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# DECOMPOSITION INTO CYCLES I: HAMILTON DECOMPOSITIONS

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## Abstract

In this part we survey the results concerning the partitions of the edge-set of a graph into Hamilton cycles or into Hamilton cycles and a single perfect matching.

By and large the terminology of [15] will be followed. The topic under discussion is the partitioning of the edge-set of a graph into Hamilton cycles. An obvious necessary condition for such a partition to exist is that the graph be regular of even degree. However, if a graph is regular of odd degree, then the closest one can come to such a partition is a partition of the edge-set into a single perfect matching (also called a 1-factor) and Hamilton cycles. The following definition is used in order to incorporate odd degree and even degree into a single definition.

**Definition 1** Let  $G$  be a regular graph with edge-set  $E(G)$ . It is said to have a *Hamilton decomposition* (or be *Hamilton decomposable*) if either

- (i)  $\deg(G) = 2d$  and  $E(G)$  can be partitioned into  $d$  Hamilton cycles, or
- (ii)  $\deg(G) = 2d + 1$  and  $E(G)$  can be partitioned into  $d$  Hamilton cycles and a perfect matching.

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The earliest results about Hamilton decompositions deal with the complete graph  $K_n$  and are attributed to Walecki in [32].

**Theorem 1** *The complete graph  $K_n$  has a Hamilton decomposition for all  $n$ .*

**Proof:** The following functional notation makes Walecki's construction easy to describe. If  $H$  is a subgraph of the graph  $K_n$  and  $f : V(H) \rightarrow V(G)$  is a 1-1 function, then  $f(H)$  denotes the subgraph of  $K_n$  whose vertex-set is  $f(V(H))$  where  $f(u)$  is adjacent to  $f(v)$  in  $f(H)$  if and only if  $u$  is adjacent to  $v$  in  $H$ . The result is trivially true for  $n = 1$  and  $n = 2$ . Let  $n = 2m + 1 \geq 3$  be odd. Let the vertices of  $K_n$  be labelled  $u_0, u_1, u_2, \dots, u_{2m}$ , let  $C$  be the Hamilton cycle  $u_0 u_1 u_2 u_{2m} u_3 u_{2m-1} u_4 u_{2m-2} \dots u_{m+3} u_m u_{m+2} u_{m+1} u_0$  and let  $\sigma$  be the permutation  $(u_0)(u_1 u_2 u_3 u_4 \dots u_{2m-1} u_{2m})$ . Then  $C, \sigma(C), \sigma^2(C), \dots, \sigma^{m-1}(C)$  is a Hamilton decomposition of  $K_n$ .

When  $n = 2m \geq 4$  is even, let  $C$  be the Hamilton cycle

$$u_0 u_1 u_2 u_{2m-1} u_3 u_{2m-2} \dots u_{m-1} u_{m+2} u_m u_{m+1} u_0$$

and let  $\sigma$  be the permutation  $(u_0)(u_1 u_2 u_3 \dots u_{2m-2} u_{2m-1})$ . Then,  $C, \sigma(C), \dots, \sigma^{m-2}(C)$  are  $m-1$  edge disjoint Hamilton cycles. The remaining edges  $u_0 u_m, u_{m-1} u_{m+1}, u_{m-2} u_{m+2}, \dots, u_1 u_{2m-1}$  form a perfect matching.  $\square$

**Corollary 1** *The complete graph  $K_{2m}$  has a decomposition into Hamilton paths.*

**Proof:** Simply take a Hamilton decomposition of  $K_{2m+1}$  and remove a vertex and its incident edges from  $K_{2m+1}$ .  $\square$

The above constructions of Walecki have been used in a wide variety of settings. One of them leads to the following problem. Consider the Walecki construction for the Hamilton decomposition of  $K_{2m+1}$ ,  $m \geq 1$ . There are  $m$  Hamilton cycles each of which may be oriented in two ways to produce a Hamilton directed cycle. If each of the  $m$  Hamilton cycles is given an orientation, the resulting regular tournament is called a *Walecki tournament*. There are  $2^m$  different orientation schemes but different orientations may produce isomorphic tournaments.

**Problem 1 (Alspach)** *Enumerate the Walecki tournaments of order  $2m + 1$ .*

The results about the complete graph lead one to consider just how large the degree of a regular graph must be before it is guaranteed to have a Hamilton decomposition.

Kotzig (unpublished) conjectured that any self-complementary regular graph of order  $4k + 1$  has a Hamilton decomposition. This is a particular case of:

**Conjecture 1 (Nash-Williams [35])** *Every  $2k$ -regular graph with at most  $4k + 1$  vertices has a Hamilton decomposition.*

A very strong result has been obtained recently by Häggkvist [20] who has proved that for any  $p < 4$  every sufficiently large  $2k$ -regular graph with at most  $pk$  vertices has a Hamilton decomposition.

Most of the remaining results that deal with Hamilton decompositions of graphs involve various products of graphs. Since terminology for products is not uniform, the definitions are included.

The *cartesian product*  $G_1 \times G_2$  of  $G_1$  and  $G_2$  (also called the cartesian sum) has vertex-set  $V(G_1) \times V(G_2)$  with  $(u_1, u_2)$  adjacent to  $(v_1, v_2)$  if and only if either  $u_1 = v_1$  and  $u_2$  is adjacent to  $v_2$  in  $G_2$  or  $u_2 = v_2$  and  $u_1$  is adjacent to  $v_1$  in  $G_1$ .

The *lexicographic product*  $G_1 \otimes G_2$  of  $G_1$  and  $G_2$  (also called the composition, or tensor product, or wreath product) is the graph with vertex-set  $V(G_1) \times V(G_2)$  and an edge joining  $(u_1, u_2)$  to  $(v_1, v_2)$  if and only if either  $u_1$  is adjacent to  $v_1$  in  $G_1$  or  $u_1 = v_1$  and  $u_2$  is adjacent to  $v_2$  in  $G_2$ .

The *conjunction*  $G_1 \cdot G_2$  of  $G_1$  and  $G_2$  (also called the cartesian product) is the graph with vertex-set  $V(G_1) \times V(G_2)$  and an edge joining  $(u_1, u_2)$  to  $(v_1, v_2)$  if and only if both  $u_1$  and  $v_1$  are adjacent in  $G_1$  and  $u_2$  and  $v_2$  are adjacent in  $G_2$ .

The cartesian product has probably attracted the most attention and the following conjecture is the prominent unsolved problem.

**Conjecture 2 (Bermond [10])** *If  $G_1$  and  $G_2$  have decompositions into Hamilton cycles, then so does  $G_1 \times G_2$ .*

The above conjecture is a general version of earlier conjectures and results. Kotzig [29] proved that  $C_r \times C_n$  can be decomposed into two Hamilton cycles and conjectured that  $C_r \times C_s \times C_n$  could be decomposed into three Hamilton cycles. The latter conjecture was solved by Foregger [19]. Kotzig also conjectured that if  $G_1, G_2, \dots, G_p$  can all be decomposed into  $n$  Hamilton cycles then  $G_1 \times G_2 \times \dots \times G_p$  can be decomposed into  $pn$  Hamilton cycles. Independent of any knowledge of Kotzig's conjecture, Myers [34] proved that  $C_r \times C_r$  and  $K_r \times K_r$  have Hamilton decompositions.

Ringel [36] earlier proved that the  $2^n$ -cube,  $n \geq 1$ , has a Hamilton decomposition and posed the problem of whether or not the  $2n$ -cube has a Hamilton decomposition for  $n \geq 1$  [36, Problem 2]. Some recent work disposes of Ringel's problem and unifies the above work.

**Theorem 2 (Aubert and Schneider [5])** *If  $C$  is a cycle and  $G$  is a 4-regular graph which is decomposable into two Hamilton cycles, then  $C \times G$  can be decomposed into three Hamilton cycles.*

The proof of Theorem 2 is constructive and has a handful of cases to consider. It has several corollaries, including the results below, which we now state.

**Corollary 2** *If  $G_1$  has a decomposition into  $n_1$  Hamilton cycles and  $G_2$  has a decomposition into  $n_2$  Hamilton cycles, where  $n_1 \leq n_2 \leq 2n_1$ , then  $G_1 \times G_2$  has a Hamilton decomposition.*

**Proof:** Let  $\oplus$  denote edge disjoint union of spanning subgraphs. Then  $G_1 = H_1 \oplus H_2 \oplus \dots \oplus H_{n_1}$  and  $G_2 = H'_1 \oplus H'_2 \oplus \dots \oplus H'_{n_2}$  where the  $H_i$ 's and  $H'_i$ 's are Hamilton cycles. Then letting  $n_2 = n_1 + d$ ,

$$G_1 \times G_2 = (H_1 \times (H'_1 \oplus H'_{n_1+1})) \oplus (H_2 \times (H'_2 \oplus H'_{n_1+2})) \oplus \dots \\ \oplus (H_d \times (H'_d \oplus H'_{n_1+d})) \oplus (H_{d+1} \times H'_{d+1}) \oplus \dots \oplus (H_{n_1} \times H'_{n_1}).$$

Each of the terms in the preceding expression has a Hamilton decomposition by either Theorem 2 or Kotzig's result. The result then follows.  $\square$

**Corollary 3** *The cartesian product  $C_{i_1} \times C_{i_2} \times \dots \times C_{i_n}$  has a Hamilton decomposition.*

**Proof:** This follows easily by induction on  $n$ . The initial values are handled by Kotzig's result and Theorem 2 or Foregger's result. For larger  $n$  write

$$C_{i_1} \times C_{i_2} \times \dots \times C_{i_n} = (C_{i_1} \times \dots \times C_{i_m}) \times (C_{i_{m+1}} \times \dots \times C_{i_n})$$

where  $m = \lfloor \frac{n}{2} \rfloor$ . □

**Proposition 1** *The  $n$ -cube has a Hamilton decomposition for all  $n$ .*

**Proof:** For even  $n$ , the  $n$ -cube is nothing more than  $C_4 \times C_4 \times \dots \times C_4$  so that the result follows from Corollary 3. For odd  $n$ , view the  $n$ -cube as two  $(n-1)$ -cubes joined by a perfect matching. Remove one edge from each Hamilton cycle of a Hamilton decomposition of the  $(n-1)$ -cube so that the removed edges have no vertices in common. Do the same for the other  $(n-1)$ -cube. Use the perfect matching to form Hamilton cycles in the  $n$ -cube. This can always be done [3] so that the  $n$ -cube has a Hamilton decomposition. □

From 9 it follows also that Kotzig's conjecture above is true.

**Corollary 4** *If  $G_1, G_2, \dots, G_p$  can all be decomposed into  $n$  Hamilton cycles, then  $G_1 \times G_2 \times \dots \times G_p$  can be decomposed into  $pn$  Hamilton cycles.*

There has been some recent progress on Conjecture 2 by Stong [38]. His results are summarized below.

**Theorem 3** *If  $G_1$  has a decomposition into  $n_1$  Hamilton cycles and  $G_2$  has a decomposition into  $n_2$  Hamilton cycles,  $n_1 \leq n_2$ , then  $G_1 \times G_2$  has a Hamilton decomposition if any one of the following is true:*

- (i)  $n_2 \leq 3n_1$ ,
- (ii)  $n_1 \geq 3$ ,
- (iii)  $G_1$  has an even number of vertices, or
- (iv)  $|V(G_2)| \geq 6 \lfloor \frac{n_2}{n_1} \rfloor - 3$ .

Aubert and Schneider also proved the following result [6].

**Theorem 4** *The cartesian product  $K_n \times K_m$  has a Hamilton decomposition for all  $n$  and  $m$ .*

Alspach and Rosenfeld [4] studied the cartesian product of a cubic graph and  $K_2$  for Hamilton decomposability. They give a necessary and sufficient condition which implies that  $G \times K_2$  has a Hamilton decomposition if  $G$  is cubic and has a perfect 1-factorization, that is,  $G$  has a 1-factorization such that the union of any pair of 1-factors is a Hamilton cycle. They asked for other conditions on a cubic graph  $G$  that guarantee that  $G \times K_2$  has a Hamilton decomposition. The following example is one condition that one might suggest.

**Question 1 (Alspach and Rosenfeld [4])** *If  $G$  is a 3-connected cubic graph having a Hamilton cycle, does  $G \times K_2$  have a Hamilton decomposition?*

A first result about the lexicographic product is the following.

**Theorem 5 (Auerbach and Laskar [8] and Hetyei [21])** *The complete multipartite graph  $K_{r;s}$  has a Hamilton decomposition.*

A general study of Hamilton decomposition of complete regular  $s$ -partite graphs has been done by Hilton and Roger [22].

The analogue of Conjecture 2 for lexicographic products asked in [10] has been solved.

**Theorem 6 (Baranyai and Szasz [9])** *If  $G_1$  can be decomposed into  $m_1$  Hamilton cycles,  $G_2$  can be decomposed into  $m_2$  Hamilton cycles and  $G_2$  has  $n$  vertices, then  $G_1 \otimes G_2$  can be decomposed into  $m_2 + m_1 n$  Hamilton cycles.*

The preceding result supercedes several special cases that had been proved earlier. For example, the statement of the theorem does not indicate what happens in the degenerate case that  $G_2 = \overline{K}_n$  (the complement of  $K_n$ ). The result is still true in this case and follows easily from the following result.

**Theorem 7 (Laskar [31] and Hetyei [21])** *The graph  $C_r \otimes \overline{K}_n$  has a Hamilton decomposition.*

However, the generalization of Theorem 7 to arbitrary Hamilton decomposable graphs is unsettled. That is, if  $G_1$  is connected and has a decomposition into Hamilton cycles and a perfect matching, it is not known whether or not  $G_1 \otimes G_2$  has a Hamilton decomposition. On the other hand, if  $G_1$  has a decomposition into one or more Hamilton cycles and  $G_2$  is Hamilton decomposable, then  $G_1 \otimes G_2$  has a Hamilton decomposition.

In [10] Bermond has shown that the following two results hold.

**Theorem 8** *The conjunction of two cycles  $C_r \cdot C_s$  can be decomposed into two Hamilton cycles if and only if at least one of  $r$  and  $s$  is odd.*

**Corollary 5** *If both  $G_1$  and  $G_2$  can be decomposed into Hamilton cycles and one of  $G_1$  or  $G_2$  has odd order, then the conjunction  $G_1 \cdot G_2$  has a Hamilton decomposition.*

Many of the special cases discussed above are, in fact, Cayley graphs on abelian groups. This led Alspach [2] to ask the following question.

**Question 2 (Alspach [2])** *Does every connected Cayley graph on an abelian group have a Hamilton decomposition?*

If the degree of the graph is 2, then the answer is obviously yes. If the degree is 3, the answer is again yes since such a graph has a Hamilton cycle as proved by many authors (see the survey of Witte and Galian [42]). The case of degree 4 recently has been solved by Bermond, Favaron and Maheo [13] and the answer is again yes. The answer is also yes for degree 5 [3]. In the case of degree 4, when the group is the additive group of integers modulo  $n$ , the Cayley graphs are known as double loop graphs and the problem was asked by Bermond, Illiades and Peyrat [14]. More generally the construction of graphs admitting a decomposition into  $k$  Hamilton cycles and having the smallest diameter possible is an open problem arising in the context of computer loop networks (see the survey by Bermond, Comellas and Hsu [12]).

There is one positive result about the more general class of vertex-transitive graphs.

**Theorem 9 (Alspach [1])** *Every connected vertex-transitive graph of order  $2p$ ,  $p$  prime,  $p \equiv 3 \pmod{4}$  has a Hamilton decomposition.*

The graph  $O_k$  (called the *odd graph*) is a  $k$ -regular graph whose vertices are indexed by the  $(k-1)$ -subsets of a  $(2k-1)$ -set. Two vertices are adjacent if and only if their indexing subsets are disjoint. For example,  $O_3$  is the Petersen graph. Meredith and Lloyd [33] proved that  $O_4$ ,  $O_5$  and  $O_6$  admit a Hamilton decomposition and more generally they conjectured.

**Conjecture 3 [33]** *The odd graph  $O_m$  has a Hamilton decomposition, for  $m \geq 4$ .*

The graph  $B_k$  (called *boolean graph*) is a  $k$ -regular bipartite graph with one part consisting of vertices by the  $(k-1)$ -subsets of a  $(2k-1)$ -set, and the other part of vertices indexed by its  $k$ -subsets. Adjacency is given by containment. Notice that in other words  $B_k$  is the subgraph induced by the middle two levels of the boolean lattice representation of a  $(2k-1)$ -cube.

The boolean graphs are conjectured to be hamiltonian. However it may be interesting to notice that, for  $k = 2^m$ ,  $m \geq 1$ , if  $O_k$  has a Hamilton decomposition, then so does  $B_k$  (see [17] for more details).

There are some more general decomposition problems involving Hamilton cycles that should be mentioned at this time.

**Conjecture 4 (Kotzig [28])** *The complete graph  $K_{2m}$  has a perfect 1-factorization for all  $m \geq 2$ .*

This conjecture for some time has been known to be true for  $2m = p+1$  and  $2p$ ,  $p$  a prime, and for  $2m = 16, 28, 244$  and  $344$ . Recently, Seah and Stinson [37] have shown it is true for 36, Ihrig, Seah and Stinson [24] have shown it is true for 50, and Kobayashi and Kiyasu-Zen'iti [27] have shown it is true for 1,332 and 6,860. The following result is proved in [18] for other reasons but it is related to Conjecture 4.

**Theorem 10** *There is a 1-factorization  $\{I_1, I_2, \dots, I_{2m-1}\}$  of  $K_{2m}$  such that  $I_1 \cup I_j$  is a Hamilton cycle for all  $j \geq 2$ .*

There are two conjectures related to Theorem 10 given in [18]. One of them conjectures that if  $I'_1, I'_2, \dots, I'_{2m-2}$  are any  $2m-2$  1-factors of  $K_{2m}$ , then there is a 1-factorization  $\{I_1, I_2, \dots, I_{2m-1}\}$  of  $K_{2m}$  such that  $I_i \cup I'_i$  is a Hamilton cycle for  $i = 1, \dots, 2m-2$ . The second, and weaker, conjecture is the same except that the conclusion is that  $I_i \cap I'_i = \emptyset$  for all  $i = 1, \dots, 2m-2$ .

**Definition 2** A *Kotzig factorization* of  $K_{2m+1}$  consists of a decomposition into Hamilton cycles  $H_1, H_2, \dots, H_m$  and a decomposition into  $m$ -matchings  $M_1, M_2, \dots, M_{2m+1}$  such that  $|E(H_i) \cap E(M_j)| = 1$  for all  $i$  and  $j$ .

**Theorem 11 (Horton [23])** *The complete graph  $K_{2m+1}$  has a Kotzig factorization for all  $m \geq 1$ .*

A Kotzig factorization is a special case of a more general factorization considered in [2] and discussed by Alspach at the "Cycles in Graphs 1982" workshop held at Simon Fraser University in July 1982. Let  $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$  be any 2-factorization of a  $2r$ -regular graph  $G$ . An  $r$ -matching  $M$  is said to be *orthogonal* to  $\mathcal{F}$  if  $|M \cap F_i| = 1$  for  $i = 1, 2, \dots, r$ . Alspach asked the following two questions at the workshop. (Also see Discrete Math. 69 (1988), p.106).

**Question 3** *If  $F_1, F_2, \dots, F_r$  is any 2-factorization of a  $2r$ -regular graph  $G$ , does there always exist an orthogonal  $r$ -matching?*

**Question 4** *What conditions on a 2-factorization  $\mathcal{F}$  of a  $2r$ -regular graph  $G$  guarantee that the edge-set of  $G$  can be partitioned into  $r$ -matchings each of which is orthogonal to  $\mathcal{F}$ ?*

Recently Kouider and Sotteau [30] gave a positive answer to Question 3 for  $2r$ -regular graphs  $G$  of order  $m \geq 3.23r$ .

Jaeger [25] proved that if  $G$  can be decomposed into an even number of Hamilton cycles, then its line graph  $L(G)$  is 1-factorable. The key of the proof lies in the following result: If  $G$  can be decomposed into two Hamilton cycles, then its line graph  $L(G)$  can be decomposed into three Hamilton cycles. Kotzig proved that if  $G$  is a cubic Hamilton graph, then its line graph can be decomposed into two Hamilton cycles. That leads to the following conjecture.

**Conjecture 5 (Bermond [11])** *If  $G$  has a Hamilton decomposition, then its line graph can be decomposed into Hamilton cycles.*

In a personal letter, Jackson has asked a question for which an affirmative answer implies Conjecture 5. A *separating transition* in a graph  $G$  is a set of two adjacent edges  $e_1$  and  $e_2$  such that  $G - \{e_1, e_2\}$  is disconnected. If  $G$  is a  $2r$ -regular graph, a necessary condition for  $L(G)$  to have a Hamilton decomposition is that  $G$  have no separating transitions. Jackson has asked whether or not this condition is also sufficient. He has also conjectured that a 3-connected,  $2r$ -regular graph has a line-graph which is Hamilton decomposable. He has verified the latter conjecture when  $r = 2$ .

Also let us mention that Thomason [39] has proved that if a  $2k$ -regular graph has a Hamilton decomposition, then it has at least  $(3k - 2)(3k - 5) \dots 7 \cdot 4$  Hamilton decompositions. Colbourn [16] gave all the non-isomorphic Hamilton decompositions of  $K_n$  for  $n = 7, 9$  and those which have a non-trivial automorphism group for  $n = 11$ .

Finally let us conclude with some results concerning Hamilton decompositions of directed graphs. Some of these decompositions are obtained from the decompositions of the underlying undirected graph in particular when the directed graph is symmetric by associating to each Hamilton cycle of the underlying graph two opposite directed Hamilton cycles. For example from Theorem 1, we have that the complete symmetric directed graph  $K_{2n+1}^*$  has a decomposition into  $2n$  directed Hamilton cycles. The following strong result cannot be obtained in that way. (For earlier results see the survey [10]).

**Theorem 12 (Tillson [41])** *If  $2n \geq 8$ ,  $K_{2n}^*$  can be decomposed into  $2n - 1$  directed Hamilton cycles.*

Aubert and Schneider [7] studied the decompositions into directed Hamilton cycles of the symmetric digraph  $G^*$  associated to a cubic graph  $G$  consisting of a cycle plus a perfect matching. In particular they proved that if the number of vertices of  $G$  (which must be even) is a multiple of 4 then  $G^*$  has no Hamilton decomposition.

Recent results have been obtained concerning products of directed graphs and tournaments.

A characterization of the Cayley digraphs of finite abelian groups which can be decomposed into two directed Hamilton cycles has been given by Keating [26]. That implies that the cartesian product of two directed cycles  $C_r$  and  $C_s$  can be decomposed into directed Hamilton cycles if and only if there exist positive integers  $u$  and  $v$  such that  $u+v = \gcd(r, s)$  and  $\gcd(uv, rs) = 1$  (result first obtained by Lindgren). In [26] can also be found results on the conjunction of two digraphs. The conjecture that if the two digraphs  $G_1$  and  $G_2$  can be decomposed into directed Hamilton cycles then their lexicographic product can also be decomposed into directed Hamilton cycles is still open (see [9] for partial results). Even the analogous version of Theorem 5 for complete multipartite digraphs is still open.

Substantial progress has been made on Kelly's conjecture that every regular tournament can be decomposed into directed Hamilton cycles by Thomassen [40] who proved that every regular tournament of order  $n$  can be covered by a collection of  $12n$  directed Hamilton cycles and by Häggkvist [20] who proved that the conjecture is true for sufficiently large regular tournaments. Furthermore he proved that every sufficiently large  $k$ -regular digraph with at most  $2k$  vertices has a Hamilton decomposition.

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**Addendum to the article**  
**Hamiltonian Decompositions of Graphs Directed Graphs and**  
**Hypergraphs**

by J-C. Bermond.

(December 1986)

1) Concerning the hamiltonian decomposition of the **cartesian sum** of graphs:

**Conjecture 1.8.** concerning the decomposition of the cartesian **sum of 3 cycles** has been proved by M.F.Foregger[8]. J.Aubert and B.Schneider [4] proved that the cartesian sum of  $G$  and  $C$  (where  $G$  is decomposable into 2 hamiltonian cycles and  $C$  is a hamiltonian cycle) can itself be decomposed into 3 hamiltonian cycles. That implies the result of M.F.Foregger, but also **Ringel's conjecture** that the **2n-cube** can be decomposed into  $n$  hamiltonian cycles and the original **Kotzig's conjecture** saying that if the graphs  $G_i$ ,  $1 \leq i \leq p$ , can be decomposed into  $n$  hamiltonian cycles then the cartesian sum of these graphs can also be decomposed into  $pn$  hamiltonian cycles (the 2n-cube is in fact the cartesian sum of  $n$   $C_4$  and Ringel's conjecture was a particular case of Kotzig's conjecture ). Indeed Aubert and Schneider's result implies that if  $G_1$  can be decomposed into  $n_1$  hamiltonian cycles and  $G_2$  can be decomposed into  $n_2$  hamiltonian cycles, where  $n_1 \leq n_2 \leq 2n_1$  then the cartesian sum of  $G_1$  and  $G_2$  can be decomposed into  $n_1 + n_2$  hamiltonian cycles; then Kotzig's conjecture follows by induction.

**Conjecture 1.10** concerning the hamiltonian decomposition of the cartesian sum of **two complete graphs** has been proved by J.Aubert and B.Schneider [5]. x

However the **general conjecture 1.11** saying that "if  $G_1$  can be decomposed into  $n_1$  hamiltonian cycles and  $G_2$  can be decomposed into  $n_2$  hamiltonian cycles, then the cartesian sum of  $G_1$  and  $G_2$  can be decomposed into  $n_1 + n_2$  hamiltonian cycles" is still open.

B. Alspach and M. Rosenfeld [3] studied when the cartesian sum of a **cubic graph**  $G$  and  $K_2$  has a hamiltonian decomposition; they give a necessary and sufficient condition which implies that it works if  $G$  has a perfect 1-factorization (a partition of the edges into 1-factors such that the union of every pair of 1-factors forms a hamiltonian cycle); they asked if it is still true when the graph  $G$  is only hamiltonian or admits a 1-factorization.

2) Concerning the hamiltonian decomposition of the **lexicographic product** of graphs:

Conjecture 1.16 (and therefore 1.15) concerning the hamiltonian decomposition of the **lexicographic product** of any two graphs has been proved by Z. Baranyai and Gy.R. Szasz [6]. Note that the particular cases where one graph is a stable set and the other a cycle (theorem 1.12.) or a complete graph (theorem 1.3.) were also proved by G. Hetyei [10].

3) Concerning the hamiltonian decomposition of the **conjunction** of two graphs:

Theorem 1.18. and corollary 1.19. as enonced are clearly false. Indeed the conjunction of two cycles is not connected if they are both of even length; theorem 1.18. is valid with the additional hypothesis that  $r$  and  $n$  are not both even. Simarly one has to add in corollary 1.19. that one of  $G_1$  or  $G_2$  is of odd order.

4) Other problems in the undirected cases:

B. Alspach asked the following problem [2]: does every connected **Cayley graph** on an abelian group with even degree admit a decomposition into hamiltonian cycles ? The particular case of degree 4 (which includes double loop or 2-jump circulant graphs) has been recently solved by J-C. Bermond, O. Favaron and M. Maheo [7]. In [1] B. Alspach proved that if  $G$  is a connected transitive graph of order  $2p$ , where  $p$  is a prime and  $p \equiv 3 \pmod{4}$  then  $G$  can be decomposed into hamiltonian cycles ( or hamiltonian cycles plus a 1-factor).

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Cayley digraphs of prime power are  
hamiltonian JCTB 40 (86)  
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F. Jaeger [11] proved that if  $G$  can be decomposed into an even number of hamiltonian cycles then its line graph  $L(G)$  is 1-factorable. The key of the proof lies in the following result: if  $G$  can be decomposed into two hamiltonian cycles then its line graph  $L(G)$  can be decomposed into three hamiltonian cycles. We propose the following conjecture: if  $G$  can be decomposed into  $k$  hamiltonian cycles then its **line graph**  $L(G)$  can be decomposed into  $2k-1$  hamiltonian cycles.

Let me recall old conjectures which as far as I know are still unsolved:

The graph  $O_k$  (called **odd graph**) is a  $k$ -regular graph whose vertices are indexed by the  $(k-1)$ -subsets of a  $(2k-1)$ -set. Two vertices are adjacent if and only if their indexing subsets are disjoint. For example  $O_3$  is the Petersen graph. G. H. J. Meredith and E. K. Lloyd [13] conjectured that  $O_{2k}$  (resp.  $O_{2k+1}$ ) can be decomposed into hamiltonian cycles (resp. hamiltonian cycles plus a 1-factor). They proved that for  $O_5$  and  $O_6$ .

A. Kotzig (1973?) conjectured that any **self-complementary regular graph** of order  $4k+1$  can be decomposed into hamiltonian cycles. This is a particular case of Nash-Williams' conjecture that every  **$2k$ -regular graph** with at most  $4k+1$  vertices has a hamiltonian decomposition. A very strong result has been recently obtained by R. Häggkvist [9] who proved that for any  $p < 4$  every sufficiently large  $2k$ -regular graph with at most  $pk$  vertices has a hamiltonian decomposition.

Finally let us mention that A. G. Thomason [14] has proved that if a  $2k$ -regular graph has a hamiltonian decomposition, then it has at least  $(3k-2)(3k-5) \dots 7 \cdot 4$  hamiltonian decompositions.

## 5) Directed case:

A characterization of the **Cayley digraphs** of finite abelian groups which can be decomposed into 2 hamiltonian circuits has been given by K. Keating [12]. That implies that the **cartesian sum of two directed** circuits  $C_r$  and  $C_s$  can be decomposed into hamiltonian circuits if and only if there exist positive integers  $u$  and  $v$  such that  $u+v = \gcd(r,s)$  and  $\gcd(uv,rs) = 1$  (result first obtained by J. Lindgren). In [12] can also be found results on the conjunction of two digraphs. The conjecture that if the two digraphs  $G_1$  and  $G_2$  can be decomposed into hamiltonian circuits then their **lexicographic product** can also be decomposed into hamiltonian circuits is still open (see [6] for partial results).

Substantial progress have been made on **Kelly's conjecture** that every **regular tournament** can be decomposed into hamiltonian circuits by C. Thomassen [15] who proved that every regular tournament of order  $n$  can be covered by a collection of  $12n$  hamiltonian circuits and by R. Häggkvist [9] who proved that the conjecture is true for sufficiently large regular tournaments. Furthermore he proved that every sufficiently large  $k$ -regular digraph with at most  $2k$  vertices has a hamiltonian decomposition.

f Jackson J. Zimla talk for NY (1989)

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