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# Global exponential stability and Input-to-State Stability of semilinear hyperbolic systems for the $L^2$ norm

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## Abstract

In this paper we study the global exponential stability in the  $L^2$  norm of semilinear 1- $d$  hyperbolic systems on a bounded domain, when the source term and the nonlinear boundary conditions are Lipschitz. We exhibit two sufficient stability conditions: an internal condition and a boundary condition. This result holds also when the source term is nonlocal. Finally, we show its robustness by extending it to global Input-to State Stability in the  $L^2$  norm with respect to both interior and boundary disturbances.

*Keywords:* Global stabilization, exponential stability, Lyapunov, hyperbolic systems; nonlinear, nonlocal, inhomogeneous

*2010 MSC:* 35F60, 35F61, 93D09, 93D15, 93D20, 93D30

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## 1. Introduction

Hyperbolic systems can be found everywhere in sciences and nature: from biology [1], to fluid mechanics, population dynamics [2], electromagnetism, networks [3, 4, 5] etc. For this reason, they are of large importance for practical applications and the question of their stability and stabilization is paramount. For linear 1- $d$  systems, studying the exponential stability or the stabilization can

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7 be achieved by looking at the eigenvalues and using spectral mapping theorems  
 8 [6, 7]. For nonlinear systems, the situation is much more tricky. For nonlinear  
 9 systems, the situation is much more tricky. In general the stabilities in different  
 10 norms are not equivalent [8]. Indeed, for the same system, stabilities in different  
 11 norms can require different criteria. For semilinear systems the spectral tools  
 12 may still work (in contrast with quasilinear systems), but the resulting expo-  
 13 nential stability may only hold locally, meaning for small enough perturbations.  
 14 Worse, most of the time spectral tools are hard to use when the system is inho-  
 15 mogeneous. Several tools were developed to deal with this situation and obtain  
 16 local exponential stability results. A first method is the characteristic analysis,  
 17 which was originally used by Li and Greenberg in 1984 in [9] then generalized in  
 18 [10, 11, 12, 13] for quasilinear homogeneous hyperbolic systems in the  $C^1$  norm.  
 19 A second method is the use of basic Lyapunov functions<sup>1</sup>. This method was,  
 20 for instance, applied in [16, Chapter 6] for general semilinear systems in the  
 21  $H^1$  norm and quasilinear systems in the  $H^2$  norm, but also in many particular  
 22 cases [17, 18, 15, 19, 20]. This will be our approach in this article. A third  
 23 method is the backstepping method, a very powerful tool originally designed  
 24 for finite-dimensional systems, modified for PDEs using a Volterra transform  
 25 in [21], <sup>2</sup> and then used in [23, 24] for quasilinear hyperbolic systems in the  
 26  $H^2$  norm. Such backstepping approach was also used to derive controllability  
 27 [25, 26] or finite-time stabilization [27, 28, 29] in both parabolic and hyperbolic  
 28 settings. Other results using a more general transform were then introduced  
 29 [30, 31]. The main drawback of this method is that it involves controls that  
 30 are usually using full-state measurements and cannot take the simple form of  
 31 output feedback controllers (see (3)). Therefore these controls might be less  
 32 convenient for practical implementation. Although sometimes observers can be  
 33 designed to tackle this issue [32]. Other methods exist, as for instance the study  
 34 of stability based on time delay systems introduced in [8] where the authors give

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<sup>1</sup>see [14, Definition 1.4.3] for a proper definition and [15] for an overview of this method

<sup>2</sup>see [22] for more details

35 criteria for exponential stability in the  $W^{2,p}$  norms for any  $p \geq 1$  (see also [33]).

36

37     So far, the nonlinear stability results for hyperbolic systems have been ob-  
38     tained in the  $H^1$  norm for semilinear systems and for the  $H^2$  norm for quasilinear  
39     systems. The  $H^1$  and  $H^2$  norm enabling to bound the nonlinear terms of the  
40     source term and of the transport term respectively, using the Sobolev embed-  
41     dings  $H^p([0, L]; \mathbb{R}) \subset C^{p-1}([0, L]; \mathbb{R})$ , for  $p \geq 1$ . Other results have been shown  
42     for the  $C^0$  and  $C^1$  norm [34, 35]. For weaker norms, such as the  $L^2$  norm, one  
43     is usually unable to derive any exponential stability result when the system is  
44     nonlinear.     However, in this paper we show that having a Lipschitz source  
45     term, with some condition on the size of the source, is enough to obtain the ex-  
46     ponential stability in the  $L^2$  norm for semilinear systems. Besides, in contrast  
47     with most of the previous analyses cited above, this result holds for a nonlocal  
48     source term. Nonlocal source terms are found in many important phenomena  
49     as population dynamics, material sciences, flocking, traffic flow [2, 36, 37], and  
50     open the door to many potential applications. Moreover, while all the above  
51     previous approaches were dealing with local exponential stability, we obtain  
52     here global exponential stability. Concerning semilinear systems with Lipschitz  
53     source terms, one should highlight the work of [38] where the authors study the  
54     exponential stability in  $C^0$  norm of a semilinear system with a diagonal and  
55     Lipschitz source term, and saturating boundary conditions. They give a poten-  
56     tially large explicit bound on the basin of attraction, and they prove in addition  
57     the well-posedness in  $L^2$ .

58

59     Finally, we show that these results can be extended to a wider notion: the  
60     Input-to-State Stability (ISS). The ISS measures the resilience of the stability of  
61     a system when adding disturbances in the boundary conditions or in the source  
62     term [39, 40]. These disturbances could have many origins such as actuator  
63     errors, quantized measurements, uncertainties of model parameters, etc. The  
64     ISS is therefore a more relevant notion from an application perspective, and is  
65     also paramount for designing observers. While exponential stability of nonlinear

hyperbolic systems has been studied for several decades now, fewer results are known concerning this wider notion of ISS. Until recently, the most up-to-date results were given in [40, Part II], for  $L^p$  norms,  $p \in \mathbb{N}^* \cup \{+\infty\}$  (see also [41] for instance for nonautonomous systems), and recently several works have been providing quite good conditions by extending exponential stabilization results obtained through Lyapunov approach to ISS results under the same conditions [42, 43, 44]. These results suffer however the same limitations as the exponential stabilization results they are generalizing: local validity and strong norms. One can also refer to [45, 46, 47, 48] for other ISS results on hyperbolic systems in particular cases, and to [40] for a more detailed review on ISS results for PDEs in general. This paper is organized as follows: in Section 2 we state some definitions and our main result, which is proven in Section 3 using a Lyapunov approach. The well-posedness and the extension to ISS are dealt with in the Appendix.

## 2. Statement of the problem and main results

A semilinear hyperbolic system can always be written in the following way [49]:

$$\partial_t \mathbf{u} + \Lambda(x) \partial_x \mathbf{u} + B(\mathbf{u}, x) = 0, \quad (1)$$

where  $\mathbf{u}(t, x) \in \mathbb{R}^n$ ,  $\Lambda(x)$  is a diagonal matrix with non vanishing eigenvalues,  $\Lambda : x \rightarrow \Lambda(x)$  belongs to  $C^1([0, L])$  and  $B \in C^0(L^2(0, L) \times [0, L], L^2(0, L))$  is the nonlinear source term, with  $B(\mathbf{0}, x) = 0$ . Note that  $B$  could be potentially non-local at it takes a function as argument, thus  $B(\mathbf{u}, x)$  refers here to  $B(\mathbf{u}(t, \cdot), x)$ . Throrough the article we will assume that  $B(\cdot, x)$  is Lipschitz with respect to  $\mathbf{u}$  with a Lipschitz constant  $C_B$  in the following sense: for  $\mathbf{u}$  and  $\mathbf{v}$  two functions of  $L^2(0, L)$ ,

$$\|B(\mathbf{u}, \cdot) - B(\mathbf{v}, \cdot)\|_{L^2} \leq C_B \|\mathbf{u} - \mathbf{v}\|_{L^2}. \quad (2)$$

Of course, this assumption is satisfied if  $B$  is local, takes argument in  $\mathbb{R}^n \times [0, L]$  and is Lipschitz with respect to the first argument, with a Lipschitz constant that might depend on  $x$  but as a  $L^2$  function. We will come back to this special

case later on in Remark 2.4. When the system is equipped with a control static and exerted at the boundaries, the boundary conditions can be written in the following way:

$$\begin{pmatrix} \mathbf{u}_+(t, 0) \\ \mathbf{u}_-(t, L) \end{pmatrix} = G \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix}, \quad (3)$$

where  $G$  is a continuous and Lipschitz function such that  $G(0) = 0$ . The notation  $\mathbf{u}_+$  is used to refer to the components of  $\mathbf{u}$  corresponding to positive propagation speeds  $\Lambda_i > 0$ , whereas the notation  $\mathbf{u}_-$  is used to refer to the components corresponding to negative propagation speeds. In the following, we assume without loss of generality that  $\Lambda_i > 0$  for  $i \in \{1, \dots, m\}$  and  $\Lambda_i < 0$  for  $i \in \{m+1, \dots, n\}$ . Note that the boundary conditions (3) are nonlinear. As  $G$  is Lipschitz, all of its components are Lipschitz, which implies that there exists a matrix  $K$  such that for any  $i \in \{1, \dots, n\}$ ,

$$\left| G_i \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \right| \leq \sum_{j=1}^m K_{ij} |u_j(t, L)| + \sum_{j=m+1}^n K_{ij} |u_j(t, 0)|. \quad (4)$$

*Remark 2.1* (Choice of  $K$ ). Of course the matrix  $K = C_G I$ , where  $I$  is the identity matrix and  $C_G$  the Lipschitz constant of  $G$  would work. However, there might be other matrices  $K$  satisfying (4) and some could lead to potentially less restrictive conditions in Theorem 2.2 than the matrix  $C_G I$  (see (11) below).

System (1), (3) with (2), (4) is well posed in  $L^2$  in the following sense:

**Theorem 2.1** (Well posedness). *For any  $T > 0$  and any  $\mathbf{u}_0 \in L^2(0, L)$  the Cauchy problem (1)–(3), with initial condition  $\mathbf{u}(0, \cdot) = \mathbf{u}_0$  has a unique solution  $\mathbf{u} \in C^0([0, T], L^2(0, L))$ . Moreover,*

$$\|\mathbf{u}(t, \cdot)\|_{L^2} \leq C(T) \|\mathbf{u}_0\|_{L^2}, \quad \forall t \in [0, T], \quad (5)$$

where  $C(T)$  is a constant depending only on  $T$ .

This theorem is shown in the Appendix. Most of the proof is a subcase of a remarkable result in [38, Theorem A.1], where the authors study the framework of saturating boundary conditions. The only differences are some slight changes

in the estimates to deal with a nonlocal functional and a density argument. These changes are indicated in Appendix A, together with a proper definition of the notion of weak solution to System (1), (3).

*Remark 2.2.* As it could be expected, the well posedness also holds for more regular solutions. In particular for any  $\mathbf{u}_0 \in H^1(0, L)$  satisfying the compatibility conditions given by (3), the Cauchy problem (1), (3) with initial condition  $\mathbf{u}(0, \cdot) = \mathbf{u}_0$  has a unique solution  $\mathbf{u} \in C^0([0, T], H^1(0, L)) \cap C^1([0, T], L^2(0, L))$ . This is also shown in Appendix A.

Before stating our main result, we recall the definition of exponential stability for the  $L^2$  norm.

**Definition 2.1** (Exponential stability). We say that System (1)–(3) is exponentially stable for the  $L^2$  norm with decay rate  $\gamma$  and gain  $C$  if there exists constants  $\delta > 0$ ,  $\gamma > 0$ , and  $C > 0$  such that for any  $T > 0$  and  $\mathbf{u}_0 \in L^2(0, L)$  such that  $\|\mathbf{u}_0\|_{L^2} \leq \delta$ , the Cauchy problem (1)–(3) with initial condition  $\mathbf{u}(0, \cdot) = \mathbf{u}_0$  has a unique solution  $\mathbf{u} \in C^0([0, T], L^2(0, L))$  and

$$\|\mathbf{u}(t, \cdot)\|_{L^2} \leq Ce^{-\gamma t} \|\mathbf{u}_0\|_{L^2}. \quad (6)$$

Moreover, if

$$\delta = +\infty, \quad (7)$$

then the system is said globally exponentially stable.

We can now state our main result.

**Theorem 2.2.** *Let a system be of the form (1), (3), where  $\Lambda \in C^1([0, L])$  and  $B$  is Lipschitz with respect to  $\mathbf{u}$ . If there exist  $K \in M_n(\mathbb{R})$  satisfying (4),  $J \in C^1([0, L]; M_n(\mathbb{R}))$  where  $J(x)$  is a diagonal matrix with positive coefficients, and  $M \in C^0([0, L]; M_n(\mathbb{R}))$ , such that the following conditions are satisfied*

1. (Interior condition)

$$-(J^2 \Lambda)' + J^2 M + M^\top J^2 \quad (8)$$

139 is positive definite and there exists  $D \in C^1([0, L]; M_n(\mathbb{R}))$  where  $D(x)$  is  
 140 a diagonal matrix with positive coefficients, such that

$$C_g < \frac{\lambda_m}{2 \max_{i,x}(D_i) \max_{i,x}(D_i J_i^2)}, \quad (9)$$

141 where  $C_g$  is the Lipschitz constant of  $g := B - M$  and  $\lambda_m$  denotes the  
 142 smallest eigenvalue of

$$-D(J^2 \Lambda)'D + DJ^2MD + DM^\top J^2D, \quad (10)$$

143

144 2. (Boundary condition) the matrix

$$\begin{pmatrix} J_+^2(L)\Lambda_+(L) & 0 \\ 0 & J_-^2(0)|\Lambda_-(0)| \end{pmatrix} - K^\top \begin{pmatrix} J_+^2(0)\Lambda_+(0) & 0 \\ 0 & J_-^2(L)|\Lambda_-(L)| \end{pmatrix} K \quad (11)$$

145 is positive semidefinite,

146 then the system is globally exponentially stable for the  $L^2$  norm. Moreover the  
 147 gain is  $\|J^{-1}\|_{L^\infty} \|J\|_{L^\infty}$  and an admissible decay rate is  $\lambda_m(2 \max_{i,x}(D_i J_i^2))^{-1} -$   
 148  $C_g \max_{i,x}(D_i)$

149 We prove this theorem in Section 3. Note that (9) does not involve directly  
 150 the Lipschitz constant of  $B$  but the Lipschitz constant of  $g = B - M$ , which is  
 151  $B$  minus a linear part that can be chosen. Of course, the Lipschitz constant  
 152 of  $B$  would be suitable by setting  $M = 0$ , but other choices of  $M$  could lead  
 153 to less restrictive conditions. Let us note that the apparent complexity of the  
 154 interior condition aims at giving a good explicit computable bound on  $C_g$  for  
 155 practical applications: indeed finding the values of  $\lambda_m$  can be numerically solved.  
 156 Besides, choosing  $D = Id$  or  $K = C_G I$  would also give a sufficient condition  
 157 that is simpler to write, but the sufficient condition would be more restrictive.

158 *Remark 2.3* (Linear case). When  $B$  is a local and linear operator we recover  
 159 the result found in [16, Proposition 5.1] (see also [50] when  $B$  is in addition



marginally diagonally stable). Indeed, we can choose  $M = B$ , then  $g = 0$  and the interior condition is reduced to the existence of  $J$ , diagonal matrix with positive coefficients such that  $-(\Lambda J^2)' + J^2 M + M^\top J^2$  is positive definite.

*Remark 2.4* (Local case). In the special case where the system is local, i.e.  $B$  is a function on  $\mathbb{R} \times [0, L]$  and  $B(\mathbf{u}, x) = B(\mathbf{u}(t, x), x)$ , the condition (9) of the previous theorem can be slightly improved as follows: assume that  $B$  is Lipschitz with respect to the first variable with a Lipschitz constant  $C(x) \in L^2(0, L)$ , then for any matrix  $M$ ,  $g = B - M$  is also Lipschitz with respect to the first variable and we can denote again its Lipschitz constant by  $C_g(x) \in L^2(0, L)$ . Then, the interior condition (9) in Theorem 2.2 can be replaced by

$$C_g < \frac{\lambda_m(x)}{\max_i(J_i^2)(x)} \text{ or } C_g < \mu_m(x) \frac{\max_i(J_i)(x)}{\inf_i(J_i)(x)}, \quad (12)$$

where  $\lambda_m(x)$  and  $\mu_m(x)$  are the smallest eigenvalues at a given  $x$  of the matrix given by (8) and (10) respectively.

### 2.1. Input-to-State Stability

In fact, this result can be extended to a more general notion: the Input-to-State Stability (ISS). This notion is more relevant when looking at practical implications as it takes into account the external disturbances that can arise. When such disturbance arise, System (1), (3) is replaced by

$$\begin{aligned} \partial_t \mathbf{u} + \Lambda(x) \partial_x \mathbf{u} + B(\mathbf{u}, x) + \mathbf{d}_1(t, x) &= 0, \\ \begin{pmatrix} \mathbf{u}_+(t, 0) \\ \mathbf{u}_-(t, L) \end{pmatrix} &= G \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} + \mathbf{d}_2(t), \end{aligned} \quad (13)$$

where  $\mathbf{d}_1$  and  $\mathbf{d}_2$  are respectively the distributed and boundary disturbances. We define the ISS as follows:

**Definition 2.2** (Input-to-State Stability). We say that System (13) is strongly Input-to-State stable (or ISS) with fading memory for the  $L^2$  norm if there exists positive constants  $\delta > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$ ,  $\gamma > 0$ , such that for any  $T > 0$  and any  $\mathbf{u}_0 \in L^2(0, L)$  with  $\|\mathbf{u}_0\|_{L^2} \leq \delta$  and  $\|\mathbf{d}_1\|_{L^2} + \|\mathbf{d}_2\|_{L^2} \leq \delta$ , there exists

183 a unique solution  $\mathbf{u} \in C^0([0, T], L^2([0, L]))$  to System (1), (3), and

$$\begin{aligned} \|\mathbf{u}(t, \cdot)\|_{L^2} &\leq C_1 e^{-\gamma t} \|\mathbf{u}_0\|_{L^2} \\ &+ C_2 \left( \|e^{-\gamma(t-s)} \mathbf{d}_1(s, x)\|_{L^2((0,t) \times (0,L))} \right. \\ &\left. + \|e^{-\gamma(t-s)} \mathbf{d}_2(s)\|_{L^2(0,t)} \right), \text{ for any } t \in [0, T]. \end{aligned} \quad (14)$$

184 Moreover, if  $\delta = +\infty$ , then the system is said to be globally strongly ISS with  
185 fading memory.

186 This defines a strong notion of ISS with an exponentially fading memory.  
187 The fading memory comes from the  $e^{-\gamma(t-s)}$  in the  $L^2$  norms of  $\mathbf{d}_1$  and  $\mathbf{d}_2$ .  
188 It means that the influence of the disturbances at a given time  $s$  decreases  
189 exponentially with time. One could have chosen other and less restrictive fading  
190 factors (see [40, Chapter 7] for a more complete description of ISS estimates  
191 with fading memory). The constants  $C_1$  and  $C_2$  are called the gains of the ISS  
192 estimate. When such notion of ISS cannot be achieved, weaker notions exist  
193 and can be found for instance in [51]. We have the following result, analogous  
194 to Theorem 2.2

195 **Theorem 2.3.** *Let a system be of the form (13) where  $\Lambda \in C^1([0, L])$ ,  $\mathbf{d}_1 \in$   
196  $L^2((0, T) \times (0, L))$ ,  $\mathbf{d}_2 \in H^1([0, T])$  and  $B$  is Lipschitz with respect to  $\mathbf{u}$ . If the  
197 condition (9) is satisfied and the matrix defined by (11) is positive definite, then  
198 the system is globally strongly ISS with fading memory for the  $L^2$  norm.*

199 The proof of this theorem is very similar to the proof of Theorem 2.2. The  
200 only difference being that the assumption on (11) has to be slightly stronger  
201 than in Theorem 2.2 (positive definite instead of positive semidefinite). A way  
202 to adapt the proof of Theorem 2.2 is given in 4. Besides, the gains can again be  
203 computed explicitly as a function of  $K$ ,  $B$  and  $\Lambda$  (see (38)).

### 204 3. Exponential stability in the $L^2$ norm

205 *Proof of Theorem 2.2.* Let a semilinear system be of the form (1), (3) with  
206  $\Lambda \in C^1([0, L], M_n(\mathbb{R}))$  and  $B$  being  $L^2$  with respect to  $\mathbf{u}$  with Lipschitz con-

stant  $C_B$ . We will first show Theorem 2.2 for  $H^1$  solutions and then re-  
cover it for  $L^2$  solutions using a density argument. Let  $T > 0$ , and let  $\mathbf{u}_0 \in$   
 $H^1(0, L)$ . From Theorem 2.1 and Remark 2.2, there exists a unique solution  
 $\mathbf{u} \in C^0([0, T], H^1(0, L)) \cap C^1([0, T], L^2(0, L))$  associated to this initial condition.  
Let us now define the following Lyapunov function candidate:

$$V(\mathbf{u}) = \int_0^L (J(x)\mathbf{u}(t, x))^T J(x)\mathbf{u}(t, x) dx, \quad (15)$$

where  $J = \text{diag}(J_1, \dots, J_n) \in C^1([0, L], \mathcal{D}_n^+(\mathbb{R}^n))$ , where  $\mathcal{D}_n^+$  is the space of  
diagonal matrices with positive coefficients. The function  $V$  is well defined on  
 $L^2(0, L)$  and equivalent to  $\|\mathbf{u}(t, \cdot)\|_{L^2}^2$ , as

$$\|\mathbf{u}(t, \cdot)\|_{L^2}^2 \|J^{-1}\|_{L^\infty}^{-2} \leq V(\mathbf{u}) \leq \|J\|_{L^\infty}^2 \|\mathbf{u}(t, \cdot)\|_{L^2}^2. \quad (16)$$

We would like to show that  $V$  decreases exponentially quickly along  $\mathbf{u}$ . Before  
going any further, let us comment on the choice of the form of this Lyapunov  
function candidate. Functions of this type are sometimes called *basic quadratic*  
*Lyapunov function* or *basic Lyapunov function for the  $L^2$  norm* because they  
can be seen as the simplest functional equivalent of the  $L^2$  norm. A commonly  
used Lyapunov function candidate for hyperbolic systems of conservation laws  
has the form (15) with  $J(x) = \text{diag}(q_i e^{-\mu s_i x})$  where  $s_i = 1$  if  $\Lambda_i > 0$  and  
 $s_i = -1$  if  $\Lambda_i < 0$  and  $q_i$  and  $\mu$  are positive constants to be chosen. In our  
case however, such function might not work. This is due to the inhomogeneity  
and this a phenomena that can be seen in balance laws in general [52]. For  
instance, in [15] is found a basic quadratic Lyapunov function that exists for  
any length  $L > 0$  provided good boundary conditions, while this could not  
happen with a basic quadratic Lyapunov function made of exponential weights.  
As  $\mathbf{u} \in C^1([0, T], L^2(0, L))$ ,  $V(\mathbf{u}(t, \cdot))$  can be differentiated with time, and we

229 have

$$\begin{aligned}
\frac{dV(\mathbf{u}(t, \cdot))}{dt} &= \int_0^L 2\mathbf{u}^\top J^2 \partial_t \mathbf{u} dx \\
&= - \int_0^L 2\mathbf{u}^\top J^2 \Lambda \partial_x \mathbf{u} dx - 2 \int_0^L \mathbf{u}^\top J^2 B(\mathbf{u}, x) dx \\
&= - [\mathbf{u}^\top J^2 \Lambda \mathbf{u}]_0^L + \int_0^L \mathbf{u}^\top (J^2 \Lambda)' \mathbf{u} dx \\
&\quad - 2 \int_0^L \mathbf{u}^\top J^2 B(\mathbf{u}, x) dx.
\end{aligned} \tag{17}$$

230 We used here that  $J$  and  $\Lambda$  commute as they are both diagonal. Now, let  $M \in$   
231  $C^0([0, L], M_n(\mathbb{R}))$  to be selected later on and set  $g(\mathbf{u}, x) = B(\mathbf{u}, x) - M(x)\mathbf{u}(t, x)$   
232 which is again Lipschitz in  $\mathbf{u}$  in the sense of (2). We have

$$\begin{aligned}
\frac{dV(\mathbf{u}(t, \cdot))}{dt} &= - [\mathbf{u}^\top J^2 \Lambda \mathbf{u}]_0^L + \int_0^L \mathbf{u}^\top (J^2 \Lambda)' \mathbf{u} dx \\
&\quad - 2 \int_0^L \mathbf{u}^\top J^2 M \mathbf{u} dx - 2 \int_0^L \mathbf{u}^\top J^2 g(\mathbf{u}, x) dx \\
&= - [\mathbf{u}^\top J^2 \Lambda \mathbf{u}]_0^L \\
&\quad - \int_0^L \mathbf{u}^\top [-(J^2 \Lambda)' + J^2 M + M^\top J^2] \mathbf{u} dx \\
&\quad - 2 \int_0^L \mathbf{u}^\top J^2 g(\mathbf{u}, x) dx
\end{aligned} \tag{18}$$

233 where we used that  $\mathbf{u}^\top J^2 M \mathbf{u} = \mathbf{u}^\top M^\top J^2 \mathbf{u}$ , as it is a scalar. Now, we set

$$\begin{aligned}
I_2 &:= [\mathbf{u}^\top J^2 \Lambda \mathbf{u}]_0^L, \\
I_3 &:= \int_0^L \mathbf{u}^\top [-(J^2 \Lambda)' + J^2 M + M^\top J^2] \mathbf{u} dx \\
&\quad + 2 \int_0^L \mathbf{u}^\top J^2 g(\mathbf{u}, x) dx
\end{aligned} \tag{19}$$

234 We would like to show that under assumptions 1. and 2. of Theorem 2.2,  $I_2$  is  
235 a nonnegative definite quadratic form with respect to the boundary conditions,  
236 and  $I_3 \geq \mu \|\mathbf{u}\|_{L^2}$  where  $\mu$  is a positive constant. We will show that this is

237 exactly the point of Assumptions 1. and 2.. Let us start with  $I_2$ . From (3),

$$\begin{aligned}
I_2 &= \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, L) \end{pmatrix}^\top J^2(L) \Lambda(L) \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, L) \end{pmatrix} \\
&\quad - \begin{pmatrix} \mathbf{u}_+(t, 0) \\ \mathbf{u}_-(t, 0) \end{pmatrix}^\top J^2(0) \Lambda(0) \begin{pmatrix} \mathbf{u}_+(t, 0) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \\
&= \sum_{i=1}^m J_i^2(L) \Lambda_i(L) u_i^2(L) - \sum_{i=m+1}^n J_i^2(0) \Lambda_i(0) u_i(0)^2 \\
&\quad + \sum_{i=m+1}^n J_i^2(L) \Lambda_i(L) \left( G_i \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \right)^2 \\
&\quad - \sum_{i=1}^m J_i^2(0) \Lambda_i(0) \left( G_i \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \right)^2.
\end{aligned} \tag{20}$$

238 We set  $x_i := 0$  if  $i \in \{1, \dots, m\}$  and  $x_i := L$  if  $i \in \{m+1, \dots, n\}$ . Then using  
239 that  $\Lambda_i > 0$  for  $i \in \{1, \dots, m\}$  and  $\Lambda_i < 0$  otherwise, and using (4),

$$\begin{aligned}
I_2 &= \sum_{i=1}^n J_i^2(L - x_i) |\Lambda_i(L - x_i)| u_i^2(L - x_i) \\
&\quad - \sum_{i=1}^n J_i^2(x_i) |\Lambda_i(x_i)| \left( G_i \begin{pmatrix} \mathbf{u}_+(t, L - x_i) \\ \mathbf{u}_-(t, L - x_i) \end{pmatrix} \right)^2 \\
&\geq \sum_{i=1}^n J_i^2(L - x_i) |\Lambda_i(L - x_i)| u_i^2(L - x_i) \\
&\quad - \sum_{i=1}^n J_i^2(x_i) |\Lambda_i(x_i)| \left( \sum_{j=1}^n K_{ij} |u_j(t, L - x_j)| \right)^2,
\end{aligned} \tag{21}$$

240 This can be rewritten as

$$I_2 \geq \mathbf{Y}^\top N \mathbf{Y}, \tag{22}$$

241 where  $\mathbf{Y}$  is a vector with components  $Y_i = |u_i(t, L - x_i)|$  and  $N$  is given by

$$\begin{aligned}
N &= \begin{pmatrix} J_+^2(L) |\Lambda_+(L)| & 0 \\ 0 & J_-^2(0) |\Lambda_-(0)| \end{pmatrix} \\
&\quad - K^\top \begin{pmatrix} J_+^2(0) |\Lambda_+(0)| & 0 \\ 0 & J_-^2(L) |\Lambda_-(L)| \end{pmatrix} K.
\end{aligned} \tag{23}$$

242 From (11) the matrix  $N$  is positive semidefinite, thus

$$I_2 \geq 0. \quad (24)$$

243 Let us now deal with  $I_3$ . Assume that the condition (9) holds. Then there exists  
 244  $D \in C^1([0, L], M_n(\mathbb{R}))$  such that  $D(x)$  is diagonal with positive coefficients for  
 245 any  $x \in [0, L]$ . Thus

$$-D(J^2\Lambda)'D + DJ^2MD + DM^\top J^2D \quad (25)$$

246 is a symmetric and definite positive matrix and we denote by  $\lambda_m$  its smallest  
 247 eigenvalue on  $[0, L]$ . We have from (19), using Cauchy-Schwarz inequality and  
 248 using the fact that  $g$  is Lipschitz with  $\mathbf{u}$  and the fact that  $g(\mathbf{0}, x) = B(\mathbf{0}, x) = 0$ ,

249

$$\begin{aligned} I_3 &\geq \int_0^L (D^{-1}\mathbf{u})^\top [-D(J^2\Lambda)'D + DJ^2MD + DM^\top J^2D](D^{-1}\mathbf{u})dx \\ &\quad - 2 \left( \int_0^L |D^{-1}\mathbf{u}|^2 dx \right)^{1/2} \left( \int_0^L |DJ^2g(\mathbf{u}, x)|^2 dx \right)^{1/2} \\ &\geq \int_0^L (D^{-1}\mathbf{u})^\top [-D(J^2\Lambda)'D + DJ^2MD + DM^\top J^2D](D^{-1}\mathbf{u})dx \\ &\quad - 2 \max_{i,x} (D_i J_i^2(x)) \left( \int_0^L |D^{-1}\mathbf{u}|^2 dx \right)^{1/2} \left( \int_0^L |g(\mathbf{u}, x)|^2 dx \right)^{1/2} \\ &\geq \int_0^L (D^{-1}\mathbf{u})^\top [-D(J^2\Lambda)'D + DJ^2MD + DM^\top J^2D](D^{-1}\mathbf{u})dx \quad (26) \\ &\quad - 2 \max_{i,x} (D_i J_i^2(x)) \left( \int_0^L |D^{-1}\mathbf{u}|^2 dx \right)^{1/2} C_g \left( \int_0^L |\mathbf{u}|^2 dx \right)^{1/2} \\ &\geq \int_0^L (D^{-1}\mathbf{u})^\top [-D(J^2\Lambda)'D + DJ^2MD + DM^\top J^2D](D^{-1}\mathbf{u})dx \\ &\quad - 2C_g \max_{i,x} (D_i J_i^2(x)) \max_{i,x} (D_i(x)) \left( \int_0^L |D^{-1}\mathbf{u}|^2 dx \right)^{1/2} \\ &\geq \lambda_m \|D^{-1}\mathbf{u}\|_{L^2}^2 - 2C_g \max_{i,x} (D_i J_i^2(x)) \max_{i,x} (D_i(x)) \|D^{-1}\mathbf{u}\|_{L^2}^2. \end{aligned}$$

250 Therefore if  $C_g < \lambda_m / (2 \max_{i,x} (D_i(x)) \max_{i,x} (D_i(x) J_i^2(x)))$  then

$$I_3 \geq \mu \|D^{-1}\mathbf{u}\|_{L^2}, \quad (27)$$

251 with  $\mu = \lambda_m - 2C_g \max_{i,x} (D_i J_i^2(x)) \max_{i,x} (D_i(x)) > 0$ . Thus from (16), the  
 252 positive definiteness of  $D$ , hence  $D^{-1}$ , (18), (24), and (27) we can set  $\gamma =$   
 253  $\mu(\max(D_i J_i^2))^{-1} > 0$  such that such that for any  $t \in [0, T]$ .

$$\frac{dV(\mathbf{u}(t, \cdot))}{dt} \leq -\gamma V, \quad (28)$$

254 and therefore

$$V(\mathbf{u}(t, \cdot)) \leq V(\mathbf{u}(s, \cdot)) e^{-\gamma(t-s)}, \quad \forall 0 \leq s \leq t \leq T. \quad (29)$$

255 From (16), this implies that

$$\|\mathbf{u}(t, \cdot)\|_{L^2} \leq \|J^{-1}\|_{L^\infty} \|J\|_{L^\infty} e^{-\frac{\gamma}{2}(t-s)} \|\mathbf{u}_0\|_{L^2}, \quad (30)$$

256 which is exactly the estimate wanted with decay rate  $\gamma/2$ . So far this estimate  
 257 is only true for  $H^1$  solutions. However, it only involves the  $L^2$  norm. Thus, as  
 258 the system is well-posed in  $C^0([0, T], L^2(0, L))$  and  $\|\cdot\|_{L^\infty((0, T); L^2(0, L))}$  is lower  
 259 semicontinuous, the estimate (30) also hold for  $L^2$  solutions by density (more  
 260 details on this argument can be found in the proof of [53, Lemma 4.2]).

261

□

#### 262 4. Adapting the proof in the ISS case

263 In this section we show how to adapt the proof of Theorem 2.2 to get The-  
 264 orem 2.3.

265 *Proof.* Let us consider System (13) and let  $T > 0$ . Let  $\mathbf{u}_0 \in H^1(0, L)$  and  
 266  $\mathbf{u} \in C^1([0, T], H^1(0, L))$  the associated solution. Then, defining  $V$  as in (15),  
 267 and differentiating along  $\mathbf{u}$ , we obtain as previously

$$\frac{dV(\mathbf{u}(t, \cdot))}{dt} = -I_2 - I_3 - 2 \int_0^L \mathbf{u}^\top J^2 \mathbf{d}_1 dx, \quad (31)$$

268 where  $I_2$  and  $I_3$  are given by (19). Thus, using Young's inequality

$$\frac{dV(\mathbf{u}(t, \cdot))}{dt} = -I_2 - I_3 + \varepsilon_0 V + \frac{\|J^2\|_{L^\infty}}{\varepsilon_0} \|\mathbf{d}_1(t, \cdot)\|_{L^2}^2, \quad (32)$$

where  $\varepsilon_0 > 0$  and can be chosen. As previously, from (9),  $I_3 \geq \mu V$  where  $\mu > 0$ .  
Therefore, choosing  $\varepsilon_0 = \mu/2$ , we have

$$-I_3 + \varepsilon_0 V \leq -\frac{\mu}{2} V. \quad (33)$$

Concerning  $I_2$ , if we denote by  $I_{2,0}$  the quantity in the absence of disturbances  
(i.e. the quantity given by the first equality of (21)) we get

$$\begin{aligned} I_2 &= I_{2,0} - \sum_{i=1}^n J_i^2(x_i) |\Lambda_i(x_i)| \left( d_{2,i}^2 + 2d_{2,i} G_i \begin{pmatrix} \mathbf{u}_+(t, L) \\ \mathbf{u}_-(t, 0) \end{pmatrix} \right) \\ &\geq \mathbf{Y}^\top N \mathbf{Y} - \sum_{i=1}^n J_i^2(x_i) |\Lambda_i(x_i)| \left( 1 + \frac{1}{\varepsilon} \right) d_{2,i}^2 \\ &\quad - \varepsilon \mathbf{Y} K^\top \begin{pmatrix} J_+^2(0) |\Lambda_+(0)| & 0 \\ 0 & J_-^2(L) |\Lambda_-(L)| \end{pmatrix} K \mathbf{Y}, \end{aligned} \quad (34)$$

where we used Young's inequality and where  $N$  is the matrix given in (23),  $\mathbf{Y}$  is  
defined as in (22), and  $\varepsilon > 0$  is to be chosen. Using the definition of  $N$  and the  
fact that  $N$  is positive definite (and not positive semidefinite in contrast with  
Theorem 2.2), we get by continuity that there exists  $\varepsilon > 0$  such that

$$N - \varepsilon K^\top \begin{pmatrix} J_+^2(0) |\Lambda_+(0)| & 0 \\ 0 & J_-^2(L) |\Lambda_-(L)| \end{pmatrix} K \quad \text{is semipositive definite.} \quad (35)$$

Therefore,  $I_2 \geq -(1 + \varepsilon^{-1}) \|J\|_\infty^2 \|\Lambda\|_\infty |\mathbf{d}_2(t)|^2$  and (32) becomes

$$\begin{aligned} \frac{dV(\mathbf{u}(t, \cdot))}{dt} &\leq -\frac{\mu}{2} V + \frac{2\|J\|_{L^\infty}^2}{\mu} \|\mathbf{d}_1(t, \cdot)\|_{L^2}^2 \\ &\quad + (1 + \varepsilon^{-1}) \|J\|_\infty^2 \|\Lambda\|_\infty |\mathbf{d}_2(s)|^2, \end{aligned} \quad (36)$$

thus, using Gronwall's Lemma,

$$\begin{aligned} V(\mathbf{u}(t, \cdot)) &\leq V(\mathbf{u}_0) e^{-\frac{\mu t}{2}} \\ &\quad + \frac{2\|J\|_{L^\infty}^2}{\mu} \int_0^t e^{-\frac{\mu}{2}(t-s)} (\|\mathbf{d}_1(s, \cdot)\|_{L^2}^2 \\ &\quad + \frac{\mu}{2} (1 + \varepsilon^{-1}) \|\Lambda\|_\infty |\mathbf{d}_2(s)|^2) ds, \end{aligned} \quad (37)$$



279 which, together with (16) and the concavity of the square root function gives

$$\begin{aligned}
\|\mathbf{u}(t, \cdot)\|_{L^2} &\leq \|J^{-1}\|_{L^\infty} \|J\|_{L^\infty} \|\mathbf{u}_0\|_{L^2} e^{-\frac{\mu t}{4}} \\
&+ \|J^{-1}\|_{L^\infty} \|J\|_{L^\infty} \sqrt{\frac{2}{\mu} \max\left(1, \frac{\mu}{2}(1 + \varepsilon^{-1})\|\Lambda\|_{L^\infty}\right)} \left(\|e^{-\frac{\mu}{2}(t-s)} \mathbf{d}_1(s, x)\|_{L^2((0,t) \times (0,L))} + \right. \\
&\left. + \|\mathbf{d}_2(t)\|_{L^2(0,t)}\right),
\end{aligned} \tag{38}$$

280 which is the ISS estimate wanted and this holds for any  $H^1$  solutions. And, by  
281 density, this holds also for any  $L^2$  solutions. Note that the gains of the estimate  
282 can again be computed explicitly. This ends the proof of Theorem 2.3  $\square$

## 283 5. Numerical simulations

284 In this section we present a numerical illustration of the previous result on a  
285 simple example. We consider a system inspired from [16, Section 5.6] and given  
286 as

$$\begin{aligned}
\partial_t u_1 + \partial_x u_1 &= cL^{-1} \sin\left(\int_0^L u_2(t, x) dx\right) \\
\partial_t u_2 - \partial_x u_2 &= cL^{-1} \sin\left(\int_0^L u_1(t, x) dx\right) \\
u_1(t, 0) - u_2(t, 0) &= 0 \\
u_1(t, L) - u_2(t, L) &= ku_1(t, L)
\end{aligned} \tag{39}$$

287 where one boundary condition can be imposed through a design parameter  $k$   
288 while the other one is imposed. Note first that in open-loop, i.e.  $k = 0$ , the null  
289 steady-state is an unstable steady-states for any  $c \in \mathbb{R}$  and any length of the  
290 domain  $L > 0$ . Indeed, there is a continuum of travelling wave solutions: for  
291 any  $\varepsilon > 0$

$$\begin{cases} u_1(x) = \varepsilon e^{\frac{2\pi i}{L}(t-x)} \\ u_2(x) = \varepsilon e^{-\frac{2\pi i}{L}(t-x)} \end{cases} \tag{40}$$

292 is a solution of (39) with  $k = 0$ . Nevertheless, Theorem 2.2 can be applied to find  
293 a feedback in closed loop as long as  $|c|L < 1/2$ : set  $M = 0$ ,  $D = Id$ ,  $\varepsilon > 0$  to be

defined, and  $k = \sqrt{1/(1 + 2L\varepsilon^{-1})}$ . Set also,  $J = \text{diag}(\sqrt{L + \varepsilon - x}, \sqrt{L + \varepsilon + x})$ ,  
one has  $\Lambda = \text{diag}(1, -1)$  therefore  $-(J^2\Lambda)' = I_d$  and therefore is positive definite  
with smallest eigenvalue 1. Besides  $\max_{i,x}(J_i^2) = \varepsilon + 2L$  and

$$\begin{aligned} \|g(U) - g(V)\|_{L^2}^2 &= \frac{1}{L^2} \int_0^L \left| \begin{pmatrix} c \sin\left(\int_0^L U_2(x)dx\right) \\ c \sin\left(\int_0^L U_1(x)dx\right) \end{pmatrix} - \begin{pmatrix} c \sin\left(\int_0^L V_2(x)dx\right) \\ c \sin\left(\int_0^L V_1(x)dx\right) \end{pmatrix} \right|^2 dx \\ &\leq |c|L^{-1} \left[ \left( \int_0^L |U_2 - V_2|dx \right)^2 + \left( \int_0^L |U_1 - V_1|dx \right)^2 \right] \\ &\leq |c|\|U - V\|_{L^2}^2. \end{aligned} \tag{41}$$

Hence, condition (9) becomes  $|c| < (\varepsilon + 2L)^{-1}$ . Now, as  $|c| < (2L)^{-1}$ , one can  
choose  $\varepsilon = 3(|c|^{-1} - 2L)/4$  such that condition (9) is satisfied. Finally, one can  
easily check that condition (11) becomes

$$(1 - k)^2 \leq \frac{\varepsilon}{\varepsilon + 2L}, \tag{42}$$

which is also satisfied from our definition of  $k$ . Thus Theorem 2.2 applies and  
the system is globally stable for the  $L^2$  norm. On Figure 1 we represent the  
 $L^2$  norm of the solution for various values of  $k$  when  $c = 1/4$  and  $L = 1$ . In  
blue is represented the open-loop situation (i.e.  $k = 0$ ), in green the closed-loop  
situation with  $k = 3/4$ , and in red  $k = 1/2$ .

## 6. Conclusion and perspective

We derived sufficient conditions for the global stability in the  $L^2$  norm of  
semilinear systems with Lipschitz boundary conditions and source term (poten-  
tially nonlocal). We also showed that a strong ISS property with respect to  
boundary and internal disturbances holds globally under the same conditions.  
This result could have many applications in practice. Knowing whether such  
conditions are optimal for the existence of a basic quadratic Lyapunov function,  
at least for  $n = 2$  as it is in the linear and local case, is an open question. An-  
other interesting direction for future works would be to try to extend, at least  
partially, these results to quasilinear but Lipschitz nonlocal systems.

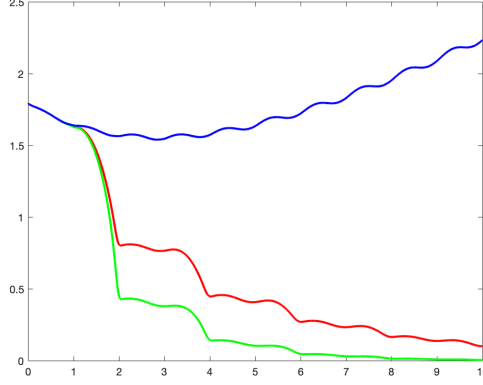


Figure 1: Stability of the system (39) in open-loop (blue) and closed-loop with  $k = 3/4$  (green) and  $k = 1/2$  (red). horizontal axis represents time, and vertical axis represents the  $L^2$  norm of the solution with initial condition  $u_{1,0}(x) = \sqrt{2\pi x}$  and  $u_{2,0}(x) = e^{-2\pi x}$ .

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### Appendix A. Well-posedness of the system

In this section we deal with the well-posedness of the system and extend [38, Theorem A.1] to get Theorem 2.1. But first, we give the definition of a weak  $L^2$  solution for System (1),(3).

**Definition Appendix A.1.** Let  $\mathbf{u}_0 \in L^2(0, L)$ . We say that  $\mathbf{u} \in C^0([0, +\infty); L^2(0, L))$  is an  $L^2$  solution of the Cauchy problem (1), (3),  $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ , if for every  $T > 0$  there exists a sequence of functions  $\mathbf{u}_{0,n} \in H^1(0, L)$  satisfying (3) and such that

$$\begin{aligned}\mathbf{u}_{0,n} &\rightarrow \mathbf{u}_0 \text{ in } L^2(0, L), \\ \mathbf{u}_n &\rightarrow \mathbf{u} \text{ in } C^0([0, T], L^2(0, L)),\end{aligned}\tag{A.1}$$

where  $\mathbf{u}_n \in C^0([0, T], H^1(0, L))$  is a weak solution of (1), (3) with initial condition  $\mathbf{u}_{0,n}$ , i.e.  $\mathbf{u}_n$  satisfies (3) and for any  $\phi \in C^1([0, T]; C_c^1((0, L); \mathbb{R}^n))$  we have

$$\begin{aligned}&\int_0^L \int_0^T \partial_t \phi^\top \mathbf{u}_n + \partial_x \phi^\top \Lambda(x) \mathbf{u}_n \\ &\quad + \phi^\top (\Lambda_x \mathbf{u}_n - B(\mathbf{u}_n, x)) dt \, dx \\ &= \int_0^L [\phi(\cdot, x)^\top \mathbf{u}_n(\cdot, x)]_0^T dx.\end{aligned}\tag{A.2}$$

333

*Remark* Appendix A.1. As noted in [38], this definition is slightly different from the definition given in [16, Definition A.3] when looking at linear systems. The reason comes from the nonlinear boundary conditions which may prevent the adjoint of the boundary operator from existing. Of course, in the linear case, a solution in the sense of [16, Definition A.3] is also a solution in the sense of Definition Appendix A.1.

With this definition in mind, we prove Theorem 2.1, by slightly adapting the proof of [38, Theorem A.1]

*Proof of Theorem 2.1.* Let  $T > 0$ . We define the operator  $\mathcal{A} = -\Lambda(x)\partial_x$  on the domain  $D(\mathcal{A})$  defined by

$$D(\mathcal{A}) = \{\mathbf{u} \in H^1(0, L) | \mathbf{u} \text{ satisfies (3)}\}.\tag{A.3}$$

We also consider  $B$  as an operator on the domain  $D(B) = L^2(0, L)$ , and in the following  $Bf$  refers to  $B(f(\cdot), x) \in L^2(0, L)$ . Observe that  $D(\mathcal{A} + B) = D(\mathcal{A})$ . First of all, we can restrict ourselves to the case where  $\Lambda$  has only positive components. Indeed, if not, we define  $\mathbf{v} = (v_i)_{i \in \{1, \dots, n\}}$  by

$$\begin{cases} v_i(t, \cdot) = u_i(t, \cdot) & \text{if } i \in \{1, \dots, m\} \\ v_i(t, \cdot) = u_i(t, L - \cdot) & \text{if } i \in \{m + 1, \dots, n\}, \end{cases}\tag{A.4}$$

348 and

$$\left\{ \begin{array}{l} \tilde{\Lambda}_i = \Lambda_i \text{ if } i \in \{1, \dots, m\} \\ \tilde{\Lambda}_i = -\Lambda_i(L - \cdot) \text{ if } i \in \{m+1, \dots, n\}, \\ \tilde{B}_i(\mathbf{v}, \cdot) = B_i(\mathbf{u}, \cdot) \text{ if } i \in \{1, \dots, m\} \\ \tilde{B}_i(\mathbf{v}, \cdot) = B_i(\mathbf{u}, L - \cdot) \text{ if } i \in \{m+1, \dots, n\}. \end{array} \right. \quad (\text{A.5})$$

349 Clearly,  $\mathbf{u}$  is a  $L^2$  solution to the system (1), (3) if and only if  $\mathbf{v}$  is an  $L^2$   
 350 solution to a system of the form (1), (3) with  $\tilde{\Lambda}$  instead of  $\Lambda$  and  $\tilde{B}$  instead of  
 351  $B$ . And now,  $\tilde{\Lambda}$  has only positive components while  $\tilde{B}$  is still Lipschitz with  
 352 respect to  $\mathbf{v}$ . Therefore, in this proof, we will assume that  $m = n$  and  $\Lambda$  has  
 353 only positive components. From [38, Appendix A.1],  $\mathcal{A} + B$  is  $\zeta$  dissipative with  
 354  $\zeta$  independent of  $n$  and is a closed operator (in  $L^2(0, L)$ ). A definition of an  
 355 operator  $\zeta$  dissipative can be found in [54, Definition 2.4 and Chapter 5, section  
 356 2]. Note that [38] study systems with a local diagonal source term and positive  
 357 and constant propagation speeds. However, the proof of these two first points in  
 358 [38, Theorem A.1] only requires  $B$  to be Lipschitz and the propagation speeds  
 359 to be positive and nonvanishing. Now we would like to show that  $\mathcal{A} + B$  satisfies  
 360 the following range condition:

$$\exists \rho_0 > 0, \forall \rho \in (0, \rho_0), D(\mathcal{A} + B) \subset Rg(Id - \rho(\mathcal{A} + B)). \quad (\text{A.6})$$

361 or equivalently that for any  $\mathbf{v} \in D(\mathcal{A} + B)$ , there exists  $\mathbf{u} \in H^1(0, L)$  such that

$$\begin{aligned} \mathbf{u} - \rho(\Lambda \partial_x \mathbf{u} + B(\mathbf{u}, \cdot)) &= \mathbf{v}, \\ \begin{pmatrix} \mathbf{u}_+(0) \\ \mathbf{u}_-(L) \end{pmatrix} &= G \begin{pmatrix} \mathbf{u}_+(L) \\ \mathbf{u}_-(0) \end{pmatrix}. \end{aligned} \quad (\text{A.7})$$

362 The difficulty comes from the nonlinearity of the equation and this was the main  
 363 point shown in [38, Theorem A.1]. In our case, all we need to do is to change  
 364 slightly their proof to take into account the nonlocal operator and the fact that  
 365  $\Lambda$  depends on  $x$ . The latter is easy to take into account by replacing  $e^{-\Lambda^{-1}x/\rho}$   
 366 by  $e^{-\int_0^x (\Lambda^{-1}(s)/\rho) ds}$  when integrating, which has a similar behavior (this holds  
 367 as  $\Lambda$  is diagonal). To take into account the nonlocal operator, we need to get  
 368 the estimate [38, (26)] while replacing the estimations in [38, 2.2.1], which hold

only for the local case when  $B$  has a Lipschitz constant independent of  $x$ . But we have from the Lipschitz behavior of  $B$  and Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| \int_0^x e^{-\int_s^x \Lambda^{-1}(v)/\rho dv} \Lambda^{-1}(s) B(\mathbf{u}, s) ds \right| \\ & \leq \|B(\mathbf{u}, \cdot)\|_{L^2} \frac{\|\Lambda^{-1}\|_\infty}{2} \rho \int_0^x \left| e^{\int_s^x 2\Lambda^{-1}(v)/\rho dv} \frac{2\Lambda^{-1}(s)}{\rho} \right| ds \\ & \leq \rho \frac{C_B \|\Lambda^{-1}\|_\infty}{2} \left| 1 - e^{-2 \int_0^L \frac{\Lambda^{-1}(v)}{\rho} dv} \right| \|\mathbf{u}\|_{L^2} \leq \rho C_2 \|\mathbf{u}\|_{L^2} \end{aligned} \quad (\text{A.8})$$

where  $C_2$  is a constant that depends only on the parameters of the system. This enables to recover the estimate [38, (26)] which is then used to apply Arzela-Ascoli Theorem and get the range condition (A.6). As in Theorem [38, Theorem A.1], from these three properties ( $\zeta$  dissipative, closed operator and range condition) and using [54, Corollary 5.13 and Remark 2 p.148],  $\mathcal{A} + B$  generates a nonlinear semigroup  $S$  of type  $\zeta$  on  $L^2(0, L)$  and the Cauchy problem has a unique integral solution  $\mathbf{u} \in C^0([0, T], L^2(0, L))$  (see [54] for a proper definition of an integral solution). Besides, let  $\mathbf{u}_{0,n} \in D(\mathcal{A})$ , then from [54, Remark 2 p.148], the unique integral solution  $\mathbf{u}_n$  of the Cauchy problem with initial condition  $\mathbf{u}_{0,n}$  belongs to  $C^1([0, T]; L^2(0, L)) \cap C^0([0, T]; H^1(0, L))$  and satisfies (3) and (A.2). We can choose a sequence  $\mathbf{u}_{0,n} \in D(\mathcal{A})$  such that  $\mathbf{u}_{0,n} \rightarrow \mathbf{u}_0 \in L^2(0, L)$ , as  $D(\mathcal{A})$  is dense in  $L^2$ . Finally, as  $S$  is a semigroup of type  $\zeta$  we have (see [54, Remark p.146])

$$\|S(t)\mathbf{u}_0 - S(t)\mathbf{u}_{0,n}\|_{L^2} \leq C e^{\zeta t} \|\mathbf{u}_0 - \mathbf{u}_{0,n}\|_{L^2}, \quad (\text{A.9})$$

which implies the convergence of  $\mathbf{u}_n$  to  $\mathbf{u}$  in  $C^0([0, T], L^2(0, L))$ . To conclude we only need to show that this is the unique solution in the sense of Definition Appendix A.1. Let assume that there is another solution  $\mathbf{u}^{(1)}$  with initial condition  $\mathbf{u}_0$ . Let  $T > 0$ . By assumption there exists a sequence  $\mathbf{u}_{0,n}^{(1)} \in D(\mathcal{A})$  such that  $\mathbf{u}_n^{(1)}$  satisfies (3) and (A.2) with initial condition  $\mathbf{u}_{0,n}^{(1)}$  and  $\mathbf{u}_n^{(1)} \rightarrow \mathbf{u}^{(1)}$  in  $C^0([0, T], L^2(0, L))$ . For any  $n \in \mathbb{N}$ ,  $\mathbf{u}_n^{(1)} \in C^1([0, T], L^2(0, L))$ , therefore  $\mathbf{u}_n^{(1)}$  is also an integral solution of the Cauchy problem with initial condition  $\mathbf{u}_{0,n}^{(1)}$  (see [54, Remark 2 p.148]). Thus, from [54, Remark p.146],

$$\|\mathbf{u}(t, \cdot) - \mathbf{u}_n^{(1)}(t, \cdot)\|_{L^2} \leq C e^{\zeta t} \|\mathbf{u}_0 - \mathbf{u}_{0,n}^{(1)}\|_{L^2}, \quad (\text{A.10})$$

and therefore  $\mathbf{u}_n^{(1)} \rightarrow \mathbf{u}$  in  $C^0([0, T], L^2)$ , which implies that  $\mathbf{u} = \mathbf{u}^{(1)}$  in  $C^0([0, T], L^2)$ . This holds for any  $T > 0$ , and ends the proof.

□

#### Well-posedness in the ISS case

The well-posedness of system (13) is again a consequence of the three properties of the operator  $(\mathcal{A} + B)$  and the results in [54]. In particular, let  $T > 0$  and  $\mathbf{u}_0 \in L^2(0, L)$ , and assume  $(\mathbf{d}_1, \mathbf{d}_2) \in L^2((0, T) \times (0, L)) \times H^1(0, T)$ , and set  $L : x \rightarrow \text{diag}(L_i(x))$  where  $L_i(x) = (L - x)/L$  if  $i \in \{1, \dots, m\}$  and  $L_i = x/L$  if  $i \in \{m + 1, \dots, n\}$ . A function  $\mathbf{u} \in C^0([0, T], L^2(0, L))$  is an  $L^2$  solution to (13),  $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ , if and only if  $\mathbf{v}(t, x) := \mathbf{u}(t, x) - L(x)\mathbf{d}_2(t)$  is an  $L^2$  solution to

$$\begin{aligned} \partial_t \mathbf{v} &= -(\mathcal{A} + B)\mathbf{v} + \mathbf{d}_1 + [(\Lambda \partial_x + B)(L\mathbf{d}_2) - L\dot{\mathbf{d}}_2] \\ \mathbf{v}(0, \cdot) &= \mathbf{u}_0 - L(x)\mathbf{d}_2(0) =: \mathbf{v}_0, \end{aligned} \tag{A.11}$$

with boundary conditions (3), and where  $B$  is seen again as an operator on  $L^2(0, L)$ . The interest of this reformulation is that the boundary conditions of  $\mathbf{v}$  are now again (3) and do not depend on time (contrary to the boundary conditions of (13)) except through  $\mathbf{v}$  (see also [42]). Using the fact that  $(\mathcal{A} + B)$  satisfies the range condition and is dissipative of type  $\zeta$ , from [54, Theorem 5.18 and Remark 2°], there exists a unique integral solution  $\mathbf{u} \in C^0([0, T]; L^2(0, L))$ , satisfying a definition analogous to Appendix A.1 (the precise definition is omitted here due to the lack of space).

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