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# On the exponential generating function of labelled trees

## *Sur la série génératrice des arbres étiquetés*

Alin Bostan<sup>a</sup> and Antonio Jiménez-Pastor<sup>\*,b</sup>

<sup>a</sup> Inria, 1 rue Honoré d'Estienne d'Orves 91120 Palaiseau, France

<sup>b</sup> Johannes Kepler University Linz, Doctoral Program Computational Mathematics,  
DK15.

*E-mails:* Alin.Bostan@inria.fr (A. Bostan), ajpastor@risc.uni-linz.ac.at  
(A. Jiménez-Pastor).

**Abstract.** We show that the generating function of labelled trees is not  $D^\infty$ -finite.

**Résumé.** Nous montrons que la série génératrice des arbres étiquetés n'est pas  $D^\infty$ -finie.

**Keywords.** Combinatorics, generating functions, differential equations.

**2020 Mathematics Subject Classification.** 05A15, 12H05, 34A34.

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**Electronic supplementary material.** Supplementary material for this article is supplied as a separate archive available from the journal's website under article's URL or from the author.

This article is a draft (not yet accepted!)

### Version française abrégée

Nous montrons que la série génératrice exponentielle des arbres étiquetés,  $T(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n$ , n'est pas  $D^\infty$ -finie. En particulier, cela implique que, bien que  $T(x)$  vérifie des équations différentielles non-linéaires, ces dernières ne peuvent pas être « trop simples ». En particulier,  $T(x)$  n'est pas un quotient de deux fonctions  $D$ -finies (vérifiant des équations différentielles à coefficients polynomiaux), et plus généralement,  $T(x)$  ne vérifie aucune équation différentielle linéaire à coefficients des fonctions  $D$ -finies. La preuve repose ultimement sur un résultat de théorie de Galois différentielle. Plusieurs questions ouvertes sont proposées, dont une sur la nature de la série génératrice ordinaire des arbres étiquetés,  $\sum_{n \geq 1} n^{n-1} x^n$ .

\* Corresponding author.

## 1. Context and main result

A formal power series  $f(x) = \sum_{n \geq 0} a_n x^n$  in  $\mathbb{C}[[x]]$  is called *differentially finite*, or simply *D-finite* [22], if it satisfies a *linear* differential equation with polynomial coefficients in  $\mathbb{C}[x]$ . Many generating functions in combinatorics and many special functions in mathematical physics are D-finite [2, 9].

DD-finite series and more generally  $D^n$ -finite series are larger classes of power series, recently introduced in [12]. DD-finite power series satisfy linear differential equations, whose coefficients are themselves D-finite power series. One of the simplest examples is  $\tan(x)$ , which is DD-finite (because it satisfies  $\cos(x)f(x) - \sin(x) = 0$ ), but is not D-finite (because it has an infinite number of complex singularities, a property which is incompatible with D-finiteness). Another basic example is the exponential generating function of the Bell numbers  $B_n$ , which count partitions of  $\{1, 2, \dots, n\}$ , namely:

$$B(x) := \sum_{n \geq 0} \frac{B_n}{n!} x^n. \quad (1)$$

Indeed, it is classical [9, p. 109] that  $B(x) = e^{e^x - 1}$ , therefore  $B(x)$  is DD-finite. On the other hand,  $B(x)$  is not D-finite: this can be proved either analytically (using the too fast growth of  $B(x)$  as  $x \rightarrow \infty$ ), or purely algebraically (using [21], and the fact that the power series  $e^x$  is not algebraic).

More generally, given a differential ring  $R$ , the set of *differentially definable* functions over  $R$ , denoted by  $D(R)$ , is the differential ring of formal power series satisfying linear differential equations with coefficients in  $R$ . In particular,  $D(\mathbb{C}[x])$  is the ring of D-finite power series,  $D^2(\mathbb{C}[x]) := D(D(\mathbb{C}[x]))$  is the ring of DD-finite power series, and  $D^n(\mathbb{C}[x]) := D(D^{n-1}(\mathbb{C}[x]))$  is the ring of  $D^n$ -finite power series. We say that a power series  $f(x) \in \mathbb{C}[[x]]$  is  $D^\infty$ -finite if there exists an  $n$  such that  $f(x)$  is  $D^n$ -finite.

It is known [13] that  $D^n$ -finite power series form a strictly increasing sequence of sets and that any  $D^\infty$ -finite power series is *differentially algebraic*, in short *D-algebraic*, that is, it satisfies a differential equation, possibly *non-linear*, with polynomial coefficients in  $\mathbb{C}[x]$ . This class, as well as its complement (of *D-transcendental* series), are quite well studied [20, 23].

Let now  $(t_n)_{n \geq 0} = (0, 1, 2, 9, 64, 625, 7776, \dots)$  be the sequence whose general term  $t_n$  counts *labelled rooted trees* with  $n$  nodes. It is well known that  $t_n = n^{n-1}$ , for any  $n$ . This beautiful and non-trivial result is usually attributed to Cayley [6], although an equivalent result had been proved earlier by Borchardt [4], and even earlier by Sylvester, see [3, Chapter 4]. Due to the importance of the combinatorial class of trees, and to the simplicity of the formula, Cayley's result has attracted a lot of interest over the time, and it admits several different proofs, see e.g., [15, §4] and [1, §30]. One of the more conceptual proofs goes along the following lines (see [9, §II. 5.1] for details). Let

$$T(x) := \sum_{n \geq 0} \frac{t_n}{n!} x^n \quad (2)$$

be the exponential generating function of the sequence  $(t_n)_n$ . The class  $\mathcal{T}$  of all rooted labelled trees is definable by a *symbolic equation*  $\mathcal{T} = \mathcal{Z} \star \text{SET}(\mathcal{T})$  reflecting their recursive definition, where  $\mathcal{Z}$  represents the atomic class consisting of a single labelled node, and  $\star$  denotes the labelled product on combinatorial classes. This symbolic equation provides, by syntactic translation, an implicit equation on the level of exponential generating functions:

$$T(x) = x e^{T(x)}, \quad (3)$$

which can be solved using *Lagrange inversion*

$$t_n = n! \cdot [x^n] T(x) = n! \cdot \left( \frac{1}{n} [z^{n-1}] (e^z)^n \right) = n^{n-1}. \quad (4)$$

From (3), it follows easily that  $T(x)$  is D-algebraic and satisfies the non-linear equation

$$x(1 - T(x))T'(x) = T(x),$$

and from there, that the sequence  $(t_n)_{n \geq 0}$  satisfies the non-linear recurrence relation

$$t_{n+1} = \frac{n+1}{n} \cdot \sum_{i=1}^n \binom{n}{i} t_i t_{n-i+1}, \quad \text{for all } n \geq 1.$$

This recurrence can also be proved using (4), by taking  $y = n$ ,  $x = w = 1$  in Abel's identity [11]

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x(x + wk)^{k-1} (y - wk)^{n-k},$$

and then by isolating the term  $k = n$  in the resulting equality.

On the other hand, it is known that the power series  $T(x)$  is not D-finite, see [10, Theorem 7], or [8, Theorem 2]. This raises the natural question whether  $T(x)$  is DD-finite, or  $D^n$ -finite for some  $n \geq 2$ . Our main result is that this is not the case:

**Theorem 1.** *The power series  $T(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n$  in (2) is not  $D^\infty$ -finite.*

To our knowledge, this is the first explicit example of a natural combinatorial generating function which is provably D-algebraic but not  $D^\infty$ -finite. In particular, Theorem 1 implies that  $T(x)$  is not equal to the quotient of two D-finite functions, and more generally, that it does not satisfy any linear differential equation with D-finite coefficients.

## 2. Proof of the main result

Our proof of Theorem 1 builds upon the following recent result by Noordman, van der Put and Top.

**Theorem 2 ([17]).** *Assume that  $u(x) \in \mathbb{C}[[x]] \setminus \mathbb{C}$  is a solution of  $u' = u^3 - u^2$ . Then  $u$  is not  $D^\infty$ -finite.*

The proof of Theorem 2 is based on two ingredients. The first one is a result by Rosenlicht [19] stating that any set of non-constant solutions (in any differential field) of the differential equation  $u' = u^3 - u^2$  is algebraically independent over  $\mathbb{C}$  (see also [17, Prop. 7.1]); the proof is elementary. The second one [17, Prop. 7.1] is that any non-constant power series solution of an autonomous first-order differential equation with this independence property cannot be  $D^\infty$ -finite; the proof is based on differential Galois theory.

**Proof of Theorem 1.** We will use Theorem 2 and a few facts about the (principal branch of the) Lambert  $W$  function, satisfying  $W(x) \cdot e^{W(x)} = x$  for all  $x \in \mathbb{C}$ .

Recall [7] that the Taylor series of  $W$  around 0 is given by

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \dots.$$

In other words, our  $T(x)$  and  $W(x)$  are simply related by  $W(x) = -T(-x)$ .

The function defined by this series can be extended to a holomorphic function defined on all complex numbers with a branch cut along the interval  $(-\infty, -\frac{1}{e}]$ ; this holomorphic function defines the principal branch of the Lambert  $W$  function.

We can substitute  $x \mapsto e^{x+1}$  in the functional equation for  $W(x)$  obtaining then

$$W(e^{x+1})e^{W(e^{x+1})} = e^{x+1},$$

or, renaming  $Y(x) = W(e^{x+1})$ , we have a new functional equation:  $Y(x)e^{Y(x)-1} = e^x$ . From this equality it follows by logarithmic differentiation that  $Y'(x) \cdot (1 + Y(x)) = Y(x)$ .

Take now  $U(x) := \frac{1}{1+Y(x)} = \frac{1}{2} - \frac{1}{8}x + \frac{1}{64}x^2 + \frac{1}{768}x^3 + \dots$ . We have that

$$U'(x) = \frac{-Y'(x)}{(1+Y(x))^2} = \frac{-Y(x)}{(1+Y(x))^3} = U(x)^3 - U(x)^2.$$

By Theorem 2,  $U(x)$  is not  $D^\infty$ -finite. By closure properties of  $D^\infty$ -finite functions (see [13, Theorem 4] and [12, §3]), it follows that  $Y(x)$  is not  $D^\infty$ -finite either.

To conclude, note that by definition, for real  $x$  in the neighborhood of 0, we have  $W(x) = Y(\log(x) - 1)$ , and by Theorem 10 in [13], it follows that  $W(x)$  and  $T(x)$  are not  $D^\infty$ -finite either, proving Theorem 1.  $\square$

### 3. Open questions

The class of  $D$ -finite power series is closed under Hadamard (term-wise) product. This is false for  $D^\infty$ -finite power series; for instance, Klazar showed in [14] that the ordinary generating function  $\sum_{n \geq 0} B_n x^n$  of the Bell numbers is not differentially algebraic, contrary to its exponential generating function (1), which is  $DD$ -finite.

Moreover, it was conjectured by Pak and Yeliussizov [18, Open Problem 2.4] that this is an instance of a more general phenomenon.

**Conjecture 3 ( [18, Open Problem 2.4]).** *If for a sequence  $(a_n)_{n \geq 0}$  both ordinary and exponential generating functions  $\sum_{n \geq 0} a_n x^n$  and  $\sum_{n \geq 0} a_n \frac{x^n}{n!}$  are  $D$ -algebraic, then both are  $D$ -finite. (Equivalently,  $(a_n)_{n \geq 0}$  satisfies a linear recurrence with polynomial coefficients in  $n$ .)*

This conjecture has been recently proven for large (infinite) classes of generating functions [5]. However, the very natural example of the generating function for labelled trees escapes the method in [5].

We therefore leave the following as an open question.

**Open question 1.** *Is the power series  $\sum_{n \geq 1} n^{n-1} x^n$   $D^\infty$ -finite? Is it at least differentially algebraic?*

According to Conjecture 3, the answer should be “no” for both questions in Open question 1.

Another natural question concerns the generating function for partition numbers:

$$\sum_{n \geq 0} p_n x^n := \prod_{n \geq 1} \frac{1}{1 - x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \dots,$$

which is known to be differentially algebraic [16].

**Open question 2.** *Is it true that  $\sum_{n \geq 0} p_n x^n$  is not  $D^\infty$ -finite?*

One may also ask for the nature of the exponential variant of the generating function for partition numbers.

**Open question 3.** *Is the power series  $\sum_{n \geq 0} \frac{p_n}{n!} x^n$   $D^\infty$ -finite, or at least differentially algebraic?*

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