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# Conditions for some non stationary random walks in the quarter plane to be singular or of genus 0

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## Abstract

We analyze the *kernel*  $K(x, y, t)$  of the basic functional equation associated with the tri-variate counting generating function (CGF) of walks in the quarter plane. In this short paper, taking  $t \in ]0, 1[$ , we provide the conditions on the step set  $\{p_{i,j}\}$  to decide whether the walks are *singular* or *regular*, as defined in [3, Section 2.3]. These conditions are independent of  $t \in ]0, 1[$  and given in terms of *step set configurations*. We also give the configurations for the kernel to be of genus 0. All these conditions are very similar to the case  $t = 1$  considered in [3]. Our results extend the work [2], which considers only very special situations, namely when  $t \in ]0, 1[$  is a transcendental number and the  $p'_{i,j}$ s are rational.

## 1 Introduction and notation

Enumeration of planar lattice walks has become a classical topic in combinatorics. For a given set  $\mathcal{S}$  of allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time, and possibly restricted to some regions of the plane.

It will be convenient to denote by  $\mathcal{S}$  the set of admissible *small steps*, included in the set of the eight nearest neighbors, so that

$$\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}.$$

Let  $f(i, j, k)$  denote the probability for a walk in  $\mathbb{Z}_+^2$  of reaching the point

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$(i, j)$  after  $k$  steps, starting from  $(0, 0)$ . Then the corresponding CGF

$$F(x, y, t) = \sum_{i, j, k \geq 0} f(i, j, k) x^i y^j t^k \quad (1.1)$$

satisfies the functional equation (see [1] for details)

$$K(x, y, t)F(x, y, t) = K(x, 0, t)F(x, 0, t) + K(0, y, t)F(0, y, t) + l(x, y, t), \quad (1.2)$$

where

$$\begin{cases} K(x, y, t) = xy[tS(x, y) - 1], \\ S(s, y) = \sum_{(i, j) \in \mathcal{S}} p_{i, j} x^i y^j, \\ l(x, y, t) = -tp_{-1, -1}F(0, 0, t) - xy. \end{cases} \quad (1.3)$$

We assume  $\sum_{(i, j) \in \mathcal{S}} p_{i, j} = 1$ , which does not restrict the generality, so that

$F(x, y, t)$  is sought to be convergent in the region  $|x| \leq 1, |y| \leq 1, |t| \leq 1$ . Usually, for fixed  $t$ , the algebraic curve corresponding to

$$\{(x, y) \in \mathbb{C}^2 : K(x, y, t) = 0\} \quad (1.4)$$

has genus 0 or 1. But this is no more true when the kernel  $K(x, y, t)$  may be factorized in  $\mathbb{C}[x, y]$ , in which case the walk is called *singular* according to the definition (see [3]).

Following the notation in [3], we define the triple  $a(x), b(x), c(x)$ , [resp.  $\tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ ] from (1.3) by

$$K(x, y, t) = xy \left( t \sum p_{i, j} x^i y^j - 1 \right) \quad (1.5)$$

$$\stackrel{\text{def}}{=} t[a(x)y^2 + b(x, t)y + c(x)] \stackrel{\text{def}}{=} t[\tilde{a}(y)x^2 + \tilde{b}(y, t)x + \tilde{c}(y)], \quad (1.6)$$

with

$$\begin{cases} a(x) = \sum_{i=0}^2 p_{i-1, 1} x^i, \\ b(x, t) = p_{-1, 0} + (p_{0, 0} - 1/t)x + p_{1, 0} x^2, \\ c(x) = \sum_{i=0}^2 p_{i-1, -1} x^i, \end{cases} \quad (1.7)$$

and similar equations for  $\tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ .

Let us introduce the discriminant of (1.6), always viewed as a polynomial in  $x$

$$D(x, t) \stackrel{\text{def}}{=} b^2(x, t) - 4a(x)c(x). \quad (1.8)$$

Then it will be convenient to write

$$D(x, t) = \sum_{i=0}^4 d_i(t)x^i, \quad (1.9)$$

where, by using (1.7),

$$\begin{cases} d_0(t) &= p_{-1,0}^2 - 4p_{-1,1}p_{-1,-1}, \\ d_1(t) &= 2[p_{-1,0}(p_{0,0} - 1/t) - 2(p_{-1,1}p_{0,-1} + p_{0,1}p_{-1,-1})], \\ d_2(t) &= (p_{0,0} - 1/t)^2 + 2p_{1,0}p_{-1,0} - 4(p_{1,1}p_{-1,-1} + p_{1,-1}p_{-1,1} + p_{0,1}p_{0,-1}), \\ d_3(t) &= 2p_{1,0}(p_{0,0} - 1/t) - 4(p_{1,1}p_{0,-1} + p_{0,1}p_{1,-1}), \\ d_4(t) &= p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \end{cases} \quad (1.10)$$

noting that  $d_0$  and  $d_4$  are in fact independant of  $t$ . It is immediate to check that  $d_2(t)$  is always strictly positive. Hence, the degree of  $D(x, t)$  in  $x$  is always  $\geq 2$ .

We shall need the following simple notion of *positivity* for an arbitrary polynomial.

**Definition 1.1.** A polynomial  $p(x)$  is said to be *positive* (resp. *negative*) to mean that its coefficients are all real positive (resp. negative) and not simultaneously 0.

## 2 The roots of the discriminant $D(x, t)$

The main result of this section resides in the following theorem.

**Theorem 2.1.** For all  $t \in ]0, 1]$ ,  $D(x, t)$  has 4 real roots, exactly two of them being inside the unit disk, and double roots can occur only at  $x = 0$  or  $x = \infty$ .

*Proof.* For all  $x$  satisfying  $a(x)c(x) \geq 0$ , we introduce

$$\phi_1(x, t) \stackrel{\text{def}}{=} b(x, t) + 2\sqrt{a(x)c(x)}, \quad \phi_2(x, t) \stackrel{\text{def}}{=} b(x, t) - 2\sqrt{a(x)c(x)}, \quad (2.1)$$

whence, according to (1.8),

$$D(x, t) = \phi_1(x, t)\phi_2(x, t).$$

Keeping in mind that  $a(x)$  and  $c(x)$  are *positive* (see Definition 1.1), they have no positive real roots. Our goal is to distribute the 4 roots of  $D(x, t)$  between  $\phi_1$  and  $\phi_2$ .

**The roots of  $\phi_1(x, t)$  and  $\phi_2(x, t)$ .** The next inequalities are immediate

$$\phi_2(1, t) \leq \phi_1(1, t) \leq 1 - 1/t < 0, \quad \phi_1(0, t) \geq 0. \quad (2.2)$$

Let  $\xi_0$  denote the largest real root of  $a(x)c(x)$ , when it exists. Then  $\xi_0 \leq 0$  and  $\phi_1(\xi_0, t) = \phi_2(\xi_0, t) = b(\xi_0, t) \geq 0$ . Consequently,  $\phi_1(x, t)$  and  $\phi_2(x, t)$  each have a root on  $]\xi_0, 1[$ . On the other hand, if  $\xi_0 \leq -1$ , then  $a(x)c(x) \geq 0$  for  $x \in [-1, 1]$  and

$$\phi_1(-1, t) \geq \phi_2(-1, t) > 0,$$

which, in tandem with (2.2), proves indeed that  $\phi_2(x, t)$  has always a root on  $]-1, 1[$ . This conclusion still holds when  $\xi_0$  does not exist, since then  $a(x)c(x)$  is positive and  $\phi_1(x, t), \phi_2(x, t)$  are real  $\forall x \in \mathbb{R}$ .

So, when degree of  $D(x, t) = 2$ , there are two real roots inside the unit disk, harmoniously distributed between  $\phi_1(x, t)$  and  $\phi_2(x, t)$ .

In addition, we note that, when degree of  $D(x, t) \geq 3$ ,

$$\lim_{x \rightarrow +\infty} \phi_1(x, t) = +\infty, \quad (2.3)$$

in which case  $\phi_1(x, t)$  has a root on  $]1, \infty[$  and a root on  $[0, 1[$ .

**Degree of  $\mathbf{D}(\mathbf{x}, \mathbf{t}) = 3$ , i.e.,  $\mathbf{d}_4(\mathbf{t}) = \mathbf{p}_{1,0}^2 - 4\mathbf{p}_{1,1}\mathbf{p}_{1,-1} = \mathbf{0}$ .** Then two roots of  $D(x, t)$  come from  $\phi_1(x, t)$ , and the third one from  $\phi_2(x, t)$ .

**Degree of  $\mathbf{D}(\mathbf{x}, \mathbf{t}) = 4$ , i.e.  $\mathbf{d}_4(\mathbf{t}) \neq \mathbf{0}$ .** Letting  $|x| \rightarrow \infty$ , there are two possibilities.

- (i)  $d_4(t) > 0$ . Then  $\lim_{x \rightarrow \infty} \phi_2(x, t) = +\infty$ , and  $\phi_2(x, t)$  has a positive real root on  $]1, +\infty[$ .
- (ii)  $d_4(t) < 0$ . Let  $\xi_1$  denote the smallest finite real root of  $a(x)c(x)$ , when it exists. Then

$$a(x)c(x) \geq 0, \quad \forall x \in ]-\infty, \xi_1], \quad \phi_2(\xi_1, t) = b(\xi_1, t) > 0,$$

and, since  $\lim_{x \rightarrow -\infty} \phi_2(x, t) = -\infty$ , it follows that  $\phi_2(x, t)$  has a negative real root on  $]-\infty, \xi_1[$ . If  $\xi_1$  does not exist, then  $\phi_2(x, t)$  has a root on  $]-\infty, -1[$ .

In conclusion, the distribution of the roots of  $D(x, t)$  between  $\phi_1(x, t)$  and  $\phi_2(x, t)$  implies that a double root exists if, and only if,

$$\phi_1(x, t) = \phi_2(x, t) = 0 \iff b(x, t) = a(x)c(x) = 0. \quad (2.4)$$

But the roots of  $b(x, t)$  are real positive, while those of  $a(x)$  and  $c(x)$  have negative real parts. Hence (2.4) can only take place if  $x = 0$  or  $x = \infty$ .

The proof of the theorem is terminated. ■

### 3 Classification of the singular random walks

**Definition 3.1.** A random walk is called *singular* (see [3]) if the associated polynomial  $K(x, y, t)$  is either reducible or of degree 1 in at least one of the variables.

We establish a useful lemma, which is of an algebraic nature and gives conditions for the factorization of the kernel  $K(x, y, t)$ .

**Lemma 3.2.** *Let  $\mathcal{A}[x]$  be the algebra of polynomials in  $x$  with coefficients in an arbitrary field  $\mathcal{A}$  containing the rational numbers  $\mathbb{Q}$ . For given  $a, b, c, p \in \mathcal{A}[x]$  satisfying*

$$b^2 - 4ac = kp^2, \quad (3.1)$$

*there exist  $\alpha, \beta, f_1, f_2 \in \mathcal{B}[x]$ , such that*

$$ay^2 + by + c = (f_1y - \alpha)(f_2y - \beta), \quad (3.2)$$

*where  $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A} + \sqrt{k}$  denotes the field  $\mathcal{A}$  to which is added the element  $\sqrt{k}$ . In addition, the relation (3.1) is necessary to have the factorization (3.2).*

*Proof.* Let  $Z_1, Z_2$  be the roots of

$$Z^2 + bZ + ac = 0, \quad (3.3)$$

so that, with an obvious notation,

$$Z_{1,2} = -\frac{b}{2} \pm \frac{p}{2}\sqrt{k}.$$

In  $\mathcal{B}[x]$ , let  $f_1$  be the greatest common divisor (g.c.d) of  $Z_1$  and  $a$ , so that

$$Z_1 = \beta f_1 \quad \text{and} \quad a = f_1 f_2,$$

where  $\beta$  and  $f_2$  are relatively prime. Then, from  $Z_1 Z_2 = ac$ , we get  $\beta Z_2 = c f_2$ . Hence, the g.c.d. of  $\beta$  and  $f_2$  being the unit element,  $f_2$  divides  $Z_2$  and we shall put  $Z_2 = \alpha f_2$ .

Setting for a while  $Z = ay$ , the announced factorization (3.2) follows directly from the chain of equalities

$$Z^2 + bZ + ac = (Z - Z_1)(Z - Z_2) = (ay - \beta f_1)(ay - \alpha f_2) = a(f_2y - \beta)(f_1y - \alpha) \quad (3.4)$$

together with the identity

$$Z^2 + bZ + ac = a(ay^2 + by + c).$$

As for the necessity of (3.1) to have (3.2), we therefore assume

$$ay^2 + by + c = (f_1y - \alpha)(f_2y - \beta).$$

Letting  $Z_1 \stackrel{\text{def}}{=} \alpha f_2, Z_2 \stackrel{\text{def}}{=} \beta f_1$ , one sees that  $Z_1, Z_2$  are the respective roots of (3.3), which both belong to  $\in \mathcal{B}[x]$  and satisfy

$$(Z_1 - Z_2)^2 = (Z_1 + Z_2)^2 - 4Z_1Z_2 = b^2 - 4ac. \quad (3.5)$$

On the other hand, as part of the hypothesis, we can set  $Z_i \stackrel{\text{def}}{=} u_i + v_i\sqrt{k} \in \mathcal{B}[x]$ , for  $i = 1, 2$ , where  $u_i, v_i \in \mathcal{A}[x]$ . Then, using the fact that  $a, b, c$  belong to  $\mathcal{A}[x]$ , together with the relations

$$\alpha f_2 + \beta f_1 = -b, \quad \text{and} \quad \alpha\beta f_1 f_2 = ac,$$

we get immediately

$$u_1 = u_2 = -b/2, \quad v_1 + v_2 = 0,$$

Hence

$$(Z_1 - Z_2)^2 = [2v_1\sqrt{k}]^2 = 4kv^2,$$

which, comparing with (3.5), yields exactly (3.1).

The proof of the lemma is terminated. ■

**Lemma 3.3.** *In Lemma 3.2, take  $\mathcal{A} = \mathbb{C}$ , the field of complex numbers. Then, among all the possible factorizations of the form (3.2), one can always choose a real one over  $\mathbb{R}[x, y]$ .*

*Proof.* Equation (3.1) shows that  $D(x, t)$  has solely double roots, which by Theorem 2.1 happens only for  $x = 0$  and  $x = \infty$ , so that  $k = d_2(t)$  and  $p = x$ . Since  $d_2(t)$  is always positive for  $t \in ]0, 1]$ , the roots  $Z_{1,2}$  of equation (3.3) have the form

$$Z_{1,2} = \frac{-b \pm \sqrt{d_2(t)x}}{2}.$$

They clearly have real coefficients, and so do  $f_1, f_2, \alpha, \beta$  which are highlighted in Lemma 3.2 in a constructive way (or algorithmic so to speak). ■

The classification of the family of random walks under study is given by the next theorem.

**Theorem 3.4.** *For  $t \in ]0, 1]$  and  $p_{i,j} \in [0, 1], p_{0,0} \neq 1$ , the random walk is singular if, and only if, one of the following conditions holds:*

- (i) *There exists  $(i, j) \in \mathbb{Z}^2, |i| \leq 1, |j| \leq 1$ , such that only  $p_{ij}$  and  $p_{-i,-j}$  are different from 0 (see figure 3.1a and the three cases obtained by rotation);*

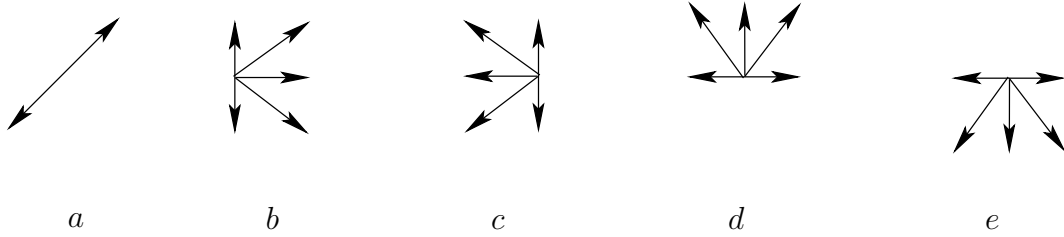


Fig. 3.1: Singular random walks.

- (ii) There exists  $i$ ,  $|i| = 1$ , such that for any  $j$ ,  $|j| \leq 1$ ,  $p_{ij} = 0$  (see figure 3.1b,c);
- (iii) There exists  $j$ ,  $|j| = 1$ , such that for any  $i$ ,  $|i| \leq 1$ ,  $p_{ij} = 0$  (see figure 3.1d,e).

*Proof.* Let us first eliminate three simple situations.

**Case a  $\equiv 0$ .** Then  $p_{1,1} = p_{0,1} = p_{-1,1} = 0$ . The kernel is of degree 1 in  $y$  and this corresponds to the walks of type  $e$  in figure 3.1 and

$$K(x, y, t) = t(by + c).$$

**Case c  $\equiv 0$ .** Then  $p_{1,-1} = p_{0,-1} = p_{-1,-1} = 0$ , giving the walks of type  $d$  in figure 3.1 and

$$K(x, y, t) = ty(ay + b).$$

By exchanging the variables  $x$  and  $y$ , and writing the kernel as a polynomial in  $x$  with coefficients  $\tilde{a}, \tilde{b}, \tilde{c}$  (see (1.5)), the cases  $\tilde{a} \equiv 0$  or  $\tilde{c} \equiv 0$  depict the walks of respective types  $c$  and  $b$  in figure (3.1).

**From now on, we shall assume  $ac \neq 0$ .**

**Case  $\mathbf{K(x, y, t) = u(x, t)v(x, y, t)}$ .** Here  $u$  (resp.  $v$ ) stands for a polynomial in  $x$  (resp.  $(x, y)$ ). It is immediate to see that the degree of  $u(x, t)$  in  $x$  is 1. But this means that  $a, b, c$  have a common real root in  $x$ , which can occur only at  $x = 0$ . Indeed, by (1.7), it appears that  $b(x, t)$  has always two positive real roots, while the roots of  $a$  and  $c$  cannot be positive. Thus the only possibility is  $p_{-1,1} = p_{-1,0} = p_{-1,-1} = 0$ , or equivalently  $\tilde{c} \equiv 0$ , which corresponds to the walks of type  $b$  in figure (3.1). Hence,

$$\tilde{c}(y) \equiv 0 \iff K(x, y, t) = tx[x\tilde{a}(y) + \tilde{b}(y, t)].$$

Similarly, by exchanging the roles of  $x$  and  $y$ , we obtain the walks of type  $d$  in figure (3.1). In particular,

$$c(x) \equiv 0 \iff K(x, y, t) = ty[ya(x) + b(x, t)].$$



Under Definition 3.1, all singular random walks satisfy (3.2) together with the system

$$\alpha f_2 + \beta f_1 = -b, \quad (3.6)$$

$$f_1 f_2 = a, \quad (3.7)$$

$$\alpha\beta = c, \quad (3.8)$$

where  $\alpha, \beta, f_1, f_2$  are polynomials of degree at most 2 in  $x$ , with real coefficients by Lemma 3.3, noting that in the course of the proof we omit the variables  $(x, t)$  for the sake of readability.

When  $a$  and  $c$  are positive (see Definition 1.1), (3.7) and (3.8) imply that  $(f_1, f_2)$  are both either positive or negative, and likewise for  $(\alpha, \beta)$ . This is true because  $a$  and  $c$  are of degree at most 2, and we shall choose  $(f_1, f_2)$  positive.

Recalling that  $b \equiv b(x, t) = p_{-1,0} + (p_{0,0} - 1/t)x + p_{1,0}x^2$ , the identity (3.6) leads to consider two different situations.

- (i)  $p_{1,0} + p_{-1,0} \neq 0$ . Here it is impossible to satisfy (3.6) for  $t \in ]0, 1]$ , since the three coefficients of the polynomial  $b$  do not have the same sign.
- (ii)  $p_{1,0} = p_{-1,0} = 0$ . Then (3.6) can hold only if  $(\alpha, \beta)$  are positive and if  $(\alpha f_2, \beta f_1)$  are proportional to  $x$ . Upon combining (3.7)-(3.8), we obtain

$$(\alpha f_2)(\beta f_1) = ac = Kx^2,$$

where  $K$  is a constant. Thus the coefficients of  $x^4, x^3, x, x^0$  of the polynomial  $a(x)c(x)$  must cancel out, yielding from (1.7) the respective equations

$$\begin{cases} p_{1,1}p_{1,-1} = 0, \\ p_{1,1}p_{0,-1} + p_{1,-1}p_{0,1} = 0, \\ p_{0,1}p_{-1,-1} + p_{-1,1}p_{0,-1} = 0, \\ p_{-1,1}p_{0,-1} = 0. \end{cases} \quad (3.9)$$

Now it is easy to check that the puzzle-like system (3.9) generates exactly the 4 following random walks:

$$\begin{cases} p_{1,-1} = p_{0,-1} = p_{-1,1} = p_{0,1} = 0, & \text{figure (3.1,a);} \\ p_{1,1} = p_{0,1} = p_{-1,-1} = p_{0,-1} = 0, & \text{figure (3.1,a) rotated by } 90^\circ; \\ p_{1,1} = p_{1,-1} = p_{-1,-1} = p_{-1,1} = 0, & \text{figure (3.1,a) rotated by } 45^\circ; \\ p_{1,1} = p_{0,1} = p_{-1,1} = p_{0,-1} = 0, & \text{sub-case of figure (3.1,e);} \end{cases} \quad (3.10)$$

By exchanging the roles of  $x$  and  $y$ , we also obtain the singular walk where only  $p_{1,0}$  and  $p_{-1,0}$  are different from zero.

The proof of the theorem is terminated. ■

**Remark 3.5.** When  $p_{0,0} = 0$ , one verifies easily that Theorem 3.4 holds in fact for all  $t \in ]0, +\infty[$ .

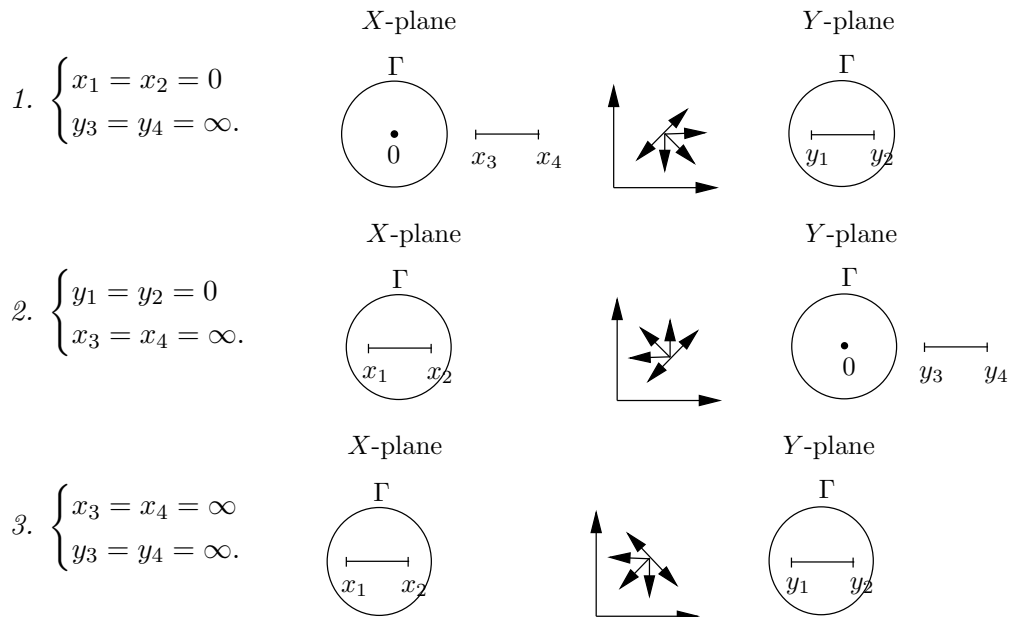
#### 4 Kernel of genus 0

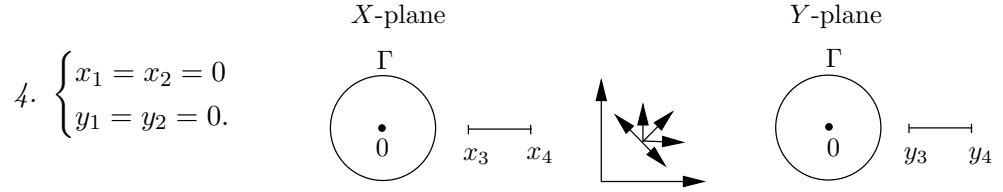
It is wellknown that, for all non-singular random walks, the Riemann surface  $\mathbf{S}$  defined by the algebraic curve (1.4) has genus 0 or 1. Moreover, the genus of  $\mathbf{S}$  is 0 if, and only if, the discriminant  $D(x, t)$  defined in (1.8) has a multiple zero in  $x$ , possibly infinite.

For the sake of completeness, we also introduce the branch points  $x_i$  (resp.  $y_i$ ),  $i = 1, \dots, 4$ , of the algebraic function  $Y(x, t)$  satisfying  $K(x, Y(x, t), t) = 0$  [resp.  $X(y, t)$  satisfying  $K(X(y, t), y, t) = 0$ ]. These branch points depend on  $t$ .

In the book [3, Section 2.3], the classification in the so-called stationary case  $t = 1$  has been dealt with in great detail. The proof was heavily using the principle of *continuity of the roots with respect to the parameters  $p_{i,j}$* , thus reducing the analysis to the case of the so-called *simple random walk*, i.e. when only  $p_{1,0}, p_{0,1}, p_{-1,0}, p_{0,-1}$  are different from 0. Indeed, for  $t \in ]0, 1[$ , exactly the same arguments hold. Nonetheless, we present hereafter a self-contained proof. The following theorem holds.

**Theorem 4.1.** *For all non singular random walks and for all  $t \in ]0, 1[$ , the algebraic curve defined by  $K(x, y, t) = 0$  has genus 0 if, and only if,  $D(x, t)$  has a double root at  $x = 0$  or  $x = \infty$ . That corresponds to the following pictures.*





In other words, multiple roots of  $D(x, t)$  can occur only in either of the two following situations:

- (a)  $x = 0$ , in which case  $d_0(t) = d_1(t) = 0$ ;
- (b)  $x = \infty$ , in which case  $d_3(t) = d_4(t) = 0$ .

*Proof.* Let us recall that, since  $d_2(t) > 0$ , the degree of  $D(x, t)$  with respect to  $x$  is always  $\geq 2$ . From the outset, we shall solve the case where this degree is exactly 2.

**Degree of  $\mathbf{D}(\mathbf{x}, \mathbf{t}) = 2$ .** This means that  $D(x, t)$  has a double root at  $x = \infty$ , whence  $d_3(t) = d_4(t) = 0$  and the genus of the kernel is 0 for all non singular random walks. Then, by using (1.10) and remarking that  $d_3(t) \leq 0$ , we get the system

$$p_{1,0} = p_{1,1}p_{0,-1} = p_{0,1}p_{1,-1} = p_{1,1}p_{1,-1} = 0. \quad (4.1)$$

It is now easy to check that the set of relations (4.1), without taking into account the singular random walk  $p_{1,0} = p_{1,1} = p_{1,-1} = 0$ , is tantamount to the pictures 2. and 3.

**Degree of  $\mathbf{D}(\mathbf{x}, \mathbf{t}) \geq 3$ .** We have proved in Theorem 2.1 that a double root can only occur at  $x = 0$ . Then  $d_0(t) = d_1(t) = 0$ , whence by using (1.10)

$$p_{-1,0} = p_{-1,1}p_{0,-1} = p_{0,1}p_{-1,-1} = p_{-1,1}p_{-1,-1} = 0,$$

which corresponds to the pictures 1. and 4.

The proof of the theorem is terminated. ■

**Remark 4.2.** When  $D(x, t)$  has a double root both at  $x = 0$  and  $x = \infty$ , we know from Lemma 3.2 that a factorization of the form (3.2) holds, and the walk is singular. On the other hand, it is worth noting that, for all  $t \in ]0, 1[$ , the discriminant  $D(x, t)$  can never have double root at  $x = 1$ . This case can occur for  $t = 1$  and corresponds random walks having zero drift vectors (see [3, Section 2.3]).

**Remark 4.3.** The approach proposed in this paper applies verbatim to functional equations pertaining to the transient distribution of two-dimensional Markov processes, such as those encountered in queueing systems (see e.g., [3, Section 8.4]).

## References

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