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Jean-Claude Bermond, Katherine Heinrich, Min-Li Yu. Existence of Resolvable Path Designs. European Journal of Combinatorics, 1990, 11 (3), pp.205-211. 10.1016/S0195-6698(13)80120-5. hal-02974661

HAL Id: hal-02974661

<https://inria.hal.science/hal-02974661>

Submitted on 21 Oct 2020

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Existence of Resolvable Path Designs

J.-C. BERMOND, KATHERINE HEINRICH AND MIN-LI YU

For $k \geq 2$ the complete multigraph λK_n has a factorization into isomorphic spanning subgraphs, each component of which is a path of length $k - 1$ iff $n \equiv 0 \pmod{k}$ and $\lambda k(n - 1) \equiv 0 \pmod{2(k - 1)}$.

1. INTRODUCTION

Let G be a multigraph. A P_k -factor in G is a spanning subgraph of G each component of which is a path (P_k) of length $k - 1$. If the edges of G can be partitioned into P_k -factors we say that G has a P_k -factorization. We will study P_k -factorizations of λK_n (the complete multigraph on n vertices in which each edge has multiplicity λ).

The problem of partitioning the edges of λK_n into paths is not new. When the edges of λK_n can be partitioned into paths of length $k - 1$, we have what has been called an (n, k, λ) -path design. Tarsi [8] has shown that an (n, k, λ) -path design exists iff $\lambda n(n - 1) \equiv 0 \pmod{2(k - 1)}$ and $n \geq k$. Some time earlier, Hell and Rosa [4] had considered the existence of resolvable, balanced (n, k, λ) -path designs. (A path design is *resolvable* if the paths can be partitioned into P_k -factors of λK_n , and is *balanced* if every vertex lies in the same number of paths. Clearly, a resolvable path design is always balanced.) In that paper they related Dudeney's problem [3, problem 272] of nine schoolboys (the solution of which is a resolvable $(9, 3, 1)$ -path design), and provided several constructions of resolvable path designs. Later, both Huang [6] and Hung and Mendelsohn [7] showed that balanced (but not necessarily resolvable) (n, k, λ) -path designs exist iff the necessary conditions (easily obtained by counting) are satisfied.

Thus the question of the existence of resolvable (n, k, λ) -path designs (or P_k -factorization of λK_n) remained. The necessary conditions were given by Hell and Rosa and we restate them in Lemma 1.1 (although our presentation is not that of Hell and Rosa). They are verified by simple counting (first on vertices, and second on edges in a P_k -factor of λK_n).

LEMMA 1.1. *If there exists a P_k -factorization of λK_n , then $n \equiv 0 \pmod{k}$ and $\lambda k(n - 1) \equiv 0 \pmod{2(k - 1)}$.*

Resolvable (n, k, λ) -path designs have also been considered by Horton [5], who obtained necessary and sufficient conditions for their existence when $k = 3$.

THEOREM 1.2 [5]. *There exists a P_3 -factorization of λK_n iff $n \equiv 0 \pmod{3}$ and $\lambda(n - 1) \equiv 0 \pmod{4}$.*

When $k = 2$ a resolvable $(n, 2, \lambda)$ -path design is simply a 1-factorization of λK_n and its existence is well known.

THEOREM 1.3. *The complete multigraph λK_n has a 1-factorization iff n is even.*

The main result in this paper is Theorem 1.4.

THEOREM 1.4. *When $k > 3$, λK_n has a P_k -factorization iff $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.*

Combined with the results of Theorems 1.2 and 1.3, the question of the existence of P_k -factorizations of λK_n will be completely resolved. Note that using a result of Ray-Chaudhuri and Wilson [2, Theorem 7.6, p. 572], Horton had already shown that for $\lambda = 1$ the necessary conditions are asymptotically sufficient.

(We remark that the question of P_k -factorizations of λK_n is analogous to the Oberwolfach problem which, in the uniform case, asks for C_k -factorizations of λK_n (C_k is a cycle of length k). This problem has been resolved (see [1]) in all cases except when λ is even and both k and n are odd.)

2. BUILDING BLOCKS

The proof of Theorem 1.4 depends on several relatively simple constructions which we present in the following lemmas. We remark that in the remainder of the paper all arithmetic calculation is modulo k on the residues $1, 2, \dots, k$.

LEMMA 2.1. (a) *If λk is even, then λK_k has a P_k -factorization.*
 (b) *If λk is odd, then $\lambda K_k - N$, where N is a set of $(k-1)/2$ independent edges, has a P_k -factorization.*

PROOF. The results follow immediately from the well known facts that K_k has a P_k -factorization when k is even, and that $K_k - N$ has a P_k -factorization when k is odd. (To prove (b) one also needs to observe that every path of length $k-1$ in K_k is the union of two disjoint sets of $(k-1)/2$ independent edges.) For completeness we now give the factorizations of K_k and $K_k - N$, where $V(K_k) = V(K_k - N) = \{1, 2, \dots, k\}$. When k is even the paths are $P(i) = [i, 1+i, k-1+i, 2+i, k-2+i, \dots, k/2+2+i, k/2-1+i, k/2+1+i, k/2+i]$, $1 \leq i \leq k/2$, and when k is odd they are $Q(i) = [i, 1+i, k-1+i, 2+i, k-2+i, \dots, (k-1)/2-1+i, (k+1)/2+1+i, (k-1)/2+i, (k+1)/2+i]$, $1 \leq i \leq (k-1)/2$. Note that we have the freedom to choose the near 1-factors in each of the λ copies of K_k so that they form $\lfloor \lambda/2 \rfloor$ paths of length $k-1$ and $\lambda - 2\lfloor \lambda/2 \rfloor$ near 1-factors. When λ is odd that near 1-factor is N . \square

LEMMA 2.2. (a) *If k is even, then $\lambda K_{k,k} - F(\lambda)$, where $F(\lambda)$ is the union of λ 1-factors of $K_{k,k}$, has a P_k -factorization.*
 (b) *If k is odd and $k \geq 3$, then $\lambda K_{k,k} - W(\lambda)$, where $W(\lambda)$ is the union of λ subgraphs of $K_{k,k}$ each consisting of $(k-1)/2$ vertex disjoint cycles of length 4 and an independent edge, has a P_k -factorization.*
 (c) *If k is odd, $k \geq 3$ and λ is even, then $\lambda K_{k,k} - C(\lambda/2)$, where $C(\lambda/2)$ is the union of $\lambda/2$ Hamilton cycles in $K_{k,k}$, has a P_k -factorization.*

PROOF. In (a) and (b) we need only consider the case $\lambda = 1$, and in (c) the case $\lambda = 2$. Let $V(K_{k,k}) = V(2K_{k,k}) = X \cup Y$, where $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$, and let $V(K_k) = \{1, 2, \dots, k\}$.

(a) By Lemma 2.1(a) we know that K_k has a P_k -factorization. Let $P = [1, 2, \dots, k]$ be one of the paths in such a factorization. In $K_{k,k}$ P defines the P_k -factor $\{\{x_1, y_2, x_3, \dots, x_{k-1}, y_k\}, \{y_1, x_2, y_3, \dots, y_{k-1}, x_k\}\}$. Repeating for each path in the P_k -factorization of K_k we obtain a P_k -factorization of $K_{k,k} - F(1)$, where $F(1) = \{\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_k, y_k\}\}$.

(b) We apply the same procedure as above but use the P_k -factorization of $K_k - N$

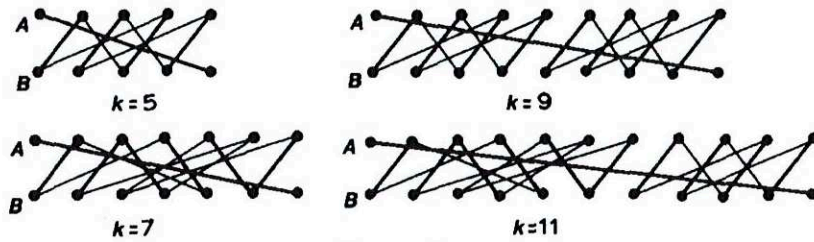


FIGURE 1.

given in Lemma 2.1(b). When all paths have been dealt with what remains in $K_{k,k}$ is the subgraph $W(1)$.

The construction technique used in (a) and (b) is well known and, for example, is described in [4].

(c) In this case a direct construction will be given. First observe that if $k = 2m + 1$, then the $2k$ edges $E = \{(x_i, y_{i+m-1}), (x_i, y_{i+m+1}) : 1 \leq i \leq k\}$ form a Hamilton cycle in $2K_{k,k}$. We now consider the cases $k = 4t + 1$ and $k = 4t + 3$, and denote the k P_k -factors by $P(1), P(2), \dots, P(k)$. In each case we give $P(1)$ from which $P(i + 1)$, $1 \leq i \leq k - 1$, is obtained as follows: $\{(x_{i+j}, y_{i+s}) \in E(P(1+i)) \text{ if } \{(x_j, y_s) \in E(P(1))\}$.

When $k = 4t + 1$, $P(1) = \{(x_1, y_1, x_{2t}, y_2, x_{2t-1}, y_3, \dots, y_t, x_{t+1}, y_{3t+1}, x_{3t+2}, y_{3t}, x_{3t+3}, \dots, y_{2t+2}, x_{4t+1}), [y_{2t}, x_2, y_{2t-1}, x_3, \dots, x_t, y_{t+1}, x_{3t+1}, y_{3t+2}, x_{3t}, \dots, y_{4t+1}, x_{2t+1}, y_{2t+1}]\}$, and when $k = 4t + 3$, $P(1) = \{(x_1, y_1, x_{2t+1}, y_2, x_{2t}, y_3, \dots, x_{t+2}, y_{t+1}, x_{3t+3}, y_{3t+2}, x_{3t+4}, y_{3t+1}, \dots, y_{2t+3}, x_{4t+3}), [y_{2t+1}, x_2, y_{2t}, x_3, \dots, y_{t+2}, x_{t+1}, y_{3t+3}, x_{3t+2}, y_{3t+4}, x_{3t+1}, \dots, x_{2t+3}, y_{4t+3}, x_{2t+2}, y_{2t+2}]\}$. \square

REMARK 2.3. Observe that in the construction given in Lemma 2.2(c) all 'vertical' edges (that is, edges $\{(x_i, y_i), 1 \leq i \leq k\}$) are contained in paths of the factorization. It is not difficult to see that in Lemma 2.2(a) we can permute the vertices of Y so that here also all vertical edges are in paths of the factorization. This is also possible in Lemma 2.2(b) provided that $k \geq 5$. When $k = 5$ and $k = 7$ permute the vertices so that W has the form shown in Figure 1, with vertex bipartition (A, B) , where $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$. Induction then takes care of all other values of k , as is shown in the cases $k = 9$ and $k = 11$. Observe that if we identify the vertices a_i and b_i , $1 \leq i \leq k$ (as shown in Figure 2), then the resulting multigraph $G(W)$ is the union of a Hamilton cycle and a Hamilton path.

We have pointed out these properties as they are features of the factorizations given in Lemmas 2.1 and 2.2, which are essential to our application of the main lemma (Lemma 2.4).

We now present the main lemma. The idea of the proof of Theorem 1.4 will be to use Lemmas 2.1 and 2.2 to find many P_k -factors so that, after their deletion from λK_n , the subgraph remaining satisfies the conditions of Lemma 2.4.

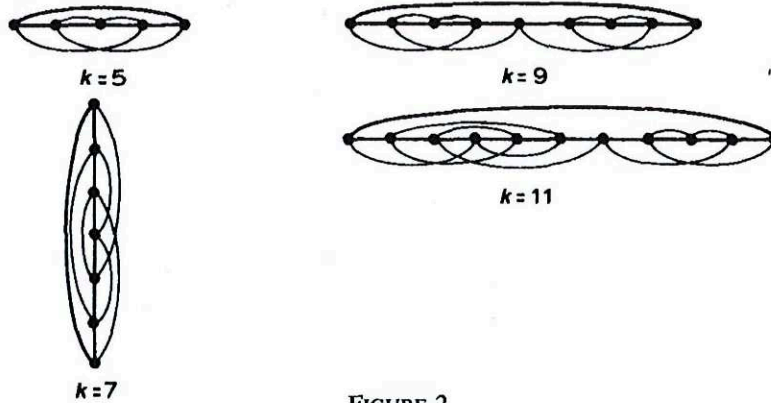


FIGURE 2.

LEMMA 2.4. Let H be a multipartite graph with $V(H) = \bigcup_{i=1}^k V_i$ so that for $1 \leq i < j \leq k$ the bipartite subgraph on vertex set $V_i \cup V_j$ with bipartition (V_i, V_j) , is $t((i, j))$ -regular, where t is a mapping from the set $\{(i, j): 1 \leq i < j \leq k\}$ to the non-negative integers. Let $G(H)$ be a graph with $V(G(H)) = \{1, 2, \dots, k\}$ in which the edge $\{i, j\}$ has multiplicity $t((i, j))$. Then H has a P_k -factorization if $G(H)$ does.

PROOF. Suppose that $G(H)$ has a P_k -factorization with paths $P(1), P(2), \dots, P(m)$, where $m = \sum_{1 \leq i < j \leq k} t((i, j))/(k-1)$. Each $P(i)$ yields a P_k -factor in H , as follows. To each edge $pq \in E(P(i))$, associate a 1-factor F_{pq}^i from the bipartite subgraph with vertex-set $V_p \cup V_q$ so that the 1-factors associated with a given edge are all distinct. (In fact, $\bigcup_{i=1}^m F_{pq}^i$ is the bipartite subgraph on the vertex set $V_p \cup V_q$, where F_{pq}^i is empty if $pq \notin P(i)$.) Clearly, $\bigcup_{pq \in E(P(i))} F_{pq}^i$ is a P_k -factor. \square

COROLLARY 2.5. Let H be as in Lemma 2.4. If $G(H)$ is one of:

- (a) a cycle of length k in which each edge has multiplicity $k-1$;
- (b) a cycle of length k , k even, in which edges alternatively have multiplicities $k/2-1$ and $k/2$; or
- (c) a cycle of length k , k odd, in which $(k+1)/2$ edges have multiplicity $(k-1)/2$, $(k-1)/2$ edges have multiplicity $(k+1)/2$ and except for two edges, alternative edges have different multiplicities, then H has a P_k -factorization.

PROOF. We need only show that the graphs $G(H)$ have P_k -factorizations. In each case $V(G(H)) = \{1, 2, \dots, k\}$ and let the cycle be $(1, 2, \dots, k)$. In each case (a) select the paths $P(i) = [i, i+1, i+2, \dots, i-1]$, $1 \leq i \leq k$, and in cases (b) and (c) select the paths $Q(i) = [2i, 2i+1, \dots, 2i-1]$, $1 \leq i \leq \lfloor k/2 \rfloor$. \square

3. PROOF OF THEOREM 1.4

The necessity of the conditions has already been shown in Lemma 1.1. We now show their sufficiency. Recall that all arithmetic calculations are modulo k on the residues $1, \dots, k$.

From the necessary conditions we know that $n = kr$ and $2(k-1) \mid \lambda k(kr-1)$. The divisibility condition implies that $(k-1) \mid \lambda(r-1)$ when λk is even, and to $(k-1) \mid \lambda(r-1)$ with odd quotient when λk is odd (as λk odd implies r odd).

Let $V(\lambda K_n) = \{(i, j): 1 \leq i \leq r, 1 \leq j \leq k\} = \bigcup_{i=1}^r H_i = \bigcup_{j=1}^k V_j$, where $H_i = \{(i, j): 1 \leq j \leq k\}$ and $V_j = \{(i, j): 1 \leq i \leq r\}$. Now λK_n is obtained from λK_r with vertex set $\{1, 2, \dots, r\}$ by replacing each vertex j by a copy of λK_k on the vertex set H_j and each edge (i, j) by a copy of $\lambda K_{k,k}$ on the vertex set $H_i \cup H_j$ with bipartition (H_i, H_j) .

Proof of sufficiency when λk is even. We begin with the case λk even, as in this case λK_k has a P_k -factorization. For even k , let R be a subgraph of λK_n with $E(R) = \bigcup_{1 \leq i < j \leq r} F_{ij}(\lambda)$, where $F_{ij}(\lambda)$ is the union of λ 1-factors in the bipartite subgraph $\lambda K_{k,k}$ with vertex bipartition (H_i, H_j) . When k is odd, R is a subgraph of λK_n with $E(R) = \bigcup_{1 \leq i < j \leq r} C_{ij}(\lambda/2)$, where $C_{ij}(\lambda/2)$ is the union of $\lambda/2$ Hamilton cycles in the bipartite subgraph $\lambda K_{k,k}$ with vertex bipartition (H_i, H_j) . (Note that R is not uniquely determined.)

PROPOSITION 3.1. The graph $\lambda K_n - R$ has a P_k -factorization.

PROOF. When r is even, K_r admits a 1-factorization with factors F_1, F_2, \dots, F_{r-1} . To each factor F_i there corresponds, in λK_n , a subgraph G_i which is the vertex-disjoint

union of $r/2$ copies of $\lambda K_{k,k}$. By Lemma 2.2, $\lambda K_{k,k} - F(\lambda)$ has a P_k -factorization when k is even, and $\lambda K_{k,k} - C(\lambda/2)$ has a P_k -factorization when k is odd (in which case λ is even). Therefore G_i contains $\lambda k/2$ P_k -factors in λK_n . Furthermore, each λK_k associated with a vertex of K_r has a P_k -factorization and so together they yield $\lambda k/2$ P_k -factors in λK_n . Removing these $\lambda k(r-1)/2 + \lambda k/2 = \lambda kr/2$ P_k -factors from λK_n leaves the subgraph R .

When r is odd, K_r admits a near 1-factorization with factors M_1, M_2, \dots, M_r . To each factor M_i there corresponds, in λK_n , a subgraph which is the vertex-disjoint union of $(r-1)/2$ copies of $\lambda K_{k,k}$ and one copy of λK_k . Applying Lemmas 2.1 and 2.2 we obtain $\lambda k/2$ P_k -factors in λK_n . Over all M_i , $1 \leq i \leq r$, this results in $\lambda kr/2$ P_k -factors which, on deletion from λK_n , leave the subgraph R . ■

Due to Proposition 3.1, all that remains is to prove that there is a graph R as described above which has a P_k -factorization. (Recall that the description did not uniquely determine R .) We consider three cases depending on the parity of k and $\lambda(r-1)$. In each case $V(\lambda K_r) = \{1, 2, \dots, r\}$.

Case 1: k odd. In R let us choose C_{ij} to be the Hamilton cycle $C_{ij} = ((i, 1), (j, 2), (i, 3), (j, 4), \dots, (i, k-1), (j, k))$, and let $C_{ij}(\lambda/2)$ be $\lambda/2$ copies of the cycle C_{ij} , $1 \leq i < j \leq r$. Consequently, all edges of R lie in the bipartite subgraphs $\lambda K_{r,r}$ on vertex sets $V_j \cup V_{j+1}$ with bipartition (V_j, V_{j+1}) , $1 \leq j \leq k$. Moreover, these subgraphs of $\lambda K_{r,r}$ are $\lambda(r-1)/2$ regular. The graph R as constructed satisfies the conditions of Lemma 2.4 with $t((i, i+1)) = \lambda(r-1)/2$ and $t((i, j)) = 0$ for all other values of i and j . Then the graph $G(R)$ is a cycle of length k and each edge has multiplicity $\lambda(r-1)/2 = s(k-1)$ (from the necessary conditions). We now apply Corollary 2.5(a) s times to obtain a P_k -factorization of R , and this completes the proof.

Let us now count the number of P_k -factors. The edges of $\lambda K_n - R$ contributed $\lambda kr/2$ P_k -factors, and those of R yield a further $sk = \lambda k(r-1)/(2k-2)$ P_k -factors. Then as $\lambda kr/2 + \lambda k(r-1)/(2k-2) = \lambda k(n-1)/(2k-2)$, we do indeed have all the P_k -factors.

Case 2: k even, $\lambda(r-1)$ even. The graph λK_r is regular of even degree and so has a 2-factorization. Arbitrarily direct the edges of each cycle in the 2-factorization to create dicycles. If (i, j) is an arc from i to j in one of the dicycles, let $F_{ij} = \{(i, 1), (j, 2)\}, \{(i, 2), (j, 3)\}, \dots, \{(i, k-1), (j, k)\}, \{(i, k), (j, 1)\}$ be a 1-factor of $\lambda K_{k,k}$ with vertex bipartition (H_i, H_j) . Since the edge $\{i, j\}$ occurs λ times in the 2-factorization of λK_r , we so define $F_{ij}(\lambda)$. The 2-factorization guarantees that in R we now have the edges of a $\lambda(r-1)/2$ -regular subgraph of $\lambda K_{r,r}$ on vertex set $V_i \cup V_{i+1}$ with vertex bipartition (V_i, V_{i+1}) , $1 \leq i \leq k$. The graph R now satisfies the requirements of Lemma 2.4 (again $G(R)$ is a cycle of length k in which each edge has multiplicity $\lambda(r-1)/2$) and by Corollary 2.5(a) we have a P_k -factorization of R . (At this point one can again easily count all P_k -factors.)

Case 3: k even, $\lambda(r-1)$ odd. In this case λK_r has odd degree but an even number of vertices and hence can be factored into one 1-factor and $(\lambda(r-1)-1)/2$ 2-factors, which we direct as in the previous case.

First we deal with the 1-factor. If $\{i, j\}$ is an edge of the 1-factor, let $F_{ij}^* = \{(i, 2), (j, 3)\}, \{(i, 3), (j, 2)\}, \{(i, 4), (j, 5)\}, \{(i, 5), (j, 4)\}, \dots, \{(i, k), (j, 1)\}, \{(i, 1), (j, k)\}$. If (i, j) is an arc in the directed cycles, let F_{ij} be as in Case 2. We have now defined $F_{ij}(\lambda)$, and our choice of F_{ij}^* and F_{ij} ensures us that here also we have a graph R satisfying the requirements of Lemma 2.4. Indeed, in this case R consists of a $(\lambda(r-1)+1)/2$ -regular subgraph of $\lambda K_{r,r}$ on vertex set $V_{2i} \cup V_{2i+1}$ with vertex

bipartition (V_{2i}, V_{2i+1}) , $1 \leq i \leq k/2$, and a $(\lambda(r-1) - 1)/2$ -regular subgraph of $\lambda K_{r,r}$ on vertex set $V_{2i-1} \cup V_{2i}$ with vertex bipartition (V_{2i-1}, V_{2i}) , $1 \leq i \leq k/2$. Since $(\lambda(r-1) + 1)/2 = (s(k-1) + 1)/2 = (k-1)(s-1)/2 + (k/2)$ and $(\lambda(r-1) - 1)/2 = (k-1)(s-1)/2 + (k/2 - 1)$, then $G(R)$ has a P_k -factorization obtained from $(s-1)/2$ applications of Corollary 2.5(a) and one application of Corollary 2.5(b). Thus we have a P_k -factorization of R and hence of λK_n . \square

Proof of sufficiency when λk is odd. We now look at the situation when λk is odd. Since both λ and k are odd it is easy to see that r is also odd. As in the case λk even, we begin by defining a subgraph S of λK_n . The edges of S are given by $E(S) = (\bigcup_{1 \leq i < j \leq r} W_{ij}(\lambda)) \cup (\bigcup_{i=1}^r N_i)$, where $W_{ij}(\lambda)$ is the union of λ subgraphs of $K_{k,k}$ (with vertex bipartition (H_i, H_j)), each isomorphic to the graph W described in Remark 2.3, and N_i is a set of $(k-1)/2$ independent edges on the vertex set H_i .

PROPOSITION 3.2. *The graph $\lambda K_n - S$ has a P_k -factorization.*

PROOF. Since r is odd, K_r has a near 1-factorization with near 1-factors M_1, M_2, \dots, M_r , and to each of these there corresponds in λK_n a subgraph which is the vertex-disjoint union of $(r-1)/2$ copies of $\lambda K_{k,k}$ and one copy of λK_k . Then Lemma 2.2(b) yields $\lambda(k-1)/2$ P_k -factors in each $\lambda K_{k,k}$. By Lemma 2.1(b) we have $(\lambda k - 1)/2$ P_k -factors in λK_k . Notice that in the copy of λK_k only $\lambda(k-1)/2$ of a possible $(\lambda k - 1)/2$ P_k -factors are used. Therefore for each near 1-factor we obtain $\lambda(k-1)/2$ P_k -factors. On each vertex set V_i , $1 \leq i \leq r$, there remains a subgraph consisting of $(\lambda - 1)/2$ paths of length $k - 1$ and a set of $(k-1)/2$ independent edges. Together (that is, over all i), these paths constitute a further $(\lambda - 1)/2$ P_k -factors of λK_n . When all these $\lambda(k-1)r/2 + (\lambda - 1)/2 = (\lambda r(k-1) + (\lambda - 1))/2$ P_k -factors are deleted from λK_n , what remains is the subgraph S . \blacksquare

As in the previous case we need now only to construct such a subgraph S so that S satisfies the conditions of Lemma 2.4, and $G(S)$ satisfies the requirements of Lemma 2.5.

We begin with a 2-factorization of K_r in which we have directed the cycles. If the edge $\{i, j\} \in E(K_r)$ is directed (i, j) , then let $W_{ij}(\lambda)$ be λ copies of the subgraph W described in Figures 1 and 2, where $A = H_i$ and $B = H_j$, and let $N_i = \{(i, 2j), (i, 2j + 1)\}: 1 \leq j \leq (k-1)/2\}$. This has now defined our subgraph S . So $G(S)$ is the union of a path of length $k - 1$ with edge multiplicity $\lambda(r-1)/2$ (corresponding to the Hamilton paths of W_{ij}), and of a cycle of length k in which the edges $\{2j, 2j + 1\}$, $1 \leq j \leq (k-1)/2$, have multiplicity $\lambda(r-1)/2 + 1$ and the others have multiplicity $\lambda(r-1)/2$ (these cycles correspond to the Hamilton cycles of W_{ij} and N_i). Since $\lambda(r-1) = s(k-1)$, $\lambda(r-1)/2 = (k-1)(s-1)/2 + (k-1)/2$ and $\lambda(r-1)/2 + 1 = (k-1)(s-1)/2 + (k+1)/2$. (Note that it is only at this point that the construction fails for $k = 3$.) Now $G(S)$ has a P_k -factorization consisting of those $\lambda(r-1)/2$ paths, and a further $(sk+1)/2$ paths resulting from $(s-1)/2$ applications of Corollary 2.5(a) and one application of Corollary 2.5(c), to the Hamilton cycles. We now apply Lemma 2.4 to complete the proof.

As a final check we observe that there are $\lambda(r-1)/2 + (sk+1)/2$ P_k -factors in S , and adding these to the previously found P_k -factors we have a total of $\lambda k(rk - 1)/(2k - 2)$ P_k -factors as required. \square

4. FURTHER REMARKS

We note that Lemma 2.4 and the procedure of the proof described in Section 3 can be used to obtain P_k -factorizations of r -partite multigraphs. Here is an example.

THEOREM 4.1. *The r -partite graph $\lambda K_{n,n,\dots,n}$ has a P_k -factorization if $r \equiv 0 \pmod{k}$ and $nk\lambda(r-1) \equiv 0 \pmod{2(k-1)}$ or if $n = kt$, rt is even, and $nk\lambda(r-1) \equiv 0 \pmod{2(k-1)}$.*

Note that for a P_k -factorization of the r -partite graph the conditions $nr \equiv 0 \pmod{k}$ and $nk\lambda(r-1) \equiv 0 \pmod{2(k-1)}$ are necessary. It is probable that these are the only necessary conditions.

ACKNOWLEDGMENTS

This paper was written while the first author was visiting the school of computing science at Simon Fraser University; the support of Advanced Systems Foundations and Simon Fraser University is gratefully acknowledged.

The second author acknowledges the financial support of the Natural Sciences and Engineering Research Council of Canada under grant A-7829.

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