly cited in the proof of:
Lemma 983 (almost order is order relation).
Lemma 980 (compatibility of almost binary relation with transitivity)
is explicitly cited in the proof of:
Lemma 981 (almost equivalence is equivalence relation),
Lemma 983 (almost order is order relation),
Lemma 993 (transitivity of almost nonnegativeness),
Lemma 994 (transitivity of almost inequality and equality).
Lemma 981 (almost equivalence is equivalence relation)
is explicitly cited in the proof of:
Lemma 982 (almost equality is equivalence relation).
Lemma 982 (almost equality is equivalence relation)
is explicitly cited in the proof of:
Lemma 1003 (compatibility of almost sum with almost equality).
Lemma 983 (almost order is order relation) is explicitly cited in the proof of:
Lemma 1001 (absolute value and almost inequality),
Lemma 1030 (essential supremum bounded by constant),
Lemma 1048 (essentially bounded is almost finite),
Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable).
Lemma 984 (compatibility of almost binary relation with operator) is explicitly cited in the proof of:
Lemma 985 (compatibility of almost equality with operator),
Lemma 1051 (nonnegative essential supremum is subadditive),
Lemma 1053 (nonnegative essential supremum is submultiplicative),
Lemma 1054 (nonnegative essential supremum is compatible with power function),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).

Lemma 985 (compatibility of almost equality with operator)
is explicitly cited in the proof of:
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1005 (absolute value is almost definite),
Lemma 1007 (power function is almost definite),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Lemma 1304 (Hölder inequality),
Lemma 1329 (first mean value inequality),
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ).

## Lemma 988 (definiteness is almost definiteness)

is explicitly cited in the proof of:
Lemma 1005 (absolute value is almost definite),
Lemma 1007 (power function is almost definite).
Lemma 992 (uniqueness of measures extended from a $\pi$-system)
is explicitly cited in the proof of:
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).
Lemma 993 (transitivity of almost nonnegativeness)
is explicitly cited in the proof of:
Lemma 995 (compatibility of almost nonnegativeness with almost equality).

## Lemma 994 (transitivity of almost inequality and equality)

is explicitly cited in the proof of:
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).
Lemma 995 (compatibility of almost nonnegativeness with almost equality)
is explicitly cited in the proof of:
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions).
Lemma 996 (almost finite)
is not yet used.
Definition 997 (summability domain)
is explicitly cited in the proof of:
Lemma 998 (summability on summability domain),
Lemma 1004 (almost sum is sum),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions).
Lemma 998 (summability on summability domain)
is explicitly cited in the proof of:
Lemma 1000 (negligibility and summability domain),
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1004 (almost sum is sum),
Lemma 1253 (integrability is closed under extension by zero).
Lemma 999 (measurability of summability domain)
is explicitly cited in the proof of:
Lemma 1002 (almost sum).

## Lemma 1000 (negligibility and summability domain)

is explicitly cited in the proof of:
Lemma 1001 (absolute value and almost inequality),
Lemma 1002 (almost sum),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions).

## Lemma 1001 (absolute value and almost inequality)

is explicitly cited in the proof of:
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 1002 (almost sum)

is explicitly cited in the proof of:
Lemma 1003 (compatibility of almost sum with almost equality),
Lemma 1004 (almost sum is sum),
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).

## Lemma 1003 (compatibility of almost sum with almost equality)

is explicitly cited in the proof of:
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ),
Lemma 1310 (Minkowski equality (nonnegative functions)),
Lemma 1312 (Minkowski equality (almost linearly dependent functions)).
Lemma 1004 (almost sum is sum)
is explicitly cited in the proof of:
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).
Lemma 1005 (absolute value is almost definite)
is explicitly cited in the proof of:
Lemma 1296 ( $N_{p}$ is almost definite).
Lemma 1006 (absolute value is almost equal to almost nonnegative function) is explicitly cited in the proof of:
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).
Lemma 1007 (power function is almost definite)
is explicitly cited in the proof of:
Lemma 1296 ( $N_{p}$ is almost definite).
Lemma 1008 (masking almost nowhere)
is explicitly cited in the proof of:
Lemma 1016 (compatibility of almost limit with measurability),
Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ),
Lemma 1312 (Minkowski equality (almost linearly dependent functions)),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 1009 (finite nonnegative part)

is explicitly cited in the proof of:
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 1010 (compatibility of almost limit with subsequence)
is explicitly cited in the proof of:
Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ).
Lemma 1011 (compatibility of almost limit with continuous function)
is explicitly cited in the proof of:
Lemma 1012 (compatibility of almost limit with power function), Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).

Lemma 1012 (compatibility of almost limit with power function)
is explicitly cited in the proof of:
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 1013 (almost limits are almost equal)

is explicitly cited in the proof of:
Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ).
Lemma 1014 (compatibility of almost limit with almost equality)
is explicitly cited in the proof of:
Lemma 1016 (compatibility of almost limit with measurability),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).
Lemma 1015 (compatibility of almost limit with almost inequality) is explicitly cited in the proof of:
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1016 (compatibility of almost limit with measurability)
is explicitly cited in the proof of:
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1017 (essential supremum)
is explicitly cited in the proof of:
Lemma 1019 (essential supremum with trivial measure),
Lemma 1020 (essential supremum is bounded by supremum),
Lemma 1021 (essential supremum is minimum),
Lemma 1022 (essential supremum is almost upper bound),
Lemma 1023 (essential supremum of constant function),
Lemma 1024 (essential supremum of extended simple functions),
Lemma 1026 (essential supremum of infinite times function),
Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1036 (equivalent definition of nonnegative essential supremum).
Lemma 1019 (essential supremum with trivial measure)
is explicitly cited in the proof of:
Lemma 1026 (essential supremum of infinite times function), Lemma 1030 (essential supremum bounded by constant).
Lemma 1020 (essential supremum is bounded by supremum) is not yet used.

## Lemma 1021 (essential supremum is minimum)

is explicitly cited in the proof of:
Lemma 1022 (essential supremum is almost upper bound),
Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1029 (essential supremum is almost monotone),
Lemma 1038 (nonnegative essential supremum is minimum).
Lemma 1022 (essential supremum is almost upper bound)
is explicitly cited in the proof of:
Lemma 1030 (essential supremum bounded by constant),
Lemma 1039 (nonnegative essential supremum is almost upper bound).

## Lemma 1023 (essential supremum of constant function)

is explicitly cited in the proof of:
Lemma 1027 (essential supremum is positive homogeneous),
Lemma 1030 (essential supremum bounded by constant),
Lemma 1040 (nonnegative essential supremum of constant function).
Lemma 1024 (essential supremum of extended simple functions)
is explicitly cited in the proof of:
Lemma 1026 (essential supremum of infinite times function),
Lemma 1041 (nonnegative essential supremum for extended simple functions).

## Lemma 1026 (essential supremum of infinite times function)

is explicitly cited in the proof of:
Lemma 1043 (nonnegative essential supremum of infinite times function).
Lemma 1027 (essential supremum is positive homogeneous)
is explicitly cited in the proof of:
Lemma 1044 (nonnegative essential supremum is positive homogeneous).
Lemma 1029 (essential supremum is almost monotone)
is explicitly cited in the proof of:
Lemma 1030 (essential supremum bounded by constant),
Lemma 1032 (compatibility of essential supremum with almost equality), Lemma 1045 (nonnegative essential supremum is almost monotone).

## Lemma 1030 (essential supremum bounded by constant)

 is explicitly cited in the proof of:Lemma 1031 (essential supremum of absolute value is nonnegative),
Lemma 1046 (nonnegative essential supremum bounded by constant).
Lemma 1031 (essential supremum of absolute value is nonnegative) is not yet used.

Lemma 1032 (compatibility of essential supremum with almost equality) is explicitly cited in the proof of: Lemma 1049 (compatibility of nonnegative essential supremum with almost equality).

## Definition 1034 (nonnegative essential supremum)

is explicitly cited in the proof of:
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1038 (nonnegative essential supremum is minimum),
Lemma 1039 (nonnegative essential supremum is almost upper bound), Lemma 1040 (nonnegative essential supremum of constant function),
Lemma 1041 (nonnegative essential supremum for extended simple functions), Lemma 1043 (nonnegative essential supremum of infinite times function),

Lemma 1044 (nonnegative essential supremum is positive homogeneous), Lemma 1045 (nonnegative essential supremum is almost monotone), Lemma 1046 (nonnegative essential supremum bounded by constant), Lemma 1049 (compatibility of nonnegative essential supremum with almost equality).

Lemma 1036 (equivalent definition of nonnegative essential supremum) is explicitly cited in the proof of:
Lemma 1037 (nonnegative essential supremum with trivial measure),
Lemma 1044 (nonnegative essential supremum is positive homogeneous), Lemma 1286 ( $p$-seminorm).

Lemma 1037 (nonnegative essential supremum with trivial measure) is explicitly cited in the proof of: Lemma 1038 (nonnegative essential supremum is minimum), Lemma 1043 (nonnegative essential supremum of infinite times function), Lemma 1044 (nonnegative essential supremum is positive homogeneous), Lemma 1045 (nonnegative essential supremum is almost monotone), Lemma 1046 (nonnegative essential supremum bounded by constant), Lemma 1049 (compatibility of nonnegative essential supremum with almost equality), Lemma 1050 (nonnegative essential supremum is almost definite), Lemma 1051 (nonnegative essential supremum is subadditive), Lemma 1053 (nonnegative essential supremum is submultiplicative), Lemma 1054 (nonnegative essential supremum is compatible with power function), Lemma 1288 ( $N_{p}$ with trivial measure), Lemma 1289 ( $N_{p}$ of indicator function).

## Lemma 1038 (nonnegative essential supremum is minimum)

 is explicitly cited in the proof of: Lemma 1044 (nonnegative essential supremum is positive homogeneous), Lemma 1050 (nonnegative essential supremum is almost definite).Lemma 1039 (nonnegative essential supremum is almost upper bound) is explicitly cited in the proof of: Lemma 1051 (nonnegative essential supremum is subadditive), Lemma 1053 (nonnegative essential supremum is submultiplicative), Lemma 1054 (nonnegative essential supremum is compatible with power function), Lemma 1293 ( $N_{\infty}$ bounded is almost bounded), Lemma 1302 (generalized Hölder inequality (with $\infty$ )).

Lemma 1040 (nonnegative essential supremum of constant function) is explicitly cited in the proof of: Lemma 1050 (nonnegative essential supremum is almost definite), Lemma 1054 (nonnegative essential supremum is compatible with power function), Lemma 1294 (compatibility of $N_{p}$ with power function), Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates).

Lemma 1041 (nonnegative essential supremum for extended simple functions) is explicitly cited in the proof of: Lemma 1289 ( $N_{p}$ of indicator function).

Lemma 1043 (nonnegative essential supremum of infinite times function) is explicitly cited in the proof of: Lemma 1044 (nonnegative essential supremum is positive homogeneous).

Lemma 1044 (nonnegative essential supremum is positive homogeneous) is explicitly cited in the proof of: Lemma 1300 ( $N_{p}$ is absolutely homogeneous).

Lemma 1045 (nonnegative essential supremum is almost monotone)
is explicitly cited in the proof of:
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1309 (Minkowski inequality in M),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ).

## Lemma 1046 (nonnegative essential supremum bounded by constant)

is explicitly cited in the proof of:
Lemma 1048 (essentially bounded is almost finite),
Lemma 1051 (nonnegative essential supremum is subadditive),
Lemma 1053 (nonnegative essential supremum is submultiplicative),
Lemma 1054 (nonnegative essential supremum is compatible with power function), Lemma 1293 ( $N_{\infty}$ bounded is almost bounded).

## Definition 1047 (essentially bounded)

is explicitly cited in the proof of: Lemma 1048 (essentially bounded is almost finite).

## Lemma 1048 (essentially bounded is almost finite)

is explicitly cited in the proof of: Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ).

## Lemma 1049 (compatibility of nonnegative essential supremum with almost equality)

is explicitly cited in the proof of:
Lemma 1050 (nonnegative essential supremum is almost definite), Lemma 1297 (compatibility of $N_{p}$ with almost equality).

Lemma 1050 (nonnegative essential supremum is almost definite) is explicitly cited in the proof of: Lemma 1296 ( $N_{p}$ is almost definite).

Lemma 1051 (nonnegative essential supremum is subadditive) is explicitly cited in the proof of: Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).

Lemma 1053 (nonnegative essential supremum is submultiplicative) is explicitly cited in the proof of: Lemma 1302 (generalized Hölder inequality (with $\infty$ )).

## Lemma 1054 (nonnegative essential supremum is compatible with power function)

 is explicitly cited in the proof of: Lemma 1294 (compatibility of $N_{p}$ with power function).
## Definition 1057 (length of interval)

 is explicitly cited in the proof of: Lemma 1058 (length is nonnegative), Lemma 1059 (length is homogeneous), Lemma 1060 (length of partition), Lemma 1066 ( $\lambda^{\star}$ is homogeneous), Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval).Lemma 1058 (length is nonnegative) is explicitly cited in the proof of: Lemma 1065 ( $\lambda^{\star}$ is nonnegative).

## Lemma 1059 (length is homogeneous)

is explicitly cited in the proof of:
Lemma 1060 (length of partition).
Lemma 1060 (length of partition) is not yet used.

Definition 1061 (set of countable cover with bounded open intervals)
is explicitly cited in the proof of:
Lemma 1062 (set of countable cover with bounded open intervals is nonempty),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval).
Lemma 1062 (set of countable cover with bounded open intervals is nonempty) is not yet used.

Definition 1063 ( $\lambda^{\star}$, Lebesgue measure candidate)
is explicitly cited in the proof of:
Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma 1067 ( $\lambda^{\star}$ is monotone),
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive),
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma 1083 (rays are Lebesgue-measurable).
Lemma 1065 ( $\lambda^{\star}$ is nonnegative) is explicitly cited in the proof of:
Lemma 1066 ( $\lambda^{\star}$ is homogeneous),
Lemma 1086 ( $\lambda^{\star}$ is measure on $\left.\mathcal{L}\right)$.
Lemma 1066 ( $\lambda^{\star}$ is homogeneous)
is explicitly cited in the proof of:
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval),
Lemma 1077 ( $\mathcal{L}$ is algebra of sets),
Lemma 1086 ( $\lambda^{\star}$ is measure on $\left.\mathcal{L}\right)$.
Lemma 1067 ( $\lambda^{\star}$ is monotone)
is explicitly cited in the proof of:
Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval), Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ), Lemma 1082 ( $\mathcal{L}$ is closed under countable union), Lemma 1083 (rays are Lebesgue-measurable).
Lemma 1068 ( $\lambda^{\star}$ is $\sigma$-subadditive) is explicitly cited in the proof of: Lemma 1073 (equivalent definition of $\mathcal{L}$ ), Lemma 1075 ( $\mathcal{L}$ is closed under finite union), Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ),
Lemma 1082 ( $\mathcal{L}$ is closed under countable union), Lemma 1083 (rays are Lebesgue-measurable).

Lemma 1069 ( $\lambda^{\star}$ generalizes length of interval) is explicitly cited in the proof of:
Lemma 1083 (rays are Lebesgue-measurable),
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ),
Lemma 1092 (Lebesgue measure generalizes length of interval),
Lemma 1094 (Lebesgue measure is $\sigma$-finite),
Lemma 1095 (Lebesgue measure is diffuse).

## Definition 1071 ( $\mathcal{L}$, Lebesgue $\sigma$-algebra)

is explicitly cited in the proof of:
Lemma 1073 (equivalent definition of $\mathcal{L}$ ),
Lemma 1074 ( $\mathcal{L}$ is closed under complement),
Lemma 1075 ( $\mathcal{L}$ is closed under finite union),
Lemma $1078\left(\lambda^{\star}\right.$ is additive on $\left.\mathcal{L}\right)$,
Lemma 1082 ( $\mathcal{L}$ is closed under countable union).

## Lemma 1073 (equivalent definition of $\mathcal{L}$ )

is explicitly cited in the proof of:
Lemma 1075 ( $\mathcal{L}$ is closed under finite union), Lemma 1082 ( $\mathcal{L}$ is closed under countable union), Lemma 1083 (rays are Lebesgue-measurable).

Lemma 1074 ( $\mathcal{L}$ is closed under complement) is explicitly cited in the proof of: Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection), Lemma 1077 ( $\mathcal{L}$ is algebra of sets), Lemma 1083 (rays are Lebesgue-measurable).

Lemma 1075 ( $\mathcal{L}$ is closed under finite union) is explicitly cited in the proof of:
Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection), Lemma 1082 ( $\mathcal{L}$ is closed under countable union).

Lemma 1076 ( $\mathcal{L}$ is closed under finite intersection)
is explicitly cited in the proof of:
Lemma 1077 ( $\mathcal{L}$ is algebra of sets),
Lemma 1084 (intervals are Lebesgue-measurable).

## Lemma 1077 ( $\mathcal{L}$ is algebra of sets)

is explicitly cited in the proof of: Lemma 1081 (disjoint union of union in $\mathcal{L}$ ), Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra).

Lemma 1078 ( $\lambda^{\star}$ is additive on $\mathcal{L}$ ) is explicitly cited in the proof of: Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ ), Lemma 1082 ( $\mathcal{L}$ is closed under countable union).

## Lemma 1080 ( $\lambda^{\star}$ is $\sigma$-additive on $\mathcal{L}$ )

is explicitly cited in the proof of:
Lemma 1086 ( $\lambda^{\star}$ is measure on $\mathcal{L}$ ).
Lemma 1081 (disjoint union of union in $\mathcal{L}$ )
is explicitly cited in the proof of:
Lemma 1082 ( $\mathcal{L}$ is closed under countable union).
Lemma 1082 ( $\mathcal{L}$ is closed under countable union) is explicitly cited in the proof of: Lemma 1083 (rays are Lebesgue-measurable), Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra).

## Lemma 1083 (rays are Lebesgue-measurable)

is explicitly cited in the proof of: Lemma 1084 (intervals are Lebesgue-measurable).

## Lemma 1084 (intervals are Lebesgue-measurable)

is explicitly cited in the proof of:
Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$.
Lemma 1085 ( $\mathcal{L}$ is $\sigma$-algebra)
is explicitly cited in the proof of:
Lemma 1086 ( $\lambda^{\star}$ is measure on $\mathcal{L}$ ).
Lemma 1086 ( $\lambda^{\star}$ is measure on $\mathcal{L}$ )
is explicitly cited in the proof of:
Lemma $1088\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{B}(\mathbb{R})\right)$.
Lemma $1087(\mathcal{B}(\mathbb{R})$ is sub- $\sigma$-algebra of $\mathcal{L})$
is explicitly cited in the proof of:
Lemma $1088\left(\lambda^{\star}\right.$ is measure on $\left.\mathcal{B}(\mathbb{R})\right)$.
Lemma 1088 ( $\lambda^{\star}$ is measure on $\mathcal{B}(\mathbb{R})$ )
is explicitly cited in the proof of:
Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ ).

## Theorem 1090 (Carathéodory, Lebesgue measure on $\mathbb{R}$ )

is explicitly cited in the proof of:
Lemma 1092 (Lebesgue measure generalizes length of interval),
Lemma 1094 (Lebesgue measure is $\sigma$-finite),
Lemma 1095 (Lebesgue measure is diffuse),
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ).

## Lemma 1092 (Lebesgue measure generalizes length of interval)

is explicitly cited in the proof of:
Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes).
Lemma 1094 (Lebesgue measure is $\sigma$-finite)
is explicitly cited in the proof of:
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ),
Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite).
Lemma 1095 (Lebesgue measure is diffuse)
is not yet used.
Definition $1098\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$, set of measurable indicator functions)
is explicitly cited in the proof of:
Lemma 1100 (indicator and support are inverse),
Lemma 1101 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is measurable),
Lemma 1102 ( $\mathcal{I F}_{\mathfrak{K}}$ is $\sigma$-additive),
Lemma 1103 ( $\mathcal{I} \mathcal{F}_{\mathbb{K}}$ is closed under multiplication),
Lemma $1105\left(\mathcal{I} \mathcal{F}_{\mathfrak{K}}\right.$ is closed under extension by zero),
Lemma $1106\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under restriction),
Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation),
Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero),
Lemma 1121 ( $\mathcal{S} \mathcal{F}_{E}$, masking is $\sigma$-additive),
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable),
Lemma 1134 (Lebesgue scheme step 2),
Lemma 1156 (integral in $\mathcal{S F}_{+}$generalizes integral in $\mathcal{I F}$ ).
Lemma 1100 (indicator and support are inverse)
is explicitly cited in the proof of:
Lemma 1107 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$, masking is $\sigma$-additive),

Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation),
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ), Lemma 1152 (integral in $\mathcal{I F}$ with counting measure), Theorem 1242 (Tonelli).

Lemma $1101\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is measurable)
is not yet used.

## Lemma $1102\left(\mathcal{I F}_{\mathbb{K}}\right.$ is $\sigma$-additive)

is explicitly cited in the proof of:
Lemma 1107 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$, masking is $\sigma$-additive),
Lemma $1117\left(\mathcal{S F}_{E}\right.$ is vector space $)$,
Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive),
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive).

## Lemma $1103\left(\mathcal{I F}_{\mathbb{K}}\right.$ is closed under multiplication)

is explicitly cited in the proof of:
Lemma $1107\left(\mathcal{I F}_{\mathfrak{K}}\right.$, masking is $\sigma$-additive), Lemma $1117\left(\mathcal{S F}_{E}\right.$ is vector space $)$, Lemma $1121\left(\mathcal{S F}_{E}\right.$, masking is $\sigma$-additive $)$, Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive).

## Lemma 1105 ( $\mathcal{I} \mathcal{F}_{\mathfrak{K}}$ is closed under extension by zero)

 is explicitly cited in the proof of: Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero), Lemma 1148 (integral in $\mathcal{I F}$ over subset).Lemma $1106\left(\mathcal{I F}_{\mathfrak{K}}\right.$ is closed under restriction) is explicitly cited in the proof of: Lemma $1120\left(\mathcal{S F} \mathcal{F}_{E}\right.$ is closed under restriction).

Lemma $1107\left(\mathcal{I F}_{\mathfrak{K}}\right.$, masking is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive).

Lemma 1108 ( $\mathcal{S F}_{E}$ simple representation) is explicitly cited in the proof of: Lemma $1111\left(\mathcal{S \mathcal { F } _ { E }}\right.$ canonical representation), Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable), Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation), Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space), Lemma $1119\left(\mathcal{S F}_{E}\right.$ is closed under extension by zero $)$, Lemma $1120\left(\mathcal{S F} \mathcal{F}_{E}\right.$ is closed under restriction), Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive), Lemma $1128\left(\mathcal{S} \mathcal{F}_{+}\right.$simple representation $)$, Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), Theorem 1180 (Beppo Levi, monotone convergence).

Lemma 1111 ( $\mathcal{S F}_{E}$ canonical representation) is explicitly cited in the proof of: Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable), Lemma 1114 ( $\mathcal{S F}_{E}$ disjoint representation), Lemma $1116\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation is subpartition of canonical representation), Lemma 1117 ( $\mathcal{S F}_{E}$ is vector space), Lemma $1126(\mathcal{S F}+$ canonical representation $)$.

## Lemma $1113\left(\mathcal{S F}_{E}\right.$ is measurable)

is explicitly cited in the proof of:
Lemma $1124\left(\mathcal{S F}_{+}\right.$is measurable), Theorem 1180 (Beppo Levi, monotone convergence).

## Lemma 1114 ( $\mathcal{S} \mathcal{F}_{E}$ disjoint representation)

is explicitly cited in the proof of:
Lemma $1116\left(\mathcal{S F}_{E}\right.$ disjoint representation is subpartition of canonical representation $)$,
Lemma $1117\left(\mathcal{S F}_{E}\right.$ is vector space $)$,
Lemma 1125 ( $\mathcal{S F}+$ disjoint representation $)$.

## Lemma $1116\left(\mathcal{S} \mathcal{F}_{E}\right.$ disjoint representation is subpartition of canonical representation)

is explicitly cited in the proof of: Lemma $1127\left(\mathcal{S F}_{+}\right.$disjoint representation is subpartition of canonical representation $)$.

## Lemma 1117 ( $\mathcal{S \mathcal { F } _ { E }}$ is vector space)

 is explicitly cited in the proof of: Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations), Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone).Lemma 1119 (SF- $\mathcal{F}_{E}$ is closed under extension by zero) is explicitly cited in the proof of: Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero).

## Lemma 1120 ( $\mathcal{S F}_{E}$ is closed under restriction)

 is explicitly cited in the proof of: Lemma $1131\left(\mathcal{S F}_{+}\right.$is closed under restriction).Lemma 1121 ( $\mathcal{S F}_{E}$, masking is $\sigma$-additive) is explicitly cited in the proof of: Lemma $1132\left(\mathcal{S F}_{+}\right.$, masking is $\sigma$-additive $)$.

Definition $1123\left(\mathcal{S} \mathcal{F}_{+}\right.$, subset of nonnegative simple functions) is explicitly cited in the proof of: Lemma $1124(\mathcal{S F}+$ is measurable), Lemma $1125\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation), Lemma $1126\left(\mathcal{S} \mathcal{F}_{+}\right.$canonical representation $)$, Lemma $1127\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation is subpartition of canonical representation $)$, Lemma $1128\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation $)$, Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations), Lemma $1130\left(\mathcal{S F}_{+}\right.$is closed under extension by zero), Lemma $1131\left(\mathcal{S F} \mathcal{F}_{+}\right.$is closed under restriction $)$, Lemma 1132 ( $\mathcal{S F}_{+}$, masking is $\sigma$-additive), Lemma 1136 ( adapted sequence in $\mathcal{M}_{+}$), Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone), Lemma 1166 (density measure in $\mathcal{S F} \mathcal{F}_{+}$), Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous).

Lemma $1124\left(\mathcal{S F}{ }_{+}\right.$is measurable) is explicitly cited in the proof of: Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero), Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$, Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive), Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function), Theorem 1242 (Tonelli).

## Lemma 1125 ( $\mathcal{S F} \mathcal{F}_{+}$disjoint representation)

is explicitly cited in the proof of:
Lemma $1127\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation is subpartition of canonical representation), Lemma $1129\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under positive algebra operations),
Lemma 1159 (integral in $\mathcal{S F}{ }_{+}$is positive linear),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1167 (integral and density measure in $\mathcal{S F}_{+}$),
Lemma 1168 (density integral in $\mathcal{S F}_{+}$).

## Lemma 1126 ( $\mathcal{S F}+$ canonical representation)

is explicitly cited in the proof of:
Lemma $1127\left(\mathcal{S} \mathcal{F}_{+}\right.$disjoint representation is subpartition of canonical representation),
Lemma 1154 (integral in $\mathcal{S F}_{+}$),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}_{+}$(disjoint)).
Lemma $1127(\mathcal{S F}+$ disjoint representation is subpartition of canonical representation)
is explicitly cited in the proof of:
Lemma 1157 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(disjoint)).
Lemma $1128\left(\mathcal{S F} \mathcal{F}_{+}\right.$simple representation)
is explicitly cited in the proof of:
Lemma 1134 (Lebesgue scheme step 2),
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero),
Lemma 1156 (integral in $\mathcal{S \mathcal { F }}+$ generalizes integral in $\mathcal{I F}$ ),
Lemma 1160 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(simple)),
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure),
Theorem 1242 (Tonelli).
Lemma $1129\left(\mathcal{S F}_{+}\right.$is closed under positive algebra operations)
is explicitly cited in the proof of:
Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1167 (integral and density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1168 (density integral in $\mathcal{S F}_{+}$),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Theorem 1242 (Tonelli).
Lemma $1130\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under extension by zero)
is explicitly cited in the proof of:
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero $)$,
Lemma 1163 (integral in $\mathcal{S F}{ }_{+}$over subset).
Lemma $1131\left(\mathcal{S} \mathcal{F}_{+}\right.$is closed under restriction)
is explicitly cited in the proof of:
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$.
Lemma 1132 ( $\mathcal{S F}_{+}$, masking is $\sigma$-additive)
is explicitly cited in the proof of:
Lemma 1164 (integral in $\mathcal{S F}_{+}$over subset is $\sigma$-additive).
Lemma 1134 (Lebesgue scheme step 2)
is explicitly cited in the proof of:
Theorem 1242 (Tonelli).
Definition 1135 (adapted sequence)
is explicitly cited in the proof of:

Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$),
Lemma 1182 (usage of adapted sequences),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
Lemma 1136 (adapted sequence in $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero),
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$,
Lemma 1141 (Lebesgue scheme step 3),
Lemma 1182 (usage of adapted sequences),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
Lemma $1137\left(\mathcal{M}_{+}\right.$is closed under extension by zero)
is explicitly cited in the proof of:
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction).
Lemma $1138\left(\mathcal{M}_{+}\right.$is closed under restriction $)$
is explicitly cited in the proof of:
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction).
Lemma 1139 ( $\mathcal{M}_{+}$, masking is $\sigma$-additive)
is explicitly cited in the proof of:
Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive),
Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive).
Lemma 1141 (Lebesgue scheme step 3)
is explicitly cited in the proof of:
Theorem 1242 (Tonelli).
Definition 1144 (integral in $\mathcal{I F}$ )
is explicitly cited in the proof of:
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ ),
Lemma 1148 (integral in $\mathcal{I F}$ over subset).
Lemma 1145 (equivalent definition of integral in $\mathcal{I F}$ )
is explicitly cited in the proof of:
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive),
Lemma 1147 (integral in $\mathcal{I F}$ is monotone),
Lemma 1152 (integral in $\mathcal{I F}$ with counting measure),
Lemma 1154 (integral in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1156 (integral in $\mathcal{S \mathcal { F }}+{ }_{+}$generalizes integral in $\mathcal{I F}$ ),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality).
Lemma 1146 (integral in $\mathcal{I F}$ is $\sigma$-additive)
is explicitly cited in the proof of:
Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive).
Lemma 1147 (integral in $\mathcal{I F}$ is monotone)
is not yet used.
Lemma 1148 (integral in $\mathcal{I F}$ over subset)
is explicitly cited in the proof of:
Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive),
Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset),
Lemma 1163 (integral in $\mathcal{S F} \mathcal{F}_{+}$over subset).

## Lemma 1149 (integral in $\mathcal{I F}$ over subset is $\sigma$-additive)

 is not yet used.Lemma 1150 (integral in $\mathcal{I F}$ over negligible subset) is explicitly cited in the proof of: Lemma 1165 (integral in $\mathcal{S F}_{+}$over negligible subset).

Lemma 1152 (integral in $\mathcal{I F}$ with counting measure) is explicitly cited in the proof of: Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure).

Lemma 1154 (integral in $\mathcal{S F}_{+}$) is explicitly cited in the proof of:
Lemma 1156 (integral in $\mathcal{S} \mathcal{F}_{+}$generalizes integral in $\mathcal{I F}$ ),
Lemma 1157 (equivalent definition of the integral in $\mathcal{S} \mathcal{F}_{+}$(disjoint)),
Lemma 1159 (integral in $\mathcal{S F}{ }_{+}$is positive linear),
Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1168 (density integral in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure),
Lemma 1172 (integral in $\mathcal{M}_{+}$),
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S F}_{+}$),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous).
Lemma 1156 (integral in $\mathcal{S F}_{+}$generalizes integral in $\mathcal{I F}$ ) is explicitly cited in the proof of: Lemma 1158 (integral in $\mathcal{S \mathcal { F }}+$ of zero is zero), Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}_{+}$(simple)),
Lemma 1163 (integral in $\mathcal{S F}{ }_{+}$over subset),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function).
Lemma 1157 (equivalent definition of the integral in $\mathcal{S F}{ }_{+}$(disjoint)) is explicitly cited in the proof of:
Lemma 1159 (integral in $\mathcal{S} \mathcal{F}_{+}$is positive linear),
Lemma 1166 (density measure in $\mathcal{S} \mathcal{F}_{+}$),
Lemma 1167 (integral and density measure in $\mathcal{S F}_{+}$).
Lemma 1158 (integral in $\mathcal{S F}_{+}$of zero is zero)
is explicitly cited in the proof of:
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear).
Lemma 1159 (integral in $\mathcal{S F}_{+}$is positive linear) is explicitly cited in the proof of:
Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}_{+}$(simple)),
Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone),
Lemma 1163 (integral in $\mathcal{S F}_{+}$over subset),
Lemma 1164 (integral in $\mathcal{S F} \mathcal{F}_{+}$over subset is $\sigma$-additive),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1183 (integral in $\mathcal{M}_{+}$is additive).
Lemma 1160 (equivalent definition of the integral in $\mathcal{S F}_{+}$(simple)) is explicitly cited in the proof of:
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous), Theorem 1180 (Beppo Levi, monotone convergence).

## Lemma 1161 (integral in $\mathcal{S F}_{+}$is monotone)

is explicitly cited in the proof of:
Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous).
Lemma 1162 (integral in $\mathcal{S F}_{+}$is continuous)
is explicitly cited in the proof of:
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S} \mathcal{F}_{+}$).
Lemma 1163 (integral in $\mathcal{S F}_{+}$over subset)
is explicitly cited in the proof of:
Lemma 1164 (integral in $\mathcal{S F}+$ over subset is $\sigma$-additive),
Lemma 1165 (integral in $\mathcal{S F}_{+}$over negligible subset),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
Lemma 1164 (integral in $\mathcal{S F}_{+}$over subset is $\sigma$-additive)
is not yet used.
Lemma 1165 (integral in $\mathcal{S F}{ }_{+}$over negligible subset)
is explicitly cited in the proof of:
Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset).
Lemma 1166 (density measure in $\mathcal{S \mathcal { F } _ { + } \text { ) }}$
is explicitly cited in the proof of:
Lemma 1167 (integral and density measure in $\mathcal{S F}_{+}$),
Lemma 1168 (density integral in $\mathcal{S F}_{+}$).
Lemma 1167 (integral and density measure in $\mathcal{S F}{ }_{+}$) is not yet used.

## Lemma 1168 (density integral in $\mathcal{S \mathcal { F } _ { + } \text { ) }}$

is not yet used.
Lemma 1169 (integral in $\mathcal{S F}_{+}$with trivial measure)
is explicitly cited in the proof of:
Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure).
Lemma 1170 (integral in $\mathcal{S F}_{+}$with counting measure) is explicitly cited in the proof of:
Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).
Lemma 1172 (integral in $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S \mathcal { F } _ { + }}$ ),
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous),
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone),
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality),
Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite),
Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable),
Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1248 (integrable is measurable),
Lemma 1249 (equivalent definition of integrability),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1256 (integral in $\mathcal{M}$ ),

Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value),
Lemma 1275 (integral over subset is $\sigma$-additive),
Lemma 1277 (integral over singleton),
Lemma 1286 ( $p$-seminorm),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma 1174 (integral in $\mathcal{M}_{+}$generalizes integral in $\mathcal{S F} \mathcal{F}_{+}$)
is explicitly cited in the proof of:
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function),
Lemma 1182 (usage of adapted sequences),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset).
Lemma 1175 (integral in $\mathcal{M}_{+}$of indicator function)
is explicitly cited in the proof of:
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite),
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1221 (measurability of measure of section (finite)),
Lemma 1227 (candidate tensor product measure is tensor product measure),
Theorem 1242 (Tonelli),
Lemma 1260 (integral in $\mathcal{M}$ of indicator function),
Lemma 1289 ( $N_{p}$ of indicator function).
Lemma 1176 (integral in $\mathcal{M}_{+}$is positive homogeneous)
is explicitly cited in the proof of:
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero),
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality),
Lemma 1244 (Tonelli for tensor product),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1300 ( $N_{p}$ is absolutely homogeneous),
Lemma 1304 (Hölder inequality).
Lemma 1177 (integral in $\mathcal{M}_{+}$of zero is zero) is explicitly cited in the proof of:
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero).
Lemma 1178 (integral in $\mathcal{M}_{+}$is monotone)
is explicitly cited in the proof of:
Theorem 1180 (Beppo Levi, monotone convergence),
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone),
Lemma 1195 (generalized Bienaymé-Chebyshev inequality),
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Theorem 1208 (Fatou's lemma),
Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),

Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1304 (Hölder inequality),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ).

## Theorem 1180 (Beppo Levi, monotone convergence)

is explicitly cited in the proof of:
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ),
Lemma 1182 (usage of adapted sequences),
Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive),
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset),
Theorem 1208 (Fatou's lemma),
Theorem 1242 (Tonelli),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ).
Lemma 1181 (integral in $\mathcal{M}_{+}$is homogeneous at $\infty$ ) is explicitly cited in the proof of: Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear), Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite), Lemma 1300 ( $N_{p}$ is absolutely homogeneous).

## Lemma 1182 (usage of adapted sequences)

 is explicitly cited in the proof of: Lemma 1183 (integral in $\mathcal{M}_{+}$is additive), Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure), Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure).Lemma 1183 (integral in $\mathcal{M}_{+}$is additive) is explicitly cited in the proof of: Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear), Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts), Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts), Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ),
Lemma 1304 (Hölder inequality),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).
Lemma 1184 (integral in $\mathcal{M}_{+}$is positive linear)
is explicitly cited in the proof of:
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1227 (candidate tensor product measure is tensor product measure), Theorem 1242 (Tonelli), Lemma 1302 (generalized Hölder inequality (with $\infty$ )), Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).

Lemma 1185 (integral in $\mathcal{M}_{+}$is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive), Lemma 1227 (candidate tensor product measure is tensor product measure).

Lemma 1186 (integral in $\mathcal{M}_{+}$of decomposition into nonnegative and nonpositive parts)
is explicitly cited in the proof of:
Lemma 1249 (equivalent definition of integrability),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value),
Lemma 1310 (Minkowski equality (nonnegative functions)).

Lemma 1187 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonpositive and nonnegative parts)
is explicitly cited in the proof of:
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1269 (integral in $\mathcal{M}$ is additive).
Lemma 1188 (integral in $\mathcal{M}_{+}$is almost definite)
is explicitly cited in the proof of:
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$),
Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function),
Lemma 1234 (negligibility of measurable section),
Lemma 1296 ( $N_{p}$ is almost definite),
Lemma 1329 (first mean value inequality).
Lemma 1189 (extended almost modus ponens for integral in $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality),
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone).
Lemma 1190 (compatibility of integral in $\mathcal{M}_{+}$with almost equality) is explicitly cited in the proof of:
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions), Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ ).
Lemma 1191 (integral in $\mathcal{M}_{+}$is almost monotone)
is explicitly cited in the proof of:
Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable),
Lemma 1302 (generalized Hölder inequality (with $\infty$ )).
Lemma 1192 (integral in $\mathcal{M}_{+}$of nonpositive part of almost nonnegative function) is explicitly cited in the proof of:
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions),
Lemma 1310 (Minkowski equality (nonnegative functions)).
Lemma 1193 (integral in $\mathcal{M}_{+}$of nonnegative and nonpositive parts of almost equal functions)
is explicitly cited in the proof of:
Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions).

Lemma 1194 (compatibility of integral in $\mathcal{M}_{+}$with addition of nonnegative parts for almost nonnegative functions)
is explicitly cited in the proof of:
Lemma 1310 (Minkowski equality (nonnegative functions)).
Lemma 1195 (generalized Bienaymé-Chebyshev inequality) is explicitly cited in the proof of: Lemma 1196 (Bienaymé-Chebyshev inequality).

## Lemma 1196 (Bienaymé-Chebyshev inequality)

 is explicitly cited in the proof of: Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite).
## Lemma 1197 (integrable in $\mathcal{M}_{+}$is almost finite)

is explicitly cited in the proof of:
Lemma 1251 (integrable is almost finite).
Lemma 1198 (almost bounded by integrable in $\mathcal{M}_{+}$is integrable)
is explicitly cited in the proof of:
Lemma 1252 (almost bounded by integrable is integrable).
Lemma 1199 (integral in $\mathcal{M}_{+}$over subset)
is explicitly cited in the proof of:
Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive),
Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset),
Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton),
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction),
Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive),
Lemma 1243 (Tonelli over subset),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1274 (integral over subset).
Lemma 1200 (integral in $\mathcal{M}_{+}$over subset is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive), Lemma 1275 (integral over subset is $\sigma$-additive).

Lemma 1201 (integral in $\mathcal{M}_{+}$over negligible subset) is explicitly cited in the proof of: Lemma 1276 (integral over negligible subset).

Lemma 1202 (integral in $\mathcal{M}_{+}$over singleton)
is explicitly cited in the proof of:
Lemma 1277 (integral over singleton).
Lemma 1203 (finite integral in $\mathcal{M}_{+}$is closed under restriction) is explicitly cited in the proof of: Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction).

Lemma 1204 (integrable in $\mathcal{M}_{+}$is closed under restriction) is explicitly cited in the proof of: Lemma 1254 (integrability is closed under restriction).

Lemma 1205 (finite integral in $\mathcal{M}_{+}$, masking is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive).

Lemma 1206 (integrable in $\mathcal{M}_{+}$, masking is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1255 (integrability, masking is $\sigma$-additive).

## Theorem 1208 (Fatou's lemma)

is explicitly cited in the proof of:
Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence), Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ), Theorem 1347 (Lebesgue, dominated convergence).

Lemma 1209 (integral in $\mathcal{M}_{+}$of pointwise convergent sequence) is not yet used.

Lemma 1210 (integral in $\mathcal{M}_{+}$with trivial measure)
is explicitly cited in the proof of:
Lemma 1280 (integral with trivial measure),
Lemma 1288 ( $N_{p}$ with trivial measure).

## Lemma 1211 (integral in $\mathcal{M}_{+}$with counting measure)

is explicitly cited in the proof of:
Lemma 1212 (integral in $\mathcal{M}_{+}$with counting measure on $\mathbb{N}$ ),
Lemma 1214 (integral in $\mathcal{M}_{+}$with Dirac measure), Lemma 1281 (integral with counting measure).

Lemma 1212 (integral in $\mathcal{M}_{+}$with counting measure on $\mathbb{N}$ ) is not yet used.

Lemma 1214 (integral in $\mathcal{M}_{+}$with Dirac measure) is not yet used.

Definition 1216 (overline-i notation to exclude term in product)
is not yet used.
Lemma 1218 (measure of section) is explicitly cited in the proof of: Lemma 1219 (measure of section of product), Lemma 1221 (measurability of measure of section (finite)), Lemma 1223 (measurability of measure of section), Lemma 1227 (candidate tensor product measure is tensor product measure), Lemma 1233 (uniqueness of tensor product measure), Theorem 1242 (Tonelli).

Lemma 1219 (measure of section of product)
is explicitly cited in the proof of: Lemma 1221 (measurability of measure of section (finite)), Lemma 1227 (candidate tensor product measure is tensor product measure).

Lemma 1221 (measurability of measure of section (finite)) is explicitly cited in the proof of: Lemma 1223 (measurability of measure of section).

Lemma 1223 (measurability of measure of section)
is explicitly cited in the proof of:
Lemma 1227 (candidate tensor product measure is tensor product measure), Theorem 1242 (Tonelli).

Definition 1224 (tensor product measure)
is explicitly cited in the proof of:
Lemma 1227 (candidate tensor product measure is tensor product measure),
Lemma 1228 (tensor product of finite measures),
Lemma 1229 (tensor product of $\sigma$-finite measures),
Lemma 1231 (uniqueness of tensor product measure (finite)),
Lemma 1233 (uniqueness of tensor product measure),
Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes).
Definition 1225 (candidate tensor product measure) is explicitly cited in the proof of:
Lemma 1227 (candidate tensor product measure is tensor product measure), Lemma 1233 (uniqueness of tensor product measure).

## Lemma 1227 (candidate tensor product measure is tensor product measure)

is explicitly cited in the proof of:
Lemma 1231 (uniqueness of tensor product measure (finite)), Lemma 1233 (uniqueness of tensor product measure).

Lemma 1228 (tensor product of finite measures) is explicitly cited in the proof of: Lemma 1231 (uniqueness of tensor product measure (finite)).

Lemma 1229 (tensor product of $\sigma$-finite measures)
is explicitly cited in the proof of:
Lemma 1233 (uniqueness of tensor product measure), Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite).

Lemma 1231 (uniqueness of tensor product measure (finite)) is explicitly cited in the proof of:
Lemma 1233 (uniqueness of tensor product measure).
Lemma 1233 (uniqueness of tensor product measure)
is explicitly cited in the proof of:
Lemma 1234 (negligibility of measurable section),
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ ),
Theorem 1242 (Tonelli).
Lemma 1234 (negligibility of measurable section)
is not yet used.
Lemma 1235 (Lebesgue measure on $\mathbb{R}^{2}$ )
is explicitly cited in the proof of:
Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes), Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite).

Lemma 1236 (Lebesgue measure on $\mathbb{R}^{2}$ generalizes area of boxes) is explicitly cited in the proof of: Lemma 1237 (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines), Lemma 1239 (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse).

Lemma 1237 (Lebesgue measure on $\mathbb{R}^{2}$ is zero on lines)
is not yet used.
Lemma 1238 (Lebesgue measure on $\mathbb{R}^{2}$ is $\sigma$-finite) is not yet used.

Lemma 1239 (Lebesgue measure on $\mathbb{R}^{2}$ is diffuse)
is not yet used.
Definition 1240 (partial function of function from product space)
is explicitly cited in the proof of:
Theorem 1242 (Tonelli),
Lemma 1243 (Tonelli over subset),
Lemma 1244 (Tonelli for tensor product).

## Theorem 1242 (Tonelli)

is explicitly cited in the proof of:
Lemma 1243 (Tonelli over subset),
Lemma 1244 (Tonelli for tensor product).
Lemma 1243 (Tonelli over subset)
is not yet used.

## Lemma 1244 (Tonelli for tensor product)

is not yet used.

## Definition 1247 (integrability)

is explicitly cited in the proof of:
Lemma 1248 (integrable is measurable),
Lemma 1249 (equivalent definition of integrability),
Lemma 1253 (integrability is closed under extension by zero),
Lemma 1254 (integrability is closed under restriction),
Lemma 1255 (integrability, masking is $\sigma$-additive),
Lemma 1256 (integral in $\mathcal{M}$ ),
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value).

## Lemma 1248 (integrable is measurable)

is explicitly cited in the proof of:
Lemma 1249 (equivalent definition of integrability).

## Lemma 1249 (equivalent definition of integrability)

is explicitly cited in the proof of:
Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$),
Lemma 1251 (integrable is almost finite),
Lemma 1252 (almost bounded by integrable is integrable),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value),
Lemma 1277 (integral over singleton),
Lemma 1280 (integral with trivial measure),
Lemma 1281 (integral with counting measure),
Lemma 1292 (integrable is finite $N_{1}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 1250 (integrability in $\mathcal{M}$ generalizes integrability in $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Lemma 1260 (integral in $\mathcal{M}$ of indicator function).

## Lemma 1251 (integrable is almost finite)

is explicitly cited in the proof of:
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 1252 (almost bounded by integrable is integrable)
is not yet used.
Lemma 1253 (integrability is closed under extension by zero) is explicitly cited in the proof of:
Lemma 1274 (integral over subset),
Lemma 1277 (integral over singleton).
Lemma 1254 (integrability is closed under restriction)
is not yet used.

## Lemma 1255 (integrability, masking is $\sigma$-additive)

is explicitly cited in the proof of:
Lemma 1275 (integral over subset is $\sigma$-additive).
Lemma 1256 (integral in $\mathcal{M}$ )
is explicitly cited in the proof of:
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$),
Lemma 1262 (compatibility of integral with almost equality),
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous),
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value),
Lemma 1280 (integral with trivial measure),
Lemma 1281 (integral with counting measure),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ).
Lemma 1258 (integral in $\mathcal{M}$ generalizes integral in $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero),
Lemma 1260 (integral in $\mathcal{M}$ of indicator function),
Lemma 1274 (integral over subset),
Lemma 1329 (first mean value inequality).
Lemma 1259 (integral in $\mathcal{M}$ of zero is zero)
is explicitly cited in the proof of:
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 1260 (integral in $\mathcal{M}$ of indicator function)

is explicitly cited in the proof of:
Lemma 1267 (integral in $\mathcal{M}$ of constant function).
Definition 1261 (merge integral in $\mathcal{M}$ and $\mathcal{M}_{+}$)
is not yet used.

## Lemma 1262 (compatibility of integral with almost equality)

is explicitly cited in the proof of:
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ ),
Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ ),
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1297 (compatibility of $N_{p}$ with almost equality),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 1263 (finite representative of integrable in $\mathcal{M}$ )
is explicitly cited in the proof of:
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ).
Lemma 1264 (extended almost modus ponens for integral in $\mathcal{M}$ )
is explicitly cited in the proof of:
Lemma 1272 (integral in $\mathcal{M}$ is almost monotone).
Lemma 1266 (integral in $\mathcal{M}$ is homogeneous)
is explicitly cited in the proof of:
Lemma 1267 (integral in $\mathcal{M}$ of constant function),
Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear),
Lemma 1274 (integral over subset).

## Lemma 1267 (integral in $\mathcal{M}$ of constant function)

is explicitly cited in the proof of:
Lemma 1328 (constant function is $\mathcal{L}^{p}$ ).

## Lemma 1268 (Minkowski inequality in $\mathcal{M}(p=1)$ )

is explicitly cited in the proof of:
Lemma 1269 (integral in $\mathcal{M}$ is additive),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1309 (Minkowski inequality in M).
Lemma 1269 (integral in $\mathcal{M}$ is additive)
is explicitly cited in the proof of:
Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear),
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone),
Lemma 1274 (integral over subset).
Lemma 1270 (integral in $\mathcal{M}_{\mathbb{R}}$ is linear)
is explicitly cited in the proof of:
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ).
Lemma 1271 (integral in $\mathcal{M}_{\mathbb{R}}$ is monotone)
is explicitly cited in the proof of:
Lemma 1272 (integral in $\mathcal{M}$ is almost monotone),
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ).
Lemma 1272 (integral in $\mathcal{M}$ is almost monotone) is explicitly cited in the proof of:
Lemma 1320 (integral in $\mathcal{L}^{1}$ is almost monotone).
Lemma 1273 (compatibility of integral in $\mathcal{M}$ with absolute value)
is explicitly cited in the proof of:
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ).
Lemma 1274 (integral over subset)
is explicitly cited in the proof of:
Lemma 1275 (integral over subset is $\sigma$-additive),
Lemma 1276 (integral over negligible subset),
Lemma 1277 (integral over singleton),
Lemma 1329 (first mean value inequality).
Lemma 1275 (integral over subset is $\sigma$-additive) is explicitly cited in the proof of: Lemma 1278 (integral over interval), Lemma 1279 (Chasles relation, integral over split intervals).

Lemma 1276 (integral over negligible subset)
is not yet used.
Lemma 1277 (integral over singleton)
is explicitly cited in the proof of: Lemma 1278 (integral over interval).

Lemma 1278 (integral over interval)
is explicitly cited in the proof of: Lemma 1279 (Chasles relation, integral over split intervals).

Lemma 1279 (Chasles relation, integral over split intervals) is not yet used.

## Lemma 1280 (integral with trivial measure)

is not yet used.
Lemma 1281 (integral with counting measure) is explicitly cited in the proof of: Lemma 1282 (integral with counting measure on $\mathbb{N}$ ), Lemma 1284 (integral with Dirac measure).

## Lemma 1282 (integral with counting measure on $\mathbb{N}$ )

 is not yet used.Lemma 1284 (integral with Dirac measure)
is not yet used.

## Definition 1285 (integral for Lebesgue measure on $\mathbb{R}$ )

 is not yet used.Lemma 1286 ( $p$-seminorm)
is explicitly cited in the proof of:
Lemma 1288 ( $N_{p}$ with trivial measure),
Lemma 1289 ( $N_{p}$ of indicator function),
Lemma 1290 ( $N_{p}$ is nonnegative),
Lemma 1291 (link between $N_{1}$ and $N_{p}$ ),
Lemma 1292 (integrable is finite $N_{1}$ ),
Lemma 1293 ( $N_{\infty}$ bounded is almost bounded),
Lemma 1294 (compatibility of $N_{p}$ with power function), Lemma 1296 ( $N_{p}$ is almost definite), Lemma 1297 (compatibility of $N_{p}$ with almost equality), Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions), Lemma 1300 ( $N_{p}$ is absolutely homogeneous), Lemma 1302 (generalized Hölder inequality (with $\infty$ )), Lemma 1304 (Hölder inequality), Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates), Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ), Lemma 1310 (Minkowski equality (nonnegative functions)), Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ), Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ), Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ), Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ), Lemma 1339 (convergence in $\mathcal{L}^{p}$ ), Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ), Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ), Theorem 1347 (Lebesgue, dominated convergence), Theorem 1349 (Lebesgue, extended dominated convergence).

## Lemma 1288 ( $N_{p}$ with trivial measure)

is explicitly cited in the proof of: Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure), Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).

## Lemma 1289 ( $N_{p}$ of indicator function)

is explicitly cited in the proof of: Lemma 1301 ( $N_{p}$ of constant function).
Lemma 1290 ( $N_{p}$ is nonnegative) is explicitly cited in the proof of: Lemma 1304 (Hölder inequality),

Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates), Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ), Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ).

## Lemma 1291 (link between $N_{1}$ and $N_{p}$ )

is explicitly cited in the proof of:
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions),
Lemma 1305 (generalized Hölder inequality),
Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates),
Lemma 1309 (Minkowski inequality in M),
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ).

## Lemma 1292 (integrable is finite $N_{1}$ )

is explicitly cited in the proof of: Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions).

Lemma 1293 ( $N_{\infty}$ bounded is almost bounded) is explicitly cited in the proof of: Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

## Lemma 1294 (compatibility of $N_{p}$ with power function)

is explicitly cited in the proof of:
Lemma 1295 (compatibility of $N_{p}$ with absolute value),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).

## Lemma 1295 (compatibility of $N_{p}$ with absolute value)

is explicitly cited in the proof of: Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions), Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value), Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ), Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

Lemma 1296 ( $N_{p}$ is almost definite)
is explicitly cited in the proof of:
Lemma 1304 (Hölder inequality), Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal).

Lemma 1297 (compatibility of $N_{p}$ with almost equality) is explicitly cited in the proof of: Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions), Lemma 1309 (Minkowski inequality in M),
Lemma 1310 (Minkowski equality (nonnegative functions)),
Lemma 1312 (Minkowski equality (almost linearly dependent functions)),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ),
Theorem 1349 (Lebesgue, extended dominated convergence).
Lemma 1298 ( $N_{p}$ is monotone for nonnegative functions)
is explicitly cited in the proof of:
Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions).
Lemma 1299 ( $N_{p}$ is almost monotone for almost nonnegative functions) is explicitly cited in the proof of: Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

## Lemma 1300 ( $N_{p}$ is absolutely homogeneous)

 is explicitly cited in the proof of: Lemma 1301 ( $N_{p}$ of constant function),Lemma 1304 (Hölder inequality),
Lemma 1311 (Minkowski equality (linearly dependent functions)),
Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space).
Lemma 1301 ( $N_{p}$ of constant function)
is explicitly cited in the proof of:
Lemma 1328 (constant function is $\mathcal{L}^{p}$ ).
Lemma 1302 (generalized Hölder inequality (with $\infty$ ))
is explicitly cited in the proof of:
Lemma 1304 (Hölder inequality),
Lemma 1305 (generalized Hölder inequality).

## Lemma 1304 (Hölder inequality)

is explicitly cited in the proof of:
Lemma 1305 (generalized Hölder inequality),
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).
Lemma 1305 (generalized Hölder inequality) is explicitly cited in the proof of:
Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ).
Lemma 1306 (compatibility of $N_{p}$ with Hölder conjugates)
is explicitly cited in the proof of:
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ ).
Lemma 1309 (Minkowski inequality in $\mathcal{M}$ )
is explicitly cited in the proof of:
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ).
Lemma 1310 (Minkowski equality (nonnegative functions))
is not yet used.
Lemma 1311 (Minkowski equality (linearly dependent functions))
is explicitly cited in the proof of:
Lemma 1312 (Minkowski equality (almost linearly dependent functions)).

## Lemma 1312 (Minkowski equality (almost linearly dependent functions))

is not yet used.
Definition 1314 ( $\mathcal{L}^{p}$, vector space of p-powered integrable functions)
is explicitly cited in the proof of:
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ ),
Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure),
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$),
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ),
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ ),
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ ),
Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication),
Lemma 1328 (constant function is $\mathcal{L}^{p}$ ),
Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ ),
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ),
Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ ),
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),

Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1349 (Lebesgue, extended dominated convergence),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1316 (equivalent definition of $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1320 (integral in $\mathcal{L}^{1}$ is almost monotone),
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ ),
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1349 (Lebesgue, extended dominated convergence),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma 1317 ( $\mathcal{L}^{p}$ with trivial measure)
is explicitly cited in the proof of:
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Lemma 1318 ( $\mathcal{L}^{p}$ and nonnegative is $\mathcal{M}_{+}$)
is explicitly cited in the proof of:
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ ),
Lemma 1329 (first mean value inequality),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1319 (integral is positive linear form on $\mathcal{L}^{1}$ )
is explicitly cited in the proof of:
Lemma 1329 (first mean value inequality),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence).
Lemma 1320 (integral in $\mathcal{L}^{1}$ is almost monotone)
is not yet used.
Lemma 1321 (link between $\mathcal{L}^{1}$ and $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Lemma 1322 ( $\mathcal{L}^{p}$ is closed under absolute value)
is explicitly cited in the proof of:
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1323 (bounded by $\mathcal{L}^{p}$ is $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1329 (first mean value inequality),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

Lemma 1324 (finite $N_{p}$ has finite representative in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).
Lemma 1326 (integrable and nonnegative has finite representative in $\mathcal{L}^{1}$ )
is explicitly cited in the proof of:
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ ).
Lemma 1327 ( $\mathcal{L}^{p}$ is closed under scalar multiplication)
is explicitly cited in the proof of:
Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space),
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal).
Lemma 1328 (constant function is $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1329 (first mean value inequality).
Lemma 1329 (first mean value inequality)
is explicitly cited in the proof of:
Lemma 1331 (first mean value theorem).
Lemma 1331 (first mean value theorem)
is not yet used.

## Lemma 1334 (generalized Hölder inequality in $\mathcal{L}^{p}$ )

 is explicitly cited in the proof of:Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ).
Lemma 1335 (interpolation inequality in $\mathcal{L}^{p}$ ) is not yet used.

## Lemma 1336 (Minkowski inequality in $\mathcal{L}^{p}$ )

is explicitly cited in the proof of: Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space), Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal), Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

Lemma 1337 ( $\mathcal{L}^{p}$ is seminormed vector space) is explicitly cited in the proof of:
Lemma 1339 (convergence in $\mathcal{L}^{p}$ ),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1349 (Lebesgue, extended dominated convergence),
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1339 (convergence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ ),
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal),
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ ),
Theorem 1347 (Lebesgue, dominated convergence),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).

Lemma 1340 (equivalent definition of convergence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ),
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Lemma 1341 (limits in $\mathcal{L}^{p}$ are almost equal)
is explicitly cited in the proof of:
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).
Theorem 1343 (Beppo Levi, monotone convergence in $\mathcal{L}^{1}$ )
is explicitly cited in the proof of:
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ).
Theorem 1345 (Fatou's lemma in $\mathcal{L}^{1}$ )
Theorem 1347 (Lebesgue, dominated convergence)
is explicitly cited in the proof of:
Theorem 1349 (Lebesgue, extended dominated convergence).
Theorem 1349 (Lebesgue, extended dominated convergence) is explicitly cited in the proof of: Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ ).

Theorem 1351 (Lebesgue, extended dominated convergence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ ),
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Theorem 1352 (Beppo Levi, monotone convergence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ ).
Theorem 1354 (Riesz-Fischer, convergence of Cauchy sequence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ ).
Lemma 1355 (subsequence of convergent sequence in $\mathcal{L}^{p}$ )
is explicitly cited in the proof of:
Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ).
Lemma 1356 (almost pointwise convergence and convergence in $\mathcal{L}^{p}$ ) is not yet used.

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[^0]:    Lemma 744 (some Borel subsets).
    Let $(X, \mathcal{T})$ be a topological space. Then, open and closed subsets are Borel subsets of $X$.
    Moreover, if the space $X$ is separable, then countable subsets of $X$ are Borel subsets of $X$.

[^1]:    Remark 1096. In this chapter, we study functions defined on a measurable space that take a finite

[^2]:    ${ }^{1}$ For instance, the case $I_{u}=\varnothing$ may correspond to $u \stackrel{\text { def. }}{=} 0_{E}$ when $0_{E} \notin\left\{u_{i} \mid i \in[0 . . n]\right\}$.

[^3]:    Remark 1245. From now on, the expressions involving integrals are taken in $\mathbb{R}$, thus functions also take almost all their values in $\mathbb{R}$.

