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# On a Rank factorisation Problem Arising in Gearbox Vibration Analysis

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**Abstract:** Given a field  $\mathbb{k}$ , a matrix  $M \in \mathbb{k}^{n \times m}$  of rank at most  $r$ ,  $r$  matrices  $D_i \in \mathbb{k}^{n \times n}$ , in this paper, we study the problem of factoring  $M$  as follows  $M = \sum_{i=1}^r D_i u v_i$ , where  $u \in \mathbb{k}^{n \times 1}$  and  $v_i \in \mathbb{k}^{1 \times m}$  for  $i = 1, \dots, r$ . This problem arises in modulation-based mechanical models studied in gearbox vibration analysis (e.g., amplitude and phase modulation). We show how linear algebra methods can be used to characterize when this polynomial problem is solvable and if so, how to explicitly compute the solutions of the problem.

*Keywords:* factorisation methods, modulation, demodulation, vibration analysis.

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## 1. INTRODUCTION

In what follows,  $\mathbb{k}$  will denote a field (e.g.,  $\mathbb{k} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ),  $M \in \mathbb{k}^{n \times m}$  a  $n \times m$  matrix with entries in  $\mathbb{k}$ ,  $D_i \in \mathbb{k}^{n \times n}$ ,  $u \in \mathbb{k}^{n \times 1}$ , and  $v_i \in \mathbb{k}^{1 \times m}$  for  $i = 1, \dots, r$ .

Let us state the main problem studied in this paper.

**The rank factorisation problem:** Let  $M \in \mathbb{k}^{n \times m}$  and  $D_1, \dots, D_r \in \mathbb{k}^{n \times n}$ . Determine – if they exist –  $u \in \mathbb{k}^{n \times 1}$  and  $v_1, \dots, v_r \in \mathbb{k}^{1 \times m}$  satisfying:

$$M = \sum_{i=1}^r D_i u v_i. \quad (1)$$

Within the framework of *vibration analysis* applied to *gearbox fault surveillance*, a new *demodulation* approach of the gearbox vibration signal was proposed in Hubert et al. (2018); Hubert (2019). It first states that the gearbox spectrum can be represented as a (structured) matrix  $M$  and then that separating the time vibration signal into its two main components amounts to estimate vectors  $u$  and  $v$  set in Problem (1). The exact case of the above mentioned factorisation problem was studied in Hubert et al. (2019) and solved in Hubert et al. (2018) for  $r = 1$  and in Hubert et al. (2019) for  $r = 2$ . In this paper, we give a proof of the general case ( $r \geq 1$ ) based on linear algebra methods.

Problem (1) corresponds to a polynomial system formed by  $n \times m$  equations in the  $n + r m$  unknowns  $\{u, v_1, \dots, v_r\}$ . Hence, methods from algebraic geometry and symbolic computation (e.g., Gröbner bases, resultants) can be used

to study Problem (1). In this paper, exploiting the algebraic structures of (1) and combining them with ideas of linear systems theory, we shall show that linear algebra methods are sufficient to solve Problem (1).

**Notation.** Let  $A \in \mathbb{k}^{r \times s}$  and the two  $\mathbb{k}$ -linear maps:

$$\begin{aligned} \cdot A : \mathbb{k}^{1 \times r} &\longrightarrow \mathbb{k}^{1 \times s} & A \cdot : \mathbb{k}^{s \times 1} &\longrightarrow \mathbb{k}^{r \times 1} \\ \lambda &\longmapsto \lambda A, & \eta &\longmapsto A \eta. \end{aligned} \quad (2)$$

We can define the finite-dimensional  $\mathbb{k}$ -vector spaces

$$\begin{aligned} \text{im}_{\mathbb{k}}(\cdot A) &:= \mathbb{k}^{1 \times r} A = \{\mu \in \mathbb{k}^{1 \times s} \mid \exists \lambda \in \mathbb{k}^{1 \times r} : \mu = \lambda A\}, \\ \text{ker}_{\mathbb{k}}(\cdot A) &:= \{\lambda \in \mathbb{k}^{1 \times r} \mid \lambda A = 0\}. \end{aligned}$$

Similarly, we can define the following  $\mathbb{k}$ -vector spaces:

$$\begin{aligned} \text{im}_{\mathbb{k}}(A \cdot) &:= A \mathbb{k}^{s \times 1} = \{\zeta \in \mathbb{k}^{r \times 1} \mid \exists \eta \in \mathbb{k}^{s \times 1} : \zeta = A \eta\}, \\ \text{ker}_{\mathbb{k}}(A \cdot) &:= \{\eta \in \mathbb{k}^{s \times 1} \mid A \eta = 0\}. \end{aligned}$$

Finally, we recall that  $A$  is said to be *full row rank* (resp., *full column rank*) if  $\text{ker}_{\mathbb{k}}(\cdot A) = 0$  (resp.,  $\text{ker}_{\mathbb{k}}(A \cdot) = 0$ ).

## 2. THE FACTORISATION PROBLEM

### 2.1 Necessary conditions for the solvability of Problem (1)

We first note that if we set

$$\begin{aligned} A(u) &:= (D_1 u \ \dots \ D_r u) \in \mathbb{k}^{n \times r}, \\ v &:= (v_1^T \ \dots \ v_r^T)^T \in \mathbb{k}^{r \times m}, \end{aligned}$$

then Problem (1) can be rewritten as follow:

$$A(u) v = M. \quad (3)$$

If  $(u, v)$  is a solution of (3), then so is  $(\lambda u, \lambda^{-1} v)$  for all  $\lambda \in \mathbb{k} \setminus \{0\}$ . Hence, if a solution exists, it is not unique. It

also shows the *projective geometrical* nature of Problem (1) (i.e.,  $u$  and the  $v_i$ 's should be considered as points of respectively the *projective spaces*  $\mathbb{P}_n(\mathbb{k})$  and  $\mathbb{P}_m(\mathbb{k})$ ).

Since the  $\mathbb{k}$ -vector space  $\text{im}_{\mathbb{k}}(A(u))$  is generated by the  $r$  vectors  $D_i u$ 's, we then have:

$$\text{rank}_{\mathbb{k}}(A(u)) := \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(A(u))) \leq r.$$

Thus, a necessary condition for the solvability of (1) is:

$$\text{rank}_{\mathbb{k}}(M) \leq r. \quad (4)$$

If Problem (3) is solvable and if  $v$  has not full row rank, then there exists  $(\alpha_1 \dots \alpha_r) \in \mathbb{k}^{1 \times r}$ , with  $\alpha_k \neq 0$  for a certain  $k \in \{1, \dots, r\}$ , such that  $\sum_{i=1}^r \alpha_i v_i = 0$ , which yields  $M = \sum_{i=1}^r (D_i - \alpha_i \alpha_k^{-1} D_k) u v_i$ . Since the above sum contains at most  $r - 1$  non-zero terms, we get  $\text{rank}_{\mathbb{k}}(M) \leq r - 1$ . Hence, a necessary condition for the solvability of Problem (3) for a matrix  $M$  of rank  $r$  is that  $v$  has full row rank, i.e., that  $v$  admits a *right inverse*  $w \in \mathbb{k}^{m \times r}$ , i.e.,  $v w = I_r$ . If so, then Problem (3) yields:

$$A(u) = M w. \quad (5)$$

In what follows, we shall suppose that  $v$  has full row rank since it is a necessary condition for the case  $\text{rank}_{\mathbb{k}}(M) = r$ .

Writing  $w = (w_1 \dots w_r)$ ,  $w_i \in \mathbb{k}^{m \times 1}$ ,  $i = 1, \dots, r$ , we get

$$A(u) = (D_1 u \dots D_r u) = (M w_1 \dots M w_r), \quad (6)$$

which shows that  $D_i u \in \text{im}_{\mathbb{k}}(M)$  for  $i = 1, \dots, r$  is a necessary condition for the solvability of Problem (3).

Let us end this section by considering a particular factorisation of the matrix  $M \neq 0$  that will be useful thereafter.

Let  $X \in \mathbb{k}^{n \times l}$  be a full column rank matrix defining a basis of  $\text{im}_{\mathbb{k}}(M)$ . To solve Problem (3), we necessarily have:

$$l := \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(M)) = \text{rank}_{\mathbb{k}}(M) \leq r. \quad (7)$$

Now, using  $\text{im}_{\mathbb{k}}(M) = \text{im}_{\mathbb{k}}(X)$  and the fact that  $X$  has full column rank, there exist a unique matrix  $Y \in \mathbb{k}^{l \times m}$  and a (non necessarily unique) matrix  $T \in \mathbb{k}^{m \times l}$  such that:

$$\begin{cases} M = X Y, \\ X = M T. \end{cases} \quad (8)$$

Since  $X$  has full column rank,  $M \eta = 0$  yields  $Y \eta = 0$ , and thus  $\ker_{\mathbb{k}}(M) = \ker_{\mathbb{k}}(Y)$ . Moreover, combining the two identities of (8), we get  $X = X Y T$ , i.e.,  $X (Y T - I_l) = 0$ , which yields  $Y T = I_l$  since  $X$  has full column rank. Hence, we have  $\text{im}_{\mathbb{k}}(Y) = \mathbb{k}^{l \times 1}$ . We have  $\ker_{\mathbb{k}}(Y) = 0$  since  $\lambda = (\lambda Y) T = 0$  for all  $\lambda \in \ker_{\mathbb{k}}(Y)$ , which shows that  $Y$  is a full row rank matrix. Hence, if  $\mu \in \ker_{\mathbb{k}}(M)$ , then  $(\mu X) Y = 0$ , which yields  $\mu X = 0$  and shows that  $\ker_{\mathbb{k}}(M) \subseteq \ker_{\mathbb{k}}(X)$  and proves  $\ker_{\mathbb{k}}(M) = \ker_{\mathbb{k}}(X)$ . Finally, since  $X$  has full column rank, there exists a *left inverse*  $Z \in \mathbb{k}^{l \times n}$  of  $X$ , i.e.,  $Z X = I_l$ . Hence, we have  $\text{im}_{\mathbb{k}}(X) = \mathbb{k}^{l \times 1}$  and  $\text{im}_{\mathbb{k}}(M) = \text{im}_{\mathbb{k}}(X Y) = \text{im}_{\mathbb{k}}(Y)$ .

If  $M = X' Y'$ , where  $\text{im}_{\mathbb{k}}(M) = \text{im}_{\mathbb{k}}(X')$ , then there exists  $V \in \text{GL}_l(\mathbb{k})$ , i.e.,  $\det(V) \neq 0$ , such that  $X = X' V$ , i.e.,  $X' = X V^{-1}$ . We get  $M = X' Y'$ , where  $Y' = V Y$ .

If  $\mathbb{k} = \mathbb{R}$  or  $\mathbb{C}$ , a way to get such a factorisation  $M = X Y$  is to compute a *Singular Value Decomposition* of  $M$  to get  $M = U \Sigma V$ , where  $U \in \mathbb{k}^{n \times n}$  (resp.,  $V \in \mathbb{k}^{m \times m}$ ) is a unitary matrix, i.e.,  $U U^* = U^* U = I_n$  (resp.,  $V V^* = V^* V = I_m$ ), where  $U^* = \overline{U}^T$  denotes the *adjoint* (i.e., the conjugate transposed) of  $U$  and  $\overline{U}$  the conjugate of  $U$ , and  $\Sigma \in \mathbb{k}^{n \times m}$  is a diagonal matrix whose diagonal entries are

the *singular values*  $\sigma_i(M)$  of  $M$  listed in descending order. Then,  $l$  is the number of non-zero singular values, i.e.:  $\sigma_i(M) \neq 0$ ,  $i = 1, \dots, l$ ,  $\sigma_i(M) = 0$ ,  $i = l+1, \dots, \min\{m, n\}$ .

Let  $U = (U_1 \ U_2)$ , where  $U_1 \in \mathbb{k}^{n \times l}$  and  $U_2 \in \mathbb{k}^{n \times (n-l)}$ ,  $V = (V_1^T \ V_2^T)^T$ , where  $V_1 \in \mathbb{k}^{l \times m}$  and  $V_2 \in \mathbb{k}^{(m-l) \times m}$ , and  $\sigma \in \mathbb{k}^{l \times l}$  be the diagonal matrix formed by the non-zero singular values of  $M$ . Then, we have

$$M = (U_1 \ U_2) \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = U_1 \sigma V_1,$$

which shows that we can take:

$$X = U_1 \in \mathbb{k}^{n \times l}, \quad Y = \sigma V_1 \in \mathbb{k}^{l \times m}.$$

Any other factorisation  $M = X' Y'$  is then of the form  $X' = X W$ ,  $Y' = W^{-1} Y$  for  $W \in \mathbb{k}^{l \times l}$  and  $\det(W) \neq 0$ .

## 2.2 Necessary and sufficient solvability condition

Let  $L \in \mathbb{k}^{p \times n}$  be a matrix whose rows generate a basis of  $\ker_{\mathbb{k}}(.M)$ , i.e.,  $L$  is a full row rank matrix satisfying:

$$\ker_{\mathbb{k}}(.M) = \text{im}_{\mathbb{k}}(.L) = \mathbb{k}^{1 \times p} L, \quad p := \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(.M)).$$

Using  $l := \text{rank}_{\mathbb{k}}(M) = \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(M)) = \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(.M))$ , we then obtain:

$$p = \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(.M)) = n - \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(.M)) = n - \text{rank}_{\mathbb{k}}(M). \quad (9)$$

*Remark 1.* With the notations of the end of Section 2.1,

$$U^{-1} = U^* = \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} \Rightarrow \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (U_1 \ U_2) = I_n,$$

which shows that  $\ker_{\mathbb{k}}(.U_1) = \text{im}_{\mathbb{k}}(.U_2^*) = \mathbb{k}^{1 \times (n-l)} U_2^*$ . For more details, see, e.g., Fabiańska et al. (2007). Hence, we get  $\ker_{\mathbb{k}}(.M) = \ker_{\mathbb{k}}(.X) = \ker_{\mathbb{k}}(.U_1) = \text{im}_{\mathbb{k}}(.U_2^*)$ , which shows that we can take  $L := U_2^*$ .

Let us now suppose that Problem (3) is solvable. Premultiplying (3) by  $L$ , we then obtain:

$$L A(u) v = L M = 0. \quad (10)$$

Since the matrix  $v$  is supposed to be full row rank, we get  $L A(u) = 0$ , i.e.,  $u$  satisfies the following  $\mathbb{k}$ -linear system:

$$\begin{cases} L D_1 u = 0, \\ \vdots \\ L D_r u = 0. \end{cases} \quad (11)$$

Then, (11) is a necessary condition for the solvability of Problem (3). Computing a basis of the  $\mathbb{k}$ -vector space

$$\mathcal{V} := \ker_{\mathbb{k}}(((L D_1)^T \dots (L D_r)^T)^T),$$

we obtain a full column rank matrix  $Z \in \mathbb{k}^{n \times d}$ , where  $d := \dim_{\mathbb{k}}(\mathcal{V})$ , such that  $\mathcal{V} = \text{im}_{\mathbb{k}}(Z)$ . Thus, all the solutions  $u \in \mathbb{k}^{n \times 1}$  of (11) are of the form:

$$\forall \psi \in \mathbb{k}^{d \times 1}, \quad u = Z \psi. \quad (12)$$

*Example 1.* Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_1 = I_3, \quad D_2 = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

We can check that  $\ker_{\mathbb{k}}(.M) = \mathbb{k} L$  with  $L = (1 \ 0 \ -1)$ .

$$(11) \Leftrightarrow \begin{cases} u_1 - u_3 = 0, \\ d_1 u_1 - d_3 u_3 = 0, \end{cases} \Leftrightarrow \begin{cases} u_3 = u_1, \\ (d_1 - d_3) u_1 = 0. \end{cases}$$

- If  $d_1 \neq d_3$ , then  $u_1 = 0$  and  $u = (0 \ u_2 \ 0)^T$  for  $u_2 \in \mathbb{k}$ , i.e., we have  $Z = (0 \ 1 \ 0)^T$  and  $\psi = u_2 \in \mathbb{k}$ .
- If  $d_1 = d_3$ , then  $u = (u_1 \ u_2 \ u_1)^T$  for  $u_1, u_2 \in \mathbb{k}$ :

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Using (12), we are now reduced to finding  $\psi \in \mathbb{k}^{d \times 1}$  such that Problem (3), i.e.,  $A(Z\psi)v = M$ , is solvable.

*Example 2.* If  $M = 0$ , then  $L = I_n$  and (11) then becomes  $D_1 u = 0, \dots, D_r u = 0$ . Hence, we get  $A(u) = 0$ , which yields  $v \in \mathbb{k}^{r \times m}$ . We note that  $u \in \mathbb{k}^{n \times 1}$  and  $v = 0$  also define a solution but  $v$  has not full row rank.

In what follows, we suppose that  $M \neq 0$  so that  $(u, v) \neq 0$ .

*Remark 2.* By definition of the matrix  $L$ , we have  $LM = 0$  and if there exists  $L' \in \mathbb{k}^{p' \times n}$  such that  $L'M = 0$ , then there exists  $L'' \in \mathbb{k}^{p' \times p}$  such that  $L' = L''L$ . In particular, we get  $\text{im}_{\mathbb{k}}(M) \subseteq \text{ker}_{\mathbb{k}}(L)$ . In fact, we have:

$$\text{ker}_{\mathbb{k}}(L) = \text{im}_{\mathbb{k}}(M). \quad (13)$$

It directly follows from the fact that the functor  $\text{hom}_{\mathbb{k}}(\cdot, \mathbb{k})$  is so-called *exact* (Rotman (2009)). Let us give the hint of the proof of this result based only on standard linear algebra results. First,  $\eta \in \text{ker}_{\mathbb{k}}(L)$ , i.e.,  $L\eta = 0$ , is equivalent to  $\lambda L\eta = 0$  for all  $\lambda \in \mathbb{k}^{1 \times p}$ , i.e.,  $(\lambda L)\eta = 0$  for all  $\lambda \in \mathbb{k}^{1 \times p}$ . Since  $\text{ker}_{\mathbb{k}}(M) = \mathbb{k}^{1 \times p}L$ , the above equivalence is equivalent to  $\mu\eta = 0$  for all  $\mu \in \text{ker}_{\mathbb{k}}(M)$ , i.e.,  $\eta \in \text{ker}_{\mathbb{k}}(M)^\perp$ , which shows that  $\text{ker}_{\mathbb{k}}(L) = \text{ker}_{\mathbb{k}}(M)^\perp$ . Then, a well-known result of linear algebra asserts that  $\text{ker}_{\mathbb{k}}(M)^\perp = \text{im}_{\mathbb{k}}(M)$ , which proves (13).

Using (13), we get that (11) is equivalent to the existence of  $w_i \in \mathbb{k}^{m \times 1}$  such that  $D_i u = M w_i$  for  $i = 1, \dots, r$ . We find again (6). Let us now suppose that  $M \neq 0$ . Using (12), we then get  $\text{im}_{\mathbb{k}}(D_i Z) \subseteq \text{im}_{\mathbb{k}}(M) = \text{im}_{\mathbb{k}}(X)$  for  $i = 1, \dots, r$  and, since  $X$  has full column rank, there exist unique matrices  $W_i \in \mathbb{k}^{l \times d}$  such that:

$$D_i Z = X W_i, \quad i = 1, \dots, r. \quad (14)$$

Combining (12) with (14), we obtain:

$$A(Z\psi) = (D_1 Z \psi \ \dots \ D_r Z \psi) = X (W_1 \psi \ \dots \ W_r \psi). \quad (15)$$

Let  $B(\psi) := (W_1 \psi \ \dots \ W_r \psi) \in \mathbb{k}^{l \times r}$ . (15) and (8) yields:

$$A(u)v = X B(\psi)v = M = XY.$$

Hence, we get  $X(B(\psi)v - Y) = 0$  and using the fact that  $X$  has full column rank matrix, we finally obtain:

$$B(\psi)v = Y. \quad (16)$$

*Remark 3.* If  $l := \text{rank}_{\mathbb{k}}(M) = r$ , then  $B(\psi)$  is square.

Hence, Problem (3) admits a solution iff there exists  $\psi \in \mathbb{k}^{d \times 1}$  such that the system  $B(Z\psi)v = Y$  is solvable.

*Example 3.* We continue Example 1. We first can check that we have the factorisation  $M = XY$ , where:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then, using (14), we obtain:

- If  $d_1 \neq d_3$ , then we have:

$$W_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}, \quad B(Z\psi) = \begin{pmatrix} 0 & 0 \\ \psi & d_2 \psi \end{pmatrix}.$$

- If  $d_1 = d_3$ , then we have:

$$W_1 = I_2, \quad W_2 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad B(Z\psi) = \begin{pmatrix} \psi_1 & d_1 \psi_1 \\ \psi_2 & d_2 \psi_2 \end{pmatrix}.$$

*Example 4.* Let us consider the case where  $D_i := D^{i-1}$  for  $i = 1, \dots, r$ , where  $D \in \mathbb{k}^{n \times n}$  satisfies  $D^r = I_n$ . As explained in Hubert (2019), this case corresponds to a so-called *planetary gearbox*. Clearly, if  $u$  is a solution of (11), then so is  $Du$ , i.e., we have the following map:

$$D : \mathcal{V} \longrightarrow \mathcal{V} \\ u \longmapsto Du.$$

Hence, if  $u = Z\psi \in \mathcal{V}$ , then there exists  $\psi' \in \mathbb{k}^{d \times 1}$  such that  $Du = Z\psi'$ , i.e.,  $(DZ)\psi = Z\psi'$  for all  $\psi \in \mathbb{k}^{d \times 1}$ . Thus, there exists  $D' \in \mathbb{k}^{d \times d}$  such that  $DZ = ZD'$ . Then, we get  $D^2 Z = DZD' = ZD'^2$ , and thus:

$$D^i Z = ZD'^i, \quad i = 0, \dots, r-1.$$

Moreover, using  $D^r = I_n$ , then  $Z = ZD'^r$ , which yields

$$D'^r = I_d$$

since  $Z$  has full column rank. Using (14) with  $D_1 = I_n$ , there exists  $W_1 \in \mathbb{k}^{l \times d}$  such that  $Z = XW_1$ . Since both  $Z$  and  $X$  have full column rank, so has  $W_1$ . Hence, we have:

$$D^i Z = X (W_1 D'^i), \quad i = 0, \dots, r-1.$$

Since the matrices  $W_i$ 's defined by (14) are unique, we get:

$$W_i = W_1 D'^i, \quad i = 0, \dots, r-1.$$

Hence, if we note

$$F(\psi) := (\psi \ D' \psi \ \dots \ D'^{r-1} \psi) \in \mathbb{k}^{d \times r}, \quad (17)$$

then we finally obtain  $B(\psi) = W_1 F(\psi)$ .

*Remark 4.* If  $Z' \in \mathbb{k}^{n \times d}$  defines another basis of the solution space of (11), then there exists  $U \in \text{GL}_d(\mathbb{k})$ , i.e.,  $\det(U) \neq 0$ , such that  $Z' = ZU$ . Then, (14) yields  $D_i Z' = D_i ZU = X(W_i U)$  for  $i = 1, \dots, r$ , and thus  $A(Z' \psi') = X B(U \psi')$ . Hence, we have to study the solvability of  $B(U \psi')v = Y$ , i.e., of (16) with  $\psi = U \psi'$ .

Now, if  $M = X'Y'$ , then it was shown in Section 2.1 that  $X = X'V$  and  $Y' = VY$  for a certain  $V \in \mathbb{k}^{l \times l}$  such that  $\det(V) \neq 0$ , which yields  $D_i Z = X'(V W_i)$ , and thus we have  $A(Z\psi) = X' B'(\psi)$ , where:

$$B'(\psi) = (V W_1 \psi \ \dots \ V W_r \psi) = V B(\psi).$$

Hence, Problem (3) is reduced to study the solvability of  $B'(\psi)v = Y'$ , i.e.,  $V B(\psi)v = VY$ , i.e., (16). Thus, (16) does not depend on particular choices for  $X, Y$  and  $Z$ .

We now state a general result on the solvability of linear systems. Even though this result seems to us to be standard, in its present form, we were not able to find a precise reference (see James (1978) for a similar result and Remark 6 below). Hence, we provide a proof. To shorten the proof, we use homological algebra methods but longer linear algebra arguments can be used instead to prove it.

*Theorem 5.* Let  $A \in \mathbb{k}^{r \times s}$  and  $y \in \mathbb{k}^{r \times t}$  be two fixed matrices. Moreover, let  $B \in \mathbb{k}^{q \times r}$  be such that the rows of  $B$  form a basis of  $\text{ker}_{\mathbb{k}}(.A)$ , i.e.,  $B$  is a full row rank matrix such that  $\text{ker}_{\mathbb{k}}(.A) = \text{im}_{\mathbb{k}}(.B)$ , where  $q = \dim_{\mathbb{k}}(\text{ker}_{\mathbb{k}}(.A))$ . Then, the inhomogeneous linear system

$$Ax = y \quad (18)$$

is solvable, i.e., (18) admits a solution  $x \in \mathbb{k}^{s \times t}$ , iff  $y$  satisfies the following compatibility condition:

$$B y = 0. \quad (19)$$

Then, all the solutions  $x \in \mathbb{k}^{s \times t}$  of  $A x = y$  are given by

$$\forall z \in \mathbb{k}^{u \times t}, \quad x = E y + C z, \quad (20)$$

where  $C \in \mathbb{k}^{s \times u}$  is a matrix whose columns form a basis of  $\ker_{\mathbb{k}}(A) := \{\eta \in \mathbb{k}^{s \times 1} \mid A \eta = 0\}$ , i.e.,  $C$  is a full column rank matrix which satisfies  $\ker_{\mathbb{k}}(A) = \text{im}_{\mathbb{k}}(C) := C \mathbb{k}^{u \times 1}$ , where  $u = \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(A))$ , and  $E \in \mathbb{k}^{s \times r}$  is a *generalized inverse* of  $A$ , namely  $A E A = A$ .

**Proof.** By definition of  $B$  and  $C$ , we have the following *exact sequences* of  $\mathbb{k}$ -vector spaces (Rotman (2009))

$$\begin{aligned} 0 &\longrightarrow \mathbb{k}^{1 \times q} \xrightarrow{B} \mathbb{k}^{1 \times r} \xrightarrow{A} \mathbb{k}^{1 \times s}, \\ \mathbb{k}^{r \times 1} &\xleftarrow{A} \mathbb{k}^{s \times 1} \xleftarrow{C} \mathbb{k}^{u \times 1} \longleftarrow 0. \end{aligned}$$

Since the *functor*  $\text{hom}_{\mathbb{k}}(\cdot, \mathbb{k})$  (duality) is *exact* (Rotman (2009)), we get the following long exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathbb{k}^{1 \times q} \xrightarrow{B} \mathbb{k}^{1 \times r} \xrightarrow{A} \mathbb{k}^{1 \times s} \xrightarrow{C} \mathbb{k}^{1 \times u} \longrightarrow 0, \\ 0 &\longleftarrow \mathbb{k}^{q \times 1} \xleftarrow{B} \mathbb{k}^{r \times 1} \xleftarrow{A} \mathbb{k}^{s \times 1} \xleftarrow{C} \mathbb{k}^{u \times 1} \longleftarrow 0. \end{aligned} \quad (21)$$

Then, the *Euler-Poincaré characteristic* (Rotman (2009)) yields  $q - r + s - u = 0$ . Moreover, since  $\mathbb{k}^{t \times 1} \otimes_{\mathbb{k}} \cdot$  and  $\cdot \otimes_{\mathbb{k}} \mathbb{k}^{1 \times t}$  are two exact functors, using  $\mathbb{k}^{l \times 1} \otimes_{\mathbb{k}} \mathbb{k}^{1 \times m} \cong \mathbb{k}^{l \times m}$ , where  $\cong$  stands for *isomorphic*  $\mathbb{k}$ -vectors, we obtain the following long exact sequences of  $\mathbb{k}$ -vector spaces:

$$\begin{aligned} 0 &\longrightarrow \mathbb{k}^{t \times q} \xrightarrow{B} \mathbb{k}^{t \times r} \xrightarrow{A} \mathbb{k}^{t \times s} \xrightarrow{C} \mathbb{k}^{t \times u} \longrightarrow 0, \\ 0 &\longleftarrow \mathbb{k}^{q \times t} \xleftarrow{B} \mathbb{k}^{r \times t} \xleftarrow{A} \mathbb{k}^{s \times t} \xleftarrow{C} \mathbb{k}^{u \times t} \longleftarrow 0. \end{aligned} \quad (22)$$

For more details, see Rotman (2009). The exact sequence (22) shows that (18) is solvable iff  $y \in \text{im}_{\mathbb{k}^{1 \times t}}(A)$ , i.e., iff  $B y = 0$ . Since the exact sequence (21) *splits* (Rotman (2009)), there exist matrices  $D \in \mathbb{k}^{r \times q}$ ,  $E \in \mathbb{k}^{s \times r}$ , and  $F \in \mathbb{k}^{u \times s}$  such that:

$$\begin{aligned} B D &= I_q, & D B + A E &= I_r, & E A + C F &= I_s, \\ F C &= I_u, & E D &= 0, & F E &= 0. \end{aligned} \quad (23)$$

Using  $A C = 0$ , the third identity of (23) yields  $A E A = A$ , i.e.,  $E$  is a generalized inverse of  $A$ .

Let us now suppose that  $y \in \mathbb{k}^{r \times t}$  is such that  $B y = 0$ . Using the second identity of (23) and  $A C = 0$ , we get  $y = A E y = A (E y + C z)$  for all  $z \in \mathbb{k}^{u \times t}$ , which shows that (20) are solutions of (18). Finally, if there exists  $x^* \in \mathbb{k}^{s \times t}$  satisfying (18), i.e.,  $A x^* = y$ , then we have

$$A (x^* - E y) = A x^* - A E A x^* = 0,$$

which shows that  $x^* - E y \in \ker_{\mathbb{k}^{1 \times t}}(A) = C \mathbb{k}^{u \times t}$ . Thus, there exists  $z \in \mathbb{k}^{u \times t}$  such that  $x^* - E y = C z$ , which shows that all the solutions of (18) are of the form of (20).

*Remark 6.* In James (1978), the solutions of (18) are given under the form:

$$\forall z' \in \mathbb{k}^{s \times t}, \quad x = E y + (I_s - E A) z'. \quad (24)$$

We note that  $u < s$  since  $\text{im}_{\mathbb{k}}(C) \subset \mathbb{k}^{s \times 1}$  and  $C$  is injective. Hence, contrary to (20) – which depends only on  $u$  arbitrary parameters  $z_i$ ,  $i = 1, \dots, u$  – the parametrization (24) is not minimal in the sense that it depends on  $s$  arbitrary parameters  $z'_i$ ,  $i = 1, \dots, s$ . In other words, the columns of the matrix  $I_s - E A$  define

a generating set but not a basis of  $\ker_{\mathbb{k}}(A)$ . This fact can be checked again by considering the third identity of (23) and using the fact that  $F$  is surjective to get  $\text{im}_{\mathbb{k}}(I_s - E A) = \text{im}_{\mathbb{k}}(C F) = \text{im}_{\mathbb{k}}(C) = \ker_{\mathbb{k}}(A)$ .

Let us now study the solvability of the inhomogeneous linear system (16). By Theorem 5, (16) is solvable iff the matrix  $Y \in \mathbb{k}^{l \times m}$  satisfies  $K Y = 0$ , where the columns of the matrix  $K \in \mathbb{k}^{k \times l}$  form a basis of  $\ker_{\mathbb{k}}(B(\psi))$ , i.e.,  $K$  is a full row rank matrix satisfying  $\ker_{\mathbb{k}}(B(\psi)) = \text{im}_{\mathbb{k}}(K)$ . But  $Y$  is a full row rank matrix, i.e.,  $\ker_{\mathbb{k}}(Y) = 0$ , which shows that (16) is solvable iff  $\ker_{\mathbb{k}}(B(\psi)) = 0$ , i.e., iff  $B(\psi)$  has full row rank, i.e., iff  $B(\psi)$  admits a right inverse  $E(\psi) \in \mathbb{k}^{r \times l}$ , i.e.,  $B(\psi) E(\psi) = I_l$ , i.e., iff  $\text{im}_{\mathbb{k}}(B(\psi)) = \mathbb{k}^{l \times 1}$ . In particular, if  $l := \text{rank}_{\mathbb{k}}(M) = r$ , then using Remark 3, Problem (3) is solvable iff:

$$\exists \psi \in \mathbb{k}^{d \times 1} \setminus \{0\} : \det(B(Z \psi)) \neq 0.$$

*Theorem 7.* Let  $M \in \mathbb{k}^{n \times m}$  and  $M = X Y$  be a factorisation such that the columns of the matrix  $X \in \mathbb{k}^{n \times l}$  form a basis of  $\text{im}_{\mathbb{k}}(M)$  with  $l = \text{rank}_{\mathbb{k}}(M) \leq r$ . Moreover, let  $u = Z \psi$  for all  $\psi \in \mathbb{k}^{d \times 1}$  be a parametrization of (11), where  $Z \in \mathbb{k}^{n \times d}$ . Finally, let  $W_i \in \mathbb{k}^{l \times d}$  be the unique matrices satisfying (14) for  $i = 1, \dots, r$ , and:

$$B(\psi) = (W_1 \psi \dots W_r \psi) \in \mathbb{k}^{l \times r}.$$

Then, Problem (3) is solvable iff there exists  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  such that  $\text{im}_{\mathbb{k}}(B(\psi)) = \mathbb{k}^{l \times 1}$ , i.e., such that  $B(\psi)$  admits a right inverse  $E(\psi) \in \mathbb{k}^{r \times l}$ . If so, then

$$u = Z \psi, \quad \forall z \in \mathbb{k}^{(r-l) \times m}, \quad v = E(\psi) Y + C(\psi) z,$$

is a solution of Problem (3), where  $C(\psi) \in \mathbb{k}^{r \times (r-l)}$  is a full column rank matrix such that:

$$\ker_{\mathbb{k}}(B(\psi)) = \text{im}_{\mathbb{k}}(C(\psi)).$$

Finally, if  $\text{rank}_{\mathbb{k}}(M) = r$  and  $E(\psi) = B(\psi)^{-1}$ , then  $C(\psi) = 0$ , i.e., a solution of Problem (3) is given by:

$$u = Z \psi, \quad v = B(\psi)^{-1} Y.$$

*Example 5.* Let  $M \in \mathbb{k}^{n \times m}$  be of rank 1 and  $D_1 = I_n$ . Let  $M = X Y$  be a factorisation defined by (8), where  $X \in \mathbb{k}^{n \times 1}$  and  $Y \in \mathbb{k}^{1 \times m}$ . Then, we have  $Z = X$ , and thus  $u = X \psi$ , where  $\psi \in \mathbb{k}$ ,  $W_1 = 1$ , which yields  $B(\psi) = \psi$ . Hence, if we consider  $\psi \neq 0$ , then  $u = X \psi$  and  $v = \psi^{-1} Y$  is a solution of Problem (3). Hence, the factorisation  $M = X Y$  is a solution of Problem (3).

*Example 6.* We continue Examples 1 and 3.

- If  $d_1 \neq d_3$ , then  $\det B(\psi) = 0$  for all  $\psi \in \mathbb{k}$ , which proves that Problem (3) is not solvable.
- If  $d_1 = d_3$ , then  $\det B(\psi) = (d_1 - d_2) \psi_1 \psi_2$ , which proves that Problem (3) is solvable iff  $d_1 \neq d_2$  and  $\psi_1 \neq 0$  and  $\psi_2 \neq 0$ . If so, using  $Z = X$ , then we get:

$$u = X \psi = (\psi_1 \quad \psi_2 \quad \psi_3)^T,$$

$$v = B(\psi)^{-1} Y = \frac{1}{d_1 - d_2} \begin{pmatrix} -\frac{d_2}{\psi_1} & \frac{d_1}{\psi_2} & \frac{d_2}{\psi_2} \\ \frac{1}{\psi_1} & -\frac{1}{\psi_2} & -\frac{1}{\psi_2} \end{pmatrix}.$$

*Example 7.* Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}, \quad D_1 = I_3, \quad D_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

We can easily check that  $\text{rank}_{\mathbb{k}}(M) = 1$  and:

$$X = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, Y = (1 \ 0 \ 1), Z = X, W_1 = 1, W_2 = 0.$$

Hence, we have  $B(\psi) = (\psi \ 0)$  and  $E(\psi) = \psi^{-1} (1 \ 0)^T$  is a right inverse of  $B(\psi)$  for  $\psi \neq 0$ . Since  $\ker_{\mathbb{k}}(B(\psi).) = (0 \ 1)^T \mathbb{k}$ , the solutions of Problem (3) are of the form:

$$\begin{aligned} u &= (1 \ -2 \ 1)^T \psi, \\ v &= \psi^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0 \ 1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (z_1 \ z_2 \ z_3) \\ &= \begin{pmatrix} \psi^{-1} & 0 & \psi^{-1} \\ z_1 & z_2 & z_3 \end{pmatrix}, \quad \forall \psi \in \mathbb{k} \setminus \{0\}. \end{aligned}$$

*Example 8.* We continue Example 4. By Theorem 7, Problem (3) is solvable iff the matrix  $B(\psi) = W_1 F(\psi)$  admits a right inverse, where the matrix  $C$  is defined by (17).

If  $f : \mathcal{E} \rightarrow \mathcal{F}$  and  $g : \mathcal{F} \rightarrow \mathcal{G}$  are two  $\mathbb{k}$ -linear maps between finite-dimensional  $\mathbb{k}$ -vector spaces with  $g$  injective, then we can prove that  $g \circ f$  is surjective iff so are  $f$  and  $g$  (and so  $g$  is invertible). Hence, using the fact that  $W_1$  has full column rank, i.e.,  $\ker_{\mathbb{k}}(W_1.) = 0$ , we obtain that  $\text{im}_{\mathbb{k}}(B(\psi).) = \mathbb{k}^{l \times 1}$  iff  $\text{im}_{\mathbb{k}}(F(\psi).) = \mathbb{k}^{d \times 1}$  and  $\text{im}_{\mathbb{k}}(W_1.) = \mathbb{k}^{l \times 1}$ , i.e., iff  $\text{im}_{\mathbb{k}}(F(\psi).) = \mathbb{k}^{d \times 1}$  and  $W_1$  is square invertible matrix with  $d = l$ . We obtain that Problem (3) is solvable iff the following 3 conditions hold:

- (1)  $d := \dim_{\mathbb{k}}(\mathcal{V}) = l := \text{rank}_{\mathbb{k}}(M)$ ,
- (2)  $W_1$  is invertible, i.e.,  $\det W_1 \neq 0$ ,
- (3) there exists  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  such that  $F(\psi) \in \mathbb{k}^{l \times r}$  defined by (17) admits a right inverse.

In particular, if  $l = r$ , then  $F(\psi) \in \mathbb{k}^{r \times r}$  and Condition 3 is reduced to the existence of  $\psi \in \mathbb{k}^{r \times 1} \setminus \{0\}$  such that  $F(\psi)$  is invertible, i.e.,  $\det F(\psi) \neq 0$ .

For instance, if we consider the following matrices

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D^2 = I_3,$$

then we obtain  $L = LD = (1 \ -2 \ 1)$  and

$$Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix},$$

$$W_1 = \frac{1}{3} \begin{pmatrix} -8 & 5 \\ 7 & -4 \end{pmatrix}, \quad W_2 = \frac{1}{3} \begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}, \quad D' = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

We can check that  $D'^2 = I_2$ . We have  $\det W_1 = -1/3 \neq 0$ ,

$$u = \begin{pmatrix} 2\psi_1 - \psi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix}, \quad F(\psi) = \begin{pmatrix} \psi_1 & \psi_1 \\ \psi_2 & 2\psi_1 - \psi_2 \end{pmatrix},$$

for all  $\psi = (\psi_1 \ \psi_2)^T \in \mathbb{k}^{2 \times 1}$ , and:

$$\det(F(\psi)) = 2\psi_1(\psi_1 - \psi_2).$$

Hence, if we consider  $\psi_1 \neq 0$  and  $\psi_1 \neq \psi_2$ , then we have:

$$E(\psi) = B(\psi)^{-1} = \det(F(\psi))^{-1} \begin{pmatrix} \psi_1 - 4\psi_2 & 2\psi_1 - 5\psi_2 \\ 7\psi_1 - 4\psi_2 & 8\psi_1 - 5\psi_2 \end{pmatrix},$$

$$v = \det(F(\psi))^{-1} \begin{pmatrix} \psi_1 - 4\psi_2 & 2\psi_1 - 5\psi_2 & 3(\psi_1 - 2\psi_2) \\ 7\psi_1 - 4\psi_2 & 8\psi_1 - 5\psi_2 & 3(3\psi_1 - 2\psi_2) \end{pmatrix}.$$

Finally, if  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  is such that  $B(\psi)$  admits a right inverse  $E(\psi)$ , then using  $B(\lambda\psi) = \lambda B(\psi)$  for  $\lambda \in \mathbb{k} \setminus \{0\}$ ,  $B(\lambda\psi)$  admits a right inverse  $E(\lambda\psi)$  (e.g.,  $\lambda^{-1} E(\psi)$ ), i.e.,  $B(\lambda\psi)E(\lambda\psi) = I_l$  for  $\lambda \in \mathbb{k} \setminus \{0\}$ . Hence, we get  $B(\psi)(\lambda E(\lambda\psi) - E(\psi)) = 0$ , which shows that there exists a unique  $q(\psi, \lambda) \in \mathbb{k}^{(r-l) \times l}$  such that:

$$\lambda E(\lambda\psi) = E(\psi) + C(\psi)q(\psi, \lambda).$$

Hence, we obtain  $E(\lambda\psi) = \lambda^{-1} E(\psi) + C(\psi)\lambda^{-1}q(\psi, \lambda)$ . In particular, setting  $\lambda = 1$ , we get  $C(\psi)q(\psi, 1) = 0$ . Now, for  $\lambda \in \mathbb{k} \setminus \{0\}$ , let  $(u = Z\psi, v = E(\psi)Y + C(\psi)z)$ , where  $z \in \mathbb{k}^{(r-l) \times m}$ , be a solution of Problem (3). Then, we have:

$$(\lambda u = Z(\lambda\psi), \lambda^{-1}v = E(\lambda\psi)Y + \lambda^{-1}C(\psi)(z - q(\psi, \lambda)Y)).$$

Since  $B(\lambda\psi)C(\psi) = \lambda B(\psi)C(\psi) = 0$ , we obtain

$$B(\lambda u)(\lambda^{-1}v) = B(\lambda Z\psi)(\lambda^{-1}v) = Y,$$

and thus, we find again that the set of solutions of Problem (3) is stable under the transformations:

$$\forall \lambda \in \mathbb{k} \setminus \{0\}, \quad \varphi_\lambda : (u, v) \mapsto (\lambda u, \lambda^{-1}v).$$

Hence, we can take  $\psi \in \mathbb{P}_d(\mathbb{k})$  to consider at once all the vectors  $\lambda\psi$  for all  $\lambda \in \mathbb{k} \setminus \{0\}$  in a single equivalent class.

### 3. A FEW MORE RESULTS

As shown in Section 2.2, a necessary condition on  $u \in \mathbb{k}^{n \times 1}$  for the solvability of Problem (3) is that  $u$  satisfies (11), i.e.,  $u$  is of the form of  $u = Z\psi$  for a certain  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$ . Hence, we have to find  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  – if it exists – such that the inhomogeneous linear system  $A(Z\psi)v = M$  admits a solution  $v \in \mathbb{k}^{m \times 1}$ . According to Theorem 5, it admits a solution iff there exists  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  such that  $L'(\psi)M = 0$ , where  $L'(\psi) \in \mathbb{k}^{p' \times n}$  satisfies  $\ker_{\mathbb{k}}(.A(Z\psi)) = \text{im}_{\mathbb{k}}(.L'(\psi))$ , i.e., iff:

$$\ker_{\mathbb{k}}(.A(Z\psi)) = \text{im}_{\mathbb{k}}(.L'(\psi)) \subseteq \ker_{\mathbb{k}}(.M) = \text{im}_{\mathbb{k}}(.L).$$

By construction,  $u = Z\psi$  satisfies  $LA(Z\psi) = 0$  for all  $\psi \in \mathbb{k}^{d \times 1}$ , which yields:

$$\forall \psi \in \mathbb{k}^{d \times 1}, \quad \ker_{\mathbb{k}}(.M) = \text{im}_{\mathbb{k}}(.L) \subseteq \ker_{\mathbb{k}}(.A(Z\psi)).$$

Hence, Problem (3) is solvable iff:

$$\exists \psi \in \mathbb{k}^{d \times 1} \setminus \{0\}, \quad \ker_{\mathbb{k}}(.A(Z\psi)) = \ker_{\mathbb{k}}(.M). \quad (25)$$

Since we have

$$\begin{aligned} \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(.A(Z\psi))) &= n - \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(.A(Z\psi))) \\ &= n - \dim_{\mathbb{k}}(\text{im}_{\mathbb{k}}(A(Z\psi).)) \\ &= n - (r - \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(A(Z\psi).))), \end{aligned}$$

using (9), i.e.,  $\dim_{\mathbb{k}}(\ker_{\mathbb{k}}(.A(Z\psi))) = n - \text{rank}_{\mathbb{k}}(M)$ , (25) holds iff we have:

$$\exists \psi \in \mathbb{k}^{d \times 1} \setminus \{0\} : \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(A(Z\psi).)) = r - \text{rank}_{\mathbb{k}}(M). \quad (26)$$

*Theorem 8.* With the above notations, Problem (3) is solvable iff (26) holds.

Using the factorisation  $A(Z\psi) = XB(Z\psi)$  (see (15)), where  $X$  has full column rank, we get:

$$\ker_{\mathbb{k}}(A(Z\psi).) = \ker_{\mathbb{k}}(B(Z\psi).).$$

Since  $\text{rank}_{\mathbb{k}}(B(Z\psi)) = r - \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(B(Z\psi).))$ , Problem (3) is solvable iff  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  exists such that

$$\begin{aligned} \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(B(Z\psi).)) &= r - \text{rank}_{\mathbb{k}}(M) \\ \Leftrightarrow \text{rank}_{\mathbb{k}}(B(Z\psi)) &= \text{rank}_{\mathbb{k}}(M) = l, \end{aligned}$$

which is equivalent to the fact that  $B(Z\psi) \in \mathbb{k}^{l \times r}$  admits a right inverse and we find again the result of Theorem 7.

Theorem 8 is a generalization of a result obtained for  $r = 2$  in Hubert et al. (2019).

To end this section, we focus on the particular case

$$\text{rank}_{\mathbb{k}}(M) = r = 2, \quad D_1 = I_2,$$

which is interesting in practice (Hubert (2019)). Using  $D_1 = I_n$ , we get  $\mathcal{V} \subseteq \ker_{\mathbb{k}}(L.)$ , which, using (9), yields:

$$\dim_{\mathbb{k}}(\mathcal{V}) \leq \dim_{\mathbb{k}}(\ker_{\mathbb{k}}(L.)) = n - p = \text{rank}_{\mathbb{k}}(M) = 2. \quad (27)$$

Moreover, (26) is equivalent to  $\ker_{\mathbb{k}}(A(Z\psi)) = 0$ , where:

$$\ker_{\mathbb{k}}(A(u)) = \{(\alpha_1 \quad \alpha_2)^T \in \mathbb{k}^{2 \times 1} \mid u\alpha_1 + D_2 u \alpha_2 = 0\}.$$

If  $\alpha_1 = 0$ , then  $D_2 u = 0$ , i.e.,  $u$  is an eigenvector of  $D_2$  with 0 as associated eigenvalue. If  $\alpha_2 \neq 0$ , then  $D_2 u = (-\alpha_1/\alpha_2)u$ , i.e.,  $u$  is an eigenvalue of  $D_2$  with  $-\alpha_1/\alpha_2$  as associated eigenvalue.

*Corollary 9.* If  $\text{rank}_{\mathbb{k}}(M) = r = 2$  and  $D_1 = I_2$ , then Problem (3) is solvable iff there exists  $\psi \in \mathbb{k}^{d \times 1} \setminus \{0\}$  such that  $u = Z\psi$  is not an eigenvalue of the matrix  $D_2$ .

*Example 9.* We consider again Example 1. If  $d_1 \neq d_3$ , then  $u = (0 \quad u_2 \quad 0)^T$  is an eigenvector of  $D_2$  for all  $u_2 \in \mathbb{k}$ , which proves that Problem (3) is not solvable. If  $d_1 = d_3$ , then  $D_2 u = (d_1 u_1 \quad d_2 u_2 \quad d_1 u_1)^T$ , where  $u = (u_1 \quad u_2 \quad u_1)^T$ . If  $d_2 = d_1$ , then  $u$  is an eigenvalue of  $D_2$ , which proves that Problem (3) is not solvable. If  $d_2 \neq d_1$ ,  $u$  is not an eigenvalue of  $D_2$  and Problem (3) is solvable. We find again the results obtained in Example 6.

*Example 10.* We consider again Example 8. The vector  $u = (2\psi_1 - \psi_2 \quad \psi_1 \quad \psi_2)^T$  is an eigenvector of  $D$  iff

$$\exists \lambda \in \mathbb{k}, \quad D u = \begin{pmatrix} \psi_2 \\ \psi_1 \\ 2\psi_1 - \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} 2\psi_1 - \psi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix},$$

i.e., iff  $\lambda = 1$  and  $\psi_2 = \psi_1$  (compare with Example 8).

*Example 11.* We consider the case of a diagonal matrix  $D_2 = \text{diag}(d_1, \dots, d_n)$  whose diagonal elements are the  $d_i$ 's. Then, using (27),  $u \in \mathbb{k}^{n \times 1} \setminus \{0\}$  must belong to the solution space of the linear system (11) of dimension less than or equal to 2. According to Corollary 9, to solve Problem (3), we must find  $0 \neq u \in \mathbb{k}^{n \times 1}$  satisfying (11) and such that  $u$  is not an eigenvector of  $D_2$ . Let  $\{f_i\}_{i \in I}$  be a basis of  $\mathcal{V}$ , where  $J = \{1\}$  if the dimension of  $\mathcal{V}$  is 1 and  $J = \{1, 2\}$  if the dimension of  $\mathcal{V}$  is 2. The vectors  $e_i$ 's of the standard basis of  $\mathbb{k}^{n \times 1}$  – where  $e_i$  is the column vector with 1 at the  $i^{\text{th}}$  position and 0 elsewhere – are the eigenvectors of  $D_2$  with  $d_i$  as an associated eigenvalue.

If all the  $d_i$ 's are distinct, then Problem (1) is solvable iff

$$\forall j \in J, \exists \gamma_j \in \mathbb{k} : u := \sum_{j \in J} \gamma_j f_j \notin \mathbb{k} e_i, \quad i = 1, \dots, n,$$

i.e., iff there exists a  $\mathbb{k}$ -linear combination of the  $f_i$ 's with at least two non-zero components.

If the multiplicity of certain of the  $d_i$ 's is strictly larger than 1, denoting by  $\{d'_1, \dots, d'_r\}$  the set of distinct  $d_i$ 's and by  $\mathcal{E}_i$  the  $\mathbb{k}$ -vector space generated by the  $e_j$ 's whose indices  $j$  correspond to the position  $j$  of  $d'_i$  in the set  $\{d_1, \dots, d_n\}$ , then a solution of Problem (1) exists iff:

$$\forall j \in J, \exists \gamma_j \in \mathbb{k} : u := \sum_{j \in I} \gamma_j f_j \notin \mathcal{E}_i, \quad i = 1, \dots, r.$$

In other words, Problem (1) admits a solution iff there exists a  $\mathbb{k}$ -linear combination of the  $f_i$ 's with a non-

zero entry at a position different from the  $j^{\text{th}}$  positions corresponding to the  $e_j$ 's defining the  $\mathcal{E}_i$ 's for  $i = 1, \dots, r$ .

*Example 12.* If  $u$  and  $D_2 u$  can be chosen to be orthogonal, then they cannot be collinear, and thus  $u$  is not an eigenvector of  $D_2$  and Problem (3) is then solvable.

## 4. CONCLUSION

In this paper, we give tractable necessary and sufficient condition for the solvability of the factorisation problem  $M = \sum_{i=1}^r D_i u v_i$  and we solve it when a solution exists.

In a forthcoming paper, the minimization problem

$$\min_{u \in \mathbb{k}^{n \times 1}, v \in \mathbb{k}^{r \times m}} \left\| \sum_{i=1}^r D_i u v_i - M \right\|_{\text{Frob}},$$

where  $\|\cdot\|_{\text{Frob}}$  stands for the *Frobenius norm*, will be studied based on algebraic and symbolic methods. This problem corresponds to the case of noisy data. The knowledge of explicit solutions as studied in this paper gives good initial conditions for optimization algorithms.

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