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1                                   **STATE-CONSTRAINED CONTROL-AFFINE**  
2                                   **PARABOLIC PROBLEMS II:**  
3                                   **SECOND ORDER SUFFICIENT OPTIMALITY CONDITIONS\***

4                                   M. SOLEDAD ARONNA<sup>†</sup>, J. FRÉDÉRIC BONNANS<sup>‡</sup>, AND AXEL KRÖNER<sup>§</sup>

5                   **Abstract.** In this paper we consider an optimal control problem governed by a semilinear heat  
6 equation with bilinear control-state terms and subject to control and state constraints. The state  
7 constraints are of integral type, the integral being with respect to the space variable. The control is  
8 multidimensional. The cost functional is of a tracking type and contains a linear term in the control  
9 variables. We derive second order sufficient conditions relying on the Goh transform.

10                   **Key words.** Optimal control of partial differential equations, semilinear parabolic equations,  
11 state constraints, second order analysis, Goh transform, control-affine problems

12                   **AMS subject classifications.** 49J20, 49K20, 35K58

13                   **1. Introduction.** This is the second part of two papers on necessary and suffi-  
14 cient optimality conditions for an optimal control problem governed by a semilinear  
15 heat equation containing bilinear terms coupling the control variables and the state,  
16 and subject to constraints on the control and state. While in the first part [5], first and  
17 second order necessary optimality conditions are shown, in this second part we derive  
18 second order sufficient optimality conditions. The control may have several compo-  
19 nents and enters the dynamics in a bilinear term and in an affine way in the cost. This  
20 does not allow to apply classical techniques of calculus of variations to derive second  
21 order sufficient optimality conditions. Therefore, we extend techniques that were re-  
22 cently established in the following articles, and that involve the Goh transform [12]  
23 in an essential way. Aronna, Bonnans, Dmitruk and Lotito [1] obtained second order  
24 necessary and sufficient conditions for bang-singular solutions of control-affine finite  
25 dimensional systems with control bounds, results that were extended in Aronna, Bon-  
26 nans and Goh [2] when adding a state constraint of inequality type. An extension of  
27 the analysis in [1] to the infinite dimensional setting was done by Bonnans [6], for a  
28 problem concerning a semilinear heat equation subject to control bounds and without  
29 state constraints. For a quite general class of linear differential equations in Banach  
30 spaces with bilinear control-state couplings and subject to control bounds, Aronna,  
31 Bonnans and Kröner [3] provided second order conditions, that extended later to the  
32 complex Banach space setting [4].

33                   There exists a series of publications on second order conditions for problems  
34 governed by control-affine ordinary differential equations, we refer to references in [5].

35                   In the elliptic framework, regarding the case we investigate here, this is, when  
36 no quadratic control term is present in the cost (or what some authors call *vanishing*  
37 *Tikhonov term*), Casas in [7] proved second order sufficient conditions for bang-bang  
38 optimal controls of a semilinear equation, and for one containing a bilinear coupling  
39 of control and state in the recent joint work with D. and G. Wachsmuth [10].

40                   Parabolic optimal control problems with state constraints are discussed in Rösch

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41 and Tröltzsch [16], who gave second order sufficient conditions for a linear equation  
 42 with mixed control-state constraints. In the presence of pure-state constraints, Ray-  
 43 mond and Tröltzsch [15], and Krumbiegel and Rehberg [13] obtained second order  
 44 sufficient conditions for a semilinear equation, Casas, de Los Reyes, and Tröltzsch [8]  
 45 and de Los Reyes, Merino, Rehberg and Tröltzsch [11] obtained sufficient second order  
 46 conditions for semilinear equations, both in the elliptic and parabolic cases. The arti-  
 47 cles mentioned in this paragraph did not consider bilinear terms, and their sufficient  
 48 conditions do not apply to the control-affine problems that we treat in the current  
 49 work.

50 It is also worth mentioning the work [9] by Casas, Ryll and Tröltzsch that provided  
 51 second order conditions for a semilinear FitzHugh-Nagumo system subject to control  
 52 constraints in the case of vanishing Tikhonov term.

53 The contribution of this paper are second order sufficient optimality conditions for  
 54 an optimal control problem for a semilinear parabolic equation with cubic nonlinear-  
 55 ity, several controls coupled with the state variable through bilinear terms, pointwise  
 56 control constraints and state constraints that are integral in space. The main chal-  
 57 lenge arises from the fact that both the dynamics and the cost function are affine  
 58 with respect to the control, hence classical techniques are not applicable to derive  
 59 second order sufficient conditions. We rely on the Goh transform [12] to derive suf-  
 60 ficient optimality conditions for bang-singular solutions. In particular, the sufficient  
 61 conditions are stated on a cone of directions larger than the one used for the necessary  
 62 conditions.

63 The paper is organized as follows. In Section 2 the problem is stated and main  
 64 assumptions are formulated. Section 3 is devoted to second order necessary conditions  
 65 and Section 4 to second order sufficient conditions.

66 **Notation.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $n \leq 3$ , with  $C^\infty$  boundary  $\partial\Omega$ . Given  
 67  $p \in [1, \infty]$  and  $k \in \mathbb{N}$ , let  $W^{k,p}(\Omega)$  be the Sobolev space of functions in  $L^p(\Omega)$  with  
 68 derivatives (here and after, derivatives w.r.t.  $x \in \Omega$  or w.r.t. time are taken in the  
 69 sense of distributions) in  $L^p(\Omega)$  up to order  $k$ . Let  $\mathcal{D}(\Omega)$  be the set of  $C^\infty$  functions  
 70 with compact support in  $\Omega$ . By  $W_0^{k,p}(\Omega)$  we denote the closure of  $\mathcal{D}(\Omega)$  with respect  
 71 to the  $W^{k,p}$ -topology. Given a horizon  $T > 0$ , we write  $Q := \Omega \times (0, T)$ .  $\|\cdot\|_p$  denotes  
 72 the norm in  $L^p(0, T)$ ,  $L^p(\Omega)$  and  $L^p(Q)$ , indistinctly. When a function depends on  
 73 both space and time, but the norm is computed only with respect of one of these  
 74 variables, we specify both the space and domain. For example, if  $y \in L^p(Q)$  and we  
 75 fix  $t \in (0, T)$ , we write  $\|y(\cdot, t)\|_{L^p(\Omega)}$ . For the  $p$ -norm in  $\mathbb{R}^m$ , for  $m \in \mathbb{N}$ , we use  $|\cdot|_p$ . We  
 76 set  $H^k(\Omega) := W^{k,2}(\Omega)$  and  $H_0^k(\Omega) := W_0^{k,2}(\Omega)$ . By  $W^{2,1,p}(Q)$  we mean the Sobolev  
 77 space of  $L^p(Q)$ -functions whose second derivative in space and first derivative in time  
 78 belong to  $L^p(Q)$ . We write  $H^{2,1}(Q)$  for  $W^{2,1,2}(Q)$  and, setting  $\Sigma := \partial\Omega \times (0, T)$ , we  
 79 define the state space as

$$80 \quad (1.1) \quad Y := \{y \in H^{2,1}(Q); y = 0 \text{ a.e. on } \Sigma\}.$$

81 If  $y$  is a function over  $Q$ , we use  $\dot{y}$  to denote its time derivative in the sense of  
 82 distributions. As usual we denote the spatial gradient and the Laplacian by  $\nabla$  and  
 83  $\Delta$ . By  $\text{dist}(t, I) := \inf\{\|t - \bar{t}\|; \bar{t} \in I\}$  for  $I \subset \mathbb{R}$ , we denote the distance of  $t$  to the  
 84 set  $I$ .

85 **2. Statement of the problem and main assumptions.** In this section we  
 86 introduce the optimal control problem and recall results on well-posedness of the state  
 87 equation and existence of solutions of the optimal control problem from [5].

88 **2.1. Setting.** The *state equation* is given as

$$89 \quad (2.1) \quad \begin{cases} \dot{y}(x, t) - \Delta y(x, t) + \gamma y^3(x, t) = f(x, t) + y(x, t) \sum_{i=0}^m u_i(t) b_i(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

90 with

$$91 \quad (2.2) \quad y_0 \in H_0^1(\Omega), \quad f \in L^2(Q), \quad b \in W^{1,\infty}(\Omega)^{m+1},$$

92  $\gamma \geq 0$ ,  $u_0 \equiv 1$  is a constant, and  $u := (u_1, \dots, u_m) \in L^2(0, T)^m$ . Lemma A.1 below  
93 shows that for each control  $u \in L^2(0, T)^m$ , there is a unique associated solution  $y \in Y$   
94 of (2.1), called the *associated state*. Let  $y[u]$  denote this solution. We consider control  
95 constraints of the form  $u \in \mathcal{U}_{\text{ad}}$ , where

$$96 \quad (2.3) \quad \mathcal{U}_{\text{ad}} = \{u \in L^2(0, T)^m; \tilde{u}_i \leq u_i(t) \leq \hat{u}_i, \quad i = 1, \dots, m\},$$

97 for some constants  $\tilde{u}_i < \hat{u}_i$ , for  $i = 1, \dots, m$ . In addition, we have finitely many linear  
98 running state constraints of the form

$$99 \quad (2.4) \quad g_j(y(\cdot, t)) := \int_{\Omega} c_j(x) y(x, t) dx + d_j \leq 0, \quad \text{for } t \in [0, T], \quad j = 1, \dots, q,$$

100 where  $c_j \in H^2(\Omega) \cap H_0^1(\Omega)$  for  $j = 1, \dots, q$ , and  $d \in \mathbb{R}^q$ .

101 We call any  $(u, y[u]) \in L^2(0, T)^m \times Y$  a *trajectory*, and if it additionally satisfies  
102 the control and state constraints, we say it is an *admissible trajectory*. The *cost*  
103 *function* is

$$104 \quad (2.5) \quad \begin{aligned} J(u, y) := & \frac{1}{2} \int_Q (y(x, t) - y_d(x))^2 dx dt \\ & + \frac{1}{2} \int_{\Omega} (y(x, T) - y_{dT}(x))^2 dx + \sum_{i=1}^m \alpha_i \int_0^T u_i(t) dt, \end{aligned}$$

105 where

$$106 \quad (2.6) \quad y_d \in L^2(Q), \quad y_{dT} \in H_0^1(\Omega),$$

107 and  $\alpha \in \mathbb{R}^m$ . We consider the optimal control problem

$$108 \quad (\text{P}) \quad \text{Min}_{u \in \mathcal{U}_{\text{ad}}} J(u, y[u]); \quad \text{subject to (2.4).}$$

109 For problem (P), assuming it in the sequel to be feasible, we consider two types  
110 of solution.

111 **DEFINITION 2.1.** *We say that  $(\bar{u}, y[\bar{u}])$  is an  $L^2$ -local solution (resp.,  $L^\infty$ -local*  
112 *solution) if there exists  $\varepsilon > 0$  such that  $(\bar{u}, y[\bar{u}])$  is a minimum among the admissible*  
113 *trajectories  $(u, y)$  that satisfy  $\|u - \bar{u}\|_2 < \varepsilon$  (resp.,  $\|u - \bar{u}\|_\infty < \varepsilon$ ).*

114 The state equation is well-posed and has a solution in  $Y$ . Furthermore, the  
115 mapping  $u \mapsto y$ ,  $L^2(0, T) \rightarrow Y$  is of class  $C^\infty$ . Since (P) has a bounded feasible set,  
116 it is easily checked that its set of solutions of (P) is non-empty. For details regarding  
117 these assertions see Appendix A.

118 **2.2. First order optimality conditions.** It is well-known that the dual of  
 119  $C([0, T])$  is the set of (finite) Radon measures, and that the action of a finite Radon  
 120 measure coincides with the Stieltjes integral associated with a bounded variation  
 121 function  $\mu \in BV(0, T)$ . We may assume w.l.g. that  $\mu(T) = 0$ , and we let  $d\mu$  denote  
 122 the Radon measure associated to  $\mu$ . Note that if  $d\mu$  belongs to the set  $\mathcal{M}_+(0, T)$  of  
 123 nonnegative finite Radon measures then we may take  $\mu$  nondecreasing. Set

$$124 \quad (2.7) \quad BV(0, T)_{0,+} := \{\mu \in BV(0, T); \mu(T) = 0; d\mu \geq 0\}.$$

125 Let  $(\bar{u}, \bar{y})$  be an admissible trajectory of problem  $(P)$ . We say that  $\mu \in BV(0, T)_{0,+}^q$   
 126 is *complementary to the state constraint* for  $\bar{y}$  if

$$127 \quad (2.8) \quad \int_0^T g_j(\bar{y}(\cdot, t)) d\mu_j(t) = \int_0^T \left( \int_{\Omega} c_j(x) \bar{y}(x, t) dx + d_j \right) d\mu_j(t) = 0, \quad j = 1, \dots, q.$$

128 Let  $(\beta, \mu) \in \mathbb{R}_+ \times BV(0, T)_{0,+}^q$ . We say that  $p \in L^\infty(0, T; H_0^1(\Omega))$  is the *costate*  
 129 *associated* with  $(\bar{u}, \bar{y}, \beta, \mu)$ , or shortly with  $(\beta, \mu)$ , if  $(p, p_0)$  is solution of (B.6). As  
 130 explained in appendix B.2,  $p = p^1 - \sum_{j=1}^q c_j \mu_j$  for some  $p^1 \in Y$ . In particular  $p(\cdot, 0)$   
 131 and  $p(\cdot, T)$  are well-defined and it can be checked that  $p_0 = p(\cdot, 0)$ .

132 **DEFINITION 2.2.** We say that the triple  $(\beta, p, \mu) \in \mathbb{R}_+ \times L^\infty(0, T; H_0^1(\Omega)) \times$   
 133  $BV(0, T)_{0,+}^q$  is a generalized Lagrange multiplier if it satisfies the following first-order  
 134 optimality conditions:  $\mu$  is complementary to the state constraint,  $p$  is the costate as-  
 135 sociated with  $(\beta, \mu)$ , the non-triviality condition  $(\beta, d\mu) \neq 0$  holds and, for  $i = 1$  to  
 136  $m$ , defining the switching function by

$$137 \quad (2.9) \quad \Psi_i^p(t) := \beta \alpha_i + \int_{\Omega} b_i(x) \bar{y}(x, t) p(x, t) dx, \quad \text{for } i = 1, \dots, m,$$

138 one has  $\Psi^p \in L^\infty(0, T)^m$  and

$$139 \quad (2.10) \quad \sum_{i=1}^m \int_0^T \Psi_i^p(t) (u_i(t) - \bar{u}_i(t)) dt \geq 0, \quad \text{for every } u \in \mathcal{U}_{\text{ad}}.$$

140 We let  $\Lambda(\bar{u}, \bar{y})$  denote the set of generalized Lagrange multipliers associated with  $(\bar{u}, \bar{y})$ .  
 141 If  $\beta = 0$  we say that the corresponding multiplier is singular. Finally, we write  $\Lambda_1(\bar{u}, \bar{y})$   
 142 for the set of pairs  $(p, \mu)$  with  $(1, p, \mu) \in \Lambda(\bar{u}, \bar{y})$ . When the nominal solution is fixed  
 143 and there is no place for confusion, we just write  $\Lambda$  and  $\Lambda_1$ .

144 We recall from [5, Lem. 6(i)] the following statement on first order conditions.

145 **LEMMA 2.3.** If  $(\bar{u}, y[\bar{u}])$  is an  $L^2$ -local solution of  $(P)$ , then the associated set  $\Lambda$   
 146 of multipliers is nonempty.

147 Consider the contact sets associated to the control bounds defined, up to null measure  
 148 sets, by  $\hat{I}_i := \{t \in [0, T]; \bar{u}_i(t) = \check{u}_i\}$ ,  $\hat{I}_i := \{t \in [0, T]; \bar{u}_i(t) = \hat{u}_i\}$ ,  $I_i := \hat{I}_i \cup \hat{I}_i$ .  
 149 For  $j = 1, \dots, q$ , the contact set associated with the  $j$ th state constraint is  $I_j^C := \{t \in$   
 150  $[0, T]; g_j(\bar{y}(\cdot, t)) = 0\}$ . Given  $0 \leq a < b \leq T$ , we say that  $(a, b)$  is a *maximal state*  
 151 *constrained arc* for the  $j$ th state constraints, if  $I_j^C$  contains  $(a, b)$  but it contains no  
 152 open interval strictly containing  $(a, b)$ . We define in the same way a *maximal (lower*  
 153 *or upper) control bound constraints arc* (having in mind that the latter are defined up  
 154 to a null measure set).

155 We will assume the following *finite arc property*:

$$156 \quad (2.11) \quad \left\{ \begin{array}{l} \text{the contact sets for the state and bound constraints are,} \\ \text{up to a finite set, the union of finitely many maximal arcs.} \end{array} \right.$$

157 In the sequel we identify  $\bar{u}$  (defined up to a null measure set) with a function whose  
 158  $i$ th component is constant over each interval of time that is included, up to a zero-  
 159 measure set, in either  $\check{I}_i$  or  $\hat{I}_i$ . For almost all  $t \in [0, T]$ , the *set of active constraints*  
 160 *at time  $t$*  is denoted by  $(\check{B}(t), \hat{B}(t), C(t))$  where

$$161 \quad (2.12) \quad \begin{cases} \check{B}(t) := \{1 \leq i \leq m; \bar{u}_i(t) = \check{u}_i\}, \\ \hat{B}(t) := \{1 \leq i \leq m; \bar{u}_i(t) = \hat{u}_i\}, \\ C(t) := \{1 \leq j \leq q; g_j(\bar{y}(\cdot, t)) = 0\}. \end{cases}$$

162 These sets are well-defined over open subsets of  $(0, T)$  where the set of active con-  
 163 straints is constant, and by (2.11), there exist time points called *junction points*  
 164  $0 =: \tau_0 < \dots < \tau_r := T$ , such that the intervals  $(\tau_k, \tau_{k+1})$  are *maximal arcs with*  
 165 *constant active constraints*, for  $k = 0, \dots, r-1$ . We may sometimes call them shortly  
 166 *maximal arcs*. For  $m = 1$  we call junction points where a *BB junction* if we have  
 167 active bound constraints on both neighbouring maximal arcs, a *CB junction* (resp.  
 168 *BC junction*) if we have a state constrained arc and an active bound constrained arc.

169 **DEFINITION 2.4.** For  $k = 0, \dots, r-1$ , let  $\check{B}_k, \hat{B}_k, C_k$  denote the set of indexes of  
 170 active lower and upper bound constraints, and state constraints, on the maximal arc  
 171  $(\tau_k, \tau_{k+1})$ , and set  $B_k := \check{B}_k \cup \hat{B}_k$ .

172 In the discussion that follows we fix  $k$  in  $\{0, \dots, r-1\}$ , and consider a maximal  
 173 arc  $(\tau_k, \tau_{k+1})$ , where the junction points. Set  $\bar{B}_k := \{1, \dots, m\} \setminus B_k$  and

$$174 \quad (2.13) \quad M_{ij}(t) := \int_{\Omega} b_i(x) c_j(x) \bar{y}(x, t) dx, \quad 1 \leq i \leq m, \quad 1 \leq j \leq q.$$

175 Let  $\bar{M}_k(t)$  (of size  $|\bar{B}_k| \times |C_k|$ ) denote the submatrix of  $M(t)$  having rows with index  
 176 in  $\bar{B}_k$  and columns with index in  $C_k$ . In the sequel we make the following assumption.

177 **Hypothesis 2.5.** We assume that  $|C_k| \leq |\bar{B}_k|$ , for  $k = 0, \dots, r-1$ , and that the  
 178 following (*uniform*) *local controllability condition* holds:

$$179 \quad (2.14) \quad \begin{cases} \text{there exists } \alpha > 0, \text{ such that } |\bar{M}_k(t)\lambda| \geq \alpha|\lambda|, \\ \text{for all } \lambda \in \mathbb{R}^{|C_k|}, \text{ a.e. on } (\tau_k, \tau_{k+1}), \text{ for } k = 0, \dots, r-1. \end{cases}$$

180 **3. Second order necessary conditions.** We start this section by recalling  
 181 some results obtained in [5], the main one being the second order necessary condition  
 182 of Theorem 3.5. We then introduce the *Goh transform* and apply it to the quadratic  
 183 form and the critical cone, and then obtain necessary conditions on the transformed  
 184 objects (see Theorem 3.12). We show later in Section 4 that these necessary conditions  
 185 can be strengthened to get sufficient conditions for optimality (see Theorem 4.5).

186 Let us consider an admissible trajectory  $(\bar{u}, \bar{y})$ .

187 **3.1. Assumptions and additional regularity.** For the remainder of the arti-  
 188 cle we make the following set of assumptions.

189 **Hypothesis 3.1.** The following conditions hold:

- 190 1. the finite maximal arc property (2.11),
- 191 2. the problem is qualified (cf. also [5, Sec. 3.2.1])

$$192 \quad (3.1) \quad \begin{cases} \text{there exists } \varepsilon > 0 \text{ and } u \in \mathcal{U}_{\text{ad}} \text{ such that } v := u - \bar{u} \text{ satisfies} \\ g_j(\bar{y}(\cdot, t)) + g'_j(\bar{y}(\cdot, t))z[v](\cdot, t) < -\varepsilon, \text{ for all } t \in [0, T], \text{ and } j = 1, \dots, q. \end{cases}$$

- 193 3. the local (uniform) controllability condition (2.14) over each maximal arc  
 194  $(\tau_k, \tau_{k+1})$ ,  
 195 4. the discontinuity of the derivative of the state constraints at corresponding  
 196 junction points, i.e.,  
 (3.2)  
 197 for some  $c > 0$ :  $g_j(\bar{y}(\cdot, t)) \leq -c \operatorname{dist}(t, I_j^C)$ , for all  $t \in [0, T]$ ,  $j = 1, \dots, q$ ,

- 198 5. the uniform distance to control bounds whenever they are not active, i.e.  
 199 there exists  $\delta > 0$  such that,

$$(3.3) \quad \operatorname{dist}(\bar{u}_i(t), \{\check{u}_i, \hat{u}_i\}) \geq \delta, \quad \text{for a.a. } t \notin I_i, \text{ for all } i = 1, \dots, m,$$

- 201 6. the following regularity for the data (we do not try to take the weakest hy-  
 202 potheses):

$$(3.4) \quad y_0, y_d T \in W_0^{1,\infty}(\Omega), \quad y_d, f \in L^\infty(Q), \quad b \in W^{2,\infty}(\Omega)^{m+1},$$

- 204 7. the control  $\bar{u}$  has left and right limits at the junction points  $\tau_k \in (0, T)$ , (this  
 205 will allow to apply [5, Lem. 3.8]).

206 *Remark 3.2.* Hypotheses 3.1 4 and 5 are instrumental for constructing feasible  
 207 perturbations of the nominal trajectory, used in the proof of Theorem 3.5 made in [5].

208 In view of point 3 above, we consider from now on  $\beta = 1$  and thus we omit the  
 209 component  $\beta$  of the multipliers.

210 **THEOREM 3.3.** *The following assertions hold.*

- 211 (i) For any  $u \in L^\infty(0, T)^m$ , the associated state  $y[u]$  belongs to  $C(\bar{Q})$ . If  $u$  re-  
 212 mains in a bounded subset of  $L^\infty(0, T)^m$  then the corresponding states form  
 213 a bounded set in  $C(\bar{Q})$ . In addition, if the sequence  $(u_\ell)$  of admissible con-  
 214 trols converges to  $\bar{u}$  a.e. on  $(0, T)$ , then the associated sequence of states  
 215  $(y_\ell := y[u_\ell])$  converges uniformly to  $\bar{y}$  in  $\bar{Q}$ .  
 216 (ii) The set  $\Lambda_1$  is nonempty and for every  $(p, d\mu) \in \Lambda_1$ , one has that  $\mu \in$   
 217  $W^{1,\infty}(0, T)^q$  and  $p$  is essentially bounded in  $Q$ .

218 *Proof.* We refer to [5, Thm. 4.2]. Note that the non-emptiness of  $\Lambda_1$  follows from  
 219 (3.1).  $\square$

220 **3.2. Second variation.** For  $(p, \mu) \in \Lambda_1$ , set  $\kappa(x, t) := 1 - 6\gamma\bar{y}(x, t)p(x, t)$ , and  
 221 consider the quadratic form

$$(3.5) \quad \mathcal{Q}[p](z, v) := \int_Q \left( \kappa z^2 + 2p \sum_{i=1}^m v_i b_i z \right) dx dt + \int_\Omega z(x, T)^2 dx.$$

223 Let  $(u, y)$  be a trajectory, and set

$$(3.6) \quad (\delta y, v) := (y - \bar{y}, u - \bar{u}).$$

225 Recall the definition of the operator  $A$  given in (B.1). Subtracting the state equation  
 226 at  $(\bar{u}, \bar{y})$  from the one at  $(u, y)$ , we get that

$$(3.7) \quad \begin{cases} \frac{d}{dt} \delta y + A \delta y = \sum_{i=1}^m v_i b_i y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3 & \text{in } Q, \\ \delta y = 0 & \text{on } \Sigma, \quad \delta y(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

228 Combining with the linearized state equation (B.2), we deduce that  $\eta$  given by  $\eta :=$   
 229  $\delta y - z$ , satisfies the equation

$$230 \quad (3.8) \quad \begin{cases} \dot{\eta} - \Delta \eta = r\eta + \tilde{r} & \text{in } Q, \\ \eta = 0 & \text{on } \Sigma, \quad \eta(\cdot, 0) = 0 & \text{in } \Omega \end{cases}$$

231 where  $r$  and  $\tilde{r}$  are defined as

$$232 \quad (3.9) \quad r := -3\gamma\bar{y}^2 + \sum_{i=0}^m \bar{u}_i b_i, \quad \tilde{r} := \sum_{i=1}^m v_i b_i \delta y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3.$$

234 See the definition of the Lagrangian function  $\mathcal{L}$  given in equation (B.5) of the  
 235 Appendix.

236 PROPOSITION 3.4. *Let  $(p, \mu) \in \Lambda_1$ , and let  $(u, y)$  be a trajectory. Then*

237

$$238 \quad (3.10) \quad \mathcal{L}[p, \mu](u, y, p) - \mathcal{L}[p, \mu](\bar{u}, \bar{y}, p) \\ 239 \quad \quad \quad = \int_0^T \Psi^p(t) \cdot v(t) dt + \frac{1}{2} \mathcal{Q}[p](\delta y, v) - \gamma \int_Q p(\delta y)^3 dx dt.$$

241 Here, we omit the dependence of the Lagrangian on  $(\beta, p_0)$  being equal to  $(1, p(\cdot, 0))$ .

242 *Proof.* We refer to [5, Prop. 4.3].  $\square$

243 **3.3. Critical directions.** Recall the definitions of  $\check{I}_i, \hat{I}_i$  and  $I_j^C$  given in Sec-  
 244 tion 2.2, and remember that we use  $z[v]$  to denote the solution of the linearized state  
 245 equation (B.2) associated to  $v$ .

246 We define the *cone of critical directions* at  $\bar{u}$  in  $L^2$ , or in short *critical cone*, by

$$247 \quad (3.11) \quad C := \left\{ \begin{array}{l} (z[v], v) \in Y \times L^2(0, T)^m; \\ v_i(t) \Psi_i^p(t) = 0 \text{ a.e. on } [0, T], \text{ for all } (p, \mu) \in \Lambda_1 \\ v_i(t) \geq 0 \text{ a.e. on } \check{I}_i, v_i(t) \leq 0 \text{ a.e. on } \hat{I}_i, \text{ for } i = 1, \dots, m, \\ \int_{\Omega} c_j(x) z[v](x, t) dx \leq 0 \text{ on } I_j^C, \text{ for } j = 1, \dots, q \end{array} \right\}.$$

248 The *strict critical cone* is defined below, and it is obtained by imposing that the  
 249 linearization of active constraints is zero,

$$250 \quad (3.12) \quad C_s := \left\{ \begin{array}{l} (z[v], v) \in Y \times L^2(0, T)^m; v_i(t) = 0 \text{ a.e. on } I_i, \text{ for } i = 1, \dots, m, \\ \int_{\Omega} c_j(x) z[v](x, t) dx = 0 \text{ on } I_j^C, \text{ for } j = 1, \dots, q \end{array} \right\}.$$

251 Hence, clearly  $C_s \subseteq C$ , and  $C_s$  is a closed subspace of  $Y \times L^2(0, T)^m$ .

252 **3.4. Second order necessary condition.** We recall from [5, Thm. 4.7].

253 THEOREM 3.5 (Second order necessary condition). *Let the admissible trajectory*  
 254  *$(\bar{u}, \bar{y})$  be an  $L^\infty$ -local solution of  $(P)$ . Then*

$$255 \quad (3.13) \quad \max_{(p, \mu) \in \Lambda_1} \mathcal{Q}[p](z, v) \geq 0, \quad \text{for all } (z, v) \in C_s.$$



256 **3.5. Goh transform.** Given a critical direction  $(z[v], v)$ , set

257 (3.14)  $w(t) := \int_0^t v(s)ds; \quad B(x, t) := \bar{y}(x, t)b(x); \quad \zeta(x, t) = z(x, t) - B(x, t) \cdot w(t).$

258 Then  $\zeta$  satisfies the initial and boundary conditions

259 (3.15)  $\zeta(x, 0) = 0 \text{ for } x \in \Omega, \quad \zeta(x, t) = 0 \text{ for } (x, t) \in \Sigma.$

260 Remembering the definition (B.1) of the operator  $A$ , we obtain that  
(3.16)

261  $\dot{\zeta} + A\zeta = \left( \dot{z} + Az - \sum_{i=1}^m v_i B_i \right) - \sum_{i=1}^m w_i (AB_i + \dot{B}_i), \quad \zeta(\cdot, 0) = 0, \quad \zeta(x, t) = 0 \text{ on } \Sigma.$

262 In view of the linearized state equation (B.2), the term between the large parentheses  
263 in the latter equation vanishes. Since  $\dot{B}_i = b_i \dot{\bar{y}}$  it follows that

264 (3.17)  $\dot{\zeta}(x, t) + (A\zeta)(x, t) = B^1(x, t) \cdot w(t), \quad \zeta(\cdot, 0) = 0, \quad \zeta(x, t) = 0 \text{ on } \Sigma,$

265 where

266 (3.18)  $B_i^1 := -fb_i + 2\nabla\bar{y} \cdot \nabla b_i + \bar{y}\Delta b_i - 2\gamma\bar{y}^3 b_i, \quad \text{for } i = 1, \dots, m.$

267 Equation (3.17) is well-posed since  $b \in W^{2,\infty}(\Omega)$ , and the solution  $\zeta$  belongs to  
268  $Y$ . We use  $\zeta[w]$  to denote the solution of (3.17) corresponding to  $w$ .

269 **3.6. Goh transform of the quadratic form.** Recall that  $(\bar{u}, \bar{y})$  is a feasible  
270 trajectory. Let  $\bar{p} = p[\bar{u}]$  be the costate associated to  $\bar{u}$ , and set

271 (3.19)  $W := Y \times L^2(0, T)^m \times \mathbb{R}^m.$

272 Let  $S(t)$  be the time dependent symmetric  $m \times m$ -matrix with generic term

273 (3.20)  $S_{ij}(t) := \int_{\Omega} b_i(x)b_j(x)p(x, t)\bar{y}(x, t)dx, \quad \text{for } 1 \leq i, j \leq m.$

274 Set

275 (3.21)  $\chi := \frac{d}{dt}(p\bar{y}) = pf + p\Delta\bar{y} - \bar{y}\Delta p + 2p\bar{y}^3 - \bar{y}(\bar{y} - y_d) - \bar{y} \sum_{j=1}^q c_j \mu_j.$

276 Observe that

277 (3.22)  $\dot{S}_{ij}(t) = \int_{\Omega} b_i b_j \frac{d}{dt}(p\bar{y})dx = \int_{\Omega} b_i b_j \chi dx.$

278 Since  $\bar{y}, p$  belong to  $L^\infty(0, T, H_0^1(\Omega))$ , and  $y_d, \bar{y}^3, \mu$  are essentially bounded, inte-  
279 grating by parts the terms in (3.21) involving the Laplacian operator and using (3.4),  
280 we obtain that  $\dot{S}_{ij}$  is essentially bounded. So we can define the continuous quadratic  
281 form on  $W$  :

282 (3.23)  $\widehat{\mathcal{Q}}[p, \mu](\zeta, w, h) := \int_0^T \hat{q}_I(t)dt + \hat{q}_T,$

283 where  
284

$$285 \quad (3.24) \quad \hat{q}_I := \int_{\Omega} \kappa \left( \zeta + \bar{y} \sum_{i=1}^m b_i w_i \right)^2 dx - w^\top \dot{S} w$$

$$286 \quad - 2 \sum_{i=1}^m w_i \int_{\Omega} \left[ \zeta (-\Delta b_i p - 2\nabla b_i \cdot \nabla p + b_i(\bar{y} - y_d) + b_i \sum_{j=1}^q c_j \dot{\mu}_j) - p B^1 \cdot w \right] dx,$$

287

288 and  
289

$$290 \quad (3.25) \quad \hat{q}_T :=$$

$$291 \quad \int_{\Omega} \left[ \left( \zeta(x, T) + \bar{y}(x, T) \sum_{i=1}^m h_i b_i(x) \right)^2 + 2 \sum_{i=1}^m h_i b_i(x) p(x, T) \zeta(x, T) \right] dx + h^\top S(T) h.$$

292

293 LEMMA 3.6 (Transformed second variation). For  $v \in L^2(0, T)^m$ , and  $w \in$   
294  $AC([0, T])^m$  given by the Goh transform (3.14), and for all  $(p, \mu) \in \Lambda_1$ , we have

$$295 \quad (3.26) \quad \mathcal{Q}[p](z[v], v) = \hat{\mathcal{Q}}[p, \mu](\zeta[w], w, w(T)).$$

296 *Proof.* We first replace  $z$  by  $\zeta + B \cdot w = \zeta + \bar{y} \sum_{i=1}^m w_i b_i$  in  $\mathcal{Q}$ , and define  
297

$$298 \quad (3.27) \quad \tilde{\mathcal{Q}} := \int_Q \left[ \kappa \left( \zeta + \bar{y} \sum_{i=1}^m w_i b_i \right)^2 + 2p \sum_{i=1}^m v_i b_i \left( \zeta + \bar{y} \sum_{j=1}^m w_j b_j \right) \right] dx dt$$

$$299 \quad + \int_{\Omega} \left( \zeta(T) + \bar{y}(T) \sum_{i=1}^m w_i(T) b_i \right)^2 dx.$$

300

301 We aim at proving that  $\tilde{\mathcal{Q}}$  coincides with  $\hat{\mathcal{Q}}$ . This will be done removing the depen-  
302 dence on  $v$  from the above expression. For this, we have to deal with the bilinear  
303 term in  $\tilde{\mathcal{Q}}$ , namely with

$$304 \quad (3.28) \quad \tilde{\mathcal{Q}}_b := \tilde{\mathcal{Q}}_{b,1} + 2 \sum_{i=1}^m \tilde{\mathcal{Q}}_{b,2i},$$

305 where, omitting the dependence on the multipliers for the sake of simplicity of the  
306 presentation,

$$307 \quad (3.29) \quad \tilde{\mathcal{Q}}_{b,1} := 2 \int_0^T v^\top S w dt \quad \text{and} \quad \tilde{\mathcal{Q}}_{b,2i} := \int_0^T v_i \int_{\Omega} b_i p \zeta dx dt, \quad \text{for } i = 1, \dots, m.$$

308 Concerning  $\tilde{\mathcal{Q}}_{b,1}$ , since  $S$  is symmetric, we get, integrating by parts,

$$309 \quad (3.30) \quad \tilde{\mathcal{Q}}_{b,1} = [w^\top S w]_0^T - \int_0^T w^\top \dot{S} w dt.$$

310 Hence  $\tilde{\mathcal{Q}}_{b,1}$  is a function of  $w$  and  $w(T)$ . Concerning  $\tilde{\mathcal{Q}}_{b,2i}$  defined in (3.29), integrating  
311 by parts, we get

$$312 \quad (3.31) \quad \tilde{\mathcal{Q}}_{b,2i} = w_i(T) \int_{\Omega} b_i p(x, T) \zeta(x, T) dx - \int_0^T w_i \int_{\Omega} b_i \frac{d}{dt} (p \zeta) dx dt.$$

313 For the derivative inside the latter integral, one has

$$314 \quad (3.32) \quad \frac{d}{dt}(p(x, t)\zeta(x, t)) = -\Delta p\zeta + p\Delta\zeta - \zeta \left( (\bar{y} - y_d) + \sum_{j=1}^q c_j \dot{\mu}_j \right) + pB^1 \cdot w.$$

315 By Green's Formula:

$$316 \quad (3.33) \quad \int_Q w_i b_i (-\Delta p\zeta + p\Delta\zeta) dx dt = \int_Q w_i (\Delta b_i p + 2\nabla b_i \cdot \nabla p) \zeta dx dt.$$

317 Using (3.32) and (3.33) in the expression (3.31) yields

318

$$319 \quad (3.34) \quad \tilde{Q}_{b,2i} = w_i(T) \int_{\Omega} b_i p(x, T) \zeta(x, T) dx + \int_Q w_i \left[ \zeta (-\Delta b_i p - 2\nabla b_i \cdot \nabla p \right. \\ 320 \quad \left. + b_i (\bar{y} - y_d) + b_i \sum_{j=1}^q c_j \dot{\mu}_j) - pB^1 \cdot w \right] dx dt. \\ 321$$

322 Hence,  $\tilde{Q}_{b,2}$  is a function of  $(\zeta, w, w(T))$ . Finally, putting together (3.27), (3.28),  
323 (3.30) and (3.34) yields an expression for  $\tilde{Q}$  that does not depend on  $v$  and coincides  
324 with  $\hat{Q}$  (in view of its definition given in (3.23)-(3.25)). This concludes the proof.  $\square$

325 *Remark 3.7.* The matrix appearing as coefficient of the quadratic term  $w$  in  $\hat{Q}$   
326 (see (3.24)) is the symmetric  $m \times m$  time dependent matrix  $R(t)$  with entries

$$327 \quad (3.35) \quad R_{ij} := \int_{\Omega} \left( \kappa b_i b_j \bar{y}^2 - \dot{S}_{ij} + p(b_i B_j^1 + b_j B_i^1) \right) dx, \quad \text{for } i, j = 1, \dots, m.$$

328 **3.7. Goh transform of the critical cone.** Here, we apply the Goh transform  
329 to the critical cone and obtain the cone  $PC$  in the new variables  $(\zeta, w, w(T))$ . We  
330 then define its closure  $PC_2$ , that will be used in the next section to prove second order  
331 sufficient conditions. In Proposition 3.11, we characterize  $PC_2$  in the case of scalar  
332 control.

333 **3.7.1. Primitives of strict critical directions.** Define the set of primitives  
334 of strict critical directions as

$$335 \quad (3.36) \quad PC := \left\{ \begin{array}{l} (\zeta, w, w(T)) \in Y \times H^1(0, T)^m \times \mathbb{R}^m; \\ (\zeta, w) \text{ is given by (3.14) for some } (z, v) \in C_s \end{array} \right\},$$

336 which is obtained by applying the Goh transform (3.14) to  $C_s$ , and let

$$337 \quad (3.37) \quad PC_2 := \text{closure of } PC \text{ in } Y \times L^2(0, T)^m \times \mathbb{R}^m.$$

338 Remember Definition 2.4 of the active constraints sets  $\tilde{B}_k, \hat{B}_k, B_k = \tilde{B}_k \cup \hat{B}_k, C_k$ .

339 **LEMMA 3.8.** *For any  $(\zeta, w, h) \in PC$ , it holds*

$$340 \quad (3.38) \quad w_{B_k}(t) = \frac{1}{\tau_{k+1} - \tau_k} \int_{\tau_k}^{\tau_{k+1}} w_{B_k}(s) ds, \quad \text{for } k = 0, \dots, r-1.$$

341 *Proof.* Immediate from the constancy of  $w_{B_k}$  a.e. on each  $(\tau_k, \tau_{k+1})$ , for any  
342  $(\zeta, w, h) \in PC$ .  $\square$

343 Take  $(z, v) \in C_s$ , and let  $w$  and  $\zeta[w]$  be given by the Goh transform (3.14). Let  
 344  $k \in \{0, \dots, r-1\}$  and take an index  $j \in C_k$ . Then  $0 = \int_{\Omega} c_j(x)z(x, t)dx$  on  $(\tau_k, \tau_{k+1})$ .  
 345 Therefore, letting  $M_j(t)$  denote the  $j$ th column of the matrix  $M(t)$  (defined in (2.13)),  
 346 one has

$$347 \quad (3.39) \quad M_j(t) \cdot w(t) = - \int_{\Omega} c_j(x)\zeta[w](x, t)dx, \quad \text{on } (\tau_k, \tau_{k+1}), \text{ for } j \in C_k.$$

348 We can rewrite (3.38)-(3.39) in the form

$$349 \quad (3.40) \quad \mathcal{A}^k(t)w(t) = (\mathcal{B}^k w)(t), \quad \text{on } (\tau_k, \tau_{k+1}),$$

350 where  $\mathcal{A}^k(t)$  is an  $m_k \times m$  matrix with  $m_k := |B_k| + |C_k|$ , and  $\mathcal{B}^k: L^2(0, T)^m \rightarrow$   
 351  $H^1(\tau_k, \tau_{k+1})^{m_k}$ . We can actually consider  $\mathcal{B} := (\mathcal{B}^1, \dots, \mathcal{B}^r)$  as a linear continuous  
 352 mapping from  $L^2(0, T)^m$  to  $\prod_{k=0}^{r-1} H^1(\tau_k, \tau_{k+1})^{m_k}$ , and  $\mathcal{A} := (\mathcal{A}^1, \dots, \mathcal{A}^r)$  as a linear  
 353 continuous mapping from  $L^2(0, T)^m$  into  $\prod_{k=0}^{r-1} L^2(\tau_k, \tau_{k+1})^{m_k}$ . For each  $t \in (\tau_k, \tau_{k+1})$ ,  
 354 let us use  $\mathcal{A}(t)$  to denote the matrix  $\mathcal{A}^k(t)$ . We have that, for a.e.  $t \in (0, T)$ ,  $\mathcal{A}(t)$  is  
 355 of maximal rank, so that there exists a unique measurable  $\lambda(t)$  (whose dimension is  
 356 the rank of  $\mathcal{A}(t)$  and depends on  $t$ ) such that

$$357 \quad (3.41) \quad w(t) = w_0(t) + \mathcal{A}(t)^\top \lambda(t), \quad \text{with } w_0(t) \in \text{Ker } \mathcal{A}(t).$$

358 Observe that  $\mathcal{A}(t)\mathcal{A}(t)^\top$  has a continuous time derivative and is uniformly invertible  
 359 on  $[0, T]$ . So,  $(\mathcal{A}(t)\mathcal{A}(t)^\top)^{-1}$  is linear continuous from  $H^1$  into  $H^1$  (with appropriate  
 360 dimensions) over each arc, and  $\mathcal{A}(t)\mathcal{A}(t)^\top \lambda(t) = (\mathcal{B}w)(t)$  a.e. We deduce that  $t \mapsto$   
 361  $(\lambda(t), w_0(t))$  belongs to  $H^1$  over each arc  $(\tau_k, \tau_{k+1})$ . So, in the subspace  $\text{Ker}(\mathcal{A} - \mathcal{B})$ ,  
 362  $w \mapsto \lambda(w)$  is linear continuous, considering the  $L^2(0, T)^m$ -topology in the departure  
 363 set, and the  $\prod_{k=0}^{r-1} H^1(\tau_k, \tau_{k+1})^{m_k}$ -topology in the arrival set. Since  $(\mathcal{A} - \mathcal{B})$  is linear  
 364 continuous over  $L^2(0, T)^m$  we have that

$$365 \quad (3.42) \quad w \in \text{Ker}(\mathcal{A} - \mathcal{B}), \quad \text{for all } (\zeta, w, h) \in PC_2.$$

366 While the inclusion induced by (3.42) could be strict, we see that for any  $(\zeta, w, h) \in$   
 367  $PC_2$ ,  $\lambda(w)$  and  $\mathcal{A}w$  are well-defined in the  $H^1$  spaces, and the following initial-final  
 368 conditions hold:

$$369 \quad (3.43) \quad \begin{cases} \text{(i)} & w_i = 0 \text{ a.e. on } (0, \tau_1), \text{ for each } i \in B_0, \\ \text{(ii)} & w_i = h_i \text{ a.e. on } (\tau_{r-1}, T), \text{ for each } i \in B_{r-1}, \\ \text{(iii)} & g'_j(\bar{y}(\cdot, T))[\zeta(\cdot, T) + B(\cdot, T) \cdot h] = 0 \text{ if } j \in C_{r-1}. \end{cases}$$

370 From the definitions of  $C_s$  (see (3.12)) and of  $PC_2$ , we can obtain additional continuity  
 371 conditions at the *bang-bang* junction points:

$$372 \quad (3.44) \quad \text{if } i \in B_{k-1} \cup B_k, \text{ then } w_i \text{ is continuous at } \tau_k, \quad \text{for all } (\zeta, w, h) \in PC_2.$$

373 *Remark 3.9.* Another example is when  $m = 1$ , the state constraint is active for  
 374  $t < \tau$  and the control constraint is active for  $t > \tau$ , then  $w$  is continuous at time  $\tau$ .  
 375 This is similar to the ODE case studied in [2, Remark 5].

376 We have seen that over each arc  $(\tau_k, \tau_{k+1})$ ,  $\lambda^k := \lambda(w)$  is pointwise well-defined,  
 377 and it possesses right limit at the entry point and left limit at the exit point, denoted  
 378 by  $\lambda(\tau_k^+)$  and  $\lambda(\tau_{k+1}^-)$ , respectively. Let  $c_{k+1} \in \mathbb{R}^m$  be such that, for some  $\nu^{k+i}$ ,

$$379 \quad (3.45) \quad c_{k+1} = \mathcal{A}^{k+i}(\tau_{k+1})^\top \nu^{k+i}, \quad \text{for } i = 0, 1,$$

380 meaning that  $c_{k+1}$  is a linear combination of the rows of  $\mathcal{A}^{k+i}(\tau_{k+1})$  for both  $i = 0, 1$ .

381 LEMMA 3.10. Let  $k = 0, \dots, r-1$ , and let  $c_{k+1}$  satisfy (3.45). Then, the junction  
382 condition

$$383 \quad (3.46) \quad c_{k+1} \cdot (w(\tau_{k+1}^+) - w(\tau_{k+1}^-)) = 0,$$

384 holds for all  $(\zeta, w, h) \in PC_2$ .

385 *Proof.* Let  $(\zeta, w, h)$  in  $PC$ , and set  $c := c_{k+1}$  and  $\tau := \tau_{k+1}$  in order to simplify  
386 the notation. Then

$$387 \quad (3.47) \quad c \cdot w(\tau) = (\nu^k)^\top \mathcal{A}^k(\tau) w(\tau) = (\nu^k)^\top \mathcal{A}^k(\tau) (\mathcal{A}^k(\tau))^\top \lambda^k(\tau).$$

388 By the same relations for index  $k+1$  we conclude that

$$389 \quad (3.48) \quad (\nu^k)^\top \mathcal{A}^k(\tau) (\mathcal{A}^k(\tau))^\top \lambda^k(\tau) = (\nu^{k+1})^\top \mathcal{A}^{k+1}(\tau) (\mathcal{A}^{k+1}(\tau))^\top \lambda^{k+1}(\tau).$$

390 Now let  $(\zeta, w, h) \in PC_2$ . Passing to the limit in the above relation (3.48) written for  
391  $(\zeta[w_\ell], w_\ell, h_\ell) \in PC$ ,  $w_\ell \rightarrow w$  in  $L^2(0, T)^m$ ,  $h_\ell \rightarrow h$  (which is possible since  $\lambda(t)$  is  
392 uniformly Lipschitz over each arc), we get that (3.48) holds for any  $(\zeta, w, h) \in PC_2$ ,  
393 from which the conclusion follows.  $\square$

394 By *junction conditions* at the junction time  $\tau = \tau_k \in (0, T)$ , we mean any relation  
395 of type (3.46). Set

$$396 \quad (3.49) \quad PC'_2 := \{(\zeta[w], w, h); w \in \text{Ker}(\mathcal{A} - \mathcal{B}), (3.46) \text{ holds, for all } c \text{ satisfying (3.45)}\}.$$

397 We have proved that

$$398 \quad (3.50) \quad PC_2 \subseteq PC'_2.$$

399 In the case of a scalar control ( $m = 1$ ) we can show that these two sets coincide.

400 **3.7.2. Scalar control case.** The following holds:

401 PROPOSITION 3.11. *If the control is scalar, then*

$$402 \quad (3.51) \quad PC_2 = \left\{ \begin{array}{l} (\zeta[w], w, h) \in Y \times L^2(0, T) \times \mathbb{R}; \quad w \in \text{Ker}(\mathcal{A} - \mathcal{B}); \\ w \text{ is continuous at } BB, BC, CB \text{ junctions} \\ \lim_{t \downarrow 0} w(t) = 0 \text{ if the first arc is not singular} \\ \lim_{t \uparrow T} w(t) = h \text{ if the last arc is not singular} \end{array} \right\}.$$

403 For a proof we refer to [2, Prop. 4 and Thm. 3].

404 **3.8. Necessary conditions after Goh transform.** The following second order  
405 necessary condition in the new variables follows.

406 THEOREM 3.12 (Second order necessary condition). *If  $(\bar{u}, \bar{y})$  is an  $L^\infty$ -local  
407 solution of problem (P), then*

$$408 \quad (3.52) \quad \max_{(p, \mu) \in \Lambda_1} \widehat{Q}[p, \mu](\zeta, w, h) \geq 0, \quad \text{on } PC_2.$$

409 *Proof.* Let  $(\zeta, w, h) \in PC_2$ . Then there exists a sequence  $(\zeta_\ell := \zeta[w_\ell], w_\ell, w_\ell(T))$   
410 in  $PC$  with

$$411 \quad (3.53) \quad (\zeta_\ell, w_\ell, w_\ell(T)) \rightarrow (\zeta, w, h), \quad \text{in } Y \times L^2(0, T) \times \mathbb{R}.$$

412 Let  $(z_\ell, v_\ell)$  denote, for each  $\ell$ , the corresponding critical direction in  $C_s$ . By Lemma  
413 3.6 and Theorem 3.5, there exists  $(p_\ell, \mu_\ell) \in \Lambda_1$  such that

$$414 \quad (3.54) \quad 0 \leq \mathcal{Q}[p_\ell](z_\ell, v_\ell) = \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta_\ell, w_\ell, h_\ell).$$

415 We have that  $(\mu_\ell)$  is bounded in  $L^\infty(0, T)$  (this is an easy variant of [5, Cor. 3.12]).  
416 Extracting if necessary a subsequence, we may assume that  $(\mu_\ell)$  weak\* converges in  
417  $L^\infty(0, T)$  to some  $d\mu$ . Consequently, the corresponding solutions  $p_\ell$  of (B.10) weakly  
418 converge to  $p$  in  $Y$ ,  $p$  being the costate associated with  $\mu$ , and  $p_\ell(T)$  converges to  $p(T)$   
419 in  $L^2(\Omega)$ . In view of the definition of  $\widehat{\mathcal{Q}}$  in (3.23), we get, by strong/weak convergence,

$$420 \quad (3.55) \quad \lim_{\ell \rightarrow \infty} \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta_\ell, w_\ell, h_\ell) = \lim_{\ell \rightarrow \infty} \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta, w, h) = \widehat{\mathcal{Q}}[p, \mu](\zeta, w, h). \quad \square$$

421 **4. Second order sufficient conditions.** In this section we derive second order  
422 sufficient optimality conditions.

423 **DEFINITION 4.1.** *We say that an  $L^2$ -local solution  $(\bar{u}, \bar{y})$  satisfies the weak qua-*  
424 *dratic growth condition if there exist  $\rho > 0$  and  $\varepsilon > 0$  such that,*

$$425 \quad (4.1) \quad F(u) - F(\bar{u}) \geq \rho(\|w\|_2^2 + |w(T)|^2),$$

426 *where  $(u, y[u])$  is an admissible trajectory,  $\|u - \bar{u}\|_2 < \varepsilon$ ,  $v := u - \bar{u}$ , and  $w(t) :=$   
427  $\int_0^t v(s) ds$ .*

428 *Remark 4.2.* Note that (4.1) is a quadratic growth condition in the  $L^2$ -norm of  
429 the perturbations  $(w, w(T))$  obtained after Goh transform.

430 The main result of this part is given in Theorem 4.5 and establishes sufficient  
431 conditions for a trajectory to be a  $L^2$ -local solution with weak quadratic growth.

432 Throughout the section we assume Hypothesis 3.1. In particular, we have by  
433 Theorem 3.3 that the state and costate are essentially bounded.

434 Consider the condition

$$435 \quad (4.2) \quad g'_j(\bar{y}(\cdot, T))(\bar{\zeta}(\cdot, T) + B(\cdot, T)\bar{h}) = 0, \text{ if } T \in I_j^C \text{ and } [\mu_j(T)] > 0, \text{ for } j = 1, \dots, q.$$

436 We define

$$437 \quad (4.3) \quad PC_2^* := \left\{ (\zeta[w], w, h) \in Y \times L^2(0, T)^m \times \mathbb{R}^m; w_{B_k} \text{ is constant on each arc;} \right\}.$$

(3.17), (3.39), (3.43)(i)-(ii), (4.2) hold.

438 Note that  $PC_2^*$  is a superset of  $PC_2$ .

439 **DEFINITION 4.3.** *Let  $W$  be a Banach space. We say that a quadratic form  $Q: W \rightarrow$*   
440  *$\mathbb{R}$  is a Legendre form if it is weakly lower semicontinuous, positively homogeneous of*  
441 *degree 2, i.e.,  $Q(tx) = t^2Q(x)$  for all  $x \in W$  and  $t > 0$ , and such that if  $x_\ell \rightharpoonup x$  and*  
442  *$Q(x_\ell) \rightarrow Q(x)$ , then  $x_\ell \rightarrow x$ .*

443 We assume, in the remainder of the article, the following strict complementarity  
444 conditions for the control and the state constraints.

445 **Hypothesis 4.4.** The following conditions hold:

$$446 \quad (4.4) \quad \left\{ \begin{array}{l} \text{(i) for all } i = 1, \dots, m : \\ \quad \max_{(p, \mu) \in \Lambda_1} \Psi_i^p(t) > 0 \text{ in the interior of } \check{I}_i, \text{ at } t = 0 \text{ if } 0 \in \check{I}_i, \text{ at } t = T \text{ if } T \in \check{I}_i, \\ \quad \min_{(p, \mu) \in \Lambda_1} \Psi_i^p(t) < 0 \text{ in the interior of } \hat{I}_i, \text{ at } t = 0 \text{ if } 0 \in \hat{I}_i, \text{ at } t = T \text{ if } T \in \hat{I}_i, \\ \text{(ii) there exists } (p, \mu) \in \Lambda_1 \text{ such that } \text{supp } d\mu_j = I_j^C, \text{ for all } j = 1, \dots, q. \end{array} \right.$$

448 THEOREM 4.5. *Let Hypotheses 3.1 and 4.4 hold. Then the following assertions*  
 449 *hold.*

450 a) *Assume that*

- 451 (i)  $(\bar{u}, \bar{y})$  *is a feasible trajectory with nonempty associated set of multipliers*  
 452  $\Lambda_1$ ;  
 453 (ii) *for each*  $(p, \mu) \in \Lambda_1$ ,  $\widehat{\mathcal{Q}}[p, \mu](\cdot)$  *is a Legendre form on the space*  
 454  $\{(\zeta[w], w, h) \in Y \times L^2(0, T)^m \times \mathbb{R}^m\}$ ; *and*  
 455 (iii) *the uniform positivity holds, i.e. there exists*  $\rho > 0$  *such that*  
 456 (4.5)

$$\max_{(p, \mu) \in \Lambda_1} \widehat{\mathcal{Q}}[p, \mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w, h) \in PC_2^*.$$

457 *Then*  $(\bar{u}, \bar{y})$  *is a*  $L^2$ -*local solution satisfying the weak quadratic growth condi-*  
 458 *tion.*

459 b) *Conversely, for an admissible trajectory*  $(\bar{u}, y[\bar{u}])$  *satisfying the growth condi-*  
 460 *tion (4.1), it holds*

$$(4.6) \quad \max_{(p, \mu) \in \Lambda_1} \widehat{\mathcal{Q}}[p, \mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w, h) \in PC_2.$$

462 The remainder of this section is devoted to the proof of Theorem 4.5. We first  
 463 state some technical results.

464 **4.1. A refined expansion of the Lagrangian.** Let  $(\bar{u}, \bar{y})$  be an admissible tra-  
 465 jectory. We start with a refinement of the expansion of the Lagrangian of Proposition  
 466 3.4.

467 LEMMA 4.6. *Let*  $(u, y)$  *be a trajectory,*  $(\delta y, v) := (u - \bar{u}, y - \bar{y})$ ,  $z$  *be the solution*  
 468 *of the linearized state equation (B.2),*  $(w, \zeta)$  *given by the Goh transform (3.14) and*  
 469  $\eta := \delta y - z$ . *Then*

$$(4.7) \quad \begin{aligned} & \text{(i) } \|z\|_{L^2(Q)} + \|z(\cdot, T)\|_{L^2(\Omega)} = O(\|w\|_2 + |w(T)|), \\ & \text{(ii.a) } \|\delta y\|_{L^2(Q)} + \|\delta y(\cdot, T)\|_{L^2(\Omega)} = O(\|w\|_2 + |w(T)|), \\ & \text{(ii.b) } \|\delta y\|_{L^\infty(0, T; H_0^1(\Omega))} = O(\|w\|_\infty), \\ & \text{(iii) } \|\eta\|_{L^\infty(0, T; L^2(\Omega))} + \|\eta(\cdot, T)\|_{L^2(\Omega)} = o(\|w\|_2 + |w(T)|). \end{aligned}$$

471 Before doing the proof of Lemma 4.6, let us recall the following property:

472 PROPOSITION 4.7. *The equation*

$$(4.8) \quad \dot{\Phi} - \Delta\Phi + a\Phi = \hat{f}, \quad \Phi(x, 0) = 0,$$

474 *with*  $a \in L^\infty(Q)$ ,  $\hat{f} \in L^1(0, T; L^2(\Omega))$ , *and homogeneous Dirichlet conditions on*  $\partial\Omega \times$   
 475  $(0, T)$ , *has a unique solution*  $\Phi$  *in*  $C([0, T]; L^2(\Omega))$ , *that satisfies*

$$(4.9) \quad \|\Phi\|_{C([0, T]; L^2(\Omega))} \leq c\|\hat{f}\|_{L^1(0, T; L^2(\Omega))}.$$

477 *Proof.* This follows from the estimate for mild solutions in the semigroup theory,  
 478 see e.g. [3, Theorem 2].  $\square$

479 *Proof of Lemma 4.6.* (i) Since  $\zeta$  is solution of (3.17), it satisfies (4.8) with

$$(4.10) \quad a := -3\gamma\bar{y}^2 + \sum_{i=0}^m \bar{u}_i b_i, \quad \hat{f} := \sum_{i=1}^m w_i B_i^1,$$

481 where  $B_i^1$  is given in (3.18). One can see, in view of Hypothesis 3.1, that  $\hat{f} \in$   
 482  $L^1(0, T; L^2(\Omega))$  since the terms in brackets in (4.10) belong to  $L^\infty(0, T; L^2(\Omega))$ . Thus,  
 483 from Proposition 4.7 we get that  $\zeta \in C([0, T]; L^2(\Omega))$  and

$$484 \quad (4.11) \quad \|\zeta\|_{L^\infty(0, T; L^2(\Omega))} = O(\|\hat{f}\|_{L^1(0, T; L^2(\Omega))}) = O(\|w\|_1).$$

485 Thus, due to Goh transform (3.14) and Lemma A.1, we get that  $z$  belongs to  $C([0, T]; L^2(\Omega))$  ■  
 486 and we obtain the estimate (i).

487 We next prove the estimate (ii) for  $\delta y$ . Set  $\zeta_{\delta y} := \delta y - (w \cdot b)\bar{y}$ . Then

$$488 \quad (4.12) \quad \dot{\zeta}_{\delta y} - \Delta \zeta_{\delta y} + a_{\delta y} \zeta_{\delta y} = \hat{f}_{\delta y},$$

489 with

$$490 \quad (4.13) \quad \begin{aligned} a_{\delta y} &:= 3\gamma\bar{y}^2 + 3\gamma\bar{y}\zeta_{\delta y} + \gamma(\zeta_{\delta y})^2 - (\bar{u} \cdot b), \\ \hat{f}_{\delta y} &:= \sum_{i=1}^m w_i [\bar{y}\Delta b_i + \nabla b_i \cdot \nabla \bar{y} - b_i(2\gamma\bar{y}^3 + f)]. \end{aligned}$$

491 By Theorem 3.3,  $\zeta_{\delta y}$  is in  $L^\infty(Q)$ , hence  $a_{\delta y}$  is essentially bounded. Furthermore,  
 492 in view of the regularity Hypothesis 3.1 and Lemma A.1,  $\hat{f}_{\delta y} \in L^1(0, T; L^2(\Omega))$ . We  
 493 then get, using Proposition 4.7,

$$494 \quad (4.14) \quad \|\zeta_{\delta y}\|_{L^\infty(0, T; H_0^1(\Omega))} \leq O(\|w\|_1).$$

495 From the latter equation and the definition of  $\zeta_{\delta y}$  we deduce (ii.a). Since

$$496 \quad (4.15) \quad \nabla(\delta y) = \nabla(\zeta_{\delta y}) + \sum_{i=1}^m w_i (\bar{y}\nabla b_i + b_i\nabla \bar{y}),$$

497 applying the  $L^\infty(0, T; L^2(\Omega))$ -norm to both sides, and using (4.14) and Lemma A.1  
 498 we get (ii.b).

499 The estimate in (iii) follows from the following consideration. To apply Proposi-  
 500 tion 4.7 to equation (3.8) we easily verify that  $r$  is in  $L^\infty(Q)$  and  $\tilde{r}$  in  $L^1(0, T; L^2(\Omega))$ .  
 501 Consequently, we have

$$(4.16) \quad \square$$

$$502 \quad \begin{aligned} \|\eta\|_{C([0, T]; L^2(\Omega))} &\leq c \left\| \sum_{i=1}^m v_i b_i \delta y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3 \right\|_{L^1(0, T; L^2(\Omega))} \\ &\leq \|v\|_2 \|b\|_\infty \|\delta y\|_2 + 3\gamma \|\bar{y}\|_\infty \|(\delta y)^2\|_{L^1(0, T; L^2(\Omega))} + \gamma \|(\delta y)^3\|_{L^1(0, T; L^2(\Omega))}. \end{aligned}$$

503 Now, since  $\|v\|_2 \rightarrow 0$  and  $\|\delta y\|_\infty \rightarrow 0$  (by similar arguments to those of the proof of  
 504 (i) in Theorem 3.3), we get (iii).

505 **PROPOSITION 4.8.** *Let  $(p, d\mu) \in \Lambda_1$ . Let  $(u_\ell) \subset \mathcal{U}_{\text{ad}}$  and let us write  $y_\ell$  for the*  
 506 *corresponding states. Set  $v_\ell := u_\ell - \bar{u}$  and assume that  $v_\ell \rightarrow 0$  a.e. Then,*

$$507 \quad (4.17) \quad \begin{aligned} \mathcal{L}[p, \mu](\bar{u} + v_\ell, y_\ell) &= \mathcal{L}[p, \mu](\bar{u}, \bar{y}) \\ &+ \int_0^T \Psi^p(t) \cdot v_\ell(t) dt + \frac{1}{2} \widehat{\mathcal{Q}}[p, \mu](\zeta_\ell, w_\ell, w_\ell(T)) + o(\|w_\ell\|_2^2 + |w_\ell(T)|^2), \end{aligned}$$

509 where  $w_\ell$  and  $\zeta_\ell$  are given by the Goh transform (3.14).  
 510



512 *Proof.* Since  $(v_\ell)$  is bounded in  $L^\infty(0, T)^m$  and converges a.e. to 0, it converges  
 513 to zero in any  $L^p(0, T)^m$ . For simplicity of notation we omit the index  $\ell$  for the  
 514 remainder of the proof. Set  $\delta y := y[\bar{u} + v] - \bar{y}$ . By Proposition 3.4 it is enough to  
 515 prove that

$$516 \quad (4.18) \quad \left| \mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p, \mu](w, w(T), \zeta) \right| = o(\|w\|_2^2 + |w(T)|^2),$$

$$517 \quad (4.19) \quad \left| \int_Q p(\delta y)^3 \right| = o(\|w\|_2^2 + |w(T)|^2).$$

518  
 519 We have, setting as before  $\eta := \delta y - z$  where  $z := z[v]$ ,

$$520 \quad (4.20) \quad \begin{aligned} & \mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p, \mu](\zeta, w, w(T)) = \mathcal{Q}[p](\delta y, v) - \mathcal{Q}[p](z, v) \\ & = 2 \int_Q (v \cdot b) p \eta dx dt + \int_Q \kappa(\delta y + z) \eta dx dt + \int_\Omega (\delta y(x, T) + z(x, T)) \eta(x, T) dx, \end{aligned}$$

521 and therefore, since the state and costate are essentially bounded:

$$522 \quad (4.21) \quad \begin{aligned} \left| \mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p, \mu](\zeta, w, w(T)) \right| & \leq 2 \left| \int_Q (v \cdot b) p \eta dx dt \right| + O(\|\delta y + z\|_2 \|\eta\|_2) \\ & \quad + O(\|(\delta y + z)(\cdot, T)\|_{L^2(\Omega)} \|\eta(\cdot, T)\|_{L^2(\Omega)}). \end{aligned}$$

523 In view of lemma 4.6, the ‘big O’ terms in the r.h.s. are of the desired order and it  
 524 remains to deal with the integral term. We have, integrating by parts in time,

$$525 \quad (4.22) \quad \int_Q (v \cdot b) p \eta dx dt = \int_\Omega (w(T) \cdot b(x)) p(x, T) \eta(x, T) dx - \int_Q (w \cdot b) \frac{d}{dt} (p \eta) dx dt.$$

526 For the first term in the r.h.s. of (4.22) we get, in view of (4.7)(ii),

$$528 \quad (4.23) \quad \left| \int_\Omega (w(T) \cdot b(x)) p(x, T) \eta(x, T) dx \right|$$

$$529 \quad \quad \quad = O(\|w(T)\| \|\eta(\cdot, T)\|_{L^2(\Omega)}) = o(\|w\|_2^2 + |w(T)|^2).$$

531 And, for the second term in the r.h.s. of (4.22), since  $b$  is essentially bounded, and  $p$   
 532 and  $\eta$  satisfy (B.10) and (3.8), respectively, we have that,

$$(4.24) \quad \frac{d}{dt} (p \eta) = \varphi_0 + \varphi_1 + \varphi_2,$$

$$533 \quad \varphi_0 := p \Delta \eta - \eta \Delta p; \quad \varphi_1 := (v \cdot b) p \delta y; \quad \varphi_2 := p e(\delta y)^2 - \eta \left( y - y_d + \sum_{j=1}^q c_j \dot{\mu}_j(t) \right).$$

534 *Contribution of  $\varphi_2$ .* Since  $y$ ,  $p$  and  $\dot{\mu}$  are essentially bounded (see Theorem 3.3), we  
 535 get

$$536 \quad (4.25) \quad \left| \int_Q (w \cdot b) \varphi_2 \right| = O(\|w(\delta y)^2 + w \eta\|_2) = o(\|w\|_2^2 + |w(T)|^2),$$

537 where the last equality follows from the estimates for  $\delta y$  and  $\eta$  obtained in Lemma 4.6.

538 *Contribution of  $\varphi_1$ .* Integrating by parts in time, we can write the contribution of  $\varphi_1$   
 539 as

$$540 \quad (4.26) \quad \frac{1}{2} \int_Q \frac{d}{dt} (w \cdot b)^2 p \delta y = \frac{1}{2} \int_\Omega (w(T) \cdot b)^2 p(x, T) \delta y(x, T) - \frac{1}{2} \int_Q (w \cdot b)^2 \frac{d}{dt} (p \delta y)$$

541 The contribution of the term at  $t = T$  is of the desired order. Let us proceed with  
 542 the estimate for the last term in the r.h.s. of (4.26). We have

$$543 \quad (4.27) \quad \begin{aligned} \frac{d}{dt}(p\delta y) &= (-\delta y\Delta p + p\Delta\delta y) \\ &+ \left( -(\bar{y} - y_d) - \sum_{j=1}^q c_j \dot{\mu}_j \right) \delta y + \left( \sum_{i=1}^m v_i b_i y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3 \right) p. \end{aligned}$$

544 For the contribution of first term in the r.h.s. of latter equation we get

$$545 \quad (4.28) \quad \int_Q (w \cdot b)^2 (-\delta y\Delta p + p\Delta\delta y) = \sum_{i,j=1}^m \int_0^T w_i w_j \int_{\Omega} \nabla(b_i b_j) \cdot (\delta y \nabla p - p \nabla \delta y).$$

546 Using [5, Lem. 2.2], since  $\nabla(b_i b_j)$  is essentially bounded for every pair  $i, j$ , it is enough  
 547 to prove that

$$548 \quad (4.29) \quad \int_{\Omega} \nabla(b_i b_j) \cdot (\delta y \nabla p - p \nabla \delta y) \rightarrow 0$$

549 uniformly in time. For this, in view of the estimate for  $\|\delta y\|_{L^\infty(0,T;H_0^1(\Omega))}$  obtained in  
 550 Lemma 4.6 item (ii.b), and since  $p$  is essentially bounded, it suffices to prove that  $p$   
 551 is in  $L^\infty(0,T;H^1(\Omega))$  which follows from Corollary B.1.

552 Let us continue with the expression in (4.27). The terms containing  $\delta y$  go to 0 in  
 553  $L^\infty(0,T;L^2(\Omega))$  and that is sufficient for our purpose. The only term that has to be  
 554 estimated is

$$555 \quad (4.30) \quad \begin{aligned} \int_Q (w \cdot b)^2 (v \cdot b) y p &= \frac{1}{3} \int_Q \frac{d}{dt} (w \cdot b)^3 y p \\ &= \frac{1}{3} \int_{\Omega} (w(T) \cdot b)^3 y(\cdot, T) p(\cdot, T) - \frac{1}{3} \int_Q (w \cdot b)^3 \frac{d}{dt} (y p). \end{aligned}$$

559 We consider the pair of state and costate equations with  $g := y - y_d$  given as

$$560 \quad (4.31) \quad \begin{aligned} \dot{y} - \Delta y + \gamma y^3 &= (u \cdot b)y + f; & y(0) &= y_0; \\ -\dot{p} - \Delta p + \gamma y^2 p &= (u \cdot b)p + g + c\dot{\mu}; & p(T) &= 0. \end{aligned}$$

561 and so for sufficiently smooth  $\varphi: \Omega \times (0, T) \rightarrow \mathbb{R}$  we have

$$562 \quad (4.32) \quad \begin{aligned} \int_Q \varphi \frac{d}{dt} (y p) &= \int_Q \varphi (\dot{y} p + y \dot{p}) \\ &= \int_Q \varphi [(\Delta y - \gamma y^3 + (u \cdot b)y + f)p + y(-\Delta p + \gamma y^2 p - (u \cdot b)p - g - c\dot{\mu})] \\ &= \int_Q \varphi [f p - y g + c\dot{\mu} y] + \nabla \varphi \cdot (-p \nabla y + y \nabla p), \end{aligned}$$

563 and, consequently, we have for  $\varphi = (w \cdot b)^3$ ,

$$564 \quad (4.33) \quad \int_Q (w \cdot b)^3 \frac{d}{dt} (y p) = \int_Q (w \cdot b)^3 [f p - y g + c\dot{\mu} y] + \nabla (w \cdot b)^3 \cdot (-p \nabla y + y \nabla p).$$

565 By Hypothesis 3.1,  $f$  and  $b$  are sufficiently smooth,  $\mu$  is essentially bounded,  $y, p \in$   
 566  $L^\infty(0, T; H_0^1(\Omega))$ . We estimate

$$567 \quad \left| \int_Q (w \cdot b)^3 \frac{d}{dt}(yp) \right| \leq \|b\|_\infty^3 \|w\|_\infty \|w\|_2^2 \|fp - yg + c\dot{\mu}y\|_{L^\infty(0, T; L^1(\Omega))} \\ + O(\|b\|_\infty^2 \|\nabla b\|_\infty) \|w\|_\infty \|w\|_2^2 \left( \|y\|_{L^\infty(0, T; H_0^1(\Omega))} \|p\|_{L^\infty(0, T; H_0^1(\Omega))} \right) = o(\|w\|^2).$$

568

569 *Contribution of  $\varphi_0$ .* Integrating by parts, we have that

$$570 \quad (4.34) \quad \int_0^T w_i \int_\Omega b_i \varphi_0 = \int_0^T w_i \int_\Omega b_i (p \Delta \eta - \eta \Delta p) = \int_0^T w_i \int_\Omega \nabla b_i \cdot (-p \nabla \eta + \eta \nabla p) \\ = \int_0^T w_i \int_\Omega (p \eta \Delta b_i + 2 \eta \nabla p \cdot \nabla b_i).$$

571 Recalling that  $b \in W^{2, \infty}(\Omega)$  (see (3.4)) and that  $p$  is essentially bounded (due to  
 572 Theorem 3.3), we get for the first term in the r.h.s. of the latter display,

$$573 \quad (4.35) \quad \left| \int_0^T w_i \int_\Omega p \eta \Delta b_i \right| \leq \|\Delta b_i\|_\infty \|w_i\|_2 \|p\|_\infty \|\eta\|_{L^2(0, T; L^2(\Omega))},$$

574 that is a small- $o$  of  $\|w\|_2^2$  in view of item (iii.a) of Lemma 4.6. For the second term in  
 575 the r.h.s. of (4.34) we get

$$576 \quad (4.36) \quad \left| \int_0^T w_i \int_\Omega \eta \nabla p \cdot \nabla b_i \right| \leq \|\nabla b_i\|_\infty \|w_i\|_2 \|\eta\|_{L^2(0, T; L^2(\Omega))} \|\nabla p\|_{L^\infty(0, T; L^2(\Omega)^n)}$$

577 Since  $p \in L^\infty(0, T; H^1(\Omega))$  as showed some lines above and in view of item (iii.a) of  
 578 Lemma 4.6, we get that the r.h.s. of latter equation is a small- $o$  of  $\|w\|_2^2$ , as desired.

579 Collecting the previous estimates, we get (4.18). Similarly, since  $\delta y \rightarrow 0$  uniformly  
 580 and the costate  $p$  is essentially bounded, with (4.7)(i) we get

$$581 \quad (4.37) \quad \left| \int_Q pb(\delta y)^3 dx dt \right| = o(\|\delta y\|_2^2) = o(\|w\|_2^2 + |w(T)|^2).$$

582 The result follows.  $\square$

583 **COROLLARY 4.9.** *Let  $u = \bar{u} + v$  be an admissible control. Then, setting  $w(t) :=$   
 584  $\int_0^t v(s) ds$ , we have the reduced cost expansion*

$$585 \quad (4.38) \quad F(u) = F(\bar{u}) + DF(\bar{u})v + O(\|w\|_2^2 + |w(T)|^2).$$

586 **PROPOSITION 4.10.** *Let  $(p, d\mu) \in \Lambda_1$ , and let  $(z, v) \in Y \times L^2(0, T)^m$  satisfy the  
 587 linearized state equation (B.2). Then,*

$$588 \quad (4.39) \quad \int_0^T \Psi^p(t) \cdot v(t) dt = DJ(\bar{u}, \bar{y})(z, v) + \sum_{j=1}^q \int_0^T g'_j(\bar{y}(\cdot, t)) z(\cdot, t) d\mu_j(t),$$

where

$$DJ(\bar{u}, \bar{y})(z, v) = \sum_{i=1}^m \int_0^T \alpha_i v_i dt + \int_Q (\bar{y} - y_d) z dx dt + \int_\Omega (\bar{y}(T) - y_{dT}) z(T) dx,$$

589 and it coincides with  $DF(\bar{u})v$ .

590 *Proof.* It follows from (B.2), (B.8) and (2.9).  $\square$

591 **4.2. Proof of Theorem 4.5.** What remains to prove is similar to what has  
 592 been done in Aronna, Bonnans and Goh [2, Theorem 5], except that here the control  
 593 variable may be multidimensional and in [2] it is scalar.

594 We start by showing item a). If the conclusion does not hold, there must exist a  
 595 sequence  $(u_\ell, y_\ell)$  of admissible trajectories, with  $u_\ell$  distinct from  $\bar{u}$ , such that  $v_\ell :=$   
 596  $u_\ell - \bar{u}$  converges to zero in  $L^2(0, T)^m$ , and

$$597 \quad (4.40) \quad J(u_\ell, y_\ell) \leq J(\bar{u}, \bar{y}) + o(\Upsilon_\ell^2),$$

where  $(w_\ell, \zeta_\ell)$  is obtained by Goh transform (3.14),  $h_\ell := w_\ell(T)$  and

$$\Upsilon_\ell := \sqrt{\|w_\ell\|_2^2 + |w_\ell(T)|^2}.$$

598 Let  $(p, d\mu) \in \Lambda_1$ . Adding  $\int_0^T g(y_\ell) d\mu \leq 0$  on both sides of (4.40) leads to

$$599 \quad (4.41) \quad \mathcal{L}[p, \mu](u_\ell, y_\ell) \leq \mathcal{L}[p, \mu](\bar{u}, \bar{y}) + o(\Upsilon_\ell^2).$$

600 Set  $(\bar{v}_\ell, \bar{w}_\ell, \bar{h}_\ell) := (v_\ell, w_\ell, h_\ell)/\Upsilon_\ell$ . Then  $(\bar{w}_\ell, \bar{h}_\ell)$  has unit norm in  $L^2(0, T)^m \times \mathbb{R}^m$ .  
 601 Extracting if necessary a subsequence, we may assume that there exists  $(\bar{w}, \bar{h})$  in  
 602  $L^2(0, T)^m \times \mathbb{R}^m$  such that

$$603 \quad (4.42) \quad \bar{w}_\ell \rightharpoonup \bar{w} \quad \text{and} \quad h_\ell \rightarrow \bar{h},$$

604 where the first limit is given in the weak topology of  $L^2(0, T)^m$ . Set  $\bar{\zeta} := \zeta[\bar{w}]$ . The  
 605 remainder of the proof is split in two parts:

606 **Fact 1:** The triple  $(\bar{\zeta}, \bar{w}, \bar{h})$  belongs to  $PC_2^*$  (defined in (4.3)).

607 **Fact 2:** The inequality (4.40) contradicts the hypothesis of uniform positiv-  
 608 ty (4.5).

609 **Proof of Fact 1.** We divide this part in four steps: (a)  $\bar{w}_i$  is constant on each  
 610 maximal arc of  $I_i$ , for  $i = 1, \dots, m$ , (b) (3.43)(i),(ii) hold, (c) (3.39) holds, and (d)  
 611 (4.2) holds.

612 (a) From Proposition 4.8, inequality (4.41), and (2.10) we have

$$613 \quad (4.43) \quad -\widehat{\mathcal{Q}}[p, d\mu](\zeta_\ell, w_\ell, h_\ell) + o(\Upsilon_\ell^2) \geq \sum_{i=1}^m \int_0^T \Psi_i^p(t) \cdot v_{\ell,i}(t) dt \geq 0.$$

614 By the continuity of the quadratic form  $\widehat{\mathcal{Q}}[p, d\mu]$  over the space  $L^2(0, T)^m \times \mathbb{R}^m$ , we  
 615 deduce that

$$616 \quad (4.44) \quad 0 \leq \int_0^T \Psi_i^p(t) v_{\ell,i}(t) dt \leq O(\Upsilon_\ell^2), \quad \text{for all } i = 1, \dots, m.$$

618 Hence, since the integrand in previous inequality is nonnegative for all  $\ell \in \mathbb{N}$ , we have  
 619 that

$$620 \quad (4.45) \quad \lim_{\ell \rightarrow \infty} \int_0^T \Psi_i^p(t) \varphi(t) \bar{v}_{\ell,i}(t) dt = 0$$

622 for any nonnegative  $C^1$  function  $\varphi: [0, T] \rightarrow \mathbb{R}$ . Let us consider, in particular,  $\varphi$   
 623 having its support  $[c, d] \subset I_i$ . Integrating by parts in (4.45) and using that  $\bar{w}_\ell$  is a  
 624 primitive of  $\bar{v}_\ell$ , we obtain

$$625 \quad (4.46) \quad 0 = \lim_{\ell \rightarrow \infty} \int_0^T \frac{d}{dt} (\Psi_i^p \varphi) \bar{w}_{\ell,i} dt = \int_c^d \frac{d}{dt} (\Psi_i^p(t) \varphi) \bar{w}_i dt.$$

627 Over  $[c, d]$ ,  $\bar{v}_{\ell,i}$  has constant sign and, therefore,  $\bar{w}_i$  is either nondecreasing or nonin-  
628 creasing. Thus, we can integrate by parts in the latter equation to get

$$629 \quad (4.47) \quad \int_c^d \Psi_i^p(t) \varphi(t) d\bar{w}_i(t) = 0.$$

630 Take now any  $t_0 \in (c, d)$ . Assume, w.l.g. that  $t_0 \in \check{I}_i$ . By the strict complementary  
631 condition for the control constraint given in (4.4), there exists a multiplier such that  
632 the associated  $\Psi^p$  verifies  $\Psi_i^p(t_0) > 0$ . Hence, in view of the continuity of  $\Psi_i^p$  on  $I_i$ ,  
633 there exists  $\varepsilon > 0$  such that  $\Psi_i^p > 0$  on  $(t_0 - 2\varepsilon, t_0 + 2\varepsilon) \subset (c, d)$ . Choose  $\varphi$  such that  
634  $\text{supp } \varphi \subset (t_0 - 2\varepsilon, t_0 + 2\varepsilon)$ , and  $\Psi_i^p \varphi \equiv 1$  on  $(t_0 - \varepsilon, t_0 + \varepsilon)$ , then  $\bar{w}_i(t_0 + \varepsilon) - \bar{x}_i(t_0 - \varepsilon) = 0$ .  
635 Since  $d\bar{w}_i \geq 0$ , we obtain  $d\bar{w}_i = 0$  a.e. on  $\check{I}_i$ . Since  $t_0$  is an arbitrary point in the  
636 interior of  $I_i$ , we get

$$637 \quad (4.48) \quad d\bar{w}_i = 0 \quad \text{a.e. on } \check{I}_i.$$

638 This concludes step **(a)**.

639 **(b)** We now have to prove (3.43)(i),(ii). Assume now that  $B_0 \neq \emptyset$  or, w.l.g., that  
640  $\bar{B}_0 \neq \emptyset$ , and let  $i \in \bar{B}_0$ . By the previous step,  $\bar{w}_i$  is equal to some constant  $\theta$  a.e.  
641 over  $(0, \tau_1)$ . Let us show that  $\theta = 0$ . By the strict complementarity condition for the  
642 control constraint (4.4) there exist  $t, \delta > 0$  and a multiplier such that the associated  
643  $\Psi^p$  satisfies  $\Psi_i^p > \delta$  on  $[0, t] \subset [0, \tau_1]$ . By considering in (4.45) a nonnegative Lipschitz  
644 continuous function  $\varphi: [0, T] \rightarrow \mathbb{R}$  being equal to  $1/\delta$  on  $[0, t]$ , with support included  
645 in  $[0, \tau_1]$ , and since  $\bar{v}_{\ell,i} \geq 0$  a.e. on  $[0, \tau_1]$ , we obtain, for any  $\tau \in [0, t]$ ,

$$646 \quad (4.49) \quad \bar{w}_{\ell,i}(\tau) = \int_0^\tau \bar{v}_{\ell,i}(s) ds \leq \int_0^t \Psi_i^p(s) \varphi(s) \bar{v}_{\ell,i}(s) ds \rightarrow 0, \quad \text{when } \ell \rightarrow \infty.$$

647 Thus  $\bar{w}_i = 0$  a.e. on  $[0, t]$ . Consequently, from (4.48) we get  $\bar{w}_i = 0$  a.e. on  $[0, \tau_1)$ .  
648 The case when  $i \in B_{r-1}$  follows by a similar argument. This yields item **(b)**.

649 **(c)** Let us prove (3.39). We have, since  $y_\ell$  is admissible and  $g$  linear,

$$650 \quad (4.50) \quad 0 \geq g_j(y_\ell(\cdot, t)) - g_j(\bar{y}(\cdot, t)) = \int_\Omega c_j(x) (y_\ell - \bar{y})(x, t) dx, \quad \text{on } [\tau_k, \tau_{k+1}],$$

651 whenever  $k, j$  are such that  $k \in \{0, \dots, r-1\}$  and  $j \in C_k$ . Let  $z_\ell$  denote the linearized  
652 state corresponding to  $v_\ell$ , and let  $\eta_\ell := y_\ell - \bar{y} - z_\ell$ . By Lemma (4.6)(iii), we deduce  
653 that

$$654 \quad (4.51) \quad \int_\Omega c_j(x) z_\ell(x, t) dx \leq - \int_\Omega c_j(x) \eta_\ell(x, t) dx \leq o(\Upsilon_\ell), \quad \text{on } [\tau_k, \tau_{k+1}].$$

655 Thus, by the Goh transform (3.14),

$$656 \quad (4.52) \quad \int_\Omega c_j(x) (\bar{\zeta}_\ell(x, t) + B(x, t) \cdot \bar{w}_\ell(t)) dx \leq o(1), \quad \text{on } [\tau_k, \tau_{k+1}],$$

657 where  $\bar{\zeta}_\ell$  is the solution of (3.17) corresponding to  $\bar{w}_\ell$ . Let  $\varphi$  be some time-dependent  
658 nonnegative continuous function with support included in  $I_j^C$ . From (4.52), we get  
659 that

$$660 \quad (4.53) \quad \int_{\tau_k}^{\tau_{k+1}} \varphi \int_\Omega c_j(\bar{\zeta}_\ell + B \cdot \bar{w}_\ell) dx dt \leq o(1).$$

661 Taking the limit  $\ell \rightarrow \infty$  yields

$$662 \quad (4.54) \quad \int_{\tau_k}^{\tau_{k+1}} \varphi \int_{\Omega} c_j(\bar{\zeta} + B \cdot \bar{w}) dx dt \leq 0,$$

663 where  $\bar{\zeta}$  is the solution of (3.17) associated to  $\bar{w}$ . Since (4.54) holds for any nonnegative  
664  $\varphi$ , we get that

$$665 \quad (4.55) \quad \int_{\Omega} c_j(\bar{\zeta}(x, t) + B(x, t) \cdot \bar{w}(t)) dx \leq 0, \quad \text{a.e. on } [\tau_k, \tau_{k+1}].$$

666 In particular, if  $T \in I_j^C$ , we get from (4.52) that

$$667 \quad (4.56) \quad \int_{\Omega} c_j(\bar{\zeta}(x, T) + B(x, T) \cdot \bar{h}) dx \leq 0.$$

668 We now have to prove the converse inequalities in (4.54) and (4.56).

669 By Proposition 4.10 and since  $u_\ell$  is admissible, we have

$$670 \quad (4.57) \quad \sum_{j=1}^q \int_0^T g'_j(\bar{y}(\cdot, t)) z(\cdot, t) d\mu_j(t) + DJ(\bar{u}, \bar{y})(z, v) = \int_0^T \Psi^p(t) \cdot v_\ell(t) dt \geq 0.$$

671 By Proposition 4.9, we have  $F(u_\ell) = F(\bar{u}) + DF(\bar{u})v_\ell + o(\Upsilon_\ell)$ . This, together with  
672 (4.57), yield

$$673 \quad (4.58) \quad 0 \leq F(u_\ell) - F(\bar{u}) + o(\Upsilon_\ell) + \sum_{j=1}^q \int_0^T g'_j(\bar{y}(\cdot, t)) z(\cdot, t) d\mu_j(t).$$

674 Using (4.40) in latter inequality implies that

$$675 \quad (4.59) \quad -o(\Upsilon_\ell) \leq \sum_{j=1}^q \int_0^T g'_j(\bar{y}(\cdot, t)) z(\cdot, t) d\mu_j(t),$$

676 thus

$$677 \quad (4.60) \quad o(1) \leq \sum_{j=1}^q \int_0^T g'_j(\bar{y}(\cdot, t)) (\bar{\zeta}_\ell(\cdot, t) + B(\cdot, t) \cdot \bar{w}_\ell(t)) d\mu_j(t).$$

678 Since, for every  $j = 1, \dots, q$ , the measure  $d\mu_j$  has an essentially bounded density over  
679  $[0, T]$  (in view of Theorem 3.3), we have that

$$680 \quad (4.61) \quad \begin{aligned} & 0 \leq \liminf_{\ell \rightarrow \infty} \sum_{j=1}^q \int_{[0, T]} g'_j(\bar{y}(\cdot, t)) (\bar{\zeta}_\ell(\cdot, t) + B(\cdot, t) \cdot \bar{w}_\ell(t)) d\mu_j \\ & = \lim_{\ell \rightarrow \infty} \sum_{j=1}^q \int_{[0, T]} g'_j(\bar{y}(\cdot, t)) (\bar{\zeta}_\ell(\cdot, t) + B(\cdot, t) \cdot \bar{w}_\ell(t)) d\mu_j. \end{aligned}$$

681 Using (4.55) and the strict complementarity for the state constraint (4.4)(ii), we get  
682 (3.39). This concludes the proof of item (c).

683 (d) Let us now prove (4.2). Assume that  $j \in \{1, \dots, q\}$  is such that  $T \in I_j^C$ . One

684 inequality was already proved in (4.56). If we further have that  $[\mu_j(T)] > 0$ , condition  
685 (4.2) follows from (4.61).

686 We conclude that the limit direction  $(\bar{\zeta}, \bar{w}, \bar{h})$  belongs to  $PC_2^*$ .

687 **Proof of Fact 2.** From Proposition 4.8 we obtain

$$688 \quad (4.62) \quad \begin{aligned} & \widehat{\mathcal{Q}}[p, d\mu](\zeta_\ell, w_\ell, h_\ell) \\ & = \mathcal{L}[p, \mu](u_\ell, y_\ell) - \mathcal{L}[p, \mu](\bar{u}, \bar{y}) - \int_0^T \Psi^p \cdot v_\ell dt + o(\Upsilon_\ell^2) \leq o(\Upsilon_\ell^2), \end{aligned}$$

689 where the last inequality follows from (4.41) and since  $\Psi^p \cdot v_\ell \geq 0$  a.e. on  $[0, T]$  in  
690 view of the first order condition (2.10). Hence,

$$691 \quad (4.63) \quad \liminf_{\ell \rightarrow \infty} \widehat{\mathcal{Q}}[p, \mu](\bar{\zeta}_\ell, \bar{y}_\ell, \bar{h}_\ell) \leq \limsup_{\ell \rightarrow \infty} \widehat{\mathcal{Q}}[p, \mu](\bar{\zeta}_\ell, \bar{w}_\ell, \bar{h}_\ell) \leq 0.$$

692 Let us recall that, in view of the hypothesis (iii) of the current theorem, the mapping  
693  $\widehat{\mathcal{Q}}[p, d\mu]$  is a Legendre form in the Hilbert space  $\{(\zeta[w], w, h) \in Y \times L^2(0, T)^m \times \mathbb{R}^m\}$ .  
694 Furthermore, for the critical direction  $(\bar{\zeta}, \bar{w}, \bar{h})$ , due to the uniform positivity condition  
695 (4.5), there is a multiplier  $(\bar{p}, \bar{\mu}) \in \Lambda_1$  such that

$$696 \quad (4.64) \quad \rho(\|\bar{w}\|_2^2 + |\bar{h}|^2) \leq \widehat{\mathcal{Q}}[\bar{p}, \bar{\mu}](\bar{\zeta}, \bar{w}, \bar{h}) = \liminf_{\ell \rightarrow \infty} \widehat{\mathcal{Q}}[\bar{p}, \bar{\mu}](\bar{\zeta}_\ell, \bar{w}_\ell, \bar{h}_\ell) \leq 0,$$

697 where the equality holds since  $\widehat{\mathcal{Q}}[\bar{p}, \bar{\mu}]$  is a Legendre form and the inequality is due to  
698 (4.62). From (4.64) we get  $(\bar{w}, \bar{h}) = 0$  and  $\lim_{k \rightarrow \infty} \widehat{\mathcal{Q}}[\bar{p}, \bar{\mu}](\bar{\zeta}_\ell, \bar{w}_\ell, \bar{h}_\ell) = 0$ . Consequently,  
699  $(\bar{w}_\ell, \bar{h}_\ell)$  converges strongly to  $(\bar{w}, \bar{h}) = 0$  which is a contradiction, since  $(\bar{w}_\ell, \bar{h}_\ell)$  has  
700 unit norm in  $L^2(0, T)^m \times \mathbb{R}^m$ . We conclude that  $(\bar{u}, \bar{y})$  is an  $L^2$ -local solution satisfying  
701 the weak quadratic growth condition.

702 Conversely, assume that the weak quadratic growth condition (4.1) holds at  $(\bar{u}, \bar{y})$   
703 for  $\rho > 0$ . Note that  $(\bar{u}, \bar{y}, \bar{w})$ , with  $\bar{w}(t) = \int_0^t \bar{u}(s) ds$ , is a  $L^2$ -local solution of the  
704 problem

$$705 \quad (4.65) \quad \begin{aligned} & \min_{u \in \mathcal{U}_{\text{ad}}} J(u, y[u]) - \rho \left( \int_0^T (w - \bar{w})^2 dt + |w(T) - \bar{w}(T)|^2 \right), \\ & \text{s.t. } \dot{w} = u, \quad w(0) = 0, \quad (2.4) \text{ holds,} \end{aligned}$$

706 Applying the second order necessary condition in Theorem 3.5 to this problem (4.65),  
707 followed by the Goh transform, yields the uniform positivity (4.6). For further de-  
708 tails we refer to the corresponding statement for ordinary differential equations in [1,  
709 Theorem 5.5].  $\square$

710 **Appendix A. Well-posedness of state equation and existence of optimal**  
711 **controls.** In this section we recall some statements from [5], for proofs we refer to  
712 the latter reference.

713 **LEMMA A.1.** *The state equation (2.1) has a unique solution  $y = y[u, y_0, f]$  in  $Y$ .  
714 The mapping  $(y, y_0, f) \mapsto y[u, y_0, f]$  is  $C^\infty$  from  $L^2(0, T)^m \times H_0^1(\Omega) \times L^2(Q)$  to  $Y$ ,  
715 and nondecreasing w.r.t.  $y_0$  and  $f$ . In addition, there exist functions  $C_i$ ,  $i = 1$  to  $2$ ,  
716 not decreasing w.r.t. each component, such that*

$$717 \quad (A.1) \quad \|y\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla y\|_2 \leq C_1(\|y_0\|_2, \|f\|_2, \|u\|_2 \|b\|_\infty),$$

$$718 \quad (A.2) \quad \|y\|_Y \leq C_2(\|y_0\|_{H_0^1(\Omega)}, \|f\|_2, \|u\|_2 \|b\|_\infty).$$

720 Moreover, the state  $y$  also belongs to  $C([0, T]; H_0^1(\Omega))$ , since  $Y$  is continuously em-  
721 bedded in that space [14, Theorem 3.1, p.23].

722 **THEOREM A.2.** *The mapping  $u \mapsto y[u]$  is of class  $C^\infty$ , from  $L^2(0, T)^m$  to  $Y$ .*

723 **THEOREM A.3.** (i) *The function  $u \mapsto J(u, y[u])$ , from  $L^2(0, T)^m$  to  $\mathbb{R}$ , is weakly*  
724 *sequentially l.s.c. (ii) The set of solutions of the optimal control problem (P) is weakly*  
725 *sequentially closed in  $L^2(0, T)^m$ . (iii) If (P) has a bounded minimizing sequence, the*  
726 *set of solutions of (P) is non empty. This is the case in particular if (P) is admissible*  
727 *and  $\mathcal{U}_{\text{ad}}$  is a bounded subset of  $L^2(0, T)^m$ .*

728 **Appendix B. First order analysis.** Here, we recall some properties from [5].

729 Throughout the section,  $(\bar{u}, \bar{y})$  is a trajectory of problem (P). We recall the  
730 hypotheses (2.2) and (2.6) on the data.

731 We fix a trajectory  $(\bar{u}, \bar{y} = y[\bar{u}])$ . Let  $A$  be linear continuous from  $L^2(0, T; H^2(\Omega))$   
732 to  $L^2(Q)$  such that, for each  $z \in L^2(0, T; H^2(\Omega))$  and  $(x, t) \in Q$ ,

$$733 \quad (\text{B.1}) \quad (Az)(x, t) := -\Delta z(x, t) + 3\gamma \bar{y}(x, t)^2 z(x, t) - \sum_{i=0}^m \bar{u}_i(t) b_i(x) z(x, t).$$

734 **B.1. The linearized state equation.** The linearized state equation at  $(\bar{u}, \bar{y})$  is  
735 given by

$$736 \quad (\text{B.2}) \quad \dot{z} + Az = \sum_{i=1}^m v_i b_i \bar{y} \quad \text{in } Q; \quad z = 0 \text{ on } \Sigma, \quad z(\cdot, 0) = 0 \text{ on } \Omega.$$

737 For  $v \in L^2(0, T)^m$ , equation (B.2) above possesses a unique solution  $z[v] \in Y$  and the  
738 mapping  $v \mapsto z[v]$  is linear from  $L^2(0, T)^m$  to  $Y$ . Particularly, the following estimate  
739 holds:

$$740 \quad (\text{B.3}) \quad \|z\|_{L^\infty(0, T; L^2(\Omega))} \leq M_1 \sum_{i=1}^m \|b_i\|_\infty \|v_i\|_1,$$

741 where  $M_1 := e^{\frac{T}{2} + \sum_{i=0}^m \|\bar{u}_i\|_1 \|b_i\|_\infty} \|\bar{y}\|_{L^\infty(0, T; L^2(\Omega))}$ .

742 **B.2. The costate equation.** For  $\mu \in BV(0, T)^q$  its distributional derivative  
743  $d\mu$  is in the space  $\mathcal{M}(0, T)$  of finite Radon measures. And, conversely, any element  
744  $d\mu \in \mathcal{M}(0, T)$  can be identified with a function  $\mu$  of bounded variation that vanishes  
745 at time  $T$ . Let us consider the set of positive finite Radon measures  $\mathcal{M}_+(0, T)$  and  
746 identify it with the set

$$747 \quad (\text{B.4}) \quad BV(0, T)_{0,+}^q := \{\mu \in BV(0, T)^q; \mu(T) = 0, d\mu \geq 0\}.$$

748 The generalized Lagrangian of problem (P) is, choosing the multiplier of the state  
749 equation to be  $(p, p_0) \in L^2(Q) \times H^{-1}(\Omega)$  and taking  $\beta \in \mathbb{R}_+$ ,  $d\mu \in \mathcal{M}_+(0, T)$ ,

$$750 \quad (\text{B.5}) \quad \begin{aligned} \mathcal{L}[\beta, p, p_0, \mu](u, y) &:= \beta J(u, y) - \langle p_0, y(\cdot, 0) - y_0 \rangle_{H_0^1(\Omega)} \\ &+ \int_Q p \left( \Delta y(x, t) - \gamma y^3(x, t) + f(x, t) + \sum_{i=0}^m u_i(t) b_i(x) y(x, t) - \dot{y}(x, t) \right) dx dt \\ &+ \sum_{j=1}^q \int_0^T g_j(y(\cdot, t)) d\mu_j(t). \end{aligned}$$



751 The *costate equation* is the condition of stationarity of the Lagrangian  $\mathcal{L}$  with respect  
752 to the state that is, for any  $z \in Y$ :

753

$$(B.6) \quad \int_Q p(\dot{z} + Az) dx dt + \langle p_0, z(\cdot, 0) \rangle_{H_0^1(\Omega)} = \sum_{j=1}^q \int_0^T \int_{\Omega} c_j z dx d\mu_j(t) \\ + \beta \int_Q (\bar{y} - y_d) z dx dt + \beta \int_{\Omega} (\bar{y}(x, T) - y_{dT}(x)) z(x, T) dx.$$

755

756

757 To each  $(\varphi, \psi) \in L^2(Q) \times H_0^1(\Omega)$ , let us associate  $z = z[\varphi, \psi] \in Y$ , the unique solution  
758 of

$$(B.7) \quad \dot{z} + Az = \varphi; \quad z(\cdot, 0) = \psi.$$

760 Since this mapping is onto, the costate equation (B.6) can be rewritten, for  $z = z[\varphi, \psi]$   
761 and arbitrary  $(\varphi, \psi) \in L^2(Q) \times H_0^1(\Omega)$ , as (see [5, Equation (3.7)])

762

$$(B.8) \quad \int_Q p \varphi dx dt + \langle p_0, \psi \rangle_{H_0^1(\Omega)} = \sum_{j=1}^q \int_0^T \int_{\Omega} c_j z dx d\mu_j(t), \\ + \beta \int_Q (\bar{y} - y_d) z dx dt + \beta \int_{\Omega} (\bar{y}(x, T) - y_{dT}(x)) z(x, T) dx.$$

764

765

766 Next consider the *alternative costates*

$$(B.9) \quad p^1 := p + \sum_{j=1}^q c_j \mu_j; \quad p_0^1 := p_0 + \sum_{j=1}^q c_j \mu_j(0),$$

768 where  $\mu \in BV(0, T)_{0,+}^q$  is the function of bounded variation associated with  $d\mu$ . By  
769 [5, Cor. 3.3 and Lem. 5],  $p^1 \in Y$  and  $p^1(\cdot, 0) = p_0^1$ . Therefore  $p(\cdot, 0)$  makes sense as  
770 an element of  $H_0^1(\Omega)$ , and it follows that  $p(\cdot, 0) = p^1(\cdot, 0) - \sum_{j=1}^q c_j \mu_j(0) = p_0$ .

771 COROLLARY B.1. *If  $\mu \in H^1(0, T)^q$  then  $p \in Y$  and*

$$(B.10) \quad -\dot{p} + Ap = \beta(\bar{y} - y_d) + \sum_{j=1}^q c_j \dot{\mu}_j.$$

773 **B.3. An example.** We recall an example from [5, Appendix B] satisfying Hy-  
774 pothesis 3.1. We will check that condition (a) of Theorem 4.5 is satisfied for this  
775 example.

776 We consider the following setting: Let  $\Omega = (0, 1)$ , and denote by  $c_1(x) :=$   
777  $\sqrt{2} \sin \pi x$  the first (normalized) eigenvector of the Laplace operator. We assume that  
778  $\gamma = 0$ , the control is scalar ( $m = 1$ ),  $b_0 \equiv 0$  and  $b_1 \equiv 1$  in  $\Omega$ , and that  $f \equiv 0$  in  $Q$ .  
779 Then the state equation with initial condition  $c_1$  reads

$$(B.11) \quad \dot{y}(x, t) - \Delta y(x, t) = u(t)y(x, t); \quad (x, t) \in (0, 1) \times (0, T), \quad y(x, 0) = c_1(x), \quad x \in \Omega.$$

781 The state satisfies  $y(x, t) = y_1(t)c_1(x)$ , where  $y_1$  is solution of

$$(B.12) \quad \dot{y}_1(t) + \pi^2 y_1(t) = u(t)y_1(t); \quad t \in (0, T), \quad y_1(0) = y_{10} = 1.$$

783 We set  $T = 3$  and consider the state constraint (2.4) with  $q = 1$  and  $d_1 := -2$ , and  
 784 the cost function (2.5) with  $\alpha_1 = 0$ . The state constraint reduces to

$$785 \quad (\text{B.13}) \quad y_1(t) \leq 2, \quad t \in [0, 3].$$

786 As target functions we take  $y_{dT} := c_1$  and  $y_d(x, t) := \hat{y}_d(t)c_1(x)$  with

$$787 \quad (\text{B.14}) \quad \hat{y}_d(t) := \begin{cases} 1.5e^t & \text{for } t \in (0, \log 2), \\ 3 & \text{for } t \in (\log 2, 1), \\ 4 - t & \text{for } t \in (1, 3). \end{cases}$$

788 We assume that the lower and upper bounds for the control are  $\check{u} := -1$  and  $\hat{u} :=$   
 789  $\pi^2 + 1$ . The optimal control is given by

$$790 \quad (\text{B.15}) \quad \bar{u}(t) := \begin{cases} \hat{u} & \text{for } t \in (0, \log 2), \\ \pi^2 & \text{for } t \in (\log 2, 2), \\ \pi^2 - 1/\hat{y}_d & \text{for } t \in (2, 3). \end{cases}$$

791 and the optimal state by

$$792 \quad (\text{B.16}) \quad \bar{y}_1(t) := \begin{cases} e^t & \text{for } t \in (0, \log 2), \\ 2 & \text{for } t \in (\log 2, 2), \\ 4 - t & \text{for } t \in (2, 3). \end{cases}$$

793 The above control is feasible. The trajectory  $(\bar{u}, \bar{y})$  is optimal. The costate equation  
 794 is

$$795 \quad (\text{B.17}) \quad -\dot{p} + Ap = c_1(\bar{y}_1 - \hat{y}_d) + c_1\dot{\mu}_1, \quad p(\cdot, T) = \bar{y}(T) - y_{dT} = 0.$$

796 Since  $\bar{y}$  and  $y_d$  are colinear to  $c_1$ , it follows that  $p(x, t) = p_1(t)c_1(x)$ , and

$$797 \quad (\text{B.18}) \quad -\dot{p}_1 + \pi^2 p_1 = \bar{u}p_1 + \bar{y}_1 - \hat{y}_d + \dot{\mu}_1; \quad p_1(3) = 0.$$

798 Over  $(2, 3)$ ,  $\dot{\mu}_1 = 0$  (state constraint not active) and  $\bar{y}_1 = \hat{y}_d$ , therefore  $p_1$  and  $p$   
 799 identically vanish. Over  $(\log 2, 2)$ ,  $\bar{u}$  is out of bounds and therefore

$$800 \quad (\text{B.19}) \quad 0 = \int_{\Omega} p(x, t)\bar{y}(x, t) = p_1(t)\bar{y}_1(t) \int_{\Omega} c_1(x)^2 = 2p_1(t).$$

801 It follows that  $p_1$  and  $p$  also vanish on  $(\log 2, 2)$  and that

$$802 \quad (\text{B.20}) \quad \dot{\mu}_1 = -(\bar{y}_1 - \hat{y}_d) > 0, \quad \text{a.a. } t \in (\log 2, 2).$$

803 Over  $(0, \log 2)$ , the control attains its upper bound, then

$$804 \quad (\text{B.21}) \quad -\dot{p}_1 = p_1 - \frac{1}{2}e^t$$

805 with final condition  $p_1(\log 2) = 0$ , so that

$$806 \quad (\text{B.22}) \quad p_1(t) = \frac{e^t}{4} - e^{-t}.$$

807 As expected,  $p_1$  is negative.

808 LEMMA B.2. *The hypothesis (a) of Theorem 4.5 holds.*

809 *Proof.* (i) This has been obtained in part I [5]. Note that the multiplier is unique.

810 (ii) We check the Legendre form condition. For this, we apply the Goh transformation  
811 to the example. For  $(v, z)$  solution of the linearized state equation we define

$$812 \quad (\text{B.23}) \quad B := \bar{y}b = \bar{y}_1(t)c_1(x); \quad \xi := z - Bw = (z_1 - \bar{y}_1w)c_1$$

813 and we observe that  $\xi = \xi_1c_1$  is solution of

$$814 \quad (\text{B.24}) \quad \dot{\xi} + A\xi = -(AB + \dot{B})w; \quad \xi(0) = 0;$$

815 where

$$816 \quad (\text{B.25}) \quad AB + \dot{B} = (\pi^2 - \bar{u})B + \dot{\bar{y}}_1c_1 = ((\pi^2 - \bar{u})\bar{y}_1 + \dot{\bar{y}}_1)c_1$$

817 so that

$$818 \quad (\text{B.26}) \quad \dot{\xi}_1 + (\pi^2 - \bar{u})\xi_1 = B^1w, \quad B^1 := (\pi^2 - \bar{u})\bar{y}_1 + \dot{\bar{y}}_1.$$

819 For checking the Legendre condition ((ii) of Theorem 4.5), we have to check the  
820 uniform positivity of the coefficient of  $w^2$  in  $\tilde{Q}$ . This trivially holds on the second and  
821 third arcs, since then  $p$  and therefore  $\chi$  vanish, so that the coefficient of  $w^2$  reduces  
822 to  $\int_{\Omega} \kappa \bar{y}^2 = \bar{y}_1(t)^2 \geq 1$ . We now detail the computation for the first arc. Replacing  $z$   
823 by  $\xi + Bw$  in the quadratic form  $\mathcal{Q}[p](v, z)$  we have

$$824 \quad (\text{B.27}) \quad \tilde{Q} = \int_Q ((\xi + Bw)^2 + pv(\xi + Bw)) dxdt + \int_{\Omega} (\xi(\cdot, T) + B(\cdot, T)w(T))^2 dx.$$

825 For the second term in the first integral we have

$$\begin{aligned} 826 \quad (\text{B.28}) \quad \int_Q pv(\xi + Bw) dxdt &= \int_0^T \left( p_1 \xi \frac{d}{dt} w + \frac{1}{2} p_1 \bar{y}_1 \frac{d}{dt} (w^2) \right) dt \\ &= - \int_0^T \left( \frac{d}{dt} (p_1 \xi) w + \frac{1}{2} \frac{d}{dt} (p_1 \bar{y}_1) w^2 \right) dt + [\text{boundary-terms}]. \\ &= - \int_0^T \left( p_1 B^1 + \frac{1}{2} \frac{d}{dt} (p_1 \bar{y}_1) \right) w^2 dt + [\text{boundary-terms}]. \end{aligned}$$

827 Finally we obtain that over the first arc, the coefficient of  $w^2$  in the integral term of  
828  $\tilde{Q}$  is  $2 + e^{2t}/4$ . It follows that  $\tilde{\mathcal{Q}}[p](w, \xi[w])$  is a Legendre form.

829 (iii) We check the uniform positivity condition. Any  $(w, h) \in PC_2^*$  is such that  $w$   
830 vanishes on the two first arcs, and since the costate vanishes on the third arc we have  
831 that, using  $\bar{y}(x, t) = (4 - t)c_1(x)$  on the third arc

$$\begin{aligned} 832 \quad (\text{B.29}) \quad \widehat{\mathcal{Q}}[p, \mu](\xi, w, h) &= \int_2^3 \int_{\Omega} (\xi(x, t) + (4 - t)c_1(x)w(t))^2 dxdt + \int_{\Omega} (\xi(x, T) + hc_1(x))^2 dx \\ &= \int_2^3 (\xi_1(t) + (4 - t)w(t))^2 dt + (\xi_1(T) + h)^2. \end{aligned}$$

833 This is a Legendre form over  $L^2(2, 3)$ , and so it is coercive iff it has positive values  
834 except at 0. If the value is zero then  $w(t) = \xi_1(t)/(t - 4)$  so that  $\xi$  vanishes identically  
835 and therefore  $w$  also, and  $h = 0$ . The conclusion follows.  $\square$

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## REFERENCES

- 837 [1] M. S. ARONNA, J. F. BONNANS, A. DMITRUK, AND P. LOTITO, *Quadratic order conditions for*  
838 *bang-singular extremals*, Numerical Algebra, Control and Optimization, AIMS Journal, 2  
839 (2012), pp. 511–546.
- 840 [2] M. S. ARONNA, J. F. BONNANS, AND B. S. GOH, *Second order analysis of control-affine problems*  
841 *with scalar state constraint*, Math. Program., 160 (2016), pp. 115–147.
- 842 [3] M. S. ARONNA, J. F. BONNANS, AND A. KRÖNER, *Optimal Control of Infinite Dimensional*  
843 *Bilinear Systems: Application to the Heat and Wave Equations*, Math. Program., 168  
844 (2018), pp. 717–757. Erratum: Math. Programming Ser. A, Vol. 170 (2018), pp. 569–570.
- 845 [4] M. S. ARONNA, J. F. BONNANS, AND A. KRÖNER, *Optimal control of PDEs in a complex*  
846 *space setting; application to the Schrödinger equation*, SIAM J. Control Optim., 57 (2019),  
847 pp. 1390–1412.
- 848 [5] M. S. ARONNA, J. F. BONNANS, AND A. KRÖNER, *State-constrained control-affine parabolic*  
849 *problems I: first and second order necessary optimality conditions*, Set-Valued Var. Anal.,  
850 (2020), <https://arxiv.org/abs/1906.00237>.
- 851 [6] J. F. BONNANS, *Singular arcs in the optimal control of a parabolic equation*, 2013, pp. 281–292.  
852 Proc 11th IFAC Workshop on Adaptation and Learning in Control and Signal Processing  
853 (ALCOSP), Caen, F. Giri ed., July 3-5, 2013.
- 854 [7] E. CASAS, *Second order analysis for bang-bang control problems of PDEs*, SIAM J. Control  
855 Optim., 50 (2012), pp. 2355–2372.
- 856 [8] E. CASAS, J. DE LOS REYES, AND F. TRÖLTZSCH, *Sufficient second-order optimality condi-*  
857 *tions for semilinear control problems with pointwise state constraints*, SIAM J. Optim., 19  
858 (2008), pp. 616–643.
- 859 [9] E. CASAS, C. RYLL, AND F. TRÖLTZSCH, *Second order and stability analysis for optimal sparse*  
860 *control of the FitzHugh-Nagumo equation*, SIAM J. Control Optim., 53 (2015), pp. 2168–  
861 2202.
- 862 [10] E. CASAS, D. WACHSMUTH, AND G. WACHSMUTH, *Second-order analysis and numerical ap-*  
863 *proximation for bang-bang bilinear control problems*, SIAM J. Control Optim., 56 (2018),  
864 pp. 4203–4227.
- 865 [11] J. DE LOS REYES, P. MERINO, J. REHBERG, AND F. TRÖLTZSCH, *Optimality conditions for state-*  
866 *constrained PDE control problems with time-dependent controls*, Control and Cybernetics,  
867 37 (2008), pp. 5–38.
- 868 [12] B. GOH, *Necessary conditions for singular extremals involving multiple control variables*, SIAM  
869 J. Control, 4 (1966), pp. 716–731.
- 870 [13] K. KRUMBIEGEL AND J. REHBERG, *Second order sufficient optimality conditions for parabolic*  
871 *optimal control problems with pointwise state constraints*, SIAM J. Control Optim., 51  
872 (2013), pp. 304–331.
- 873 [14] J.-L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications. Vol. 1*,  
874 Dunod, Paris, 1968.
- 875 [15] J.-P. RAYMOND AND F. TRÖLTZSCH, *Second order sufficient optimality conditions for nonlinear*  
876 *parabolic control problems with state constraints*, Discrete Contin. Dynam. Systems, 6  
877 (2000), pp. 431–450.
- 878 [16] A. RÖSCH AND F. TRÖLTZSCH, *Sufficient second-order optimality conditions for a parabolic*  
879 *optimal control problem with pointwise control-state constraints*, SIAM J. Control Optim.,  
880 42 (2003), pp. 138–154.