



# Mixed Nondeterministic-Probabilistic Interfaces

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# Mixed Nondeterministic-Probabilistic Interfaces

Albert Benveniste, Kim G. Larsen, Jean-Baptiste Raclet

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# Mixed Nondeterministic-Probabilistic Interfaces<sup>\*</sup>

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Project-Team Hycomes

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**Abstract:** Interface theories are powerful frameworks supporting incremental and compositional design of systems through refinements and constructs for conjunction, and parallel composition. In this report we present a first Interface Theory—Modal Mixed Interfaces—for systems exhibiting both non-determinism and randomness in their behaviour. The associated component model—Mixed Markov Decision Processes—is also novel and subsumes both ordinary Markov Decision Processes and Probabilistic Automata.

**Key-words:** Interface theories, Probabilistic interfaces, Probabilistic systems, Nondeterministic systems

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## Interfaces Mixtes Probabilistes-Nondéterministes

**Résumé :** Les théories d'interfaces sont des formalismes de spécification. Elles permettent une spécification incrémentale grâce à une algèbre riche d'opérateurs tels que le raffinement, la conjonction et la composition parallèle. Dans ce rapport, on propose une théorie d'interfaces, les Interfaces Modales Mixtes, qui permettent de spécifier des systèmes combinant étroitement des aspects probabilistes et non-déterministes. Les Interfaces Modales Mixtes sont construites au-dessus du modèle de composant des Automates Mixtes, qui étend à la fois les Processus de Décision Markoviens et les Automates Probabilistes.

**Mots-clés :** Théories d'interfaces, interfaces probabilistes, systèmes probabilistes, systèmes non-déterministes

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## 1 Introduction

Contract or Interface Theories are powerful frameworks for the incremental and compositional design of systems. Their essence is to handle *components* (for capturing actual designs) and *contracts* or *interfaces* (for capturing specifications). At their heart sits the notion of satisfaction, stating that a design suitably implements a specification. To achieve this, frameworks of components must be equipped with a parallel composition, and interface theories need a richer algebra to support the incremental and compositional design of systems, namely: refinement, conjunction, and parallel composition [3, 5]. Different styles of frameworks for interfaces include de Alfaro-Henzinger *Interface Automata* [10] Larsen et al. *Modal Automata* [1] and their variants, and trace based *Assume/Guarantee contracts* [4].

There are several cases where the underlying class of systems involves a mix of nondeterminism and randomness. Faults and their possible propagation through a system are naturally modelled probabilistically, whereas lack of knowledge of scheduling principles or incomplete information must be modelled through by nondeterminism. Since faults may be affected by scheduling policies, frameworks supporting the joint handling of nondeterminism and randomness are needed. In this paper we thus ask the following natural questions:

- *Question 1.* Can we develop a framework for components able to blend probabilistic and nondeterministic behaviors in a compositional way?
- *Question 2.* Can we develop a theory of interfaces blending probabilistic and nondeterministic aspects to serve as specifications of the former?

Regarding Question 1 about components, Markov Decision Processes (MDP) [20] provide a natural framework for capturing randomness. Runs of an MDP proceed as follows: from a state  $q$  some action  $\alpha$  can be selected, which brings the system into a probabilistic state  $\pi$  (a probability), from which the next state  $q'$  is drawn at random. MDP compose by synchronizing over common actions, whereas probabilistic state-choice is made independently. As a first contribution of this paper we propose a non-deterministic extension of MDP called *Mixed Markov Decision Processes* (MMDP). Runs of MMDP proceed as follows: from a state  $q$  some action  $\alpha$  can be selected, which brings the system into a *mixed* state  $S$  (a system blending nondeterminism and probability, albeit with no dynamics, as illustrated in Figure 1-left), from which

the next state  $q'$  is drawn in a mixed nondeterministic/probabilistic way. MMDP compose by synchronizing on their common actions: from  $(q_1, q_2)$ , performing action  $\alpha$  brings the composed MMDP to mixed state  $S_1 \times S_2$ , from which the next state  $(q'_1, q'_2)$  is drawn. One major issue is here the definition of mixed systems and their composition (see Figure 1-mid and right, latter explained in Section 2). Our proposed MMDP component model subsumes that of Probabilistic Automata [19].

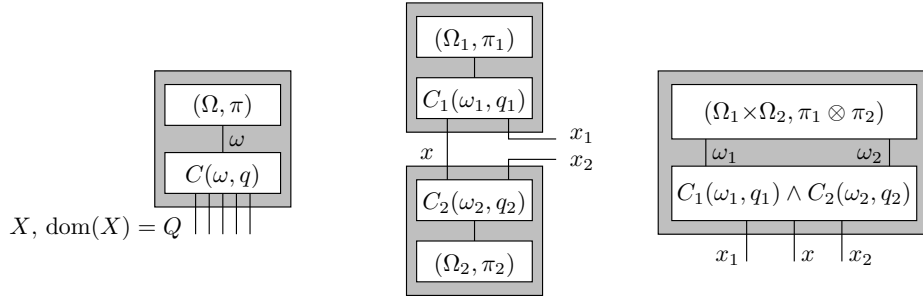


Figure 1: Left: a mixed system; a probability space  $(\Omega, \pi)$  produces the private random outcome  $\omega$  subject to the constraint  $C(\omega, q)$ , where  $q$  is the visible value taken by a tuple of variables  $X$ . Mid: parallel composition of two mixed systems, the intuition: the two systems interact through their shared variables (here  $x$ ). Right: the actual formal result as another mixed system.

Regarding Question 2 about interfaces, there exist few attempts in this direction, but no complete answer that we are aware of. Caillaud et al. [7, 8] propose the framework of Constrained Markov Chains (CMC) as an extension of Interval Markov Chains [15]. By imposing constraints on transition probabilities, CMCs are a specification theory for discrete time Markov Chains: refinement relations are proposed, as well as constructs for conjunction and parallel composition. The framework is made effective by restricting constraints on transition probabilities to be polynomial. *Abstract Probabilistic Automata* (APA) [11, 12, 13] is a proposal for an interface theory for Probabilistic Automata. APA borrows from the CMC model the idea of setting polynomial constraints on the transition probabilities attached to the probabilistic states, and offer the same algebra as CMC does.

The second and major contribution of this paper is the novel framework of Modal Mixed Interfaces (or Mixed Interfaces for short), which is an interface theory for MMDP. Our approach consists in lifting, to MMDP, the construction of Modal Interfaces [21] on top of automata. Mixed Interfaces offer the usual algebra of interface theories, namely: satisfaction



(also named implementation), refinement, conjunction, and parallel composition. We show that Mixed Interfaces extend CMC regarding satisfaction and refinement, while offering a much cleaner notion of parallel composition.

The paper is organized as follows. In Section 2 we develop the model of Mixed Systems sketched in Figure 1, on top of which MMDP are built in Section 3 to serve as model of component. In Section 4 we show how to embed in MMDP Segala's Probabilistic Automata with nondeterministic transition relations. Section 5 introduces Mixed Modal Interfaces as a specification framework for MMDP and we show in Section 6 how to embed in it the specification framework of Constraint Markov Chains. All proofs are deferred to appendices.

## 2 Mixed Probabilistic Nondeterministic systems

For  $(\Omega, \pi)$  a finite or countable probability space,  $\pi$  is entirely determined by its associated *weighting function*  $w(\omega) =_{\text{def}} \pi(\{\omega\})$ . By abuse of notation, we denote by  $\pi(\omega)$  the weighting function associated to  $\pi$ . Also, for a subset  $W \subseteq \Omega$  such that  $\pi(W) > 0$ , we define the *conditional probability*  $\pi(\cdot | W)$  by the formula  $\pi(V|W) =_{\text{def}} \frac{\pi(V \cap W)}{\pi(W)}$ , which is well defined since  $\pi(W) > 0$ . The *support* of  $\pi$ , denoted by  $\text{supp}(\pi)$ , is the set of all  $\omega$  such that  $\pi(\omega) > 0$ . Throughout this paper and unless otherwise specified we consider only finite or countable probability spaces.<sup>1</sup>

We are now ready to define Mixed Probabilistic Nondeterministic systems and give their semantics. This was illustrated in Figure 1.

**Definition 1** *A Mixed Nondeterministic Probabilistic system or Mixed System for short is a tuple:  $S = ((\Omega, \pi), X, C)$ , where  $(\Omega, \pi)$  is a probability space;  $X$  is a finite set of variables having finite or countable domain  $Q = \prod_{x \in X} Q_x$ ; and  $C \subseteq \Omega \times Q$  is a relation.*

*A system  $S$  is called inconsistent if  $\pi(\exists q.C) = 0$ , otherwise it is said consistent. If  $S$  is consistent, its operational semantics consists in:*

1. *drawing  $\omega \in \Omega$  at random according to  $\pi(\cdot | \exists q.C)$ , and*
2. *nondeterministically selecting  $q \in Q$  such that  $\omega C q$ .*

*This two-step procedure is denoted by  $S \rightsquigarrow q$  and, for  $\mathcal{S}$  a set of mixed systems, we write  $\mathcal{S} \rightsquigarrow q$  if  $S \rightsquigarrow q$  holds for some  $S \in \mathcal{S}$ .*

<sup>1</sup> The restriction that  $\Omega$  is at most countable is technically important in the above material. For the general case, we must abandon conditional probabilities and use the notion of *conditional expectation*, which is defined in full generality. Conditional distributions require additional topological assumptions for their definition, and so does the notion of *support*.

In the sequel and unless otherwise specified, we only consider consistent systems.

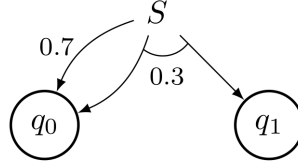


Figure 2: Example 1: a mixed system

**Example 1** The following mixed system  $S$  is depicted in Figure 2:

- $\Omega = \{\omega_1, \omega_2\}$  with  $\pi(\omega_1) = 0.7$  and  $\pi(\omega_2) = 0.3$ ;
- $X = \{x\}$  over  $Q_x = \{0, 1\}$ . Call  $q_i$  for  $i \in Q_x$  the state for which  $x = i$ ;
- $C = \{(\omega_1, q_0), (\omega_2, q_0), (\omega_2, q_1)\}$ .

Intuitively,  $S$  may evolve to  $q_0$  with probability 0.7. It may also nondeterministically evolve to  $q_0$  or  $q_1$  with probability 0.3.  $\square$

**Example 2** In a mixed system, the randoms are hidden that is, only their effect on the visible system variables is of interest. Now suppose that  $\Omega = \Omega_1 \times \Omega_2$  and the constraint  $C$  has the form  $C(\omega_1, q)$ . In this case,  $\Omega_2$  is not needed and can be removed, e.g., by replacing  $(\Omega, \pi)$  by its marginal  $(\Omega_1, \pi_1)$ , where  $\pi_1(\omega_1) = \sum_{\omega_2} \pi(\omega_1, \omega_2)$ .  $\square$

This was just a simple case and we now discuss the operation of compression, on top of which a notion of equivalence between systems can be defined. This material is borrowed from [6].

**Definition 2 (compression)** For  $S = ((\Omega, \pi), X, C)$  a Mixed System, we define the following equivalence relation on  $\Omega$ :

$$\omega \sim \omega' \quad \text{iff} \quad \forall q : (\omega, q) \in C \Leftrightarrow (\omega', q) \in C \quad (1)$$

The compression of  $S$ , denoted by  $[S] = ([\Omega], [\pi], X, [C])$ , is defined as follows:  $[\Omega] = \Omega / \sim$  (its elements are written  $[\omega]$ ),  $[C]([\omega], \cdot) = C(\omega, \cdot)$  for  $\omega \in [\omega]$  and  $[\pi]([\omega]) = \sum_{\omega \in [\omega]} \pi(\omega)$ . Say that  $S$  is compressed if it coincides with its compression.

Distinguishing  $\omega$  and  $\omega'$  is impossible if  $\omega \sim \omega'$ . Compressing  $\Omega$  is thus natural. We say that two systems are equivalent if their compressed forms are isomorphic.

**Definition 3 (equivalence)** *Two compressed mixed system  $S$  and  $S'$  are called equivalent, written  $S \equiv S'$ , if they possess identical sets of variables  $X = X'$  and isomorphic operational semantics, i.e., if, when setting  $C_\pi = \{(\omega, q) \in C \mid \pi(\omega) > 0\}$ , there exists a bijective map:  $\varphi : C_\pi \mapsto C'_{\pi'}$  such that, for every  $(\omega, q) \in C_\pi$ , we have  $\pi(\omega) = \pi'(\omega')$  and  $q = q'$ , where  $(\omega', q') =_{\text{def}} \varphi(\omega, q)$ . Say that arbitrary systems  $S$  and  $S'$  are equivalent if their compressions are equivalent.*

Mixed Systems are equipped with a parallel composition by intersection in which probabilistic choices remain local and independent, conditionally to the satisfaction of synchronization constraints.

**Definition 4 (parallel composition)** *For  $S_i, i = 1, 2$  two mixed systems, we define their parallel composition  $S = S_1 \times S_2$  as the following Mixed System:*

$$\begin{aligned} X &= X_1 \cup X_2, \Omega = \Omega_1 \times \Omega_2, \text{ and } \pi = \pi_1 \otimes \pi_2 \\ C &= \{(\omega, q) \mid \omega_1 C_1 \mathbf{Pr}_1(q) \wedge \omega_2 C_2 \mathbf{Pr}_2(q)\} \end{aligned}$$

where  $\mathbf{Pr}_i(q)$  denotes the projection of the state  $q$  over the variables  $X_i$ .

The definition of  $C$  expresses that the two systems must agree on their shared variables  $X_1 \cap X_2$ . For the next definition,  $\mathbf{Pr}_{12}(\cdot)$  denotes the projection over the shared variables  $X_1 \cap X_2$ . We write  $q_1 \bowtie q_2$  and say that  $q_1$  and  $q_2$  are *compatible* if  $\mathbf{Pr}_{12}(q_1) = \mathbf{Pr}_{12}(q_2)$ . If  $q_1 \bowtie q_2$ , we define the *join*  $q_1 \sqcup q_2$  as the unique  $q$  projecting over  $q_1$  and  $q_2$ . Using this notation,  $C$  in Definition 4 rewrites

$$C = \{(\omega, q_1 \sqcup q_2) \mid q_1 \bowtie q_2 \wedge \omega_1 C_1 q_1 \wedge \omega_2 C_2 q_2\} \quad (2)$$

Observe that the composition of two consistent systems may be inconsistent.

**Lemma 1** *For mixed systems, equivalence is a congruence, i.e.,  $S_i \equiv S'_i$  for  $i = 1, 2$  implies  $S_1 \times S_2 \equiv S'_1 \times S'_2$ .*

**Proof** See Appendix A.1. □

Let  $\mathcal{S}(X)$  denote the collection of all mixed systems having  $X$  as set of variables.

**Definition 5 (lifting relations)** Relation  $\rho^S \subseteq \mathcal{S}(X_1) \times \mathcal{S}(X_2)$  is called the lifting of relation  $\rho \subseteq Q_1 \times Q_2$  if there exists a weighting function  $w : \Omega_1 \times \Omega_2 \rightarrow [0, 1]$  such that:

1. For every triple  $(\omega_1, \omega_2; q_1)$  such that  $w(\omega_1, \omega_2) > 0$  and  $\omega_1 C_1 q_1$ , there exists  $q_2$  such that  $\omega_2 C_2 q_2$ , and  $q_1 \rho q_2$ ;
2.  $\sum_{\omega_2} w(\omega_1, \omega_2) = \pi_1(\omega_1)$  and  $\sum_{\omega_1} w(\omega_1, \omega_2) = \pi_2(\omega_2)$ .

Note the existential quantifier in Condition 1. By Condition 2,  $w$  induces a probability on  $\Omega_1 \times \Omega_2$ . We write  $S_1 \rho^S S_2$  to mean  $(S_1, S_2) \in \rho^S$ .

**Example 3** Consider the mixed systems  $S_1$  and  $S_2$  depicted in Figure 3. We can lift the relation  $\rho$  such that  $\rho = \{(q_{10}, q_{20}), (q_{11}, q_{20}), (q_{11}, q_{21})\}$  and see that  $S_1 \rho^S S_2$  by considering the weighting function shown in red. However, the relation  $\rho'$  such that  $\rho' = \{(q_{10}, q_{20}), (q_{11}, q_{21})\}$  cannot be lift as a witness  $w$  does not exist.  $\square$

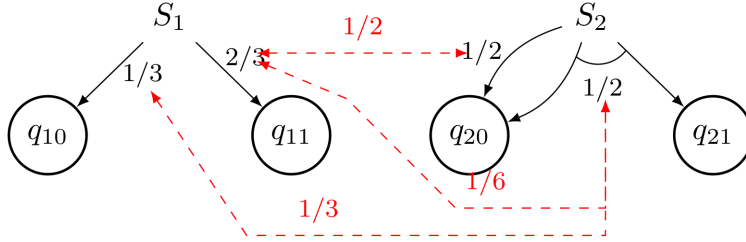


Figure 3: Example 3: lifted relation

**Lemma 2**  $S_1 \rho^S S_2$  and  $S'_1 \equiv S_1$  together imply  $S'_1 \rho^S S_2$ .

**Proof** See Appendix A.2.  $\square$

Lemma 2 expresses that mixed system equivalence is also a congruence with respect to the lifting of relations.

**Definition 6** Given  $\rho \subseteq Q_1 \times Q_2$  and  $\mathcal{S}_i \subseteq \mathcal{S}(Q_i)$  for  $i = 1, 2$ , define

$$\mathcal{S}_1 \subseteq^\rho \mathcal{S}_2 \text{ iff } \forall S_1 \in \mathcal{S}_1, \exists S_2 \in \mathcal{S}_2 : S_1 \rho^S S_2,$$

and define  $\mathcal{S}_1 \supseteq^\rho \mathcal{S}_2$  as being  $\mathcal{S}_2 \subseteq^{\tilde{\rho}} \mathcal{S}_1$ , where  $\tilde{\rho}$  denotes the transpose of  $\rho$ . Write  $S \in^\rho \mathcal{S}$  to mean  $\{S\} \subseteq^\rho \mathcal{S}$  and  $\mathcal{S} \ni^\rho S$  to mean  $\mathcal{S} \supseteq^\rho \{S\}$ .

For  $Q_1$ ,  $Q_2$ , and  $Q_3$  three finite or countable sets, and  $\rho_{12} \subseteq Q_1 \times Q_2$  and  $\rho_{23} \subseteq Q_2 \times Q_3$  two relations, define:

$$\rho_{12} \bullet \rho_{23} \stackrel{\text{def}}{=} \mathbf{Pr}_{Q_1 \times Q_3}(\rho_{12} \wedge \rho_{23}) \quad (3)$$

that is,  $\rho_{12} \bullet \rho_{23} \subseteq Q_1 \times Q_3$  and  $q_1(\rho_{12} \bullet \rho_{23})q_3$  iff  $q_1 \rho_{12} q_2$  and  $q_2 \rho_{23} q_3$  for some  $q_2 \in Q_2$ .

**Lemma 3** We have  $(\rho_{12} \bullet \rho_{23})^S = \rho_{12}^S \bullet \rho_{23}^S$  and  $\subseteq^{\rho_{12} \bullet \rho_{23}} = \subseteq^{\rho_{12}} \bullet \subseteq^{\rho_{23}}$ .

**Proof** See Appendix A.3.

**The set algebra of mixed systems.** We have introduced in Definition 3 the notion of equivalence  $\equiv$  for mixed systems. Lemmas 1 and 2 show that this equivalence is a congruence with respect to both mixed systems composition and the lifting of relations from state spaces to mixed systems. We now define the set algebra induced by this equivalence. For  $\mathcal{S}$  and  $\mathcal{S}'$  two sets of mixed systems:

$$\begin{aligned} \mathcal{S} \subseteq \mathcal{S}' & \quad \text{iff} \quad \forall S \in \mathcal{S}, \exists S' \in \mathcal{S}' : S' \equiv S \\ \mathcal{S} = \mathcal{S}' & \quad \text{iff} \quad \mathcal{S} \subseteq \mathcal{S}' \text{ and } \mathcal{S}' \subseteq \mathcal{S} \\ \mathcal{S}_1 \cap \mathcal{S}_2 & = \bigcup \{ \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{S}_1 \text{ and } \mathcal{S} \subseteq \mathcal{S}_2 \} \end{aligned} \quad (4)$$

Thanks to this redefinition, we shall freely use the usual set theoretic notations for sets of mixed systems.

### 3 Mixed Markov Decision Processes

Probabilistic automata have been introduced in [23] for the study of randomization in concurrency theory. They are labeled transitions systems where transitions are from states not to a single target state but to a target state determined by a probability measure. *Markov Decision Processes* [2, 14] exist in mathematics for quite some time. They correspond to *deterministic* probabilistic automata in the following sense: from each state, each action identifies a unique probability measure. In this paper we consider extensions of MDP in which the target of a transition is a mixed probabilistic/nondeterministic system as defined in Section 2:

**Definition 7 (mmdp)** A Mixed Markov Decision Process (MMDP) is a tuple  $M = (\Sigma, X, r_0, \rightarrow)$ , where:

- $\Sigma$  is a finite alphabet of actions;

- $X$  is a finite set of variables having finite or countable domain  $R = \prod_{x \in X} R_x$ , and  $r_0 \in R$  is the initial state;
- $\rightarrow \subseteq R \times \Sigma \times \mathcal{S}(X)$  is the transition relation; we write  $r \xrightarrow{\alpha} S$  (or  $r \xrightarrow{\alpha}_M S$ ) to mean  $(r, \alpha, S) \in \rightarrow$ , and  $r \xrightarrow{\alpha}$  if  $r \xrightarrow{\alpha} S$ .

Let  $\mathcal{S}(X)$  be the set of all mixed systems  $S$  over  $X$ , possibly inconsistent. We require that  $M$  shall be deterministic: for any pair  $(r, \alpha) \in R \times \Sigma$ ,  $r \xrightarrow{\alpha} S$  and  $r \xrightarrow{\alpha} S'$  implies  $S = S'$ .  $M$  is said to be live if all its transitions target consistent systems.

A run  $\sigma$  of  $M$  is a finite or infinite sequence of states  $r_0, r_1, r_2, \dots$  starting from initial state  $r_0$  and then progressing by a sequence of steps of the form  $r_k \xrightarrow{\alpha} S \Rightarrow r_{k+1}$ , where  $S \leadsto r'$  is the operational semantics of system  $S$  following Definition 1.

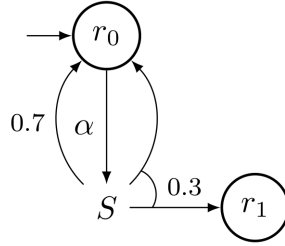


Figure 4: Example of an MMDP ( $S$  is the mixed system of Figure 2)

**Example 4** An MMDP is depicted in Figure 4 with  $\Sigma = \{\alpha\}$ ; it has only one transition  $(r_0, \alpha, S)$  where  $S$  is the mixed system of Figure 2.  $\square$

**Definition 8 (simulation)** Given two MMDP  $M_1$  and  $M_2$  over  $\Sigma$ ,  $M_2$  simulates  $M_1$ , written  $M_1 \leq M_2$ , if there exists a relation  $\leq \subseteq R_1 \times R_2$  such that:

- $r_{0,1} \leq r_{0,2}$  and,
- for  $r_1 \leq r_2$  and for each transition  $r_1 \xrightarrow{\alpha}_{M_1} S_1$ , there exists a transition  $r_2 \xrightarrow{\alpha}_{M_2} S_2$  such that  $S_1 \leq^S S_2$ .

$M_1$  and  $M_2$  are called simulation equivalent if they simulate each other.

The composition of two MMDP having identical alphabets is introduced next. A transition labeled  $\alpha$  is available in the product if and only if the components are ready to do simultaneously a transition labeled  $\alpha$ .

**Definition 9 (composition)** For  $M_1$  and  $M_2$  two MMDP having identical alphabet  $\Sigma$  and compatible initial states  $r_{0,1} \bowtie r_{0,2}$ , their composition  $M_1 \times M_2$  has alphabet  $\Sigma$ , set of variables  $X_1 \cup X_2$ , and initial state  $r_{0,1} \sqcup r_{0,2}$ . Its transition relation is the minimal relation satisfying:

$$r_i \xrightarrow{\alpha}_{M_i} S_i \text{ for } i = 1, 2 \text{ and } r_1 \bowtie r_2 \implies r_1 \sqcup r_2 \xrightarrow{\alpha}_M S_1 \times S_2$$

where  $S_1 \times S_2$  has been defined in Definition 4.

Parallel composition preserves simulation:

**Lemma 4** Let  $M_i, i = 1, 2$  be two MMDP and let  $M'_i \leq M_i, i = 1, 2$ . Then, we have  $M'_1 \times M'_2 \leq M_1 \times M_2$ .

**Proof** See Appendix B.1. □

## 4 Link to Probabilistic Automata

Probabilistic Automata (PA) [19] are a nondeterministic extension of MDPs. We show here that MMDP can capture this nondeterminism by making use of the nondeterminism involved in mixed systems. We discuss here the version of PA with no consideration of internal actions.

**Definition 10** A Probabilistic Automaton PA is a tuple  $P = (\Sigma, Q, q_0, \rightarrow)$ , where  $\Sigma$  is the finite alphabet of actions,  $Q$  is a finite state space,  $q_0 \in Q$  is the initial state, and  $\rightarrow \subseteq Q \times \Sigma \times \mathcal{P}(Q)$  is a probabilistic transition relation where  $\mathcal{P}(Q)$  is the set of all probability distributions over  $Q$ .

The operational semantics of  $P$  is as follows: if  $P$  is in state  $q \in Q$ , performing  $\alpha \in \Sigma$  leads to some target set of probability distributions over  $Q$ , of which one is selected, nondeterministically, and then used to draw the next state  $q'$ . We can reinterpret this operational semantics as follows: performing  $\alpha \in \Sigma$  while being in state  $q \in Q$  leads to the same target set of probability distributions over  $Q$ , that we use differently. We form the direct product of all distributions belonging to the target set and we perform one trial according to this distribution, i.e., we perform independent random trials for all probabilities belonging to the target set. This yields a tuple of candidate values for the next state, of which we select one, nondeterministically.

Clearly, these two operational semantics produce identical outcomes. Now, the latter is the operational semantics of the MMDP  $M_P = (\Sigma, \xi, q_0, \rightarrow_P)$ , defined as follows:  $\Sigma$  is as before,  $\xi$  is the system variable with domain  $Q$ ,  $q_0$

is as before, and  $\rightarrow_P$  is the transition relation defined as follows:  $\rightarrow_P$  maps a pair  $(q, \alpha) \in Q \times \Sigma$  to the mixed system  $S = ((\Omega, \Pi), \xi, C)$  defined as follows. Let  $n$  be the cardinality of the set  $\{\pi \mid (q, \alpha, \pi) \in \rightarrow\}$ . Take for  $\Omega$  the product of  $n$  copies of  $Q$ , so that  $\omega$  is an  $n$ -tuple of states:  $\omega = (q_1, \dots, q_n)$ . Take for  $\Pi$  the product of all probabilities belonging to set  $\{\pi \mid (q, \alpha, \pi) \in \rightarrow\}$ . Finally,  $(\omega, q) \in C$  if and only if  $q \in \{q_1, \dots, q_n\}$ . The following theorem holds, for which the definitions of simulation and composition of PA are available in [19]:

**Theorem 1** *Let  $P_1, P_2$  be two PA and  $M_{P_1}, M_{P_2}$  be the corresponding MMDP. The mapping  $P \rightarrow M_P$  preserves both simulation and product:  $P_1 \leq P_2$  if and only if  $M_{P_1} \leq M_{P_2}$ , and  $M_{P_1 \times P_2}$  and  $M_{P_1} \times M_{P_2}$  are simulation equivalent.*

A reverse mapping also exists. The PA associated to the MMDP of Fig. 4 is easily guessed: performing  $\alpha$  leads to the family of two probability spaces over  $R$ :  $(R, \pi_1)$  where  $\pi_1(r_0) = 1$  and  $(R, \pi_2)$  where  $\pi_2(r_0) = 0.7$  and  $\pi_2(r_1) = 0.3$ . Theorem 1 holds for this inverse mapping as well. So, what is the point in preferring MMDP? The rich algebra developed in Section 2 (with the two key notions of compression and lifting) is essential in supporting a flexible notion of parallel composition. In particular, when extending PA with labeling using sets of atomic propositions (AP), it is required, for the parallel composition to be defined, that the two sets are disjoint. Our MMDP offer the expressive power of AP-labeling without setting any restriction on the parallel composition. See Section 6 for a detailed study of the same issue, for Constraint Markov Chains.

## 5 Modal Mixed Interfaces

In this section we develop the first part of our agenda, namely a framework of *Modal Mixed Interfaces* (or *Mixed Interfaces* for short) which allow to specify sets of MMDP called the *models* of the interface. Note that in this section sets of probabilities associated to Mixed Interfaces are manipulated by not paying attention to effectiveness. Mixed Interfaces extend to a mixed probabilistic-nondeterministic setting the formalism of Modal Specifications [18, 16, 1]. In this paper we develop our framework for the case of a fixed alphabet  $\Sigma$  of actions. Following [21], alphabet extension techniques allow to handle the general case.

**Definition and Semantics.** For  $X$  a finite set of variables,  $\mathcal{S}(X)$  denotes the class of all mixed systems  $S$  over  $X$  and we call *mixed state* a subset



$\mathcal{S} \subseteq \mathcal{S}(X)$ .

**Definition 11** A Mixed Interface is defined as a tuple  $\mathcal{C} = (\Sigma, X, q_0, \rightarrow, \dashrightarrow)$ , where:

- $\Sigma$  is the finite alphabet of actions;
- $X$  is a finite set of variables having finite domain  $Q =_{\text{def}} \prod_{x \in X} Q_x$ ;
- $q_0$  is the initial state (we do not require that  $q_0 \in Q$ );
- $\rightarrow, \dashrightarrow \subseteq Q \times \Sigma \times 2^{\mathcal{S}(X)}$  are the must and may transition relations.

We require that  $\mathcal{C}$  is deterministic in the following sense: for any pair  $(q, \alpha) \in Q \times \Sigma$ ,  $(q, \alpha, \mathcal{S}) \in \rightarrow$  and  $(q, \alpha, \mathcal{S}') \in \rightarrow$  imply  $\mathcal{S} = \mathcal{S}'$ , and similarly for  $\dashrightarrow$ .

We write  $q \xrightarrow{\alpha} \mathcal{S}$  to mean  $(q, \alpha, \mathcal{S}) \in \rightarrow$ ;  $q \dashrightarrow^{\alpha} \mathcal{S}$  is defined similarly. We write  $q \not\xrightarrow{\alpha}$  if there exists no mixed state  $\mathcal{S}$  such that  $q \xrightarrow{\alpha} \mathcal{S}$ ;  $q \not\dashrightarrow^{\alpha}$  is defined similarly. Finally, we write  $\mathcal{S} \Rightarrow q'$  to mean that  $S \Rightarrow q'$  holds for some  $S \in \mathcal{S}$ . Note that  $q_0 \notin Q$  will typically arise when the subset  $Q$  of states is empty; it will be useful to model *unsatisfiable* interfaces. Whenever convenient, we shall write  $\mathcal{S}^{\square}$  and  $\mathcal{S}^{\diamond}$  when referring to mixed states targeted by *must* and *may* transitions, respectively.

**Example 5** The following Mixed Interface is depicted in Figure 5-right:

- $\Sigma = \{\alpha\}$ ;
- $X = \{x\}$  over  $Q_x = \{0, 1\}$  with  $x = 0$  in  $q_{0,1}$  and  $x = 1$  in  $q_{0,1}$ ;
- $q_{0,2} \dashrightarrow^{\alpha} \{S_1, S_2\}$  with  $S_1$  and  $S_2$  two mixed systems.

Note that in the Mixed Interface of Figure 5-left, we have  $q_{0,1} \dashrightarrow^{\alpha} \{S\}$  and  $q_{0,1} \xrightarrow{\alpha} \{S\}$  but only the plain arrow corresponding to the must transition is depicted in order to lighten the figure.  $\square$

The intuitive semantics is the following: a *must* transition labeled by  $\alpha$  must be available in any model with an associated system  $S$  selected from  $\mathcal{S}^{\square}$  and then a next state  $q'$  is selected according to the operational semantics of  $S$ . The same holds for a *may* transition except that in this case, the occurrence of the action is allowed but not required and the selected system belongs to  $\mathcal{S}^{\diamond}$ .

We now formally define the notion of *model* of a Mixed Interface over  $\Sigma$  in terms of MMDP over the same alphabet; we make use of Definition 1 for the notion of *consistent system*, Definition 6 for the meaning of  $\models$  and Definition 7 for *live* MMDP:

**Definition 12 (satisfaction)** For  $\mathcal{C}$  a Mixed Interface such that  $q_0 \in Q$  and  $M$  a live MMDP, a relation  $\models \subseteq R \times Q$  is a satisfaction relation iff,

for any  $(r, q)$  such that  $r \models q$ , the following holds:

$$\left. \begin{array}{l} \text{only may transitions} \\ \text{of } \mathcal{C} \text{ are allowed for } M \end{array} \right\} \forall \alpha : r \xrightarrow{\alpha}_M S_M \Rightarrow \left[ q \xrightarrow{\alpha}_{\mathcal{C}} \mathcal{S}^{\diamond} \text{ and } S_M \in \models \mathcal{S}^{\diamond} \right] \quad (5)$$

$$\left. \begin{array}{l} \text{must transitions of } \mathcal{C} \\ \text{are mandatory for } M \end{array} \right\} \forall \alpha : q \xrightarrow{\alpha}_{\mathcal{C}} \mathcal{S}^{\square} \Rightarrow \left[ r \xrightarrow{\alpha}_M S_M \text{ and } S_M \in \models \mathcal{S}^{\square} \right] \quad (6)$$

$M$  is a model of  $\mathcal{C}$ , written  $M \models \mathcal{C}$ , if  $r_0 \models q_0$ . A Mixed Interface  $\mathcal{C}$  such that  $q_0 \notin Q$  does not admit any model.

The set of models of a Mixed Interface is closed under the simulation equivalence of Definition 8. Observe moreover that the condition (6) makes only sense because we consider deterministic interfaces, since the system  $S_M$  reached by performing action  $\alpha$  is unique in this case.

Note that, by definition,  $r \models q$  induces constraints on the set of systems associated to the *must* and *may* transitions stemming from  $q$ . More precisely, for any  $\alpha$  and  $\mathcal{S}^{\diamond}$  and  $\mathcal{S}^{\square}$  as in (5) and (6), the intersection  $\mathcal{S}^{\square} \cap \mathcal{S}^{\diamond}$  necessarily contains at least one consistent system. In this statement and in the sequel, we stress that the set algebra over sets of Mixed Systems is the one defined in (4).

**Definition 13** A state  $q$  is called inconsistent if  $q \xrightarrow{\alpha}_{\mathcal{C}} \mathcal{S}^{\square}$ , and either  $q \xrightarrow{\alpha}_{\mathcal{C}}$ , or  $q \xrightarrow{\alpha}_{\mathcal{C}} \mathcal{S}^{\diamond}$  but the intersection  $\mathcal{S}^{\square} \cap \mathcal{S}^{\diamond}$  contains no consistent system.

The subset of consistent systems of  $\mathcal{S}^{\square} \cap \mathcal{S}^{\diamond}$  entirely specifies the set of models of the considered Mixed Interface. This leads to the operation of pruning that we introduce next. The *pruning* of  $\mathcal{C}$ , written  $[\mathcal{C}]$ , is obtained as follows:

1. Let  $\mathcal{C}'$  the Mixed Interface obtained from  $\mathcal{C}$  by thinning  $\mathcal{S}^{\square}$  down to the intersection  $\mathcal{S}^{\square} \cap \mathcal{S}^{\diamond}$ ;
2. Apply repeatedly the following transformation until fixed point, with initial value  $k = 0$  and  $\mathcal{C}_0 = \mathcal{C}'$ :
  - (a) Let  $Q_{k,\text{incon}}$  be the set of states  $q$  of  $\mathcal{C}_k$  such that all inconsistent states of the state space  $Q_k$  and set  $Q_{k+1} = Q_k - Q_{k,\text{incon}}$ ; by construction, replacing  $Q_k$  by  $Q_{k+1}$  does not modify the set of models of  $\mathcal{C}$ ;
  - (b) Performing this step may create new inconsistent states, however; and, thus, we set  $k \leftarrow k + 1$  and return to step 2a.

Let  $[\mathcal{C}]$  be the Mixed Interface obtained at fixed point.

**Lemma 5** By construction,  $[\mathcal{C}]$  and  $\mathcal{C}$  possess identical sets of models.

**Proof** See Appendix D.1.  $\square$

Note that by considering that Mixed Interfaces have finite sets of states, the pruning procedure is terminating. A Mixed Interface  $\mathcal{C}$  is called *inconsistent* iff it has no model, i.e. iff the initial state  $q_0$  does not belong to the set of states of  $[\mathcal{C}]$ . Unless otherwise specified, we assume in the sequel that:

$$\begin{aligned} &\text{Pruning has been applied to every} \\ &\text{considered Mixed Interface: } [\mathcal{C}] = \mathcal{C}. \end{aligned} \quad (7)$$

**Refinement.** We now consider refinement which aims at comparing interfaces at different stages of their design. Intuitively, it allows to check if an interface is a more detailed version of an initial one. More precisely, refining an interface amounts to exclude some potential models from its set of models.

**Definition 14 (modal refinement)** Let  $\mathcal{C}_i, i = 1, 2$  be two Mixed Interfaces over  $\Sigma$ , a relation  $\preceq \subseteq Q_1 \times Q_2$  is a modal refinement iff, for all  $(q_1, q_2)$  such that  $q_1 \preceq q_2$  and for every  $\alpha \in \Sigma$ :

$$\begin{aligned} q_1 \xrightarrow{\alpha}_1 \mathcal{S}_1^\diamond &\Rightarrow q_2 \xrightarrow{\alpha}_2 \mathcal{S}_2^\diamond \text{ and } \mathcal{S}_1^\diamond \subseteq \mathcal{S}_2^\diamond \\ q_2 \xrightarrow{\alpha}_2 \mathcal{S}_2^\square &\Rightarrow q_1 \xrightarrow{\alpha}_1 \mathcal{S}_1^\square \text{ and } \mathcal{S}_1^\square \subseteq \mathcal{S}_2^\square \end{aligned} \quad (8)$$

Say that  $\mathcal{C}_1$  is a modal refinement of  $\mathcal{C}_2$ , written  $\mathcal{C}_1 \preceq \mathcal{C}_2$ , if for  $q_{0,1} \in Q_1$  and  $q_{0,2} \in Q_2$ , we have  $q_{0,1} \preceq q_{0,2}$ .

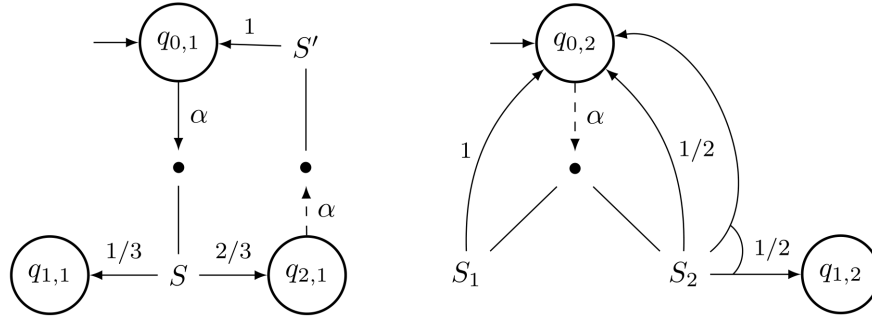


Figure 5: Example of refinement, see Examples 5 and 6.

**Example 6** Figure 5 shows an example of refinement. The Mixed Interface on the left is a refinement of the one on the right. Observe in particular that the Mixed Interface on the left still encompasses probabilistic aspects but no longer has nondeterministic select for the next state. This is allowed by the lifting operation on mixed systems as already seen in Example 3.  $\square$

**Lemma 6** *The modal refinement on Mixed Interfaces is a preorder.*

**Proof** See Appendix D.2.  $\square$

**Theorem 2** *For  $\mathcal{C}_i, i = 1, 2$  two Mixed Interfaces, if  $\mathcal{C}_1 \preceq \mathcal{C}_2$  then every model of  $\mathcal{C}_1$  is also a model of  $\mathcal{C}_2$ .*

**Proof** See Appendix D.3.  $\square$

Despite Mixed Interfaces are taken deterministic in Definition 11, modal refinement is correct but not fully abstract as for Modal Automata [17]: the following counterexample shows that Theorem 2 cannot be strengthened to an if-and-only-if statement. The reason for this is the nondeterminism that sits in the mixed systems themselves.

**Counterexample 1** Consider the two “purely non-probabilistic” Mixed Interfaces over  $\Sigma = \{a\}$  depicted in Figure 6. They are purely non-probabilistic as any associated random follows a Dirac probability.  $\mathcal{C}_1$  has only models that can perform at most two consecutive  $\alpha$ -actions. Any such implementation is also an implementation of  $\mathcal{C}_2$ . However, it is not true that  $\mathcal{C}_1 \preceq \mathcal{C}_2$  in the sense of modal refinement.  $\square$

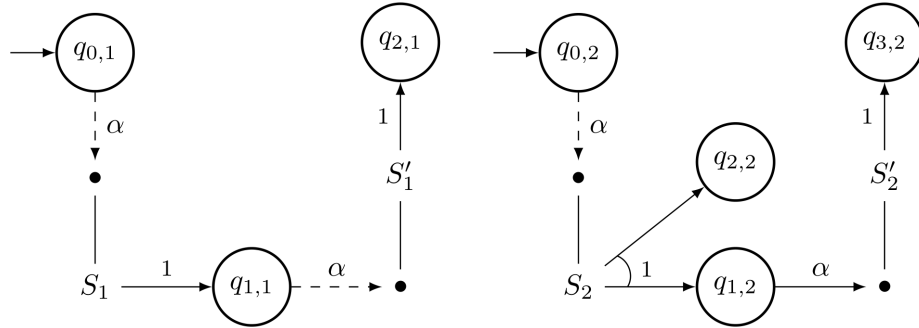


Figure 6: Counterexample 1 showing that modal refinement is not fully abstract

**Conjunction.** Consider  $\mathcal{C}_i, i = 1, 2$  two Mixed Interfaces over  $\Sigma$  with respective sets of variables  $X_1$  and  $X_2$  and state spaces  $Q_1$  and  $Q_2$ .

$$\mathcal{S}_1 \times \mathcal{S}_2 = \{S_1 \times S_2 \mid S_i \in \mathcal{S}_i\} \quad (9)$$

where  $S_1 \times S_2$  is defined in Definition 4.

We are now able to define the conjunction of two Mixed Interfaces.

**Definition 15 (conjunction)** Let  $\mathcal{C}_i, i = 1, 2$  be two Mixed Interfaces over  $\Sigma$ , their pre-conjunction  $\mathcal{C}_1 \underline{\wedge} \mathcal{C}_2$  has alphabet  $\Sigma$ , set of variables  $X_1 \cup X_2$ , initial state  $(q_{1,0}, q_{2,0})$ , and its may and must transition relations are the minimal relations satisfying the following rules:

$$\begin{aligned}
[\text{ConjMay}] & : q_1 \overset{\alpha}{\dashrightarrow}_1 \mathcal{S}_1^\diamond \text{ and } q_2 \overset{\alpha}{\dashrightarrow}_2 \mathcal{S}_2^\diamond \Rightarrow (q_1, q_2) \overset{\alpha}{\dashrightarrow} \mathcal{S}_1^\diamond \times \mathcal{S}_2^\diamond \\
[\text{ConjMust0}] & : q_1 \overset{\alpha}{\longrightarrow}_1 \mathcal{S}_1^\square \text{ and } q_2 \overset{\alpha}{\longrightarrow}_2 \mathcal{S}_2^\square \Rightarrow (q_1, q_2) \overset{\alpha}{\longrightarrow} \mathcal{S}_1^\square \times \mathcal{S}_2^\square \\
[\text{ConjMust1}] & : q_1 \overset{\alpha}{\longrightarrow}_1 \mathcal{S}_1^\square \text{ and } q_2 \overset{\alpha}{\dashrightarrow}_2 \Rightarrow (q_1, q_2) \overset{\alpha}{\longrightarrow} \mathcal{S}_1^\square \times \mathcal{S}(X_2) \\
[\text{ConjMust2}] & : q_1 \overset{\alpha}{\dashrightarrow}_1 \text{ and } q_2 \overset{\alpha}{\longrightarrow}_2 \mathcal{S}_2^\square \Rightarrow (q_1, q_2) \overset{\alpha}{\longrightarrow} \mathcal{S}(X_1) \times \mathcal{S}_2^\square
\end{aligned}$$

Pruning for consistency the pre-conjunction  $\mathcal{C}_1 \underline{\wedge} \mathcal{C}_2$  yields the conjunction  $\mathcal{C}_1 \wedge \mathcal{C}_2$ .

Inconsistency may result from the rules [ConjMust1] and [ConjMust2].

**Theorem 3** For any Mixed Interface  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , any model of  $\mathcal{C}_1 \wedge \mathcal{C}_2$  is also a model of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .

**Proof** See Appendix D.4. □

**Parallel composition.** Quite often in the literature, an issue of *compatibility* arises along with the parallel composition of interfaces [10, 21]. As clarified in [21], the issue of compatibility is due to the different roles played by the component and its environment in dealing with inputs and outputs. As we do not distinguish inputs and outputs here, compatibility is not an issue for us.

**Definition 16 (composition)** Let  $\mathcal{C}_i, i = 1, 2$  be two Mixed Interfaces over  $\Sigma$ , their composition  $\mathcal{C}_1 \otimes \mathcal{C}_2$  has alphabet  $\Sigma$ , set of variables  $X_1 \cup X_2$ , and initial state  $(q_{1,0}, q_{2,0})$ . Its transition relations are the minimal relations satisfying the following rules:

$$\begin{aligned}
q_1 \overset{\alpha}{\dashrightarrow}_1 \mathcal{S}_1^\diamond \text{ and } q_2 \overset{\alpha}{\dashrightarrow}_2 \mathcal{S}_2^\diamond & \Rightarrow (q_1, q_2) \overset{\alpha}{\dashrightarrow} \mathcal{S}_1^\diamond \times \mathcal{S}_2^\diamond \\
q_1 \overset{\alpha}{\longrightarrow}_1 \mathcal{S}_1^\square \text{ and } q_2 \overset{\alpha}{\longrightarrow}_2 \mathcal{S}_2^\square & \Rightarrow (q_1, q_2) \overset{\alpha}{\longrightarrow} \mathcal{S}_1^\square \times \mathcal{S}_2^\square
\end{aligned}$$

Parallel Composition does not raise any issue of consistency.

**Theorem 4** The parallel composition  $\otimes$  satisfies the following properties:

1.  $\otimes$  is commutative and associative.
2. For  $\mathcal{C}_1$  and  $\mathcal{C}_2$  two Mixed Interfaces, we have:

$$\forall M_i, i = 1, 2 : M_i \models \mathcal{C}_i \Rightarrow M_1 \times M_2 \models \mathcal{C}_1 \otimes \mathcal{C}_2 \quad (10)$$

$$\mathcal{C}_1 \preceq \mathcal{C}_2 \Rightarrow \forall \mathcal{C} : \mathcal{C} \otimes \mathcal{C}_1 \preceq \mathcal{C} \otimes \mathcal{C}_2 \quad (11)$$

**Proof** See Appendix D.5.  $\square$

Last, let us mention that no quotient exists for Mixed Interfaces. This is inherently due to the nondeterminism involved in Mixed Systems. Probabilistic specification models already suffer from the same limitation.

## 6 Link to Constraint Markov Chains

Constraint Markov Chains have been proposed in [8] as a specification formalism with Markov Chains as models. Let us first recall their basic definitions.

Let  $A, B$  be sets of propositions with  $A \subseteq B$ . The restriction of  $W \subseteq B$  to  $A$  is given by  $W_{\downarrow A} = W \cap A$ . If  $T \subseteq 2^B$ , then  $T_{\downarrow A} = \{W_{\downarrow A} \mid W \in T\}$ . Let  $\mathcal{P}(Q)$  denote the set of all probabilities over the set  $Q$ . For  $R$  and  $Q$  two at most denumerable state spaces, a *transition probability*  $\Delta$ , from  $R$  to  $Q$ , is a map  $\Delta : R \times Q \rightarrow [0, 1]$  such that, for every  $r \in R$ ,  $\Delta(r, \cdot)$  is a probability over  $Q$ . If  $\pi_R$  is a probability distribution over  $R$ , then  $\pi_R \Delta$  denotes the probability distribution over  $Q$  defined by:

$$\pi_R \Delta(q) = \sum_{r \in R} \pi_R(r) \Delta(r, q). \quad (12)$$

A *transition sub-probability*  $\Delta$  from  $R$  to  $Q$  is a map  $\Delta : R \times Q \rightarrow [0, 1]$  such that, for every  $(r, q) \in R \times Q$ ,  $\Delta(r, q) \geq 0$  and, for every  $r \in R$ ,  $\sum_{q \in Q} \Delta(r, q) \leq 1$ .

**Definition 17** A Markov Chain (MC) is a tuple  $\mathbf{P} = (R, r_0, \Pi, A, v)$ , where  $R$  is a set of states containing the initial state  $r_0$ ,  $A$  is a set of atomic propositions,  $v : R \rightarrow 2^A$  is a state valuation, and  $\Pi : R \times R \rightarrow [0, 1]$  is a transition probability.

**Definition 18** A Constraint Markov Chain (CMC) is a tuple

$$\mathbf{S} = (Q, q_0, \varphi, A, V),$$

where  $Q$  is a set of states containing the initial state  $q_0$ ,  $A$  is a set of atomic propositions,  $V : Q \rightarrow 2^{2^A}$  is a set of admissible state valuations, and  $\varphi : Q \rightarrow 2^{\mathcal{P}(Q)}$  is a constraint function, mapping states to sets of probability distributions over states.

In practice, constraint functions will be only partially specified, in that a function mapping  $Q$  to  $[0, 1]^Q$  will be implicitly complemented by the additional constraints to make the target being a probability. This consideration

is only practical and does not need to be taken into account for our subsequent development. Whenever needed to avoid confusion, we will denote by  $A_{\mathbf{P}}$  and  $v_{\mathbf{P}}$ , and  $A_{\mathbf{S}}$  and  $V_{\mathbf{S}}$ , the elements  $A$  and  $V$  of MC  $\mathbf{P}$  and CMC  $\mathbf{S}$ .

**Definition 19 (satisfaction)** *Let  $\mathbf{P}$  and  $\mathbf{S}$  be respectively an MC and a CMC such that  $A_{\mathbf{S}} \subseteq A_{\mathbf{P}}$ . A satisfaction relation between  $\mathbf{P}$  and  $\mathbf{S}$  is a relation  $\rho \subseteq R \times Q$  such that, whenever  $r \rho q$ :*

1.  $v_{\mathbf{P}}(r) \downarrow_{A_{\mathbf{S}}} \in V_{\mathbf{S}}(q)$ ;
2. *there exists a transition sub-probability  $\Delta$ , from  $R$  to  $Q$ , such that:*
  - (a) *for all  $r' \in R$  such that  $\Pi(r, r') > 0$ ,  $\Delta(r', q)$  is a transition probability from  $R$  to  $Q$ , and;*
  - (b)  $\Pi(r, \cdot) \Delta \in \varphi(q)$ , *and;*
  - (c) *if  $\Delta(r', q') \neq 0$ , then  $r' \rho q'$  holds.*

$\mathbf{P}$  satisfies  $\mathbf{S}$  if and only if there exists a satisfaction relation between  $\mathbf{P}$  and  $\mathbf{S}$  that contains the two initial states.

**Definition 20 (weak refinement)** *Let  $\mathbf{S}_1$  and  $\mathbf{S}_2$  be two CMC such that  $A_2 \subseteq A_1$ . The relation  $\rho \subseteq Q_1 \times Q_2$  is a weak refinement iff, whenever  $q_1 \rho q_2$ :*

1.  $V_1(q_1) \downarrow_{A_2} \subseteq V_2(q_2)$ ;
2. *for any probability distribution  $\pi_1 \in \varphi_1(q_1)$ , there exists a transition sub-probability  $\Delta$ , from  $Q_1$  to  $Q_2$ , such that:*
  - (a) *for all  $q_1$  such that  $\pi_1(q_1) > 0$ ,  $\Delta(q_1, \cdot)$  is a probability over  $Q_2$ ;*
  - (b)  $\pi_1 \Delta \in \varphi_2(q_2)$ ;
  - (c) *if  $\Delta(q'_1, q'_2) > 0$ , then  $q'_1 \rho q'_2$  holds.*

We say that  $\mathbf{S}_1$  weakly refines  $\mathbf{S}_2$ , written  $\mathbf{S}_1 \preceq \mathbf{S}_2$ , if  $q_{0,1} \rho q_{0,2}$ .

We now show that Mixed Interfaces subsume CMC. First, we define the embedding of MC in MMDP. Given  $\mathbf{P} = (R, r_0, \Pi, A, v)$  a Markov Chain, we associate the MMDP  $M_{\mathbf{P}} = (\Sigma, X, r_0, \rightarrow)$ , where:

- $\Sigma = \{\alpha\}$  (no need to mention the only action labeling transitions);
- $X = \{\xi, v\}$  collects a variable  $\xi$  with domain  $R$ , and the variable  $v$ ;
- $r_0 \in R$  is the initial condition for  $\xi$ ; no initial condition is given for  $v$ ;
- the transition relation is  $r \rightarrow S$ , where the mixed system  $S = ((\Omega, \pi), X, C)$  is such that:

$$\Omega = R; \pi = \Pi(r, \cdot) \text{ and } C = \{(r, r, v(r)) \mid r \in R\} \subseteq \Omega \times (R \times 2^A) \quad (13)$$

**Lemma 7** *Let  $\mathbf{P}$  be an MC. Then,  $\mathbf{P}$  and  $M_{\mathbf{P}}$  possess identical semantics.*

**Proof** See Appendix E.1.  $\square$

Consider now the embedding of CMC in Mixed Interfaces. For any CMC  $\mathbf{S} = (Q, q_0, \varphi, A, V)$ , we associate a Mixed Interface  $\mathcal{C}_{\mathbf{S}} = (\Sigma, X, q_0, \rightarrow, \dashrightarrow)$ , where:

- $\Sigma = \{\alpha\}$  (no need to mention the only action labeling transitions);
- $X = \{\xi, v\}$  collects a variable  $\xi$  with domain  $Q$ , and a variable  $v$  with domain  $2^A$ ;
- $q_0 \in Q$  is the initial condition for  $\xi$ ; no initial condition is given for  $V$ ;
- the *must* transition relation  $\rightarrow$  is empty;
- the *may* transition relation is  $q \dashrightarrow \mathcal{S}$ , where  $\mathcal{S}$  is the set of mixed systems of the form  $S_v = ((\Omega, \pi), X, C_v)$ , where  $v(q)$  ranges over  $V(q)$  and:

$$\Omega = Q; \pi \in \varphi(q) \text{ and } C_v = \{(q, q, v(q)) \mid q \in Q\} \subseteq \Omega \times (R \times 2^A) \quad (14)$$

Whenever needed, we will use subscripts to relate items of  $\mathbf{P}$  and  $\mathbf{S}$  to their respective host entities.

**Theorem 5** *Let  $\mathbf{S}$  be a CMC. Then,  $\mathbf{S}$  and  $\mathcal{C}_{\mathbf{S}}$  possess identical semantics.*

The previous Theorem decomposes into the two following lemmas.

**Lemma 8** *Let  $\mathbf{P}$  and  $\mathbf{S}$  be respectively an MC and a CMC such that  $A_{\mathbf{S}} \subseteq A_{\mathbf{P}}$ . Then,  $\mathbf{P}$  satisfies  $\mathbf{S}$  iff  $M_{\mathbf{P}}$  is a model of  $\mathcal{C}_{\mathbf{S}}$ .*

**Proof** See Appendix E.2.  $\square$

**Lemma 9** *Let  $\mathbf{S}_1$  and  $\mathbf{S}_2$  be two CMCs such that  $A_2 \subseteq A_1$ . Then,  $\mathbf{S}_1$  weakly refines  $\mathbf{S}_2$  iff  $\mathcal{C}_{\mathbf{S}_1}$  refines  $\mathcal{C}_{\mathbf{S}_2}$ .*

**Proof** See Appendix E.3.  $\square$

## 7 Conclusion

We have proposed the first interface theory that allows to mix probabilities and nondeterminism. Our component model is that of Mixed Markov Decision Processes (MMDP) which subsume Probabilistic Automata. Our specification formalism is that of Mixed Interfaces. It offers a complete algebra for interfaces, namely: satisfaction, refinement, conjunction, and



parallel composition. No quotient exists for Mixed Interfaces. This is inherently due to the nondeterminism involved in Mixed Systems. We presented our framework for the case of a fixed alphabet of actions. Following [21], alphabet extension techniques allow to handle the general case, this will be reported in the extended version of this work.

Mixed Interfaces extend and clarify the satisfaction and refinement relations defined for Constraint Markov Chains. The same holds for Abstract Probabilistic Automata (APA) [11]. CMC and APA differ from Mixed Interfaces regarding the parallel composition, however. The parallel composition for Mixed Interfaces is general (system variables can be shared), whereas the one for CMC or APA requires that the specifications for composition have disjoint sets of atomic propositions. Also, a subclass of Mixed Interfaces can be defined that tightly emulates the networks of *Price Timed Automata* (PTA) equipped with their stochastic semantics [9]; a complete emulation, however, requires the consideration of some non-compositional priority policy for closed systems in this subclass. Due to lack of space, these additional results were not presented here.

This paper sets the theoretical foundations of formalisms that we plan to apply to safety and vulnerability analysis as ongoing works. To make it effective and amenable of tool development, one step further is needed, namely a finitary syntax for specifying and manipulating sets of Mixed Systems.

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## A Proofs regarding Mixed Systems

### A.1 Proof of Lemma 1

**Proof** It is enough to prove the result for compressed systems. For  $i = 1, 2$ , let  $S_i \equiv S'_i$  and let  $\varphi_i$  be the bijections defining the two equivalences. With reference to (2), we define

$$\varphi(\omega, q_1 \sqcup q_2) = ((\omega'_1, \omega'_2), q'_1 \sqcup q'_2) \text{ where } (\omega'_i, q'_i) = \varphi_i(\omega_i, q_i), i = 1, 2$$

and we have to verify that  $\varphi$  defines the desired equivalence between  $S =_{\text{def}} S_1 \times S_2$  and  $S' =_{\text{def}} S'_1 \times S'_2$ . Using the expression (2) for  $C$  and the fact that  $\pi = \pi_1 \otimes \pi_2$ , we get

$$\begin{aligned} C_\pi &= \{(\omega, q_1 \sqcup q_2) \mid q_1 \bowtie q_2 \wedge \omega_1 C_1 q_1 \wedge \pi_1(\omega_1) > 0 \wedge \omega_2 C_2 q_2 \wedge \pi_2(\omega_2) > 0\} \\ &= \{(\omega, q_1 \sqcup q_2) \mid q_1 \bowtie q_2 \wedge (\omega_1, q_1) \in C_{1\pi} \wedge (\omega_2, q_2) \in C_{2\pi}\} \end{aligned}$$

Thus, for every  $(\omega, q_1 \sqcup q_2) \in C_\pi$ , we have  $q'_1 = q_1 \bowtie q_2 = q'_2$  and  $(\omega'_i, q'_i) \in C_{i\pi}, i = 1, 2$ , whence  $(\omega', q') \in C'_\pi$  and  $\varphi$  is a bijection. Since  $\pi' = \pi'_1 \otimes \pi'_2$  we get  $\pi'(\omega') = \pi(\omega)$ , which finishes the proof.  $\square$

### A.2 Proof of Lemma 2

**Proof** The result is immediate if both  $S_1$  and  $S'_1$  are compressed, see Definition 2. It is thus sufficient to prove the lemma for the following two particular cases:  $S_1$  compresses to  $S'_1$ , and the converse.

Consider first the case:  $S_1$  compresses to  $S'_1$ . Let  $w(\omega_1, \omega_2)$  be the weighting function associated to the lifting  $S_1 \rho^S S_2$ , and let  $\pi'_1(\omega'_1) = \sum_{\omega_1 \in \omega'_1} \pi_1(\omega_1)$  be the relation between  $\pi'_1$  and  $\pi_1$  in the compression of  $S_1$  to  $S'_1$ . Then  $w'(\omega'_1, \omega_2) = \sum_{\omega_1 \in \omega'_1} w(\omega_1, \omega_2)$  defines the weighting function associated to the lifting  $S'_1 \rho^S S_2$ . The other properties required to deduce  $S'_1 \rho^S S_2$  are immediate to prove.

Now, consider the alternative case:  $S'_1$  compresses to  $S_1$ , with relation

$$\pi_1(\omega_1) = \sum_{\omega'_1 \in \omega_1} \pi'_1(\omega'_1) \tag{15}$$

between  $\pi'_1$  and  $\pi_1$ , where  $\omega'_1 \in \omega_1$  means that  $\omega_1$  is the equivalence class of  $\omega'_1$  with respect to relation  $\sim$  defined in (1) when compressing  $S'_1$ . This case is more involved since the construction of the weighting function  $w'(\omega'_1, \omega_2)$

is nontrivial. We need  $w'(\omega'_1, \omega_2)$  to satisfy the following relations:

$$\begin{aligned} \forall \omega'_1 : \quad \pi'_1(\omega'_1) &= \sum_{\omega_2} w'(\omega'_1, \omega_2) \\ \forall \omega_2 : \quad \pi_2(\omega_2) &= \sum_{\omega'_1} w'(\omega'_1, \omega_2) \\ \forall(\omega'_1, \omega_2; q_1) : \quad &\left[ \begin{array}{c} w'(\omega'_1, \omega_2) > 0 \\ \omega'_1 \ C'_1 \ q_1 \end{array} \right] \Rightarrow \exists q_2 : \left[ \begin{array}{c} \omega_2 \ C_2 \ q_2 \\ q_1 \ \rho \ q_2 \end{array} \right] \end{aligned} \quad (16)$$

Focus first on the first two lines of (16). We claim that to find a solution  $w'$  to the first two lines of (16), it is enough to find a solution to the following system of equations where the unknowns are the values  $w'(\omega'_1, \omega_2)$ :

$$\begin{aligned} \forall \omega_1, \omega_2 : \quad \sum_{\omega'_1 \in \omega_1} w'(\omega'_1, \omega_2) &= w(\omega_1, \omega_2) \\ \forall \omega'_1 : \quad \sum_{\omega_2} w'(\omega'_1, \omega_2) &= \pi'_1(\omega'_1) \end{aligned} \quad (17)$$

Observe that  $\sum_{\omega'_1} w'(\omega'_1, \omega_2) = \sum_{\omega_1} \sum_{\omega'_1 \in \omega_1} w'(\omega'_1, \omega_2) = \sum_{\omega_1} w(\omega_1, \omega_2) = \pi_2(\omega_2)$  since  $w(\omega_1, \omega_2)$  is the weighting function of the lifting  $S_1 \rho^S S_2$ . Our claim is thus justified.

To solve (17), we observe that it splits into the following independent subsystems in which  $\omega_1$  is seen as a parameter ranging over  $\Omega_1$ :

$$\begin{aligned} \forall \omega_2 : \quad \sum_{\omega'_1 \in \omega_1} w'(\omega'_1, \omega_2) &= w(\omega_1, \omega_2) \\ \forall \omega'_1 \in \omega_1 : \quad \sum_{\omega_2} w'(\omega'_1, \omega_2) &= \pi'_1(\omega'_1) \end{aligned} \quad (18)$$

The rows of System (18) are linked by the following relation: summing over all  $\omega_2$  the first set of equations yields  $\sum_{\omega_2} \sum_{\omega'_1 \in \omega_1} w'(\omega'_1, \omega_2) = \sum_{\omega_2} w(\omega_1, \omega_2) = \pi_1(\omega_1)$ , whereas summing over all  $\omega'_1 \in \omega_1$  the second set of equations yields  $\sum_{\omega'_1 \in \omega_1} \sum_{\omega_2} w'(\omega'_1, \omega_2) = \sum_{\omega'_1 \in \omega_1} \pi'_1(\omega'_1) = \pi_1(\omega_1)$ , and the two resulting equations are identical, by Fubini theorem.

Let  $K_2$  be the cardinal of  $\Omega_2$  and  $L_1$  the cardinal of the set  $\{\omega'_1 \mid \omega'_1 \in \omega_1\}$ . We distinguish the three cases  $L_1 = 1$ ,  $K_2 = 1$ , and  $L_1, K_2 > 1$ .

If  $L_1 = 1$ , setting  $\forall \omega_2 : w'(\omega'_1, \omega_2) = w(\omega_1, \omega_2)$  yields a solution to (18) since the last equation of (18) is trivially satisfied.

Case  $K_2 = 1$  is trivial either, since  $w'(\omega'_1, \omega_2) = \pi'(\omega'_1)$  is the unique solution.

For the third case  $L_1, K_2 > 1$ , the system (18) has more unknowns ( $K_2 \times L_1$ ) than equations ( $K_2 + L_1$ ). To prove that it indeed has solutions, we reorganize the unknowns  $w'(\omega'_1, \omega_2)$  into a row matrix by listing as a submatrix

the  $w'(\omega'_1, \omega_2)$  for every fixed value of  $\omega'_1$  and  $\omega_2$  ranging over  $\Omega_2$ :

$$\begin{aligned} & [w'(\omega'_{11}, \omega_{21}), \dots, w'(\omega'_{11}, \omega_{2K_2}), \\ & w'(\omega'_{12}, \omega_{21}), \dots, w'(\omega'_{12}, \omega_{2K_2}), \\ & \vdots \\ & w'(\omega'_{1L_1}, \omega_{21}), \dots, w'(\omega'_{1L_1}, \omega_{2K_2})] \end{aligned}$$

We arrange the equations as indicated in (18): we put on top the  $K_2$  equations parameterized by  $\omega_2$  followed by the  $L_1$  equations parameterized by  $\omega'_1$ . For  $A$  and  $A'$  two matrices, of respective sizes  $m \times n$  and  $m' \times n'$ , we denote by  $A \otimes A'$  their *Kronecker product* obtained by replacing the  $a_{ij}$  entry of  $A$  by the matrix  $a_{ij} \cdot A'$ , thus obtaining a matrix of size  $(m \times m') \times (n \times n')$ . With these conventions and notations, the matrix of the linear system (18) takes the following form, where  $\mathbb{I}_m$  denotes the identity matrix of size  $m \times m$ :

$$M = \begin{bmatrix} \overbrace{[1 \dots 1]}^{L_1 \text{ times}} \otimes \mathbb{I}_{K_2} \\ \mathbb{I}_{L_1} \otimes \underbrace{[1 \dots 1]}_{K_2 \text{ times}} \end{bmatrix}, \quad (19)$$

of size  $(K_2 + L_1) \times (K_2 \times L_1)$ . The proof that the first two lines of (16) are satisfied rests on the two lemmas 10 and 11 below.

We move to the third line of (16). The conditions  $w'(\omega'_1, \omega_2) > 0$  and  $\omega'_1 C'_1 q_1$  together imply  $w(\omega_1, \omega_2) > 0$  and  $\omega_1 C_1 q_1$  where  $\omega_1$  is the equivalence class of  $\omega'_1$ , i.e.,  $\omega'_1 \in \omega_1$ . The right hand side then follows since we have  $S_1 \rho^S S_2$ . This finishes the proof.

**Lemma 10** *If  $L_1, K_2 > 1$ , then the matrix  $M$  defined in (19) has row rank equal to  $L_1 + K_2 - 1$ .*

**Proof** We proceed by double induction over  $L_1, K_2$ . The base case is  $L_1 = K_2 = 2$ , for which matrix  $M$  is equal to

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$M$  is singular but the submatrix obtained by erasing the first row and the last column in  $M$  (the latter are shown in green) is regular. This is proved by observing that this submatrix possesses only one traversal,<sup>2</sup> shown in

<sup>2</sup>A *traversal* of a  $p \times p$ -matrix  $B$  is a selection of  $p$  non-zero entries of  $B$  visiting all columns and rows of  $B$ .

red, hence its determinant equals  $\pm 1$  and cannot be zero.

In the rest of the proof, we use the convention that symbols written in boldface denote a matrix of suitable sizes filled with the indicated symbol. For example,  $\mathbf{0}$  denotes a matrix filled with zeros, the sizes of which depend on the context.

For the induction argument, let  $M(L_1, K_2)$  denote the matrix defined in (19) with the values  $L_1, K_2$  and  $\overline{M}(L_1, K_2)$  the square submatrix of  $M(L_1, K_2)$  obtained by erasing the first row in  $M(L_1, K_2)$  and then selecting columns accordingly. Using these notations, the invariant of the induction argument is the following:

$$\text{The number of traversals of } \overline{M}(L_1, K_2) \text{ equals } 1. \quad (20)$$

Increasing  $L_1$  by 1: matrix  $M(L_1, K_2)$  becomes

$$M(L_1+1, K_2) = \begin{bmatrix} M(L_1, K_2) & \begin{bmatrix} \mathbb{I}_{K_2} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{0} & \underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{K_2 \text{ times}} \end{bmatrix} \quad (21)$$

where the added part is highlighted in red. We construct  $\overline{M}(L_1+1, K_2)$  by adding, to  $\overline{M}(L_1, K_2)$ , one row below and one among the  $K_2$  new columns shown on the right part of  $M(L_1+1, K_2)$ . For this case the number of traversals keeps constant.

Increasing  $K_2$  by 1: matrix  $M(L_1, K_2)$  becomes

$$M(L_1, K_2+1) = \begin{bmatrix} \underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{L_1 \text{ times}} \otimes \begin{bmatrix} \mathbb{I}_{K_2} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \\ \mathbb{I}_{L_1} \otimes \underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{K_2 \text{ times}} & 1 \end{bmatrix}$$

where the additional entries are shown in red. We move the new row

$$\underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{L_1 \text{ times}} \otimes [\mathbf{0} \ 1]$$

to the last line of the matrix. The new columns arising from

$$\underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{L_1 \text{ times}} \otimes \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix}$$



are all shifted to the right to become the last ones of the matrix while keeping the same order. Having done this, we end up with a reorganized matrix that has the following form:

$$M(L_1, K_2+1) = \begin{bmatrix} M(L_1, K_2) & \begin{bmatrix} \star \end{bmatrix} \\ \mathbf{0} & \underbrace{\begin{bmatrix} 1 \dots 1 \end{bmatrix}}_{K_2 \text{ times}} \end{bmatrix}$$

where the added part is highlighted in red. Again the number of traversals remains constant.

In the following, for  $X$  a matrix,  $X^T$  denotes its transpose. Also, we take the convention that vectors identify with column matrices.

**Lemma 11** *Let  $A$  be an  $m \times n$  matrix with  $m \leq n$  such that  $A$  has rank  $m - 1$ , and there exists a non-zero  $m$ -vector  $v$  such that  $v^T A = [0 \dots 0]$ . Then, for every  $m$ -vector  $y$  such that  $v^T y = 0$ , the linear system  $Ax = y$  possesses a solution.*

**Proof** We complete  $v$  with  $m - 1$  vectors to get a basis of  $\mathbb{R}^m$  and denote by  $C$  the  $m \times m$ -matrix obtained by taking this basis as its columns,  $v$  being the first one. Premultiplying the linear system  $Ax = y$  by  $C^T$  yields  $C^T Ax = C^T y$ . Vector  $C^T y$  has a 0 as its first entry, completed by an  $m - 1$ -vector that we denote by  $z$ . Similarly, matrix  $C^T A$  has its first row equal to zero, and we denote by  $B$  the matrix obtained by erasing the first row of  $C^T A$ . Our original linear system is then equivalent to the reduced linear system  $Bx = z$ . By assumption,  $B$  has rank  $m - 1$ , i.e., full row rank, which ensures that a solution to  $Bx = z$  exists (possibly not unique).

To prove that the first two lines of (16) are satisfied, we apply Lemma 10 to the matrix  $M$  defined in (19), and then Lemma 11 to the matrix  $M$  with

$$v^T = \left[ \underbrace{1 \dots 1}_{K_2 \text{ times}} \quad \underbrace{-1 \dots -1}_{L_1 \text{ times}} \right]$$

### A.3 Proof of Lemma 3

**Proof** By definition,  $S_1 (\rho_{12}^S \bullet \rho_{23}^S) S_3$  iff there exists  $S_2 \in \mathcal{S}(Q_2)$  such that  $S_1 \rho_{12}^S S_2$  and  $S_2 \rho_{23}^S S_3$ , that is, there exists two weighted functions  $w_{12}$  over  $\Omega_1 \times \Omega_2$  and  $w_{23}$  over  $\Omega_2 \times \Omega_3$ , such that

- $w_{12}$  projects to  $\pi_1$  and  $\pi_2$ , and  $w_{23}$  projects to  $\pi_2$  and  $\pi_3$ , and

- $w_{12}(\omega_1, \omega_2) > 0$  and  $\omega_1 C_1 q_1$  together imply the existence of a  $q_2$  such that  $\omega_2 C_2 q_2$  and  $q_1 \rho_{21} q_2$ ;
- $w_{23}(\omega_2, \omega_3) > 0$  and  $\omega_2 C_2 q_2$  together imply the existence of a  $q_3$  such that  $\omega_3 C_3 q_3$  and  $q_2 \rho_{23} q_3$ .

On the other hand,  $S_1 (\rho_{12} \bullet \rho_{23})^S S_3$  iff there exists a weighted function  $w$  over  $\Omega_1 \times \Omega_3$  projecting to  $\pi_1$  and  $\pi_3$  and such that:  $w(\omega_1, \omega_3) > 0$  and  $\omega_1 C_1 q_1$  together imply the existence of a  $q_3$  such that  $\omega_3 C_3 q_3$  and  $q_1 (\rho_{12} \bullet \rho_{23}) q_3$ .

We thus construct the following function  $w$  defined over  $\Omega_1 \times \Omega_3$ :

$$w(\omega_1, \omega_3) = \sum_{\omega_2 \in \Omega_2} w_{12}(\omega_1, \omega_2) \cdot w_{23}(\omega_2, \omega_3) \quad (22)$$

To show that  $S_1 (\rho_{12} \bullet \rho_{23})^S S_3$ , we have to prove the following regarding  $w$ :

- if  $w(\omega_1, \omega_3) > 0$  and  $\omega_1 C_1 q_1$  hold, then we can find  $q_3$  such that  $\omega_3 C_3 q_3$  and  $q_1 (\rho_{12} \bullet \rho_{23}) q_3$ . To show this, note that if  $w(\omega_1, \omega_3) > 0$  then, by (22), we can find an  $\omega_2$  such that  $w_{12}(\omega_1, \omega_2) \cdot w_{23}(\omega_2, \omega_3) > 0$ . Since  $w_{12}(\omega_1, \omega_2) > 0$ , there exists some  $q_2$  such that  $\omega_2 C_2 q_2$  and  $q_1 \rho_{12} q_2$ . Since  $w_{23}(\omega_2, \omega_3) > 0$ , there exists some  $q_3$  such that  $\omega_3 C_3 q_3$  and  $q_2 \rho_{23} q_3$ . Now, we have  $q_1 \rho_{12} q_2$  and  $q_2 \rho_{23} q_3$ , which implies  $q_1 (\rho_{12} \bullet \rho_{23}) q_3$ ;
- $w$  projects to  $\pi_3$ :

$$\begin{aligned} \sum_{\omega_1} w(\omega_1, \omega_3) &= \sum_{\omega_1} \sum_{\omega_2} w_{12}(\omega_1, \omega_2) \cdot w_{23}(\omega_2, \omega_3) \\ \text{by Fubini} &= \sum_{\omega_2} \sum_{\omega_1} w_{12}(\omega_1, \omega_2) \cdot w_{23}(\omega_2, \omega_3) \\ &= \sum_{\omega_2} w_{23}(\omega_2, \omega_3) \underbrace{\sum_{\omega_1} w_{12}(\omega_1, \omega_2)}_{=1} \\ &= \pi_3(\omega_3) \end{aligned}$$

- $w$  projects to  $\pi_1$ : this is proved similarly.

Therefore,  $S_1 (\rho_{12}^S \bullet \rho_{23}^S) S_3$  iff  $S_1 (\rho_{12} \bullet \rho_{23})^S S_3$ .

## B Proofs regarding MMDPs

### B.1 Proof of Lemma 4

**Proof** Set  $M' =_{\text{def}} M'_1 \times M'_2$  and  $M =_{\text{def}} M_1 \times M_2$ . Define the relation  $\leq$  between  $R'$  and  $R$  by:  $r' \leq r$  iff  $r'_1 \leq_1 r_1$  and  $r'_2 \leq_2 r_2$ . Let us prove that  $\leq$  is a simulation.

Let  $r'$  be such that  $r' \xrightarrow{\alpha}_{M'} S'$  for some consistent  $S'$ . Then,  $r' = r'_1 \sqcup r'_2$  and  $S' = S'_1 \times S'_2$ . By definition of the parallel composition, we have  $r'_i \xrightarrow{\alpha}_{M'_i} S'_i$  for  $i = 1, 2$ . Since  $r'_i \leq r_i$ , we derive the existence (and uniqueness) of consistent systems  $S_i, i = 1, 2$  such that  $r_i \xrightarrow{\alpha}_{M_i} S_i$ . Since  $r = r_1 \sqcup r_2$  we have  $r_1 \bowtie r_2$  and, thus, by definition of the parallel composition, we deduce  $r \xrightarrow{\alpha}_M S_1 \times S_2$ .

It remains to show that  $S_1 \times S_2$  is consistent. To prove this, remember that  $S' = S'_1 \times S'_2$  is consistent. Thus, there exist compatible  $r'_1$  and  $r'_2$  such that  $S'_i \rightsquigarrow r'_i, i = 1, 2$ . By definition of the simulations  $\leq_i$ , we deduce that  $S_i \rightsquigarrow r_i, i = 1, 2$ , which shows that  $S_1 \times S_2$  is consistent.

## C Proofs regarding Probabilistic Automata

### C.1 Proof of Theorem 1

Defining simulation relations for PA requires lifting relations, from states to distributions over states. The formal definition for this lifting, as given in Section 4.1 of [22], corresponds to our Definition 5, when restricted to purely probabilistic mixed systems.

The same holds for the strong simulation relation defined in Section 4.2 of the same reference: it is verbatim our Definition 8, when restricted to purely probabilistic mixed systems. This proves the part of Theorem 1 regarding simulation.

We move to parallel composition, for which the reader is referred to [19], Section 3. For  $P_1 = (\Sigma, Q_1, q_{0,1}, \rightarrow_1)$  and  $P_2 = (\Sigma, Q_2, q_{0,2}, \rightarrow_2)$  two PA, their parallel composition is  $P = P_1 \times P_2 = (\Sigma, Q_1 \times Q_2, (q_{0,1}, q_{0,2}), \rightarrow)$ , where

$$(q_1, q_2) \xrightarrow{\alpha} \pi_1 \otimes \pi_2 \quad \text{iff} \quad q_i \xrightarrow{\alpha}_i \pi_i \text{ for } i = 1, 2 \quad (23)$$

So, on one hand we consider the MMDP  $M_P$ . On the other hand, we consider the parallel composition of the mappings  $M_{P_1}$  and  $M_{P_2}$ , that is  $M = M_{P_1} \times M_{P_2} = (\Sigma, \{\xi_1, \xi_2\}, (q_{0,1}, q_{0,2}), \rightarrow_{12})$ , so that the state space is the domain

of the pair  $(\xi_1, \xi_2)$ , namely  $Q_1 \times Q_2$ , and, since there is no shared variable between the two MMDP, the transition relation  $\rightarrow_{12}$  is given by:

$$(q_1, q_2) \xrightarrow{\alpha}_{12} S_1 \times S_2 \quad \text{iff} \quad q_i \xrightarrow{\alpha}_i S_i \text{ for } i = 1, 2 \quad (24)$$

We thus need to show that

$$M_P \text{ and } M \text{ are simulation equivalent.} \quad (25)$$

We will actually show that the identity relation between the two state spaces (both are equal to  $Q_1 \times Q_2$ ) is a simulation relation in both directions.

Observe first that (23) and (24) differ in that the former involves a non-deterministic transition relation, whereas the latter involves a deterministic transition function, mapping states to mixed systems.

Pick  $(q_1, q_2) \in Q_1 \times Q_2$  and consider a transition for  $M_P$ :

$$(q_1, q_2) \xrightarrow{\alpha}_{M_P} S = ((\Omega, \Pi), \xi, (q_{0,1}, q_{0,2}), C)$$

where we have, for  $S$ :

- $\Omega$  is the product of  $n_1$  copies of  $Q_1$  and  $n_2$  copies of  $Q_2$ , where, for  $i = 1, 2$ ,  $n_i$  is the cardinality of the set  $\{\pi_i \mid (q_i, \alpha, \pi_i) \in \rightarrow_i\}$ , so that  $\omega$  identifies  $n_1 \times n_2$ -tuple of states:  $\omega = (q_{11}, \dots, q_{1n_1}; q_{21}, \dots, q_{2n_2})$ ;
- $\Pi$  is the product of all probabilities belonging to set

$$\{\pi_1 \otimes \pi_2 \mid (q_i, \alpha, \pi_i) \in \rightarrow_i\}$$

- $\xi$  has domain  $Q_1 \times Q_2$ ;
- $(\omega, (q_1, q_2)) \in C$  if and only if

$$(q_1, q_2) \in \{(q_{1i_1}, q_{2i_2}) \mid i_1 \in \{1, \dots, n_1\} \text{ and } i_2 \in \{1, \dots, n_2\}\}.$$

Next, pick  $(q_1, q_2) \in Q_1 \times Q_2$  and consider a transition for  $M$ , see (24). We need to detail what  $S_1 \times S_2 = ((\Omega', \Pi'), \xi', (q'_{0,1}, q'_{0,2}), C')$  is. We have, for  $S_1 \times S_2$ :

- $\Omega'$  is still the product of  $n_1$  copies of  $Q_1$  and  $n_2$  copies of  $Q_2$ ;
- $\Pi'$  is the product  $\Pi_1 \otimes \Pi_2$ , where  $\Pi_i$  is the product of all probabilities belonging to set  $\{\pi_i \mid (q_i, \alpha, \pi_i) \in \rightarrow_i\}$ ;
- $\xi'$  has domain  $Q_1 \times Q_2$ ;

- $(\omega, (q_1, q_2)) \in C'$  if and only if

$$(q_1, q_2) \in \{(q_{1i_1}, q_{2i_2}) \mid i_1 \in \{1, \dots, n_1\} \text{ and } i_2 \in \{1, \dots, n_2\}\}.$$

By associativity of  $\otimes$ ,  $\Pi' = \Pi$ , whereas other items for  $S$  on the one hand and other items for  $S_1 \times S_2$  on the other hand, are syntactically identical. Thus (25) follows.

## D Proofs regarding Mixed Interfaces

### D.1 Proof of Lemma 5

#### Proof

- We remove from  $\mathcal{C}$  inconsistent states  $q$ ;
  - if  $q \not\stackrel{\alpha}{\rightarrow}_{\mathcal{C}}$  and  $q \stackrel{\alpha}{\rightarrow}_{\mathcal{C}} \mathcal{S}^\square$  then  $q$  cannot be involved in a simulation relation allowing to state that  $M$  is a model of  $\mathcal{C}$  because of (5) in the definition of the model relation.
  - if  $q \stackrel{\alpha}{\rightarrow}_{\mathcal{C}}$  and  $q \not\stackrel{\alpha}{\rightarrow}_{\mathcal{C}} \mathcal{S}^\diamond$  but  $\mathcal{S}^\square \cap \mathcal{S}^\diamond$  contains no consistent system in the sense of Definition 1  $q$  cannot be involved in a simulation relation allowing to state that  $M$  is a model of  $\mathcal{C}$  because of (6) in the definition of the model relation.

As a result,  $q$  plays no role in the semantics of  $\mathcal{C}$  and its lack in  $[\mathcal{C}]$  does not change the semantics.

- We remove from  $\mathcal{C}$  some may transitions to inconsistent states which could not be realized by any model  $\mathcal{C}$ .  $\square$

### D.2 Proof of Lemma 6

**Proof** The reflexivity of  $\preceq$  follows immediately from Definition 14.

Now for the transitivity, assume that  $\mathcal{C}_1 \preceq \mathcal{C}_2$  and  $\mathcal{C}_2 \preceq \mathcal{C}_3$ . with the respective refinement relations  $\preceq_{12} \subseteq Q_1 \times Q_2$  and  $\preceq_{23} \subseteq Q_2 \times Q_3$ .

Define now using notation (3):

$$\preceq_{13} = \preceq_{12} \bullet \preceq_{23} . \quad (26)$$

Let  $q_1$  and  $q_3$  such that  $q_1 \preceq_{13} q_3$ . By 26, we have  $q_1 \preceq_{12} q_2$  and  $q_2 \preceq_{23} q_3$  for some  $q_2$ . Thus, for all  $\alpha$  such that  $q_1 \stackrel{\alpha}{\rightarrow}_1 \mathcal{S}_1^\diamond$ , we have  $q_2 \stackrel{\alpha}{\rightarrow}_2 \mathcal{S}_2^\diamond$  and

$\mathcal{S}_1^\diamond \subseteq^{\preceq_{12}} \mathcal{S}_2^\diamond$ . Moreover,  $q_3 \xrightarrow{\alpha}_3 \mathcal{S}_3^\diamond$  and  $\mathcal{S}_2^\diamond \subseteq^{\preceq_{23}} \mathcal{S}_3^\diamond$ . By Lemma 3, we have  $\mathcal{S}_1^\diamond \subseteq^{\preceq_{13}} \mathcal{S}_3^\diamond$ .

Similarly for must transitions, for all  $\alpha$  such that  $q_3 \xrightarrow{\alpha}_3 \mathcal{S}_3^\square$ , we have  $q_2 \xrightarrow{\alpha}_2 \mathcal{S}_2^\square$  and  $\mathcal{S}_2^\square \subseteq^{\preceq_{23}} \mathcal{S}_3^\square$ . Moreover,  $q_1 \xrightarrow{\alpha}_1 \mathcal{S}_1^\square$  and  $\mathcal{S}_1^\square \subseteq^{\preceq_{12}} \mathcal{S}_2^\square$ . By Lemma 3, we have  $\mathcal{S}_1^\square \subseteq^{\preceq_{13}} \mathcal{S}_3^\square$ . As a result, we have  $\mathcal{C}_1 \preceq \mathcal{C}_3$ .  $\square$

### D.3 Proof of Theorem 2

**Proof** Assume  $\mathcal{C}_1 \preceq \mathcal{C}_2$  and consider the refinement relation  $\preceq \subseteq Q_1 \times Q_2$ . Let  $M$  be a model of  $\mathcal{C}_1$  and let  $(r, q_1) \in R \times Q_1$  satisfy  $r \models_1 q_1$ . Focus first on the *may* transition relation. By (5) applied to  $\models_1$ , for any  $\alpha$  such that  $r \xrightarrow{\alpha}_M S_M$

$$q_1 \xrightarrow{\alpha}_{\mathcal{C}_1} \mathcal{S}_1^\diamond \text{ and } S_M \in \models^1 \mathcal{S}_1^\diamond \text{ both hold.} \quad (27)$$

Let  $q_2 \in Q_2$  be such that  $q_1 \preceq q_2$ . Using the first condition of (8), we get

$$q_2 \xrightarrow{\alpha}_2 \mathcal{S}_2^\diamond \text{ and } \mathcal{S}_1^\diamond \subseteq^{\preceq} \mathcal{S}_2^\diamond \quad (28)$$

Define the relation:  $r \models_2 q_2 \Leftrightarrow \exists q_1 \in Q_1 : r \models_1 q_1 \text{ and } q_1 \preceq q_2$ . Using notation (3), we have

$$\models_2 = \models_1 \bullet \preceq. \quad (29)$$

Now, let  $q_2$  be such that  $r \models_2 q_2$ . Combining (27) and (28) yields

$$q_2 \xrightarrow{\alpha}_2 \mathcal{S}_2^\diamond \text{ and } S_M \in \models^1 \mathcal{S}_1^\diamond \subseteq^{\preceq} \mathcal{S}_2^\diamond \quad (30)$$

which, by (29) and Lemma 3, yields  $S_M \in \models^2 \mathcal{S}_2^\diamond$ . Combining this and (30) shows that  $r \models_2 q_2$ . Focus next on the *must* transition relation. Since  $M$  is a model of  $\mathcal{C}_1$ , (6) applied to  $\models_1$  yields the existence of  $S_M \in \models^1 \mathcal{S}_1^\square \subseteq^{\preceq} \mathcal{S}_2^\square$  such that  $r \xrightarrow{\alpha}_M S_M$ , which implies that (6) holds for  $\models_2$  by the same reasoning as before.

### D.4 Proof of Theorem 3

**Proof** Using Theorem 2, the previous statements follow from  $\mathcal{C}_1 \wedge \mathcal{C}_2 \preceq \mathcal{C}_i$  for  $i = 1, 2$ . Take the first projection as the candidate refinement relation, namely:  $(q_1, q_2) \preceq q_1$  for  $(q_1, q_2)$  and  $q_1$  reachable from their respective initial states. Using the four rules of Definition 15, we get  $\mathcal{C}_1 \triangle \mathcal{C}_2 \preceq \mathcal{C}_1$ , and thus  $\mathcal{C}_1 \wedge \mathcal{C}_2 = [\mathcal{C}_1 \triangle \mathcal{C}_2] \preceq \mathcal{C}_1$  since  $\mathcal{C}_1$  possesses no inconsistent state. The same holds for  $\mathcal{C}_2$  by symmetry.

### D.5 Proof of Theorem 4

**Proof** We successively prove the two statements. Regarding Statement 1), the same proof holds as for associativity and commutativity of the conjunction. Regarding Statement 2), Property (10) is an immediate consequence of Definitions 9, 12 and 16. Focus next on (11). Assume

$$\begin{aligned} (q, q_2) &\xrightarrow{\alpha}_{\mathcal{C} \otimes \mathcal{C}_2} \mathcal{S}^\diamond \times \mathcal{S}_2^\diamond \\ (q, q_1) &\xrightarrow{\alpha}_{\mathcal{C} \otimes \mathcal{C}_1} \mathcal{S}^\square \times \mathcal{S}_1^\square \end{aligned}$$

By the rules of the composition, we deduce that the premises of (8) holds, so we can apply rule (8) since  $\mathcal{C}_1 \preceq \mathcal{C}_2$ , which yields

$$\begin{aligned} q_1 &\xrightarrow{\alpha}_1 \mathcal{S}_1^\diamond \text{ and } \mathcal{S}_2^\diamond \subseteq^{\preceq} \mathcal{S}_1^\diamond \\ q_2 &\xrightarrow{\alpha}_2 \mathcal{S}_2^\square \text{ and } \mathcal{S}_2^\square \subseteq^{\preceq} \mathcal{S}_1^\square \end{aligned}$$

which implies

$$\begin{aligned} (q, q_1) &\xrightarrow{\alpha}_{\mathcal{C} \otimes \mathcal{C}_1} \mathcal{S}^\diamond \times \mathcal{S}_1^\diamond \text{ and } \mathcal{S}^\diamond \times \mathcal{S}_2^\diamond \subseteq^{\preceq'} \mathcal{S}^\diamond \times \mathcal{S}_1^\diamond \\ (q, q_2) &\xrightarrow{\alpha}_{\mathcal{C} \otimes \mathcal{C}_2} \mathcal{S}^\square \times \mathcal{S}_2^\square \text{ and } \mathcal{S}^\square \times \mathcal{S}_2^\square \subseteq^{\preceq'} \mathcal{S}^\square \times \mathcal{S}_1^\square \end{aligned}$$

where  $\preceq'$  is defined by  $(q, q_2) \preceq' (q, q_1)$  iff  $q_2 \preceq q_1$ . This shows that  $\preceq'$  is a refinement.

## E Proofs regarding CMC

### E.1 Proof of Lemma 7

**Proof** Let us detail the semantics of mixed system  $S$ , see Definition 1. First, we draw  $r' \in \Omega = R$  according to the probability  $\Pi(r, \cdot)$ : this corresponds to the drawing of the next state in Markov Chain  $\mathbf{P}$ . Second, we nondeterministically select  $(r'', v(r''))$  in the state space  $R \times 2^A$  of  $S$  so that  $(r', r'', v(r'')) \in C$ . The only solution is  $(r', r', v(r'))$ , which provides us with the second component  $v(r')$  of the state. The two semantics coincide.  $\square$

### E.2 Proof of Lemma 8

**Proof** To the satisfaction relation  $\rho \subseteq R \times Q$  following Definition 19, we associate the relation  $\models_\rho \subseteq (R \times 2^{A_P}) \times (Q \times 2^{A_S})$ , defined by

$$(r, \bar{r}) \models_\rho (q, \bar{q}) \quad \text{iff} \quad \begin{cases} r \rho q \\ \bar{r} = v_P(r) \\ \bar{q} = v_S(q) \\ \bar{r} \downarrow_{A_S} = \bar{q} \end{cases} \quad (31)$$

Observe that, vice versa, we recover  $\rho$  from  $\models_\rho$  by keeping only the first condition of it. We have to prove that

$$\rho \text{ is a satisfaction relation for CMC if and only if } \models_\rho \text{ is a satisfaction relation for Mixed Interface.} \quad (32)$$

**We first prove the “only if” part of (32)** Let  $(r, q)$  satisfy  $r \rho q$ . By (31),  $\models_\rho$  is a relation between the states of MMDP  $M_{\mathbf{P}}$  and Mixed Interface  $\mathcal{C}_{\mathbf{S}}$ . With reference to Definition 12, to show that  $\models_\rho$  is a satisfaction relation, it is enough to show that only *may* transitions of  $\mathcal{C}_{\mathbf{S}}$  are allowed for  $M_{\mathbf{P}}$ —the condition related to the *must* transitions is vacuously satisfied.

Let  $(r, \bar{r}) \models_\rho (q, \bar{q})$  and  $(r, \bar{r}) \rightarrow_{M_{\mathbf{P}}} S_{\mathbf{P}}$ , where  $S_{\mathbf{P}} = ((\Omega_{\mathbf{P}}, \pi_{\mathbf{P}}), X_{\mathbf{P}}, C_{\mathbf{P}})$  is defined by applying (13) to  $M_{\mathbf{P}}$ . We must prove that the latter transition is allowed by the *may* transitions of Mixed Interface  $\mathcal{C}_{\mathbf{S}}$ , i.e., the target mixed system  $S_{\mathbf{P}}$  satisfies condition (5), meaning that

$$(q, \bar{q}) \dashrightarrow_{\mathcal{C}_{\mathbf{S}}} S_{\mathbf{S}} \text{ and there exists } S_{\mathbf{S}} \in \mathcal{S}_{\mathbf{S}} \text{ such that } S_{\mathbf{P}} \models_\rho^S S_{\mathbf{S}}. \quad (33)$$

To construct a mixed system  $S_{\mathbf{S}}$  satisfying (33), we start from  $r \rho q$ , which provides us with a transition sub-probability  $\Delta$  satisfying the conditions 2) of Definition 19. We then consider the mixed system  $S_{\mathbf{S}} = ((\Omega_{\mathbf{S}}, \pi_{\mathbf{S}}), X_{\mathbf{S}}, C_{\mathbf{S}})$ , where:

- $\Omega_{\mathbf{S}} = Q$ ;
- $\pi_{\mathbf{S}} = \Pi(r, \cdot) \Delta$ , which belongs to  $\varphi(q)$  by Definition 19;
- $C_{\mathbf{S}} \subseteq \Omega_{\mathbf{S}} \times (Q \times 2^{A_{\mathbf{S}}})$  consists of the triples  $(q', (q', \bar{q}'))$ , where  $q'$  ranges over  $Q$ ,  $\bar{q}' = v_{\mathbf{S}}(q')$ , and  $v_{\mathbf{S}}$  relates to  $v_{\mathbf{P}}$  by  $v_{\mathbf{S}}(q') = v_{\mathbf{P}}(r') \downarrow_{A_{\mathbf{S}}}$ . By Condition 1) of Definition 19, we get  $v_{\mathbf{P}}(r') \downarrow_{A_{\mathbf{S}}} \in V_{\mathbf{S}}(q')$ .

Let us prove that the so constructed mixed system  $S_{\mathbf{S}}$  satisfies  $S_{\mathbf{P}} \models_\rho^S S_{\mathbf{S}}$ . We must find a weighting function  $w : R \times Q \rightarrow [0, 1]$  satisfying the conditions of Definition 5. We claim that the wanted weighting function is

$$w(r', q') = \Pi(r, r') \Delta(r', q').$$

We now prove that Conditions 1) and 2) of Definition 5 are satisfied by  $w$ . We begin with Condition 2). We have  $\sum_{r'} w(r', q') = \sum_{r'} \Pi(r, r') \Delta(r', q') = \Pi(r, \cdot) \Delta(q')$  using (12). On the other hand,

$$\sum_{q'} w(r', q') = \sum_{q'} \Pi(r, r') \Delta(r', q') = \Pi(r, r') \sum_q \Delta(r', q') = \Pi(r, r')$$



by Condition 2a of Definition 19.

Focus next on Condition 1) of Definition 5. Pick  $(r', q'; (r', v_{\mathbf{P}}(r')))$  such that  $w(r', q') > 0$ , which implies  $\Delta(r', q') > 0$ . Then by Condition 2c of Definition 19,  $r' \rho q'$  holds. On the other hand, we have  $(q', q', v_{\mathbf{P}}(r') \downarrow_{A_{\mathbf{S}}}) \in C$ , showing that  $(q', v_{\mathbf{P}}(r') \downarrow_{A_{\mathbf{S}}})$  is the state of  $S_{\mathbf{S}}$  wanted in Condition 1) of Definition 5. Hence, the so constructed mixed system  $S_{\mathbf{S}}$  satisfies  $S_{\mathbf{P}} \models_{\rho}^{\mathcal{S}} S_{\mathbf{S}}$ . This proves the “only if” part of (32).

**We now move to the “if” part of (32)** Let  $(r, \bar{r}) \models_{\rho} (q, \bar{q})$ . Then by the definition (31) of relation  $\models_{\rho}$ , we deduce that  $r \rho q$  holds and we must prove that  $\rho$  is a satisfaction relation for CMC. To this end we use the fact that  $\models_{\rho}$  is a satisfaction relation for Mixed Interface, namely: if  $(r, \bar{r}) \longrightarrow_{M_{\mathbf{P}}} S_{\mathbf{P}}$ , then there exists  $S_{\mathbf{S}} \in \mathcal{S}_{\mathbf{S}}$  such that  $S_{\mathbf{P}} \models_{\rho}^{\mathcal{S}} S_{\mathbf{S}}$ . The target system  $S_{\mathbf{S}}$  takes the form  $S_{\mathbf{S}} = ((\Omega, \pi), X, C)$ , where:

- $\Omega = Q$ ;
- $\pi(q') = \sum_{r' \in R} w(r', q')$ , where  $w(r', q')$  is the weighting function associated to the lifting of relation  $\models_{\rho}$ ;
- $C \subseteq \Omega \times (Q \times 2^{A_{\mathbf{S}}})$  consists of the triples of the form  $(q', q', v_{\mathbf{P}}(r') \downarrow_{A_{\mathbf{S}}})$ , where  $r'$  ranges over  $R$  and  $r' \rho q'$ .

In proving that the relation  $\rho$  inferred from  $\models_{\rho}$  is a satisfaction relation for CMC, we must find the  $\Delta$  occurring in Definition 19. We define it as

$$\Delta(r', q') = \begin{cases} \frac{w(r', q')}{\Pi(r, r')} & \text{if } \Pi(r, r') > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The conditions of Definition 19 are satisfied. This finishes the proof of the “if” part and the lemma is proved.

### E.3 Proof of Lemma 9

**Proof** The proof follows the same lines as for Lemma 8. To the refinement relation  $\rho \subseteq Q_2 \times Q_1$  following Definition 20, we associate the relation

$$\preceq_{\rho} \subseteq (Q_2 \times 2^{A_2}) \times (Q_1 \times 2^{A_1})$$

defined by

$$(q_2, \bar{q}_2) \preceq_\rho (q_1, \bar{q}_1) \quad \text{iff} \quad \begin{cases} q_2 \rho q_1 \\ \bar{q}_2 = v_2(q_2) \\ \bar{q}_1 = v_1(q_1) \\ \bar{q}_1 = \bar{q}_2 \downarrow_{A_1} \end{cases} \quad (34)$$

By (34),  $\preceq_\rho$  is a relation between the states of Mixed Interface  $\mathcal{C}_{\mathbf{S}_2}$  and  $\mathcal{C}_{\mathbf{S}_1}$ . Observe that, vice versa, we recover  $\rho$  from  $\preceq_\rho$  by keeping only the first condition of it. We have to prove that

$$\begin{aligned} \rho &\text{ is a weak refinement relation for CMC if and only if} \\ \preceq_\rho &\text{ is a refinement relation for Mixed Interfaces.} \end{aligned} \quad (35)$$

**We first prove the “only if” part of (35)** Let  $(q_2, q_1)$  satisfy  $q_2 \rho q_1$ . With reference to Definition 14, to show that  $\preceq_\rho$  is a refinement relation, it is enough to show the first condition of (8)—the condition related to the *must* transitions is vacuously satisfied.

From  $(q_2, \bar{q}_2) \preceq_\rho (q_1, \bar{q}_1)$  and  $q_2 \dashrightarrow_2 \mathcal{S}_2^\diamond$ , we have to deduce

$$q_1 \dashrightarrow_1 \mathcal{S}_1^\diamond \text{ and } \mathcal{S}_2^\diamond \subseteq \preceq_\rho \mathcal{S}_1^\diamond,$$

which translates as

$$\begin{aligned} &\text{for every } S_2 \in \mathcal{S}_2^\diamond \text{ we can find} \\ &S_1 \in \mathcal{S}_1^\diamond \text{ such that } S_2 \preceq_\rho^S S_1. \end{aligned} \quad (36)$$

Let  $S_{2,v}$  have the form  $S_{2,v} = ((\Omega_2, \pi_2), X_2, C_{2,v})$  following (14). To construct a mixed system  $S_1$  satisfying (36) we start from  $q_2 \rho q_1$ , which provides us with a transition sub-probability  $\Delta$  satisfying the Conditions 2) of Definition 20. We then consider the mixed system  $S_1 = ((\Omega_1, \pi_1), X_1, C_1)$ , where:

- $\Omega_1 = Q_1$ ;
- $\pi_1 = \pi_2 \Delta$ , which belongs to  $\varphi_1(q_1)$  by Definition 20;
- $C_1 \subseteq \Omega_1 \times (Q_1 \times 2^{A_1})$  consists of the triples of the form  $(q'_1, q'_1, v(q'_2) \downarrow_{A_1})$ , where  $v$  is the one arising in the definition of  $S_{2,v}$  and  $q'_1$  ranges over  $Q_1$ . By Condition 1) of Definition 20, we get  $v(q'_2) \downarrow_{A_1} \subseteq V_1(q'_1)$ .

Let us prove that the mixed system  $S_1$  satisfies  $S_{2,v} \preceq_\rho^S S_1$ . We must find a weighting function  $w : Q_2 \times Q_1 \rightarrow [0, 1]$  satisfying the conditions of Definition 5. We claim that the wanted weighting function is

$$w(q'_2, q'_1) = \pi_2(q'_2) \Delta(q'_2, q'_1).$$

Let us prove that Conditions 1) and 2) of Definition 5 are satisfied by  $w$ . We begin with Condition 2). We have  $\sum_{q'_2} w(q'_2, q'_1) = \sum_{q'_2} \pi_2(q'_2) \Delta(q'_2, q'_1) = \pi_1(q'_1)$  by definition of  $\pi_1$ . On the other hand,

$$\sum_{q'_1} w(q'_2, q'_1) = \sum_{q'_1} \pi_2(q'_2) \Delta(q'_2, q'_1) = \pi_2(q'_2)$$

by Condition 2a of Definition 20.

Focus next on Condition 1) of Definition 5. Pick  $(q'_2, q'_1; (q'_2, v(q'_2)))$  such that  $w(q'_2, q'_1) > 0$ , which implies  $\Delta(q'_2, q'_1) > 0$ . Then by Condition 2c of Definition 20,  $q'_2 \rho q'_1$  holds. On the other hand, we have  $(q'_1, q'_1, v(q'_2)_{\downarrow A_1}) \in C_1$ , showing that  $(q'_1, v(q'_2)_{\downarrow A_1})$  is the state of  $S_1$  wanted by Condition 1) of Definition 5. Hence, the so constructed mixed system  $S_1$  satisfies  $S_{2,v} \preceq_\rho^S S_1$ . This proves the “only if” part of (35).

**We next move to the “if” part of (35)** Let  $(q_2, \bar{q}_2) \preceq_\rho (q_1, \bar{q}_1)$ . Then by the definition (34) of relation  $\preceq_\rho$ , we deduce that  $q_2 \rho q_1$  holds and we must prove that  $\rho$  is a weak refinement relation for CMC. To this end we use the fact that  $\preceq_\rho$  is a modal refinement relation for Mixed Interface, namely: if  $q_2 \dashrightarrow_2 S_2^\diamond$ , then  $q_1 \dashrightarrow_1 S_1^\diamond$  and  $S_2^\diamond \subseteq \preceq_\rho S_1^\diamond$ . That is, for any  $S_{2,v_2} \in S_2^\diamond$ , of the form  $S_{2,v_2} = ((\Omega_2, \pi_2), X_2, C_{2,v_2})$  following (14), there exists  $S_{1,v_1} = ((\Omega_1, \pi_1), X_1, C_{1,v_1}) \in S_1^\diamond$  such that

$$S_{2,v_2} \preceq_\rho^S S_{1,v_1}. \quad (37)$$

Condition (37) and Definition 5 of the lifting of a relation together imply the existence of a weighting function  $w(q'_2, q'_1)$  satisfying the following conditions:

1. For every triple  $(q'_2, q'_1; (q'_2, v_2(q'_2)))$  such that

$$w(q'_2, q'_1) > 0 \text{ and } (q'_2, (q'_2, v_2(q'_2))) \in C_{2,v_2},$$

there exists  $(q'_1, v_1(q'_1))$  such that

$$(q'_1, (q'_1, v_1(q'_1))) \in C_{1,v_1} \text{ and } (q'_2, \bar{q}'_2) \preceq_\rho (q'_1, \bar{q}'_1).$$

2.  $\sum_{q'_2} w(q'_2, q'_1) = \pi_1(q'_1)$  and  $\sum_{q'_1} w(q'_2, q'_1) = \pi_2(q'_2)$ .

In proving that the relation  $\rho$  inferred from  $\preceq_\rho$  is a weak refinement relation for CMC, we must find the  $\Delta$  occurring in Definition 20. We define it as

$$\Delta(q'_2, q'_1) = \begin{cases} \frac{w(q'_2, q'_1)}{\pi_2(q'_2)} & \text{if } \pi_2(q'_2) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The conditions of Definition 20 are satisfied. This finishes the proof of the “if” part and the lemma is proved.



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