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Approximation rate in Wasserstein distance of probability measures on the real line by deterministic empirical measures

O. Bencheikh and B. Jourdain*

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Abstract

We are interested in the approximation in Wasserstein distance with index $\rho \geq 1$ of a probability measure μ on the real line with finite moment of order ρ by the empirical measure of N deterministic points. The minimal error converges to 0 as $N \rightarrow +\infty$ and we try to characterize the order associated with this convergence. In [17], Xu and Berger show that, apart when μ is a Dirac mass and the error vanishes, the order is not larger than 1 and give a sufficient condition for the order to be equal to this threshold 1 in terms of the density of the absolutely continuous with respect to the Lebesgue measure part of μ . They also prove that the order is not smaller than $1/\rho$ when the support of μ is bounded and not larger when the support is not an interval. We complement these results by checking that for the order to lie in the interval $(1/\rho, 1)$, the support has to be bounded and by stating a necessary and sufficient condition in terms of the tails of μ for the order to be equal to some given value in the interval $(0, 1/\rho)$, thus precisising the sufficient condition in terms of moments given in [17]. We also give a necessary condition for the order to be equal to the boundary value $1/\rho$. In view of practical application, we emphasize that in the proof of each result about the order of convergence of the minimal error, we exhibit a choice of points explicit in terms of the quantile function of μ which exhibits the same order of convergence.

Keywords: deterministic empirical measures, Wasserstein distance, rate of convergence.

AMS Subject Classification (2010): 49Q22, 60-08

Introduction

Let $\rho \geq 1$ and μ be a probability measure on the real line with finite moment of order ρ . We are interested in the rate of convergence to 0 in terms of $N \in \mathbb{N}^*$ of

$$(0.1) \quad e_N(\mu, \rho) := \inf \left\{ \mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \mu \right) : -\infty < x_1 \leq x_2 \leq \cdots \leq x_N < +\infty \right\},$$

where $\mathcal{W}_\rho$ denotes the Wasserstein distance with index ρ . The Hoeffding-Fréchet or comonotone coupling between two probability measures on the real line is optimal for $\mathcal{W}_\rho$ so that when $-\infty < x_1 \leq x_2 \leq \cdots \leq x_N < +\infty$,

$$(0.2) \quad \mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \mu \right) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |x_i - F^{-1}(u)|^\rho du,$$

where F^{-1} denotes the quantile function of μ defined by $F^{-1}(u) = \inf \{x \in \mathbb{R} : F(x) \geq u\}$ for $u \in (0, 1)$ with $F(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$. The motivation for our study is the approximation of the probability measure μ by finitely supported probability measures. Examples of application are provided by the optimal initialization

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of systems of particles with mean-field interaction [14, 2], where, to preserve the mean-field feature, it is important to get N points with equal weight $1/N$ (of course, nothing prevents several of these points to be equal) and by the numerical analysis of restricted Monte Carlo methods which may only use random bits instead of random numbers [11]. The optimal approximation in the quadratic case $\rho = 2$ has been shown by Baker [1] to preserve the convex order. This is not the case for the optimal quantization [12] which is obtained by also optimizing over the weights and considering :

$$\inf \left\{ \mathcal{W}_\rho \left(\sum_{i=1}^N p_i \delta_{x_i}, \mu \right) : -\infty < x_1 \leq x_2 \leq \dots \leq x_N < +\infty, (p_1, \dots, p_N) \in [0, 1]^N, \sum_{i=1}^N p_i = 1 \right\}.$$

The optimal quantization was introduced in signal processing [4] but has turned out since to be useful in scientific computing [16]. Both the minimization problem with respect to the locations $x_1 \leq x_2 \leq \dots \leq x_N$ under prescribed but non necessarily uniform weights $(p_1, \dots, p_N)$ and with respect to the weights $(p_1, \dots, p_N)$ under prescribed locations $x_1 \leq x_2 \leq \dots \leq x_N$ have been studied by Xu and Berger [17]. The case when the weights are not prescribed but only satisfy a size constraint has been studied in [5] while [15] addresses the minimization with respect to the locations under prescribed uniform weights when the Wasserstein distance is replaced by the energy distance.

According to Corollary 5.12 [17], the fact that $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$ ensures that $e_N(\mu, \rho)$ goes to 0 as $N \rightarrow \infty$. The main purpose of the paper is to study the rate at which this convergence occurs. In particular, we would like to give necessary conditions on μ , which, when possible, are also sufficient, to ensure convergence at a rate $N^{-\alpha}$ with $\alpha > 0$ called the order of convergence.

One of course has

$$e_N(\mu, \rho) \leq \mathcal{W}_\rho(\mu_N, \mu) \text{ and } e_N(\mu, \rho) \leq \mathbb{E}^{1/\rho} [\mathcal{W}_\rho^\rho(\mu_N, \mu)] \text{ where } \mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}$$

is the usual empirical measure of random variables $(X_i)_{i \geq 1}$ i.i.d. according to μ . In the one-dimensional setting of the present paper, the convergence rate of $\mathcal{W}_\rho(\mu_N, \mu)$ has been studied in [8] for $\rho = 1$ and in [9] in the quadratic case $\rho = 2$, the one of $\mathbb{E}^{1/\rho} [\mathcal{W}_\rho^\rho(\mu_N, \mu)]$ for $\rho \geq 1$ in the book [6] by Bobkov and Ledoux. In general dimension, estimations of $\mathbb{E}^{1/\rho} [\mathcal{W}_\rho^\rho(\mu_N, \mu)]$ and concentration inequalities for $\mathcal{W}_\rho^\rho(\mu_N, \mu)$ are given in [10]. In the random case, the largest possible order of convergence (apart from the case when μ is a Dirac mass and the error vanishes) is $\alpha = 1/2$, which matches the rate of convergence in the standard strong law of large numbers given by the central limit theorem under square integrability.

The rate of convergence of $e_N(\mu, \rho)$ has already been addressed by Xu and Berger [17] in the one-dimensional setting of the present paper, by Chevallier [7] in higher finite dimension and by [11] along the subsequence $(N = 2^n)_{n \in \mathbb{N}}$ in the quadratic case $\rho = 2$ when μ is a Gaussian measure on an Hilbert space or the law of the solution to a scalar autonomous stochastic differential equation. In particular, according to Theorem 5.21 (ii) [17], when the support of μ is bounded, then $\sup_{N \geq 1} N^{\frac{1}{\rho}} e_N(\mu, \rho) < +\infty$, while Remark 5.22 (ii) [17] ensures that $\limsup_{N \rightarrow +\infty} N^{\frac{1}{\rho}} e_N(\mu, \rho) > 0$ when F^{-1} is discontinuous. According to Theorem 5.20 [17], apart when μ is a Dirac mass and $e_N(\mu, \rho)$ vanishes for all $N, \rho \geq 1$, $\limsup_{N \rightarrow \infty} N e_N(\mu, \rho) > 0$ so that the order of convergence cannot exceed 1. It is equal to 1 when the density f of the absolutely continuous with respect to the Lebesgue measure part of μ is dx a.e. positive on $\{x \in \mathbb{R} : 0 < F(x) < 1\}$ (or equivalently F^{-1} is absolutely continuous), since Theorem 5.15 [17] then ensures that

$$\lim_{N \rightarrow +\infty} N e_N(\mu, \rho) = \frac{1}{2(\rho + 1)^{1/\rho}} \left(\int_{\mathbb{R}} \frac{\mathbf{1}_{\{0 < F(x) < 1\}}}{f^{\rho-1}(x)} dx \right)^{1/\rho},$$

where, by Theorem 2.4 [3], the right-hand side is not smaller than $\liminf_{N \rightarrow +\infty} N e_N(\mu, \rho)$ even without the positivity assumption on the density. In [7], Chevallier addresses the multidimensional setting and proves in Theorem 3 that for a probability measure μ on $\mathbb{R}^d$ with support bounded by r , there exist points $x_1, \dots, x_N \in \mathbb{R}^d$ such that $\frac{1}{4r} \mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \mu \right) \leq f_{\rho, d}(N)$ where $f_{\rho, d}(N)$ is respectively equal to $\left(\frac{d}{d-\rho} \right)^{\frac{1}{\rho}} N^{-\frac{1}{d}}, \left(\frac{1+\ln N}{N} \right)^{\frac{1}{d}}$, and $\zeta(p/d) N^{-\frac{1}{\rho}}$ with ζ denoting the zeta Riemann function when $\rho < d$, $\rho = d$ and $\rho > d$.

The case when the support of μ is not bounded is also considered by Xu and Berger [17] in the one-dimensional setting of the present paper and by [7] in the multidimensional setting. In Corollary 1 [7],

Chevallier proves that $\lim_{N \rightarrow \infty} (f_{\rho,d}(N))^{-\alpha\rho} \mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \mu \right) = 0$ when $\int_{\mathbb{R}^d} |x|^{\frac{\rho}{1-\alpha\rho}} \mu(dx) < +\infty$ for some $\alpha \in (0, 1/\rho)$. This generalizes the one-dimensional statement of Theorem 5.21 (i) [17] : under the same moment condition, $\lim_{N \rightarrow \infty} N^\alpha e_N(\mu, \rho) = 0$.

In the main contribution of the present paper, Theorem 2.2, we refine this result by stating the following necessary and sufficient condition

$$\forall \alpha \in (0, 1/\rho), \quad \lim_{x \rightarrow +\infty} x^{\frac{\rho}{1-\alpha\rho}} \left(F(-x) + 1 - F(x) \right) = 0 \Leftrightarrow \lim_{N \rightarrow +\infty} N^\alpha e_N(\mu, \rho) = 0.$$

We also check that

$$\forall \alpha \in (0, 1/\rho), \quad \sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} \left(F(-x) + 1 - F(x) \right) < +\infty \Leftrightarrow \sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty,$$

a condition under which, the order of convergence α of the minimal error $e_N(\mu, \rho)$ is preserved by choosing $x_1 = F^{-1} \left(\frac{1}{N} \right) \wedge (-N^{\frac{1}{\rho}-\alpha})$, $x_N = F^{-1} \left(\frac{N-1}{N} \right) \vee N^{\frac{1}{\rho}-\alpha}$ and any $x_i \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$. We also show that for $(N^\alpha e_N(\mu, \rho))_{N \geq 1}$ to be bounded for $\alpha > 1/\rho$, the support of μ has to be bounded. Then we address the boundary case $\alpha = 1/\rho$: among Weibull distributions, we exhibit probability measures μ with unbounded support such that, for $\rho > 1$, $\lim_{N \rightarrow +\infty} N^{1/\rho} e_N(\mu, \rho) = 0$ and give a necessary condition for $(N^{1/\rho} e_N(\mu, \rho))_{N \geq 1}$ to be bounded, which unfortunately is not sufficient but ensures the boundedness of $\left(\frac{N^{1/\rho}}{1+\ln N} e_N(\mu, \rho) \right)_{N \geq 1}$. These results are summarized together with the ones obtained by Xu and Berger [17] in Table 1. They are stated and proved in the second section of the paper.

The first section is devoted to preliminary results. We first recall that, when $\int_{\mathbb{R}} |x|^\rho \mu(dx) < \infty$, then the infimum in (0.1) is attained:

$$(0.3) \quad e_N(\mu, \rho) = \mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}, \mu \right) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |x_i^N - F^{-1}(u)|^\rho du$$

for some points $x_i^N \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})] \cap \mathbb{R}$ which are unique as long as $\rho > 1$ and explicit when $\rho \in \{1, 2\}$. To circumvent the otherwise lack of explicit formula for the optimal point x_i^N when studying the order of convergence of $e_N(\mu, \rho)$ to 0, we also derive bounds where x_i^N does not appear for the i -th contribution in the right-hand side of (0.3). We then give an alternative expression of each contribution in terms of the cumulative distribution function F in place of the quantile function F^{-1} , before taking advantage of the induced alternative formula for $e_N(\mu, \rho)$ to recover that the error goes to 0 as $N \rightarrow \infty$ when $\int_{\mathbb{R}} |x|^\rho \mu(dx) < \infty$. We also precise Theorem 5.21 (ii) [17] which states that the boundedness of the support of μ implies that of the sequence $(N^{1/\rho} e_N(\mu, \rho))_{N \geq 1}$, by remarking that when F^{-1} is moreover continuous, then the sequence goes to 0 as $N \rightarrow \infty$ for $\rho > 1$, with an order of convergence arbitrarily low as exemplified by the beta distributions. Last, when μ is not a Dirac mass, we complement in a non-asymptotic way the positivity of $\limsup_{N \rightarrow \infty} N e_N(\mu, \rho)$ proved by Xu and Berger in Theorem 5.20 [17].

The proofs of two technical lemmas are given in the appendix.

Notation :

- We set $F(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$ and denote $F^{-1}(u) = \inf \{x \in \mathbb{R} : F(x) \geq u\}$ for $u \in (0, 1)$. We have $u \leq F(x) \Leftrightarrow F^{-1}(u) \leq x$. The quantile function F^{-1} is left-continuous and non-decreasing and we denote by $F^{-1}(u+)$ its right-hand limit at $u \in [0, 1)$ (in particular $F^{-1}(0+) = \lim_{u \rightarrow 0+} F^{-1}(u) \in [-\infty, +\infty)$) and set $F^{-1}(1) = \lim_{u \rightarrow 1-} F^{-1}(u) \in (-\infty, +\infty]$.
- We respectively denote by $x \wedge y$ and $x \vee y$ the minimum and the maximum of two real numbers x and y .
- We denote by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) the integer j such that $j \leq x < j+1$ (resp. $j-1 < x \leq j$) and by $\{x\} = x - \lfloor x \rfloor$ the fractional part of $x \in \mathbb{R}$.

α	Necessary condition	Sufficient condition
$\alpha = 1$	$\int_{\mathbb{R}} \frac{\mathbf{1}_{\{f(x)>0\}}}{f^{\rho-1}(x)} dx < +\infty$ (Thm. 2.4 [3])	$f(x) > 0$ dx a.e. on $\{x \in \mathbb{R} : 0 < F(x) < 1\}$ and $\int_{\mathbb{R}} \frac{\mathbf{1}_{\{f(x)>0\}}}{f^{\rho-1}(x)} dx < +\infty$ (Thm. 5.15 [17])
$\alpha \in \left(\frac{1}{\rho}, 1\right)$ when $\rho > 1$	F^{-1} continuous (Remark 5.22 (ii) [17]) and μ with bounded support (Prop. 2.1)	related to the modulus of continuity of F^{-1} (Example 1.8)
$\alpha = \frac{1}{\rho}$	$\exists \lambda > 0, \forall x \geq 0, F(-x) + 1 - F(x) \leq \frac{e^{-\lambda x}}{\lambda}$ (Prop. 2.7)	μ with bounded support (Thm. 5.21 (ii) [17]) For $\rho > 1$, Example 2.6 with unbounded supp.
$\alpha \in \left(0, \frac{1}{\rho}\right)$	$\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) < +\infty$ (Thm. 2.2)	$\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) < +\infty$ (Thm. 2.2)

Table 1: Conditions for the convergence of $e_N(\mu, \rho)$ with order α : $\sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty$.

- For two sequences $(a_N)_{N \geq 1}$ and $(b_N)_{N \geq 1}$ of real numbers with $b_N > 0$ for $N \geq 2$ we denote $a_N \asymp b_N$ when $0 < \inf_{N \geq 2} \left(\frac{a_N}{b_N}\right)$ and $\sup_{N \geq 2} \left(\frac{a_N}{b_N}\right) < +\infty$.

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1 Preliminary results

In the present section, we state preliminary results which will prove useful for the study of the order of convergence of $e_N(\mu, \rho)$ to 0 when $N \rightarrow \infty$ in the next section devoted to the case when the support of μ is not bounded. We first recall that the infimum over $(x_1, \dots, x_N)$ is attained in (0.1) and bound the first (resp. last) coordinate of the minimizer from below (resp. above). Next, we give estimations from below and above not involving x_i^N of each contribution in the sum in the right-hand side of (0.3). We also rewrite these contributions in terms of the cumulative distribution function F in place of the quantile function F^{-1} . We recall that the finiteness of the ρ -th order moment of μ is a necessary and sufficient condition for the error $e_N(\mu, \rho)$ to go to 0 as $N \rightarrow \infty$ before checking that when μ has a bounded support and a continuous quantile function, then $\lim_{N \rightarrow \infty} N^{1/\rho} e_N(\mu, \rho) = 0$ with an order of convergence arbitrarily low as exemplified by the beta distributions. We finally derive a non-asymptotic lower bound for $N e_N(\mu, \rho) + (N+1) e_{N+1}(\mu, \rho)$.

1.1 Infimum is attained in (0.1)

When $\rho = 1$ (resp. $\rho = 2$), $\mathbb{R} \ni y \mapsto N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |y - F^{-1}(u)|^\rho du$ is minimal for y belonging to the set $[F^{-1}(\frac{2i-1}{2N}), F^{-1}(\frac{2i-1}{2N} +)]$ of medians (resp. equal to the mean $N \int_{\frac{i-1}{N}}^{\frac{i}{N}} F^{-1}(u) du$) of the image of the uniform law on $[\frac{i-1}{N}, \frac{i}{N}]$ by F^{-1} .

For general $\rho > 1$, the function $\mathbb{R} \ni y \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} |y - F^{-1}(u)|^\rho du$ is strictly convex and continuously differentiable with derivative

$$(1.1) \quad \rho \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\mathbf{1}_{\{y \geq F^{-1}(u)\}} (y - F^{-1}(u))^{\rho-1} - \mathbf{1}_{\{y < F^{-1}(u)\}} (F^{-1}(u) - y)^{\rho-1} \right) du$$

non-positive for $y = F^{-1}(\frac{i-1}{N} +)$ when either $i = 1$ and $F^{-1}(0+) > -\infty$ or $i \geq 2$ and non-negative for $y = F^{-1}(\frac{i}{N})$ when either $i \leq N-1$ or $i = N$ and $F^{-1}(1) < +\infty$. Since the derivative has a positive limit as

$y \rightarrow +\infty$ and a negative limit as $y \rightarrow -\infty$, we deduce that $\mathbb{R} \ni y \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} |y - F^{-1}(u)|^\rho du$ admits a unique minimizer $x_i^N \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})] \cap \mathbb{R}$, as already stated in Corollary 4.4 [17] (to keep notations simple, we do not explicit the dependence of x_i^N on ρ). Therefore

$$(1.2) \quad e_N^\rho(\mu, \rho) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |x_i^N - F^{-1}(u)|^\rho du \text{ with } \left[F^{-1}\left(\frac{i-1}{N}\right), F^{-1}\left(\frac{i}{N}\right) \right] \ni x_i^N = \begin{cases} F^{-1}\left(\frac{2i-1}{2N}\right) & \text{if } \rho = 1, \\ N \int_{\frac{i-1}{N}}^{\frac{i}{N}} F^{-1}(u) du & \text{if } \rho = 2, \\ \text{not explicit otherwise.} \end{cases}$$

We will see that when the support of μ is unbounded, the contributions with $i \in \{1, N\}$ in the right-hand side of (0.3) are dominant. To prove our main result, Theorem 2.2 below which characterizes the convergence of $e_N(\mu, \rho)$ to 0 with order $\alpha \in (0, \frac{1}{\rho})$, we will need the following estimates on x_1^N and x_N^N .

Lemma 1.1. *Let $\rho \geq 1$ and $\alpha \in (0, \frac{1}{\rho})$. There is a finite constant C only depending on ρ and α such that the two extremal points in the optimal sequence $(x_i^N)_{1 \leq i \leq N}$ for $e_N(\mu, \rho)$ satisfy*

$$\forall N \geq 1, x_1^N \geq CN^{\frac{1}{\rho}-\alpha} \inf_{u \in (0, \frac{1}{N})} u^{\frac{1}{\rho}-\alpha} F^{-1}(u) \text{ and } x_N^N \leq CN^{\frac{1}{\rho}-\alpha} \sup_{u \in (0, \frac{1}{N})} u^{\frac{1}{\rho}-\alpha} F^{-1}(1-u).$$

If $\sup_{u \in (0, 1/2]} u^{\frac{1}{\rho}-\alpha} (F^{-1}(1-u) - F^{-1}(u)) < +\infty$, then $\sup_{N \geq 1} N^{\alpha-\frac{1}{\rho}} (x_N^N \vee (-x_1^N)) < +\infty$.

The proof which relies on the expression (1.1) of the derivative when $\rho \notin \{1, 2\}$ is postponed to the appendix.

1.2 Bounds on $e_N(\mu, \rho)$

To circumvent the lack of explicit formula for the optimal point x_i^N (unless $\rho \in \{1, 2\}$) when studying the order of convergence of $e_N(\mu, \rho)$ to 0, we are now going to derive bounds where x_i^N does not appear for the i -th contribution in the right-hand side of (0.3).

When needing to bound $e_N(\mu, \rho)$ from above (see in particular the derivation of the order of convergence of $e_N(\mathbf{1}_{\{x>0\}} \beta x^{\beta-1} \exp(-x^\beta) dx, \rho)$ for $\beta > 0$ in Example 2.6 below), we may replace the optimal point x_i^N by $F^{-1}(\frac{2i-1}{2N})$:

$$(1.3) \quad \forall i \in \{1, \dots, N\}, \quad \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i^N|^\rho du \leq \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| F^{-1}(u) - F^{-1}\left(\frac{2i-1}{2N}\right) \right|^\rho du,$$

a simple choice particularly appropriate when linearization is possible since $\left[\frac{i-1}{N}, \frac{i}{N} \right] \ni v \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} |u-v|^\rho du$

is minimal for $v = \frac{2i-1}{2N}$.

To bound $e_N(\mu, \rho)$ from below, we can use that, by Jensen's inequality and the minimality of $F^{-1}(\frac{2i-1}{2N})$ for $\rho = 1$,

$$(1.4) \quad \begin{aligned} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i^N|^\rho du &\geq N^{\rho-1} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i^N| du \right)^\rho \geq N^{\rho-1} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| F^{-1}(u) - F^{-1}\left(\frac{2i-1}{2N}\right) \right| du \right)^\rho \\ &\geq N^{\rho-1} \left(\frac{1}{4N} \left(F^{-1}\left(\frac{2i-1}{2N}\right) - F^{-1}\left(\frac{4i-3}{4N}\right) + F^{-1}\left(\frac{4i-1}{4N}\right) - F^{-1}\left(\frac{2i-1}{2N}\right) \right) \right)^\rho \\ &\geq \frac{1}{4\rho N} \left(F^{-1}\left(\frac{4i-1}{4N}\right) - F^{-1}\left(\frac{4i-3}{4N}\right) \right)^\rho. \end{aligned}$$

This estimate will be used in the proof that the order of convergence of $e_N(\mu, \rho)$ cannot exceed $1/\rho$ when the support of μ is not bounded (Proposition 2.1 below), in the derivation of the order of convergence of $e_N(\beta \mathbf{1}_{[0,1]}(x) x^{\beta-1} dx, \rho)$ and $e_N(\mathbf{1}_{\{x>0\}} \beta x^{\beta-1} \exp(-x^\beta) dx, \rho)$ for $\beta > 0$ in Examples 1.8 and 2.6 below and in the derivation of the necessary condition for convergence with boundary order $1/\rho$ (Proposition 2.7 below).

1.3 Alternative formula in terms of the cumulative distribution function

In the proof of our main result (Theorem 2.2 below which characterizes the convergence of $e_N(\mu, \rho)$ to 0 with order $\alpha \in (0, \frac{1}{\rho})$) and in the derivation of our necessary condition for the convergence of $e_N(\mu, \rho)$ with boundary order $1/\rho$ (Proposition 2.7 below), we will need to rewrite in terms of the cumulative distribution function F in place of the quantile function F^{-1} contributions in the decomposition over $i \in \{1, \dots, N\}$ in the right-hand side of (0.3). This is possible thanks to the next lemma.

Lemma 1.2. *Assume that $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$ with $\rho \geq 1$. For $i \in \{1, \dots, N\}$ and $x \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})] \cap \mathbb{R}$ (with convention $F^{-1}(0) = F^{-1}(0+)$), we have:*

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} |x - F^{-1}(u)|^\rho du = \rho \int_{F^{-1}(\frac{i-1}{N})}^x (x-y)^{\rho-1} \left(F(y) - \frac{i-1}{N} \right) dy + \rho \int_x^{F^{-1}(\frac{i}{N})} (y-x)^{\rho-1} \left(\frac{i}{N} - F(y) \right) dy,$$

and the right-hand side is minimal for $x = x_i^N$.

Remark 1.3. *Under the convention $F^{-1}(0) = -\infty$, when, for some $i \in \{1, \dots, N\}$, $F^{-1}(\frac{i-1}{N}+) > F^{-1}(\frac{i-1}{N})$, then $F(y) = \frac{i-1}{N}$ for $y \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i-1}{N}+))$ and $\int_{F^{-1}(\frac{i-1}{N})}^{F^{-1}(\frac{i-1}{N}+)} (x-y)^{\rho-1} \left(F(y) - \frac{i-1}{N} \right) dy = 0$ so that the lower integration limit in the first integral in the right-hand side of (1.5) may be replaced by $F^{-1}(\frac{i-1}{N}+)$. In a similar way, the upper integration limit in the second integral may be replaced by $F^{-1}(\frac{i}{N}+)$ under the convention $F^{-1}(1+) = +\infty$.*

Plugging this equality written for $x = x_i^N$ in the right-hand side of (0.3), we immediately deduce the following alternative formulation of $e_N(\mu, \rho)$ in terms of the cumulative distribution function F :

Proposition 1.4.

(1.5)

$$e_N^\rho(\mu, \rho) = \rho \sum_{i=1}^N \left(\int_{F^{-1}(\frac{i-1}{N}+)}^{x_i^N} (x_i^N - y)^{\rho-1} \left(F(y) - \frac{i-1}{N} \right) dy + \int_{x_i^N}^{F^{-1}(\frac{i}{N})} (y - x_i^N)^{\rho-1} \left(\frac{i}{N} - F(y) \right) dy \right).$$

Remark 1.5. *When $\rho = 1$, the equality (1.5) follows from the interpretation of $\mathcal{W}_1(\nu, \eta)$ as the integral of the absolute difference between the cumulative distribution functions of ν and η (equal, as seen with a rotation with angle $\frac{\pi}{2}$, to the integral of the absolute difference between their quantile functions) and the integral simplifies into:*

(1.6)

$$e_N(\mu, 1) = \sum_{i=1}^N \left(\int_{F^{-1}(\frac{i-1}{N}+)}^{F^{-1}(\frac{2i-1}{N})} \left(F(y) - \frac{i-1}{N} \right) dy + \int_{F^{-1}(\frac{2i-1}{N})}^{F^{-1}(\frac{i}{N})} \left(\frac{i}{N} - F(y) \right) dy \right) = \frac{1}{N} \int_{\mathbb{R}} \min_{j \in \mathbb{N}} |NF(y) - j| dy.$$

For $\rho > 1$, this equality can be deduced from the general formula for $\mathcal{W}_\rho^\rho(\nu, \eta)$ in terms of the cumulative distribution functions of μ and η (see for instance Lemma B.3 [13]).

Proof of Lemma 1.2. Let $i \in \{1, \dots, N\}$ and $x \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})] \cap \mathbb{R}$. We have $\frac{i-1}{N} \leq F(x)$ and $F(x-) \leq \frac{i}{N}$. Since $F^{-1}(u) \leq x \Leftrightarrow u \leq F(x)$ and $F^{-1}(u) = x$ for $u \in (F(x-), F(x)]$, we have:

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} |x - F^{-1}(u)|^\rho du = \int_{\frac{i-1}{N}}^{F(x)} (x - F^{-1}(u))^\rho du + \int_{F(x)}^{\frac{i}{N}} (F^{-1}(u) - x)^\rho du.$$

Using the well-known fact that the image of $\mathbf{1}_{[0,1]}(v) dv \mu(dz)$ by $(v, z) \mapsto F(z-) + v\mu(\{z\})$ is the Lebesgue measure on $[0, 1]$ and that $\mathbf{1}_{[0,1]}(v) dv \mu(dz)$ a.e., $F^{-1}(F(z-) + v\mu(\{z\})) = z$, we obtain that:

$$\begin{aligned} \int_{\frac{i-1}{N}}^{F(x)} (x - F^{-1}(u))^\rho du &= \int_{v=0}^1 \int_{z \in \mathbb{R}} \mathbf{1}_{\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\}) \leq F(x)\}} (x - z)^\rho \mu(dz) dv \\ &= \int_{v=0}^1 \int_{z \in \mathbb{R}} \mathbf{1}_{\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\}) \leq F(x)\}} \int \rho(x - y)^{\rho-1} \mathbf{1}_{\{z \leq y \leq x\}} dy \mu(dz) dv \\ &= \rho \int_{y=-\infty}^x (x - y)^{\rho-1} \int_{v=0}^1 \int_{z \in \mathbb{R}} \mathbf{1}_{\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\})\}} \mathbf{1}_{\{z \leq y\}} \mu(dz) dv dy. \end{aligned} \tag{1.7}$$

For $v > 0$, $\{z \in \mathbb{R} : F(z-) + v\mu(\{z\}) \leq F(y)\} = (-\infty, y] \cup \{z \in \mathbb{R} : z > y \text{ and } F(z) = F(y)\}$ with $\mu(\{z \in \mathbb{R} : z > y \text{ and } F(z) = F(y)\}) = 0$ and therefore

$$\int_{z \in \mathbb{R}} \mathbf{1}_{\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\})\}} \mathbf{1}_{\{z \leq y\}} \mu(dz) = \int_{z \in \mathbb{R}} \mathbf{1}_{\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\}) \leq F(y)\}} \mu(dz).$$

Plugging this equality in (1.7), using again the image of $\mathbf{1}_{[0,1]}(v) dv \mu(dz)$ by $(v, z) \mapsto F(z-) + v\mu(\{z\})$ and the equivalence $\frac{i-1}{N} \leq F(y) \Leftrightarrow F^{-1}(\frac{i-1}{N}) \leq y$, we deduce that:

$$\int_{\frac{i-1}{N}}^{F(x)} (x - F^{-1}(u))^\rho du = \rho \int_{y=-\infty}^x (x - y)^{\rho-1} \int_{u=0}^1 \mathbf{1}_{\{\frac{i-1}{N} \leq u \leq F(y)\}} du dy = \rho \int_{F^{-1}(\frac{i-1}{N})}^x (x - y)^{\rho-1} \left(F(y) - \frac{i-1}{N} \right) dy.$$

In a similar way, we check that:

$$\int_{F(x)}^{\frac{i}{N}} (F^{-1}(u) - x)^\rho du = \rho \int_x^{F^{-1}(\frac{i}{N})} (y - x)^{\rho-1} \left(\frac{i}{N} - F(y) \right) dy,$$

which concludes the proof. $\square$

1.4 Convergence of the error to 0

According to Corollary 5.12 [17], the finiteness of the moment $\int_{\mathbb{R}} |x|^\rho \mu(dx)$ with order ρ implies that the error $e_N(\mu, \rho)$ goes to 0 as $N \rightarrow \infty$. Since, by the inverse transform sampling, the moment clearly has to be finite for $e_N(\mu, \rho)$ to be finite for some $N \geq 1$, the following equivalence holds.

Proposition 1.6. *For each $\rho \geq 1$, we have $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty \Leftrightarrow \lim_{N \rightarrow +\infty} e_N(\mu, \rho) = 0$.*

The direct implication can also be deduced from the inequality $e_N(\mu, \rho) \leq \mathcal{W}_\rho\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu\right)$ and the almost sure convergence to 0 of $\mathcal{W}_\rho\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu\right)$ for $(X_i)_{i \geq 1}$ i.i.d. according to μ deduced from the strong law of large numbers and stated for instance in Theorem 2.13 [6]. For the sake of completeness, we give an alternative simple argument based on (1.5).

Proof. According to the introduction, the finiteness of $e_N(\mu, \rho)$ for some $N \geq 1$ implies that $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$. So it is enough to check the zero limit property under the finite moment condition.

When respectively $F^{-1}(\frac{i}{N}) \leq 0$, $F^{-1}(\frac{i-1}{N}+) < 0 < F^{-1}(\frac{i}{N})$ or $F^{-1}(\frac{i-1}{N}+) \geq 0$, then, by Lemma 1.2, the term with index i in (1.5) is respectively bounded from above by

$$\begin{aligned} & \int_{F^{-1}(\frac{i-1}{N}+)}^{F^{-1}(\frac{i}{N})} \left(F^{-1}\left(\frac{i}{N}\right) - y \right)^{\rho-1} \left(F(y) - \frac{i-1}{N} \right) dy \leq \int_{F^{-1}(\frac{i-1}{N}+)}^{F^{-1}(\frac{i}{N})} (-y)^{\rho-1} \left(\frac{1}{N} \wedge F(y) \right) dy, \\ & \int_{F^{-1}(\frac{i-1}{N}+)}^0 (-y)^{\rho-1} \left(\frac{1}{N} \wedge F(y) \right) dy + \int_0^{F^{-1}(\frac{i}{N})} y^{\rho-1} \left(\frac{1}{N} \wedge (1 - F(y)) \right) dy, \\ & \int_{F^{-1}(\frac{i-1}{N}+)}^{F^{-1}(\frac{i}{N})} \left(y - F^{-1}\left(\frac{i-1}{N}+\right) \right)^{\rho-1} \left(\frac{i}{N} - F(y) \right) dy \leq \int_{F^{-1}(\frac{i-1}{N}+)}^{F^{-1}(\frac{i}{N})} y^{\rho-1} \left(\frac{1}{N} \wedge (1 - F(y)) \right) dy. \end{aligned}$$

After summation, we deduce that:

$$e_N^\rho(\mu, \rho) \leq \rho \int_{-\infty}^0 (-y)^{\rho-1} \left(\frac{1}{N} \wedge F(y) \right) dy + \rho \int_0^{+\infty} y^{\rho-1} \left(\frac{1}{N} \wedge (1 - F(y)) \right) dy.$$

Since, by Fubini's theorem, $\rho \int_{-\infty}^0 (-y)^{\rho-1} F(y) dy + \rho \int_0^{+\infty} y^{\rho-1} (1 - F(y)) dy = \int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$, Lebesgue's theorem ensures that the right-hand side and therefore $e_N(\mu, \rho)$ go to 0 as $N \rightarrow +\infty$. $\square$

According to Theorem 5.21 (ii) [17], when the support of μ is bounded, then $\sup_{N \geq 1} N^{\frac{1}{\rho}} e_N(\mu, \rho) < +\infty$, while Remark 5.22 (ii) [17] ensures that $\limsup_{N \rightarrow +\infty} N^{\frac{1}{\rho}} e_N(\mu, \rho) > 0$ when F^{-1} is discontinuous. We complement these results by the following lemma.

Lemma 1.7. *If F^{-1} is continuous and the support of μ bounded, then for each $\rho > 1$, $\lim_{N \rightarrow +\infty} N^{1/\rho} e_N(\mu, \rho) = 0$.*

Proof. By (1.3),

$$\begin{aligned} e_N^\rho(\mu, \rho) &\leq \frac{1}{2N} \sum_{i=1}^N \left\{ \left(F^{-1} \left(\frac{2i-1}{2N} \right) - F^{-1} \left(\frac{i-1}{N} + \right) \right)^\rho + \left(F^{-1} \left(\frac{i}{N} \right) - F^{-1} \left(\frac{2i-1}{2N} \right) \right)^\rho \right\} \\ &\leq \frac{1}{2N} (F^{-1}(1) - F^{-1}(0+)) \max_{1 \leq j \leq 2N} \left(F^{-1} \left(\frac{j}{2N} \right) - F^{-1} \left(\frac{j-1}{2N} \right) \right)^{\rho-1}, \end{aligned}$$

where we use the convention $F^{-1}(0) = F^{-1}(0+)$ in $\max_{1 \leq j \leq 2N} (F^{-1}(\frac{j}{2N}) - F^{-1}(\frac{j-1}{2N}))^{\rho-1}$. When the support of μ is bounded then $F^{-1}(1) - F^{-1}(0+) < \infty$ and when moreover F^{-1} is continuous, then this function is uniformly continuous on $(0, 1)$ and the conclusion follows. $\square$

The next example shows that when μ is compactly supported with F^{-1} continuous then, for each $\rho > 1$, the rate of convergence of $N^{1/\rho} e_N(\mu, \rho)$ to 0 as $N \rightarrow +\infty$ may be arbitrarily small.

Example 1.8. *Let $\mu_\beta(dx) = \beta \mathbf{1}_{[0,1]}(x) x^{\beta-1} dx$ with $\beta > 0$. Then $F^{-1}(u) = u^{1/\beta}$. Let us suppose that $\rho > 1$ and $\beta \geq \frac{\rho}{\rho-1}$. Using (1.4) with $i = 1$ for the second inequality, we obtain that*

$$e_N^\rho(\mu_\beta, \rho) \geq \int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du \geq \frac{1}{4^\rho N^{1+\frac{\rho}{\beta}}} \left(\left(\frac{3}{4} \right)^{\frac{1}{\beta}} - \left(\frac{1}{4} \right)^{\frac{1}{\beta}} \right)^\rho.$$

On the other hand, under the convention $F^{-1}(0) = 0$:

$$\begin{aligned} e_N^\rho(\mu_\beta, \rho) &\leq \mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{F^{-1}(\frac{i-1}{N})}, \mu_\beta \right) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(u^{\frac{1}{\beta}} - \left(\frac{i-1}{N} \right)^{\frac{1}{\beta}} \right)^\rho du \\ &\leq \int_0^{\frac{1}{N}} u^{\frac{\rho}{\beta}} du + \frac{1}{N^{1+\frac{\rho}{\beta}}} \sum_{i=2}^N \left(i^{\frac{1}{\beta}} - (i-1)^{\frac{1}{\beta}} \right)^\rho \\ (1.8) \quad &\leq \frac{\beta}{\beta + \rho} \times \frac{1}{N^{1+\frac{\rho}{\beta}}} + \frac{1}{\beta^\rho N^{1+\frac{\rho}{\beta}}} \sum_{i=2}^N (i-1)^{\frac{\rho}{\beta}-\rho}. \end{aligned}$$

When $\beta > \frac{\rho}{\rho-1}$, the last sum is smaller than $\sum_{j \in \mathbb{N}^*} j^{-(\rho-\frac{\rho}{\beta})}$ which is finite since $\rho - \frac{\rho}{\beta} > 1$ and $e_N(\mu_\beta, \rho) \asymp N^{-\frac{1}{\rho}-\frac{1}{\beta}}$. Notice that according Theorem 5.15 [17],

$$\forall \beta > 0, \forall \rho \geq 1, \quad \lim_{N \rightarrow +\infty} N e_N(\mu_\beta, \rho) = \frac{1}{2\beta(\rho+1)^{1/\rho}} \left(\int_0^1 u^{\frac{\rho}{\beta}-\rho} du \right)^{1/\rho},$$

with the right-hand side finite if and only if $\rho = 1$ or $\rho > 1$ and $\beta < \frac{\rho}{\rho-1}$ and then equal to $\frac{1}{2\beta(\rho+1)^{1/\rho}} \left(\frac{\beta}{\rho+\beta-\rho\beta} \right)^{1/\rho}$.

When $\rho > 1$, for the limiting value $\beta = \frac{\rho}{\rho-1}$, one has $\frac{1}{\rho} + \frac{1}{\beta} = 1$ and, by (1.8), $e_N^\rho(\mu_\beta, \rho) \leq \frac{1}{\rho} + \frac{1}{\beta^\rho N^\rho} \sum_{i=2}^N \frac{1}{i-1} \sim \frac{\ln N}{\beta^\rho N^\rho}$ as $N \rightarrow +\infty$. On the other hand, according to (1.4),

$$e_N^\rho(\mu_\beta, \rho) \geq \frac{1}{4^{2\rho-1} N^\rho} \sum_{i=1}^N \left((4i-1)^{\frac{1}{\beta}} - (4i-3)^{\frac{1}{\beta}} \right)^\rho \geq \frac{2^\rho}{4^{2\rho-1} \beta^\rho N^\rho} \sum_{i=1}^N \frac{1}{4i-1} \geq \frac{2^\rho}{4^{2\rho} \beta^\rho N^\rho} \sum_{i=1}^N \frac{1}{i} \sim \frac{2^\rho \ln N}{4^{2\rho} \beta^\rho N^\rho}$$

so that $e_N(\mu_\beta, \rho) \asymp N^{-1}(\ln N)^{\frac{1}{\rho}}$.

According to Corollary 6.15 [6], for $(X_i)_{i \geq 1}$ i.i.d. according to μ_β , $\mathbb{E}^{1/\rho} \left[\mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu_\beta \right) \right] \asymp N^{-\frac{1}{\rho} - \frac{1}{\beta}}$ if

$\rho > 2$ and $\beta > \frac{2\rho}{\rho-2}$ and $\mathbb{E}^{1/\rho} \left[\mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu_\beta \right) \right] \asymp N^{-1/2}$ if $\rho \leq 2$ and $\beta \geq 1$ or $\rho > 2$ and $\beta \in [1, \frac{2\rho}{\rho-2})$.

Note that apart from the restriction $\beta \geq 1$ made in [6] to ensure that the distribution is log-concave, the results concerning the optimal deterministic choice and the random choice share the same structure with different maximal orders of convergence 1 and 1/2. When $\rho > 2$ and $\beta > \frac{2\rho}{\rho-2}$, the deterministic and random orders of convergence are both equal to $\frac{1}{\rho} + \frac{1}{\beta}$.

1.5 Non-asymptotic lower bound for the error

According to Theorem 5.20 [17] and its proof, $\limsup_{N \rightarrow \infty} N e_N(\mu, \rho) \geq \frac{1}{2} \int_0^{\frac{1}{2}} (F^{-1}(u + \frac{1}{2}) - F^{-1}(u)) du$ with the right-hand side positive apart when μ is a Dirac mass and $e_N(\mu, \rho)$ vanishes for all $N, \rho \geq 1$. This result may be complemented by the following non-asymptotic bound.

Lemma 1.9. $\forall \rho \geq 1, \forall N \geq 1, \quad N e_N(\mu, \rho) + (N+1) e_{N+1}(\mu, \rho) \geq \frac{1}{2} \int_{\mathbb{R}} F(x) \wedge (1 - F(x)) dx.$

Remark 1.10. • One has

$$\int_{\mathbb{R}} F(x) \wedge (1 - F(x)) dx = \int_{-\infty}^{F^{-1}(\frac{1}{2})} F(x) dx + \int_{F^{-1}(\frac{1}{2})}^{+\infty} \left(\frac{1}{2} - \left(F(x) - \frac{1}{2} \right) \right) dx = \int_0^{\frac{1}{2}} \left(F^{-1}\left(u + \frac{1}{2}\right) - F^{-1}(u) \right) du$$

as easily seen since the sum and the last integral correspond to the area of the points at the right to $(F(x))_{-\infty \leq x \leq F^{-1}(\frac{1}{2})}$, at the left to $(F(x) - \frac{1}{2})_{F^{-1}(\frac{1}{2}) < x < +\infty}$, above 0 and below $\frac{1}{2}$ respectively computed by integration with respect to the abscissa and to the ordinate.

- The analogous result in the random case is stated in Theorem 3.1 [6]: when $(X_i)_{i \geq 1}$ are i.i.d. according to μ , $\mathbb{E} \left[\mathcal{W}_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu \right) \right] \geq \frac{1}{2\sqrt{2N}} \mathbb{E} [|X_1 - F^{-1}(1/2)|]$. In other words, unless μ is a Dirac mass, the random rate cannot be quicker than the usual Monte Carlo rate $\frac{1}{\sqrt{N}}$.

Proof. Note that when $\rho \geq \tilde{\rho} \geq 1$ and $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$, with $(x_i^N)_{1 \leq i \leq N}$ denoting the optimal points for $\rho \geq 1$,

$$(1.9) \quad e_N(\mu, \rho) = \mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}, \mu \right) \geq \mathcal{W}_{\tilde{\rho}} \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}, \mu \right) \geq e_N(\mu, \tilde{\rho}).$$

Hence $\rho \mapsto e_N(\mu, \rho)$ is non-decreasing and it is enough to check the statement for $\rho = 1$. It follows by plugging into (1.6) the following inequality written for $v = F(x)$:

$$\forall v \in [0, 1], \forall N \geq 1, \quad \min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq \frac{v \wedge (1-v)}{2},$$

To prove this inequality, we remark that for $v \in (0, 1)$, there are two possibilities

- Either $\lfloor Nv \rfloor \leq Nv < (N+1)v \leq \lfloor Nv \rfloor + 1$, which implies that $(Nv - \lfloor Nv \rfloor) \vee (\lfloor Nv \rfloor + 1 - (N+1)v) \geq \frac{1-v}{2}$ while $\lfloor Nv \rfloor + 1 - Nv = \lfloor Nv \rfloor + 1 - (N+1)v + v \geq v$ and $(N+1)v - \lfloor Nv \rfloor = Nv - \lfloor Nv \rfloor + v \geq v$ so that

$$\min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq v \wedge \frac{1-v}{2}.$$

- Or $\lfloor Nv \rfloor \leq Nv < \lfloor Nv \rfloor + 1 \leq (N+1)v$, which implies that $(\lfloor Nv \rfloor + 1 - Nv) \vee ((N+1)v - (\lfloor Nv \rfloor + 1)) \geq \frac{v}{2}$ while $Nv - \lfloor Nv \rfloor = (N+1)v - (\lfloor Nv \rfloor + 1) + 1 - v \geq 1 - v$ and $\lfloor Nv \rfloor + 2 - (N+1)v = \lfloor Nv \rfloor + 1 - Nv + 1 - v > 1 - v$ so that

$$\min_{j \in \mathbb{N}} |Nv - j| \vee \min_{j \in \mathbb{N}} |(N+1)v - j| \geq \frac{v}{2} \wedge (1-v).$$

Synthesising the two cases and remarking that the inequality still holds for $v \in \{0, 1\}$, we conclude. $\square$

2 The case when the support of μ is unbounded

According to Theorem 5.21 (ii) [17], when the support of μ is bounded, $\sup_{N \geq 1} N^{1/\rho} e_N(\mu, \rho) < +\infty$ and, when moreover F^{-1} is continuous, then $\lim_{N \rightarrow \infty} N^{1/\rho} e_N(\mu, \rho) = 0$ for $\rho > 1$ by Lemma 1.7. In this section, we first show that in the unbounded support case, the order of convergence cannot exceed the minimal order $1/\rho$ in the bounded support case. Next we state our main result : a necessary and sufficient condition for convergence with order $\alpha \in (0, \frac{1}{\rho})$ which precises the implication stated in Theorem 5.21 (i) [17] :

$$\int_{\mathbb{R}^d} |x|^{\frac{\rho}{1-\alpha\rho}} \mu(dx) < +\infty \Rightarrow \lim_{N \rightarrow \infty} N^\alpha e_N(\mu, \rho) = 0.$$

We finally address the boundary case $\alpha = 1/\rho$ where we only obtain a necessary condition and illustrate the possibility that, when $\rho > 1$, $\lim_{N \rightarrow \infty} N^{1/\rho} e_N(\mu, \rho) = 0$ for some probability measure μ with unbounded support.

2.1 The order of convergence cannot exceed $1/\rho$

According to the next result, the order of convergence of $e_N(\mu, \rho)$ cannot exceed $\frac{1}{\rho}$ when the support of μ is not bounded.

Proposition 2.1. *Let $\rho > 1$. Then $\exists \alpha > \frac{1}{\rho}$, $\sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty \implies F^{-1}(1) - F^{-1}(0+) < +\infty$.*

Proof. Let $\rho > 1$ and $\alpha > \frac{1}{\rho}$ be such that $\sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty$ so that, by (1.2),

$$\sup_{N \geq 1} N^{\alpha\rho} \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \right) < +\infty.$$

By (1.4) for $i = 1$ and $N \geq 1$, we have:

$$\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du \geq \frac{1}{4^\rho N} \left(F^{-1}\left(\frac{1}{2N}\right) - F^{-1}\left(\frac{1}{4N}\right) \right)^\rho.$$

Therefore $C := \sup_{N \geq 1} (2N)^{\alpha-\frac{1}{\rho}} \left(F^{-1}\left(\frac{1}{2N}\right) - F^{-1}\left(\frac{1}{4N}\right) \right) < +\infty$. For $k \in \mathbb{N}^*$, we deduce that $F^{-1}(2^{-(k+1)}) - F^{-1}(2^{-k}) \geq -C 2^{\frac{1-\alpha\rho}{\rho}k}$, and after summation that:

$$(2.1) \quad \forall k \in \mathbb{N}^*, F^{-1}(2^{-k}) \geq F^{-1}(1/2) - \frac{C}{2^{\alpha-\frac{1}{\rho}-1}} \left(1 - 2^{\frac{1-\alpha\rho}{\rho}(k-1)} \right).$$

When $k \rightarrow +\infty$, the right-hand side goes to $\left(F^{-1}\left(\frac{1}{2}\right) - \frac{C}{2^{\alpha-\frac{1}{\rho}-1}} \right) > -\infty$ so that $F^{-1}(0+) > -\infty$. In a symmetric way, we check that $F^{-1}(1) < +\infty$ so that μ is compactly supported. $\square$

2.2 Necessary and sufficient condition for convergence with order $\alpha \in (0, \frac{1}{\rho})$

Our main result is the following necessary and sufficient condition for $e_N(\mu, \rho)$ to go to 0 with order $\alpha \in (0, \frac{1}{\rho})$.

Theorem 2.2. *Let $\rho \geq 1$ and $\alpha \in (0, \frac{1}{\rho})$. We have*

$$\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} \left(F(-x) + 1 - F(x) \right) < +\infty \Leftrightarrow \sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty \Leftrightarrow \sup_{N \geq 2} \sup_{x_{2:N-1}} N^\alpha \mathcal{W}_\rho(\mu_N(x_{2:N-1}), \mu) < +\infty$$

where $\mu_N(x_{2:N-1}) = \frac{1}{N} \left(\delta_{F^{-1}(\frac{1}{N}) \wedge (-N^{\frac{1}{\rho}-\alpha})} + \sum_{i=2}^{N-1} \delta_{x_i} + \delta_{F^{-1}(\frac{N-1}{N}) \vee N^{\frac{1}{\rho}-\alpha}} \right)$ and $\sup_{x_{2:N-1}}$ means the supremum over the choice of $x_i \in [F^{-1}(\frac{i-1}{N}+), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$. Moreover,

$$\lim_{x \rightarrow +\infty} x^{\frac{\rho}{1-\alpha\rho}} \left(F(-x) + 1 - F(x) \right) = 0 \Leftrightarrow \lim_{N \rightarrow +\infty} N^\alpha e_N(\mu, \rho) = 0.$$

Remark 2.3. • Let us relate the order of convergence to the maximal integrable power $\hat{\beta} := \sup\{\beta \geq 0 : \int_{\mathbb{R}} |x|^\beta \mu(dx) < \infty\}$ of μ . For $\beta \in (0, \hat{\beta})$, $\int_{\mathbb{R}} |x|^\beta \mu(dx) < \infty$ while when $\hat{\beta} < \infty$, for $\beta > \hat{\beta}$, $\int_{\mathbb{R}} |x|^{\frac{\beta+\hat{\beta}}{2}} \mu(dx) = +\infty$, so that, by Lemma 2.4 below, $\sup_{x \geq 0} x^\beta (F(-x) + 1 - F(x)) = +\infty$. Let $\rho \geq 1$. If $\hat{\beta} > \rho$, we deduce from Theorem 5.21 (i) [17] that for each $\alpha \in (0, \frac{1}{\rho} - \frac{1}{\hat{\beta}})$, $\lim_{N \rightarrow \infty} N^\alpha e_N(\mu, \rho) = 0$ and, when $\hat{\beta} < +\infty$, Theorem 2.2 ensures that for each $\alpha > \frac{1}{\rho} - \frac{1}{\hat{\beta}}$, $\sup_{N \geq 1} N^\alpha e_N(\mu, \rho) = +\infty$ since $\frac{\rho}{1-\alpha\rho} > \hat{\beta}$. In this sense, when $\rho < \hat{\beta} < +\infty$ the order of convergence of $e_N(\mu, \rho)$ to 0 is $\frac{1}{\rho} - \frac{1}{\hat{\beta}}$. Moreover, the boundedness and the vanishing limit at infinity for the sequence $(N^{\frac{1}{\rho} - \frac{1}{\hat{\beta}}} e_N(\mu, \rho))_{N \geq 1}$ are respectively equivalent to the same property for the function $\mathbb{R}_+ \ni x \mapsto x^{\hat{\beta}} (F(-x) + 1 - F(x))$. Note that $\limsup_{x \rightarrow +\infty} x^{\hat{\beta}} (F(-x) + 1 - F(x))$ can be $+\infty$, in which case, $\sup_{N \geq 1} N^{\frac{1}{\rho} - \frac{1}{\hat{\beta}}} e_N(\mu, \rho) = +\infty$.

- In the proof (see (2.4) below), we estimate $\sup_{N \geq 2} \sup_{x_{2:N-1}} N^{\alpha\rho} \mathcal{W}_\rho^\rho(\mu_N(x_{2:N-1}), \mu)$ in terms of $C = \sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x))$.
- According to Theorem 7.16 [6], for $(X_i)_{i \geq 1}$ i.i.d. according to μ ,

$$\sup_{N \geq 1} N^{\frac{1}{2\rho}} \mathbb{E}^{1/\rho} \left[\mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu \right) \right] \leq \left(\rho 2^{\rho-1} \int_{\mathbb{R}} |x|^{\rho-1} \sqrt{F(x)(1-F(x))} dx \right)^{1/\rho}$$

with $\exists \varepsilon > 0$, $\int_{\mathbb{R}} |x|^{2\rho+\varepsilon} \mu(dx) < +\infty \Rightarrow \int_{\mathbb{R}} |x|^{\rho-1} \sqrt{F(x)(1-F(x))} dx < +\infty \Rightarrow \int_{\mathbb{R}} |x|^{2\rho} \mu(dx) < +\infty$ (the reverse implications fail) by the discussions just after this theorem and after Theorem 3.2 [6]. The condition $\sup_{x \geq 0} x^{2\rho} (F(-x) + 1 - F(x)) < +\infty$ equivalent to $\sup_{N \geq 1} N^{\frac{1}{2\rho}} e_N(\mu, \rho) < +\infty$ is slightly weaker than $\int_{\mathbb{R}} |x|^{2\rho} \mu(dx) < +\infty$, according to Lemma 2.4 just below. Moreover, we address similarly any order of convergence α with $\alpha \in (0, \frac{1}{\rho})$ for $e_N(\mu, \rho)$, while the order $\frac{1}{2\rho}$ seems to play a special role for $\mathbb{E}^{1/\rho} \left[\mathcal{W}_\rho^\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu \right) \right]$ in the random case. When $\rho = 1$, the order of convergence α for $\alpha \in (0, 1/2)$ is addressed in the random case in Theorem 2.2 [8] where the finiteness of $\sup_{x \geq 0} x^{\frac{1}{1-\alpha}} (F(-x) + 1 - F(x))$ is stated to be equivalent to the stochastic boundedness of the sequence $\left(N^\alpha \mathcal{W}_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu \right) \right)_{N \geq 1}$. When $\alpha = 1/2$, the stochastic boundedness property is, according to Theorem 2.1 (b) [8], equivalent to $\int_{\mathbb{R}} \sqrt{F(x)(1-F(x))} dx < +\infty$.

The proof of Theorem 2.2 relies on the next lemma, the proof of which is postponed to the appendix.

Lemma 2.4. For $\beta > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}} |y|^\beta \mu(dy) < +\infty &\implies \lim_{x \rightarrow +\infty} x^\beta (F(-x) + 1 - F(x)) = 0 \\ &\implies \sup_{x \geq 0} x^\beta (F(-x) + 1 - F(x)) < +\infty \implies \forall \varepsilon \in (0, \beta], \int_{\mathbb{R}} |y|^{\beta-\varepsilon} \mu(dy) < +\infty \end{aligned}$$

and $\sup_{x \geq 0} x^\beta (F(-x) + 1 - F(x)) < +\infty \Leftrightarrow \sup_{u \in (0, 1/2]} u^{\frac{1}{\beta}} (F^{-1}(1-u) - F^{-1}(u)) < +\infty$ with

$$(2.2) \quad \sup_{u \in (0, 1/2]} u^{\frac{1}{\beta}} (F^{-1}(1-u) - F^{-1}(u)) \leq \left(\sup_{x \geq 0} x^\beta F(-x) \right)^{\frac{1}{\beta}} + \left(\sup_{x \geq 0} x^\beta (1 - F(x)) \right)^{\frac{1}{\beta}}.$$

Last, $\lim_{x \rightarrow +\infty} x^\beta (F(-x) + 1 - F(x)) = 0 \Leftrightarrow \lim_{u \rightarrow 0+} u^{\frac{1}{\beta}} (F^{-1}(1-u) - F^{-1}(u)) = 0$.

Proof of Theorem 2.2. Since by Lemma 2.4,

$$\sup_{u \in (0, 1/2]} u^{\frac{1}{\rho} - \alpha} (F^{-1}(1-u) - F^{-1}(u)) < +\infty \implies \sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) < +\infty$$

and, by (1.2),

$$e_N^\rho(\mu, \rho) \geq \int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du$$

to prove the equivalence, it is enough to check that

$$\begin{aligned} \sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} \left(F(-x) + 1 - F(x) \right) < +\infty &\implies \sup_{N \geq 1} N^{\alpha\rho} e_N^\rho(\mu, \rho) < +\infty \text{ and that} \\ \sup_{N \geq 1} N^{\alpha\rho} \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \right) < +\infty \\ &\implies \sup_{u \in (0, 1/2]} u^{\frac{1}{\rho}-\alpha} (F^{-1}(1-u) - F^{-1}(u)) < +\infty. \end{aligned}$$

We are now going to do so and thus prove that the four suprema in the two last implications are simultaneously finite or infinite.

Let us first suppose that $C := \sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) < +\infty$ and set $N \geq 2$. Let $x_i \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$. We have

$$e_N^\rho(\mu, \rho) \leq \mathcal{W}_\rho^\rho(\mu_N(x_{2:N-1}), \mu) = L_N + M_N + U_N$$

with $L_N = \int_0^{\frac{1}{N}} \left| F^{-1}(u) - F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha}) \right|^\rho du$, $U_N = \int_{\frac{N-1}{N}}^1 \left| F^{-1}(u) - F^{-1}\left(\frac{N-1}{N}\right) \vee N^{\frac{1}{\rho}-\alpha} \right|^\rho du$ and

$$\begin{aligned} M_N &= \sum_{i=2}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i|^\rho du \leq \sum_{i=2}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(F^{-1}\left(\frac{i}{N}\right) - F^{-1}\left(\frac{i-1}{N}\right) \right)^\rho du \\ &\leq \frac{1}{N} \sum_{i=1}^N \left(F^{-1}\left(\frac{N-1}{N}\right) - F^{-1}\left(\frac{1}{N}\right) \right)^{\rho-1} \left(F^{-1}\left(\frac{i}{N}\right) - F^{-1}\left(\frac{i-1}{N}\right) \right) \\ (2.3) \quad &= \frac{1}{N} \left(F^{-1}\left(\frac{N-1}{N}\right) - F^{-1}\left(\frac{1}{N}\right) \right)^\rho \\ &\leq 2^\rho C^{1-\alpha\rho} N^{-\alpha\rho}, \end{aligned}$$

where we used (2.2) applied with $\beta = \frac{\rho}{1-\alpha\rho}$ for the last inequality. Let $x_+ = 0 \vee x$ denote the positive part of any real number x . Applying Lemma 1.2 with $x = F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha})$, we obtain that

$$\begin{aligned} L_N &= \rho \int_{-\infty}^{F^{-1}(\frac{1}{N}) \wedge (-N^{\frac{1}{\rho}-\alpha})} \left(F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha}) - y \right)^{\rho-1} F(y) dy \\ &\quad + \rho \int_{F^{-1}(\frac{1}{N}) \wedge (-N^{\frac{1}{\rho}-\alpha})}^{F^{-1}(\frac{1}{N})} \left(y - F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha}) \right)^{\rho-1} \left(\frac{1}{N} - F(y) \right) dy \\ &\leq \rho \int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1} F(-y) dy + \frac{1}{N} \left(N^{\frac{1}{\rho}-\alpha} + F^{-1}\left(\frac{1}{N}\right) \right)_+^\rho. \end{aligned}$$

In a symmetric way, we check that $U_N \leq \rho \int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1} (1 - F(y)) dy + \frac{1}{N} \left(N^{\frac{1}{\rho}-\alpha} - F^{-1}\left(\frac{N-1}{N}\right) \right)_+^\rho$ so that

$$\begin{aligned} L_N + U_N &\leq \rho C \int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{-1-\frac{\alpha\rho^2}{1-\alpha\rho}} dy + \frac{1}{N} \left(\left(N^{\frac{1}{\rho}-\alpha} + F^{-1}(1/2) \right)_+^\rho + \left(N^{\frac{1}{\rho}-\alpha} - F^{-1}(1/2) \right)_+^\rho \right) \\ &\leq \frac{1-\alpha\rho}{\alpha\rho} C N^{-\alpha\rho} + (1 + 2^{\rho-1}) N^{-\alpha\rho} + 2^{\rho-1} |F^{-1}(1/2)|^\rho N^{-1}. \end{aligned}$$

Since $N^{-1} \leq 2^{\alpha\rho-1}N^{-\alpha\rho}$, we conclude that with $\sup_{x_{2:N-1}}$ denoting the supremum over $x_i \in [F^{-1}(\frac{i-1}{N}+), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$,

(2.4)

$$\sup_{N \geq 2} N^{\alpha\rho} e_N^\rho(\mu, \rho) \leq \sup_{N \geq 2} \sup_{x_{2:N-1}} N^{\alpha\rho} \mathcal{W}_\rho^\rho(\mu_N(x_{2:N-1}), \mu) \leq 2^\rho C^{1-\alpha\rho} + \frac{1-\alpha\rho}{\alpha\rho} C + 1 + 2^{\rho-1} + 2^{\rho+\alpha\rho-2} |F^{-1}(1/2)|^\rho.$$

We may replace $\sup_{N \geq 2} N^{\alpha\rho} e_N^\rho(\mu, \rho)$ by $\sup_{N \geq 1} N^{\alpha\rho} e_N^\rho(\mu, \rho)$ in the left-hand side, since, applying Lemma 1.2 with $x = 0$, then using that for $y \geq 0$, $F(-y) + 1 - F(y) = \mu((-\infty, -y] \cup (y, +\infty)) \leq 1$, we obtain that

$$e_1^\rho(\mu, \rho) \leq \rho \int_0^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy \leq \rho \int_0^1 y^{\rho-1} dy + \rho C \int_1^{+\infty} y^{-1-\frac{\alpha\rho^2}{1-\alpha\rho}} dy = 1 + \frac{1-\alpha\rho}{\alpha\rho} C.$$

Let us next suppose that $\sup_{N \geq 1} N^{\alpha\rho} \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \right) < +\infty$. Like in the proof of Proposition 2.1, we deduce (2.1). With the monotonicity of F^{-1} , this inequality implies that

$$\exists C < +\infty, \forall u \in (0, 1/2], \quad F^{-1}(u) \geq F^{-1}(1/2) - \frac{C}{1 - 2^{\alpha-\frac{1}{\rho}}} \left(u^{\alpha-\frac{1}{\rho}} - 1 \right),$$

and therefore that $\inf_{u \in (0, 1/2]} \left(u^{\frac{1}{\rho}-\alpha} F^{-1}(u) \right) > -\infty$. With a symmetric reasoning, we conclude that

$$\sup_{u \in (0, 1/2]} u^{\frac{1}{\rho}-\alpha} \left(F^{-1}(1-u) - F^{-1}(u) \right) < +\infty.$$

Let us now assume that $\limsup_{x \rightarrow +\infty} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) \in (0, +\infty)$, which, in particular implies that $\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} (F(-x) + 1 - F(x)) < +\infty$ and check that $\limsup_{N \rightarrow \infty} N^\alpha e_N(\mu, \rho) > 0$. For $x > 0$, we have, on the one hand

$$\begin{aligned} x^{\frac{\alpha\rho^2}{1-\alpha\rho}} \int_x^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy &\geq x^{\frac{\alpha\rho^2}{1-\alpha\rho}} \int_x^{2x} x^{\rho-1} (F(-2x) + 1 - F(2x)) dy \\ &= x^{\frac{\rho}{1-\alpha\rho}} (F(-2x) + 1 - F(2x)). \end{aligned}$$

On the other hand, still for $x > 0$,

$$\begin{aligned} x^{\frac{\alpha\rho^2}{1-\alpha\rho}} \int_x^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy &\leq x^{\frac{\alpha\rho^2}{1-\alpha\rho}} \sup_{y \geq x} y^{\frac{\rho}{1-\alpha\rho}} (F(-y) + 1 - F(y)) \int_x^{+\infty} y^{-\frac{\alpha\rho^2}{1-\alpha\rho}-1} dy \\ (2.5) \quad &= \frac{1-\alpha\rho}{\alpha\rho^2} \sup_{y \geq x} y^{\frac{\rho}{1-\alpha\rho}} (F(-y) + 1 - F(y)) \end{aligned}$$

Therefore $\limsup_{x \rightarrow +\infty} x^{\frac{\alpha\rho^2}{1-\alpha\rho}} \int_x^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy \in (0, +\infty)$ and, by monotonicity of the integral,

$$(2.6) \quad \limsup_{N \rightarrow +\infty} y_N^{\frac{\alpha\rho^2}{1-\alpha\rho}} \int_{y_N}^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy \in (0, +\infty)$$

along any sequence $(y_N)_{N \in \mathbb{N}}$ of positive numbers increasing to $+\infty$ and such that $\limsup_{N \rightarrow +\infty} \frac{y_{N+1}}{y_N} < +\infty$.

By Lemmas 2.4 and 1.1, we have $\kappa := \sup_{N \geq 1} N^{\alpha-\frac{1}{\rho}} (x_1^N \vee (-x_1^N)) < +\infty$ (notice that since $x_1^N \leq x_N^N$, $\kappa \geq 0$). With (1.5), we deduce that:

$$\begin{aligned} \frac{e_N^\rho(\mu, \rho)}{\rho} &\geq \int_{-\infty}^{x_1^N} (x_1^N - y)^{\rho-1} F(y) dy + \int_{x_N^N}^{+\infty} (y - x_N^N)^{\rho-1} (1 - F(y)) dy \\ &\geq \int_{-\infty}^{-\kappa N^{\frac{1}{\rho}-\alpha}} \left(-\kappa N^{\frac{1}{\rho}-\alpha} - y \right)^{\rho-1} F(y) dy + \int_{\kappa N^{\frac{1}{\rho}-\alpha}}^{+\infty} \left(y - \kappa N^{\frac{1}{\rho}-\alpha} \right)^{\rho-1} (1 - F(y)) dy \\ &\geq 2^{1-\rho} \int_{2\kappa N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1} (F(-y) + 1 - F(y)) dy. \end{aligned}$$

Applying (2.6) with $y_N = 2\kappa N^{\frac{1}{\rho}-\alpha}$, we conclude that $\limsup_{N \rightarrow +\infty} N^{\alpha\rho} e_N^\rho(\mu, \rho) > 0$. If $x^{\frac{\rho}{1-\alpha\rho}}(F(-x) + 1 - F(x))$ does not go to 0 as $x \rightarrow +\infty$ then either $\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}}(F(-x) + 1 - F(x)) = +\infty = \sup_{N \geq 1} N^\alpha e_N(\mu, \rho)$ or $\limsup_{x \rightarrow +\infty} x^{\frac{\rho}{1-\alpha\rho}}(F(-x) + 1 - F(x)) \in (0, +\infty)$ and $\limsup_{N \rightarrow +\infty} N^\alpha e_N(\mu, \rho) \in (0, +\infty)$ so that, synthesizing the two cases, $N^\alpha e_N(\mu, \rho)$ does not go to 0 as $N \rightarrow +\infty$. Therefore, to conclude the proof of the second statement, it is enough to suppose $\lim_{x \rightarrow +\infty} x^{\frac{\rho}{1-\alpha\rho}}(F(-x) + 1 - F(x)) = 0$ and deduce $\lim_{N \rightarrow +\infty} N^\alpha e_N(\mu, \rho) = 0$, which we now do. By Lemma 2.4, $\lim_{N \rightarrow \infty} N^{\alpha-\frac{1}{\rho}}(F^{-1}(\frac{N-1}{N}) - F^{-1}(\frac{1}{N})) = 0$. Since, reasoning like in the above derivation of (2.3), we have $\sum_{i=2}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i^N|^\rho du \leq \frac{1}{N} (F^{-1}(\frac{N-1}{N}) - F^{-1}(\frac{1}{N}))^\rho$ for $N \geq 3$, we deduce that $\lim_{N \rightarrow \infty} N^{\alpha\rho} \sum_{i=2}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F^{-1}(u) - x_i^N|^\rho du = 0$. Let

$$S_N = \sup_{x \geq N^{\frac{1}{2\rho}-\frac{\alpha}{2}}} \left(x^{\frac{\rho}{1-\alpha\rho}}(F(-x) + 1 - F(x)) \right)^{\frac{1-\alpha\rho}{2\alpha\rho^2}} \text{ and } y_N = F^{-1}\left(\frac{N-1}{N}\right) \vee N^{\frac{1}{2\rho}-\frac{\alpha}{2}} \vee (S_N N^{\frac{1}{\rho}-\alpha}).$$

Using Lemma 1.2 for the first inequality, (2.5) for the second one, then the definition of y_N for the third, we obtain that

$$\begin{aligned} & N^{\alpha\rho} \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \\ & \leq \rho N^{\alpha\rho} \int_{F^{-1}(\frac{N-1}{N})}^{y_N} (y_N - y)^{\rho-1} \left(F(y) - \frac{N-1}{N} \right) dy + \rho N^{\alpha\rho} \int_{y_N}^{+\infty} (y - y_N)^{\rho-1} (1 - F(y)) dy \\ & \leq N^{\alpha\rho-1} \int_{F^{-1}(\frac{N-1}{N})}^{y_N} \rho (y_N - y)^{\rho-1} dy + \frac{1-\alpha\rho}{\alpha\rho^2} N^{\alpha\rho} y_N^{-\frac{\alpha\rho^2}{1-\alpha\rho}} \sup_{y \geq y_N} y^{\frac{\rho}{1-\alpha\rho}} (F(-y) + 1 - F(y)) \\ & \leq N^{\alpha\rho-1} \left(N^{\frac{1}{2\rho}-\frac{\alpha}{2}} \vee (S_N N^{\frac{1}{\rho}-\alpha}) - F^{-1}\left(\frac{N-1}{N}\right) \right)_+^\rho \\ & \quad + \frac{1-\alpha\rho}{\alpha\rho^2} N^{\alpha\rho} \left(S_N N^{\frac{1}{\rho}-\alpha} \right)^{-\frac{\alpha\rho^2}{1-\alpha\rho}} \sup_{y \geq N^{\frac{1}{2\rho}-\frac{\alpha}{2}}} y^{\frac{\rho}{1-\alpha\rho}} (F(-y) + 1 - F(y)) \\ & \leq \left(N^{\frac{\alpha}{2}-\frac{1}{2\rho}} \vee S_N - N^{\alpha-\frac{1}{\rho}} F^{-1}\left(\frac{N-1}{N}\right) \right)_+^\rho + \frac{1-\alpha\rho}{\alpha\rho^2} S_N^{\frac{\alpha\rho^2}{1-\alpha\rho}}. \end{aligned}$$

Since $\lim_{N \rightarrow \infty} N^{\alpha-\frac{1}{\rho}} F^{-1}(\frac{N-1}{N}) = 0 = \lim_{N \rightarrow \infty} N^{\frac{\alpha}{2}-\frac{1}{2\rho}} = \lim_{N \rightarrow +\infty} S_N$, we deduce that $\lim_{N \rightarrow \infty} N^{\alpha\rho} \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du = 0$. Dealing in a symmetric way with $N^{\alpha\rho} \int_0^{\frac{1}{N}} |F^{-1}(u) - x_N^N|^\rho du$, we conclude that $\lim_{N \rightarrow \infty} N^{\alpha\rho} e_N^\rho(\mu, \rho) = 0$. $\square$

Example 2.5. let $\mu_\beta(dx) = f(x) dx$ with $f(x) = \beta \frac{\mathbf{1}_{\{x \geq 1\}}}{x^{\beta+1}}$ be the Pareto distribution with parameter $\beta > 0$. Then $F(x) = \mathbf{1}_{\{x \geq 1\}}(1 - x^{-\beta})$ and $F^{-1}(u) = (1 - u)^{-\frac{1}{\beta}}$. To ensure that $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$, we suppose that $\beta > \rho$. Since $\frac{\rho}{1-\rho(\frac{1}{\rho}-\frac{1}{\beta})} = \beta$ we have $\lim_{x \rightarrow +\infty} x^{\frac{\rho}{1-\rho(\frac{1}{\rho}-\frac{1}{\beta})}}(F(-x) + 1 - F(x)) = 1$. Replacing $\limsup$ by $\liminf$ in the last step of the proof of Theorem 2.2, we check that $\liminf_{N \rightarrow +\infty} N^{\frac{1}{\rho}-\frac{1}{\beta}} e_N(\mu_\beta, \rho) > 0$ and deduce with the statement of this theorem that $e_N(\mu_\beta, \rho) \asymp N^{-\frac{1}{\rho}+\frac{1}{\beta}} \asymp \sup_{x_{2:N-1}} \mathcal{W}_\rho(\mu_N(x_{2:N-1}), \mu_\beta)$.

2.3 The boundary case $\alpha = 1/\rho$

Before giving a necessary condition for convergence with the boundary order $1/\rho$, we show in the next example that for $\beta > 0$, $e_N(\mathbf{1}_{\{x > 0\}} \beta x^{\beta-1} \exp(-x^\beta) dx, \rho)$ converges to 0 with this order up to some logarithmic factor. In particular, the case $\beta > 1$ illustrates the possibility that, when $\rho > 1$, $\lim_{N \rightarrow +\infty} N^{1/\rho} e_N(\mu, \rho) = 0$ for some probability measures μ with unbounded support. Of course, F^{-1} is then continuous on $(0, 1)$, since, by Remark 5.22 (ii) [17], $\limsup_{N \rightarrow +\infty} N^{1/\rho} e_N(\mu, \rho) > 0$ otherwise.

Example 2.6. For the Weibull distribution $\mu_\beta(dx) = f(x) dx$ with $f(x) = \mathbf{1}_{\{x>0\}} \beta x^{\beta-1} \exp(-x^\beta)$ with $\beta > 0$ (the exponential distribution case $\beta = 1$, was addressed in Example 5.17 and Remark 5.22 (i) [17]), we have that $F(x) = \mathbf{1}_{\{x>0\}} (1 - \exp(-x^\beta))$, $F^{-1}(u) = (-\ln(1-u))^{\frac{1}{\beta}}$ and $f(F^{-1}(u)) = \beta(1-u)(-\ln(1-u))^{1-\frac{1}{\beta}}$. The density f is decreasing on $[x_\beta, +\infty)$ where $x_\beta = \left(\frac{(\beta-1)\vee 0}{\beta}\right)^{\frac{1}{\beta}}$. Using (1.4), the equality $F^{-1}(w) - F^{-1}(u) = \int_u^w \frac{dv}{f(F^{-1}(v))}$ valid for $u, w \in (0, 1)$ and the monotonicity of the density, we obtain that for N large enough so that $\lceil NF(x_\beta) \rceil \leq N-1$,

$$(2.7) \quad \begin{aligned} e_N^\rho(\mu_\beta, \rho) &\geq \frac{1}{4^\rho N} \sum_{i=\lceil NF(x_\beta) \rceil+1}^N \left(\int_{\frac{4i-3}{4N}}^{\frac{4i-1}{4N}} \frac{du}{f(F^{-1}(u))} \right)^\rho \geq \frac{1}{8^\rho N^{\rho+1}} \sum_{i=\lceil NF(x_\beta) \rceil+1}^N \frac{1}{f^\rho(F^{-1}(\frac{4i-3}{4N}))} \\ &\geq \frac{1}{(8N)^\rho} \sum_{i=\lceil NF(x_\beta) \rceil+2}^N \int_{\frac{i-2}{N}}^{\frac{i-1}{N}} \frac{du}{f^\rho(F^{-1}(u))} = \frac{1}{(8N)^\rho} \int_{\frac{\lceil NF(x_\beta) \rceil}{N}}^{\frac{N-1}{N}} \frac{du}{f^\rho(F^{-1}(u))}. \end{aligned}$$

Using (1.3) for the first inequality, Hölder's inequality for the second inequality, then Fubini's theorem for the third, we obtain that

$$\begin{aligned} e_N^\rho(\mu_\beta, \rho) - \int_{\frac{N-1}{N}}^1 |x_N^N - F^{-1}(u)|^\rho du &\leq \sum_{i=1}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| \int_{\frac{2i-1}{2N}}^u \frac{dv}{f(F^{-1}(v))} \right|^\rho du \\ &\leq \sum_{i=1}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| u - \frac{2i-1}{2N} \right|^{\rho-1} \left| \int_{\frac{2i-1}{2N}}^u \frac{dv}{f^\rho(F^{-1}(v))} \right| du \leq \frac{1}{(2N)^\rho \rho} \int_0^{\frac{N-1}{N}} \frac{dv}{f^\rho(F^{-1}(v))}. \end{aligned}$$

We have $F(x_\beta) < 1$ and, when $\beta > 1$, $F(x_\beta) > 0$. By integration by parts, for $\rho > 1$,

$$\begin{aligned} (\rho-1) \int_{F(x_\beta)}^{\frac{N-1}{N}} \frac{\beta^\rho du}{f^\rho(F^{-1}(u))} &= \int_{F(x_\beta)}^{\frac{N-1}{N}} (\rho-1)(1-u)^{-\rho} (-\ln(1-u))^{\frac{\rho}{\beta}-\rho} du \\ &= \left[(1-u)^{1-\rho} (-\ln(1-u))^{\frac{\rho}{\beta}-\rho} \right]_{F(x_\beta)}^{\frac{N-1}{N}} + \left(\frac{\rho}{\beta} - \rho \right) \int_{F(x_\beta)}^{\frac{N-1}{N}} (1-u)^{-\rho} (-\ln(1-u))^{\frac{\rho}{\beta}-\rho-1} du \\ &= N^{\rho-1} (\ln N)^{\frac{\rho}{\beta}-\rho} + o \left(\int_{F(x_\beta)}^{\frac{N-1}{N}} (1-u)^{-\rho} (-\ln(1-u))^{\frac{\rho}{\beta}-\rho} du \right) \sim N^{\rho-1} (\ln N)^{\frac{\rho}{\beta}-\rho}, \end{aligned}$$

as $N \rightarrow +\infty$. We obtain the same equivalent when replacing the lower integration limit $F(x_\beta)$ in the left-hand side by $\frac{\lceil NF(x_\beta) \rceil}{N}$ or 0 since $\lim_{N \rightarrow +\infty} \int_{F(x_\beta)}^{\frac{\lceil NF(x_\beta) \rceil}{N}} \frac{du}{f^\rho(F^{-1}(u))} = 0$ and $\int_0^{F(x_\beta)} \frac{du}{f^\rho(F^{-1}(u))} < +\infty$. On the other hand,

$$\int_{\frac{N-1}{N}}^1 |x_N^N - F^{-1}(u)|^\rho du \leq \int_{\frac{N-1}{N}}^1 \left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \right)^\rho du.$$

When $\beta < 1$, for $u \in [\frac{N-1}{N}, 1]$, $(-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \leq \frac{1}{\beta} (-\ln(1-u))^{\frac{1}{\beta}-1} (-\ln(1-u) - \ln N)$ so that

$$(2.8) \quad \begin{aligned} \int_{\frac{N-1}{N}}^1 \left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \right)^\rho du &\leq \frac{1}{\beta^\rho} \int_{\frac{N-1}{N}}^1 (-\ln(1-u))^{\frac{\rho}{\beta}-\rho} (-\ln(N(1-u)))^\rho du \\ &= \frac{1}{\beta^\rho N} \int_0^1 (\ln N - \ln v)^{\frac{\rho}{\beta}-\rho} (-\ln(v))^\rho dv \\ &\leq \frac{2^{(\frac{\rho}{\beta}-\rho-1)\vee 0}}{\beta^\rho N} \left((\ln N)^{\frac{\rho}{\beta}-\rho} \int_0^1 (-\ln(v))^\rho dv + \int_0^1 (-\ln(v))^{\frac{\rho}{\beta}} dv \right). \end{aligned}$$

When $\beta \geq 1$, for $N \geq 2$ and $u \in [\frac{N-1}{N}, 1]$, $(-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \leq \frac{1}{\beta} (\ln N)^{\frac{1}{\beta}-1} (-\ln(1-u) - \ln N)$ so that

$$(2.9) \quad \int_{\frac{N-1}{N}}^1 \left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \right)^\rho du \leq \frac{(\ln N)^{\frac{\rho}{\beta}-\rho}}{\beta^\rho N} \int_0^1 (-\ln(v))^{\frac{\rho}{\beta}} dv.$$

We conclude that for $\rho > 1$ and $\beta > 0$, $e_N(\mu_\beta, \rho) \asymp N^{-\frac{1}{\rho}} (\ln N)^{\frac{1}{\beta}-1} \asymp \mathcal{W}_\rho(\frac{1}{N}(\sum_{i=1}^{N-1} \delta_{F^{-1}(\frac{2i-1}{2N})} + \delta_{F^{-1}(\frac{N-1}{N})}), \mu_\beta)$. In view of Theorem 5.20 [17], this rate of convergence does not extend continuously to $e_N(\mu_\beta, 1)$, at least for $\beta > 1$. In fact, by Remark 2.2 [14], for $\beta > 0$, $e_N(\mu_\beta, 1) \asymp N^{-1} (\ln N)^{\frac{1}{\beta}}$, which in view of (2.8) and (2.9), implies that $\sum_{i=1}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |x_i^N - F^{-1}(u)|^\rho du \asymp N^{-1} (\ln N)^{\frac{1}{\beta}}$.

In the Gaussian tail case $\beta = 2$, $e_N(\mu_2, \rho) \asymp N^{-\frac{1}{\rho}} (\ln N)^{-\frac{1}{2}+1_{\{\rho=1\}}}$ for $\rho \geq 1$, like for the true Gaussian distribution, according to Example 5.18 [17]. This matches the rate obtained when $\rho > 2$ in Corollary 6.14 [6] for $\mathbb{E}^{1/\rho} \left[\mathcal{W}_\rho \left(\frac{1}{N} \sum_{i=1}^N \delta_{X_i}, \mu \right) \right]$ where $(X_i)_{i \geq 1}$ are i.i.d. with respect to some Gaussian distribution μ with positive variance. When $\rho = 2$, still according to this corollary the random rate is $N^{-1/2} (\ln \ln N)^{1/2}$ (of course worse than the standard Monte Carlo rate $N^{-1/2}$).

The next proposition gives a necessary condition for $e_N(\mu, \rho)$ to go to 0 with order $\alpha = \frac{1}{\rho}$.

Proposition 2.7. For $\rho \geq 1$,

$$\begin{aligned} & \sup_{N \geq 1} N^{1/\rho} e_N(\mu, \rho) < +\infty \\ \Rightarrow & \sup_{N \geq 1} N \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \right) < +\infty \\ \Leftrightarrow & \sup_{u \in (0, 1/2]} (F^{-1}(1-u/2) - F^{-1}(1-u) + F^{-1}(u) - F^{-1}(u/2)) < +\infty \\ \Rightarrow & \sup_{u \in (0, 1/2]} \frac{F^{-1}(1-u) - F^{-1}(u)}{\ln(1/u)} < +\infty \\ \Leftrightarrow & \exists \lambda \in (0, +\infty), \forall x \geq 0, (F(-x) + 1 - F(x)) \leq e^{-\lambda x} / \lambda \\ \Rightarrow & \sup_{N \geq 2} \sup_{x_{2:N-1}} \frac{N^{1/\rho}}{1 + \ln N} \mathcal{W}_\rho(\mu_{N,\lambda}(x_{2:N-1}), \mu) < +\infty \\ \Rightarrow & \sup_{N \geq 1} \frac{N^{1/\rho}}{1 + \ln N} e_N(\mu, \rho) < +\infty, \end{aligned}$$

where $\mu_{N,\lambda}(x_{2:N-1}) = \frac{1}{N} \left(\delta_{F^{-1}(\frac{1}{N}) \wedge (-\frac{\ln N}{\lambda})} + \sum_{i=2}^{N-1} \delta_{x_i} + \delta_{F^{-1}(\frac{N-1}{N}) \vee \frac{\ln N}{\lambda}} \right)$ and $\sup_{x_{2:N-1}}$ means the supremum over the choice of $x_i \in [F^{-1}(\frac{i-1}{N}), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$.

Remark 2.8. • The first implication is not an equivalence for $\rho = 1$. Indeed, in Example 2.6, for $\beta \geq 1$, $\lim_{N \rightarrow +\infty} N e_N(\mu, 1) = +\infty$ while $\sup_{N \geq 1} N \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N| du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N| du \right) < +\infty$.

- The second implication is not an equivalence as exemplified by $F^{-1}(u) = g(u) \ln(u)$ where $g(u) = \sum_{k \in \mathbb{N}} \frac{k}{k+1} \mathbf{1}_{(2^{-(k+1)^3}, 2^{-k^3}]}(u)$. The function $(0, 1) \ni u \mapsto g(u) \ln(u)$ is a quantile function since it is left-continuous and non-decreasing as the product of the left-continuous, non-increasing and non-negative function g by the continuous, non-decreasing and non-positive logarithm function. Moreover, since g is bounded by 1, one easily checks that $\sup_{u \in (0, 1/2]} \frac{F^{-1}(1-u) - F^{-1}(u)}{\ln(1/u)} < +\infty$. On the other hand,

$$\begin{aligned} F^{-1} \left(2^{1-(k+1)^3} \right) - F^{-1} \left(2^{-(k+1)^3} \right) &= \frac{k}{k+1} (1 - (k+1)^3) \ln 2 + \frac{k+1}{k+2} (k+1)^3 \ln 2 \\ &= \frac{k}{k+1} \ln 2 + \frac{(k+1)^2}{k+2} \ln 2 \end{aligned}$$

goes to ∞ with k .

- The exponential tail condition $\exists \lambda \in (0, +\infty), \forall x \geq 0, (F(-x) + 1 - F(x)) \leq e^{-\lambda x} / \lambda$ is not equivalent to $\sup_{N \geq 1} \frac{N^{1/\rho}}{1 + \ln N} e_N(\mu, \rho) < +\infty$ when $\rho > 1$ since in Example 2.6, for $\beta \in [1/2, 1)$, $\sup_{N \geq 1} \frac{N^{1/\rho}}{1 + \ln N} e_N(\mu, \rho) < +\infty$ while the exponential tail condition fails.

Proof. The first implication is an immediate consequence of (1.2).

To prove the first equivalence, we first suppose that:

$$(2.10) \quad \sup_{N \geq 1} N^{\frac{1}{\rho}} \left(\int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du + \int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du \right)^{\frac{1}{\rho}} < +\infty.$$

and denote by C the finite supremum in this equation. By (1.4) for $i = 1, \forall N \geq 1$, $F^{-1}(\frac{1}{2N}) - F^{-1}(\frac{1}{4N}) \leq 4C$. For $u \in (0, 1/2]$, there exists $N \in \mathbb{N}^*$ such that $u \in [\frac{1}{2(N+1)}, \frac{1}{2N}]$ and, by monotonicity of F^{-1} and since $4N \geq 2(N+1)$, we get

$$\begin{aligned} F^{-1}(u) - F^{-1}(u/2) &\leq F^{-1}\left(\frac{1}{2N}\right) - F^{-1}\left(\frac{1}{4(N+1)}\right) \\ &\leq F^{-1}\left(\frac{1}{2N}\right) - F^{-1}\left(\frac{1}{4N}\right) + F^{-1}\left(\frac{1}{2(N+1)}\right) - F^{-1}\left(\frac{1}{4(N+1)}\right) \leq 8C. \end{aligned}$$

Dealing in a symmetric way with $F^{-1}(1 - u/2) - F^{-1}(1 - u)$, we obtain that

$$\sup_{u \in (0, 1/2]} \left(F^{-1}(1 - u/2) - F^{-1}(1 - u) + F^{-1}(u) - F^{-1}(u/2) \right) \leq 16C.$$

On the other hand, for $N \geq 2$, by Lemma 1.2 applied with $x = F^{-1}(\frac{1}{N})$,

$$\begin{aligned} \frac{1}{\rho} \int_0^{\frac{1}{N}} |F^{-1}(u) - x_1^N|^\rho du &\leq \sum_{k \in \mathbb{N}} \int_{F^{-1}(\frac{1}{2^{k+1}N})}^{F^{-1}(\frac{1}{2^k N})} \left(F^{-1}\left(\frac{1}{N}\right) - y \right)^{\rho-1} F(y) dy \\ &\leq \sum_{k \in \mathbb{N}} \frac{F^{-1}(\frac{1}{2^k N}) - F^{-1}(\frac{1}{2^{k+1}N})}{2^k N} \left(\sum_{j=0}^k \left(F^{-1}\left(\frac{1}{2^j N}\right) - F^{-1}\left(\frac{1}{2^{j+1}N}\right) \right) \right)^{\rho-1} \\ &\leq \frac{1}{N} \left(\sup_{u \in (0, 1/2]} (F^{-1}(u) - F^{-1}(u/2)) \right)^\rho \sum_{k \in \mathbb{N}} \frac{(k+1)^{\rho-1}}{2^k}, \end{aligned}$$

where the last sum is finite. Dealing in a symmetric way with $\int_{\frac{N-1}{N}}^1 |F^{-1}(u) - x_N^N|^\rho du$, we conclude that (2.10) is equivalent to the finiteness of $\sup_{u \in (0, 1/2]} \left(F^{-1}(1 - u/2) - F^{-1}(1 - u) + F^{-1}(u) - F^{-1}(u/2) \right)$. Under (2.10) with C denoting the finite supremum, for $k \in \mathbb{N}^*$, $F^{-1}(2^{-(k+1)}) - F^{-1}(2^{-k}) \geq -4C$ and, after summation,

$$F^{-1}(2^{-k}) \geq F^{-1}(1/2) - 4C(k-1).$$

With the monotonicity of F^{-1} , we deduce that:

$$\forall u \in (0, 1/2], F^{-1}(u) \geq F^{-1}(1/2) + \frac{4C}{\ln 2} \ln u$$

and therefore that $\sup_{u \in (0, 1/2]} \frac{-F^{-1}(u)}{\ln(1/u)} < +\infty$. With the inequality $F^{-1}(F(x)) \leq x$ valid for $x \in \mathbb{R}$, this implies that $\sup_{\{x \in \mathbb{R}: 0 < F(x) \leq 1/2\}} \frac{-x}{\ln(1/F(x))} < +\infty$ and therefore that $\exists \lambda \in (0, +\infty)$, $\forall x \leq 0$, $F(x) \leq e^{\lambda x}/\lambda$. Under the latter condition, since $u \leq F(F^{-1}(u))$ and $F^{-1}(u) \leq 0$ for $u \in (0, F(0)]$, we have $\sup_{u \in (0, F(0))} \frac{-F^{-1}(u)}{\ln(1/u)} < \infty$ and even $\sup_{u \in (0, 1/2]} \frac{-F^{-1}(u)}{\ln(1/u)} < \infty$ since when $F(0) < \frac{1}{2}$, $\sup_{u \in (F(0), 1/2]} \frac{-F^{-1}(u)}{\ln(1/u)} \leq 0$. By a symmetric reasoning, we obtain the two equivalent tail properties $\sup_{u \in (0, 1/2]} \frac{F^{-1}(1-u) - F^{-1}(u)}{\ln(1/u)} < +\infty$ and $\exists \lambda \in (0, +\infty)$, $\forall x \geq 0$, $\left(F(-x) + 1 - F(x) \right) \leq e^{-\lambda x}/\lambda$.

Let us finally suppose these two tail properties and deduce that $\sup_{N \geq 2} \sup_{x_{2:N-1}} \frac{N^{1/\rho}}{1 + \ln N} \mathcal{W}_\rho(\mu_{N,\lambda}(x_{2:N-1}), \mu) < +\infty$. We use the decomposition $\mathcal{W}_\rho^\rho(\mu_{N,\lambda}(x_{2:N-1}), \mu) = L_N + M_N + \bar{U}_N$ introduced in the proof of Theorem

2.2 but with $F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right)$ and $F^{-1}\left(\frac{N-1}{N}\right) \vee \left(\frac{\ln N}{\lambda}\right)$ respectively replacing $F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha})$ and $F^{-1}\left(\frac{N-1}{N}\right) \vee (N^{\frac{1}{\rho}-\alpha})$ in L_N and U_N . By (2.3), we get:

$$\forall N \geq 3, M_N \leq \frac{1}{N} \left(F^{-1}\left(\frac{N-1}{N}\right) - F^{-1}\left(\frac{1}{N}\right) \right)^\rho \leq \left(\sup_{u \in (0, 1/2]} \frac{F^{-1}(1-u) - F^{-1}(u)}{\ln(1/u)} \right)^\rho \frac{(\ln N)^\rho}{N}.$$

Applying Lemma 1.2 with $x = F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right)$ then the estimation of the cumulative distribution function, we obtain that for $N \geq 2$,

$$\begin{aligned} L_N &\leq \rho \int_{-\infty}^{F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right)} \left(F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right) - y \right)^{\rho-1} F(y) dy \\ &\quad + \rho \int_{F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right)}^{F^{-1}\left(\frac{1}{N}\right)} \left(y - F^{-1}\left(\frac{1}{N}\right) \wedge \left(-\frac{\ln N}{\lambda}\right) \right)^{\rho-1} \left(\frac{1}{N} - F(y) \right) dy \\ &\leq \frac{\rho}{\lambda} \int_{-\infty}^{-\frac{\ln N}{\lambda}} (-y)^{\rho-1} e^{\lambda y} dy + \frac{1}{N} \left(\frac{\ln N}{\lambda} + F^{-1}\left(\frac{1}{N}\right) \right)_+^\rho \\ &\leq \frac{\rho}{\lambda} \sum_{k \geq 1} \int_{k \frac{\ln N}{\lambda}}^{(k+1) \frac{\ln N}{\lambda}} \left((k+1) \frac{\ln N}{\lambda} \right)^{\rho-1} e^{-\lambda y} dy + \frac{1}{N} \left(\frac{\ln N}{\lambda} + F^{-1}\left(\frac{1}{2}\right) \right)_+^\rho \\ &\leq \frac{\rho(\ln N)^\rho}{\lambda^{\rho+1} N} \sum_{k \geq 1} \frac{(k+1)^{\rho-1}}{2^{k-1}} + \frac{1}{N} \left(\frac{1}{\lambda} \ln N + F^{-1}\left(\frac{1}{2}\right) \right)_+^\rho, \end{aligned}$$

where we used that $N^k \geq N 2^{k-1}$ for the last inequality. Dealing in a symmetric way with U_N , we conclude that $\sup_{N \geq 2} \sup_{x_{2:N-1}} \frac{N \mathcal{W}_\rho^\rho(\mu_N, \lambda(x_{2:N-1}), \mu)}{1 + (\ln N)^\rho} < +\infty$. $\square$

Conclusion

In the present paper, we have characterized the convergence of $e_N(\mu, \rho)$ to 0 with order $\alpha \in (0, \frac{1}{\rho})$ and also studied the convergence with boundary order $\alpha = 1/\rho$ between the unbounded support case and the bounded support case. In view of Example 2.6, it would be nice to investigate whether the leading factor remains $N^{-1/\rho}$ for μ with unbounded support and superpolynomial but subexponential tails. Characterizing the order of convergence when the support of μ is bounded and its quantile function F^{-1} is continuous is another open question.

Of course, generalizing our results to higher dimension would be of great interest. This appears to be challenging since our approach heavily relies on one-dimensional tools like the cumulative distribution function and the quantile function.

Appendix

Proof of Lemma 1.1. Since the finiteness of $\sup_{u \in (0, 1/2]} u^{\frac{1}{\rho}-\alpha} (F^{-1}(1-u) - F^{-1}(u))$ implies the finiteness of both $\sup_{u \in (0, 1)} u^{\frac{1}{\rho}-\alpha} F^{-1}(1-u)$ and $\inf_{u \in (0, 1)} u^{\frac{1}{\rho}-\alpha} F^{-1}(u)$, the second statement is a consequence of the first one, that we are now going to prove. When $\rho = 1$ (resp. $\rho = 2$), then the conclusion easily follows from the explicit form $x_1^N = F^{-1}\left(\frac{1}{2N}\right)$ and $x_N^N = F^{-1}\left(\frac{2N-1}{2N}\right)$ (resp. $x_1^N = N \int_0^{\frac{1}{N}} F^{-1}(u) du$ and $x_N^N = N \int_{\frac{1}{N}}^1 F^{-1}(u) du$). In the general case $\rho > 1$, we are going to take advantage of the expression

$$f(y) = \rho \int_0^{\frac{1}{N}} \left(\mathbf{1}_{\{y \geq F^{-1}(1-u)\}} (y - F^{-1}(1-u))^{\rho-1} - \mathbf{1}_{\{y < F^{-1}(1-u)\}} (F^{-1}(1-u) - y)^{\rho-1} \right) du$$

of the derivative of the function $\mathbb{R} \ni y \mapsto \int_{\frac{N-1}{N}}^1 |y - F^{-1}(u)|^\rho du$ minimized by x_N^N . Since this function is strictly convex $x_N^N = \inf\{y \in \mathbb{R} : f(y) \geq 0\}$. Let us first suppose that $S_N := \sup_{u \in (0, \frac{1}{N})} u^{\frac{1}{\rho}-\alpha} F^{-1}(1-u) \in$

$(0, +\infty)$. Since for fixed $y \in \mathbb{R}$, $\mathbb{R} \ni x \mapsto (\mathbf{1}_{\{y \geq x\}}(y-x)^{\rho-1} - \mathbf{1}_{\{y < x\}}(x-y)^{\rho-1})$ is non-increasing, we deduce that $\forall y \in \mathbb{R}$, $f(y) \geq \rho S_N^{\rho-1} g(\frac{y}{S_N})$ where

$$g(z) = \int_0^{\frac{1}{N}} \left(\mathbf{1}_{\{z \geq u^{\alpha-\frac{1}{\rho}}\}} \left(z - u^{\alpha-\frac{1}{\rho}} \right)^{\rho-1} - \mathbf{1}_{\{z < u^{\alpha-\frac{1}{\rho}}\}} \left(u^{\alpha-\frac{1}{\rho}} - z \right)^{\rho-1} \right) du.$$

For $z \geq (4N)^{\frac{1}{\rho}-\alpha}$, we have $z^{\frac{\rho}{\alpha\rho-1}} \leq \frac{1}{4N}$ and $z - (2N)^{\frac{1}{\rho}-\alpha} \geq (1 - 2^{\alpha-\frac{1}{\rho}})z$ so that

$$\begin{aligned} g(z) &= \int_{z^{\frac{\rho}{\alpha\rho-1}}}^{\frac{1}{N}} \left(z - u^{\alpha-\frac{1}{\rho}} \right)^{\rho-1} du - \int_0^{z^{\frac{\rho}{\alpha\rho-1}}} \left(u^{\alpha-\frac{1}{\rho}} - z \right)^{\rho-1} du \geq \int_{\frac{1}{2N}}^{\frac{1}{N}} \left(z - (2N)^{\frac{1}{\rho}-\alpha} \right)^{\rho-1} du - \int_0^{z^{\frac{\rho}{\alpha\rho-1}}} u^{(\rho-1)\frac{\alpha\rho-1}{\rho}} du \\ &\geq \left(1 - 2^{\alpha-\frac{1}{\rho}} \right)^{\rho-1} z^{\rho-1} \int_{\frac{1}{2N}}^{\frac{1}{N}} du - \frac{\rho z^{\frac{\rho}{\alpha\rho-1}+\rho-1}}{1 + (\rho-1)\alpha\rho} = z^{\rho-1} \left(\frac{\left(1 - 2^{\alpha-\frac{1}{\rho}} \right)^{\rho-1}}{2N} - \frac{\rho z^{\frac{\rho}{\alpha\rho-1}}}{1 + (\rho-1)\alpha\rho} \right). \end{aligned}$$

The right-hand side is positive for $z > (\kappa N)^{\frac{1}{\rho}-\alpha}$ with $\kappa := \frac{2\rho}{\left(1 - 2^{\alpha-\frac{1}{\rho}} \right)^{\rho-1} (1 + (\rho-1)\alpha\rho)}$. Hence for $z > ((\kappa \vee 4)N)^{\frac{1}{\rho}-\alpha}$,

$g(z) > 0$ so that for $y > ((\kappa \vee 4)N)^{\frac{1}{\rho}-\alpha} S_N$, $f(y) > 0$ and therefore

$$x_N^N \leq ((\kappa \vee 4)N)^{\frac{1}{\rho}-\alpha} S_N.$$

Clearly, this inequality remains valid when $S_N = +\infty$. It holds in full generality since $S_N \geq 0$ and $S_N = 0 \Leftrightarrow F^{-1}(1) \leq 0$ a condition under which $x_N^N \leq 0$ since $x_N^N \in [F^{-1}(\frac{N-1}{N}+), F^{-1}(1)] \cap \mathbb{R}$. By a symmetric reasoning, we check that $x_1^N \geq ((\kappa \vee 4)N)^{\frac{1}{\rho}-\alpha} \inf_{u \in (0, \frac{1}{N})} u^{\frac{1}{\rho}-\alpha} F^{-1}(u)$. $\square$

Proof of Lemma 2.4. Let $\beta > 0$. For $x > 0$, using the monotonicity of F for the first inequality then that for $y \in [\frac{x}{2}, x]$, $y^{\beta-1} \geq (\frac{x}{2})^{\beta-1} \wedge x^{\beta-1} = \frac{x^{\beta-1}}{2^{(\beta-1) \vee 0}}$, we obtain that

$$F(-x) + 1 - F(x) \leq \frac{2}{x} \int_{x/2}^x \left(F(-y) + 1 - F(y) \right) dy \leq \frac{2^{\beta \vee 1}}{x^\beta} \int_{x/2}^{+\infty} y^{\beta-1} \left(F(-y) + 1 - F(y) \right) dy.$$

Since $\int_0^{+\infty} y^{\beta-1} \left(F(-y) + 1 - F(y) \right) dy = \frac{1}{\beta} \int_{\mathbb{R}} |y|^\beta \mu(dy)$, the finiteness of $\int_{\mathbb{R}} |y|^\beta \mu(dy)$ implies by Lebesgue's theorem that $\lim_{x \rightarrow \infty} x^\beta (F(-x) + 1 - F(x)) = 0$. Since $x \mapsto x^\beta (F(-x) + 1 - F(x))$ is right-continuous with left-hand limits on $[0, +\infty)$,

$$\sup_{x \geq 0} x^\beta \left(F(-x) + 1 - F(x) \right) < +\infty \Leftrightarrow \limsup_{x \rightarrow \infty} x^\beta \left(F(-x) + 1 - F(x) \right) < +\infty,$$

with the latter property clearly implied by $\lim_{x \rightarrow \infty} x^\beta (F(-x) + 1 - F(x)) = 0$.

For $\varepsilon \in (0, \beta)$, using that for $y \geq 0$, $F(-y) + 1 - F(y) = \mu((-\infty, -y] \cup (y, +\infty)) \leq 1$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} |x|^{\beta-\varepsilon} \mu(dx) &= (\beta - \varepsilon) \int_0^{+\infty} y^{\beta-\varepsilon-1} (F(-y) + 1 - F(y)) dy \\ &\leq (\beta - \varepsilon) \int_0^1 y^{\beta-\varepsilon-1} dy + (\beta - \varepsilon) \sup_{x \geq 0} x^\beta \left(F(-x) + 1 - F(x) \right) \int_1^{+\infty} y^{-\varepsilon-1} dy \\ &= 1 + \frac{\beta - \varepsilon}{\varepsilon} \sup_{x \geq 0} x^\beta \left(F(-x) + 1 - F(x) \right). \end{aligned}$$

Therefore $\sup_{x \geq 0} x^\beta \left(F(-x) + 1 - F(x) \right) < +\infty \implies \forall \varepsilon \in (0, \beta)$, $\int_{\mathbb{R}} |x|^{\beta-\varepsilon} \mu(dx) < +\infty$.

Let us next check that

$$(2.11) \quad \sup_{x \geq 0} x^\beta \left(F(-x) + (1 - F(x)) \right) < +\infty \Leftrightarrow \sup_{u \in (0, 1/2]} u^{\frac{1}{\beta}} (F^{-1}(1-u) - F^{-1}(u)) < +\infty.$$

For the necessary condition, we set $u \in (0, 1/2]$. Either $F^{-1}(u) \geq 0$ or, since for all $v \in (0, 1)$, $F(F^{-1}(v)) \geq v$, we have $(-F^{-1}(u))^\beta u \leq \sup_{x \geq -F^{-1}(u)} x^\beta F(-x)$ and therefore $F^{-1}(u) \geq -(\sup_{x \geq 0} x^\beta F(-x))^\frac{1}{\beta} u^{-\frac{1}{\beta}}$. Either $F^{-1}(1-u) \leq 0$ or, since for all $v \in (0, 1)$, $F(F^{-1}(v)-) \leq v$, we have $(F^{-1}(1-u))^\beta u \leq \sup_{x \geq F^{-1}(1-u)} x^\beta (1 - F(x-))$ and therefore $F^{-1}(1-u) \leq (\sup_{x \geq 0} x^\beta (1 - F(x)))^\frac{1}{\beta} u^{-\frac{1}{\beta}}$. Hence (2.2) holds.

For the sufficient condition, we remark that the finiteness of $\sup_{u \in (0, 1/2]} u^\frac{1}{\beta} (F^{-1}(1-u) - F^{-1}(u))$ and the inequality $F^{-1}(1-u) - F^{-1}(u) \geq (F^{-1}(1/2) - F^{-1}(u)) \vee (F^{-1}(1-u) - F^{-1}(1/2))$ valid for $u \in (0, 1/2]$ imply that $\inf_{u \in (0, 1/2]} u^\frac{1}{\beta} F^{-1}(u) > -\infty$ and $\sup_{u \in (0, 1/2]} u^\frac{1}{\beta} F^{-1}(1-u) < +\infty$. With the inequality $x \geq F^{-1}(F(x))$ valid for $x \in \mathbb{R}$ such that $0 < F(x) < 1$, this implies that $\inf_{x \in \mathbb{R}: F(x) \leq 1/2} (F(x))^\frac{1}{\beta} x > -\infty$ and therefore that $\sup_{x \geq 0} x^\beta F(-x) < +\infty$. With the inequality $x \leq F^{-1}(F(x)+)$ valid for $x \in \mathbb{R}$ such that $0 < F(x) < 1$, we obtain, in a symmetric way $\sup_{x \geq 0} x^\beta (1 - F(x)) < +\infty$.

Let us finally check that $\lim_{x \rightarrow +\infty} x^\beta (F(-x) + 1 - F(x)) = 0 \Leftrightarrow \lim_{u \rightarrow 0+} u^\frac{1}{\beta} (F^{-1}(1-u) - F^{-1}(u)) = 0$. For the necessary condition, we remark that either $F^{-1}(1) < +\infty$ and $\lim_{u \rightarrow 0+} u^\frac{1}{\beta} F^{-1}(1-u) = 0$ or $F^{-1}(1-u)$ goes to $+\infty$ as $u \rightarrow 0+$. For u small enough so that $F^{-1}(1-u) > 0$, we have $(F^{-1}(1-u))^\beta u \leq \sup_{x \geq F^{-1}(1-u)} x^\beta (1 - F(x-))$, from which we deduce that $\lim_{u \rightarrow 0+} u(F^{-1}(1-u))^\beta = 0$. The fact that $\lim_{u \rightarrow 0+} u^\frac{1}{\beta} F^{-1}(u) = 0$ is deduced by a symmetric reasoning.

For the sufficient condition, we use that $x(1 - F(x))^\frac{1}{\beta} \leq \sup_{u \leq 1-F(x)} u^\frac{1}{\beta} F^{-1}((1-u)+)$ and $x(F(x))^\frac{1}{\beta} \geq \inf_{u \leq F(x)} u^\frac{1}{\beta} F^{-1}(u)$ for $x \in \mathbb{R}$ such that $0 < F(x) < 1$. □

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