



De Bruijn and Kautz networks: a competitor for the hypercube?

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► To cite this version:

Jean-Claude Bermond, Claudine Peyrat. De Bruijn and Kautz networks: a competitor for the hypercube?. First European Workshop on Hypercube and Distributed Computers, Oct 1989, Rennes, France. pp 279-283. hal-03013384

HAL Id: hal-03013384

<https://inria.hal.science/hal-03013384>

Submitted on 18 Nov 2020

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ABSTRACT : In this paper we give a survey of the known properties of de Bruijn and Kautz networks showing that their behavior is as good as the hypercube's. As they have the advantage of connecting considerably more processors, they are good competitors for the hypercube and might constitute the next generation of parallel architectures.

I. Introduction.

The Connection Machine (see D. Hillis' book [30]) has demonstrated the viability of constructing massively parallel supercomputers using current hardware technology. Let us quote some sentences from chapter 3.2 : "The most difficult technical problem in the design of a Connection Machine is the design of the general interconnection network through which the processors communicate. The communications network represents most of the cost of the machine, most of the power dissipation, most of the wiring, and most of the performance limitations. This is in part because we have relatively little experience in designing such networks, so our methods are far from optimal. But it is also because designing such networks is fundamentally hard; the communication network is doing most of the computation."

Therefore the choice of the topology of the interconnection network is critical in the design of parallel processors. Different goals can be pursued to increase the performance and minimize the cost. These goals can be expressed on the graph (or digraph) that represents the interconnection network. The vertices of the graph correspond to processors and the edges (or arcs) correspond to communication links between processors.

For a complete review of the topic, see the forthcoming books by D. Ameter and Max de Gree[2], and by Tom Leighton [45].

We list here some of the major requirements :

- small communication delay, that is, small diameter. The diameter of a graph is the maximum distance between any pair of vertices and corresponds to the maximum number of times a message has to be forwarded when traveling from one processor to another (one can also consider the average distance instead of the diameter).
- small and fixed degree. Each physical connection costs money and a small degree corresponds to reduced wiring (if the network is regular, the two parameters are proportional); furthermore if the degree is fixed, only one processor design has to be built.
- easy routing algorithms. The choice of a routing algorithm is important (see [30, chapter 3.5]) and depends on the chosen topology. For example it is interesting to have an algorithm that only needs a knowledge of the destination address.

- uniformity or symmetry. One can desire that all processors behave in the same manner and that they communicate in similar ways, or at least that there is some balance in the traffic patterns. This implies at least some regularity and some symmetric properties on the graph.

- fault tolerance. The network must continue to work in case of node or link failures. Different notions of vulnerability (and fault tolerance) exist, the simplest one corresponding to the connectivity of the graph (that is, the minimum number of vertices (or edges) which must be deleted in order to destroy all the paths between a pair of vertices).

- extendability. It should be possible to build a network of any given size, or at least to build arbitrarily large versions of the network. Furthermore, it would be nice to construct large networks from small ones.

- embeddability of other topologies. That would enable one to use algorithms originally designed for another topology (for example, algorithms on a ring, a binary tree or a grid).

- efficient layout

Note that the wish list contains contradictions. For example, if one fixes a diameter D and a maximum degree Δ , then the maximum number of processors that can be interconnected is bounded by a function of D and Δ , called the Moore bound, equal to $1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \dots + \Delta(\Delta-1)^{D-1}$. Similarly, one cannot have both a small degree and a big number of disjoint paths. Therefore any decision will be a compromise.

In the past, designers have restricted their attention to simple interconnection topologies such as complete networks or circular rings. Now the literature offers a rich choice of possible interconnection topologies ("the topology zoo"). Here we restrict our attention to point to point topologies which seem to be adapted to form very large systems to support connectionist models of computation (see Hwang and Ghosh [33] and the references included). We do not consider here multistage or bus networks.

The most popular non trivial such topology, currently in use, is the k -cube or hypercube. Its popularity stems from the fact that it represents a good compromise between the different goals and that its topology is well understood.

One way to describe the k -cube is to consider the vertices as binary sequences of length k (or k -tuples), two vertices being joined if the sequences they represent differ in exactly one coordinate. The k -cube has 2^k vertices, is regular of degree k and has diameter k . Its topology is completely symmetric (the k -cube is vertex transitive). The k -cube has a very simple routing algorithm. It has a lot of fault-tolerant properties, in particular there are k disjoint paths between any pair of vertices. The k -cube can also be defined recursively as the sum of two $(k-1)$ -cubes. Furthermore, a large number of algorithms have now been designed to run on it.

The aim of this article is to put in a single paper a survey of all the properties of two other families of networks called **de Bruijn and Kautz networks**. These properties show that these networks are good candidates for the next generation of machines, after the hypercubes. Indeed, their known properties show that these networks behave like the hypercubes, having most of the same good properties. But they allow interconnecting more processors for the same values of the other parameters (or for the same number of processors, they have a considerably smaller degree or diameter). For example, a 8-cube interconnects 256 processors while a de Bruijn network of diameter 8 and degree 8 can interconnect 65,536 processors and the Kautz network with same parameters can interconnect 81,920 processors ! Another advantage on the k -cube is that the degree and the diameter are not related; if the designer insists on a small degree or a small diameter, he can find an adequate graph. Their generalizations are furthermore defined for any

number of vertices. Finally, their definition is based on digraphs, and they are quasi optimal networks for unidirectional links.

Here again, if we insist on some specific parameters, better topologies can be found: for example, if we specify only the maximum degree and the diameter, graphs having a larger number of vertices are known; they solve the so-called (Δ, D) -graph problem (see [10] for a survey). But like the k -cube, de Bruijn and Kautz graphs represent good compromises. Some of their properties are still to be investigated. One of their disadvantages is that they are considerably less known than the k -cube, therefore as far as we know, there is no hardware prototype or commercialized machine based on it, although some teams are planning to use these topologies to interconnect transputers. Furthermore, there exist only a few algorithms designed on them. However, in view of some recent work, we believe that most of the algorithms proposed for the k -cube can be easily transposed on them. Furthermore some general basis distributed algorithms designed for any topology have their message complexity in $O(Dm)$ where m is the number of links, which gives for appropriate Kautz or de Bruijn networks an optimal complexity $O(n \log_2 n)$, (see [7]). Finally let us note that another survey concerning mainly de Bruijn graphs with degree 4 has been recently written by Samatham and Pradhan [54] and the reader is referred to it.

II. de Bruijn and Kautz digraphs.

The most natural way to define these networks is as digraphs (unidirectional links), in particular the de Bruijn graphs also called Good graphs, see [27]. These digraphs have been frequently rediscovered in relation with the existence of particular sequences called de Bruijn sequences or as models for shift registers or in cryptography. For historical references (going back to 1894) and more details, see the surveys of Fredericksen [25], Ralston [50] or Stein's book [56, chapter 10].

It is possible to give several definitions of the de Bruijn and Kautz digraphs, named according to [15, 39]. We begin with the one which is the closest to the definition of the k -cube.

1. Definitions using words on an alphabet.

The de Bruijn digraph $B(d, D)$ is the digraph whose vertices are the words of length D on an alphabet of d letters (or d -ary D -tuples). There is an arc from a vertex x to a vertex y if the $D-1$ first letters of y are equal to the $D-1$ last letters of x . That is, there is an arc from (x_1, x_2, \dots, x_D) to all the vertices $(x_2, \dots, x_D, \alpha)$, where α is any letter of the alphabet (shifting property).

Therefore $B(d, D)$ is d -regular (that is, each vertex has in-degree and out-degree equal to d). It is easy to check that its diameter is D (from a given vertex all vertices are attained after D shiftings) and its number of vertices is d^D . Figure 1 shows $B(2, 1)$, $B(2, 2)$ and $B(2, 3)$.

Similarly, the Kautz digraph $K(d, D)$ is the digraph whose vertices are the words of length D on an alphabet on $d+1$ letters, such that any two consecutive letters are different. Again, there is an arc from x to y if the $D-1$ first letters of y are equal to the $D-1$ last letters of x . That is there is an arc from (x_1, x_2, \dots, x_D) to all the vertices $(x_2, \dots, x_D, \alpha)$, where α is any letter of the alphabet different from x_D .

$K(d, D)$ is d -regular. Its diameter is D and its number of vertices is $d^D + d^{D-1}$. Figure 2 shows $K(2, 1)$, $K(2, 2)$ and $K(2, 3)$.

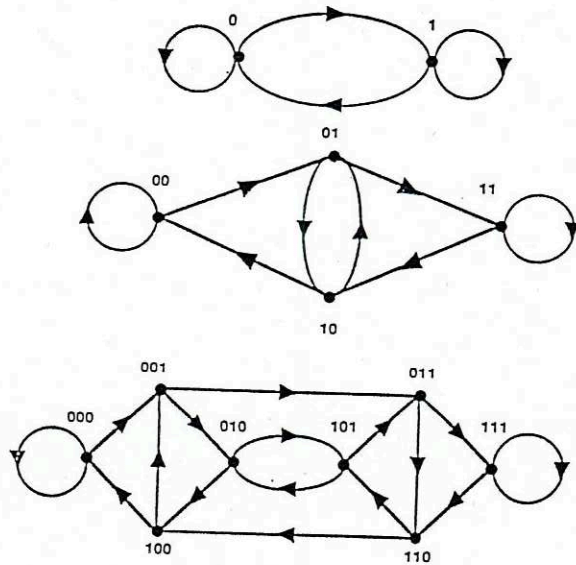


Figure 1

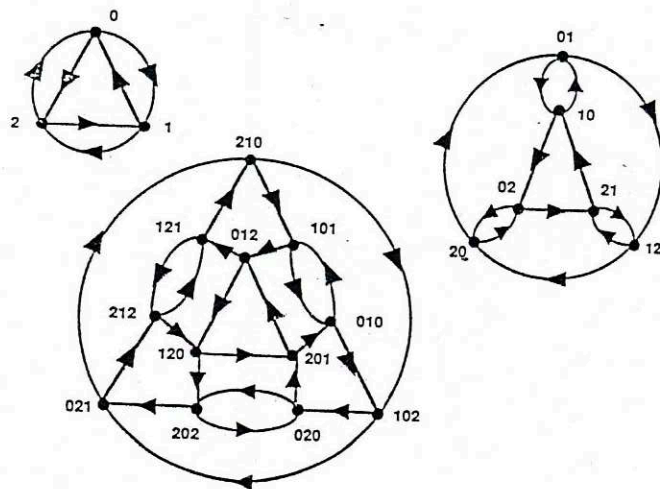


Figure 2

From these definitions, it is clear that $K(d,D)$ is an induced subgraph of $B(d+1,D)$ if $d \geq 2$, that $B(d,D)$ is a subgraph of $B(d+1,D)$ and that $K(d,D)$ a subgraph of $K(d+1,D)$, (for more details, and more relations see [14]).

One can note that, for a fixed D , the number of vertices of these digraphs is of the same asymptotic order as the directed Moore bound (which is equal to $1+d+d^2+\dots+d^D$). When $D=2$, $K(d,2)$ has order $d+d^2$ and is an optimal graph (since it has been proved that it is impossible to reach the directed Moore bound if $d>1$ and $D>1$).

Remark. One can argue that in de Bruijn digraphs, the self loops are not useful; however, they can be deleted and one can use the free links to communicate with the environment (work stations, printer,...). Reddy Pradhan and Khul [51] proposed to modify the de Bruijn digraphs by deleting the d self loops and by adding a directed cycle between the vertices of in- or out-degree $d-1$. They alternatively proposed to add a new vertex joined in both directions to all the vertices of degree $d-1$. One obtains therefore a digraph with d^D+1 vertices and the same parameters.

2. Definitions using line digraph iterations

These definitions are due to Fiol Yebra and Alegre [23].

First let us recall the definition of a line digraph. The line digraph of a digraph G is the digraph LG whose vertices represent the arcs of G and such that there is an arc from e to f in LG if e represents an arc whose terminal endvertex is the initial vertex of the arc represented by f (e represents (x,y) and f represents (y,z)).

The line digraph operation has the following property [23]: If G is a d -regular digraph with diameter D , then LG is also a d -regular digraph, and its diameter is $D+1$ except if $d=1$ ($G=LG$, in that case).

Let K_d^+ be the complete digraph on d vertices, each vertex having a self loop, then $L^{D-1}(K_d^+)$ (that is, the result of $D-1$ line digraph operations on K_d^+) is a d -regular digraph of diameter D and order d^D . In fact, it is isomorphic to $B(d,D)$ (see Figure 1 for K_2^+ and its first and second line digraphs).

Let K_{d+1}^* be the complete digraph on $d+1$ vertices, without loops, then $L^{D-1}(K_{d+1}^*)$ is isomorphic to $K(d,D)$ (see Figure 2 for K_3^* and its first and second line digraphs).

The equivalence with the definitions using words and alphabet is immediate if the arc from x to y in G (the starting graph) is represented by a vertex denoted by (x,y) in LG and more generally the arc between (x_1,x_2,\dots,x_k) and (x_2,\dots,x_k,x_{k+1}) in $L^{k-1}(G)$ is represented by the vertex $(x_1,x_2,\dots,x_k,x_{k+1})$ in $L^k(G)$.

Remark. This recursive definition is the most useful for proving theorems on de Bruijn (di)graphs, and can be seen as playing the same role as the recursive definition of the k -cube (obtained by joining the identical vertices of two $(k-1)$ -cube by a perfect matching).

3. Definitions from congruences.

These definitions have been introduced by Imase and Itoh [35,34] and Reddy Pradhan and Kuhl [51].

The vertices of $B(d,D)$ are the integers modulo d^D . There is an arc from x to y if and only if there exists an integer a , $0 \leq a \leq d-1$, such that $y \equiv dx+a \pmod{d^D}$. For instance, if $d=2$, then x is joined to $2x$ and

to $2x+1$. This definition is equivalent to the definition using words on alphabet, the vertex (x_1, x_2, \dots, x_D) corresponding to the integer $\sum_{i=0}^{D-1} x_{D-i}d^i$. In other words, (x_1, \dots, x_D) is the d -ary representation of x .

The vertices of $K(d, D)$ are the integers modulo $d^D + d^{D-1}$. There is an arc from x to y if and only if there exists an integer b , $1 \leq b \leq d$, such that $y \equiv -dx - b$. The equivalence of these definitions and the preceding ones is proved for instance in [23] and [31]. Figure 3 shows the two numberings of $K(2, 3)$.

Remark. The advantage of this definition, is that it can immediately be extended to any number of vertices (see paragraph V.).

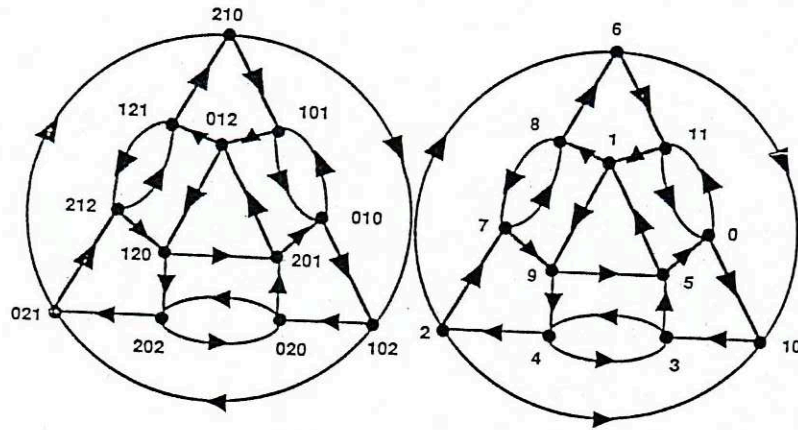


Figure 3

4. A property of the adjacency matrix.

The adjacency matrices of these digraphs have nice properties. Indeed, in the de Bruijn digraph $B(d, D)$, there exists exactly one path of length D between any two (not necessarily different) vertices. Therefore the adjacency matrix M satisfies $M^D = J$, where J is the matrix whose entries are all equal to one. Similarly, in the Kautz digraph $K(d, D)$ there exists exactly one path of length D or $D-1$ between any two vertices. Therefore, the adjacency matrix satisfies $M^D + M^{D-1} = J$.

III. Undirected graphs.

The undirected de Bruijn graph $UB(d, D)$ is the undirected graph obtained from the de Bruijn digraph by forgetting the orientations of the arcs, removing the self loops, and replacing each double edge by a single edge. Now the vertex (x_1, x_2, \dots, x_D) is adjacent to all the vertices $(x_2, \dots, x_D, \alpha)$ and $(\alpha, x_1, \dots, x_{D-1})$, where α is any letter of the alphabet. Therefore the maximum degree Δ is equal to $2d$. Note that $UB(d, D)$ is not regular. Indeed, some vertices have degree $2d-2$ (vertices with self loop in $B(d, D)$, that is, vertices such that all the letters are equal) and some vertices have degree $2d-1$ (vertices of the form (a, b, a, b, a, \dots)).

The number of vertices of $UB(d, D)$ may be expressed in terms of Δ and D as $\left\lceil \frac{\Delta}{2} \right\rceil^D$.

Similarly, the undirected Kautz graph $UK(d, D)$ is obtained from $K(d, D)$ by removing the orientations of the arcs and replacing each double edge by a single edge. The vertex (x_1, \dots, x_D) is adjacent to all the vertices $(x_2, \dots, x_D, \alpha)$ and $(\beta, x_1, \dots, x_{D-1})$, where α is any letter of the alphabet different from x_D and β is any letter of the alphabet different from x_1 . Therefore the maximum degree of $UK(d, D)$ is $2d$. As before, note that this graph is not regular, its minimum degree being equal to $2d-1$. The number of vertices of $UK(d, D)$, in terms of Δ and D , is equal to $\left\lceil \frac{\Delta}{2} \right\rceil^D + \left\lceil \frac{\Delta}{2} \right\rceil^{D-1}$.

Remark. If the designer insists on constructing regular graphs, it is easy to modify these graphs to obtain $2d$ -regular graphs (here again, free links could be useful). For instance, in the undirected Kautz graph, it suffices to add a perfect matching (set of disjoint edges) between the vertices of degree $2d-1$. A particular case of this construction has been considered by Kumar and Reddy [42] (they wanted the induced subgraph, on the vertices originally of degree $2d-1$, to form a cycle; however, any matching gives nice properties).

For de Bruijn graphs, the same modification can be applied on the modified de Bruijn digraphs described above (remark at the end of II.1.).

It is interesting to compare the order of de Bruijn graphs, Kautz graphs and k -cubes for given values of Δ and D . As the k -cube is defined only for $\Delta = D = k$, we give the following table for $\Delta = D = 4, 6, 8, 10$.

	$\Delta=D=4$	$\Delta=D=6$	$\Delta=D=8$	$\Delta=D=10$
k -cube	16	64	256	1 024
de Bruijn	16	729	65,536	9,765,625
Kautz	24	972	81,920	11,718,750

From this table, it is clear that for the same values of Δ and D , de Bruijn and Kautz graphs have many more vertices than the k -cubes.

Furthermore, for a same number of vertices, one can choose between different values of Δ and D . For example, with 256 vertices there is a unique 8-cube of degree and diameter 8. But there are also the de Bruijn graphs $UB(2, 8)$ of maximum degree 4 and diameter 8, $UB(4, 4)$ of maximum degree 8 and diameter 4 and $UB(16, 2)$ of maximum degree 32 and diameter 2.

Remark. One might object that de Bruijn or Kautz graphs are defined only for a given even maximum degree. In fact similar graphs have been defined for any odd degree. For instance one can define a graph whose vertices are words of length $2k$ on two alphabets X and Y , two successive letters in a word belonging to different alphabets. A vertex represents therefore a word $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$. The vertex $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$ is joined to the vertices $(x_{k+1}, y_k, x_k, y_1, \dots, y_2, x_2, y_1)$, where x_{k+1} is any letter from X , and to the vertices $(x_k, y_{k-1}, x_{k-1}, \dots, x_2, y_1, x_1, y_0)$, where y_0 is any letter from Y . This graph has maximum degree $\Delta = |X| + |Y|$. Therefore if one chooses $|X| = d$ and $|Y| = d+1$, the graph has maximum degree $\Delta = 2d+1$. The diameter is equal to $D = 2k$ and the number of vertices is

$$2 d^k (d+1)^k \text{ or } 2 \left\lfloor \frac{\Delta}{2} \right\rfloor^{\left\lfloor \frac{D}{2} \right\rfloor} \left\lceil \frac{\Delta}{2} \right\rceil^{\left\lceil \frac{D}{2} \right\rceil}.$$

It is also possible to define graphs of even diameter and odd degree (see Fiol, Yebra, Fabrega, and Gomez [24, 26]).

IV. Properties of de Bruijn and Kautz networks.

We have seen that these networks have, for a given degree and diameter, a large number of vertices. In fact they are families having the greatest number of vertices for a given degree and diameter, among the known families. We will now look at their properties according to the goals we defined in the introduction.

- **easy routings.** There exist very easy routings which are consequences of the definitions. Suppose one wants to send a message, in a de Bruijn digraph, from the vertex (x_1, x_2, \dots, x_D) to the vertex (y_1, y_2, \dots, y_D) , then one can use the routing given by the following path: $(x_1, x_2, \dots, x_D), (x_2, \dots, x_D, y_1), \dots, (x_D, y_1, \dots, y_{D-1}), (y_1, y_2, \dots, y_D)$. In a Kautz digraph, the path is the same except that, if $x_D = y_1$, the path goes directly from (x_1, x_2, \dots, x_D) to (x_2, \dots, x_D, y_2) . These routings are very simple, however they are not routings of shortest paths. A shortest path can easily be obtained by searching in (x_1, x_2, \dots, x_D) for the longest terminal substring (x_i, \dots, x_D) equal to (y_1, \dots, y_{D-i+1}) . Suppose it is attained for some j , then we have a path of length $j-1$: $(x_1, \dots, x_D) = (x_1, \dots, x_{j-1}, y_1, \dots, y_{D-j+1}), (x_2, \dots, x_{j-1}, y_1, \dots, y_{D-j+2}), \dots, (y_1, \dots, y_D)$. In the undirected case, we can use the simple routing induced by the directed case, obtaining two simple routings, one corresponding to left shiftings and the other to right shiftings. Indeed, in a de Bruijn networks, there are at least two paths of length D between x and y namely $x = (x_1, \dots, x_D), (x_2, \dots, x_D, y_1), \dots, (y_1, \dots, y_D) = y$ and $(x_1, \dots, x_D), (y_D, x_1, \dots, x_{D-1}), \dots, (y_2, \dots, y_D, x_1), (y_1, \dots, y_D)$.

One can also consider the routings associated to the shortest paths in the digraph but they are not routings of shortest paths in the undirected graph, (see Pradhan and Reddy [48]). There is no known simple way to generate a routing of shortest paths in the undirected graph. But, in fact, it seems that, for most algorithms, it is better to consider the simple routing algorithms of paths of length D . Indeed, the mean distance in these graphs is not far from the diameter, therefore the gain obtained by using routings of shortest paths is small.

Remark. Simple routings for the variations for odd degrees have been given by Escudero, Fabrega and Fiol [21].

- **Loads and forwarding indices.** Following Chung, Coffman, Reiman and Simon's paper [17] let us define, for a network with a given routing, the forwarding index as the maximum number of paths passing through a given vertex. That corresponds to the maximum amount of forwarding done by any vertex. They consider the problem of finding networks that minimize the forwarding index for a given number of vertices, each having a fixed degree. They show that de Bruijn networks are asymptotically optimal. A similar study concerning the edge forwarding index has been done by Heydemann, Meyer and Sotteau [29] who showed that de Bruijn networks are also asymptotically optimal for the edge forwarding index. In that case, the parameter corresponds to the load of the links. The load is nearly the same for any link and it is almost the best one can hope for.

- **Broadcasting.** Another way of diffusing information in a network is broadcasting, where a message is to be sent from a single originator to the whole network by placing calls over the communication links of the network. At a given time a node can place only one call. The broadcast time of a vertex corresponds to the minimum number of time units required to complete broadcasting from this vertex. The broadcast time of the network is the maximum broadcast time of any vertex. k -cubes are known to have the best broadcast time, that is, $\log_2 n$. In [8] we designed simple broadcasting algorithms on de Bruijn (and Kautz graphs) in time $\frac{d+1}{2}D + \frac{d}{2}$. For instance, the broadcasting time on $UB(2, D)$ is of order $1.5 \log_2 n$. These

results have been recently improved by Hahn, Heydeman, Opatny and Sotteau [28]. They gave algorithms for broadcasting in time $2 \log_2 n$ (for any d and any D) and in time $\log_2 n$ if $D=2$. Furthermore, if we restrict our attention to networks with bounded degree, de Bruijn networks appear to be near the theoretical bound and give rise to the best known minimum broadcast networks of bounded degree (see Bermond, Hell, Liestman, Peters [5]).

- **Uniformity and algebraic structure.** de Bruijn and Kautz digraphs are regular. The associated undirected graphs are quite regular (it is possible, as we have already seen to make them perfectly regular if needed). Unfortunately, they are not fully symmetric. Indeed, they are neither vertex-transitive nor Cayley graphs of some group (this is one point where the k -cube is better). However, recent studies indicate that they have nice algebraic properties, for instance, they can be viewed as permutation graphs associated with the wreath product or as coset graphs associated with the butterfly networks (see [3]). Recently J.L. Villar [57] determined the automorphism groups of these graphs. Furthermore if one uses the simple routings mentioned above, the vertices have the same behavior; for instance, the load of the edges is almost uniform. Generalizations of these networks have been introduced by Delorme and Fahri [18] as C_s graphs whose vertices are of the form $(a; x_1, \dots, x_k)$, where a is an integer modulo m and the vertex $(a; x_1, \dots, x_k)$ is joined to the vertices $(a+1; x_2, \dots, x_k, \alpha)$ or $(a-1; \alpha, x_1, \dots, x_{k-1})$. If $m=1$, then the parameter a is not useful and these graphs correspond to de Bruijn graphs. It is shown in [18] that if $m \geq k$, these graphs are vertex transitive. The particular case $m=k+1$ was described before by Memmi and Raillard [47].

- **fault tolerance.** The connectivity of the de Bruijn or Kautz networks is always the best possible, that is, equal to the minimum degree of the considered network (Recall that the connectivity is at most the minimum degree). For digraphs, the result follows from the fact that the vertex connectivity κ of LG is equal to the edge connectivity λ of G . For undirected graphs, results can be found in [12] and proofs in [11] or [22]. For example the connectivity of de Bruijn graph $UB(d, D)$ is $\Delta-2$. In that sense these networks are optimally fault-tolerant. We refer the reader to the survey [12] for more details and references. Note that it is possible to modify slightly these networks to obtain graphs which are regular and of connectivity equal to their degree, see [55]. Other notions of fault tolerance have been considered in the literature, like diameter vulnerability or surviving route graph. Again these networks have very good behavior concerning all these notions. For instance Imase, Soneoka and Okada [36] have proved that in any de Bruijn digraph $B(d, D)$, there exist $d-1$ vertex disjoint paths of length at most $D+1$ and that in $K(d, D)$, there exist d disjoint paths of length at least $D+2$. Again, we refer the reader to [12] for more details.

- **extendability.** We have already seen that $UB(d, D)$ is an induced subgraph of $UB(d+1, D)$ and $UK(d, D)$ is an induced subgraph of $UK(d+1, D)$ (it suffices to add a letter to the alphabet). If $D=2$, it is possible to

make the extension starting from $UB(d,2)$ by adding the vertices one by one while keeping the diameter equal to 2. Furthermore, this extension can be done in several ways (see [4]). One can also extend $UB(2,3)$ to $UB(3,3)$; but for $d > 4$, it is not possible to find an extension adding the vertices one by one and keeping the diameter equal to D . However we can add the vertices by packets and keep the diameter or, if one insists on adding the vertices one by one, it is possible to keep the diameter smaller than $3\frac{D}{2}$ (it even seems experimentally possible to keep the diameter smaller than $D+1$ or $D+2$).

- **embeddability of other topologies.** It is easy to show that de Bruijn or Kautz digraphs and therefore the associated graphs have a hamiltonian cycle. That is, they contain a uni- or bi-directional ring containing all the vertices. Indeed as de Bruijn and Kautz digraphs are d -regular digraphs, they are eulerian (that is, they contain a directed cycle containing all the arcs). Therefore they are hamiltonian as the line digraph of any eulerian digraph is a hamiltonian digraph. In fact the existence of Hamiltonian cycles of $B(d,D)$ is equivalent to the existence of "de Bruijn sequences", that is, circular sequences of length d^D , on d symbols, and such that every subsequence of length D appears exactly once in the sequence. These sequences where the first motivation for the study of de Bruijn graphs and play a role analogous to that of Gray codes in the k -cube. Furthermore, de Bruijn graphs contain cycles of any length, as proved by Yoeli [58]. In the case of the k -cube, it has been proved that the edges of the k -cube can be partitioned into $\frac{k}{2}$ hamiltonian cycles if k is even (see [1]). The existence of a large number of disjoint hamiltonian cycles in de Bruijn or Kautz graphs is still an unsolved problem, see [13].

By drawing all the paths of length D from a vertex to all the other vertices, one obtains a d -ary (directed) tree of height D in $UB(d,D)$, where all the vertices are leaves of the tree. Similarly $UK(d,D)$ contains a d -ary tree of height D . In particular, $UB(2,D)$ contains a spanning binary tree of height D , which is best possible. Recently Hahn, Heydemann, Opatny and Sotteau [28] proved that $UB(d,D)$ contains a spanning binary tree rooted at any vertex, of height $\lceil \log_2 d \rceil D$, which is very close to the best possible value $\lceil \log_2 n \rceil = \lceil D \log_2 d \rceil$ (the case when d is a power of 2 was proved by de la Vega, private communication).

Embeddability of other interesting networks, such as grids or cubes seem to have been studied only recently in the case $d=2$. For example, recent work of Rosenberg [3, 52, 53] and of Koch et al [40] shows that one can easily simulate butterfly networks on de Bruijn networks with $d=2$ (and vice versa) in a work preserving manner. As the butterfly networks can simulate the hypercube with little or no slowdown on a large important class of computational problems, de Bruijn networks, with $d=2$ can be viewed as being of the same algorithmic power as the hypercubes. We refer the reader to the papers for precise definitions and results. These results led Rosenberg [53] to propose a new interconnection network based on products of de Bruijn networks which share properties of the de Bruijn and the butterfly networks and contain cycles, meshes, complete binary trees and meshes of trees as subgraphs.

Finally, let us note that the de Bruijn digraph $UB(2,D)$ is very similar to the shuffle exchange graph (in fact it is obtained from it by contracting all the "exchange edges"). They can simulate each other with dilatation at most 2. Therefore, one can use the work already done on shuffle exchanges to embed structures or design algorithms.

- **Design of algorithms.** Most algorithms designed for the hypercube use its symmetric and recursive structure, hence they are not easily transported to the de Bruijn networks. However this apparent distinction between the two networks is not definitive. Recent work by Koch et al [40] or Samatham and

Pradhan [54] yield very efficient sorting algorithms (even in case of faults). Recently Rosenberg [52] proved that $B(2,D)$ can simulate the k -cube on any generalized ascend-descend algorithm with no time loss. In a generalized ascend-descend algorithm, the data are exchanged on some dimension of the cube, the dimension used at time $t+1$ differs from the one used at time t by one (see also the results of [49, 54, 45]). Algorithms like the Fast Fourier Transform also seem to adapt well to de Bruijn networks.

A large class of distributed algorithms is formed by the "consensus protocols". One wants to compute a function or a predicate whose arguments are distributed on the nodes of the network (for instance the data are distributed among the nodes, or one wants to perform an election). In order to do that, the nodes must cooperate and there is no node with a privileged behavior. Efficient algorithms have been designed (see Bermond, Konig, Raynal [6] and [7]) whose message complexity is $O(Dm)$ (where m is the number of links), that is, $O(\Delta Dn)$ on a Δ -regular graph. Lakshman and Agrawala [43] used networks of diameter 2 and degree roughly \sqrt{n} to obtain a complexity of $O(n^{\frac{3}{2}})$. They conjectured that one can find networks on which the complexity will be $O(n^{1+\frac{1}{k}})$ for any k . This is obtained with the de Bruijn or Kautz graphs of parameters Δ and D . Indeed $n = \left(\frac{\Delta}{2}\right)^D$, therefore $O(D\Delta n) = O(n^{1+\frac{1}{D}})$. In fact, if we consider an $UB(2,D)$ or $UK(2,D)$ (that is, $\Delta = 4$), the complexity is in $O(n \log_2 n)$, which is the best possible in most of the cases (see Bermond, Konig, [7, 41]).

- **VLSI Layout.** For this part, we refer the reader to the survey of Samatham and Pradhan [54]. They consider this problem in details, showing that the de Bruijn graph $UB(2,D)$ has an asymptotically optimal area layout (using its relation with the shuffle exchange graph). It might be possible to extend this result to any degree by using the techniques described in Leiserson [46] and Leighton [44].

V. Generalized de Bruijn or Kautz graphs.

Like the k -cube, de Bruijn and Kautz graphs are defined only for some values of the number of nodes, namely d^D or $d^D + d^{D-1}$. But in the case of de Bruijn and Kautz networks the third definition using congruences can be extended to any number of vertices. The generalization of de Bruijn digraph has been introduced by Reddy Pradhan and Kuhl [51] and Imase and Itoh [34].

The vertices of the digraph denoted by $RPK_{d,n}$ are the integers modulo n , the vertex x being joined to the vertices $y \equiv dx+a$ with $0 \leq a \leq d-1$. In the case $n=d^D$, this is exactly the definition of de Bruijn digraphs. Similarly, Imase and Itoh [35] generalized Kautz networks. The vertices of the network denoted by $\Pi_{d,n}$ are the integers modulo n , the vertex x being joined to the vertices $y \equiv -dx-b$, $1 \leq b \leq d$. If $n = d^D + d^{D-1}$, we find exactly the Kautz digraph.

These digraphs are d -regular, their diameter is $\lceil \log_2 n \rceil$ for $RPK_{d,n}$ (see [51]) and for $\Pi_{d,n}$ it is either $\lceil \log_2 n \rceil$ or $\lceil \log_2 n \rceil - 1$ (see [35]), which is the best that can be done (in view of the Moore bound). By forgetting the orientation it is possible to obtain undirected graphs.

The generalized networks enjoy much of the properties of de Bruijn or Kautz digraphs. Most of these properties have been investigated only recently and the proofs are more complicated. Sometimes only quasi optimal results can be proved. We survey here, very quickly, what is known.

Routings are still easy and need only the knowledge of the source and destination addresses. For example let us show how to route a message in $RPK_{d,n}$ from a node x to a node y . Let $D = \lceil \log_2 n \rceil$ and let (a_1, a_2, \dots, a_D) be the representation of $y - x d^D$ modulo n in base d . Then there exists a path $x = z_0, z_1, \dots, z_D = y$ from x to y where $z_i \equiv d z_{i-1} + a_i$ (see [34, 51, 20]).

The connectivity and fault tolerance of these digraphs is almost always the best possible (see [37, 32, 31]). For example the connectivity of $\Pi_{d,n}$ is equal to the minimum degree $d-1$ when $d+1$ does not divide n (in that case, the digraph contains some self loops). When $d+1$ divides n , the minimum degree is d , and the connectivity is d if and only if d and n are not coprime (in that case the connectivity is $d-1$). Modifications of these generalized networks are described in [55]; they are d -connected and d -regular digraphs with a quasi-optimal diameter. For results on fault tolerance, see the survey [12].

Recently Du, Hsu, Hwang, and Zhang [19] have proved that $RPK_{d,n}$ is hamiltonian unless $d=2$ and n is odd. They have also proved that $\Pi_{d,n}$ is hamiltonian unless $d=2$ and n is odd and different from 3^k . The embeddability of a ring is therefore almost always assured, embeddability of d -ary trees is shown in Du and Hwang [20].

VI. Conclusion.

Many other alternatives for the k -cubes have been proposed in the literature. We hope to have shown that the de Bruijn and Kautz graphs and their generalizations have for most properties the same good behavior as the hypercubes, and represent a good compromise among the different goals a designer can have in mind. Furthermore, they allow interconnecting considerably more processors than the hypercubes and the degree does not increase with the diameter. Therefore they are good candidates for the next generation after the hypercube. However more studies of their properties, in particular concerning embeddability, VLSI layout and design of adapted algorithms have to be pursued. It is also interesting to note that these networks play an important role in constructing networks with bounded degree like cubic networks, either by using folding techniques (see the survey of Chung [16]) or compounding techniques (see Jerrum and Skyum [38]). For example, replacing each vertex of a $UB(2,D)$ by a 3-star gives rise to the best known family of graphs with degree 3 (in terms of diameter). These networks can also be used to construct the best among known families of bus networks (see [9]), and appear closely related to classical multistage networks.

Acknowledgments

This paper was written while the authors were visiting the School of Computing Science at Simon Fraser University, Burnaby, V5A 1S6 Canada; support of Advanced Systems Foundations and Simon Fraser University is gratefully acknowledged as well as support from the French GRECO PRC C³. We also wish to thank all the persons who helped us in preparing this paper, in particular the group called "D. Ameter and Max de Gree".

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