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# Tile-transitive tilings of the Euclidean and hyperbolic planes by ribbons

Benedikt Kolbe<sup>\*†‡</sup>      Vanessa Robins<sup>§¶</sup>

## Abstract

We present a method to enumerate tile-transitive crystallographic tilings of the Euclidean and hyperbolic planes by unbounded ribbon tiles up to equivariant equivalence. The hyperbolic case is relevant to self-assembly of branched polymers. This is achieved by combining and extending known methods for enumerating crystallographic disk-like tilings. We obtain a natural way of describing all possible stabiliser subgroups of tile-transitive tilings using a topological viewpoint of the tile edges as a graph embedded in an orbifold, and a group theoretical one derived from the structure of fundamental domains for discrete groups of planar isometries.

## 1 Introduction

Patterns built from repeating motifs appear in all cultures and have long been studied in art, mathematics, engineering and science. Most mathematical work has focussed on patterns in the Euclidean plane (the book [18] contains a comprehensive survey of the field up to the mid 1980s) but the importance of hyperbolic geometry as a model for natural forms is increasingly recognised [23, 37, 24, 42]. An example that inspires the work in this paper is the discovery that star co-polymer systems consisting of three mutually immiscible arms can self-assemble into structures modelled by stripes on the gyroid triply-periodic minimal surface [26, 8]. The gyroid surface has genus three in its smallest side-preserving translational unit cell, and therefore has the hyperbolic plane as its simply-connected Riemann covering space. Its 3d space-group symmetries induce a non-Euclidean crystallographic group generated by hyperbolic isometries that are known explicitly [41, 40]. Stripe patterns on the gyroid lift via the covering map to tilings of the hyperbolic plane by infinitely long strips, or ribbons. The defining property of a *ribbon tile* is the existence of a translation isometry that maps a given tile back onto itself, along with the restriction that the tile is simply connected. See figure 1 for some ribbon tilings and their projections to the gyroid.

The Euclidean case of striped patterns is described in Section 6.5 of [18], citing earlier work by Wollny [47]; there are 26 distinct types of crystallographic ribbon tilings of the Euclidean plane, a result that can readily be proven using the enumerative methods we present here. Corresponding results for the hyperbolic plane require different mathematical techniques to those of [18]. Our methods derive from the classification of 2d discrete groups of isometries up to isomorphism via their quotient spaces [34, 46, 45, 6]; from the enumeration of polygonal fundamental domains for these groups via graphs on their quotient spaces [30, 32]; and from the results of Dress *et al.* who developed the field now known as combinatorial tiling theory [11, 10]. See [20] for a detailed introduction to combinatorial tiling theory for the Euclidean and hyperbolic planes, [9] for algorithms and [48] for a very recent implementation leading to a database of tilings of the sphere, and the Euclidean and hyperbolic planes.

Combinatorial tiling theory treats periodic (crystallographic) tilings of a simply connected space where the tiles are compact topological disks and defines an invariant called the Delaney-Dress symbol or *D-symbol* as a weighted and coloured graph. From the D-symbol it is possible to reconstruct the

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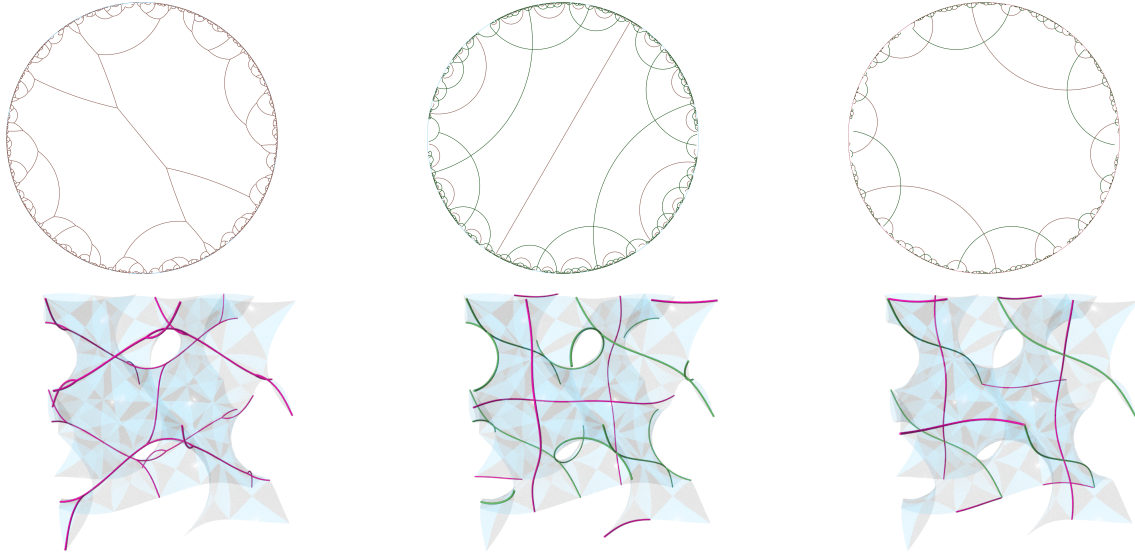


Figure 1: Nets in  $\mathbb{R}^3$  on the gyroid from ribbon tilings with 22222 symmetry in  $\mathbb{H}^2$ .

tile shapes and adjacencies and the isomorphism class of the symmetry group for the tiling. In the 2D setting, a D-symbol encodes a finite triangulation derived from the tiling by barycentric subdivision. The underlying space of the triangulation is the quotient of the plane by a discrete group of isometries that preserve the tiling. This quotient space is a 2-orbifold and can be viewed as a compact surface with (possibly) a finite number of boundary components and a finite number of isolated marked points. When the orbifold is an orientable manifold with no boundary or singular points, then the D-symbol encodes the same information as the rotation system for a 2-cell embedding of a graph [36]. Although the full theory of D-symbols does not directly generalise to our setting with unbounded ribbon tiles, the correspondence between crystallographic patterns and graphs on 2-orbifolds does (see Section 3).

The main issues to overcome are

1. characterising the possible stabiliser subgroups for unbounded tiles,
2. enumerating the tile-transitive unbounded ribbon tiling compatible with a given symmetry group, and
3. constructing a canonical triangulation from such as tiling.

Solutions to the above challenges are the main contributions of this paper. For Euclidean tilings by ribbons the possible stabiliser groups are the seven frieze groups; for the hyperbolic case infinitely many such stabilizer groups, i.e., non-Euclidean frieze groups are possible. These are described further in Section 2. Definitions and notation for combinatorial tiling theory are covered in Section 3. The enumeration of tile-transitive ribbon tilings is achieved by deleting edges from a fundamental domain tiling. The results needed to characterise the existence and structure of ribbon tiles are detailed in Section 4. An algorithm for constructing a canonical D-symbol from a ribbon tiling is described in Section 5, with examples illustrating all of the above in Section 6.

Previous work related to this paper includes Huson's paper on tile-transitive partial tilings of the Euclidean plane [21], where the untiled part of the plane could consist of parallel ribbons. The exploration of crystallographic line and tree patterns in the hyperbolic plane goes back to Hyde et al. [22, 13, 14, 12]. Our approach to enumeration using graphs on orbifolds and D-symbols allows us to adapt the methods of isotopic tiling theory [28, 27] to systematically enumerate the distinct ways in which tilings fit on compact surfaces.

## 2 Groups of isometries and the orbifold fundamental group

Throughout this paper, let  $\mathcal{X}$  be either the Euclidean ( $\mathbb{E}^2$ ) or hyperbolic ( $\mathbb{H}^2$ ) plane, and let  $\Gamma$  be a discrete group of isometries of  $\mathcal{X}$  with compact quotient space. If  $\mathcal{X} = \mathbb{E}^2$  then  $\Gamma$  is one of the 17 wallpaper groups of crystallography [6]. If  $\mathcal{X} = \mathbb{H}^2$ , then  $\Gamma$  is known as a NEC group (non-Euclidean crystallographic group). A key result in geometric group theory is that these groups of isometries are completely classified up to isomorphism by an associated *orbifold*. The homeomorphism class of the orbifold is in turn specified by Conway's *orbifold symbol* [7], a highly-readable version of Macbeath's group signature [34], as described below.

**Definition 1.** Let  $\Gamma$  be a wallpaper or NEC group. A geometric (good) 2-orbifold,  $\mathcal{O} = \mathcal{X}/\Gamma$ , is a quotient space obtained by identifying points of  $\mathcal{X}$  under the action of  $\Gamma$ . The orbifold retains the metric information carried by the particular isometries of  $\Gamma$  by specifying an atlas of charts compatible with the  $\Gamma$  action on the topological space and the branch points of the projection map  $p : \mathcal{X} \rightarrow \mathcal{X}/\Gamma$ .

For more detailed definitions of the concepts involved, refer to [39, Chapter 13].

It is well-known [45] that 2-orbifolds have the topology of a finite-area 2-manifold with a finite number of boundary components. Boundaries in a 2-orbifold arise from the fixed lines of reflection isometries. Other special points arise as the fixed points of rotational isometries; these are called *cone points* if they lie in the interior of the orbifold, and *corner points* if they lie on a boundary. The branching number,  $N$ , of a cone or corner point is the order of the rotational isometry,  $\sigma$ , that fixes that point i.e.  $\sigma^N = id$ . The boundaries, corner and cone points are collectively referred to as the *singular locus*,  $\Sigma$ , of the orbifold. The topology of a 2-orbifold ( $\mathcal{O}$ ) is therefore specified by a symbol as follows:

1. The number of handles,  $h$ , if the orbifold is orientable, or the number of cross-caps,  $k$ , if non-orientable. Handles are denoted by  $\circ$  at the beginning of the orbifold symbol. Cross-caps are denoted by  $\times$  at the end of the orbifold symbol.
2. The branching number for each cone point, listed in arbitrary order after any handles.
3. The number of boundary components,  $q$ . Each boundary component is represented by a  $*$  in the symbol. Branching numbers for the corner points lying on each boundary component are listed in cyclic order, such that each boundary component has a consistent orientation for the manifold. The ordering of the boundary components is arbitrary.

For example, the orbifold symbol for the patterns in Figure 1 is 22222, or  $2^5$ , telling us that there are five distinct conjugacy classes of 2-fold rotation for this symmetry group.

It is known [45, 7] that any orbifold symbol will correspond to a group of isometries of  $\mathbb{E}^2$ ,  $\mathbb{H}^2$ , or  $\mathbb{S}^2$ , except for the symbols A, \*A, AB, and \*AB, with  $A \neq B$ . Moreover, the plane geometry associated with an orbifold is determined by computing a curvature-related quantity (the orbifold Euler characteristic) directly from the group symbol.

The symmetry group associated with a geometric orbifold also has an interpretation as a type of fundamental group defined by equivalence classes of loops. Intuitively, *orbifold loops* are piecewise lifts to  $\mathcal{X}$  of closed curves in the quotient space  $\mathcal{O}$ . The general definition of orbifold loops in  $\mathcal{O}$  and their homotopies is quite involved, but it is sufficient for our purposes to view them as closed curves in  $\mathcal{O}$ , disjoint from the singular locus except possibly at isolated points of mirror boundaries. We also require that homotopies of simple curves in  $\mathcal{O}$  do not change the incidence relation of a curve to any point of the singular locus; that the group  $\Gamma$  is fixed; and that paths in  $\mathcal{O}$  meeting a mirror boundary transversally will lift to paths that cross the mirror line in  $\mathcal{X}$ , to eliminate the ambiguity of the lift in this situation. See [39, Chapter 13] or [7].

The *orbifold fundamental group*  $\pi_1^{orb}(\mathcal{O}, x)$ ,  $x \in \mathcal{O} \setminus \Sigma$ , is then defined as the set of orbifold loops based at  $x_0 \in \mathcal{X}$  up to homotopy equivalence, where  $x_0 \in p^{-1}(x)$ . It is known that for the geometric 2-orbifolds considered in this paper,  $\Gamma \simeq \pi_1^{orb}(\mathcal{X}/\Gamma)$  in the natural way [39, Theorem 13.3.2]. In other words,  $\pi_1^{orb}(\mathcal{X}/\Gamma)$  is the group of deck transformations for the branched covering map  $p : \mathcal{X} \rightarrow \mathcal{O}$ .

Other discrete groups of isometries  $\Gamma$  will be important when we discuss the internal symmetries of a tile. For a bounded tile in  $\mathcal{X}$ ,  $T$  homeomorphic to a closed disk, the possible symmetry groups include

those that fix a single point, namely the cyclic  $C_N$  and dihedral  $D_N$  groups for  $N \geq 2$ , and  $D_1$  the symmetry group generated by a single reflection. When these groups are viewed as acting on the tile  $T$ , we can form the quotient space  $T/C_N$  or  $T/D_N$  and describe the topology of these quotient spaces similarly to the orbifold symbol above, using *signatures* for rosette patterns:  $\mathbf{N}\bullet$  and  $\ast\mathbf{N}\bullet$  for the cyclic and dihedral groups respectively [6]. The symbol  $\bullet$  represents for us a section of tile boundary in the quotient space. Note that the group  $D_1$  is abstractly isomorphic to  $C_2$ , but their associated quotient spaces have different singular loci and different signatures.

We next consider isometries of a tile  $T$  that is homeomorphic to  $[0, 1] \times \mathbb{R}$ , i.e. a *ribbon*. The possible discrete symmetry groups for such a tile are  $D_1$ ,  $D_2$ ,  $C_2$  or one of the seven frieze groups. Again, we can specify the quotient space structure of  $T/\Gamma$  using a descriptive signature or symbol for the symmetry group. In Table 1 we give the signatures (from [6]) and Hermann-Mauguin (IUCr) name of each of the frieze groups. In these signatures the  $\ast$  symbol represents a single boundary component of  $T/\Gamma$  due to fixed points of reflection isometries, and in our setting the  $\infty$  symbol represents a segment of tile boundary in  $T/\Gamma$ . So the signature  $\ast\infty\infty$  implies that  $T/\Gamma$  is a disk with a mirror boundary interrupted by two tile boundary segments and so combinatorially it is a quadrilateral. Note that in other contexts, the  $\infty$  symbol can represent an orbifold puncture as might be generated by a parabolic isometry of  $\mathbb{H}^2$ . The different contexts are just other ways of obtaining a geometric representation of the same abstract group. It is important to note that the above rosette and frieze groups can be realised using isometries of either the Euclidean or hyperbolic plane. This is in contrast to the geometric 2-orbifolds which can be realised by isometries in exactly one of the three plane geometries. Figures 1a) and c) show tilings for which each ribbon tile has symmetry group  $22\infty$ . Figure 1b) shows an example with symmetry group  $\infty\infty$ .

Table 1: Signature of  $T/\Gamma$  and IUCr name for the seven frieze groups, the index of the translation isometry in  $\Gamma$ , and a presentation that makes the translation isometry  $t$  explicit in each case.

| Group              | name   | index | group presentation  |
|--------------------|--------|-------|---|
| $\infty\infty$     | $p1$   | 1     | $\langle t \rangle$   |
| $\infty\ast$       | $p11g$ | 2     | $\langle g, t \mid g^2 = t \rangle$   |
| $\infty\ast$       | $p11m$ | 2     | $\langle r, t \mid r^2, rtr = t \rangle$  |
| $\ast\infty\infty$ | $p1m1$ | 2     | $\langle r_1, r_2, t \mid r_1^2, r_2^2, r_1r_2 = t \rangle$                                     |
| $22\infty$         | $p2$   | 2     | $\langle q_1, q_2, t \mid q_1^2, q_2^2, q_1q_2 = t \rangle$                                     |
| $2\ast\infty$      | $p2mg$ | 4     | $\langle r, q, t \mid r^2, q^2, (qr)^2 = t \rangle$   |
| $\ast22\infty$     | $p2mm$ | 4     | $\langle r_1, r_2, r_3, t \mid r_1^2, r_2^2, r_3^2, (r_1r_2)^2, (r_2r_3)^2, r_1r_3 = t \rangle$ |

In the hyperbolic plane we naturally encounter unbounded simply connected tiles with branching structure homeomorphic to a neighbourhood of an infinite tree embedded in  $\mathbb{H}^2$ . See Figure 8c for an example. We call such tiles *branched ribbons*. The isometries of such a tile are isomorphic to a group action on a tree, and are covered by the theory of Bass-Serre [43]. The simplest examples of group actions on trees are those that have a line segment as fundamental domain. These groups are a free product with amalgamation of the subgroups that fix the vertices, amalgamated via the subgroup that fixes the edge (viewed as an oriented line segment). For example, if the line segment generating the tree has end points on rotation centers of order A and B, and the edge group is trivial, then the group is  $\Gamma = C_A \ast C_B = \langle q_1, q_2, t \mid q_1^A, q_2^B, q_1q_2 = t \rangle$  and  $T/\Gamma$  has the signature  $\mathbf{AB}\infty$ .

### 3 Combinatorial tiling theory

Standard or classical *combinatorial tiling theory* describes the adjacency structure of periodic tilings of a simply connected metric space for which each tile is homeomorphic to a closed bounded disk [20, 10]. In this paper, we employ a more general definition of tiling.

**Definition 2.** A crystallographic tiling of  $\mathcal{X}$  is a locally finite, countable set  $\mathcal{T}$  of connected closed domains, with  $\text{cl}(\text{int}(T)) = T$  for all tiles  $T$ , such that every point  $x \in \mathcal{X}$  belongs to at least one tile, all tiles have pairwise disjoint interiors, and such that  $\mathcal{T}$  is invariant under a discrete group of isometries of  $\mathcal{X}$ .

For emphasis, we call a simply connected compact tile a *disk*, and a tiling where every tile is a disk a *disk-like tiling*. Similarly, a tiling in which every tile is a ribbon or branched ribbon is called a *ribbon tiling*. A simple example of a ribbon tiling is a covering of  $\mathbb{E}^2$  by vertical strips,  $T_k = [k, k + 1] \times \mathbb{R}$ , for  $k \in \mathbb{Z}$ . The trivial ribbon tiling consists of one tile that is all of  $\mathcal{X}$ . Other examples are the partial tilings of the Euclidean plane, enumerated by Huson in [21], if we treat the complementary regions of types 2 and 3 in that paper as tiles.

As in [20], we define the vertices and edges of a tile topologically rather than using the geometry of straight lines and corners. A *vertex* is a point that is contained in at least three tiles, and an *edge* is a connected segment of the intersection of two tiles. In the classical setting, an edge is a compact simple curve joining two vertices, but in the general setting an edge can be an unbounded 1-dimensional sub-manifold embedded in  $\mathcal{X}$ , possibly with no vertices.

**Definition 3.** Let  $\mathcal{T}$  be a tiling of  $\mathcal{X}$  and let  $\Gamma$  be a discrete group of isometries. If  $\mathcal{T} = \gamma\mathcal{T} := \{\gamma T \mid T \in \mathcal{T}\}$  for all  $\gamma \in \Gamma$  then we call the pair  $(\mathcal{T}, \Gamma)$  an *equivariant tiling*.

Note that this definition does not require  $\Gamma$  to be the maximal symmetry group for the tiling  $\mathcal{T}$ . The *orbit* of a tile  $T$  (or an edge or a vertex) is the subset of  $\mathcal{T}$  given by images of  $T$ :  $\Gamma.T = \{\gamma T \mid \gamma \in \Gamma\}$ . Given a particular tile  $T \in \mathcal{T}$ , the *stabiliser subgroup*  $\Gamma_T$  is the subgroup of  $\Gamma$  that fixes  $T$ , i.e.  $\Gamma_T = \{\gamma \in \Gamma \mid \gamma T = T\}$ . A tile is called *fundamental* if  $\Gamma_T$  is trivial and we call the whole tiling fundamental if this is true for all tiles. An equivariant tiling is called *tile- $k$ -transitive*, when  $k$  is the number of distinct orbits of tiles under the action of  $\Gamma$ . A fundamental tile-1-transitive equivariant tiling  $(\mathcal{T}, \Gamma)$  has a single type of tile that is a fundamental domain for  $\Gamma$ . Conversely, any fundamental domain for  $\Gamma$  homeomorphic to a bounded disk also gives rise to such a tiling.

The following is a central definition for combinatorial tiling theory, and is also the notion of equivalence among tilings we consider.

**Definition 4.** Two equivariant tilings  $(\mathcal{T}_1, \Gamma_1)$  and  $(\mathcal{T}_2, \Gamma_2)$  of a simply connected space  $\mathcal{X}$  are *equivariantly equivalent* if there is a homeomorphism,  $\phi$ , of  $\mathcal{X}$  and a group isomorphism  $h : \Gamma_1 \rightarrow \Gamma_2$ , such that  $\phi(T) \in \mathcal{T}_2$  for all  $T \in \mathcal{T}_1$  and  $h(\gamma)[\phi(T)] = \phi(\gamma[T])$  for all  $\gamma \in \Gamma_1$ .

Intuitively, this definition means that two tilings are considered to be equivalent if there is a self-map of  $\mathcal{X}$  that preserves the tile adjacencies and their symmetries. A natural question is whether there is an invariant that detects when two tilings are equivariantly equivalent. Dress *et al.*, [11, 10] show that a complete invariant, *the D-symbol*, is indeed possible for disk-like tilings of simply connected manifolds.

First, recall that a *flag* in this context is a triple of (tile, edge, vertex) where each lower-dimensional element is a face of the higher dimensional ones, and that two flags are adjacent if they differ by a single one of their elements. We picture the each flag as a triangle in the ( $\Gamma$ -compatible) barycentric subdivision of the tiling, so that each triangle spans a tile centre-point, edge mid-point and tiling vertex. The D-symbol consists of a graph whose nodes are the  $\Gamma$ -equivalence classes of flags. Each node has three edges, one each of three colours to record adjacencies of the flag, and two weights, corresponding to the number of edges of the tile and the degree of the vertex the flag represents [9].

**Theorem 5** ([20, Lemma 1],[11]). *Two equivariant disk-like tilings of  $\mathcal{X}$  are equivariantly equivalent if and only if their D-symbols are isomorphic as weighted and coloured graphs.*

The properties of D-symbols are exploited in [20, 9, 48] to achieve a fully algorithmic approach to the enumeration and identification up to equivariant equivalence of disk-like tilings of  $\mathbb{S}^2$ ,  $\mathbb{E}^2$  and  $\mathbb{H}^2$ . In particular, it is established that one can enumerate equivariant equivalence classes of fundamental domains in a purely combinatorial way, by producing the D-symbols that represent them. This enumeration proceeds by first specifying the number of edges and vertex degrees of a tile; the isomorphism class of the group (i.e. its orbifold symbol) is then computed from the D-symbol. An alternative approach due to [30, 31, 32] starts with the orbifold symbol and finds all possible combinatorial types of fundamental domain, as described in the next paragraphs.

The edges and vertices of any tiling  $(\mathcal{T}, \Gamma)$  map onto a graph  $G = (V, E)$  embedded in the orbifold under the covering map  $p : \mathcal{X} \rightarrow \mathcal{X}/\Gamma$ . In general, vertices of  $G$  come from vertices of  $\mathcal{T}$ , except in the case that an edge midpoint in  $T$  is a cone point of order 2. Such an edge of  $T$  maps onto an edge of  $G$  with a vertex of degree 1 on the cone point. Lucic *et al.* [30, 31] have characterised the graphs associated

with all possible combinatorial types of polygonal fundamental domain, and present an algorithm for enumerating these from the group signature (i.e. orbifold symbol) in [32].

**Theorem 6** ([31, Theorem 4.1]). *A generalised polygon  $F$  is a fundamental domain for a wallpaper or NEC group  $\Gamma$  if and only if the boundary of  $F \subset \mathcal{X}$  maps under  $p : \mathcal{X} \rightarrow \mathcal{X}/\Gamma = \mathcal{O}$  to a connected graph  $G$  embedded in  $\mathcal{O}$  with the following properties:*

1.  $\mathcal{O} \setminus G$  is an open disk.
2. Each cone point is a vertex of  $G$  with at least one incident edge in  $G$ .
3. Let  $\tilde{\mathcal{O}}$  be the closed surface of genus  $g$  obtained from  $\mathcal{O}$  by capping each boundary component of  $\mathcal{O}$  with a disk. Then  $G$  is contractible in  $\tilde{\mathcal{O}}$  to the graph  $\tilde{G}$  with one vertex and  $2g$  loops if  $\tilde{\mathcal{O}}$  is orientable and  $g$  loops if  $\tilde{\mathcal{O}}$  is non-orientable.
4. Each cone point is a vertex of  $G$  with at least one incident edge in  $G$ .
5. Each (mirror) boundary component of  $\mathcal{O}$  lies in a subgraph  $G_i$  of  $G$ . This means each corner point is a vertex of  $G$  with at least two incident edges. Moreover when  $G_i$  is contracted in  $\tilde{\mathcal{O}}$ , it becomes a vertex with at least one incident edge in  $G \setminus G_i$ .
6. Any vertex of  $G$  that does not lie on a cone point, corner point or boundary must have at least three incident edges in  $G$ .

See [31, 32] for general figures, and Figure 3 for a simple Euclidean example.

The definition of combinatorial equivalence in [30, 31] is made for fundamental domain tilings, but extends to a general tiling as per Definition 2.

**Definition 7.** Two equivariant tilings are *combinatorially equivalent* if there is a bijection mapping one onto the other which preserves the incidence relations of vertices and edges, their cyclic order, the  $\Gamma$ -equivalence of vertices, directed edges, tiles, and the isomorphism class of the stabiliser subgroups of each of these.

We show in the appendix, Section A, that two tilings are combinatorially equivalent if and only if they are equivariantly equivalent. It follows straight from the definition that equivariant equivalence implies combinatorial equivalence. The converse result extends combinatorial tiling theory of D-symbols by showing that they can be viewed as topological equivalence classes of graphs embedded in the orbifold  $\mathcal{O}$  associated to  $\Gamma$ .

The final result from geometric group theory that we require in order to enumerate ribbon tilings is a classical method for obtaining a group presentation from the intersection pattern of a covering of  $\mathcal{X}$  by the  $\Gamma$ -orbit of fundamental domains [46, 33].

**Theorem 8.** *Let  $(\mathcal{T}, \Gamma)$  be a fundamental tile-1-transitive disk-like tiling and  $T_0$  a tile. Then the adjacency between tiles defines a presentation for  $\Gamma$ , where each edge of  $T_0$  corresponds to a generator and each vertex to a relation. If  $e_i = T_0 \cap T_i$  is an edge, then the edge-generator is an element,  $[e_i] \in \Gamma$ , such that  $T_i = [e_i]T_0$ . The relation at vertex  $v$  is found by listing the generators associated with each edge crossing when making a clockwise circuit around  $v$ .*

We call the group elements associated with edges the *Wilkie generators* (although according to Macbeath [33], the idea goes back to Fricke and Klein's enumeration of Fuchsian groups.) Wilkie's proof is given for polygonal Dirichlet (Voronoi) fundamental domains [46], but it holds more generally as detailed in [33].

## 4 Orbifold paths and tile glueing

In this section, we develop a relationship between closed paths in the orbifold  $\mathcal{O} = \mathcal{X}/\Gamma$ , their lifts in  $\mathcal{X}$ , and equivariant tilings  $(\mathcal{T}, \Gamma)$  that contain unbounded tiles. Furthermore, we study how the topological structure of tiles relates to their stabiliser subgroup and how to construct the possible stabiliser subgroups of tiles by edge deletions from fundamental tilings.

Recall that a non-fundamental tile is one that has a non-trivial stabiliser. In the D-symbol approach to the enumeration of disk-like tilings, it is shown in [20] that any non-fundamental disk-like tiling  $(\mathcal{T}, \Gamma)$  is obtained from a fundamental one  $(\mathcal{F}, \Gamma)$  by performing *tile glueing*. Since the tiles are homeomorphic to a closed bounded disk, their stabiliser group must be a finite cyclic or dihedral group. We characterise the possible tile glueing operations as erasing edges from the graph  $G$  on the orbifold  $\mathcal{O}$  associated to  $(\mathcal{F}, \Gamma)$ .

Given a tile-1-transitive fundamental tiling by disks  $(\mathcal{F}, \Gamma)$ , let  $G$  be its corresponding graph on  $\mathcal{O}$ . By Theorem 6,  $\mathcal{O} \setminus G$  is a disk. *Tile glueing* erases at most two edges from  $G$  to get  $G'$  so that  $G'$  is connected,  $\mathcal{O} \setminus G'$  is still a disk and contains some part of the singular locus of  $\mathcal{O}$ , and the corresponding tiling  $(\mathcal{T}, \Gamma)$  has one class of non-fundamental tile. The edges of  $G$  that can be erased are of three types:

1. An edge of  $G$  that has a vertex of degree 1 at a cone point of order  $N$ . This glues  $N$  copies of the fundamental tile into one new one with stabiliser group  $\mathbf{N}\bullet$ .
2. A pair of edges of  $G$  that lie in a mirror boundary and meet at a vertex of degree 2 on a corner point of order  $N$ . This glues  $2N$  copies of a fundamental tile into one new one with stabiliser group  $*\mathbf{N}\bullet$ .
3. A single edge of  $G$  that is a segment of mirror boundary and has vertices of degree at least 3 (or degree 2 on a corner point). This glues two copies of a tile together into one with stabiliser group  $*\bullet$ .

We now work through the possibilities for tile glueing to obtain tile-transitive ribbon tilings.

**Lemma 9.** *The stabiliser subgroup,  $H$ , for a non-fundamental tile  $T$  is infinite if and only if it contains a translation isometry.*

*Proof.* First note that if  $H$  contains a translation (or glide) isometry then it must be infinite.

Now assume  $H$  is an infinite group of isometries of  $\mathbb{H}^2$ . If  $H$  is abelian, then it must be generated by a single isometry of infinite order, i.e., a translation, glide or parabolic rotation (an isometry with a single fixed point at infinity) [17, Section 4.5]. This last case can be ruled out as our tiling group  $\Gamma$  is assumed to have a compact orbifold. If  $H$  is non-abelian, then it is known that such a group must contain a translation [17, Section 4.5].

Finally, we consider the case that  $H$  is an infinite discrete subgroup of isometries of  $\mathbb{E}^2$ :  $H \subset O(2) \rtimes \mathbb{R}^2$ . If  $H$  consists entirely of rotations and/or reflection isometries and is discrete, it must be a discrete subgroup of  $O(2)$ , and therefore closed, since discrete subgroups of topological groups are well-known to be closed. By compactness this would imply  $H$  is finite. But we assumed  $H$  to be infinite so not all its elements can be rotations and reflections and we see that it must contain a translation or glide.  $\square$

Recall from section 3 that the edges of a tiling map to an embedded graph  $G = (V, E)$  in  $\mathcal{O}$ . Let  $S = \{e_1, \dots, e_k\} \subset E$  be the edges to be erased and  $R = E \setminus S$  be the edges that remain. Let  $G_R = (V_R, R) \subset G$  be the subgraph obtained from  $G$  by erasing  $S$  and any vertices left isolated. We also avoid “dangling ends”, so for all  $e \in R$  with a vertex  $v$  of degree 1 in  $G_R$  with  $v \notin \Sigma$  we add  $e$  to  $S$ .

**Theorem 10.** *Let  $\mathcal{X} = \mathbb{E}^2$  or  $\mathbb{H}^2$  and suppose  $\Gamma$  is a wallpaper or NEC group. Suppose we are given a fundamental tile-1-transitive tiling  $(\mathcal{F}, \Gamma)$  whose edges map onto a graph  $G$  embedded in  $\mathcal{O} = \mathcal{X}/\Gamma$ . Let  $S = \{e_1, \dots, e_k\}$  be a subset of edges of  $G$  whose removal avoids dangling ends. Then erasing all preimages of these edges from  $\mathcal{F}$  results in a non-fundamental tile-1-transitive tiling  $(\mathcal{T}, \Gamma)$ , such that the stabiliser group of each tile  $T \in \mathcal{T}$  is isomorphic to the subgroup of  $\Gamma$  generated by the erased edges.*

*Proof.* We use the correspondence between edges of a fundamental domain,  $F \in \mathcal{F}$ , and generators for  $\Gamma$  as described in Theorem 8. If  $f$  is an edge of  $F$ , then  $f = F \cap \gamma(F)$  for an element  $\gamma \in \Gamma$ , and we use the notation  $[f]$  for this Wilkie generator. We also know that each edge  $f$  is mapped to another edge  $f' \in F$  (possibly itself) by the element  $[f']$ , so that  $[f'] = [f]^{-1}$ . The image of  $f$  in  $\mathcal{O}$  is an edge  $e \in G$ , and  $p(f) = p(f') = e$ .

Choose a point  $x_0 \in \text{int}(F)$ , and for each  $e_i \in G$  choose a single tile edge  $f_i \in p^{-1}(e_i) \cap F$ . Then for each  $f_i$ , there is a simple orbifold loop  $\alpha_{e_i}$  based at  $x = p(x_0)$ , with  $\alpha_{e_i}(0) = x_0$  and  $\alpha_{e_i}(1) = [f_i](x_0)$ , such that  $\alpha_{e_i}([0, 1])$  is a connected curve in  $\mathcal{X}$  that intersects the boundary of  $F$  in a single point of



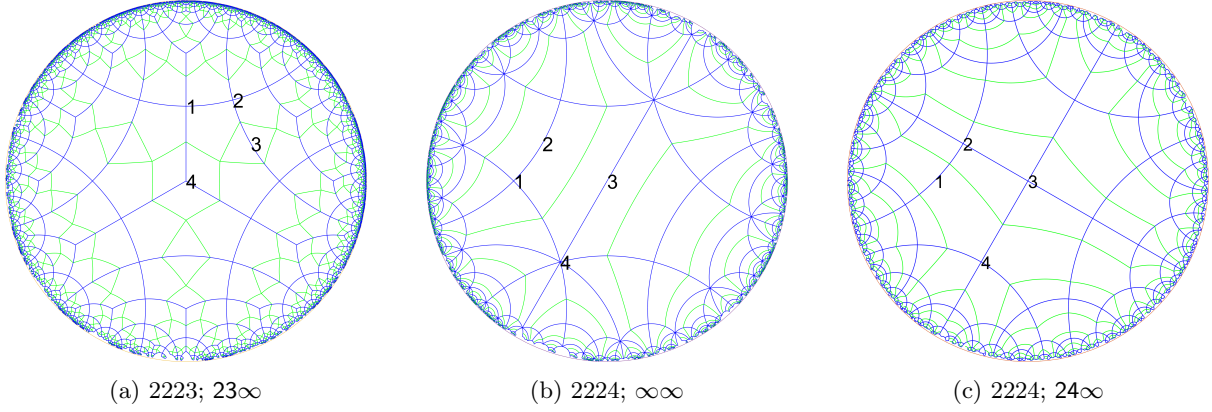


Figure 2: Fundamental tile-1-transitive tilings in  $\mathbb{H}^2$  are drawn with blue edges, and the full set of lifts of orbifold loops are drawn in green. A set of generators of each symmetry group is labelled 1, 2, 3, 4 according to their order in the given Conway symbol. The symbols given in each case are the orbifold of the fundamental tiling followed by the ribbon tile stabiliser group.

$f_i$ . This is possible because  $F$  is path-connected. In the deck-transformation correspondence between  $\pi_1^{orb}(\mathcal{O}, x)$  and  $\Gamma$ , we then have that  $[\alpha_{e_i}] \sim [f_i]$ .

Now let  $H$  be the subgroup of  $\Gamma$  generated by the group elements associated with edges  $e_i \in S$  and let  $T = \bigcup_{\eta \in H} \eta(F)$ . By this definition  $H$  is the stabiliser subgroup of the tile  $T$ .  $T$  is path connected by the following argument. Each orbifold loop  $\alpha_{e_i}$  has a connected representative from  $x \in F$  to  $[f_i](x)$  in  $\mathcal{X}$ , and any other such connected representative of  $\alpha_{e_i}$  has the end-points  $\gamma(x)$  and  $\gamma[f_i](x)$  for some  $\gamma \in \Gamma$ . Therefore, for each  $\eta \in H$ , there is a path in  $\mathcal{X}$  from  $x$  to  $\eta(x)$  that lies entirely within  $T$ , obtained by writing  $\eta$  as a word in  $[f_1], \dots, [f_k]$ , and forming the corresponding concatenation of the lifts of the  $\alpha_{e_i}$  loops.

If  $H$  is infinite,  $T$  must be unbounded as it is the union of infinitely many distinct copies of  $F$ . If  $H$  is finite, then  $T$  will be bounded.

Next consider the action of an isometry,  $\gamma \in \Gamma$  on the tile  $T$ . We use the facts that the construction of  $T$  began with a particular choice of fundamental domain tile  $F \in \mathcal{F}$ , and that  $\Gamma$  acts transitively on the tiling  $\mathcal{F}$ . If  $\gamma \in H$ , then  $\gamma F \subset T$ ,  $\gamma\eta \in H$  for all  $\eta \in H$  and so  $\gamma T = T$ . If  $\gamma \notin H$ , then  $\gamma F \not\subset T$ , and in particular,  $\gamma\eta F \not\subset T$  for any  $\eta \in H$ , so that  $\gamma(\text{int}(T)) \cap \text{int}(T) = \emptyset$ . It follows that for any two  $\gamma, \gamma' \notin H$ , that either  $\gamma T = \gamma' T$  or  $\gamma(\text{int}(T)) \cap \gamma'(\text{int}(T)) = \emptyset$ . Let  $\mathcal{T}$  be the union of all distinct images of  $T$ . It then follows from the tile-transitivity of  $(\mathcal{F}, \Gamma)$ , that  $(\mathcal{T}, \Gamma)$  is also a tile-transitive tiling of  $\mathcal{X}$ .  $\square$

**Theorem 11.** *Let  $(\mathcal{T}, \Gamma)$  be a tiling obtained via edge deletion from a fundamental tile-1-transitive tiling as in Theorem 10. If the stabiliser group  $H$  is infinite, then the tile  $T$  is simply connected: it is a ribbon or branched ribbon. If  $H$  is finite, then  $T$  is a disk.*

*Proof.* Assume first that  $H$  is infinite. From Theorem 10, it follows that  $\mathcal{X}$  is the union of path-connected, unbounded tiles of the form  $\gamma T$  for  $\gamma \in \Gamma$  and that all of these tiles have disjoint interiors. By Lemma 9,  $H$  contains a translation. If  $T$  is not simply connected, then its complement includes a bounded component of  $\mathcal{X}$  that is covered by isometric copies of  $T$ , which is clearly a contradiction.

If  $H$  is finite, then  $T$  will be bounded. The tile  $T$  must be simply connected: If it were not, then it contains a boundary component that is homeomorphic to a compact circle  $C$  bounding a disk  $D$ . Now,  $(\mathcal{T}, \Gamma)$  is tile-transitive, so  $D$  must contain isometric copies of  $T$ , which is clearly impossible, since such any copy would itself contain yet more copies of  $T$ , which would eventually contradict the local finiteness of the tiling  $\mathcal{T}$ . Therefore, in this case, the erased edges must be one of the three cases discussed in Section 3.  $\square$

Figure 2 illustrates how orbifold loops are associated with unbounded tiles. The tile edges are drawn in blue and the all lifts of the loops in green. Edges crossed by an orbifold loop are deleted to obtain ribbon tiles, and the green graphs collapse to trees embedded in these tiles. Isometries of the ribbon tiles are isomorphic to a group action on this tree.

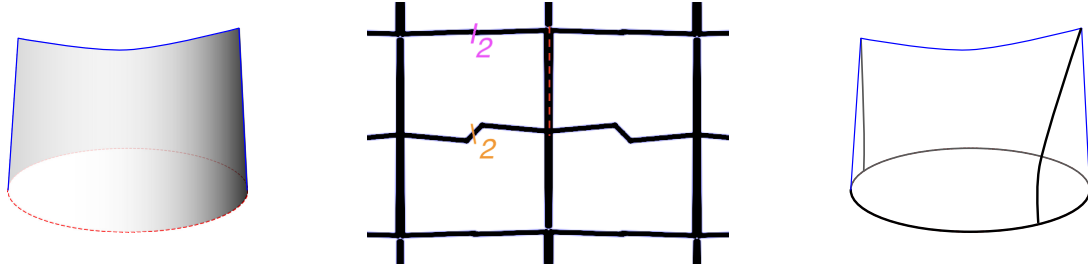


Figure 3: Left to right: A Euclidean orbifold quotient space with symbol  $22^*$ . A slightly deformed standard fundamental domain (the vertical lines of this tiling lie along the mirrors and the two inequivalent centres of rotation are marked). The corresponding edge graph for this tiling.

**Proposition 12.** *Let  $(\mathcal{T}, \Gamma)$  be a tile-1-transitive tiling. Let  $T$  be the tile produced via edge deletion of two non-adjacent edges and their orbits under  $\Gamma$ , avoiding dangling ends, of a fundamental tile  $F$  in  $\mathcal{T}$ . Then the produced tile  $T$  is a ribbon tile.*

*Proof.* The stabiliser subgroup  $H$  of  $T$  is generated by the deleted edges by Theorem 10. If  $H$  is finite, then by Theorem 11, it must have resulted from a classical gluing procedure as described in Section 3. This does not include deleting two non-adjacent edges from a tiling. Therefore,  $H$  must be infinite and contain a translation by Lemma 9.  $\square$

Of course this is not a complete characterisation of edge deletions that create ribbon tiles. For example, deleting two neighbouring edges from a fundamental domain tiling can result in either a disk-like or a ribbon tiling.

## 5 Enumeration and classification of crystallographic tilings

We are now in a position to enumerate equivariant equivalence classes of crystallographic tile-transitive tilings obtained via the generalised glue operation described in the previous section. The steps are as follows:

1. Select a symmetry group of interest, and construct its orbifold.
2. Enumerate the finitely many possible tile-1-transitive fundamental tilings with methods described in [31, 32] or [20], and represent these as graphs embedded in the orbifold.
3. Systematically delete subsets of edges from the embedded graphs as described in the section above to derive all tile-1-transitive non-fundamental tilings, both regular ones with bounded tiles and general ones with unbounded ribbon or branched-ribbon tiles.
4. Keep only one representative of any set of equivariantly equivalent tilings.

Steps 1–3 are illustrated for the wallpaper group  $pmg$  with Euclidean orbifold  $22^*$  in Figures 3–6, reproducing the corresponding results from [18, Table 6.5.1] with our methods.

The resulting list of tilings from the above enumeration procedure up until the last step will naturally contain equivariantly equivalent duplicates. For the tile-1-transitive cases where the orbifold topology and singular locus is not too complex, it is possible to determine the equivalence classes by eye because combinatorial and equivariant equivalence are the same. For a more systematic approach, it is desirable to have a computable invariant. As already stated, the D-symbol provides such an invariant for disk-like tilings. It is clear that by introducing edges, any tiling can be made into a disk-like tiling. On the other hand, there are many ways to introduce these extra edges. A proper treatment of how to find a canonical choice lies outside the scope of this paper. We only outline the idea here: starting with the graph  $G$  of tile edges on  $\mathcal{O}$ , we insert some number of coloured edges. The algorithm for enumerating the possible ways of doing this adapts that for fundamental domains as given in [32]. We compare the resulting disk-like tilings using an ordering for D-symbols described in [48, 9]. This ordering allows us to find a unique, minimally complex representative for the tiling that we call the *coloured D-symbol*. The

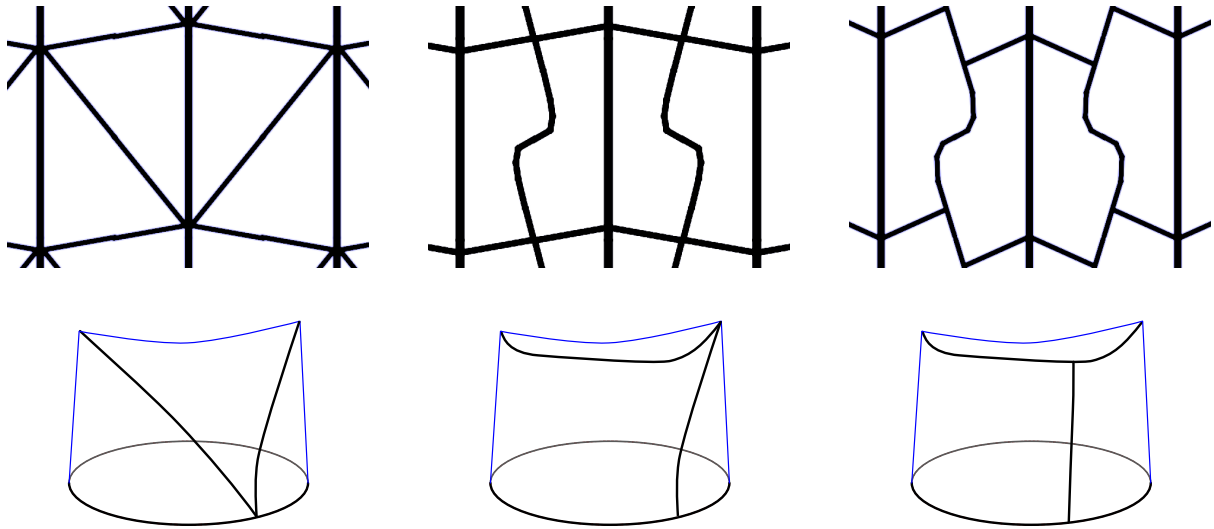


Figure 4: The other combinatorial types of fundamental domain for  $22^*$  depicted as tilings and their edge graphs below. From left to right, the tilings are equivariantly equivalent to triangular, quadrilateral, and pentagonal polygonal regions.

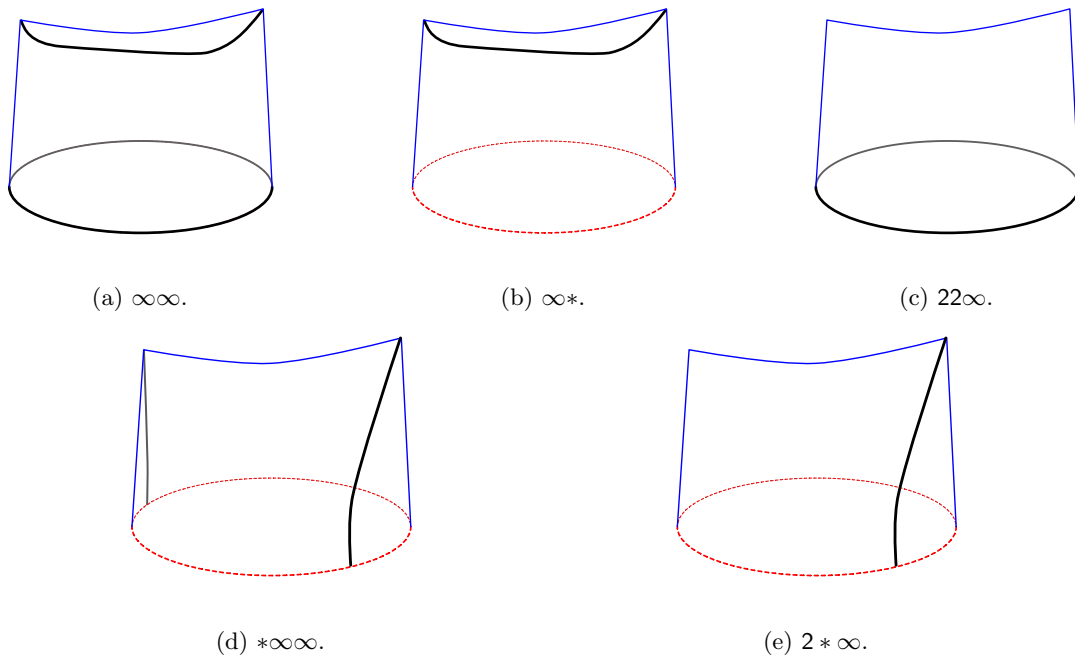


Figure 5: By deleting subsets of edges from the four fundamental domain edge graphs shown in Figures 3 and 4, we obtain five possible equivariant equivalence classes of non-fundamental tilings of  $22^*$ . The frieze group stabilisers are given by their quotient space symbol. There are also five possible non-fundamental disk-like tilings, not illustrated.

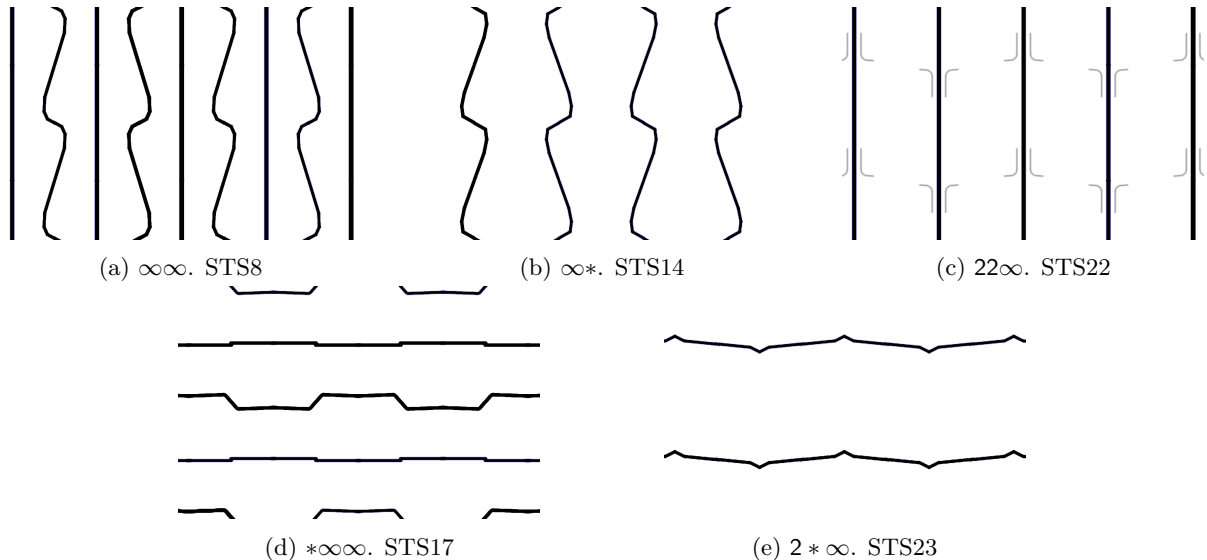


Figure 6: These ribbon tiling patterns correspond directly to the quotient graph diagrams in Figure 5. Each drawing shows 2 by 2 translational unit cells in the symmetry group  $22^*$ . The symmetry of the ribbon in (c) can only be geometrically depicted by marking the tiles, e.g. with an ‘L’ motif. The STSN labels refer to [18, Table 6.5.1].

isomorphism class of the coloured D-symbol completely determines the equivariant equivalence class of the ribbon tiling, when the D-symbol isomorphisms preserve the colouring of the edges (i.e. the original versus inserted ones).

**Remark 13.** The enumeration of non-fundamental tile-1-transitive tilings, in both the disk-like and more general tiling cases effectively constructs those subgroups that are generated by a subset of the parent group Wilkie generators for a fundamental tiling. The subgroups are the stabiliser groups of the non-fundamental tile while the parent group generators are dictated by the particular fundamental domain that we start with. We observe that some stabiliser subgroups are derivable from each fundamental domain (examples (c) and (e) in Figures 5, 6), but this does not hold in all cases. In fact, there are some non-fundamental tilings that arise from edge deletion in just one combinatorial class of fundamental domain (Figure 5d for example).

## 6 Hyperbolic tiling examples

In this section we look at hyperbolic examples illustrating the enumeration in Section 5 and the theory developed above for symmetry group  $2224$ . There are nine combinatorially distinct fundamental domain tilings with this symmetry, shown in Figure 7. More information about these tilings, coloured pictures and their D-symbols can be found in [38].

Non-fundamental disk-like tilings are built from these fundamental domains by deleting a single edge that terminates at a cone point. This yields five combinatorially distinct tilings, each with stabiliser group either  $2\bullet = C_2$  or  $4\bullet = C_4$ .

To construct the tile-1-transitive ribbon tilings, we need to delete at least two edges from one of the nine fundamental domains. After performing all possible combinations, we find that just three combinatorially distinct ribbon tilings are possible, shown in Figure 8. The example in Figure 8a can be generated by deleting suitable subsets of edges from six fundamental domains, namely those of Figures 7c and 7e–7i. The one in Figure 8b can be built from each of the nine fundamental domain tilings with multiple distinct edge deletions from some domains giving the same tile-class. The branched ribbon tiling in Figure 8c can be found in eight of the fundamental domain tilings; only the minimal triangular fundamental domain outlined in Figure 7a cannot support this branched ribbon. Figure 9 illustrates that this triangular fundamental domain supports only one type of ribbon tile, the one with stabiliser  $22\infty$  shown in Figure 8b.

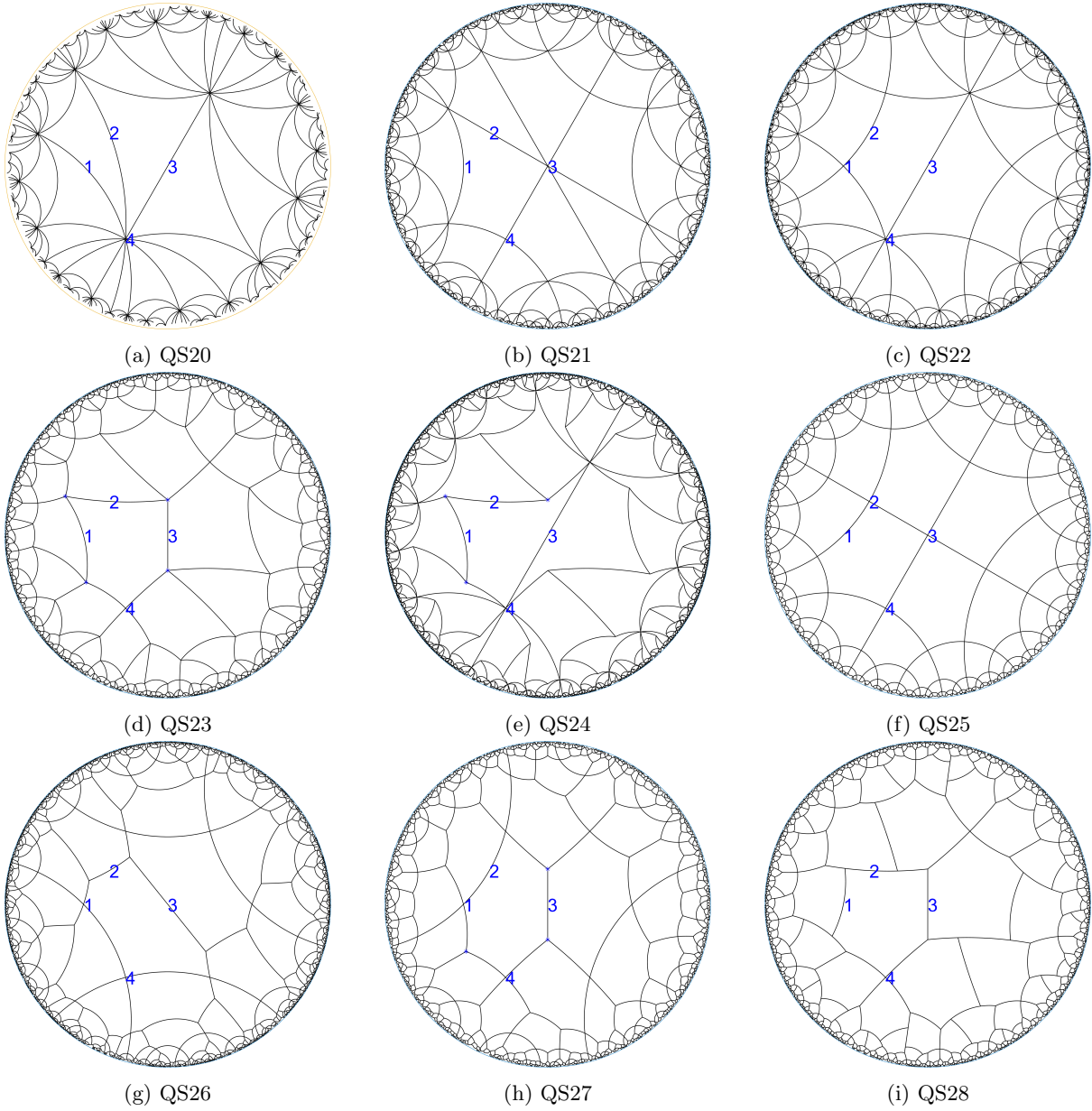
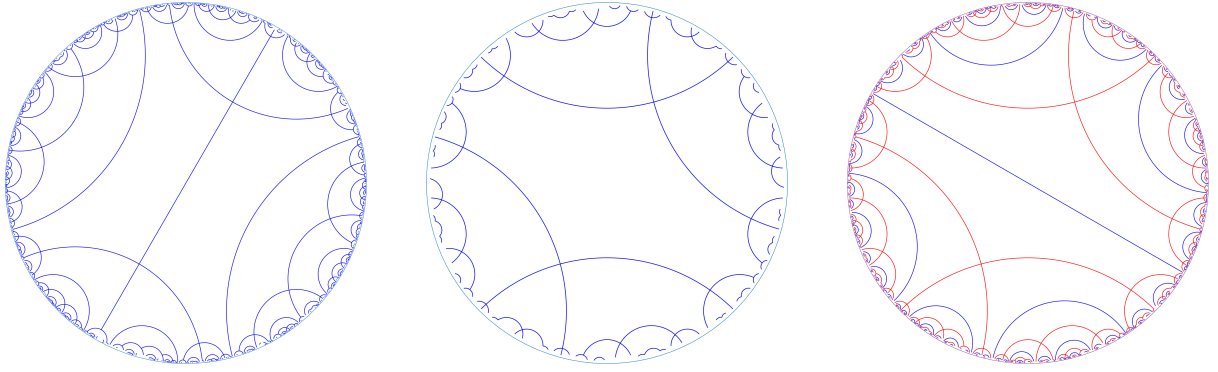


Figure 7: Fundamental tilings with symmetry group  $2224$  with the 4-fold rotation located at vertex 4 in each case. The naming  $QS_n$  is that used in the epinet database [38].



(a) Ribbon tiling with stabiliser group  $\infty\infty$ .

(b) Ribbon tiling with stabiliser group  $22\infty$ .

(c) Branched ribbon tiling with stabiliser group  $24\infty$ . The medial axis in red shows the branching structure.

Figure 8: The three distinct classes of ribbon tilings with symmetry group  $2224$  with blue edges.

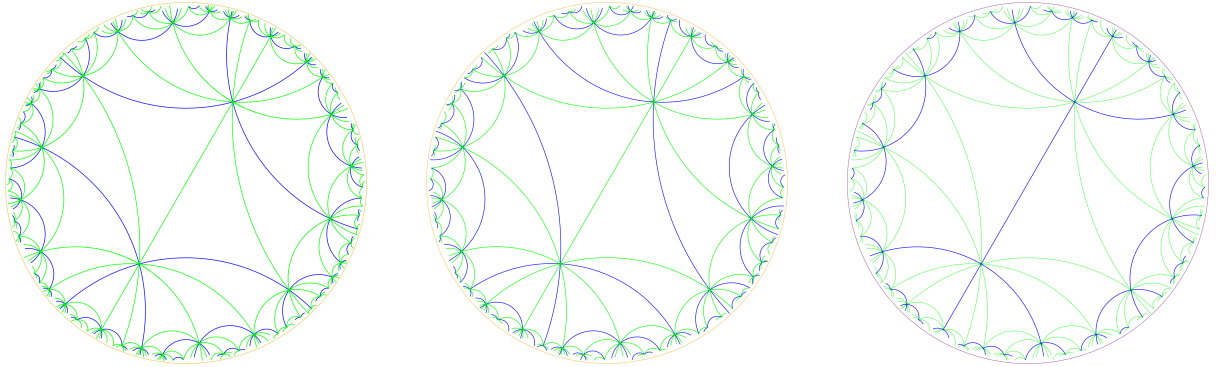


Figure 9: Deleting any pair of edges from the triangular fundamental domain of figure 7a leads to a ribbon tiling with stabiliser  $22\infty$ . Each of these three tilings is equivalent to that in figure 8b.

Unlike the Euclidean  $22^*$  example, all three classes of ribbon tilings in figure 8 can be created by erasing edges from single fundamental domain tiling; both the fundamental domains in Figures 7c and 7f.

Note that the combinatorial classification of fundamental domains and ribbon tilings is ultimately obtained by graphs embedded in an orbifold quotient space. This means symmetry groups with orbifold symbols of the same form have the same number of possible ribbon tilings. For example, any group of the form  $222a$ , for  $a > 2$  will have just as many equivariantly inequivalent fundamental domains and ribbon tilings as for  $2224$ .

## 7 Summary and Outlook

In this paper, we showed how to enumerate periodic, locally-finite tile-transitive tilings of the Euclidean or hyperbolic plane  $\mathcal{X}$ , by unbounded ribbon tiles. Returning to the initial inspiration of this work—the question of how to enumerate stripe patterns on the gyroid—this can now be achieved by finding the (branched) ribbon tilings in the symmetry groups compatible with the covering map that wraps the hyperbolic plane onto this periodic surface. Moreover, since we fix the symmetry group of the tilings under investigation, the methods of [28, 27] give a natural approach to using the theory developed here for enumerations and investigations into isotopy classes of ribbon tilings on non-simply connected surfaces.

The next step in this work is to enumerate tile- $k$ -transitive tilings to include tiles with an infinite stabiliser group. One of the main technical difficulties in extending our results to this case concerns the correct definition of the class of tilings under consideration. The standard split and glue methods to enumerate tile- $k$ -transitive D-symbols can lead to tiles for which  $\text{cl}(\text{int}(T)) \neq T$ , and tile-intersections

$T_1 \cap T_2$  that are not connected. While these technicalities are not insurmountable, they make the definition of the coloured D-symbol and proofs of results substantially more involved.

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## References

- [1] William Abikoff. *The real analytic theory of Teichmüller space*. Berlin, Springer-Verlag, 1980.
- [2] L. Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume I. *Math. Ann.*, 70:297–336, 1911.
- [3] L. Bieberbach. Über die Bewegungsgruppen der Euklidischen Räume II. *Canadian Journal of Mathematics*, 72:400–412, 1912.
- [4] Svend Bundgaard and Jakob Nielsen. On normal subgroups with finite index in  $f$ -groups. *Matematisk tidsskrift. B*, pages 56–58, 1951.
- [5] T. C. Chau. A note concerning fox’s paper on fenchel’s conjecture. *Proc. Amer. Math. Soc.*, 88:584–586, 1983.
- [6] J H Conway, H Burgiel, and C Goodman-Strauss. *The Symmetries of Things*. AK Peters Ltd., 2008.
- [7] John H. Conway and Daniel H. Huson. The orbifold notation for two-dimensional groups. *Structural Chemistry*, 13(3-4):247–257, 2002.
- [8] Liliana de Campo, Toen Castle, and Stephen T. Hyde. Optimal packings of three-arm star polyphiles: from tricontinuous to quasi-uniformly striped bicontinuous forms. *Interface Focus*, 7(4):20160130, 2017. URL: <https://royalsocietypublishing.org/doi/abs/10.1098/rsfs.2016.0130>, doi: 10.1098/rsfs.2016.0130.
- [9] Olaf Delgado-Friedrichs. Data structures and algorithms for tilings I. *Theoretical Computer Science*, 303(2-3):431–445, 2003. doi:10.1016/S0304-3975(02)00500-5.
- [10] Andreas W. M. Dress and Daniel Huson. On tilings of the plane. *Geometriae Dedicata*, 24(3):295–310, Dec 1987.
- [11] Andreas W.M Dress. Presentations of discrete groups, acting on simply connected manifolds, in terms of parametrized systems of Coxeter matrices — A systematic approach. *Advances in Mathematics*, 63(2):196 – 212, 1987.
- [12] Myfanwy E. Evans and Stephen T. Hyde. Periodic entanglement III: tangled degree-3 finite and layer net intergrowths from rare forests. *Acta Crystallographica Section A*, 71(6):599–611, Nov 2015. doi:10.1107/S2053273315014710.
- [13] Myfanwy E. Evans, Vanessa Robins, and Stephen T. Hyde. Periodic entanglement I: networks from hyperbolic reticulations. *Acta Crystallographica Section A*, 69(3):241–261, 2013. URL: <http://dx.doi.org/10.1107/S0108767313001670>, doi:10.1107/S0108767313001670.
- [14] Myfanwy E. Evans, Vanessa Robins, and Stephen T. Hyde. Periodic entanglement II: weavings from hyperbolic line patterns. *Acta Crystallographica Section A*, 69(3):262–275, 2013. URL: <http://dx.doi.org/10.1107/S0108767313001682>, doi:10.1107/S0108767313001682.
- [15] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2012. URL: <http://www.jstor.org/stable/j.ctt7rkjw>.

- [16] Ralph H. Fox. On fenchel’s conjecture about f-groups. *Matematisk Tidsskrift. B*, pages 61–65, 1952. URL: <http://www.jstor.org/stable/24530069>.
- [17] Thomas Friedrich and Ilka Agricola. *Elementary Geometry*. American Mathematical Society, 2008. doi:10.1038/066438b0.
- [18] Branko Grünbaum and G. C. Shephard. *Tilings and patterns*. Freeman and Co., 1987.
- [19] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.
- [20] Daniel H. Huson. The generation and classification of tile-k-transitive tilings of the Euclidean plane, the sphere and the hyperbolic plane. *Geometriae Dedicata*, 47(3):269–296, Sep 1993. doi:10.1007/BF01263661.
- [21] Daniel H. Huson. Tile-transitive partial tilings of the plane. *Contributions to Algebra and Geometry*, 34(1):87–118, 1993.
- [22] S. T. Hyde and C. Oguey. From 2D hyperbolic forests to 3D Euclidean entangled thickets. *European Physical Journal B*, 16(4):613–630, 2000. doi:10.1007/PL00011063.
- [23] Stephen T. Hyde, T. Landh, S. Lidin, B.W. Ninham, K. Larsson, and S. Andersson. *The Language of Shape*. Elsevier Science, 1996.
- [24] Sebastian C. Kapfer, Stephen T. Hyde, Klaus Mecke, Christoph H. Arns, and Gerd E. Schröder-Turk. Minimal surface scaffold designs for tissue engineering. *Biomaterials*, 32(29):6875 – 6882, 2011. URL: <http://www.sciencedirect.com/science/article/pii/S0142961211006776>, doi: <https://doi.org/10.1016/j.biomaterials.2011.06.012>.
- [25] Svetlana Katok. *Fuchsian Groups*. University of Chicago Press, 1992.
- [26] Jacob J K Kirkensgaard, Myfanwy E Evans, Liliana de Campo, and Stephen T Hyde. Hierarchical self-assembly of a striped gyroid formed by threaded chiral mesoscale networks. *Proceedings of the National Academy of Sciences of the United States of America*, 111(4):1271–6, 2014.
- [27] Benedikt Kolbe and Myfanwy Evans. Enumerating isotopy classes of tilings guided by the symmetry of triply-periodic minimal surfaces. 2020. In preparation.
- [28] Benedikt Kolbe and Myfanwy E. Evans. Isotopic tiling theory for hyperbolic surfaces. *Geometriae Dedicata*, 2020. arXiv:1808.00721, doi:10.1007/s10711-020-00554-2.
- [29] Benedikt Maximilian Kolbe. *Structures in three-dimensional Euclidean space from hyperbolic tilings*. Doctoral thesis, Technische Universität Berlin, Berlin, 2020. URL: <http://dx.doi.org/10.14279/depositonce-10476>, doi:10.14279/depositonce-10476.
- [30] Z. Lučić and E. Molnár. Combinatorial classification of fundamental domains of finite area for planar discontinuous isometry groups. *Archiv der Mathematik*, 54(5):511–520, 1990. doi:10.1007/BF01188679.
- [31] Z. Lučić and E. Molnár. Fundamental domains for planar discontinuous groups and uniform tilings. *Geometriae Dedicata*, 40(2):125–143, Nov 1991. doi:10.1007/BF00145910.
- [32] Zoran Lučić, Emil Molnár, and Nebojša Vasiljević. An algorithm for classification of fundamental polygons for a plane discontinuous group. In Marston D. E. Conder, Antoine Deza, and Asia Ivić Weiss, editors, *Discrete Geometry and Symmetry*, pages 257–278, Cham, 2018. Springer International Publishing.
- [33] A. M. Macbeath. Groups of Homeomorphisms of a Simply Connected Space. *The Annals of Mathematics*, 79(3):473, May 1964. doi:10.2307/1970405.
- [34] A. M. Macbeath. The classification of non-Euclidean plane crystallographic groups. *Canadian Journal of Mathematics*, 19:1192–1205, 1967. URL: <http://www.cms.math.ca/10.4153/CJM-1967-108-5>, doi:10.4153/CJM-1967-108-5.



- [35] Albert Marden. Isomorphisms between fuchsian groups. In *Advances in Complex Function Theory*, pages 56–78. Springer Verlag, 1976. URL: <http://link.springer.com/10.1007/BFb0081099>, doi: 10.1007/BFb0081099.
- [36] Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces*. The Johns Hopkins University Press, Baltimore, 2001.
- [37] Martin Cramer Pedersen and Stephen T. Hyde. Polyhedra and packings from hyperbolic honeycombs. *Proceedings of the National Academy of Sciences*, 115(27):6905–6910, July 2018. URL: <http://www.pnas.org/lookup/doi/10.1073/pnas.1720307115>, doi:10.1073/pnas.1720307115.
- [38] S.J. Ramsden, V. Robins, and S.T. Hyde. Euclidean patterns in non-Euclidean tilings. <http://epinet.anu.edu.au/>. Accessed: 2020-03-20.
- [39] J. Ratcliffe. *Foundations of Hyperbolic Manifolds*. Graduate Texts in Mathematics. Springer New York, 2006. URL: <https://books.google.com.au/books?id=FHCGU1zRZ4cC>.
- [40] V. Robins, S. J. Ramsden, and S. T. Hyde. A note on the two symmetry-preserving covering maps of the gyroid minimal surface. *European Physical Journal B*, 48(1):107–111, 2005.
- [41] J. Sadoc and J. Charvolin. Infinite periodic minimal surfaces and their crystallography in the hyperbolic plane. *Acta Crystallogr A*, 45(1):10–20, 1989.
- [42] Eran Sharon, Benoît Roman, Michael Marder, Gyu-Seung Shin, and Harry L. Swinney. Buckling cascades in free sheets. *Nature*, 419(6907):579–579, 2002. doi:10.1038/419579a.
- [43] J. Stilwell and J.P. Serre. *Trees*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002.
- [44] A. Szczepański. *Geometry of Crystallographic Groups*. Algebra and discrete mathematics. World Scientific, 2012.
- [45] William Thurston. *Geometry and Topology of Three-Manifolds*. Princeton lecture notes, 1980.
- [46] H C Wilkie. On non-Euclidean crystallographic groups. *Math. Zeitschr.*, 91:87–102, 1966.
- [47] Wolfgang Wollny. Dualität von Cayley-diagramm und fundamentalbereichsparkett bei den diskreten 1-dimensionalen bewegungsgruppen. *Geometriae Dedicata*, 11(1):31–46, Mar 1981. doi:10.1007/BF00183188.
- [48] Rüdiger Zeller, Friedrichs Olaf-Delgado, and Daniel Huson. Tegula – exploring a galaxy of two-dimensional periodic tilings. 2020. arXiv:2007.10625.

## A Proof of equivalence

This section is dedicated to proving to that the notion of combinatorial equivalence of Definition 7 is equivalent to equivariant equivalence from definition 4.

Our techniques can be used to show the equivalence of these two definitions for a more general class of tilings than the closed domain tilings of definition 2. We start by defining these.

**Definition 14.** A tiling  $\mathcal{T}$  is a locally finite, countable set  $\mathcal{T}$  of *connected domains*,  $T_i = \text{int}(T_i)$ , such that every point  $x \in \mathcal{X}$  belongs to the closure  $\text{cl}(\text{int}(T_i))$  of at least one tile, the  $\text{int}(T_i)$  are pairwise disjoint, and any bounded disk in  $\mathcal{X}$  intersects finitely many tiles, such that  $\mathcal{T}$  is invariant under a discrete group of symmetries  $\Gamma$  of  $\mathcal{X}$ , with compact quotient space  $\mathcal{X}/\Gamma$ . We define *edges* in this case as maximally connected segments of tile boundary that touches the same at most two tiles. A *vertex* is a component of the boundary of edges. Then the last requirement is that  $\mathcal{T}$  does not contain vertices of degree one.

The condition on the vertices means, in particular, that there are no edges protruding into a tile and no isolated points or stray edges in a tile that do not belong to it.

Note that the interior of a connected closed domain is a tile according to Definition 14. Note also that Definition 14 allows for the case of a tile that is topologically an open disk whose closure is an annulus.

**Lemma 15.** *Let  $(\mathcal{T}, \Gamma)$  be a disk-like equivariant tiling. Then the tile edges of  $\mathcal{T}$  project to a 2-cell embedding of a graph  $G$  in  $O = \mathcal{X}/\Gamma$ , i.e. an embedding of  $G$  s. t.  $O - G$  is a union of open disks containing at most one of the following a cone point, one corner point (and incident mirror segments), or part of a mirror segment of the singular locus.*

A loop in a space is called *contractible* if it is homotopic into a simply connected neighborhood containing at most one of the following of the singular locus: a cone point, one corner point (and incident mirror segments), or part of a mirror segment.

*Proof.* Assume there was a tiling by disks with 1-skeleton  $\tilde{G}$  in  $\mathcal{X}$  that projects to a graph  $G$  in  $O$  s. t.  $O - G$  has a component  $C$  that is not a disk. Let  $c$  be a noncontractible loop in  $C$  that we can assume avoids the singular locus of  $O$ , except possibly at isolated points of mirror boundary components not coinciding with cone points. Then  $c$  cannot be homotopic to a single point in  $O$  because  $G$  in  $O$  is connected. Thus,  $c$  is noncontractible in  $O$  and corresponds to a deck transformation a power of which is a translation (it may generally correspond to a glide reflection). To see this, consider a finite covering surface  $S$  of the orbifold  $O$  that is a closed surface, similarly to the previous section, and consider a maximal connected lift  $c_S$  of  $c$  to  $S$ . Then  $c_S$  is closed in  $S$  because otherwise it would be dense in some region of  $S$  by compactness, which would contradict the lifting property, as  $c$  is nowhere dense in  $O$ . Away from the singular locus of  $O$ , the covering  $S \rightarrow O$  is a usual covering, so  $c_S$  is noncontractible in  $S$  [19, Proposition 1.31]. Note that if  $c$  touches a mirror boundary component, then the condition on lifts of curves with nontrivial intersection with mirror boundaries ensures that homotopies of  $O$  lift to homotopies of  $S$  and therefore  $c_S$  is noncontractible. It is well-known that a noncontractible curve on the closed surface  $\pi_1(S)$  corresponds to a translation in  $\mathcal{X}$  [1, Lemma 1]. Denote by  $\tilde{c}$  a maximal lift of  $c$  to the universal cover  $\mathcal{X}$ . Now,  $\tilde{c}$  separates  $\mathcal{X}$  into two pieces and is entirely contained in a component of  $\mathcal{X} - \tilde{G}$ , which contradicts the assumption on the tiling in  $\mathcal{X}$ .  $\square$

A homeomorphism between orbifolds is a homeomorphism between the labelled quotient spaces that preserves the labels of each point. As mentioned previously, by the uniqueness of the homeomorphism class of a 2-orbifold described by a Conway symbol, the existence of such a homeomorphism is equivalent to the existence of an isomorphism of the symmetry groups. This is equivalent to the existence of an equivariant homeomorphism of the universal covering space mapping the symmetry groups to each other, as in definition 4 [34, 44, 2, 3].

As a tool to prove the equivalence of the two notions of equivalence for tilings, we will introduce a slightly generalised class of orbifolds, as is carried out in more detail in [29, Section 2.3]. This relies on well-known techniques in the field, relying on similar arguments found in [34]. We consider orbifolds that, in addition to the features discussed in Section 2, have non-mirror boundary components, i.e., they are the result of cutting an orbifold along a collection of embedded closed curves in the orbifolds quotient space. We impose a further technical requirement: We assume that the graph of curves embedded in the orbifolds quotient space lifts to a tiling with only simply connected tiles.

We shall call a component of an orbifold cut open along a set of such curves with empty intersection with the singular locus a *bordered orbifold*. We note that this restriction allows orbifolds with mirror boundaries. Examples of bordered orbifolds are hyperbolic bordered surfaces, Euclidean surfaces, and disks, possibly containing parts of the singular locus of the original orbifold in their interior. We emphasize that we will use bordered orbifolds solely as a tool to prove our results. The tilings in both Definition 14 and Definition 2 allow for non-empty intersections of the singular locus and the boundary of tiles. Bordered orbifolds also have associated symmetry groups corresponding to subgroups of the symmetry groups of the original orbifold, and have covering bordered orbifolds in the same way as orbifolds do. There is then a subspace of  $\mathcal{X}$  that corresponds to their universal covering space, a manifold with boundary components that cover the boundary components of the boundary of the bordered orbifold, along with deck transformation without fixed points associated to it. We call such transformations *boundary transformations*. We also define homeomorphisms of bordered orbifolds in the same way as for orbifolds, as projections of equivariant homeomorphisms of the universal covering space.

To distinguish between hyperbolic and Euclidean bordered orbifolds, one can use the orbifold Euler characteristic [6], with one technical addition: We allow boundary components as features of the orbifold. In the Euler characteristic, these get assigned a value of 1, similarly to the classical case for surfaces. The orbifold Euler characteristic for bordered orbifolds has the following property: Its value is equal to zero if and only if the symmetry group of the corresponding orbifold is Euclidean; it is negative if and

only if the symmetry group is hyperbolic. This means, for our purposes, that the universal covering space of each of these cases has a ‘natural’ interpretation as a subspace of  $\mathbb{E}^2$  or  $\mathbb{H}^2$ , respectively, with geodesic boundary. These facts can be verified easily for the orbifold characteristic for this case of bordered orbifolds because the characteristic is multiplicative under orbifold coverings [29, Section 2.3], so that a classical theorem [15, Theorem 1.2] (see [25] for a detailed treatment) for surfaces yields the result, by considering a finite covering that is a classical surface. Such a covering exists by a theorem of Fox [16, 4, 5] for the hyperbolic case and by the Bieberbach theorems [44, 2, 3] in the Euclidean case. The two cases are mutually exclusive. However, as discussed in Section 2, that having a ‘natural’ interpretation as the symmetry of a Euclidean bordered orbifold does not mean that we cannot find a tile with that symmetry group as a stabiliser in  $\mathbb{H}^2$ . This occurs for example when there are sufficiently many symmetries in the full orbifold that are not present in the stabiliser of the tile and implies that the boundary components of the tile cannot be geodesics in  $\mathbb{H}^2$ . On the other hand, a symmetry group of a hyperbolic bordered orbifold cannot act as a group of isometries on  $\mathbb{E}^2$ . This again follows from the fact that such a group is covered finitely by a hyperbolic closed surface with boundary components, for which the fundamental group gives rise to a non-Abelian group of translations, which does not exist in  $\mathbb{E}^2$ .

The next theorem is perhaps of independent interest, which is why we present it in the stated form, even though we need a slightly more general, but more technical statement in the proof of Theorem 18 below.

**Theorem 16.** *Let  $\Psi : \Gamma \rightarrow \Gamma'$  be an isomorphism of bordered hyperbolic orbifold groups with the property that when  $\Psi(b)$  is a boundary transformation,  $b$  is too. Then  $\Psi$  is realized geometrically, i.e., there is a homeomorphism  $h : \mathbb{H}^2 \rightarrow \mathbb{H}^2$  such that for all  $g \in \Gamma$ ,  $\Psi(g) = hgh^{-1}$ .*

*Proof.* First consider a finite index subgroup  $N$  of  $\Gamma$  that contains only orientation preserving elements, is torsion free, and normal in  $\Gamma$ . This is possible by a theorem of Fox [16, 4, 5]. Now,  $N$  and  $N' := \Psi(N)$  are both fundamental groups of orientable surfaces of finite topological type<sup>1</sup>, so by [35], there is a homeomorphism  $\tau$  of  $\mathbb{H}^2$  that induces the isomorphism  $\psi := \Psi|_N$  from  $N$  to  $N'$ . Define  $\varphi : N \ni n \mapsto \tau n \tau^{-1}$ . Then  $\psi \circ \varphi^{-1}$  is an automorphism of  $N$  that takes the transformations associated to boundaries to likewise transformations, respectively. Again by [35],  $\psi \circ \varphi^{-1}$  is realized geometrically, and thus also  $\psi = \psi \circ \varphi^{-1} \circ \varphi$  is. Now, in the nonempty set of all homeomorphisms  $t : \mathbb{H}^2 \rightarrow \mathbb{H}^2$  satisfying  $\psi(n) = tnt^{-1} \forall n \in N$ , by the results of [1], there is a unique extremal quasi-conformal homeomorphism  $h$ . Now define for arbitrary  $g \in \Gamma$

$$h' := \Psi(g)hg^{-1}.$$

Observe that  $h'$  has the same maximal dilation as  $h$ , since both  $g$  and  $\Psi(g)$  act as isometries of  $\mathbb{H}^2$ . For  $n \in N$  we obtain

$$h'nh'^{-1} = \Psi(g)h(g^{-1}ng)h^{-1}\Psi^{-1}(g) = \Psi(g)\Psi(g^{-1}ng)\Psi(g^{-1}) = \Psi(gg^{-1}ngg^{-1}) = \Psi(n), \quad (1)$$

where we used that  $N$  was normal in the second step. Thus, by the uniqueness of extremal quasi-conformal maps,  $h' = h$ , whence  $\Psi(g) = hgh^{-1}$  for all  $g \in \Gamma$ .  $\square$

**Remark 17.** Theorem 16 is not true for Euclidean bordered orbifolds. In this case, we need to consider the Frieze groups  $\infty\infty$ ,  $*\infty\infty$ ,  $\infty\bowtie$ , and  $22\infty$  from table 1, since we only consider the cases where there are no points of increased symmetry on the boundary. Both  $22\infty$  and  $*\infty\infty$  are not isomorphic to the other two,  $\infty\bowtie$  and  $\infty\infty$ . However, both of these pairs are isomorphic as groups, but topologically distinct. See also [6, Chapter 18].

Theorem 16 implies, in particular, that the universal covering space of a hyperbolic bordered orbifold is a simply connected totally geodesic subspace of  $\mathcal{X}$  that is uniquely determined up to topological equivalence in  $\mathcal{X}$ . Note that the proof of theorem 16 implies, in particular, that isomorphisms of hyperbolic orbifold symmetry groups are realized geometrically.

**Theorem 18.** *Combinatorial equivalence of equivariant tilings is equivalent to equivariant equivalence.*

<sup>1</sup>A surface is said to have finite topology if it is homeomorphic to a compact surface with finite genus with finitely many points removed.

*Proof.* The proof will rely to a large part on Theorem 16. A  $\Gamma_1$ - and a  $\Gamma_2$ -equivariant tiling  $\mathcal{T}_1$  and  $\mathcal{T}_2$  project to graphs  $G_1$  and  $G_2$  on  $\Gamma_1$ 's and  $\Gamma_2$ 's orbifold's underlying quotient space  $O_1$  and  $O_2$ , respectively. We will show that these are related by a homeomorphism of the orbifolds if and only if the tilings are combinatorially equivalent. The existence of such a homeomorphism of orbifolds is equivalent to the existence of an isomorphism of the associated symmetry groups, which is equivalent to the existence of a homeomorphism giving rise to an equivariant equivalence, as noted in Section 2. It is clear that a homeomorphism of orbifolds that maps  $G_1$  to  $G_2$  gives rise to combinatorial equivalent tilings of  $\mathcal{X}$  by simply lifting the graphs to obtain the tilings, so we only need to focus on proving that combinatorial equivalent tilings yield homeomorphisms of the orbifolds. Essentially, this comes down to proving that the isomorphisms of the stabiliser subgroups of the tiles, edges, and vertices mean that the isomorphism is induced by a topological map of the universal covering spaces, or, equivalently, that they induce an isomorphism of  $\Gamma_1$  and  $\Gamma_2$ . If the tiling were classical, i.e. by compact disks, then this statement is relatively straightforward, because then, by Lemma 15,  $O_i \setminus G_i$  ( $i = 1, 2$ ) is a collection of disks as specified in the Lemma. These disks are in a 1-to-1 correspondence that preserves the types of vertices on the boundaries of these disks. Since the action of the symmetry group of every tile uniquely determines an embedding into  $\mathcal{X}$  up to topological equivalence, this dictates which generators of the symmetry groups get mapped to each other, and therefore produces an isomorphism of the symmetry groups with a corresponding homeomorphism that realizes the combinatorial equivalence of these tilings.

The idea is the same in the general case, but instead of just compact disks, the tiles can have a more complicated geometry. As in the above case, because combinatorial equivalence preserves incidence relations, orientations, orders and the stabilisers of vertices, directed edges and tiles, it suffices to show that every tile that is not a compact disk has a realization in  $\mathcal{X}$  that is uniquely determined by the restrictions combinatorial equivalence places enforces. For empty tilings with no edges, this is a result of the equivalence of algebraic isomorphisms of symmetry groups and homeomorphisms of orbifolds already mentioned. Since edges with vertices of degree one are prohibited, the existence of an edge implies the existence of at least two tiles in  $\mathcal{T}_i$ . We assume for now that all tiles are simply connected. Then, if a tile's boundaries contains no nontrivial stabilisers, each tile gives rise to a bordered orbifold in the above sense. If the associated stabiliser groups are hyperbolic, then by theorem 16, the tile's shape in  $\mathbb{H}^2$  is uniquely determined (up to homeomorphism). If the symmetry group of the bordered orbifold in this case is Euclidean, then the combinatorial equivalence differentiates between isomorphic groups. For example, for the two groups  $\infty \times$  and  $\infty \infty$  of Remark 17, since  $\times$  corresponds to the existence of a glide transformation in the tile, the action of the group on the boundary of the tile differentiates the two arising situations.

In case the singular locus has non-empty intersection with the boundary of the tile  $T_i$  in  $\mathcal{T}_i$ , note first that the stabiliser subgroup  $H_i$  of  $T_i$  is either Euclidean or hyperbolic, since  $\partial T_i$  has at least two components. If  $H_i$  is Euclidean, then it has to be one of the Frieze groups in Table 1 and the action of  $H_i$  on the boundary  $\partial T_i$  again suffices to pin down the tiles  $T_i$  up to homeomorphisms. Further points of the singular locus of  $O_i$  on the boundary of  $T_i$  that do not give rise to elements of  $H_i$  can be treated by passing over to the subgroup  $S_i \subset \Gamma_i$  generated by  $H_i$  and the stabiliser subgroups of the vertices and edges on  $\partial T_i$ . Then  $S_i$  is either all of  $\Gamma_i$  and the combinatorial equivalence induces an isomorphism of  $\Gamma_i$ , in which case we are done, or  $S_i$  is hyperbolic, the case we treat below, or it is again a Frieze group, which concludes the case that  $H_i$  is Euclidean.

In case  $H_i$  is hyperbolic, we pass over to finite index, normal subgroups  $N_i$  of  $H_i$  that are torsion-free with the same index, similar to the discussion before the proof of Theorem 16. Since  $T_i$  normal, it is easy to see that the combinatorial equivalence of  $(\mathcal{T}_i, \Gamma_i)$  implies the combinatorial equivalence of  $(\mathcal{T}_i, N_i)$ . For  $(\mathcal{T}_i, N_i)$ , we can argue like in the case with no points of the singular locus on the boundary to deduce that they are equivariantly equivalent, from which the same construction as (1) yields equivariant equivalence of the  $(\mathcal{T}_i, H_i)$ .

In case there are further points of the singular locus of  $O_i$  on the boundary of  $T_i$  that do not give rise to elements of  $H_i$  we can similarly pass over to the subgroup  $S_i \subset \Gamma_i$  generated by the stabiliser subgroups  $H_i$  of the tiles  $T_i$  and the vertices and edges on  $\partial T_i$ . Now, either  $S_i = \Gamma_i$  and the combinatorial equivalence has produced an isomorphism of  $\Gamma_1$  and  $\Gamma_2$ , or it is again bordered hyperbolic orbifold, in which case we are in a case already discussed. This concludes the proof of the theorem in case all tiles are simply connected.

The last step of the proof is to consider the situation where a tile may not be simply connected. In such a case, by ignoring the set  $E_i$  of all edges in  $O_i$  that lift to disks in the interior of a noncontractible

curve in tile in  $\mathcal{X}$ , we can argue like above and then subsequently reinsert the removed edges  $E_i$ . Note for this that the removed boundary component any tile necessarily bounds a compact disk in  $\mathcal{X}$ , whose realization in  $\mathcal{X}$  is again, by virtue of combinatorial equivalence, uniquely determined up to equivariant equivalence.  $\square$

Theorem 18 allows equivariant equivalence classes of equivariant tilings to be equivalently viewed as topological equivalence classes of graphs on the labelled quotient space of the orbifold of the symmetry group of the tiling.