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Exponential Synchronization of Nonlinear Oscillators Under Sampled-Data Coupling

Jijju Thomas[1], Erik Steur[2], Christophe Fiter[3], Laurentiu Hetel[4], and Nathan van de Wouw[5].

***Abstract*— This paper presents a novel approach towards synchronization analysis of nonlinear oscillatory systems, bidirectionally coupled via a networked communication channel. The system under consideration is a two-agent nonlinear system, under the constraint that information is transmitted between the two systems using a sampled-data communication strategy that could be periodic or aperiodic. The networked system dynamics is remodelled as a feedback-interconnection of a continuous-time system, and an operator that accounts for the communication constraints. By studying the properties of this feedback-interconnection in the framework of dissipativity theory, we provide a novel criterion that guarantees exponential synchronization. The provided criterion also aids in deciding the trade-off between a bound on the sampling intervals, the coupling gain, and the desired transient rate of synchronization. Finally, the theoretical results are illustrated using a two-agent Fitzhugh-Nagumo system.**

I. INTRODUCTION

In many natural and engineering systems, the phenomenon of synchronization has been investigated by researchers and scientists from various fields. Typical examples of synchronization include flashing fireflies, firing neurons, cooperative control of multi-agent systems, etc. [1]. In control theory, synchronization of networked nonlinear oscillators is a topic that has specifically garnered attention owing to its significance in neural processes, communication systems, electronic circuits, etc. [2], [3]. Such networked scenarios often result in individual sub-systems interacting bidirectionally. In comparison to weaker definitions of synchronization, i.e., in the form of consensus or synchronization to a set of equilibria, synchronization in the sense of asymptotic matching of time-varying (oscillatory) solutions, has been proven to be a more complex and generic problem [4]. For nonlinear oscillators with networked bidirectional coupling, such synchronization problems become more complex due to uncertainties introduced via networked communication channels. For example, individual systems could be connected via sampled-data coupling, thereby increasing the complexity of the associated synchronization problem [5], [6], [7].

Synchronization problems in sampled-data systems have been studied in recent years, and different approaches have been proposed to study the relation between sampling period, coupling strength, and synchronization properties [8], [9]. In existing results, it is typically considered that individual systems have the same sampling frequency, which is either constant or time-varying [7], [6]. However, in realistic settings, individual systems usually transmit information at different frequencies over a network, depending upon the communication channel, data traffic, etc. In this work, we consider a bidirectionally coupled, sampled-data, two-agent nonlinear system, wherein individual systems transmit information over a networked communication channel, asynchronously at possibly different (time-varying and uncertain) sampling intervals.

The main contribution of this paper is a novel approach towards synchronization analysis of nonlinear oscillatory systems, bidirectionally coupled via aperiodic sampled-data communication. By using tools based on input-output methods and dissipativity theory previously used to analyse stability of sampled-data systems [10], we provide a novel criterion that guarantees exponential synchronization of a two-agent nonlinear system, bidirectionally coupled via sampled-data synchronizing laws. In practical scenarios, it is desirable to have a certain measure of system performance. In synchronization problems, this implies achieving a specific transient rate of synchronization, which also depends on the oscillator dynamics. The result provided in this paper also takes the rate of synchronization into account, and aids in computing the trade-offs between coupling gain, rate of synchronization, and the maximum bound on the (time-varying, uncertain) sampling intervals.

The remainder of this paper has been structured as follows. In Section II, we introduce the problem setting under consideration, which consists of a two-agent nonlinear system, bidirectionally coupled via sampled-data coupling laws. In Section II-A, we introduce assumptions and existing results guaranteeing synchronization in the absence of sampled-data effects. In Section III, we remodel the system under consideration as a feedback-interconnection between a continuous-time system, and an operator that captures the sampled-

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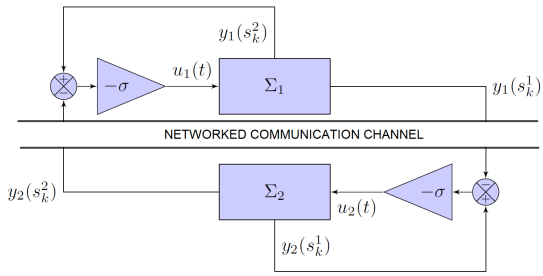

Fig. 1: Bidirectionally coupled agents Σ_1 and Σ_2 under (asynchronous) sampled-data transmission.

data/network effects. In Section IV, we provide the main result of this paper, where we use dissipativity theory based tools to study the feedback interconnection introduced in Section III, and provide a novel criterion that guarantees exponential synchronization. In Section V, we provide a numerical example illustrating the application of our result.

Notation: $\mathbb{R}$ is the set of all real numbers, implying $\mathbb{R}^n$ is the set of all n-dimensional real vectors. The set of all natural numbers is denoted by $\mathbb{N}$. For an integer $n \geq 1$, I_n denotes the $n \times n$ unit matrix. The Euclidean norm of a vector $z \in \mathbb{R}^n$ is denoted by $\|z\| = \sqrt{z^T z}$, where z^T denotes the transposition of z.

II. Problem Statement

In this paper, we consider a 2-agent interconnected system as shown in Figure 1. Let the individual sub-system dynamics be given by

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \quad i = 1, 2, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i, y_i \in \mathbb{R}^m$ are the state, input, and output, of the ith system, respectively. The function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector field, and B and C are matrices with appropriate dimensions, with $CB =: b \in \mathbb{R}^{m \times m}$ and b being positive definite and, without loss of generality, diagonal. This implies the system is considered to be of relative degree one. The output of the ith sub-system is transmitted to the jth sub-system only at instants given by the sampling sequence

$$s_{k+1}^i = s_k^i + h_k^i, h_k^i \in [\underline{h}_i, \bar{h}_i], k \in \mathbb{N}, \tag{2}$$

where $0 \leq \underline{h}_i \leq \bar{h}_i$. Without loss of generality, we consider $s_0^i = 0$, $i = 1, 2$.

***Assumption** 1:* The ith sub-system has access to local output information at time instants $t = s_k^j, j \neq i, k \in \mathbb{N}$.

The aforementioned assumption represents scenarios in which individual sub-systems could be synchronously sampling at identical high frequencies, but transmit information asynchronously at different (and time-varying) frequencies, due to constraints imposed by the communication network. The local information, which is not transmitted over the network, can then be considered to be instantaneously available locally, and can hence be used for local feedback, when the output of the other system is received over the network. A second scenario justifying Assumption 1 is that local measurements could be performed in an event-based fashion, as soon as the sampled output of the other system is received. Under Assumption 1, the coupling laws can be expressed as follows:

$$\begin{aligned} u_1(t) &= -\sigma(y_1(s_k^2) - y_2(s_k^2)), \forall t \in [s_k^2, s_{k+1}^2), \\ u_2(t) &= -\sigma(y_2(s_k^1) - y_1(s_k^1)), \forall t \in [s_k^1, s_{k+1}^1), \end{aligned} \tag{3}$$

where $\sigma \in \mathbb{R}_+$ is the constant coupling strength. We care to stress that Assumption 1 is key in making exact synchronization feasible. This is evident when we discuss the role of this assumption in the positive invariance of the synchronization manifold in the next sub-section. In the absence of such an assumption, typically, only approximate synchronization is feasible.

***Definition** 2:* The bidirectionally coupled system given by (1), (3) is said to exponentially synchronize if

$$\|e_x(t)\| \leq \rho e^{-\alpha t} \|e_x(0)\|, \forall t \geq 0, \tag{4}$$

where $\rho, \alpha > 0$, $e_x(0) \in \mathbb{R}^n$, and $e_x(t) := x_1(t) - x_2(t)$ is the state synchronization error.

The goal of this paper is to analyse the exponential synchronization of system (1), (3).

A. Assumptions and Preliminary Results

In this section, in order to establish exponential synchronization of the coupled system (1), (3), in the sense of Definition 2, we introduce a few assumptions.

***Assumption** 3:* The solutions of the closed-loop system (5), (3) are ultimately bounded.

Since the individual systems under consideration are of relative degree one, we know that there exists a well-defined coordinate transformation $z_i = \Phi(x_i)$, $z_i \in \mathbb{R}^{n-m}$, $i = 1, 2$, such that the ith sub-system dynamics are given by

$$\begin{aligned} \dot{z}_i(t) &= q(z_i(t), y_i(t)), \\ \dot{y}_i(t) &= a(z_i(t), y_i(t)) + bu_i(t), i = 1, 2, \end{aligned} \tag{5}$$

where $z_i \in \mathbb{R}^{n-m}$, $u_i, y_i \in \mathbb{R}^m$, $q\colon \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, and $a\colon \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ [4]. In this paper, we will analyse the exponential synchronization of system (5), (3), which implies the exponential synchronization of system (1), (3). The remainder of this paper deals with system (5), (3), and all the results provided for this system, hold for system (1), (3) as well.

First, we introduce a definition of the convergence property, which will be exploited later to ensure certain synchronization properties.

***Definition** 4:* [11] The dynamics $\dot{x} = g(x, u(t))$, with $u(t)$ a bounded time-varying input, are said to be convergent if

- there exists a solution $\bar{x}(t)$ which is bounded for all $t \in \mathbb{R}$,
- $\bar{x}(t)$ is a globally asymptotically stable solution.

The solution $\bar{x}(t)$ is called the steady-state solution. If $\bar{x}(t)$ is in addition exponentially stable, then the system is called *exponentially convergent*.

Next, we consider the following assumption guaranteeing exponential convergence of internal dynamics of (5).

***Assumption** 5:* (Demidovich Condition [12]) There exists a positive definite matrix $P_z \in \mathbb{R}^{(n-m)\times(n-m)}$, such that the internal state dynamics given by $\dot{z}_i(t) = q(z_i(t), y_i(t))$, $i = 1, 2$, satisfies

$$P_z \frac{\partial q}{\partial z_i}(z_i, y_i) + \frac{\partial q^T}{\partial z_i}(z_i, y_i) P_z \leq -\delta I_{n-m}, \tag{6}$$

where $P_z = P_z^T > 0$, and $\delta > 0$.

Assumption 5 implies that the z_i-dynamics, for $i = 1, 2$, are exponentially convergent and establishes detectability-type conditions on individual sub-systems. This implies that on the synchronization manifold, when $y_1(t) = y_2(t) \equiv y_s(t)$, the internal dynamics $\dot{z}_i(t) = q(z_i(t), y_i(t)), i = 1, 2$, converge to a unique, bounded solution only dependent on $y_s(t)$ and not on the (typically different) initial conditions of the sub-systems (see [13], Theorem 2.29).

In the absence of sampling, the corresponding ideal synchronizing input is given by

$$\begin{aligned} u_1^\star(t) &= -\sigma(y_1(t) - y_2(t)), \\ u_2^\star(t) &= -\sigma(y_2(t) - y_1(t)), \forall t \geq 0. \end{aligned} \tag{7}$$

Now, we introduce a preliminary result on the bidirectionally coupled system (5), in the absence of sampling, i.e., with $u_i(t) = u_i^\star(t)$, given by the coupling law in (7).

***Theorem** 6:* [4] Consider system (5) with $u_i(t) = u_i^\star(t)$ given by (7), and let Assumption 3 hold for system (5), with $u_i(t) = u_i^\star(t)$, and let Assumption 5 hold. Then, there exists a constant $\bar{\sigma}$ such that for all $\sigma > \bar{\sigma}$, the bidirectionally coupled system given by (5) with $u_i(t) = u_i^\star(t)$, achieves exponential synchronization.

Under the conditions of Theorem 6, there exists a block-diagonal matrix $P_y = P_y^T > 0$, such that the storage function

$$V(e(t)) = \begin{bmatrix} y_1(t) - y_2(t) \\ z_1(t) - z_2(t) \end{bmatrix}^T \begin{bmatrix} P_y & 0 \\ 0 & P_z \end{bmatrix} \begin{bmatrix} y_1(t) - y_2(t) \\ z_1(t) - z_2(t) \end{bmatrix} \tag{8}$$

characterizes the exponential stability properties of the synchronization manifold. This implies that using such a storage function, it can be shown that $\dot{V} + \alpha V \leq 0$, where $\alpha > 0$ [4]. While the matrix P_z results from Assumption 5, it has previously been shown that with the simple choice of $P_y = I_m$, a candidate storage function of the form given in (8), can be used to characterize the exponential stability properties of the synchronization manifold (see [4], proof of Theorem 1). Later, we will exploit such type of storage function as a candidate storage function to show synchronization for the case with sampled-data coupling.

In the following lemma, we establish positive invariance of the synchronization manifold for system (5), (3), i.e., for the system with sampled-data coupling.

***Lemma** 7:* The synchronization manifold given by

$$\begin{aligned} \mathcal{M} = \big\{ col(z_1, z_2, y_1, y_2) \in \mathbb{R}^{2n} \\ : z_1 = z_2, y_1 = y_2 \big\} \end{aligned} \tag{9}$$

is positively invariant under the bidirectionally coupled system (5), with $u_i(t), i \in \{1, 2\}$ given by (3).

Proof: From (7), we know that on the synchronization manifold, $u_1(t) = u_2(t) = 0$ for all $t \in \mathbb{R}$, since $y_1(t) = y_2(t)$. Consequently, from (5), since on the synchronization manifold $\mathcal{M}$, $y_1(t) = y_2(t)$ and $z_1(t) = z_2(t)$, we have that $\dot{y}_1(t) = \dot{y}_2(t)$ and $\dot{z}_1(t) = \dot{z}_2(t)$. Therefore, we have that $\mathcal{M}$ is positively invariant. ■

Given the positive invariance of the synchronization manifold, we proceed to analyse the exponential synchronization of system (5), (3), which relates to the exponential stability properties of this manifold. In order to do so, in the next section, we will consider the effects of sampling as perturbations to the continuous-time networked system given by (5), with $u_i(t) = u_i^\star(t)$ in (7).

III. System Reformulation

In this section, we remodel the networked system, given by (5), (3), as the feedback interconnection of a system operator $\mathbf{G}$ representing the dynamics of the networked system with continuous-time coupling, i.e., with $u_i(t) = u_i^\star(t)$ given by (7), and an operator $\mathbf{\Delta}$ that captures the error introduced by asynchronous sampling.

Consider an operator $\Delta_i \colon \mathbb{R}^m \mapsto \mathbb{R}^m$, such that for any signal ρ, it follows that

$$(\Delta_i \rho)(t) = -\int_{s_k^i}^{t} \rho(s) ds, \forall t \in [s_k^i, s_{k+1}^i), k \in \mathbb{N}, i = 1, 2. \tag{10}$$

In the following lemma, we will show how the sampled-data synchronizing input (3) is equivalent to the ideal continuous-time synchronizing input (7), subject to a sampling-induced perturbation captured by the operator Δ_i.

***Lemma** 8:* Consider the operator Δ_i given in (10). Then, the sampled-data coupling law given by (3) can be expressed as

$$u_i(t) = u_i^\star(t) + \sigma \Delta_j(\dot{y}_j - \dot{y}_i)(t), i = 1, 2, j \neq i, \tag{11}$$

where $u_i^\star(t)$ is given by (7).

Proof: Recalling the definition of the sampled-data synchronizing input in (3), we have, for $i = 1, 2$,

$$\begin{aligned} u_i(t) &= -\sigma(y_i(s_k^j) - y_j(s_k^j)), \forall t \in [s_k^j, s_{k+1}^j), j \neq i, \\ &= -\sigma(y_i(s_k^j) - y_j(s_k^j) + y_i(t) - y_i(t) \\ &\qquad + y_j(t) - y_j(t)) \\ &= -\sigma\Big((y_i(t) - y_j(t)) + (y_j(t) - y_j(s_k^j)) \\ &\qquad - (y_i(t) - y_i(s_k^j)) \Big). \end{aligned} \tag{12}$$

From (7), we know that, for $i = 1, 2$, $j \neq i$, the term $-\sigma(y_i(t) - y_j(t))$ in (12) is the ith synchronizing input without sampling, denoted by $u_i^\star(t)$. Consequently,

$$u_i(t) = u_i^\star(t) + \sigma(y_j(s_k^j) - y_j(t)) - \sigma(y_i(s_k^j) - y_i(t)). \tag{13}$$

Using the definition of operator Δ_i in (10), we obtain

$$\begin{aligned} u_i(t) &= u_i^\star(t) + \sigma(\Delta_j \dot{y}_j)(t) - \sigma(\Delta_j \dot{y}_i)(t), \\ &= u_i^\star(t) + \sigma \Delta_j(\dot{y}_j - \dot{y}_i)(t). \end{aligned} \tag{14}$$

■

Based on Lemma 8, we now proceed to remodel the bidirectionally coupled system (5), (3). In order to do so, we define the output synchronization error by

$$e_y(t) = y_1(t) - y_2(t), \tag{15}$$

and the internal dynamics synchronization error by

$$e_z(t) = z_1(t) - z_2(t). \tag{16}$$

Using the aforementioned notations, in the following lemma, we will formulate the synchronization error dynamics for system (5), (3). The stability property of these error dynamics are directly related to those of the synchronization manifold and hence will be investigated later (in Section IV) to formulate conditions for synchronization.

Lemma 9: Consider a system operator $\mathbf{G}$, representing the synchronization error dynamics given by

$$\begin{aligned}\dot{e}_y(t) &= a(e_z(t) + z_2(t), e_y(t) + y_2(t)) - a(z_2(t), y_2(t)) \\ &\quad -2b\sigma e_y(t) + b\sigma w(t), \\ \dot{e}_z(t) &= q(e_z(t) + z_2(t), e_y(t) + y_2(t)) - q(z_2(t), y_2(t)),\end{aligned} \tag{17}$$

and the operator $\mathbf{\Delta}$ such that

$$\begin{aligned}w(t) &:= (\mathbf{\Delta}\dot{e}_y)(t) := -(\Delta_2\dot{e}_y)(t) - (\Delta_1\dot{e}_y)(t), \\ &= w_1(t) + w_2(t),\end{aligned} \tag{18}$$

where $w_1(t) := -(\Delta_2\dot{e}_y)(t)$ and $w_2(t) := -(\Delta_1\dot{e}_y)(t)$, with $\Delta_i, i \in \{1, 2\}$, given by (10). Then, the feedback interconnection of $\mathbf{G}$ and $\mathbf{\Delta}$, given by (17), (18), represents the synchronization error dynamics of the bidirectionally coupled system (5), (3).

Proof: The proof is given in the report [14]. ■

IV. Main Result

In this section, we provide conditions that guarantee exponential synchronization of the bidirectionally coupled systems (5), (3). First, we provide a generic result that holds for the feedback-interconnection $\mathbf{G} - \mathbf{\Delta}$. By studying the properties of the feedback-interconnection $\mathbf{G} - \mathbf{\Delta}$, we provide conditions that guarantee exponential stability of the origin of the error dynamics concerning $e_y(t)$ and $e_z(t)$, i.e., exponential synchronization of the system (5), (3).

Theorem 10: Consider the feedback interconnection $\mathbf{G} - \mathbf{\Delta}$ given by (17), (18), and assume that

1) there exists a supply function $\mathcal{S}\colon \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ so that
$$\int_0^t \mathcal{S}(\theta, \dot{e}_y(\theta), w_1(\theta), w_2(\theta))d\theta \leq 0, \forall t \geq 0, \tag{19}$$
where $\dot{e}_y$ is given by (17), and w_1, w_2 are defined below (18).
2) there exists a continuously differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}^+$ and scalars $0 < c_1 < c_2$, $\alpha > 0$ such that
$$c_1\|e\|^2 \leq V(e) \leq c_2\|e\|^2, \tag{20}$$
and
$$\dot{V}(e(t)) + \alpha V(e(t)) \leq e^{-\alpha t}\mathcal{S}(t, \dot{e}_y(t), w(t)), \forall t \geq 0, \tag{21}$$

where $e(t) = \begin{bmatrix} e_y^T(t) & e_z^T(t) \end{bmatrix}^T$ is governed by (17). Then, the equilibrium $e = 0$ of system (17), (18) is globally exponentially stable with a decay rate of at least $\alpha/2$.

Proof: Consider the function

$$\begin{aligned}W(t) &= e^{\alpha t}V(e(t)) \\ &\quad - \textstyle\int_0^t \mathcal{S}(\theta, \dot{e}_y(\theta), w_1(\theta), w_2(\theta))d\theta, \forall t \geq 0.\end{aligned} \tag{22}$$

From condition (21), we can conclude $\dot{W}(t) \leq 0, \forall t \geq 0$, implying $W(t) \leq W(0)$, i.e.,

$$e^{\alpha t}V(e(t)) - \int_0^t \mathcal{S}(\theta, \dot{e}_y(\theta), w_1(\theta), w_2(\theta))d\theta \leq V(e(0)). \tag{23}$$

Therefore,

$$e^{\alpha t}V(e(t)) \leq V(e(0)) + \int_0^t \mathcal{S}(\theta, \dot{e}_y(\theta), w_1(\theta), w_2(\theta))d\theta, \tag{24}$$

and using condition (19), we have

$$V(e(t)) \leq e^{-\alpha t}V(e(0)). \tag{25}$$

Consequently, using (20), we obtain

$$\|e(t)\|^2 \leq \frac{c_2}{c_1}e^{-\alpha t}\|e(0)\|^2, \tag{26}$$

i.e.,

$$\|e(t)\| \leq \sqrt{\frac{c_2}{c_1}}e^{\frac{-\alpha}{2}t}\|e(0)\|, \tag{27}$$

implying that the equilibrium $e = 0$, of system (17), (18) is exponentially stable with a decay rate of at least $\alpha/2$. ■

Theorem 10 holds for any operator $\mathbf{\Delta}$, as long as its properties can be characterized using a supply function satisfying (19). In the case of sampling-induced perturbations captured by the operator $\mathbf{\Delta}$, as defined in (18), we characterize the properties of the operator $\mathbf{\Delta}$ as shown in the following lemma.

Lemma 11: Consider the operator $\mathbf{\Delta}$ defined in (18), (10). Then, for any $R_1 \in \mathbb{R}^{m\times m}$ and $R_2 \in \mathbb{R}^{m\times m}$, with $R_1 = R_1^T > 0$ and $R_2 = R_2^T > 0$, $\mathbf{\Delta}$ satisfies

$$\int_0^t S(\theta, \dot{e}_y(\theta), w_1(\theta), w_2(\theta)) \leq 0, \forall t \geq 0, \tag{28}$$

where $\alpha > 0$, and the function $S : \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ is given by

$$\begin{aligned}&\mathcal{S}(t, \dot{e}_y(t), w_1(t), w_2(t)) \\ &\quad = e^{\alpha t}\left(\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}\right. \\ &\qquad \left. -\dot{e}_y^T(t)(\bar{h}_2^2 e^{\alpha\bar{h}_2}R_1 + \bar{h}_1^2 e^{\alpha\bar{h}_1}R_2)\dot{e}_y(t)\right),\end{aligned} \tag{29}$$

with $\bar{h}_1$, $\bar{h}_2$ given by (2), and $w_1(t)$, $w_2(t)$ defined as in Lemma 9.

Proof: Consider the term $w_1(t)$ in (18), and the definition of operator Δ_2 from (10). We have,

$$\begin{aligned}w_1(t) &= -\textstyle\int_{s_k^2}^t \dot{e}_y(s)ds. \forall t \in [s_k^2, s_{k+1}^2), k \in \mathbb{N}, \\ &= e_y(s_k^2) - e_y(t).\end{aligned} \tag{30}$$

Using Jensen's inequality [15], we obtain

$$\begin{aligned} w_1^T(t)R_1w_1(t) &\leq (t-s_k^2)\int_{s_k^2}^{t}\dot{e}_y^T(\zeta)R_1\dot{e}_y(\zeta)d\zeta \\ &\leq \bar{h}_2\int_{s_k^2}^{t}\dot{e}_y^T(\zeta)R_1\dot{e}_y(\zeta)d\zeta. \end{aligned} \tag{31}$$

Using the change of variable $s=\zeta-t$, we obtain

$$\begin{aligned} w_1^T(t)R_1w_1(t) &\leq \bar{h}_2\int_{s_k^2-t}^{0}\dot{e}_y^T(t+s)R_1\dot{e}_y(t+s)ds \\ &\leq \bar{h}_2\int_{-\bar{h}_2}^{0}\dot{e}_y^T(t+s)R_1\dot{e}_y(t+s)ds. \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} &\textstyle\int_0^t e^{\alpha\theta}w_1^T(\theta)R_1w_1(\theta)d\theta \leq \\ &\quad \bar{h}_2\textstyle\int_0^t e^{\alpha\theta}\left(\int_{-\bar{h}_2}^{0}\dot{e}_y^T(\theta+s)R_1\dot{e}_y(\theta+s)ds\right)d\theta. \end{aligned} \tag{33}$$

Substituting $u=\theta+s$, we have that

$$\begin{aligned} &\textstyle\int_0^t e^{\alpha\theta}w_1^T(\theta)R_1w_1(\theta)d\theta \\ &\leq \bar{h}_2\textstyle\int_{-\bar{h}_2}^{0}\left(\int_s^{t+s}e^{\alpha(u-s)}\dot{e}_y^T(u)R_1\dot{e}_y(u)du\right)ds. \end{aligned} \tag{34}$$

Since the inner integral in the right-hand side of the inequality in (34) is always positive because R_1 is positive definite, we can upper bound the left-hand side in (34) using the limits of s and obtain

$$\begin{aligned} &\textstyle\int_0^t e^{\alpha\theta}w_1^T(\theta)R_1w_1(\theta)d\theta \\ &\quad\leq \bar{h}_2\textstyle\int_{-\bar{h}_2}^{0}\left(\int_{-\bar{h}_2}^{t+0}e^{\alpha(u+\bar{h}_2)}\dot{e}_y^T(u)R_1\dot{e}_y(u)du\right)ds \\ &\quad= \bar{h}_2e^{\alpha\bar{h}_2}\textstyle\int_{-\bar{h}_2}^{0}\left(\int_0^t e^{\alpha u}\dot{e}_y^T(u)R_1\dot{e}_y(u)du\right)ds \\ &\quad= \bar{h}_2^2e^{\alpha\bar{h}_2}\textstyle\int_0^t e^{\alpha\theta}\dot{e}_y^T(\theta)R_1\dot{e}_y(\theta)d\theta. \end{aligned} \tag{35}$$

Similarly, for the term $w_2(t)=-(\Delta_1\dot{e}_y)(t)$, we can obtain

$$\begin{aligned} &\textstyle\int_0^t e^{\alpha\theta}w_2^T(\theta)R_2w_2(\theta)d\theta \\ &\quad\leq \bar{h}_1^2e^{\alpha\bar{h}_1}\textstyle\int_0^t e^{\alpha\theta}\dot{e}_y^T(\theta)R_2\dot{e}_y(\theta)d\theta. \end{aligned} \tag{36}$$

Summation of (35) and (36) gives

$$\int_0^t \mathcal{S}(\theta,\dot{e}_y(\theta),w(\theta))\leq 0, \tag{37}$$

where

$$\begin{aligned} &\mathcal{S}(t,\dot{e}_y(t),w_1(t),w_2(t)) \\ &\quad= e^{\alpha t}\left(\begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}^T\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}\right. \\ &\qquad\left.-\dot{e}_y^T(t)(\bar{h}_2^2e^{\alpha\bar{h}_2}R_1+\bar{h}_1^2e^{\alpha\bar{h}_1}R_2)\dot{e}_y(t)\right). \end{aligned} \tag{38}$$

This concludes the proof. ■

Remark: The aforementioned lemma is an extension of the result provided in [10], wherein an operator was used to capture the effects of sampling in a single-loop input-affine nonlinear system.

We now proceed to provide a dissipativity-based criterion that guarantees exponential synchronization of the system (5), (3), by using the supply function given in Lemma 11, in conjunction with the result given in Theorem 10.

***Theorem** 12:* Consider the bidirectionally coupled system (5), (3) satisfying Assumption 3 and Assumption 5, the feedback-interconnection (17), (18), the supply function in (29), and the storage function $V:\mathbb{R}^n\to\mathbb{R}^+$ of the form given in (8). Then, for a given coupling gain σ and sampling interval bounds $\bar{h}_1$, $\bar{h}_2$, the system (5), (3) exponentially synchronizes with a decay-rate of at least $\alpha/2$, if there exist matrices $P_y=P_y^T>0$, $P_z=P_z^T>0$, $R_1=R_1^T>0$ and $R_2=R_2^T>0$ such that

$$\begin{aligned} &\dot{e}_y^TP_ye_y+\dot{e}_z^TP_ze_z+e_y^TP_y\dot{e}_y+e_z^TP_z\dot{e}_z+\alpha(e_y^TP_ye_y \\ &+e_z^TP_ze_z)-w_1^TR_1w_1-w_2^TR_2w_2+\dot{e}_y^T(\bar{h}_2^2e^{\alpha\bar{h}_2}R_1 \\ &\quad+\bar{h}_1^2e^{\alpha\bar{h}_1}R_2)\dot{e}_y\leq 0. \end{aligned} \tag{39}$$

Proof: From the result given in Lemma 11, we know that the supply function in (29) satisfies condition (19) in Theorem 10.

Additionally, from Theorem 6, we know that the storage function V in (8) characterizes the exponential stability properties of the synchronization manifold, in the absence of sampling-induced effects. Using the block-diagonal structure of this storage function, condition (20) in Theorem 10 can be easily shown.

Now, using the storage function in (8), and the supply function in (29), the inequality condition (21) in Theorem 10, corresponds to (39).

Therefore, if condition (39) is satisfied for $\alpha>0$, and a given coupling gain σ and sampling-interval bounds $\bar{h}_1$, $\bar{h}_2$, then, as a consequence of Theorem 10, we know that system (5), (3) exponentially synchronizes with a decay-rate of at least $\alpha/2$. ■

Remark: As a result of Theorem 6, the storage function V in (8), will satisfy inequality (21) in the absence of sampling, i.e., when $\mathcal{S}(\theta,\dot{e}_y(\theta),w_1(\theta),w_2(\theta))=0$. Additionally, in the continuous-time case, the decay-rate α in (21) (with $\mathcal{S}(t,\dot{e}_y(t),w_1(t),w_2(t))=0$) will depend on P_z, P_y, δ (in Assumption 5) and σ (given $\sigma>\bar{\sigma}$).

In the next section, we will illustrate how condition (39) aids in deciding the trade-off between the sampling interval bounds $\bar{h}_1$ and $\bar{h}_2$, the coupling gain σ, and the exponential synchronization rate $\alpha/2$.

V. Numerical Example

In this section, we consider a two-agent Fitzhugh-Nagumo system ([16], [17]) to illustrate the application of the result given in Theorem 10. The individual sub-system dynamics is given by

$$\begin{aligned} \dot{y}_i &= y_i-\tfrac{y_i^3}{3}-z_i+I+u_i, \\ \dot{z}_i &= \zeta(y_i+a-bz_i), i\in\{1,2\}, \end{aligned} \tag{40}$$

where $I=0.5, a=0.7, b=0.8, \zeta=0.08$, and the synchronizing coupling laws are given by (3). Considering the error terms $e_y=y_1-y_2$, $e_z=z_1-z_2$, and using the fact that

$$y_1^3-y_2^3 = \tfrac{1}{4}\left((y_1-y_2)^3+3(y_1+y_2)^2(y_1-y_2)\right), \tag{41}$$

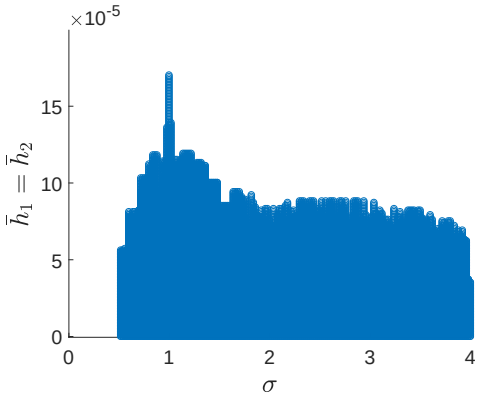

Fig. 2: Trade-off between coupling strength σ and maximum sampling interval $\bar{h}_1 = \bar{h}_2$, for the system (40), with $\alpha = 0.01$.

the error dynamics are given by

$$\begin{aligned}\dot{e}_y &= \dot{y}_1 - \dot{y}_2 \\ &= e_y - e_z - 2\sigma e_y - \tfrac{1}{12}\left(e_y^3 + 3e_y(e_y + 2y_2)^2\right) \\ &\quad + \sigma(w_1 + w_2),\end{aligned} \tag{42}$$

and

$$\dot{e}_z = \zeta(e_y - be_z) = 0.08(e_y - e_z), \tag{43}$$

where w_1 and w_2 are defined as shown in (18).

We know that by choosing the storage function V in (8), and considering the supply function given in (29), inequality (21) leads to (39). For the system (40), using the definition of $\dot{e}_y$ and $\dot{e}_z$ in (42) and (43), respectively, condition (39) leads to

$$\begin{aligned}&2P_y e_y \dot{e}_y + 2P_z e_z \dot{e}_z + \alpha P_y e_y^2 + \alpha P_z e_z^2 - R_1 w_1^2 \\ &- R_2 w_2^2 + (\bar{h}_1^2 e^{\alpha \bar{h}_1} R_2 + \bar{h}_2^2 e^{\alpha \bar{h}_2} R_1)\dot{e}_y^2 \leq 0.\end{aligned} \tag{44}$$

Additionally, the Demidovich condition introduced in Assumption 5, leads to

$$2P_z(-\zeta b) \leq -\delta, \delta > 0. \tag{45}$$

Inequality (44) can be used to analyse the trade-off between $\bar{h}_1$, $\bar{h}_2$, α and σ, for system (40). An exemplary feasibility region is computed using SOSTOOLS, by fixing $\alpha = 0.01$, considering $\bar{h}_1 = \bar{h}_2$, and solving inequalities (44) and (45). The synchronization diagram giving the trade-off between coupling gain σ and the maximum sampling bounds $\bar{h}_1 = \bar{h}_2$, is given in Figure 2. The profile of the synchronization diagram in Figure 2, is similar to the profile obtained for two Fitzhugh-Nagumo systems with time-delayed coupling [18]. When $\alpha = 0.01$, the critical value of σ, below which synchronization is not possible, was found to be $\sigma = \bar{\sigma} = 0.51$.

VI. Conclusion

In this paper, a novel dissipativity-based approach towards synchronization analysis of nonlinear oscillatory systems with bidirectional sampled-data coupling, is provided. The approach builds on remodelling the sampled-data system as a feedback-interconnection of a continuous-time system operator that captures the bidirectionally coupled system dynamics in the absence of sampling, and an operator that captures the effects of sampling. The properties of this feedback-interconnection are then studied to provide dissipativity-based conditions that guarantee exponential synchronization of the nonlinear oscillatory systems with bidirectional sampled-data coupling. Finally, a two-agent Fitzhugh-Nagumo system is used to illustrate the theoretical results.

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