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Towards a combinatorial algorithm for the enumeration of isotopy classes of tilings on hyperbolic surfaces

Benedikt Kolbe^{*†‡}

Abstract

Based on the mathematical theory of isotopic tilings on hyperbolic surfaces and mapping class groups, we present the, to the best of our knowledge, first algorithms for the enumeration of isotopy classes of tilings by compact disks with a given symmetry group on hyperbolic surfaces, which is moreover combinatorial in nature. This enumeration is relevant for crystallography and materials science. Using the theory of automatic groups, we give some results on the computational tractability of the presented algorithm. We also extend data structures for combinatorial classes of tilings to isotopy classes and give an implementation of the proposed algorithm for certain classes of tilings and illustrate the enumeration with examples.

1 Introduction and motivation

The use of repeating motifs to tile space has a long and involved history in mathematics, engineering, art and sciences. Most efforts have been directed towards patterns in Euclidean spaces but lately, the role of hyperbolic geometry for the natural sciences and also in Euclidean tilings is becoming increasingly acknowledged. It has been recognized more recently that assemblies of atoms (and molecules) in crystalline arrangements that are energetically favourable involve (intrinsic) curvature [42], and many chemical structures such as metal-organic frameworks were found to reticulate triply-periodic minimal surfaces (TPMS) [40, 41, 42, 13], which are minimal surfaces invariant under three linearly independent translations.

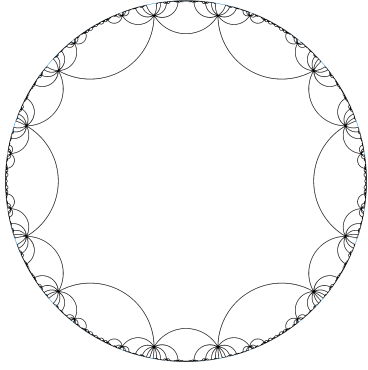
These observations and ideas have led to the development of the EPINET [1] (Euclidean patterns in non-euclidean tilings) project, which is based on a novel investigation of 3-dimensional Euclidean networks, where TPMS are used as a convenient route to the enumeration of crystallographic nets and polyhedra in \mathbb{R}^3 [74, 66, 72, 43, 70, 12], with results integrated into the Reticular Chemistry Structure Resource database [4]. By investigating how graphs embed on TPMS, the EPINET project aims to produce and analyse arising chemical structures. TPMS are inherently hyperbolic in the sense that the closed Riemannian surface obtained by gluing the unit cell's (the fundamental domain of all translations under which the TPMS is invariant) faces to itself according to its translations is inherently hyperbolic [64]. In this paper, a *hyperbolic surface* is a closed surface that is universally covered by and locally isometric to the hyperbolic plane \mathbb{H}^2 . Every closed surface with genus $g > 1$ is conformally equivalent to such a surface by the celebrated uniformization theorem [5], in such a way that the symmetries of the original surface are preserved [56, Lemma 2.2.1]. Figure 1(a) shows the covering of the closed surface S_D giving rise to the diamond TPMS by the hyperbolic plane \mathbb{H}^2 . The fundamental group $\pi_1(S_D)$ of S_D is generated by the hyperbolic translations that map opposite edges to each other. The in-surface symmetries of TPMS manifest as ambient Euclidean symmetries of \mathbb{R}^3 [69, Theorem 2.2.5], so that symmetric tilings of TPMS give rise to embedded triply-periodic connected graphs, or *nets* in \mathbb{R}^3 , important for crystallography. In Figure 1(b) and (c), the symmetries of S_D and the diamond TPMS are illustrated by the tiling by triangles with symmetry group $*246$, in \mathbb{H}^2 and in \mathbb{R}^3 . Note that we adhere to Conway's notation [15] for orbifold symmetry groups in this paper, explained in Section A in the appendix.

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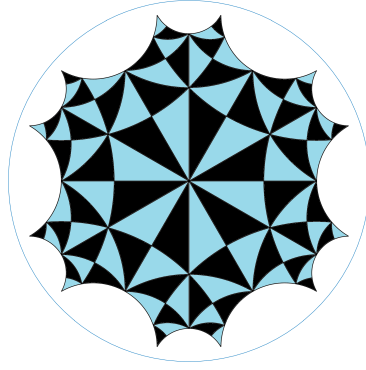
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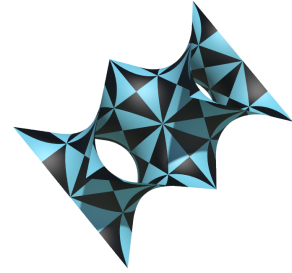
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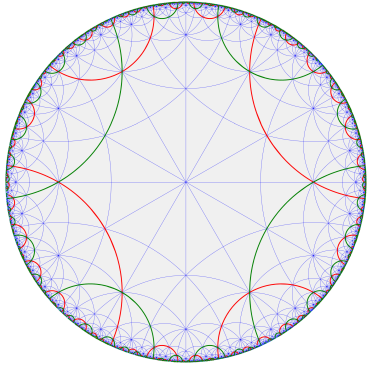
(a) Tiling of \mathbb{H}^2 by dodecagons corresponding to the genus 3 Riemann surface that gives rise to the diamond TPMS.



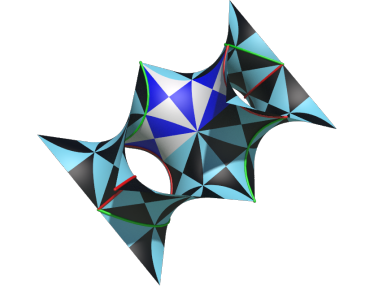
(b) The symmetries of the diamond TPMS within a unit cell in \mathbb{H}^2 . Each line represents a mirror symmetry.



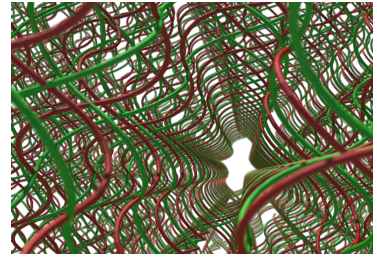
(c) A section of the diamond TPMS in \mathbb{R}^3 , together with its smallest asymmetric triangle patches.



(d) A tiling of the hyperbolic plane with symmetry group 22222, represented by the green and red edges, with *246 triangles in blue in the background.



(e) The decoration from (d) and a fundamental domain of 22222 shown as a collection of dark blue and grey triangles on the diamond TPMS.



(f) The resulting net in \mathbb{R}^3 when the tile boundaries are taken as trajectories in Euclidean 3-space, after enforcing the translational symmetries.

Figure 1: (a)-(c) shows symmetries of the diamond TPMS in \mathbb{R}^3 and its uniformization in \mathbb{H}^2 . (d)-(f) shows the progression from a tiling of the hyperbolic plane to a 3-dimensional net via a decoration of the Diamond triply-periodic minimal surface.

The EPINET approach can be summarized as in Figure 1(d)-(f), where (d) shows a hyperbolic tiling that corresponds to a graph embedded in the *orbifold* associated to the symmetry group of the tiling. The tiling with 1-skeleton consisting of the red and green edges is invariant under the symmetries in $\pi_1(S_D)$, illustrated in Figure 1(a), of the covering map of \mathbb{H}^2 onto the diamond TPMS. When the tile boundaries are considered as trajectories in \mathbb{R}^3 rather than curves on the surface, we obtain a net in \mathbb{R}^3 , shown in (f).

Several special cases of hyperbolic tilings and their isotopy classes have been explored [47, 70, 44, 39, 46, 48, 45, 68, 67, 23, 24, 22], and resulting structures have been used in analysis in real physical systems [52].

Research relating to EPINET is inherently interdisciplinary. The output is aimed at material science and chemistry. On the other hand, there is deep mathematics involved, pertaining to orbifolds [76] and their *mapping class groups* (MCGs) [54, 53]. The mathematics at the core of these developments is a field now known as combinatorial tiling theory [20, 19]. Combinatorial tiling theory has been investigated from a point of view of enumerations [38, 17, 78], introducing data structures and algorithms that allow for analyses and manipulations of combinatorial objects, called *D-symbols*, complete invariants for equivariant equivalence classes of tilings (defined below) of simply connected spaces by disks.

With the recent advent of new techniques arising from isotopic tiling theory [54] for the study of isotopy classes of tilings (defined below) by closed disks on hyperbolic surfaces that generalize the afore-

mentioned theory for equivariant equivalence classes, there is a need to extend existing algorithms to accommodate the more refined equivalence classes of isotopic tilings on surfaces. This paper focusses on algorithmic aspects of the theory, proposing a natural data structure for isotopy classes of tilings by extending, in a compatible way, the existing framework for combinatorial tiling theory. We use this to present the, to the best of our knowledge, first algorithm for the enumeration of isotopy classes of tilings with a given symmetry group on hyperbolic surfaces. Our approach shares intimate relations with other areas of maths and computer science, which we highlight along the way (and in the appendix). We derive results, some of them of a more experimental, that facilitate applications of algorithms of computational group theory and discuss their implementability with finite deterministic automata. We also illustrate the algorithm by presenting an array of examples that are the result of an implementation for the special case where all symmetries of the structure are generated by rotations, which constitutes a promising case for applications [52, 53]. We also present a natural data structure to represent tilings for this case. We focus exclusively on the hyperbolic case, which is the relevant case for applications, although our results also hold for Euclidean orbifolds.

2 Isotopic tiling theory

Isotopic tiling theory [54] describes the structure of tilings, up to deformations, on a hyperbolic surface for which each tile is homeomorphic to a closed and bounded disk. We recall the definitions in this context. A countable set \mathcal{T} of compact disks in a metric space X is called a *tiling* if every point $x \in X$ belongs to at least one tile $T \in \mathcal{T}$, every two tiles T_1 and T_2 of \mathcal{T} have disjoint interior, and the collection of tiles is locally finite. A tiling \mathcal{T} is locally finite if any compact set in X meets only a finite number of tiles. The vertices and edges of a tile are defined topologically, so a *vertex* is a point that is contained in at least three tiles, and an *edge* is a connected segment of tile boundary joining two vertices.

Let \mathcal{T} be a tiling of X and let Γ be a discrete group of isometries. If $\mathcal{T} = \gamma\mathcal{T} := \{\gamma T \mid T \in \mathcal{T}\}$ for all $\gamma \in \Gamma$ then we call Γ a symmetry group of \mathcal{T} and the pair (\mathcal{T}, Γ) an *equivariant tiling*. Two tiles $T_1, T_2 \in \mathcal{T}$ are *equivalent* or symmetry-related if there exists $\gamma \in \Gamma$ such that $\gamma T_1 = T_2$. The *orbit* of a tile is the subset of \mathcal{T} given by images of T : $\Gamma.T = \{\gamma T \text{ for } \gamma \in \Gamma\}$. Given a particular tile $T \in \mathcal{T}$, the *stabilizer subgroup* Γ_T is the subgroup of Γ that fixes T , i.e. $\Gamma_T = \{\gamma \in \Gamma \mid \gamma T = T\}$. A tile is called *fundamental* if Γ_T is trivial and we call the whole tiling fundamental if this is true for all tiles. An equivariant tiling is called *tile- k -transitive*, when k is the number of equivalence classes (i.e. distinct orbits) of tile under the action of Γ . Note that these definitions do not require Γ to be the maximal symmetry group for the tiling \mathcal{T} .

For the rest of this paper, let \mathcal{O} be a compact (*hyperbolic*) orbifold with symmetry group $\Gamma \subset \text{Iso}(\mathbb{H}^2)$, and O its underlying topological space, i.e. $O = \mathbb{H}^2/\Gamma$. The symmetry group Γ is also known as the *orbifold fundamental group* of \mathcal{O} and the set of branch points in \mathcal{O} and O of the covering by \mathbb{H}^2 is known as the *singular locus*. For our purposes, we define an orbifold as a quotient space \mathbb{H}^2/Γ of orbits of a symmetry group Γ with each point in the singular locus marked by the isomorphism class of its stabilizer subgroup in Γ . See Section A in the appendix for further details.

Definition 1. Two equivariant tilings $(\mathcal{T}_1, \Gamma_1)$ and $(\mathcal{T}_2, \Gamma_2)$ of \mathbb{H}^2 are *equivariantly equivalent* if there is a homeomorphism ϕ of \mathbb{H}^2 s. t. $\phi(T_1) \in \mathcal{T}_2$ and ϕ equivariant, i.e. if there is a group isomorphism $h : \Gamma_1 \rightarrow \Gamma_2$ s. t. $h(\gamma_1)[\phi(T_1)] = \phi(\gamma_1[T_1])$ for all $\gamma_1 \in \Gamma_1$ and $T_1 \in \mathcal{T}_1$. If $\Gamma_1 = \Gamma_2$ and ϕ is isotopic to the identity through a (continuous) path of equivariant homeomorphisms, the two tilings are called *isotopically equivalent*.

For equivariant tilings of hyperbolic surfaces, to be isotopically equivalent w.r.t. any group of isometries of the surface means exactly that the tilings can be deformed into one another on the surface [54, Proposition 3].

Dress [20] showed that a complete invariant for equivariant equivalence, called the D-symbol, exists for tilings of simply connected manifolds. The D-symbol consists of a finite graph that records adjacencies between tiles and their faces, augmented by weights that encode the group action of Γ on \mathcal{T} . We will not need the exact definition of D-symbols for our purposes, only that it has as many vertices as orbits of flags in the tiling, and that two equivariant tilings are equivariantly equivalent if and only if their D-symbols are isomorphic as weighted graphs [38, Lemma 1.1].

D-symbols can be exploited for a fully algorithmic approach to the enumeration and identification up to equivariant equivalence of 2D tilings of \mathbb{H}^2 (and \mathbb{E}^2 and \mathbb{S}^2) by assigning a *size*, a measure of complexity to D-symbols [38, 17]. We use the word enumeration to refer to an unambiguous and complete list in this paper, and only mention these properties for emphasis and clarity in the following.

A fundamental tile-1-transitive equivariant tiling (\mathcal{T}, Γ) has a single tile orbit, and every representative is the closure of a fundamental domain for Γ . Conversely, any closed fundamental domain for Γ also gives rise to such a tiling. One can enumerate fundamental tile-1-transitive tilings by constructing all such tilings as representative D-symbols [38]. The same paper then shows that tile- k -transitive fundamental tilings can be derived from those of transitivity $(k - 1)$ using the operation of *tile splitting*. Non-fundamental tile- k -transitive tilings are obtained from fundamental ones by *tile glueing*, where one glues equivalent tiles in a tiling by removing edges that separate them. The glueing and splitting operations can also be used in enumerations of isotopy classes of equivariant tilings [54, Proposition 1].

The mapping class group (MCG) of an orbifold with hyperbolic symmetry group Γ can be defined as the group of homeomorphisms $\{f\}$ of \mathbb{H}^2 such that $f\Gamma f^{-1} = \Gamma$, modulo isotopies preserving Γ [54]. Such a homeomorphism f projects to a homeomorphism of \mathbb{H}^2/Γ , but not all homeomorphisms of \mathcal{O} give rise to a homeomorphism of orbifolds [54, Section 5]. We shall refer to a MCG of a closed orientable surface, possibly with punctures, as a *classical MCG*. These have been extensively studied [26]. We will explain the role of MCGs for our purposes in the next section.

Proposition 2. [54, Section 5] *MCGs of symmetry groups without reflection symmetries are finite index subgroups of classical MCGs of surfaces with punctures, while those with mirrors boundaries are split extensions of these by finitely generated abelian groups.*

3 The algorithm for enumerations of isotopy classes of tilings

For a given symmetry group Γ with orbifold \mathcal{O} , the MCG $\text{Mod}(\mathcal{O})$ acts transitively on the set of isotopy classes of equivariant tiles with symmetry group Γ and given D-symbol [54]. However, it is not always the case that there are no ambiguities. A stabilizer subgroup of the action of the MCG on an isotopy class of a tiling (\mathcal{T}, Γ) (by disks) is always finite [54, Proposition 5]. Furthermore, there exists a nontrivial stabilizer subgroup if and only if there is a tiling (\mathcal{T}', Γ') equivariantly equivalent to (\mathcal{T}, Γ) s. t. Γ' is not the maximal symmetry group of \mathcal{T}' [54, Section 7]. This situation can be identified by the existence of a nontrivial graph automorphism of the D-symbol encoding the equivariant equivalence class of the tiling [19]. Results on the structure of finite subgroups of MCGs can be used to identify candidates of subgroups of the MCG that account exactly for the ambiguities in the list of isotopy classes of a given tiling on a hyperbolic surface, as explained in [54, Section 7].

Lemma 3. *Given an equivariant tiling (\mathcal{T}, Γ) , the set $E = \text{Mod}(\mathcal{O})/M_{\mathcal{T}}$, where $M_{\mathcal{T}} \subset \text{Mod}(\mathcal{O})$ is the stabilizer subgroup in $\text{Mod}(\mathcal{O})$ of \mathcal{T} , is a complete unambiguous enumeration of isotopy classes of tilings with symmetry group Γ and combinatorial structure equivalent to that of \mathcal{T} .*

Proof. Since $\text{Mod}(\mathcal{O})$ acts transitively on the set of tilings, we only need to show that the enumeration is unambiguous. For $[m_1], [m_2] \in E$, if $[m_1]T = [m_2]T$, then $[m_1]^{-1}[m_2] \in M_{\mathcal{T}}$, so $[m_1] = [m_2]$. \square

Lemma 3 implies that for an enumeration of isotopy classes of tilings in \mathbb{H}^2 with symmetry group Γ using the MCG, one needs to enumerate cosets of the MCG w.r.t. finite subgroups, by listing a representative of each class.

By Hurwitz's theorem, any hyperbolic surface admits a finite group of isometries that contains all isometries of the surface [26, Theorem 7.4, Proposition 7.7]. Therefore, for a given hyperbolic surface $S = \mathbb{H}^2/\text{Tr}$ with fundamental group Tr , there is a maximal symmetry group Γ_M with $\text{Tr} \subset \Gamma_M$ such that any tiling on S lifts to a tiling in \mathbb{H}^2 with symmetry group Γ s. t. $\text{Tr} \subset \Gamma \subset \Gamma_M$. Conversely, if a tiling \mathcal{T} has Γ as its symmetry group, then it is *commensurate* with S , i.e. invariant under Tr , and therefore it will project to a tiling \mathcal{T}_S on S such that vertices of \mathcal{T}_S are projections of vertices of the same degree in \mathcal{T} . Figure 3 shows an example of three isotopically distinct tilings that invariant under the same symmetry group that all fit onto the surface obtained by identifying opposite edges of the dodecagon shown in green on the left.

In case S is the hyperbolic surface associated to a TPMS, Tr is normal in Γ_M [53, Section 2.3]. Indeed, in this case Tr is characterized as the preimage of the group $\widetilde{\text{Tr}}$ of all translations under which

the TPMS is invariant, as discussed in the introduction, and $\widetilde{\text{Tr}}$ is clearly normal in the symmetry group of the TPMS. By normality, each subgroup Γ in-between Tr and Γ_M corresponds uniquely to a subgroup of the finite group Γ_M/Tr [33, Lemma 2.7.5]. Given a finite presentation of Γ_M/Tr , its subgroups can be enumerated using existing algorithms and their implementations available in the GAP programming language [27].

We can now formulate an algorithm for the enumeration of isotopy class of tilings by disks with a given symmetry group up to a particular complexity. As input, the algorithm takes a symmetry group, the highest complexity of the D-symbol up to which we enumerate, and word length up to which we enumerate the tilings as representatives of cosets of the MCG of the orbifold associated to the symmetry group. Step 1 has been treated in the literature, in particular in relation to EPINET, and can be accomplished

Algorithm 1 Enumerating isotopy classes of a given symmetry group up to given threshold

1. List D-symbols up to the given complexity.
 2. Check for the existence of graph automorphisms of each D-symbol, as these lead to ambiguities in the action of the MCG.
 3. Find the finite subgroup H in the MCG corresponding to the graph automorphism group for each D-symbol, using theoretical results on their structures.
 4. Enumerate the cosets in $\text{Mod}(\mathcal{O})/H$ by listing representatives.
 5. Find a realization of each D-symbol as a tiling in \mathbb{H}^2 with symmetry group Γ and apply to it each element of the list of step 4.
-

using existing enumerations of D-symbols [38], as well as algorithms and their implementations [17] and has recently led to a database of equivariant equivalence classes of tilings [78]. Step 2 can be solved using algorithms 9 and 10 in [17], which have expected quadratic time complexity in the number of vertices of the D-symbol. To find the finite subgroups corresponding to automorphisms of D-symbols in MCGs, there is a wide mathematical literature on MCGs investigating related questions [50]. Depending on the symmetry group of interest, there exist full classifications, for example for *stellate groups*, i.e. symmetry groups that can be generated solely by rotations [57, Theorem 4.4]. These represent an important class in different contexts in the natural sciences [44, 45, 23, 24, 22, 68, 52] and also the main case for the examples produced here. After explaining the data structure for tilings with stellate symmetry groups in Section 4, our main objective in the remainder of this paper will be the investigation of the enumeration of cosets, for which we make use of the two, to the best of our knowledge, only (semi-)algorithms for this type of problem, that work under certain conditions. There are two implementations available for both, in the package KBMAG [34] for GAP and in C++ in MAF [77].

Note that to apply the representatives of cosets to tilings as in step 5, one can substitute elements of the MCG by algebraic transformations of a set of generators of Γ [54, Theorem 2, Section 3], which can be explicitly derived for special sets of generators [53]. The tilings in \mathbb{H}^2 can be built from combinatorial information involving a set of generators for Γ , see Section 4 for further details. Thus, each step of algorithm 1 solves a problem that is combinatorial in nature, by-passing the inherently geometric nature of the problem and transforming the geometric problem into a purely combinatorial and group theoretical one.

4 Data structure for isotopy classes of tilings

We call a *geodesic representative* of an equivariant tiling (\mathcal{T}, Γ) a tiling in the same isotopy class as (\mathcal{T}, Γ) with geodesic edges between the vertices. Some isotopy classes of tilings do not admit geodesic representatives, see figure 2.

Theorem 4. *Assume that for an isotopy class of tilings \mathcal{T} , there is a representative tiling \mathcal{V} with only convex tiles. Then \mathcal{V} is a geodesic representative. Any isotopy class of fundamental tile-1-transitive tilings with symmetry group Γ has a geodesic representative.*

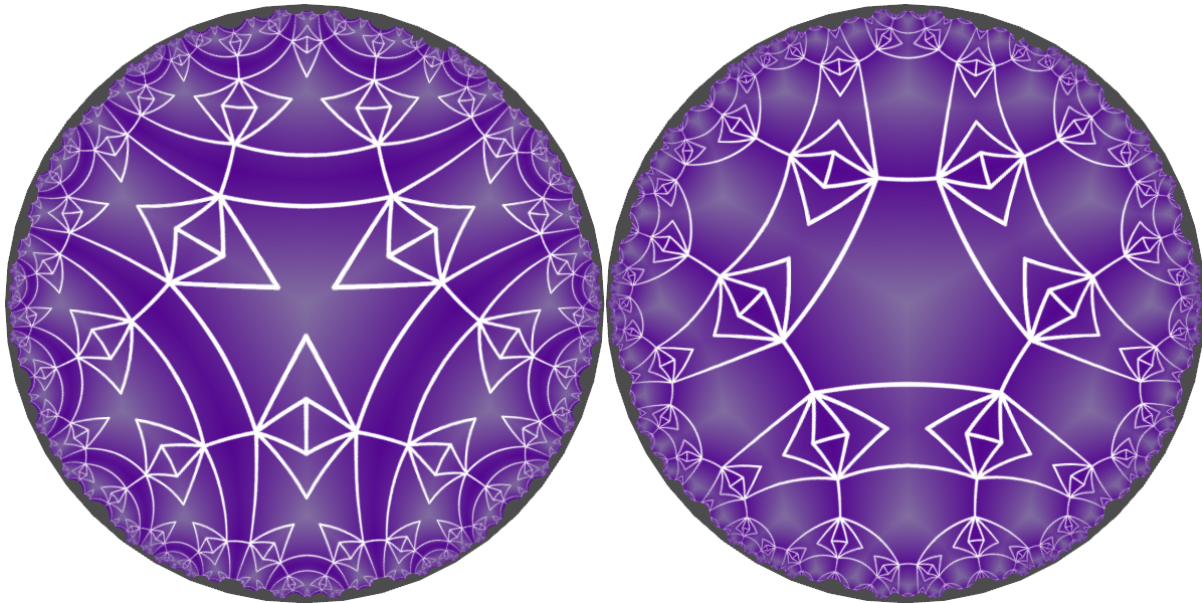


Figure 2: Two examples of tilings that cannot be transformed into tilings entirely made up of geodesics, with breaks occurring only at vertices. Tilings from the EPINET database [1].

Proof. For every vertex of the tiling, by applying a Moebius transformation to the tiling in \mathbb{H}^2 that sends the vertex to the origin in the Poincaré model, we see that the edges are given by line segments through the origin. This establishes the first statement.

For the second statement, it is known [59, 58] that the graph G obtained from projecting the tile-1-transitive tiling \mathcal{T} to the orbifold \mathcal{O} associated to Γ can be built iteratively. One starts with a set of αg closed curves ($\alpha = 2$ if \mathcal{O} is orientable and $\alpha = 1$ otherwise) only intersecting at some point $v \in \mathcal{O}$, where g is the genus of \mathcal{O} . Then, a small disk D around v containing the singular locus is removed to produce $2\alpha g$ disjoint curves on \mathcal{O} . A tree is subsequently embedded into D that spans the $2\alpha g$ vertices and a point on each component of the singular locus. See [59, 60] for representative figures of the graph G . By applying an isotopy, we can assume that D is a convex disk with hyperbolic metric induced from the covering of \mathcal{O} by \mathbb{H}^2 . By applying another isotopy, we can assume that the tree is embedded in D with geodesic edges, without creating intersections, since hyperbolic geodesics minimize intersections between endpoints, see [26, Propositions 1.3 and 1.9] and [26, Section 1.2.7]. Then, using again that geodesics minimize intersections, when applying an isotopy that turns each edge of the original graph G into a geodesic, one does not create extra intersections. \square

There is a version of Tutte’s embedding theorem for graphs that are 1-skeletons of triangulations of hyperbolic surfaces [14], which shows that within each isotopy class of such a graph with symmetry group corresponding to the fundamental group of a classical hyperbolic surface, there is a geodesic representative.

Theorem 4 suggests a (theoretical) data structure for fundamental tile-1-transitive tilings, similar to the data structure of SYSTRE [18] and in Delaunay triangulations in CGAL [49]. We can reconstruct any such tiling from the locations of the vertices of some tile in the tiling along with an adjacency matrix specifying which (copies of) these vertices to join by geodesics. This leads to a natural structure and similar to that of .poly or .obj files, used in geometry processing. Given a set of generators, one finds the vertices of a tile such that its edges connect copies of these vertices that arise from simple words (combinations) in the generators. The same simple words in the generators will then describe edges for all sets of generators that are a result of applying the MCG to them [54, Section 3].

We explain the proposed data structure in more detail and more concretely for the case where Γ is a stellate group. In this case, the underlying topological space \mathcal{O} of the associated orbifold is a sphere with some marked points $\{m_i\}$. As used in the proof of Theorem 4, every fundamental tile-1-transitive tiling (\mathcal{F}, Γ) is the result of lifting, to \mathbb{H}^2 , a tree t on \mathcal{O} that spans the marked points and possibly other vertices $\{v_j\}$ of degree at least 3. Every fundamental domain in \mathbb{H}^2 for Γ has vertices $\{\tilde{m}_n, \tilde{v}_k\}$

in \mathbb{H}^2 that project to the points in $\{m_i, v_j\}$. A given set of generators for Γ corresponds to a set of points $\{\tilde{m}_i\} \subset \{\tilde{m}_n\}$ in \mathbb{H}^2 that correspond to the centers of rotation that generate Γ . A fundamental domain can then be described by a set of edges that connect the generators $\{\tilde{m}_i\}$ according to the tree t representing the fundamental domain. Each edge \tilde{e} of a fundamental domain will be equivalent, under Γ , to an edge described in this way. The vertices $\{\tilde{v}_j\}$ in \mathbb{H}^2 that project to $\{v_j\}$ in O are not fixed in isotopies of tilings with symmetry group Γ . By Theorem 4, there is a placement of these in \mathbb{H}^2 for every fundamental domain such that the tiling of \mathbb{H}^2 resulting by applying to the fundamental domain the symmetries of Γ has non-intersecting geodesic edges. Figure 3 shows an example of how a different set of four generators, corresponding to three two-fold rotations and one four-fold rotation, respectively, of the group 2224, gives rise to different geodesic representatives of isotopy classes of tilings. In each case, the tiling is a result of combining the geodesics from the generator marked by 1 to the generators marked by 2 and 4, and from the generator marked by 4 to the generator marked by 3. Figures 4, 5, 6 show more complicated examples.

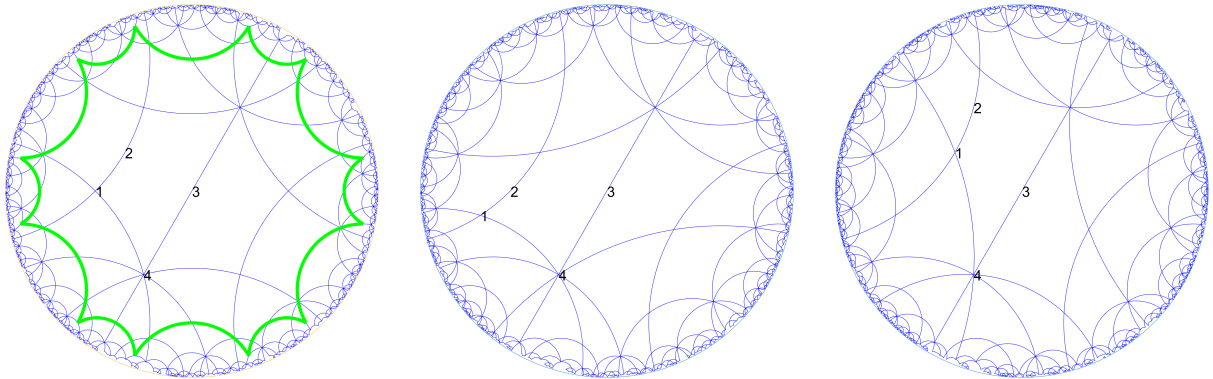


Figure 3: Three different isotopy classes of fundamental tile-1-transitive tilings from a fixed way of connecting generators with geodesics. They all project to tilings of the surface obtained by gluing opposite edges of the dodecagon shown on the left.

Since all isotopy classes of tiling can be constructed from fundamental tile-1-transitive tilings by glue and split operations [54]. The split operation adds vertices that can be described relative to the positions of the vertices present before its application, so if we keep the vertices added during every split operation, we have a candidate for the data structure of general tilings, extending that for D-symbols from [17] by adding the locations of generators.

5 Computational group theory and coset enumeration

This section deals with notions from computational group theory, needed to discuss the properties of the groups involved in algorithm 1 that allow its execution, in particular step 4 and the creation of the tilings from fundamental domains. For an introduction to the subject, refer to [35], or the more comprehensive treatment in [21]. The central results of this section show that MCGs carry natural automatic structures, and may allow for algorithms for coset enumerations, for a complete and unambiguous enumeration of isotopy classes of tilings on hyperbolic surfaces. To establish this, we need results on the structure of MCGs, some of their subgroups, and the notion of automatic structures on groups.

It is well-known that there are finitely presented groups with undecidable word problem [9], meaning that there cannot exist an algorithm that computes a unique representative, or *normal form*, for an element of such a group in terms of a given set of generators. Since coset enumeration generalizes the enumeration of group elements, it is also undecidable in general. For word-hyperbolic (Gromov-hyperbolic) groups [35, Theorem 6.3.6] in the sense of Gromov [31], which are essentially groups that act cocompactly on some hyperbolic space [10, theorem 3.6] and include our symmetry groups, the situation is much better. We explain the set-up.

Let G be a group generated by a finite set $A_G \subset G$, which we assume, in this paper, to be closed under taking inverses. We denote the set of *words* over the *alphabet* A_G by $A_G^* := \bigcup_{k=0}^{\infty} A^k$ and call a

word $a \in A_G^*$ *geodesic* if it is a word of minimal length in the set of all words in A_G^* that represent the same element in G . For words g, h in A_G^* , we write $g \sim_G h$ to indicate that they are equal in G . We denote the length of $g \in G$ by $|a|$ when a is geodesic word with $a \sim_G g$, defining what is known as a *word norm* $|\cdot|$. Note that any given $g \in G$ may have several representatives in A_G^* that are geodesic words. A subset L of A_G^* is called a *language* for G if for every $g \in G$ there is at least one representative word in L . We call L a *normal form* for G if L contains precisely one representative word for every element in G . Most algorithms make use of the *shortlex normal form* [35, Section 5.3], but we shall also make use of more general geodesic languages, where every word is geodesic.

To define the shortlex normal form, we define shortlex orderings as usual. The starting point is any total ordering $<$ of the alphabet A_G . Then, for words $u, v \in A^*$ with $u = a_1 \cdots a_n$ and $v = \tilde{a}_1 \cdots \tilde{a}_m$, where $\{a_i\}_{i=1}^n, \{\tilde{a}_i\}_{i=1}^m \subset A_G$, we define $u < v$ if either (i) $n < m$, or (ii) $n = m$ and for some $k \in \{1, \dots, n\}$, $a_i = \tilde{a}_i \forall i < k$ and $a_k < \tilde{a}_k$. The shortlex normal form for a group consists of the smallest representative words in A_G^* for every element of G , w.r.t the shortlex ordering. We shall also write $u(i)$ for the starting i factors of u , i.e. $u(i) = a_1 \cdots a_i$ if $i \leq n$ and $u(i) = u$ otherwise. A convenient feature of the shortlex normal form for words is that it is *prefix closed*, i.e. if a is part of the language, then $a(i)$ is as well for all i .

Remark 5. There are prominent generators for $\text{Mod}(\mathcal{O})$ called *half-twists* and *Dehn twists* [26] which we will use as generators. For the enumerations of tilings and its applicability in the natural sciences, the shortlex ordering is very convenient, as it vaguely captures our intuition of a complexity ordering of structures. Figure 3 shows three tilings that are separated by the application of a half-twist and its inverse.

For G , consider its Cayley graph $\text{CG}(G, A_G)$ w.r.t. the generating set A_G , with unit edge lengths. Recall that a Cayley graph is a graph representing a presentation of a group G with labeled edges corresponding to generators and their inverses, and vertices corresponding to group elements [35]. Two elements $a, b \in G$ are joined by an edge labelled g if $ag = b$. We regard the edges labelled by inverse elements of A_G that connect the same two vertices as a single undirected edge. Observe that $\text{CG}(G, A_G)$ is naturally a length space and denote its distance by d . Words in A_G^* can now also be interpreted as paths in $\text{CG}(G, A_G)$ and it is easy to see that geodesic words correspond to geodesic paths.

Automatic structures are a very convenient tool in the computation of normal forms for group elements. To define these, we need the notion of *regular languages* and *finite state automotons* [37].

Definition 6. ([35],[21]) An automatic structure for a group $G = \langle A_G \rangle$, with A_G like above, is a *regular language* L for G over A_G s.t. there exists a constant K with the property that for all $v, w \in L$ with $d(v, w) = |v^{-1}w| \leq 1$, one has $d(v(i), w(i)) \leq K$ for all $i \geq 1$. This condition on v, w is expressed by saying that v and w *K-fellow travel*. A biautomatic group is an automatic group w.r.t. the generating set A_G , with the additional requirement that, if $v, w \in L$ with $wv^{-1} \sim_G a \in A_G$, then av and w *K-fellow travel*.

While biautomatic groups are automatic, it is unknown whether or not the converse is true [35]. These notions do not depend on the finite generating set A_G of G [21, Theorems 2.4.1 and 7.33], so we call a group G (bi)automatic if it admits a generating set A_G w.r.t. which it is (bi)automatic. Note also that a finite generating set always exists for automatic groups [35, Theorem 5.2.13].

It is known that if a group admits a (bi)automatic structure, then it also admits a (bi)automatic structure such that the language is in normal form [35, Theorem 5.2.7]. Unfortunately, however, a given group can be automatic w.r.t. some generating set A_G without being geodesically automatic w.r.t. A_G [21, Section 3.5]. Whether or not every automatic group admits a geodesic or shortlex automatic structure is an open problem [35, 5.3.2] that has ties to different areas of mathematics [25, Question 3.13].

Automatic groups are important for us because they are flexible enough to be used in a variety of situations and because they are especially amenable to algorithms, as they can be handled effectively with deterministic finite state automata [35, Theorem 5.2.3]. Moreover, their word problem is solvable in quadratic time [35, Corollary 5.2.12].

Word-hyperbolic groups are biautomatic w.r.t. the language containing all geodesic words and shortlex biautomatic for any shortlex ordering for any finite generating set [6, 35]. As mentioned above, word-hyperbolic groups include all symmetry groups of closed surfaces giving rise to TPMS. Algorithms for

automatic groups have been implemented in KBMAG [34] using GAP [27], and more recently, with a more robust build, in MAF [77]. The algorithm attempts to produce a shortlex automatic structure for a group with a given presentation, using the set of generators given in the presentation [35, Section 5.4]. Using KBMAG and MAF, we implemented an algorithm that creates tilings of \mathbb{H}^2 using the data structure described in Section 4. Starting from a tile-1-transitive tiling, a single representative of each edge of the tiling is stored, relative to a set of generators. These generators are expanded to a finite group presentation of the symmetry group, from which the algorithm creates a list \mathcal{L} of group elements, with each entry corresponding to a unique shortest word in the generators. The elements in \mathcal{L} map the given edges to copies of themselves exactly once.

While the above algorithm also provides an efficient way to produce tilings, it is also important for us because we can utilize the uniqueness property to ensure that each edge of a tile is uniquely represented in a tiling. This means we can more easily analyse the resulting net in \mathbb{R}^3 after mapping the edges of a tiling in \mathbb{H}^2 to \mathbb{R}^3 through surface embeddings. The outlined algorithm can be seen as a related but more general version of implementations of algorithms for related problems in the computation of Delaunay triangulations on the Bolza surface used in CGAL [49], which uses Dehn's algorithm. Note that MCGs of classical orientable surfaces are not word-hyperbolic [63]. This implies that Dehn's algorithm cannot be used for the word problem in MCGs of orbifolds [35, Theorem 6.6.1], illustrating the need for more general techniques.

We are now ready to prove a theorem on the algorithmic properties of MCGs of orbifolds. MCGs of punctured surfaces, orientable or non-orientable, admit an automatic [65]. Suppose that H is a finite index subgroup of G . Then, H is automatic if and only if G is [21, Theorem 4.1.4]. By proposition 2, we thus have the following.

Theorem 7. *The MCGs of orbifolds without mirror boundaries admit a automatic structure.*

We remark that MCGs of orientable punctured surfaces admit a biautomatic structure [32] that is inherited by finite-index subgroups [21, Theorem 4.1.4]. Therefore, MCGs of orbifolds without mirror boundaries with orientable quotient space, such as those associated to stellate symmetry groups, admit a biautomatic structure. See Section B in the appendix for a treatment of the case where the orbifold has mirror boundaries.

The above sums up what we can say with certainty about the automatic properties of MCGs of orbifolds. However, since, as mentioned above, the available algorithms for automatic groups depend on shortlex languages, we want to assume at least a geodesic automatic structure for the groups we deal with. We discuss the ensuing theory in the appendix, see Theorem 13. Note, though, that to the best of our knowledge all known automatic structures for MCGs are not geodesic, see also [25, Question 3.13]. On the other hand, recent work shows promising results in this direction [7]. Let $L \subset A_G^*$ be a regular

language for a group $G = \langle A_G \rangle$. A subgroup $H \subset G$ is called L -quasiconvex if there is a constant k such that if $w \in L$ satisfies $w \sim_G h \in H$, then w viewed as a path in the Cayley graph $\text{CG}(G, A_G)$ lies in a k -neighbourhood of the vertices of $\text{CG}(G, A_G)$ corresponding to H [31, 28]. A subgroup $H \subset G$ is L -quasiconvex if and only if the subset $L' \subset L$ of words that map onto H is a regular language for H [28, Theorem 2.2]. Moreover, L -quasiconvex subgroups of G are finitely generated [28, Proposition 2.3], and if L is a (bi)automatic structure for G and H is L -quasiconvex, then H is (bi)automatic [28, Theorem 3.1]. It follows straight from the fellow traveler property that finite subgroups of L -automatic groups are L -quasiconvex, since then there are only finitely many representative words for each element in the group. For examples, see [21, Section 8.3]. Note that this notion of quasiconvexity depends on a particular language L for G . In case L is the language of geodesic words, we call the subgroup quasiconvex. In this case, a subgroup H is quasiconvex if and only if every geodesic in $\text{CG}(G, A_G)$ connecting two points in H is contained in a k -neighbourhood of $\text{CG}(H, A_H)$, since H acts on $\text{CG}(G, A_G)$ as isometries, where we assume $A_H \subset A_G$.

There are similar notions to automaticity in groups for cosets, again depending on finite state automata [36]. Of these notions, we will note one which allows for generalizations of algorithms to compute shortlex automatic on groups [35]. See also [35, Section 8.2] for more details on the following. Let G be a group with generating set $\langle A \rangle$ and corresponding metric Cayley graph $\text{CG} := (\text{CG}(X, A), d_{\text{CG}})$, let H be a subgroup and L a regular language. Assume L contains a representative of each (right) coset Hg of H in G and moreover, that there exists a constant K such that if $v, w \in L$ and $h \in H$ satisfy $d_{\text{CG}}(v, hw) \leq 1$, then $d_{\text{CG}}(v(i), hw(i)) \leq K$ for all $i \geq 1$. Then we say that the pair (G, H) is *strongly*

coset automatic. This notion does not depend on the finite generating set A . It is an open problem how this notion relates to other related notions. For strongly coset automatic pairs (G, H) , there exists an *automatic coset system*, which is a certain automaton that makes possible an efficient coset enumeration in quadratic time [35]. One of the central results concerning automatic coset structures is the following.

Theorem 8 ([35, Theorem 8.2.8]). *If H is a quasiconvex subgroup of a word-hyperbolic group G , then (G, H) is strongly shortlex coset automatic w.r.t. any finite generating set A of G .*

A semi-algorithm that, given a shortlex automatic structure on G , tries to find shortlex coset automatic structures for a pair (G, H) , where $H \subset G$ is a subgroup, has been implemented in KBMAG [34]. This algorithm is the basis of our creation and enumeration of isotopy classes of tilings in \mathbb{H}^2 . As previously mentioned, KBMAG actually offers two approaches to the solving the problem. One that derives from automatic structures, and another, which relies on the same assumptions on the groups involved, but tries to derive a confluent rewriting system for words representing elements of a group and also relies heavily on the fellow traveler property. The second can actually be used as part of an attempt to find a shortlex automatic structure [35]. Note that, as mentioned above, the stabilizer subgroups in $\text{Mod}(\mathcal{O})$ are quasiconvex because they are finite.

Remark 9. Our own investigations showed that coset enumeration of MCGs w.r.t. not only finite subgroups, but more generally quasiconvex subgroups seemed to work reasonably well. From private communication, it seems that experts in the field also believe that theorem 8 is just a special case of a much stronger statement. This leads us to ask the following question: If G is a group with geodesic automatic structure and H is a finitely generated undistorted subgroup, then is it always possible to construct a coset automatic structure for H in G ?

6 Conclusion and outlook

The theoretical approach and results outlined in this paper show that a fully algorithmic approach to enumerations of isotopy classes of tilings of hyperbolic surfaces is feasible. While there are many open problems, the results here are promising.

To illustrate the enumeration, we show a collection of examples of tilings produced in the enumeration. Figure 4 shows 12 tilings that are the starting point of our enumeration, corresponding to the 12 D-symbols that a tile-1-transitive tiling with symmetry group 22222 can have [3]. Figures 5 and 6 show the third and tenth isotopy class in our enumeration of isotopy classes. To generate the list of coset representatives, we implemented the finite presentation of the MCG of 22222 in [53, Theorem 2].

Figure 8 shows a sample of an enumeration of an isotopy class corresponding to a single D-symbol, QS99 (naming from [2]), with symmetry group 22223. These tilings are commensurate with the Schwarz H surface, another TPMS important for crystallographic structures [73].

To conclude the illustration of our enumeration of isotopy classes, we present an array of examples that show another aspect of the enumeration presented here. One can consider more general tilings of \mathbb{H}^2 than those by bounded disks, obtained by these by deleting edge orbits appropriately. These are relevant to self-assembly of branched polymers [52]. The stabilizer subgroup in $\text{Mod}(\mathcal{O})$ of isotopy classes of such tiles are generally not finite. Although more involved, we can establish similar results to the theory developed in this paper so that by Lemma 3, isotopy classes of such tiles can be enumerated by cosets, explained in more detail in Section D in the appendix. To illustrate this, figure 7 shows a selection of the ensuing enumeration of isotopy classes of a selection of these, again commensurate with the diamond TPMS and symmetry group 22222.

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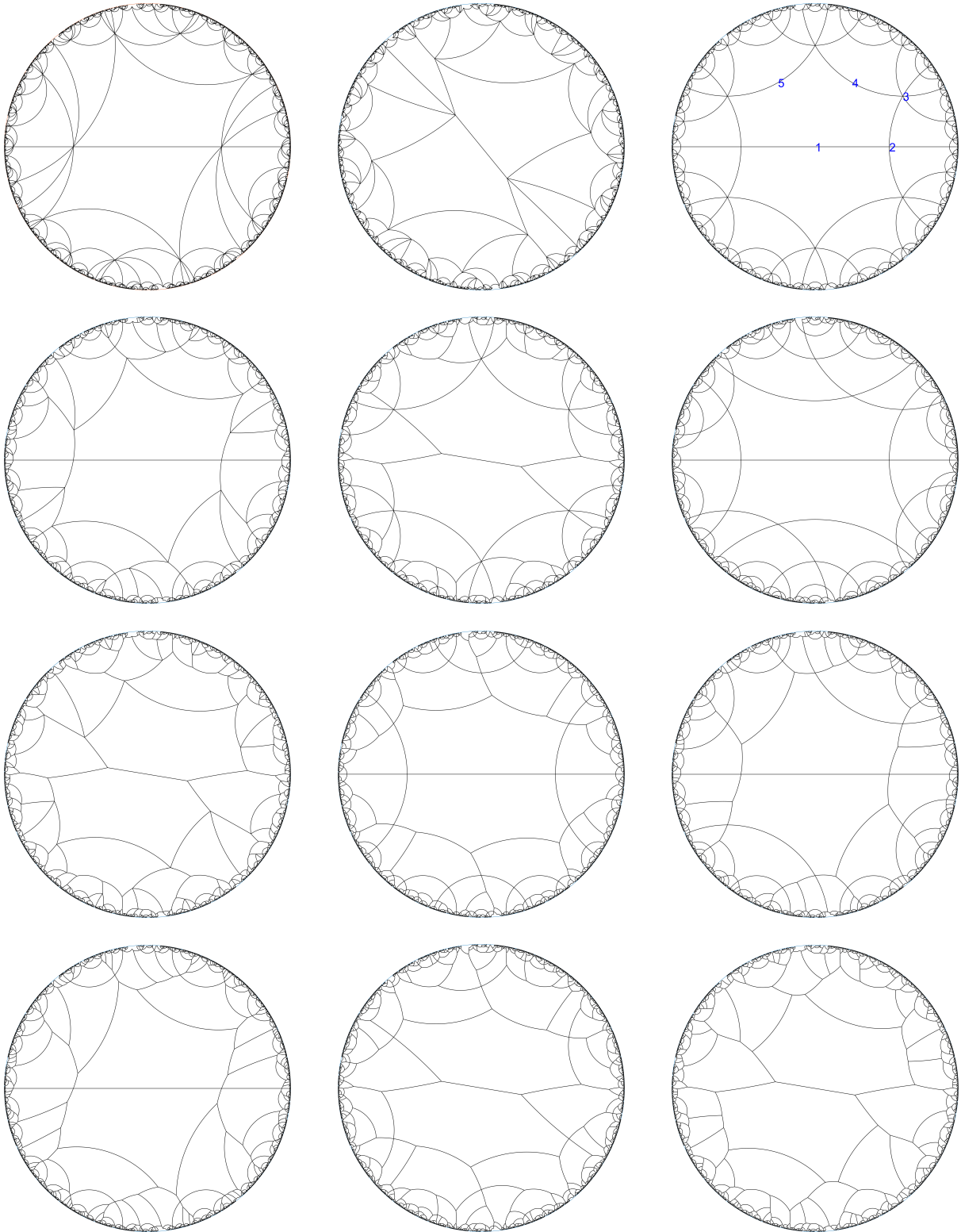


Figure 4: The different combinatorial classes of fundamental tile-1-transitive tilings with symmetry group 22222 , commensurate with the diamond TPMS. The tilings correspond to, from left to right and top to bottom, the classes QS53 to QS64 with the nomenclature from EPINET [3]. The third tiling shows the placements of the five centers of rotations, corresponding to the generators of 22222 .

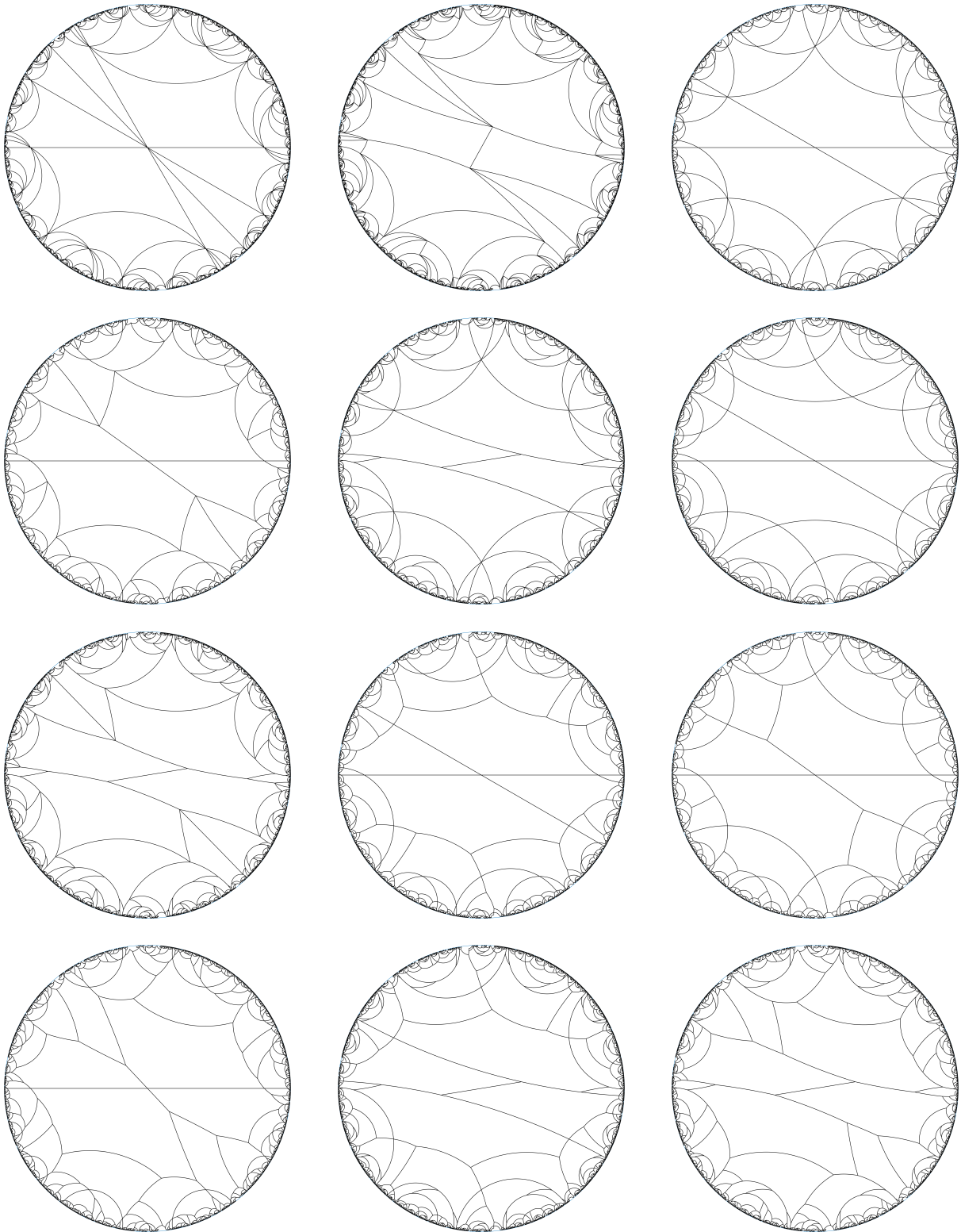


Figure 5: Tilings with the same D-symbols as the respective tilings in Figure 4. The second tilings in the enumeration of isotopy classes with symmetry group 22222 .

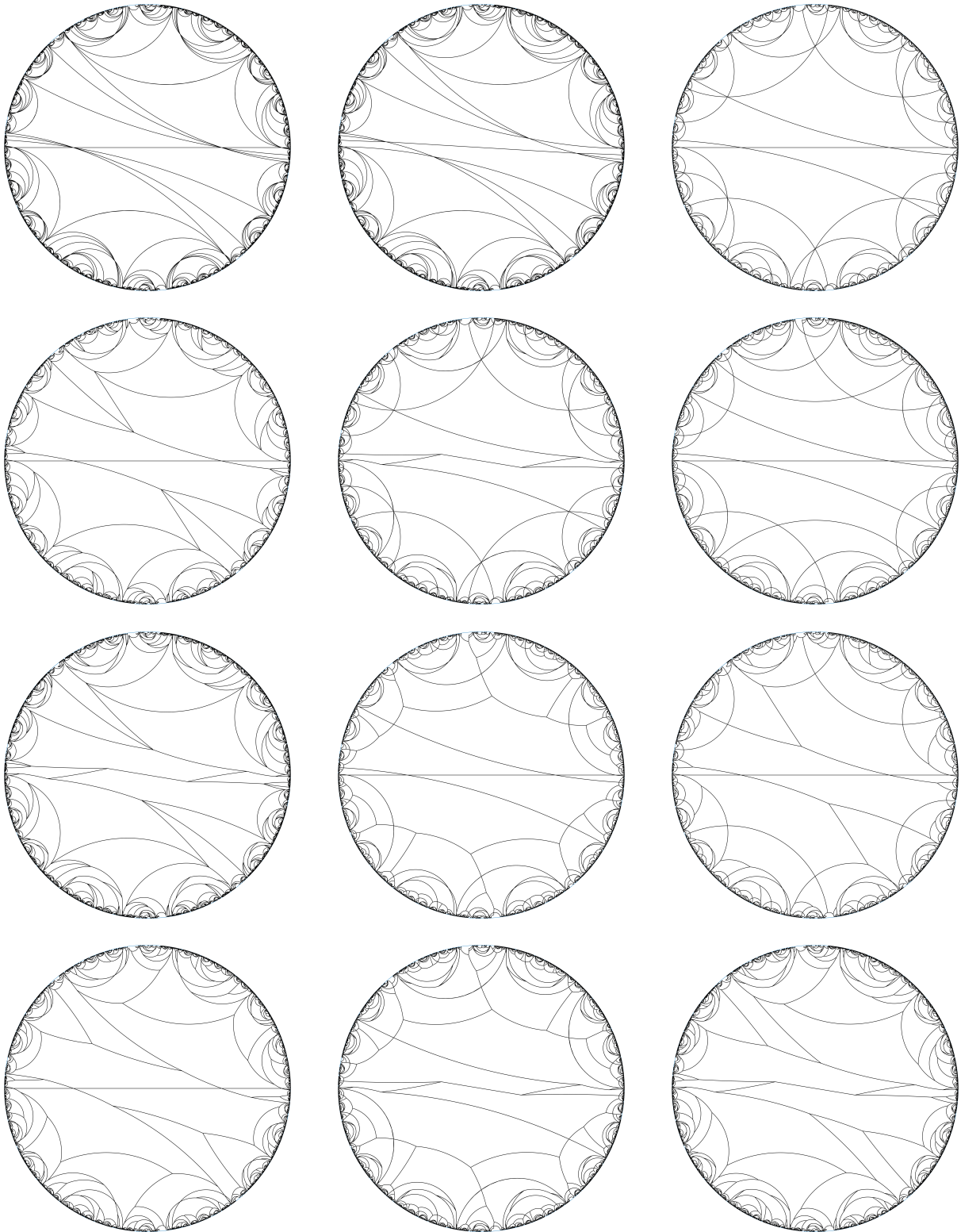


Figure 6: Tilings with the same D-symbols as the respective tilings in Figure 4. The tenth tilings in the enumeration of isotopy classes with symmetry group 22222 .

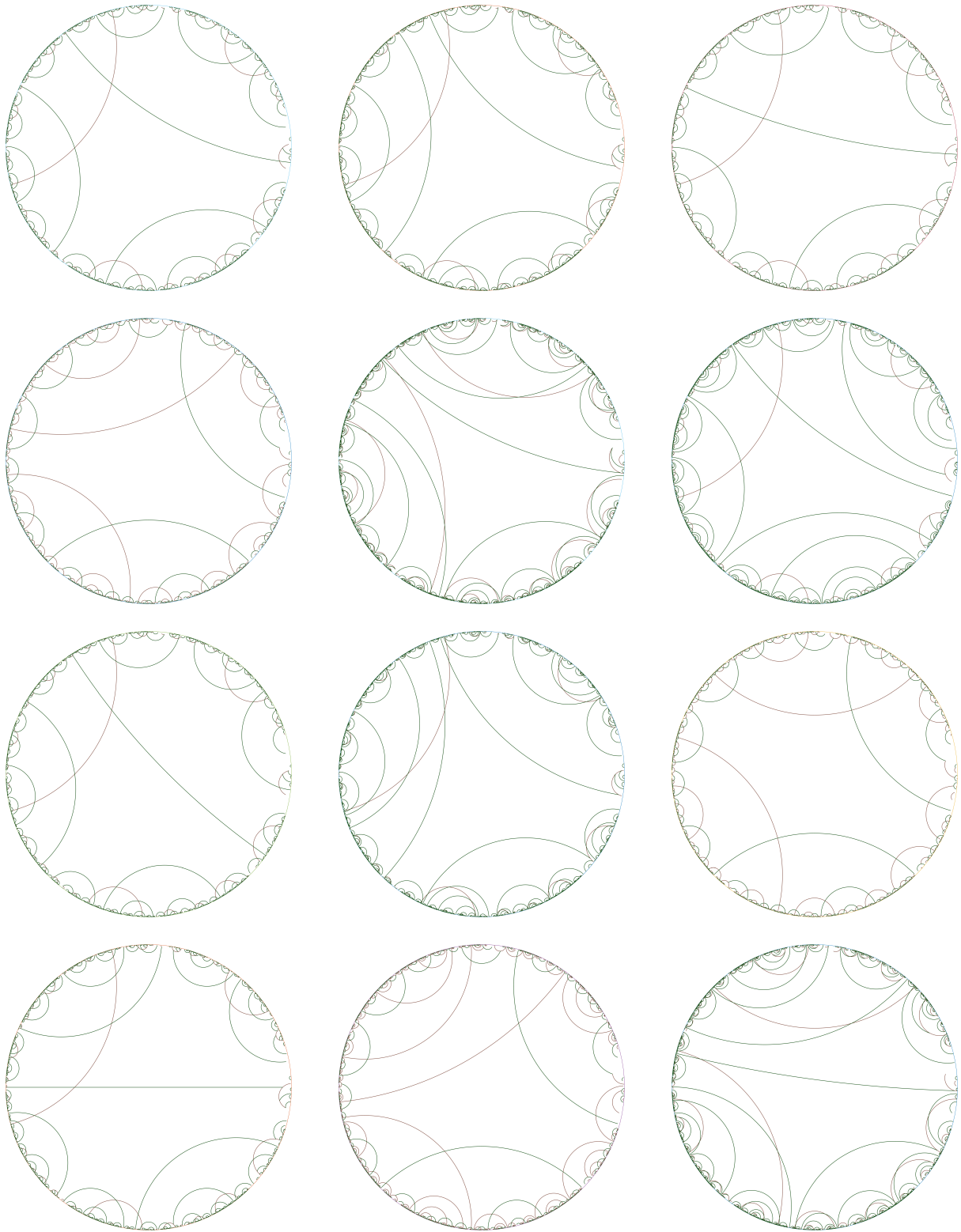


Figure 7: The first 12 isotopy classes of tilings, from left to right and top to bottom, in a fixed the combinatorial class, with symmetry group isomorphic to 22222 . The two edge orbits are are colored differently.

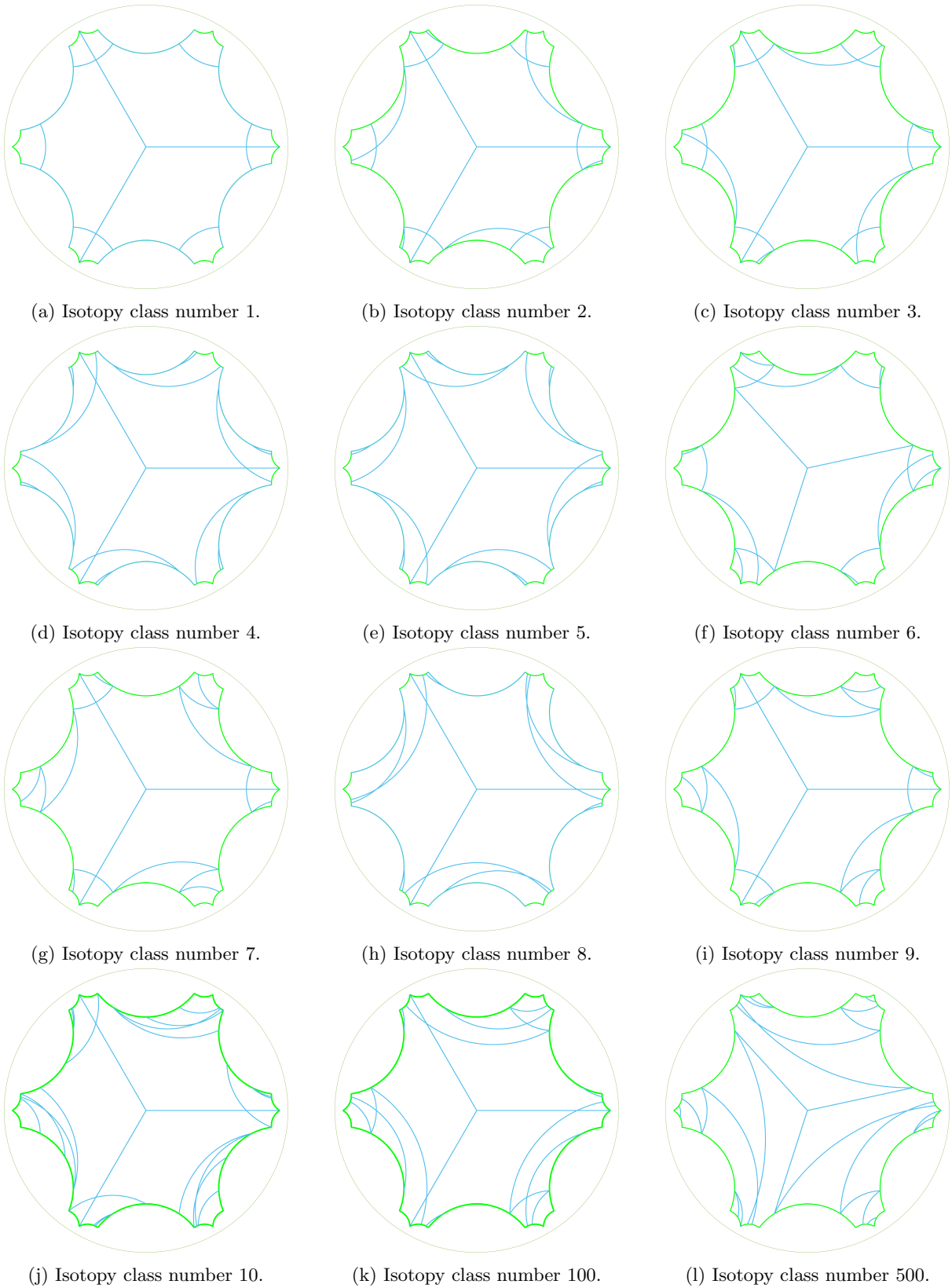


Figure 8: Tilings commensurate with symmetry group 22223. Each is in the combinatorial class QS99, but are isotopically distinct. The tilings are only shown on a fundamental domain for the commensurate translational subgroup that gives rise to the hyperbolic surface associated to the Schwarz H surface.

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A Orbifolds

In the literature, a symmetry group Γ of \mathbb{H}^2 , i.e. a discrete group of isometries of \mathbb{H}^2 having a compact fundamental domain is known as a NEC group (non-Euclidean crystallographic group). We identify the isomorphism class of a group using Conway’s *orbifold symbol* [15], a highly-readable version of Macbeath’s group signature [61], as described below.

For the purposes of this paper an (2-)orbifold, $\mathcal{O} = \mathbb{H}^2/\Gamma$, is a quotient space obtained by identifying points of \mathbb{H}^2 that are equivalent under the action of Γ , i.e. $x \sim y$ if $y = \gamma x$ for some $\gamma \in \Gamma$. The quotient space can be endowed with the quotient topology.

The difference between \mathbb{H}^2/Γ as a topological space and as an orbifold is that the full orbifold structure retains the metric information carried by the particular isometries of Γ and an atlas of charts compatible with the Γ action. We require both the topological view point and orbifold structure of the quotient space in this paper, and will use the script notation \mathcal{O} for the orbifold with the additional structure and plain O for its underlying topological space.

It is well-known [76] that 2-orbifolds are topologically equivalent to finite-area 2-manifold with a finite number of boundary components and marked points. Boundaries in a 2-orbifold arise from the fixed lines of reflection isometries while marked points arise as the fixed points of rotational isometries. The marked points are called *cone points* if they lie in the interior of the orbifold, and *corner points* if they lie on a boundary. The order, N , of a cone or corner point is the order of the rotational isometry, σ , that fixes that point i.e. $\sigma^N = id$. The boundaries and marked points are collectively referred to as the *singular locus* of the orbifold. The topology of a 2-orbifold (O) is therefore specified by a symbol as follows:

1. The number of handles, h , if the orbifold is orientable, or the number of cross-caps, k , if non-orientable. Handles are denoted by \circ at the beginning of the orbifold symbol. Cross-caps are denoted by \times at the end of the orbifold symbol.
2. The order for each cone point, listed in arbitrary order after any handles.
3. The number of boundary components, q . Each boundary component is represented by a $*$ in the symbol. Branching numbers for the corner points lying on each boundary component are listed in cyclic order, such that each boundary component has a consistent orientation for the manifold. The ordering of the boundary components is arbitrary.

Simple examples include the group of isometries for a square tiling of \mathbb{E}^2 , which is $*244$ ($p4m$ in Hermann-Mauguin notation). Note that for a Euclidean pattern with only translational symmetries the same example has the symmetry group $\circ(p1)$.

It is well-known [76, 15] that any orbifold symbol corresponds to a discrete group of isometries of \mathbb{E}^2 , \mathbb{H}^2 , or \mathbb{S}^2 , with the exception of the symbols A , $*A$, AB , and $*AB$, with $A \neq B$. The Conway orbifold symbol is useful as it allows the computation of the orbifold Euler characteristic, which determines the plane geometry associated with an orbifold. An orbifold is hyperbolic if its orbifold Euler characteristic is negative [15].

The symmetry group associated with a geometric orbifold also has an interpretation as a type of fundamental group defined by equivalence classes of loops. Intuitively, *orbifold loops* are piecewise lifts to \mathcal{X} of closed curves in the quotient space \mathcal{O} . The general definition of orbifold loops in \mathcal{O} and their homotopies is quite involved, but it is sufficient for our purposes to view them as closed curves in \mathcal{O} , disjoint from the singular locus except possibly at isolated points of mirror boundaries. We also require that homotopies of simple curves in \mathcal{O} do not change the incidence relation of a curve to any point of the singular locus; that the group Γ is fixed; and that paths in \mathcal{O} meeting a mirror boundary transversally will lift to paths that cross the mirror line in \mathcal{X} , to eliminate the ambiguity of the lift in this situation. See [71, Chapter 13] or [15].

The *orbifold fundamental group* $\pi_1^{orb}(\mathcal{O}, x)$, $x \in \mathcal{O} \setminus \Sigma$, is then defined as the set of orbifold loops based at $x_0 \in \mathcal{X}$ up to homotopy equivalence, where $x_0 \in p^{-1}(x)$. It is known that for the geometric 2-orbifolds considered in this paper, $\Gamma \simeq \pi_1^{orb}(\mathcal{X}/\Gamma)$ in the natural way [71, Theorem 13.3.2]. In other words, $\pi_1^{orb}(\mathcal{X}/\Gamma)$ is the group of deck transformations for the branched covering map $p : \mathcal{X} \rightarrow \mathcal{O}$.

B An asynchronous automatic structure for orbifolds with mirror boundary

In the following, we trace the steps needed to conclude that MCGs of orbifolds with mirror boundaries have favorable algorithmic properties, akin, but different to the situation for orbifolds without mirror boundaries. Recall that by Proposition 2, such MCGs are split extensions of MCGs of orbifolds without mirrors. A combing for a group G is a language for G that satisfies the fellow traveler condition, but is not necessarily regular. An asynchronous combing refers to the situation where in the fellow traveler condition for automaticity the condition $d(v(i), w(i)) \leq K$ is replaced by $d(v(\alpha(i)), w(\beta(i))) \leq K$, where $\alpha, \beta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ are non-decreasing surjective functions, and v and w are said to *asynchronously fellow travel* [35, Section 5.1.2]. An asynchronous automatic structure is an asynchronous combing that is regular. For more detailed definitions of the concepts involved, refer to [21, Section 7], which also includes examples of asynchronously automatic but not automatic groups and discusses the adjustments to the automata needed to deal with the asynchronous case.

If N is a finitely generated abelian group or a word-hyperbolic group and H is asynchronously combable, then any split extension G of Q by N (so that G is the semi direct product of Q and N) is also asynchronously combable [11]. Note that [11] at first defines asynchronous combings somewhat differently than above, but the last section and its closing remarks outline a generalization that includes our case. Furthermore, as already remarked in [30, P. 133], the construction used in the proof of this result is inherently about asynchronous combings and the proof generalizes to yield usual combings satisfying the fellow traveler property on split extensions if and only if G has a subgroup of finite index that is the direct product of groups with combings. By the slight generalization of the pertinent results of [11] in [30, Proposition 4.5], the language of the semidirect product G is the concatenation of the languages for the constituent parts N and Q . Since regular languages are closed under concatenation, the results there show that in fact G inherits an asynchronous automatic structure from such structures on the constituent parts. It is furthermore well-known that finitely generated abelian groups are automatic [21, Theorem 4.3.1].

Summarizing, the above shows that if G is a semidirect product of either a word-hyperbolic group, or a finitely generated abelian group, and an automatic group, then G admits an asynchronously automatic structure. We know of no results that improve these findings to (asynchronously) biautomatic structures. Note that the semidirect product of an automatic group H with a finite group is a supergroup of finite index, and therefore automatic. Since MCGs of orbifolds with mirrors are extensions of MCGs of orbifolds without mirrors by finitely generated abelian groups, we have the following.

Proposition 10. *The MCGs of orbifolds with mirror boundaries admit asynchronously automatic structures.*

C Geodesic automatic structures and undistorted subgroups

Given finite generating sets $A_{G'}$ and A_G of groups G' and G , respectively, where $G' \subset G$ is a subgroup and A_G is an extension of $A_{G'}$, there are word norms $|\cdot|$ on G and $|\cdot|'$ on G' , making the Cayley graphs

$\text{CG}(G, A_G)$ and $\text{CG}(G', A_{G'})$ into metric spaces. We call G' *undistorted* if the inclusion $\text{CG}(G', A_{G'}) \rightarrow \text{CG}(G, A_G)$ is a quasi-isometric embedding, i.e. if and only if there exists a constant k such that $|g'| \leq k|g|$ for all $g \in G'$ [32]. Note that this condition is independent of the finite generating set for G' , since the quasi-isometry class of metric Cayley graphs of finitely generated groups does not depend on the finite set of generators chosen for the construction of the Cayley graph and the identity map is a quasi-isometry for different sets of generators [26, Corollary 8.3]. One can therefore say that $G' \subset G$ is undistorted if and only if it is finitely generated and the inclusion $G' \hookrightarrow G$ is a quasi-isometrical embedding. As an example, any finite subgroup of G is necessarily undistorted.

Lemma 11. *Any finite index subgroup H of an undistorted subgroup G' of a finitely generated group G is itself undistorted.*

This fact can be considered as folklore that, to the best of our knowledge, does not appear in the literature, except as the corollary of an exercise problem in a section devoted to the Svarc-Milnor Lemma [62, Exercise 8.20(1)].

Proof. We will apply the Svarc-Milnor Lemma [62, Proposition 8.19] to conclude that H is quasi-isometric to G' meaning that their Cayley graphs are quasi-isometric. We view the group action of H on $\text{CG}(G', A_{G'})$, where the finite set $A_{G'}$ extends a given finite set of generators A_H of H by (left) multiplication. Note for this that it follows straight from Schreier's lemma [75, Lemma 4.2.1] that H is finitely generated when G' is. The action of H is proper in the sense that if $(g'_i)_i$ is a sequence in G' and $(h_i)_i$ a sequence in H such that both $(g'_i)_i$ and $(h_i g'_i)_i$ converge, then $(h_i)_i$ must converge. Indeed, g'_i converging means that it is eventually a constant $g' \in G'$, since the vertices of Cayley graphs are isolated. Therefore, if $(h_i g'_i)_i$ converges, so does $(h_i g'_i g'^{-1})_i$, which is eventually equal to $(h_i)_i$. By construction of the Cayley graphs, the action of H by multiplication is an isometry on $\text{CG}(G', A_{G'})$. Therefore, by the Svarc-Milnor Lemma, H is quasi-isometric to G' . \square

The following is well-known in the case where G is a word-hyperbolic group [62, Chapter 3, Corollary 3.6].

Proposition 12. *Let G be (bi)automatic w.r.t. a geodesic language $L \subset A_G^*$ and $H \subset G$ a subgroup. Then H is quasiconvex if and only if H is undistorted. In either case, H inherits a geodesically (bi)automatic structure from G .*

Proof. Assume that H is quasiconvex. Then H is finitely generated and quasi-isometrically embedded in G [62, Chapter 3, Lemma 3.5]. As mentioned above, $H \subset G$ being undistorted means that the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding. Assume now that H is undistorted in G . Then, $\text{CG}(H, A_H)$, with A_H finite, can be interpreted as a subgraph of $\text{CG}(G, A_G)$, but, since L is a fixed geodesic language over A_G , we cannot assume that $A_H \subset A_G$. Note that even though (bi)automaticity does not depend on the generating set, as mentioned above, the property of being geodesic of a language can. On the other hand, since each of the finitely many elements in A_H can be expressed as a finite concatenation of elements in A_G , $\text{CG}(H, A_H)$ is quasi-isometrically embedded in $\text{CG}(G, A_G)$. Thus, a geodesic γ_h connecting the identity element e to $h \in H$ in $\text{CG}(H, A_H)$ is contained in some K -neighborhood of any geodesic connecting the same points in $\text{CG}(G, A_G)$. By the definition of the fellow traveler property for automatic groups, any two words in the geodesic language L for G that map to the element h , when interpreted as paths in $\text{CG}(G, A_G)$, are contained in a K' -neighborhood of each other. Therefore, γ_h is contained in a uniform neighborhood of a geodesic in $\text{CG}(G, A_G)$ from e to h . Since L is regular, H is quasiconvex.

In either case, by their equivalence, H inherits a (bi)automatic structure from G [28, Theorem 3.1]. \square

Proposition 12 establishes an important property of undistorted subgroups in our setting that can be salvaged in more general settings, and is closely related to the very recent notion of *stable subgroups* [51].

While MCGs of classical orientable surfaces are not word-hyperbolic, they are relatively hyperbolic w.r.t. to some subgroups [63, Theorem 1.3]. Moreover, MCGs of orientable punctured surfaces admit a biautomatic structure [32]. Suppose that H is a finite index subgroup of G . Then a biautomatic structure on G induces a similar one on H , which is geodesic if that on G is [21, Theorem 4.1.4].

Theorem 13. *If MCGs of orientable surfaces admit a geodesic (bi)automatic structure, then the MCGs of orbifolds admit a geodesic (bi)automatic structure.*

Note that the differences between the two treatment of MCGs with mirrors and those without disappear in this case.

Proof. We will show that MCGs of orbifolds are undistorted subgroups of MCGs of orientable classical surfaces, which by Proposition 12 implies the theorem. The proof will consist of combining ideas from different proofs in the mathematical literature that together yield the result. For this reason, we will not give all the details, but give a comprehensive account of where to find the corresponding statements and how they apply.

Let \mathcal{O} be an orbifold with symmetry group Γ . The proof of Theorem 4 in [54] constructs a finite covering space $S = \mathbb{H}^2/\Gamma'$ of \mathcal{O} that is a closed surface such that $\text{Mod}(\mathcal{O})$ injects into $\text{Mod}(S)$, s.t. all homeomorphisms of \mathcal{O} lift to homeomorphisms of S and Γ' is normal in Γ . Since Γ' is normal, $\Gamma_N := \Gamma/\Gamma'$ is a finite group of isometries of S . In particular, Γ_N is isomorphic to a subgroup of $\text{Mod}(S)$, since a nontrivial isometry maps at least one closed curve on S to a homotopically different curve. Indeed, note that closed curves have a unique representative in their homotopy class that is a geodesic everywhere and with minimal length. A nontrivial isometry must therefore map at least one curve to another homotopy class.

In the proof of Proposition 4.3 in [32], citing [8], a short exact sequence is constructed, which applied to our situation, reads:

$$1 \rightarrow \Gamma_N \rightarrow \mathcal{N}(\Gamma_N) \rightarrow \text{Mod}(\mathcal{O}) \rightarrow 1 \quad (1)$$

where $\mathcal{N}(\Gamma_N) \subset \text{Mod}(S)$ denotes the normalizer of Γ_N and the right hand side is equal to $\text{Mod}(\mathcal{O})$ because by our construction above, all homeomorphisms of \mathcal{O} lift to S . Note that the argument for the validity of (1) rests on the contractibility of Teichmüller space, which is also true for orbifolds [76, Corollary 13.3.7] and the positive resolution of the Nielsen realization problem [50], where an explicit proof that the classical case implies the solution for orbifolds is known [54, Theorem 4].

In particular, $\text{Mod}(\mathcal{O})$ has finite index in $\mathcal{N}(\Gamma_N)$. Thus, since $\mathcal{N}(\Gamma_N)$ is undistorted [32, Theorem 3], $\text{Mod}(\mathcal{O})$ is undistorted too, by Lemma 11. \square

The method in [32] to show the existence of a biautomatic structure on MCGs has the additional feature of being constructive, it is not clear how to effectively implement all steps in the construction. Moreover, the constructed biautomatic structure is not geodesic.

D The stabilizer subgroup of ribbon tiles is undistorted

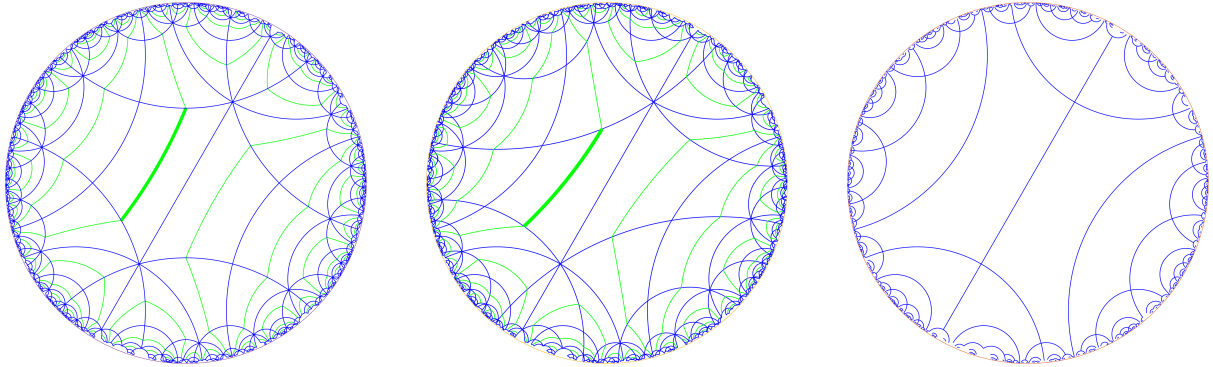
We discuss another implication of the existence of geodesic automatic structures on MCGs. The theory for the enumeration of isotopy class of tilings developed thus far can be extended to yield enumerations of tilings that result from deleting edge orbits from equivariant tilings. This is motivated by recent extensions of combinatorial tiling theory to include such tilings [55]. The stabilizer subgroups of tilings arising in this way include all finite symmetry groups of classical tilings by disks, but also many more, as illustrated in figure 9.

By Lemma 3, we need to identify the stabilizer subgroup $\text{Stab}_{\mathcal{T}}$ of the isotopy class of a given tiling (\mathcal{T}, Γ) in the MCG $\text{Mod}(\mathcal{O})$ of the orbifold associated to Γ for an unambiguous enumeration of such tilings in a given equivariant equivalence class. Figure 9 illustrates that $\text{Stab}_{\mathcal{T}}$ is not finite in this case. We will study properties of $\text{Stab}_{\mathcal{T}}$ that make the problem of coset enumerations computationally tractable.

In [32, Theorem 2], it is proved that the MCGs of *essential subsurfaces* inject into the MCG of the original surface as undistorted subgroups. An essential subsurface of S is a bordered surface S_0 embedded in S as a closed subset that satisfies the following two requirements.

- The homomorphism $\pi_1(S_0) \rightarrow \pi_1(S)$ induced by the inclusion is injective.
- Each boundary component of S_0 is an essential simple closed curve in S , i.e. a simple closed curve neither contractible nor freely homotopic into a neighbourhood of a puncture.

Color the boundaries of S_0 that are not boundary components of S red and define $\text{Mod}_0(S_0)$ similarly to $\text{Mod}(S)$, but with the additional requirement that homeomorphisms fix the red boundary components pointwise. Then, $\text{Mod}_0(S_0)$ can be identified with the subgroup of $\text{Mod}(S)$ of all elements that fix $S - S_0$ pointwise.



(a) Fundamental tiling of 2224. Any edge that crosses a green curve is subject to removal.

(b) Isotopically distinct fundamental tiling with same combinatorial structure as (a). Any edge that crosses a green curve is subject to removal.

(c) The same tiling results from removing the edges indicated in (a) and (b).

Figure 9: Distinct isotopy classes of fundamental tilings with symmetry group 2224 sharing the same associated tiling.

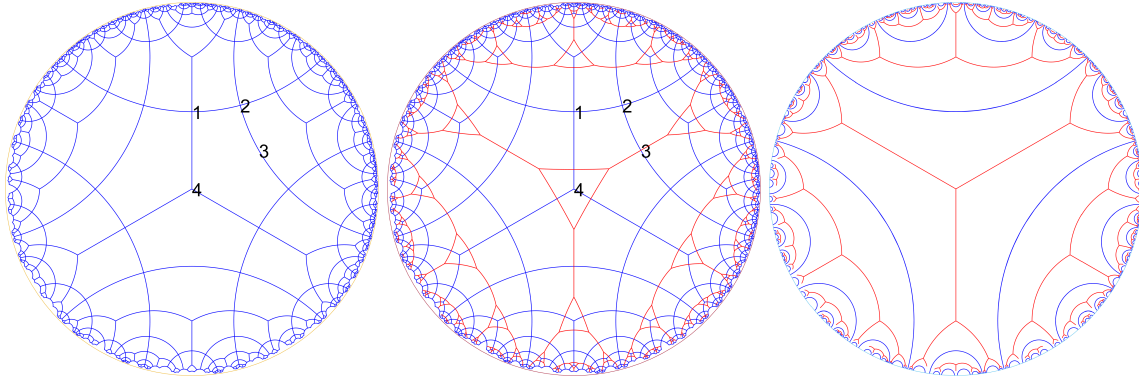
To list the possible stabilizer subgroups $M \subset G$ of a ribbon tile, observe that it came from a classical tiling and that each deleted edge corresponds to an element of M and G , and therefore to a closed curve in O , by section A. These closed curves can be pictured as intersecting at some base point in the interior of O , as a subgraph of the 1-skeleton of the dual tiling. Consider, for simplicity, a tile-1-transitive ribbon tiling \mathcal{T} and the bordered subsurface S_0 of O that is a (small) regular neighbourhood of the closed curves belonging to deleted edges that generate M and contains no parts of non-deleted edges and also no further elements of the singular locus of O than those that were associated to the deleted edges. We want to avoid creating boundaries for S_0 that, in O , are freely homotopic into a neighbourhood of a puncture. Every time this situation arises, we cap such a boundary component with a punctured disc. Clearly, S_0 is connected and satisfies property 1 and 2 of essential subsurfaces above.

It is clear that $\text{Mod}_0(S_0)$ is contained in the stabilizer subgroup $M_R \subset \text{Mod}(S)$ of \mathcal{T} . Consider the case where there is an element of the stabilizer of \mathcal{T} not in $\text{Mod}_0(S_0)$ and first observe that $S - S_0$ is a bordered surface. Consider a regular neighbourhood U of a component of the edge graph Γ of \mathcal{T} in $S - S_0$. If a homeomorphism f fixes Γ , with edge orientations, then we can assume it is the identity on ∂U . Notice that ∂U is isotopic to a boundary component B of S_0 , by construction. Therefore, $A := S - (S_0 \cup U)$ is an annulus and the orientable MCG $\text{Mod}(A)$ is generated by a Dehn twist T around a component of ∂A [26]. Clearly, T induces the same Dehn twist in $\text{Mod}(S)$ as the Dehn twist around B does. Therefore, if f fixes Γ , with orientations, f is isotopic to an element of $\text{Mod}(S_0)$.

By the previous paragraph, the stabilizer subgroup $M_R \subset \text{Mod}(S)$ of the tiling with stabilizer group M is equal to a finite extension of $\text{Mod}_0(S_0)$. We sum up the above discussion as follows, recalling that both a finite index subgroup and a finite extension are quasi-isometric to the original group.

Proposition 14. *For every equivariant tiling \mathcal{T} with symmetry group $G = \pi_1(O)$ without mirrors, the stabilizer subgroup M_R in $\text{Mod}(O)$ of \mathcal{T} is quasiconvex.*

We present an example of the above construction of S_0 , illustrated in the hyperbolic plane \mathbb{H}^2 . Consider the tiling with symmetry group 2223 in figure 10. Figure 10a shows a fundamental tiling with the locations of the generators of the group on the boundary of one tile, the starting tile, D . Figure 10b shows, in red, closed curves in the quotient space corresponding to edge deletions. The picture is slightly incorrect because of the red curve from the center of the starting tile to the point of the singular locus marked with a 3, which should really represent a slightly moved closed curve in the quotient space that avoids this point. The point of this is that we want to focus on what happens around the origin, marked with a 4. One sees that a regular neighbourhood of the red graph would contain a boundary component violating the assumptions of an essential subsurface. Capping the boundary component around the origin with a punctured disc essentially leads to the situation in figure 10c, which illustrates our procedure of finding an essential subsurface with a MCG that corresponds to the stabilizer subgroup in the MCG



(a) Fundamental tile-1-transitive tiling with symmetry group 2223 and generators around starting tile labelled in order.

(b) Red edges corresponding to closed curves in the quotient space representing elements of 2223 that correspond to edge traversals.

(c) Tiling with infinite stabilizer subgroup, along with embedded graph in red, whose regular neighbourhood gives rise to an essential subsurface of the underlying space of 2223.

Figure 10: Tilings and essential subsurfaces

$\text{Mod}(2223)$ of the tile created when deleting the blue edges from the tiling in figure 10a that cross the red edges. Notice that the red graph in figure 10b corresponds to the Cayley graph of the stabilizer subgroup of the tile resulting from the edge deletions w.r.t. the edge traversals as a set of generators.

E Discussion of experimental results and possible limitations

Investigations into questions like that in Remark 9 are made difficult by the fact that the only general algorithms available to study problems of this nature are those in KBMAG. These work only for shortlex automatic structures of the supergroup G , and, as mentioned above, it is unknown how much of a restriction this is. As a result of the relationship between braid groups and MCGs [26], a natural starting point for such investigations for MCGs are the braid groups, which are geodesically biautomatic, by [16], where it is shown that all Garside groups (see the paper for definitions) are geodesically biautomatic w.r.t. their Garside generators. See [35, Section 1.10.6] for a definition of these generators. Finding coset automatic structures for braid groups is already a challenging problem. To make the algorithms work, one often has to fine tune the parameters on a case-by-case basis. Even then, there are some examples that we have not been able to complete successful runs of the algorithm with. For the braid groups, on the other hand, as mentioned above, the algorithms used to investigate this problem use shortlex automatic structures, and therefore using Garside generators for the braid groups is natural. However, there are $n! - 1$ Garside generators for the braid group B_n on n strands [35, Section 1.10.6], which means that computations quickly become very involved for large n and it becomes unclear whether the algorithm fails or the parameters just need to be tuned better. We encountered similar difficulties in almost all of our investigations. However, even in cases where the algorithms of KBMAG do not terminate successfully, they return a list of words, corresponding to a subset of a language for the group G , but the words are not guaranteed to be in normal form. With a little experimentation, it is relatively simple to generate a list of millions of words.

On the other hand, it is unknown whether or not the algorithms in KBMAG can always accomplish this enumeration, because, for one, as mentioned above, it is unknown whether automatic structures can be improved to be shortlex automatic structures. We found that in experiments, the algorithms in KBMAG work well, at least in simple cases. However, in the case of the MCG of a genus 3 surface in the so-called Gervais presentation [29], a rather complicated presentation whose generators are rather symmetric, even after prolonged experimenting, we failed to get the algorithms to successfully solve, comprehensively, even the word problem. This means that, although the algorithm outputs a list of non-equivalent words, the words are not guaranteed to be in normal form.