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► **To cite this version:**

Jean-Claude Bermond, Claudine Peyrat. Broascasting in de Bruijn networks. *Congressus Numerantium*, 1988, 66, pp.283-292. hal-03013393

**HAL Id: hal-03013393**

**<https://inria.hal.science/hal-03013393>**

Submitted on 18 Nov 2020

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# BROADCASTING IN DE BRUIJN NETWORKS.

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**Abstract:** Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of a network. The broadcast time of a vertex is the minimum number of time units required to complete broadcasting from this vertex. The broadcast time of a graph is the maximum broadcast time of any vertex. Several previous papers have investigated methods to construct sparse networks in which this process can be completed in minimum time from any originator. But the graphs produced by these methods contain high degree vertices. If these graphs are to be used in the design of actual communication networks (for example for parallel computers) other considerations can override the need for minimum broadcasting time. So we are interested in networks with a fixed maximum degree, in which every vertex can broadcast quickly. Here, we study the broadcasting time of de Bruijn and Kautz networks (which are among the best known families of interconnection networks for given degree and diameter). These networks have a simple broadcasting scheme and for small degree they have a very good broadcasting time of order  $\log n$ . For example for maximum degree 4, we obtain networks on  $2^D$  or  $2^D + 2^{D-1}$  vertices, with broadcasting time at most  $1.5 D$ , improving on earlier results.

## 1. Definitions and previous results.

Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of a network. This is to be completed as quickly as possible, subject to the constraints that each call involves only two vertices, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call a vertex to which it is adjacent.

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This article was written while the authors were visiting the School of Computing Science at Simon Fraser University, Burnaby, Canada, V5A 1S6; support of Advanced Systems Foundations and Simon Fraser University is gratefully acknowledged.



Given a connected graph  $G$  and a message originator, vertex  $u$ , we define the **broadcast time of vertex  $u$** ,  $b(u)$ , to be the minimum number of time units required to complete broadcasting from vertex  $u$ . It is easy to see that for any vertex  $u$  in a connected graph with  $n$  vertices,  $b(u) \geq \lceil \log_2 n \rceil$ , since during each time unit the number of vertices informed can at most double.

We define the **broadcast time of a graph  $G$** ,  $b(G)$ , to be the maximum broadcast time of any vertex  $u$  in  $G$ . For the complete graph  $K_n$ ,  $b(K_n) = \lceil \log_2 n \rceil$ , yet  $K_n$  is not minimal with respect to this property. That is we can remove edges from  $K_n$  and still have a graph  $G$  with  $n$  vertices and  $b(G) = \lceil \log_2 n \rceil$ .

So far, the emphasis in research in this subject has been on obtaining graphs  $G$  with  $n$  vertices, minimum broadcasting time  $b(G) = \lceil \log_2 n \rceil$  and having the smallest possible number of edges. Let us call such a graph a **minimum broadcast graph** if it has the minimum number of edges and broadcasting time  $\lceil \log_2 n \rceil$  (for a given  $n$ ).

The only known general family of minimum broadcast graphs is the family of the  $k$ -cubes. The vertices of a  $k$ -cube are labelled with the  $k$ -tuples on a binary alphabet and two vertices are adjacent if their associated  $k$ -tuples differ in exactly one coordinate.

The  $k$ -cube has  $2^k$  vertices and  $k2^{k-1}$  edges. Since the  $k$ -cube is the cartesian sum of two  $(k-1)$ -cubes, it is easy to show by induction that its broadcasting time is  $k = \log_2 n$ . It is a minimum broadcast graph because in a graph on  $2^k$  vertices with broadcasting time  $k$  the originating vertex must be involved in at least  $k$  calls and so the minimum degree is at least  $k$ . Except for the  $k$ -cubes, only a few minimum broadcast graphs have been found, namely for  $n \leq 15$  by Farley, Hedetniemi, Mitchell and Prokurovski [9],  $n = 17$  by Mitchell and Hedetniemi [18],  $n = 18$  by Wang (see [4]) and  $n = 19$  and  $n = 30$  by Bermond, Hell, Liestman and Peters [4]. A variety of techniques have been used to construct sparse broadcast graphs with minimum broadcast time and a small but not necessary minimum number of edges (see [7, 8, 19, 4]).

From an applications perspective, minimum broadcast graphs represent the cheapest possible communication networks (having the fewest links) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible. However if these graphs are to be used in the design of actual networks (for example in parallel computers) other considerations may override the need for minimum broadcasting time. In particular the constructions of Farley [8] and of Chau and Liestman [7] result in networks with average degree  $O(\log_2 n)$  while the construction of Peleg [19] yields networks with some vertices of degree  $O(n)$ . It may be more realistic to use a graph with a fixed maximum degree in

which every vertex can broadcast "quickly". Indeed, the maximum degree is an important parameter in the design of interconnection networks, for cost reasons (small number of lines) and is also important for VLSI layout problems.

For other considerations on broadcasting we refer the reader to the survey of Hedetniemi, Hedetniemi and Liestman [11] and for considerations in the design of interconnection networks to the survey of Bermond, Delorme and Quisquater [2] or the forthcoming book of D.I. Ameter and Max de Gree [1].

In a recent paper, Liestman and Peters [17] investigated broadcast networks with maximum degrees  $\Delta = 3$  and  $\Delta = 4$ . They gave lower bounds on the time required to broadcast in such graphs and presented constructions which produce good broadcast networks.

The lower bound follows from the fact that each vertex can participate in at most  $\Delta$  calls. So, if each vertex informs its neighbors as quickly as possible after it has received the message, the only vertices that can place calls at time  $t$  are those that first became informed at times  $i$  where  $t - \Delta + 1 \leq i \leq t - 1$ . So, if we let  $a_t$  denote the maximum number of nodes which could be informed after  $t$  units of time in any graph with maximum degree  $\Delta$  we have  $a_t = 2a_{t-1} - a_{t-\Delta}$ .

Let  $b(n, \Delta)$  denote the minimum of  $b(G)$  over all graphs  $G$  with  $n$  vertices and maximum degree  $\Delta$ .

Liestman and Peters [17] showed in particular that  $1.4404 \log_2 n \leq b(n, 3)$  and that if  $n$  is power of 2,  $b(n, 3) \leq 2 \log_2 n + 1$ . The upper bound is achieved by constructing folded shuffle-exchange graphs. In the case of maximum degree 4, they showed that  $1.1374 \log_2 n \leq b(n, 4)$  and that if  $n$  is a power of 4,  $b(n, 4) \leq 1.625 \log_2 n + 2.25$ . The upper bound is achieved by constructing folded 4-shuffle-exchange graphs.

Here we consider general families of networks, namely the de Bruijn and Kautz networks, which are the general families of graphs of bounded maximum degree and diameter having the largest known number of vertices. These networks might be an alternative for the  $k$ -cube for designing new parallel architectures (see Bermond and Peyrat [5]). We determine in this article an upper bound on their broadcasting time. By considering the special case of de Bruijn or Kautz networks of degree 4, we show that  $b(n, 4) \leq 1.5 \log_2 n + 1$  when  $n$  is a power of 2 or 3 times a power of 2, which improves on the bound given in [17]. Furthermore compounds of de Bruijn graphs appear to give the best upper bounds known for  $b(n, \Delta)$  (see the article of Bermond, Hell, Liestman and Peters [3]).

## 2. Broadcasting in de Bruijn graphs.

The **de Bruijn digraph  $B(d, D)$**  (defined in [6]) with in- and out-degree  $d$  and diameter  $D$  is the digraph whose vertices are the words of length  $D$  on an alphabet of  $d$  letters. There



is an arc from a vertex  $x$  to a vertex  $y$  if and only if the last  $D-1$  letters of  $x$  are the same as the first  $D-1$  letters of  $y$ , that is, there is an arc from  $(x_1, x_2, \dots, x_D)$  to the vertices  $(x_2, \dots, x_D, \lambda)$ , where  $\lambda$  is any letter of the alphabet. This digraph has  $d^D$  vertices. We will denote by  $UB(d, D)$  the associated undirected de Bruijn graph of maximum degree  $\Delta = 2d$ , and diameter  $D$ . That is,  $UB(d, D)$  is the graph whose vertices are the words of length  $D$  on an alphabet of  $d$  letters in which the vertex  $(x_1, \dots, x_D)$  is adjacent to the vertices  $(\lambda, x_1, \dots, x_{D-1})$  and  $(x_2, \dots, x_D, \lambda)$ .

**Theorem:** Let  $UB(d, D)$  be the undirected de Bruijn graph of maximum degree  $2d$  and diameter  $D$ . Then  $b(UB(d, D)) = \frac{(d+1)}{2} D + \frac{d}{2}$ .

*First proof:* Suppose the message originates at vertex  $(x_1, x_2, \dots, x_D)$ . First broadcast from this vertex successively to the vertices  $(x_2, x_3, \dots, x_D, 0)$ ,  $(x_2, x_3, \dots, x_D, 1)$ , ...,  $(x_2, x_3, \dots, x_D, d-1)$ , and then to the vertices  $(d-1, x_1, x_2, \dots, x_{D-1})$ , ...,  $(0, x_1, x_2, \dots, x_{D-1})$ . So, vertex  $(x_2, x_3, \dots, x_D, i)$  is informed at time  $i+1$  and vertex  $(j, x_1, x_2, \dots, x_{D-1})$  is informed at time  $2d-j$ . For any other vertex  $(y_1, y_2, \dots, y_D)$  that first receives the message at time  $t$ , broadcast with the following rules :

- i) If the message arrives from a left neighbor  $(\lambda, y_1, \dots, y_{D-1})$ , broadcast it to the right neighbors in the order  $(y_2, \dots, y_D, 0)$ ,  $(y_2, \dots, y_D, 1)$ , ...,  $(y_2, \dots, y_D, d-1)$ . So, vertex  $(y_2, \dots, y_D, i)$  receives the message at time  $t+i+1$ .
- ii) If the message arrives from a right neighbor  $(y_2, \dots, y_D, \lambda)$ , broadcast it to the left neighbors in the order  $(d-1, y_1, \dots, y_{D-1})$ ,  $(d-2, y_1, \dots, y_{D-1})$ , ...,  $(0, y_1, \dots, y_{D-1})$ . So vertex  $(j, y_1, \dots, y_{D-1})$  will be informed at time  $t+d-j$ .

Using these simple rules we obtain that vertex  $(z_1, z_2, \dots, z_D)$  is informed

a) at time at most  $\sum_1 (z_i+1)$

b) at time at most  $d + \sum_1 (d-z_i)$ .

To prove a), consider the following canonical path of length  $D$  from  $(x_1, x_2, \dots, x_D)$  to  $(z_1, z_2, \dots, z_D)$  :  $(x_1, \dots, x_D)$ ,  $(x_2, \dots, x_D, z_1)$ , ...,  $(x_i, \dots, x_D, z_1, \dots, z_{i-1})$ , ...,  $(z_1, \dots, z_D)$ . The vertex  $(x_2, \dots, x_D, z_1)$  is informed at time  $t_1 = z_1 + 1$ , by rule i), the vertex  $(x_3, \dots, x_D, z_1, z_2)$  is informed at time  $t_2 = t_1 + z_2 + 1$ , and so on. Therefore the vertex  $(z_1, \dots, z_D)$  is informed at time at most  $\sum_1 (z_i+1)$ .

Similarly, for b) consider the other canonical path  $(x_1, \dots, x_D)$ ,  $(z_D, x_1, \dots, x_{D-1})$ , ...,  $(z_i, \dots, z_D, x_1, \dots, x_{i-1})$ , ...,  $(z_1, \dots, z_D)$ . The vertex  $(z_D, x_1, \dots, x_{D-1})$  is informed at time  $d+d-z_D$ . By applying rule ii), if the vertex  $(z_{j+1}, \dots, z_D, x_1, \dots, x_j)$  is informed at time  $t_j$ , then the vertex  $(z_j, \dots, z_D, x_1, \dots, x_{j-1})$  is informed at time at most  $t_j + d - z_j$ . So

$(z_1, \dots, z_D)$  is informed at time at most  $d + \sum_1 (d-z_i)$ .

By a) and b), vertex  $(z_1, \dots, z_D)$  is informed at time  $t = \min(\sum_1 (z_i+1), d + \sum_1 (d-z_i))$ .

Therefore  $2t \leq d + \sum_1 ((z_i+1) + (d-z_i)) = (d+1)D + d$ , and  $b(UB(d, D)) \leq \frac{d+1}{2} D + \frac{d}{2}$ .  $\square$

*Second proof:* We now give a second proof that uses a different broadcasting scheme and was inspired by the work of Jerrum and Skyum [15]. It appears to be more useful than the first proof for computing broadcast time in compounds of de Bruijn graphs (see [3]). Any vertex (including the originator of the message) broadcasts only to its right neighbors (so a vertex will receive messages only from its left neighbors). The order of broadcasting will be determined according to the  $d$ -arity  $\alpha$  of  $(y_1, \dots, y_D)$ , that is  $\alpha \equiv \sum_1 y_i \pmod{d}$

and  $0 \leq \alpha \leq d-1$ . Vertex  $(z_1, \dots, z_D)$  will broadcast to its right neighbors, in the order  $(y_2, \dots, y_D, \alpha)$ ,  $(y_2, \dots, y_D, \alpha+1)$ , ...,  $(y_1, \dots, y_D, \alpha-1)$ . The last digit is taken modulo  $d+1$ .

Now, consider the following  $d$  paths of length  $D+1$  from  $(x_1, \dots, x_D)$  to  $(z_1, \dots, z_D)$ , where  $0 \leq k \leq d-1$  :

$P_k : (x_1, \dots, x_D), (x_2, \dots, x_D, k), (x_3, \dots, x_D, k, z_1), \dots, (x_{i+2}, \dots, x_D, k, z_1, z_i), \dots,$

$(k, z_1, \dots, z_{D-1}), (z_1, \dots, z_D)$ .

Let  $t_k^0$  denote the number of units of time required to broadcast from  $(x_1, x_2, \dots, x_D)$  to  $(x_2, \dots, x_D, k)$  and let  $t_k^i$  denote the number of units of time to broadcast from  $(x_{i+1}, \dots, x_D, k, z_1, \dots, z_{i-1})$  to  $(x_{D+2}, \dots, x_D, k, z_1, \dots, z_i)$ . Note that the  $d$ -arity of all corresponding intermediate vertices in any two different paths are different (all the coordinates are the same, except the coordinate containing the value  $k$ ). Therefore, for a fixed  $i$ ,  $0 \leq i \leq D$ , when  $k$  varies from  $0$  to  $d-1$ , the  $t_k^i$ 's take all the values between  $1$  and  $d$ . So,

$\sum_k t_k^i = \frac{d(d+1)}{2}$ . Therefore if  $t_k$  denotes the time to broadcast the message via  $P_k$ ,  $\sum_k t_k = (D+1)(1+2+\dots+d) = \frac{(D+1)d(d+1)}{2}$ . As there are  $d$  different paths the message will reach

$(z_1, \dots, z_D)$ , on one of these paths at a time at most  $\frac{d+1}{2}(D+1)$ .

*Remark.* The two proofs give two easy algorithms to broadcast the messages. The algorithm from the first proof is completely independent of the vertex receiving the message. The first proof cannot be generalized to the other known general family of networks : the Kautz networks. The second proof can be generalized with some refinements. We will give here a third proof which can also be applied to the de Bruijn graphs (with a greater additive constant). This gives a third broadcasting scheme.

The **Kautz digraph**  $K(d, D)$  (defined in [16]) with in- and outdegree  $d$  and diameter  $D$  is the subgraph of the de Bruijn digraph  $B(d+1, D)$  induced by the set of vertices without



two consecutive identical letters. In other words the vertices are labeled with words  $(x_1, \dots, x_D)$ , where  $x_i$  belongs to an alphabet of  $d+1$  letters, and  $x_i \neq x_{i+1}$ , for  $1 \leq i \leq D-1$ . The vertex  $(x_1, \dots, x_D)$  is joined to the  $d$  vertices  $(x_2, \dots, x_D, \lambda)$ , where  $\lambda$  is any letter different from  $x_D$ . This digraph has  $d^D + d^{D-1}$  vertices.

We denote by  $UK(d,D)$  the undirected Kautz graph of maximum degree  $\Delta = 2d$ , diameter  $D$ . That is,  $UK(d,D)$  is the graph whose vertices are the words of length  $D$  from an alphabet of  $d+1$  letters, with no two consecutive identical letters, in which the vertex  $(x_1, \dots, x_D)$  is adjacent to the vertices  $(\lambda, x_1, \dots, x_{D-1})$  and  $(x_2, \dots, x_D, \lambda)$ .

**Theorem:** Let  $UK(d,D)$  be the Kautz graph of maximum degree  $2d$  and diameter  $D$ . Then  $b(UK(d,D)) = \frac{(d+1)}{2} D + \frac{2d-1}{2}$ .

*Proof:* The proof is similar to the first one given for de Bruijn graphs, but the broadcasting algorithm now depends on the last coordinate of the current vertex.

When a message arrives for the first time at time  $t$  at vertex  $(y_1, y_2, \dots, y_D)$  then the vertex broadcasts with the following rules:

- i) If the message arrives from a left neighbor  $(\lambda, y_1, \dots, y_{D-1})$ , the vertex broadcasts it to the right neighbors in the order  $(y_2, \dots, y_D, y_D+1)$ ,  $(y_2, \dots, y_D, y_D+2)$ , ...,  $(y_2, \dots, y_D, y_D-1)$  (the value of the last coordinate is taken modulo  $d+1$ ). So, vertex  $(y_2, \dots, y_D, i)$  receives the message at time  $t+i-y_D$  if  $i > y_D$  or at time  $t+d+1+i-y_D$  if  $i < y_D$ .
- ii) If the message arrives from a right neighbor  $(y_2, \dots, y_D, \lambda)$ , the vertex broadcasts it to the left neighbors in the order  $(y_1+1, y_1, \dots, y_{D-1})$ ,  $(y_1+2, y_1, \dots, y_{D-1})$ , ...,  $(y_1-1, y_1, \dots, y_{D-1})$ . So, the vertex  $(i, y_1, \dots, y_{D-1})$  will be informed at time  $t+i-y_1$  if  $i > y_1$  and at time  $t+d+1+i-y_1$  if  $i < y_1$ .

The rules above are also used by the originator of the message to broadcast first to its right neighbors then to its left neighbors.

a) Using rule i) and the canonical path from  $(x_1, \dots, x_D)$  to  $(z_1, \dots, z_D)$ :

$(x_1, \dots, x_D), (x_2, \dots, x_D, z_1), \dots, (x_D, z_1, \dots, z_{D-1}), (z_1, \dots, z_D)$  if  $x_D \neq z_1$ , or the path  $(x_1, \dots, x_D), (x_2, \dots, x_D, z_2), \dots, (x_D, z_2, \dots, z_D)$  if  $x_D = z_1$ ,

we obtain that vertex  $(z_1, z_2, \dots, z_D)$  is informed at time at most:

$$(z_1 - x_D + \epsilon_1) + (z_2 - z_1 + \epsilon_2) + \dots + (z_i - z_{i-1} + \epsilon_i) + \dots + (z_D - z_{D-1} + \epsilon_D) = z_D - x_D + \sum_1 \epsilon_i$$

where  $\epsilon_1 = 0$  if  $z_1 \geq x_D$  and  $\epsilon_1 = d+1$  otherwise, and  $\epsilon_i = 0$  if  $z_i > z_{i-1}$  and  $\epsilon_i = d+1$  otherwise, for  $2 \leq i \leq D$ .

b) Using rule ii) and the other canonical path from  $(x_1, \dots, x_D)$  to  $(z_1, \dots, z_D)$ :

$(x_1, \dots, x_D), (z_D, x_1, \dots, x_{D-1}), \dots, (z_2, \dots, z_D, x_1), (z_1, \dots, z_D)$  if  $z_D \neq x_1$  or the path

$(x_1, \dots, x_D), (z_{D-1}, x_1, \dots, x_{D-1}), \dots, (z_1, z_2, \dots, z_{D-1}, x_1)$  if  $\bar{x}_1 = z_D$ , we obtain that vertex  $(z_1, \dots, z_D)$  is informed at time at most:

$$d + (z_D - x_1 + \epsilon'_1) + (z_{D-1} - z_D + \epsilon'_D) + \dots + (z_{i-1} - z_i + \epsilon'_i) + \dots + (z_1 - z_2 + \epsilon'_2) = d + z_1 - x_1 + \sum_1 \epsilon'_i$$

where  $\epsilon'_1 = 0$  if  $z_D \geq x_1$  and  $\epsilon'_1 = d+1$  otherwise, and  $\epsilon'_i = 0$  if  $z_{i-1} > z_i$  and  $\epsilon'_i = d+1$  otherwise, for  $2 \leq i \leq D$ .

So the vertex  $(z_1, z_2, \dots, z_D)$  is informed at time at most  $\frac{1}{2} [d + z_D - x_D + z_1 - x_1 + \sum_1 (\epsilon_i + \epsilon'_i)]$ .

Note that  $\epsilon_i + \epsilon'_i = d+1$ , for  $2 \leq i \leq D$ ; and that  $z_1 - x_D + \epsilon_1 \leq d$  and  $z_D - x_1 + \epsilon'_1 \leq d$ .

$$\text{So, } b(UK(d,D)) \leq \frac{3d + (D-1)(d+1)}{2} = \frac{d+1}{2} D + \frac{2d-1}{2}.$$

**Corollary:** For  $n = d^k$ ,  $b(n, 2d) \leq \frac{(d+1)}{2} \log_d n + \frac{d}{2}$ , and for  $n = d^k + d^{k-1}$ ,  $b(n, 2d) \leq \frac{(d+1)}{2} (\log_d n + 1 - \log_d(d+1)) + \frac{2d-1}{2}$ .

*Proof:* Use either  $UB(d,D)$  or  $UK(d,D)$  and note that, for  $UB(d,D)$ ,  $n = d^D$ , so  $D = \log_d n$ , and for  $UK(d,D)$ ,  $n = d^D + d^{D-1}$ , so  $D = \log_d \frac{n}{d+1} + 1$ .

In particular for  $d = 2$  we obtain that for  $n = 2^k$  or  $n = 3 \cdot 2^{k-1}$ ,  $b(n, 4) \leq 1.5 \log_2 n + 1$ , improving on the result of Liestman and Peters [17].

So far, we have obtained some "good" broadcast networks for even maximum degrees. For odd degrees we can use networks similar to de Bruijn or Kautz networks for example the sequence graphs introduced by Fiol, Yebra and Fabrega [10] or some other generalizations (see Gomez, Fiol and Yebra [14]).

Let the diameter  $D$  be even and equal to  $2k$ ; the vertices of the sequence graph  $S(d, d+1, k)$  are labelled by the  $2k$ -tuples  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$  where  $x_i$  belongs to an alphabet  $X$  of  $d$  letters and  $y_i$  belongs to an alphabet  $Y$  of  $d+1$  letters; vertex  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$  is adjacent to vertices  $(\lambda, y_k, x_k, \dots, y_2, x_2, y_1)$  where  $\lambda$  can be any of the  $d$  letters of  $X$ , and  $(x_k, y_{k-1}, \dots, y_2, x_2, y_1, x_1, \mu)$  where  $\mu$  can be any of the  $d+1$  letters of  $Y$ .

$S(d, d+1, k)$  has degree  $2d+1$ , diameter  $2k$  and  $d^k(d+1)^k$  vertices.

Another way to define  $S(d, d+1, k)$  is to consider words on a two alphabets of the form  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$  or  $(y_1, x_1, y_2, x_2, \dots, y_k, x_k)$  where  $x_i$  belongs to an alphabet  $X$  of  $d$  letters and  $y_i$  belongs to an alphabet  $Y$  of  $d+1$  letters; we will consider the two words  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$  and  $(y_k, x_k, \dots, y_2, x_2, y_1, x_1)$  as identical and represent them by a unique vertex. Now vertex  $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$  is (like in de Bruijn graphs) adjacent to vertices  $(y_1, x_2, \dots, x_k, y_k, \lambda)$  where  $\lambda$  can be any of the  $d$  letters of  $X$ , and  $(\mu, x_1, y_1, \dots, x_k)$  where  $\mu$  can be any of the  $d+1$  letters of  $Y$ .

With the last definition, a proof similar to that of the first proof of the theorem concerning de Bruijn networks gives:

**Theorem:** Let  $S(d,d+1,k)$  be the sequence graph with degree  $2d+1$  and diameter  $k$ . Then

$$b(S(d,d+1,k)) = \frac{(2d+3)}{2} k + \frac{d}{2}.$$

**Corollary:** For  $n = d^k(d+1)^k$ ,  $b(n,2d+1) \leq \frac{(2d+3)}{2} \log_{d(d+1)} n + \frac{d}{2}$ .

### 3. Conclusion and Problems.

The following table indicates the asymptotic upper bounds on  $b(n, \Delta)$ , for  $4 \leq \Delta \leq 10$ , obtained with de Bruijn or sequence networks. We have also indicated the lower bounds. We have not indicated the additive constants (which are all less than 2.5).

$n$	$\Delta$	upper bound on $b(n,\Delta)$	lower bound on $b(n,\Delta)$
$2^D$	4	$1.5 \log_2 n = 1.500 \log_2 n$	$1.137 \log_2 n$
$6^{\frac{D}{2}}$	5	$3.5 \log_6 n = 1.354 \log_2 n$	$1.056 \log_2 n$
$3^D$	6	$2 \log_3 n = 1.262 \log_2 n$	$1.025 \log_2 n$
$12^{\frac{D}{2}}$	7	$4.5 \log_{12} n = 1.255 \log_2 n$	$1.012 \log_2 n$
$4^D$	8	$2.5 \log_4 n = 1.250 \log_2 n$	$1.006 \log_2 n$
$20^{\frac{D}{2}}$	9	$5.5 \log_{20} n = 1.273 \log_2 n$	$1.003 \log_2 n$
$5^D$	10	$3 \log_5 n = 1.292 \log_2 n$	$1.001 \log_2 n$

As can be seen in the table above, de Bruijn graphs and their generalizations appear to have a good broadcasting time at least for  $\Delta \leq 8$ . For  $\Delta > 8$ , the upper bound  $b(n, \Delta)$  obtained increases with  $\Delta$ , which is clearly not good. However, as we already mentioned, another interest of these networks is to use them in compound graphs. By replacing the

vertices of well chosen de Bruijn networks by ad hoc graphs, Bermond, Hell, Liestman and Peters [3] have obtained the best known broadcasting networks with a given maximum degree.

An other fact is that de Bruijn or Kautz networks exist only for some sparse values of  $n$ ; so these techniques give upper bound on  $b(n,\Delta)$  only for some values of  $n$ . Therefore it will be interesting to determine the broadcasting time of the generalized de Bruijn graphs introduced by Reddy Pradhan and Kuhl [20] and Imase and Itoh [12] or the generalized Kautz graphs introduced by Imase and Itoh [13]. Generalized de Bruijn digraphs are defined as follows. The vertices are the integers modulo  $n$ . There is an arc from the vertex  $i$  to the vertex  $j$  if there exists  $a$ ,  $0 \leq a \leq d-1$ , such that  $j \equiv di+a [n]$ . For example, in the case when  $d=2$ , the successors of the vertex labelled by  $i$  are the two vertices labelled by  $2i$  and  $2i+1$ . It might be still true that the broadcasting time in these generalized de Bruijn graphs is still at most  $\frac{(d+1)}{2} \log_2 n$  but the proofs given above do not seem to work.

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