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Abstract

We extend a sliding mode control methodology to linear evolution equations with uncertain but bounded inputs and noise in observations. We first describe the reachability set of the state equation in the form of an infinite-dimensional ellipsoid, and then steer the minimax center of this ellipsoid toward a finite-dimensional sliding surface in finite time by using the standard sliding mode output-feedback controller in equivalent form. We demonstrate that the designed controller is the best (in the minimax sense) in the class of all measurable functionals of the output. Our design is illustrated by two numerical examples: output-feedback stabilization of linear delay equations, and control of moments for an advection-diffusion equation in 2D.

Keywords: evolution equations, sliding mode, minimax, Riccati equations,

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1. Introduction

Robust output-based feedback control algorithms are required for many practical applications. The output-based sliding mode control design methodology is well-developed for finite dimensional systems (see, for example, [1], [2], [3], [4] and references therein). Infinite-dimensional (distributed parameter) systems

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are widely used in practise, e.g., to model flexible robots, controlled turbulent flows, combustion, and chemical processes. The sliding mode methodology can also be used to design controllers for such complicated systems [5], [6]. We refer the reader to [7, 8, 9, 10, 11, 12] for an extensive overview of the recent achievements in this field.

We stress that, in practice, it is quite difficult to apply the state of the art sliding mode methods in the case of noisy measurements (see, [13, 14]) and/or mismatched disturbances (see, [15, 16, 17]). The aim of this paper is to propose a mathematically sound extension of the sliding mode control methodology allowing one to deal with the aforementioned cases efficiently. Specifically, we consider conventional (first-order) sliding mode control principles and study the problem of observer-based sliding mode control design for a plant described by a linear evolution equation in a Hilbert space with additive exogenous disturbances and L^2-bounded deterministic measurement noise. Note that, in this case, the solution of the classical sliding model control problem does not exist, i.e., it is impossible to ensure the ideal/exact sliding mode (even in the finite-dimensional case [18]) due to the noise in the measurements. Following [18, 19, 20, 21] we propose to generalize the notion of the solution of the classical sliding mode control problem for linear evolution equations, i.e., to construct a control law u steering the state's motion as close as possible (in the minimax sense) to the selected sliding surface. To design such u we first provide a dual description of the reachability set for a linear evolution equation, and then solve the following minimax control problem: find a feedback control u steering the minimax center of the reachability set towards the sliding surface. The dual description of the reachability set relies upon the minimax framework [22, 23, 24, 25] and a duality argument [26, 27]. As it turns out, the optimal control for this problem combines a linear observer whose gain is given by the solution of a differential Riccati equation, with a linear, memoryless, but time-varying feedback law. In order to implement the proposed sliding mode control design, we approximate the solution of the differential Riccati equation, and we discuss the convergence of the approximating sequence. We apply the theory presented in this paper to two examples. The first example is a delayed differential equation, where the infinite dimensionality is caused by the delay-operator, which is discretized by using averaging (over time). The second example is an advection-diffusion equation over two spatial dimensions, where we use a spectral-element method for discretization. We solve the resulting large-scale finite-dimensional differential Riccati equation using Bernoulli substitutions and an implicit midpoint rule.

The paper is organized as follows. The next section presents the problem statement and basic assumptions. The minimax observer for linear systems is discussed in Section III. The problem of control design is studied in Section IV. Next the numerical simulation results and conclusions are provided.

Throughout the paper the following notations are used: H, H_u, H_d, H_y are abstract Hilbert spaces, $\langle x, y\rangle_H$ denotes the canonical inner product of H, $\|x\|_H^2 := \langle x, x\rangle_H$, $\mathscr{L}(H, H)$ denotes the space of linear continuous operators from H to H, $A^\star$ denotes the adjoint of a linear operator A, $\mathscr{D}(A)$ denotes the domain of A, I denotes the identity operator of the corresponding space, $L^2(0, T, H)$ denotes the space of square-integrable functions on $(0, T)$ with values in H.

2. Problem statement

Consider a linear evolution equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Dd(t)\,, t \geq 0\,, x(0) = x_0\,, \tag{1}$$

where $A : \mathscr{D}(A) \subset H \to H$ generates a strongly continuous semigroup $G(t)$ on a Hilbert space H (see [28] or [29]), $x_0 \in H$ is a given initial condition, $u \in L^2(0, T, H_u)$ is a control function, $d \in L^2(0, T, H_d)$ an uncertain disturbance, and $B \in \mathscr{L}(H_u, H)$, $D \in \mathscr{L}(H_d, H)$ are given bounded operators. Then

$$x(t) = G(t)x_0 + \int_0^t G(t-s)(Dd(s) + Bu(s))ds \tag{2}$$

is the mild solution of (1) and is continuous on $[0, T]$ (see [28, p.104, Lem. 3.1.5]). Note that this mild solution is unique and it coincides with a so-called weak solution used in the study of partial differential equations (see [28, p.106, Thm. 3.1.7]).

The output of (1), $y(t) \in H_y$ is measured in the following form:

$$y(t) = Cx(t) + w(t)\,, t \in [0, T)\,, \tag{3}$$

where $C \in \mathscr{L}(H, H_y)$ is an observation operator, which represents a mathematical model of a gauge, and $w \in L^2(0, T, H_y)$ is unknown deterministic measurement noise.

We further assume that x_0, d, w are uncertain and belong to the following bounding set:

$$\mathscr{E}(T) := \{(x_0, d, w) : \rho_T(x_0, d, w; S, Q, R) \leq 1\}\,, \tag{4}$$

where

$$\rho_t(x, d, w; S, Q, R) := \langle Sx, x\rangle_H + \int_0^t \langle Qd(s), d(s)\rangle_{H_d} ds + \int_0^t \langle Rw(s), w(s)\rangle_{H_y} ds\,,$$

and S, Q, R are given self-adjoint positive definite bounded linear operators in H, H_d and H_y respectively. Clearly, $\mathscr{E}(T) \subset H \times L^2(0, T, H_d) \times L^2(0, T, H_y)$, and ρ_T defines a new norm in the space $H \times L^2(0, T, H_d) \times L^2(0, T, H_y)$, and $\mathscr{E}(T)$ represents the unit ball of this space w.r.t. to ρ_T. In what follows we suppose that H_u and H_y are abstract Hilbert spaces.

The aim of this paper is, for a given finite-rank linear operator $F : H \to H_u$ and any (fixed) time $T < +\infty$, to design a control law $u \in L^2(0, T, H_u)$ in the form of a functional of the output which, for all $(x_0, d, w) \in \mathscr{E}(T)$, steers the state vector of (1) *towards the null-space of F (as close as possible in H_u).* More specifically, given F such that $FB : H_u \to H_u$ is a linear bounded invertible operator, we aim at finding u as a solution of a minimax version of the classical Mayer optimal control problem:

$$\begin{gathered}\inf_u \sup_{(x_0,d,w)\in\mathscr{E}(T)} \|Fx(T)\|_{H_u} \\ \text{s.t. (1) - (3)}\end{gathered} \tag{5}$$

We recall that the classical sliding mode control problem is (see, [3, 10]) to find a feedback control law u which (i) steers the state of (1) towards a given linear hyperplane $Fx = 0$, and (ii) guarantees that the state does not leave this plane, provided FB is a linear bounded invertible operator. It is worth noting [2, 3] that the latter condition is necessary (in the finite-dimensional case) for existence of a control law, which ensures sliding mode on the null-space of F. We stress that reaching $\{x \mid Fx = 0\}$ exactly may not be possible (as it is demonstrated by our examples below) due to the presence of generic L^2-disturbances, instead (5) guarantees that the state will be "as close as possible."

3. Dual description of the reachability set

According to the classical methodology of the sliding mode control design, the precise knowledge of the so-called sliding variable $\sigma(t) := Fx(t)$ is required in order to ensure the motion of the system (1) on the surface $\{x \mid Fx = 0\}$. In the considered case this information is not available as the output $y(t)$ is incomplete and subject to deterministic noise, and the state equation is subject to uncertain deterministic disturbances. In the following proposition we construct the a priori reachability set of the evolution equation (1), i.e., the set of all the states of (1) which are compatible with all possible outputs y and uncertainty description $\mathscr{E}(T)$. This representation is then used to solve (5).

Theorem 1. *Assume that x is a mild solution of (1) for some $(x_0, d, w) \in \mathscr{E}(T)$. Then, for any $t^* \in [0, T]$ the following estimate holds true:*

$$\sup_{(x_0,d,w)\in\mathscr{E}(t^*)} |\langle l, x(t^*) - \hat{x}(t^*)\rangle_H| = \langle l, P(t^*)l\rangle_H^{\frac{1}{2}} \qquad \forall l \in H\,, \tag{6}$$

where

- *the linear bounded self-adjoint positive definite operator P is the unique solution of the infinite-dimensional differential Riccati equation*

$$\begin{aligned}\frac{d}{dt}\langle P(t)v, q\rangle_H &= \langle P(t)A^\star v, q\rangle_H + \langle P(t)v, A^\star q\rangle_H + \\ &+ \langle DQ^{-1}D^\star v, q\rangle_H - \langle P(t)C^\star RCP(t)v, q\rangle_H \end{aligned} \tag{7}$$

  with $P(0) = S^{-1}$ (for all $v, q \in \mathscr{D}(A)$), and

- $\hat{x}$ *is the unique mild solution of the following evolution equation:*

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = A\hat{x}(t) + P(t)C^\star R(y(t) - C\hat{x}(t)) + Bu(t), \\ \hat{x}(0) = 0\,, \end{cases} \tag{8}$$

Proof. *A. Existence of solutions*

Let us first note that (7) has the unique solution P, i.e., $P(t)$ is a linear bounded self-adjoint positive definite operator on H for any finite $t \in [0,T]$, and $t \mapsto P(t)x$ is a continuous vector-valued function for any $x \in H$ [29, p.393,Thm. 2.1]. This fact allows us to represent the unique mild solution of (8) by means of an evolution operator $\Phi(t,s)$ generated by $A - P(t)C^\star RC$ (see [29, p.138]). Indeed, it has been shown in [29, p.135, Lem. 3.2] that the unique mild solution of the evolution equation $u' = Au + F(t)u + f$, $u(0) \in H$ exists, provided $f \in L^2(0,T,H)$ and $F(t)$ is a strongly continuos function with values in $\mathscr{L}(H)$, and it coincides with the mild solution. Moreover, the strong solution of the aforementioned equation can be represented in terms of a so-called evolution operator generated by $A + F(t)$ (see [29, p.139, f.(3.20)]). Since $t \mapsto P(t)$ is a strongly continuous function with values in $\mathscr{L}(H)$ and $t \mapsto P(t)C^\star Ry(t) + Bu(t) \in L^2(0,T,H)$ it follows that there exists an evolution operator $\Phi(t,s)$ generated by $A - P(t)C^\star RC$ such that:

- $\Phi(t,s)$ is a bounded linear operator in H, strongly continuous for all $s \le t$ and

$$\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s)\,, \quad s \le \tau \le t\,, \quad \Phi(t,t) = I$$

- the unique mild solution of the evolution equation (8), $\hat{x}(0) = 0$ and

$$\frac{d}{dt}\langle \hat{x}(t), v\rangle_H = \langle \hat{x}(t), (A^\star - C^\star RCP(t))\,v\rangle_H + \langle P(t)C^\star Ry(t) + Bu(t), v\rangle_H$$

is given by

$$\hat{x}(t) = \int_0^t \Phi(t,s)(P(s)C^\star Ry(s) + Bu(s))ds\,. \tag{9}$$

In what follows we will be using the following representation: $\hat{x} = \hat{x}_n + x_u$, where $\dfrac{d\hat{x}_n}{dt} = A\hat{x}_n + PC^\star R(y_n(t) - C\hat{x}_n)$, $\hat{x}_n(0) = 0$ and $\dot{x}_u = Ax_u + Bu$, $x_u(0) = 0$, provided $y_n(t) = y(t) - Cx_u(t)$.

B. Optimality of the estimate

In order to prove (6), we first fix $t^* \in (0,T]$. Then, for any $\tilde{v} \in L^2(0,t^*,H_y)$, we define

$$J_{t^*}(\tilde{v}) := \sup_{(x_0,d,w)\in\mathscr{E}(t^*)} |\langle l, x(t^*)\rangle_H - \int_0^{t^*} \langle \tilde{v}(t), y_n(t)\rangle_{H_y} dt - c_{t^*}(u)| \qquad (10)$$

where $c_{t^*}(u) := \langle l, x_u(t^*)\rangle_H$. To prove (6) we will first show that

$$\inf_{\tilde{v}\in L^2(0,t^*,H_y)} J_{t^*}(\tilde{v}) = \langle l, P(t^*)l\rangle^{\frac{1}{2}}, \forall l \in H, \qquad (11)$$

and then we will prove that there exist a unique $\hat{v} \in \operatorname{arginf}_{\tilde{v}\in L^2(0,t^*,H_y)} J_{t^*}(\tilde{v})$ such that:

$$\langle l, \hat{x}_n(t^*)\rangle_H = \int_0^{t^*} \langle \hat{v}(t), y_n(t)\rangle_{H_y} dt, \qquad (12)$$

which implies (6) by virtue of (11) and the equality: $\int_0^{t^*} \langle \hat{v}(t), y_n(t)\rangle_{H_y} dt + c_{t^*}(u) = \langle l, \hat{x}(t^*)\rangle_H$.

Let us prove (11). We note that $x = x_n + x_u$ where $\dot{x}_n = Ax_n + Dd$, $x_n(0) = x_0$ and so

$$\langle l, x(t^*)\rangle_H - \int_0^{t^*} \langle \tilde{v}(t), y_n(t)\rangle_{H_y} dt - c_{t^*}(u) = \langle l, x_n(t^*)\rangle_H - \int_0^{t^*} \langle \tilde{v}(t), y_n(t)\rangle_{H_y} dt. \qquad (*)$$

Let us further transform the latter formula. By using the semigroup representation (2) for x_n, namely $x_n(t) = G(t)x_0 + L_t Dd$, where $L_t q := \int_0^t G(t-s)q(s)ds$, we compute:

$$\langle l, x_n(t^*)\rangle_H = \langle G^\star(t^*)l, x_0\rangle_H + \int_0^{t^*} \langle G^\star(t^*-s)l, Dd(s)\rangle_H ds,$$

and

$$\begin{aligned} \int_0^{t^*} \langle \tilde{v}(t), y_n(t)\rangle_{H_y} dt = \int_0^{t^*} \langle G^\star(t)C^\star\tilde{v}(t), x_0\rangle_H dt + \\ \int_0^{t^*} \langle C^\star\tilde{v}(t), L_t Dd(t)\rangle_H dt + \int_0^{t^*} \langle \tilde{v}, w\rangle_{H_y} ds. \end{aligned} \qquad (**)$$

We note that

$$\begin{aligned} \int_0^{t^*} \langle C^\star\tilde{v}(t), L_t Dd(t)\rangle_H dt = \langle C^\star\tilde{v}, L_t Dd\rangle_{L^2(0,T,H)} = \\ \langle L_t^\star C^\star\tilde{v}, Dd\rangle_{L^2(0,T,H)} = \\ \int_0^{t^*} \left\langle \int_t^{t^*} G^\star(s-t)C^\star\tilde{v}(s)ds, Dd(t)\right\rangle_H dt. \end{aligned} \qquad (***)$$

Now, let us define the adjoint variable

$$z(t) = G^\star(t^* - t)l - \int_t^{t^*} G^\star(s-t)C^\star\tilde{v}(s)ds\,. \tag{13}$$

By subtracting $(**)$ from $(*)$ and taking into account $(***)$ and the definition of z we get:

$$\left(\langle l, x_n(t^*)\rangle_H - \int_0^{t^*} \langle\tilde{v}(t), y_n(t)\rangle_{H_y} dt\right)^2$$
$$= \left(\langle z(0), x_0\rangle_H + \int_0^{t^*}\langle D^\star z(t), d(t)\rangle_H dt - \int_0^{t^*}\langle\tilde{v}(t), w(t)\rangle_{H_y} dt\right)^2.$$

Hence, we find that

$$\begin{aligned} J^2_{t^*}(\tilde{v}) &= \sup_{(x_0,d,w)\in\mathscr{E}(t^*)}\left(\langle l, x_n(t^*)\rangle_H - \int_0^{t^*}\langle\tilde{v}(t), y_n(t)\rangle_{H_y} dt\right)^2 \\ &= \sup_{(x_0,d,w)\in\mathscr{E}(t^*)} \langle \begin{bmatrix} z(0) \\ D^\star z \\ -\tilde{v}\end{bmatrix}, \begin{bmatrix} x_0 \\ d \\ w\end{bmatrix}\rangle^2_{H\times L^2(0,t^*,H_d)\times L^2(0,T,H_y)}\,. \end{aligned} \tag{14}$$

Now, the 2nd line of the latter formula represents the support functional of the strictly convex bounded set $\mathscr{E}(t^*)$, and hence, for any $\tilde{v} \in L^2(0, t^*, H_y)$, there exists the unique tuple $(\bar{x}_0, \bar{d}, \bar{w}) \in \mathscr{E}(t^*)$ such that the sup is attained. Indeed, we can compute the latter sup by applying the generalized Cauchy-Schwartz inequality:

$$\left\langle \begin{bmatrix} z(0) \\ D^\star z \\ -\tilde{v}\end{bmatrix}, \begin{bmatrix} x_0 \\ d \\ w\end{bmatrix}\right\rangle^2_{H\times L^2(0,t^*,H_d)\times L^2(0,T,H_y)} \le$$
$$\rho_{t^*}(z(0), D^\star z, -v; S^{-1}, Q^{-1}, R^{-1})\rho_{t^*}(x_0, d, w; S, Q, R)\,.$$

Since $\rho_{t^*}(x_0, d, w; S, Q, R) \le 1$ for any $(x_0, d, w) \in \mathscr{E}(t^*)$ we find that $\sup_{\mathscr{E}(t^*)}$ in (14) is attained at

$$\begin{aligned} &\bar{x}_0 := \tfrac{S^{-1}z(0)}{\sqrt{\bar{\rho}_{t^*}}}\,, \bar{d} := \tfrac{Q^{-1}D^\star z}{\sqrt{\bar{\rho}_{t^*}}}\,, \bar{w} := \tfrac{-R^{-1}\tilde{v}}{\sqrt{\bar{\rho}_{t^*}}}\,, \\ &\bar{\rho}_{t^*} := \rho_{t^*}(z(0), D^\star z, -v; S^{-1}, Q^{-1}, R^{-1}), \end{aligned} \tag{15}$$

and

$$\begin{aligned} &J^2_{t^*}(\tilde{v}) = \rho_{t^*}(z(0), D^\star z, -\tilde{v}; S^{-1}, Q^{-1}, R^{-1}) = \\ &\langle S^{-1}z(0), z(0)\rangle_H + \int_0^{t^*}\langle Q^{-1}D^\star z, D^\star z\rangle_H + \langle R^{-1}\tilde{v}, \tilde{v}\rangle_H dt\,. \end{aligned} \tag{16}$$

This latter representation shows that to find $\hat{v} \in \operatorname{arginf}_{\tilde{v}\in L^2(0,t^*,H_y)} J_{t^*}(\tilde{v})$ one needs to solve an LQ-control problem with cost $J^2_{t^*}(\tilde{v})$ along mild solutions of the adjoint equation (13). Following [30, p.263,5.2] we can represent the unique solution $\hat{v}$ of the latter LQ-control problem as follows: $\hat{v}(t) = RCP(t)z(t)$, provided $P(t)$ solves (7). By substituting $\hat{v}$ into (13) we get the following evolution equation:

$$\frac{d}{dt}\langle \hat{z}(t), q\rangle_H = \langle \hat{z}, (-A + PC^\star RC)q\rangle_H\,, \langle \hat{z}(t^*) - l, q\rangle_H = 0, \\ \forall q \in \mathscr{D}(A)\,. \tag{17}$$

Following [30, p.255] we can represent the unique mild solution of this equation in the following form: $\hat{z}(t) = \Phi^\star(t^*, t)l$ where the evolution operator Φ has been defined above in subsection A. The validity of (11) follows from the identity: $J^2_{t^*}(\hat{v}) = \langle l, P(t^*)l\rangle_H$ (see [30, p.268]).

To conclude the proof we need to show (12). By using the operator Φ defined above we can represent $\hat{x}$ as follows:

$$\hat{x}_n(t) = \int_0^t \Phi(t,s)P(s)C^\star R y_n(s)ds$$

and so

$$\begin{aligned}\langle l, \hat{x}_n(t^*)\rangle_H &= \int_0^{t^*} \langle \Phi^\star(t^*, s)l, P(s)C^\star R y_n(s)\rangle_H ds \\ &= \int_0^{t^*} \langle RCP(s)\hat{z}(s), y_n(s)\rangle_{H_y} ds \\ &= \int_0^{t^*} \langle \hat{v}(t), y_n(t)\rangle_{H_y} dt\,.\end{aligned}$$

■

Using [30, p.339, Th.6.8.3] the following corollary can also be proven.

Corollary 1. *Assume that* (A, D) *and* $(A^\star, C^\star)$ *are exponentially stabilizable. Then*

$$\lim_{t\to+\infty} |\langle l, x(t) - \hat{x}(t)\rangle_H| \le \langle l, P^\infty l\rangle_H^{\frac{1}{2}}\,, \qquad \forall l \in H\,, \tag{18}$$

where P^∞ *is the unique self-adjoint solution of the algebraic Riccati equation:*

$$\langle P^\infty v, A^\star v\rangle_H + \langle A^\star v, P^\infty v\rangle_H + \\ \langle Q^{-1}D^\star v, D^\star v\rangle_H - \langle RCP^\infty v, CP^\infty v\rangle_H = 0\,. \tag{19}$$

In addition, $A - P^{\infty}C^{\star}RC$ generates an exponentially stable semigroup.

It is worth noting that (6) is describing an ellipsoid, which is centered at vector $\hat{x}(T)$ with axes defined by eigenfunctions of $P(T)$. This ellipsoid is, in fact, the worst-case realisation of the reachability set of (1), i.e., it takes into account all $(x_0, d, w) \in \mathscr{E}(T)$. The estimate (18) describes an ellipsoid which contains all the states of (1) in the limit $t \to \infty$. Finally, we stress that $P(t)$ does not depend on the control $u(t)$. This suggests to design the controller u as a function of the center of the ellipsoid, $\hat{x}$.

4. Control Design

Denoting the sliding variable by $\sigma = Fx$ we derive

$$\sigma(T) = Fx(T) = \hat{\sigma}(T) + Fe(T),$$
$$|\langle l, e(T)\rangle_H| \le \langle l, P(T)l\rangle_H^{\frac{1}{2}}, \quad \forall l \in H,$$

where $\hat{\sigma}(T) = F\hat{x}(T)$, and $\hat{x}$ satisfies (8).

Theorem 2. *If the control u verifies the following equality:*

$$\hat{\sigma}(T) = 0 \tag{20}$$

then it solves the minimax control problem (5)*.*

Proof. Let us first transform the cost function

$$\tilde{J}(u) := \sup_{(x_0,d,w)\in\mathscr{E}(T)} \|Fx(T)\|_{H_u}. \tag{21}$$

Since $\|Fx(T)\|_{H_u} = \sup_{\|\ell\|_{H_u}=1} \langle \ell, Fx(T)\rangle_{H_u}$ we can substitute this latter representation into the right hand side of (21) and swap the sup operations, i.e., we can write:

$$\tilde{J}(u) = \sup_{\|\ell\|_{H_u}=1} \sup_{(x_0,d,w)\in\mathscr{E}(T)} \langle \ell, Fx(T)\rangle_{H_u}.$$

Now, recall from (10)-(16) that for $t^* = T$

$$\begin{gathered} J_T^2(\hat{v}) = \sup\nolimits_{(x_0,d,w)\in\mathscr{E}(T)} \Big(\langle l, x(T)\rangle_H - \int_0^T \langle \hat{v}(t), y_n(t)\rangle_{H_y} dt - c_T(u) \Big)^2 = \\ \rho_T(\hat{z}(0), D^{\star}\hat{z}, -\hat{v}; S^{-1}, Q^{-1}, R^{-1}) = \langle l, P(T)l\rangle_H, \end{gathered} \tag{22}$$

where $\hat{v} = RCP\hat{z}$ and $\hat{z}$ is the unique mild solution of (17). Moreover, according to (15) we have that the sup in (22) is attained at:

$$\begin{gathered}\bar{x}_0 := \frac{S^{-1}\hat{z}(0)}{\sqrt{\hat{\rho}_T}}\,, \quad \bar{d} := \frac{Q^{-1}D^\star\hat{z}}{\sqrt{\hat{\rho}_T}}\,, \quad \bar{w} := \frac{-R^{-1}\hat{v}}{\sqrt{\hat{\rho}_T}}\,, \\ \hat{\rho}_T := \rho_T(\hat{z}(0), D^\star\hat{z}, -\hat{v}; S^{-1}, Q^{-1}, R^{-1})\,.\end{gathered} \tag{23}$$

Denote by y' the output $y' = Cx' + \bar{w}$, which corresponds to the solution x' of (1) with initial condition $\bar{x}_0$, disturbance $\bar{d}$ and noise $\bar{w}$, and let $\hat{x}(\cdot, y')$ denote the unique mild solution of (8) which corresponds to y'. It then follows that

$$\begin{gathered}\sup_{(x_0,d,w)\in\mathscr{E}(T)} \langle l, x(T) - \hat{x}(T)\rangle_H^2 = \\ \langle l, x'(T) - \hat{x}(T,y')\rangle_H^2 = \langle l, P(T)l\rangle_H\,.\end{gathered} \tag{24}$$

Thus, we can write

$$\begin{gathered}\sup_{(x_0,d,w)\in\mathscr{E}(T)} \langle \ell, Fx(T)\rangle_{H_u} \geq \langle \ell, Fx'(T)\rangle_{H_u} = \\ \langle \ell, F\hat{x}(T,y')\rangle_{H_u} + \langle \ell, Fx'(T) - F\hat{x}(T,y')\rangle_{H_u} \\ \overset{\text{by (24)}}{=} \langle \ell, F\hat{x}(T,y')\rangle_{H_u} + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}\end{gathered}$$

and so

$$\tilde{J}(u) \geq \sup_{\|\ell\|_{H_u}=1} \left(\langle F^\star\ell, \hat{x}(T,y')\rangle_H + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}} \right) \tag{25}$$

for any $u \in L^2(0,T,H_u)$. Let u_0 be chosen so that $F\hat{x}(T) = 0$. Then

$$\tilde{J}(u_0) = \sup_{\|\ell\|_{H_u}=1} \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}} = \|(FP(T)F^\star)^{\frac{1}{2}}\|_{H_u}\,. \tag{26}$$

Indeed, recalling (24) we get:

$$\begin{gathered}\tilde{J}(u_0) = \sup_{\|\ell\|_{H_u}=1} \sup_{(x_0,d,w)\in\mathscr{E}(T)} \langle \ell, Fx(T) - F\hat{x}(T)\rangle_{H_u} = \\ \sup_{\|\ell\|_{H_u}=1} \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}} = \|(FP(T)F^\star)^{\frac{1}{2}}\|\,,\end{gathered}$$

where $\|(FP(T)F^\star)^{\frac{1}{2}}\|$ denotes the induced operator norm of the square-root of the finite-rank non-negative operator $FP(T)F^\star$. We claim that $\tilde{J}(u) > \tilde{J}(u_0)$ for any u such that $F\hat{x}(T) \neq 0$. To prove this it is enough to show that

$$\begin{gathered}\sup_{\|\ell\|_{H_u}=1} \left(\langle \ell, F\hat{x}(T,y')\rangle_{H_u} + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}} \right) \geq \\ \|(FP(T)F^\star)^{\frac{1}{2}}\|_{H_u} = \tilde{J}(u_0)\end{gathered} \tag{27}$$

and then apply (25). Let us prove (27): $\langle F^\star\ell,\, P(T)F^\star\ell\rangle_H \ge 0$ for any ℓ, but $\langle F^\star\ell, \hat{x}(T)\rangle_H$ can be either positive or negative, depending on ℓ. There exists $\tilde{\ell}$ such that $\|\tilde{\ell}\|_{H_u} = 1$ and $\langle F^\star\tilde{\ell}, \hat{x}(T)\rangle_H \ge 0$. Denote the set of all such $\tilde{\ell}$ by $\mathcal{S}_+$. We get:

$$\begin{aligned}\sup_{\|\ell\|_{H_u}=1}\left(\langle \ell, F\hat{x}(T,y')\rangle_{H_u} + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}\right) =\\ \sup_{\ell\in\mathcal{S}_+}(\langle F^\star\ell, \hat{x}(T,y')\rangle_H + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}) \ge\\ \langle F^\star\ell, \hat{x}(T,y')\rangle_H + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}} \ge\\ \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}, \forall \ell\in\mathcal{S}_+\,.\end{aligned}$$

and so

$$\begin{aligned}\sup_{\|\ell\|_{H_u}=1}\left(\langle \ell, F\hat{x}(T,y')\rangle_{H_u} + \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}\right) \ge\\ \sup_{\ell\in\mathcal{S}_+}\langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}\}\,.\end{aligned}$$

On the other hand, $\mathcal{S}_+$ is the intersection of the sphere $\{\ell : \|\ell\| = 1\}$ and the (closed) half-space $H_+ := \{\ell : \phi(\ell) \ge 0\}$ where $\phi(\ell) = \langle F^\star\ell, \hat{x}(T,y')\rangle_H$. Clearly, if $\ell \in H_+$ and $\phi(\ell) > 0$ then $-\ell \in H_-$ where H_- is the (open) half-space complement to H_+, i.e., $H_- \cup H_+ = H_u$ and $H_- \cap H_+ = \varnothing$. Since the functional $\ell \mapsto q(\ell) := \langle F^\star\ell, P(T)F^\star\ell\rangle_H^{\frac{1}{2}}$ is even, i.e., $q(\ell) = q(-\ell)$, we conclude that $\sup_{\mathcal{S}_+} q = \sup_{\|\ell\|=1} q$. This proves (27) which in turn proves that $\tilde{J}(u) > \tilde{J}(u_0)$ for any u such that $F\hat{x}(T) \ne 0$. The latter proves (20). ■

Usually (see, e.g. [10], [9]), additional technical considerations are required in order to apply a discontinuous sliding mode control and prove the existence of solutions in this case. In contrast, the theorem allows us to construct a continuous feedback control verifying the condition (20). To this end without loss of generality assume that the range of F is spanned by orthonormal vectors $\phi_1 \dots \phi_N$ so that $Fx = \sum_{i=1}^N \langle Fx, \phi_i\rangle\phi_i$.

Corollary 2. *If $F^\star\phi_i \in \mathscr{D}(A^\star)$, $i = 1\dots N$ then the following control functional*

$$u_{eq}(t) = -(FB)^{-1}F\left[A\hat{x}(t) + P(t)C^\star R(y(t) - C\hat{x}(t))\right] \tag{28}$$

solves (5)*. The minimal possible worst-case deviation of the state vector of* (1) *from the sliding hyperplane is given by*

$$\max_{(x_0,d,w)\in\mathscr{E}(T)} \|Fx(T)\| = \|FP(T)F^\star\|_{H_u}\,. \tag{29}$$

and the maximum is attained at the worst-case realizations of x_0, d, w given in (15).

Proof. Note that by plugging u_{eq} into (8) one obtains a perturbed operator $(I - B(FB)^{-1}F)(A - PC^\star RC)$. The term involving P is uniformly continuous, hence the perturbed operator will be a generator provided so is the term involving A. Clearly, $B(FB)^{-1}FAx = \sum_{i=1}^{N}\langle x, A^\star F^\star\phi_i\rangle B(FB)^{-1}\phi_i$ hence $B(FB)^{-1}FA \in \mathscr{L}(H)$ by assumption, and thus $A - B(FB)^{-1}FA$ generates a C_0-semigroup.

Now, $\forall v \in H_u$ the feedback u_{eq} ensures

$$\begin{aligned}&\langle F\hat{x}(t), v\rangle_{H_u} = \\ &\quad\int_0^t \langle FA\hat{x}(s) + FP(s)C^\star R(y(s) - C\hat{x}(s)) + FBu_{eq}(s), v\rangle_{H_u} ds = \\ &\int_0^t \langle (F - FB(FB)^{-1}F)[A\hat{x}(s) + P(s)C^\star R(y(s) - C\hat{x}(s))], v\rangle_{H_u} ds = 0.\end{aligned}$$

Since $\hat{x}$ starts on the linear sliding hypersurface $\{F\hat{x} = 0\}$ as $\hat{x}(0) = 0$ it follows that the minimax center of the reachability set stays on the hyperplane $F\hat{x} = 0$ and the actual state $x(t)$ fluctuates in the ellipsoid centered at $\hat{x}$, i.e.,

$$|\langle Fx(t), v\rangle_{H_u}| \le \langle F^\star v, P(t)F^\star v\rangle_H^{\frac{1}{2}}\,, \quad \forall v \in H_u\,.$$

Moreover, (21) and (26) imply (29). The very last claim can be easily deduced from (22) and (23). ■

Remark 1. *In fact, the feedback u_{eq} is an infinite-dimensional analog of what is known as "equivalent control" in sliding mode control, which can be found explicitly: indeed, u_{eq} depends on P and $\hat{x}$ which can be computed numerically (or even analytically in some cases). Note that the speed at which $x(t)$ approaches the sliding hyperplane is proportional to the speed of the decay of the eigenvalues of $FPF^\star$. In the infinite-horizon case, the actual state of the plant reaches the sliding surface exactly, provided $\langle F^\star v, P^\infty F^\star v\rangle_H^{\frac{1}{2}} = 0$ for any $v \in H_u$ (see* (18)*).*

5. Approximation of Solutions of Infinite-dimensional Operator Differential Riccati Equations (ODRE)

To implement the proposed sliding mode control design and perform numerical experiments, one needs to approximate $P(t)$, the unique solution of

$$\frac{d}{dt}\langle P(t)v,q\rangle_H = \langle P(t)A^\star v,q\rangle_H + \langle P(t)v,A^\star q\rangle_H + \\ \langle DQ^{-1}D^\star v,q\rangle_H - \langle P(t)C^\star RCP(t)v,q\rangle_H, \qquad P(0)=P_0, \quad (30)$$

where $v,q\in\mathscr{D}(A^\star)$, $t\in[0,T]$, and P_0 is nonnegative and selfadjoint. Note that this is equation (7) with $P_0 = S^{-1}$. Without loss of generality, take $Q = I$ and $R = I$. We follow the lines from [31, 32, 33] to construct $(\mathcal{P}^N(t))_N$, a sequence approximating $P(t)$, such that strong convergence (uniformly in time) is obtained assuming that certain conditions are satisfied.

Consider the system $(H,\mathcal{A},\mathcal{D},\mathcal{C},P_0)$. Let $(H^N)_N$, $N\in\mathbb{N}$, be a sequence of subspaces of H of finite dimension, $(\Pi^N)_N$ the corresponding sequence of orthogonal projections $\Pi^N : H\to H^N$ satisfying $\lim\limits_{N\to\infty}\|\Pi^N x - x\| = 0$, $\forall x\in H$. Let also $(\mathcal{A}^N)_N$, $(\mathcal{D}^N)_N$ and $(\mathcal{C}^N)_N$ be the sequences of approximating linear and bounded operators where $\mathcal{A}^N : H^N\to H^N$, $\mathcal{D}^N : H_d\to H^N$ and $\mathcal{C}^N : H^N\to H_y$. Consider also a sequence $({P_0}^N)_N$ of nonnegative and selfadjoint initial conditions. Denote by $T^N(t)$ the semigroup generated by $\mathcal{A}^N$. The system $(H^N,\mathcal{A}^N,\mathcal{D}^N,\mathcal{C}^N,{P_0}^N)$ is the N-th approximating system for $(H,\mathcal{A},\mathcal{D},\mathcal{C},P_0)$ with the corresponding Riccati equation

$$\begin{aligned}&\dot{\mathcal{P}}^N(t) = \mathcal{P}^N(t)(\mathcal{A}^N)^\star + \mathcal{A}^N\mathcal{P}^N(t) + \mathcal{D}^N(\mathcal{D}^N)^\star \\ &-\mathcal{P}^N(t)(\mathcal{C}^N)^\star\mathcal{C}^N\mathcal{P}^N(t),\ t\in(0,T], \qquad \mathcal{P}^N(0) = P_0^N.\end{aligned} \quad (31)$$

Consider the following assumptions:

Assumption 1 (convergence conditions).
For every $x\in H$, every $y\in H_y$ and every $d\in H_d$

(i) $T^N(t)\Pi^N x\to T(t)x$ as $N\to\infty$, uniformly in t on bounded subintervals of $[0,T]$,

(ii) $(T^N(t))^\star \Pi^N x \to T^\star(t)x$ *as* $N \to \infty$*, uniformly in* t *on bounded subintervals of* $[0,T]$*,*

(iii) $\mathcal{C}^N \Pi^N x \to \mathcal{C}x$ *as* $N \to \infty$*,*

(iv) $(\mathcal{C}^N)^\star y \to \mathcal{C}^\star y$ *as* $N \to \infty$*,*

(v) $(\mathcal{D}^N)d \to \mathcal{D}d$ *as* $N \to \infty$*,*

(vi) $(\mathcal{D}^N)^\star \Pi^N x \to \mathscr{D}^\star x$ *as* $N \to \infty$*,*

(vii) $P_0^N \Pi^N x \to P_0 x$ *as* $N \to \infty$*.*

These assumptions are of the same type as in [32, Assumption (H1) and (H2)] (see also [33, (H2)]) but now on the finite interval $[0,T]$. Note that we also added assumption (vii) on the convergence of the nonnegative initial conditions.

The following convergence result is a direct consequence of [31] and [32].

Theorem 3. *Consider* $(H^N, \mathcal{A}^N, \mathcal{D}^N, \mathcal{C}^N, {P_0}^N)$ *the* N*-th approximating systems of* $(H, \mathcal{A}, \mathcal{D}, \mathcal{C}, P_0)$ *such that Assumption 1 is satisfied. If*

(viii) the family of pairs $(\mathcal{A}^N, \mathcal{C}^N)_N$ *is uniformly detectable, and*

(ix) the family of pairs $(\mathcal{A}^N, \mathcal{D}^N)_N$ *is uniformly stabilizable,*

then the sequence $(\mathcal{P}^N(t))_N$ *of unique and non-negative solutions of (31) converges strongly to* $P(t)$ *uniformly in* t *on bounded subintervals of* $[0,T]$*, where* $P(t)$ *is the unique non-negative solution of the Riccati equation (30). Moreover,* $(T^N(t))_N$ *converges strongly to* $T(t)$ *uniformly in* t *on bounded subintervals of* $[0,T]$*.*

Proof. Using similar reasoning as in the proof of [34, Theorem 6.9] (see pg. 165), the Riccati equation (30) can be written as an integral operator Riccati equation similar to [31, (3.28)]. If $(i)-(ix)$ hold and are satisfied, the theorem follows from [31, Theorem 5.1] (or [33, Theorem 2.2]) and [32], now restricted to the finite-time interval $[0,T]$. ■

Note that [33, Theorem 2.2] is contained in [31, Theorem 5.1], but the main difference is that in [33, Theorem 2.2] the finite dimensional state problems are defined in the projection subspaces. The uniform detectability condition imposed in [32] can be seen as a relaxation of the coercivity assumption from [33, Theorem 2.2]. The assumptions are satisfied for averaging approximations of hereditary systems and Galerkin approximation of parabolic systems ([32, 33]), so we can implement the proposed sliding mode control design and perform numerical experiments.

6. Examples

In this section we implement the proposed sliding mode control design and perform numerical experiments on two examples: a delay system (particular hereditary system) and a linear parabolic equation in two spatial dimensions.

6.1. Delay systems

We consider the system of delayed differential equations that was used to illustrate the *infinite* horizon controller in [35]; namely the delayed differential equation with point-delay

$$
\begin{aligned}
&\dot{z}(t) = A_0 z(t) + A_1 z(t-h) + B_0 u(t) + D_0 d(t), && t \geq 0,\\
&z(0) = r, \quad z(\theta) = f(\theta), && -h \leq \theta < 0,\\
&y(t) = C_0 x(t) + w(t).
\end{aligned}
$$

On the space M_2 (see e.g. [28, Chapter 2]), the system can be represented as the abstract evolution equation

$$
\begin{aligned}
\frac{dx}{dt}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{D}d(t), \quad t \geq 0, \quad x(0) = x_0,\\
y(t) &= \mathcal{C}x(t) + w(t),
\end{aligned}
\tag{32}
$$

with the state vector $x(t) = \begin{bmatrix} z(t) \\ z(t+\cdot) \end{bmatrix}$. As $M_2([-h,0];\mathbb{R}^n)$ is isometrically isomorphic to $\mathbb{R}^n \times L^2(-h,0;\mathbb{R}^n)$ (see also [36]), one may define $H =$

$\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$ and $\mathcal{A}$, the infinitesimal generator of the corresponding C_0-semigroup, as

$$\mathcal{A}\begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} A_0 r + A_1 f(-h) \\ \frac{df}{d\theta}(\cdot) \end{bmatrix},$$

with domain

$$\mathscr{D}(\mathcal{A}) = \left\{ \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} \in H \mid f \text{ abs. cont.}, \frac{df}{d\theta}(\cdot) \in L^2(-h, 0; \mathbb{R}^m) \text{ and } f(0) = r \right\}.$$

The disturbance operator $\mathcal{D} : \mathbb{R}^d \to H$, measurement operator $\mathcal{C} : H \to \mathbb{R}^d$, and input operator $\mathcal{B} : \mathbb{R}^u \to H$ are defined by

$$\mathcal{D}d := \begin{bmatrix} D_0 d \\ 0 \end{bmatrix}, \quad \mathcal{C}x := \mathcal{C}\begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = C_0 r \quad \text{and } \mathcal{B}u := \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}.$$

To approximate P, the unique non-negative solution of the operator Riccati equation (30), consider the sequence of finite dimensional spaces $(H^N_{AVE})_N$, and $(A^N_{AVE})_N$, $(B^N_{AVE})_N$, $(D^N_{AVE})_N$ and $(C^N_{AVE})_N$, the sequences of approximating linear and bounded operators obtained using averaging approximations (AVE) as in [37]:

Let $t^N_j := \frac{jh}{N}$, for $j = 0, ..., N$, and χ^N_j the normalized characteristic functions on $[-t^N_j, -t^N_{j-1})$ such that $\|\chi^N_j\|_{L^2} = 1$. The sequence of finite-dimensional approximating spaces is then

$$H^N := \left\{ [\xi, \phi^N] \in H \mid \phi^N(\tau) = \sum_{j=1}^{N} v^N_j \chi^N_j(\tau),\, v^N_j \in \mathbb{R}^n \right\},$$

and the projection $\Pi^N : H \to H^N$ is

$$\Pi^N[\xi, \phi] := \left[\xi, \sum_{j=1}^{N} \phi^N_j \chi^N_j \right], \quad \phi^N_j := \sqrt{\frac{N}{h}} \int_{-jh/N}^{-(j-1)h/N} f(\tau) \mathrm{d}\tau.$$

The approximating operators on those spaces are

$$A^N_{AVE}[\xi, \phi^N] := \Big[A_0 \xi + \sqrt{\frac{N}{h}} A_1 v^N_N,\ \frac{N}{h} \sum_{j=1}^{N} (v^N_{j-1} - v^N_j) \chi^N_j \Big], \tag{33a}$$

$$B^N_{AVE} u := \Pi^N \mathcal{B}u = \mathcal{B}u, \qquad D^N_{AVE} d := \Pi^N \mathcal{D}d = \mathcal{D}d, \tag{33b}$$

$$C^N_{AVE}[\xi, \phi]^T := \mathcal{C}\Pi^N[\xi, \phi]^T = \mathcal{C}[\xi, \phi]^T, \tag{33c}$$

where we take $v_0^N = \sqrt{h/N}\xi$. The operators $\mathcal{C}$ and $\mathcal{D}$ are compact. The semigroup $T(t)$, and the operators $\mathcal{D}$ and $\mathcal{C}$ in combination with their averaging approximations, satisfy Assumption 1 and Theorem 3.

As a concrete example, we take $A_0 = \left[\begin{smallmatrix} -1 & 1 \\ 1 & -3 \end{smallmatrix}\right]$ and $A_1 = -\left[\begin{smallmatrix} 0.5 & 0 \\ 0 & 0.5 \end{smallmatrix}\right]$, $B_0 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $C_0 = [1\ 0]$, $D_0 = B_0$, and $h = 2$. We discretise the infinite-dimensional part using the AVE scheme [37] as laid out above, with $N = 128$. The matrix representation of the operator A^N_{AVE} on H^N in the orthonormal basis of characteristic functions $\chi_j^N(t) = \sqrt{N/h} \cdot \mathbf{1}_{[-\frac{jh}{N}, -\frac{(j-1)h}{N}]}(t)$, $j = 1, \ldots, N$ for the "L^2-part," is given by

$$A^N_{AVE} = \begin{bmatrix} A_0 & 0 & \cdots & \cdots & 0 & \sqrt{\frac{N}{h}} A_1 \\ \sqrt{\frac{N}{h}} I_2 & -\frac{N}{h} I_2 & 0 & \cdots & 0 & 0 \\ 0 & \frac{N}{h} I_2 & -\frac{N}{h} I_2 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & & & \frac{N}{h} I_2 & -\frac{N}{h} I_2 \end{bmatrix},$$

and $B^N_{AVE} = [B_0^\intercal\ 0 \cdots]^\intercal$, $C^N_{AVE} = [C_0\ 0 \cdots]$. The initial state is given by

$$\xi^N = \begin{bmatrix} r \\ \sqrt{\frac{N}{h}} \int_{-h/N}^{0} f(\tau)\mathrm{d}\tau \\ \vdots \\ \sqrt{\frac{N}{h}} \int_{-h}^{-(N-1)h/N} f(\tau)\mathrm{d}\tau \end{bmatrix},$$

and we denote the vector representing the state $[\xi, \phi^N]$ by x^N.

We let the sliding surface be defined by $\{Fx = 0\}$, where F is such that $F^N_{AVE} = [\sqrt{1/2}\ \sqrt{1/2}\ 0 \cdots 0]$ and simulate the full system, consisting of the plant (32) with input $u = u^N_{eq,AVE}$ and the filter (8). We have

$$u^N_{eq,AVE} = -(F^N_{AVE} B^N_{AVE})^{-1} F^N_{AVE} \left[A^N_{AVE} \hat{x}^N(t) - P^N(t)(y^N(t) - C^N_{AVE} \hat{x}^N(t)) \right],$$

where $\hat{x}^N$ is the state of the filter (8) and $P^N(t)$ is the solution of the DRE (31). The plant and filter equations are solved by discretisation using the AVE scheme and a symplectic mid-point integrator with time step $dt = 0.002$, and the DRE is solved by conversion into a linear Hamiltonian system which is then solved by means of the Mobius integrator (a combination of reinitialization and implicit midpoint rule, as reported in [38]). Results are shown in Figure 1, see the caption for an interpretation.

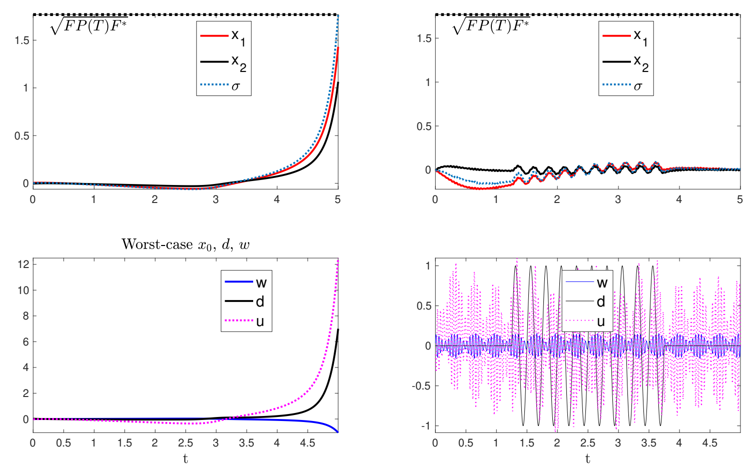

Figure 1: Closed-loop behaviour of the delayed differential equation (32) when coupled with the proposed controller. The top panels show the trajectories of x_1 and x_2 as well as the sliding variable σ resulting if the disturbance d and measurement noise w are as shown in the bottom panels. The control input u_{eq} is also shown in the bottom panels. The panels on the left demonstrate the *worst-case* d and w (and also initial state x_0, which is not shown) in the ellipsoid (4), as given in (15); i.e. $\sigma(T)$ should equal its largest possible value. As can be seen, $\sigma(T) = \sqrt{FP(T)F^\star}$, as claimed in (29) in Corollary 2. The panels to the right demonstrate the case of an arbitrary, non-worst-case realization of w, d and x_0; we see that the control u_{eq} is effective in steering σ close to zero, and that the actual σ is way below the worst-case bound (29).

6.2. Advection-diffusion equation in 2D

Assume that $\Omega = (0,1)^2 \subset \mathbb{R}^2$, set $H = L^2(\Omega)$ and $H_u = \mathbb{R}^3$, and let $x(t) \in H$ solve the following linear evolution equation:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Dd(t), x(0) = x_0\,, \tag{34}$$

where A is a strongly elliptic differential operator with domain $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ (the specific expression of A is given below), $D = B$ and $Bu(t) = u_1(t) + 2\xi_1\xi_2 u_2(t) + 3\xi_1^2\xi_2^2 u_3(t)$, where $\xi_i \in (0,1)$ denote the spatial variable. Since $B \in \mathscr{L}(H_u, H)$, and A is strongly elliptic it follows that (34) has a unique solution $x \in L^2(0,T,\mathscr{D}(A))$ such that $\frac{dx}{dt} \in L^2(t_0,T,H)$, provided $x(0) \in \mathscr{D}(A)$.

To specify A let $\mathbf{1}_{(a,b)\times(c,d)}$ denote the indicator function of $(a,b)\times(c,d)$ and ∂_{ξ_i} denote the partial derivative with respect to ξ_i. Then

$$\begin{aligned} Ax &= \sum_{i=1}^{2} \partial_{\xi_i}(K(\xi_1,\xi_2)\partial_{\xi_i} x - a_i(\xi_1,\xi_2)x) \\ a_1(\xi_1,\xi_2) &= \alpha(\xi_1,\xi_2)\sin(4\pi\xi_1), a_2 = \alpha(\xi_1,\xi_2)\cos(4\pi\xi_2 + 0.2) \\ \alpha(\xi_1,\xi_2) &= 5\mathbf{1}_{(0,0.5)\times(0,1.8)}(\xi_1,\xi_2) + \frac{5}{100}\mathbf{1}_{(0.5,1)\times(0,0.2)}(\xi_1,\xi_2), \\ K(\xi_1,\xi_2) &= 0.1\mathbf{1}_{(0,0.5)\times(0,1.8)}(\xi_1,\xi_2) + 0.01/5. \end{aligned} \tag{35}$$

Finally, let C be the multiplication by a H-function, $\tilde{c}$: $Cx(t) = \tilde{c}(\xi_1,\xi_2)x(\xi_1,\xi_2,t)$, and take $Fx(t) = \begin{bmatrix} 2\int_\Omega x(\xi_1,\xi_2,t)d\xi_1 d\xi_2 \\ \int_\Omega \xi_1 x(\xi_1,\xi_2,t)d\xi_2 d\xi_2 \\ \int_\Omega \xi_1\xi_2 x(\xi_1,\xi_2,t)d\xi_2 d\xi_2 \end{bmatrix}$, i.e. that the sliding surface is defined by three linear functionals, namely the mean and two mixed moments of the state vector $x(t)$. In this specific case, the minimax control problem (5) is to steer to 0 (as close as possible) the mean and two mixed moments of a distribution (e.g. concentration of a non-reactive chemical quantity) which verifies the advection-diffusion equation (34) with A defined by (35) subject to homogeneous Dirichlet boundary conditions ($x(t,\xi_1,\xi_2) = 0$ on $\partial\Omega$), and a bounded unknown time-varying disturbance f, which belongs to $\text{span}\{1, \xi_1\xi_2, \xi_1^2\xi_2^2\}$, and observation noise with values in H.

For the numerical simulations, the operators A, B, C, D were discretized by means of the spectral element method. The results are shown in Figure 2, see again the caption for interpretations.

7. Conclusion

The minimax sliding mode control which solves (5) generalizes the conventional sliding mode control: it steers the state of (1) as close as possible in the

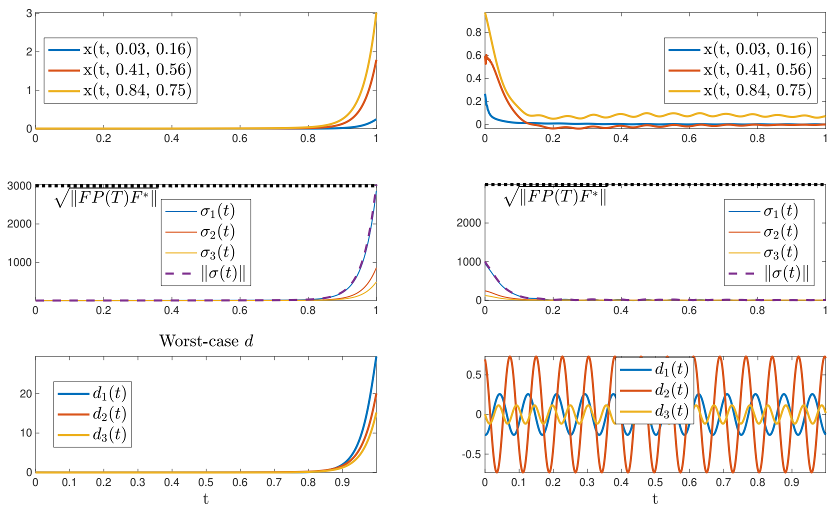

Figure 2: As in Figure 1, the left column demonstrates the worst case, whereas the right column demonstrates some arbitrary values for d, w, x_0. The top panels show the values of $x(t)$ in 3 locations within Ω, whereas the middle shows the components $\sigma_i(t)$, $i = 1, 2, 3$, of the sliding variable, as well as its norm, and we see that also in this case, the worst-case bound is reached as claimed, and the arbitrary case stays well below it. The observations were generated setting $\tilde{c} = 1$ in every 2nd grid-point of Ω.

minimax sense) towards the hyperplane $\{x \mid Fx = 0\}$ (as the exact reaching $Fx(T) = 0$, required in the definition of the conventional sliding mode control, cannot be guaranteed due to *unknown measurement noise* and *uncertain model disturbances*. We conjecture that the exact reaching may be guaranteed provided the model disturbance and measurement noise "disappear" after a given time instant T^*. This latter question will require a modification of the differential Riccati equation and is left for the future research. We stress that the exact "numerical" reaching, i.e., making the distance between the actual state x and the sliding hyperplane negligible, is possible, provided the eigenvalues of the Riccati operator $P(t)$ rapidly decay to zero (see (6)), and the null-space of the algebraic Riccati operator P^∞ contains the sliding hyperplane.

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