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STABILIZATION FOR A PERTURBED CHAIN OF INTEGRATORS IN PRESCRIBED TIME

YACINE CHITOUR, ROSANE USHIROBIRA AND HASSAN BOUHEMOU

Abstract. In this paper, we consider issues relative to prescribed time stabilisation of a chain of integrators of arbitrary length, either pure (i.e., where there is no disturbance) or perturbed. In a first part, we revisit the proportional navigation feedback (PNF) approach and we show that it can be appropriately recasted within the framework of time-varying homogeneity. As a first consequence, we recover all previously obtained results on PNF with simpler arguments. We then apply sliding mode inspired feedbacks to achieve prescribed stabilisation with uniformly bounded gains. However, all these feedbacks are robust to matched uncertainties only. In a second part, we provide a feedback law yet inspired by sliding mode which not only stabilises the pure chain of integrators in prescribed time but also exhibits some robustness in the presence of measurement noise and unmatched uncertainties.

1. Introduction

In this paper, we consider the following problem: for n positive integer and T positive real number, the perturbed chain of integrators is the control system given by

$$\dot{x}(t) = J_n x(t) + (d(t) + b(t)u(t))\ e_n, \qquad t \in [0, T), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R} \tag{1}$$

where $(e_i)_{1\leq i\leq n}$ denotes the canonical basis of $\mathbb{R}^n$, J_n denotes the n-th Jordan block (i.e., $J_n e_i = e_{i-1}$ for $1 \leq i \leq n$ with the convention $e_0 = 0$), $d(\cdot)$ and $b(\cdot)$ denote respectively a matched uncertainty and the uncertainty on the control respectively. Moreover we assume that there exists $\underline{b} > 0$ such that

$$b(t) \geq \underline{b}, \quad \forall t \in [0, T). \tag{2}$$

Our goal consists in designing a feedback control u that renders the system fixed-time input-to-state stable in any time $T > 0$ (prescribed-time stabilisation) and possibly convergent to zero (PT+ISS+C) (cf. [18], and Definition 2 given below). Note also that one may ask robustness properties in the presence of noise measurement d_1, for instance, if the feedback control u is static, it takes the form $u = F(x + d_1))$ and unmatched uncertainty d_2, i.e., $\dot{x} = J_n x + (d + bu)e_n + d_2$. Here, the feedback law $F(x)$ stabilises the n-th order pure chain of integrators.

It is clearly more difficult to address the issue of prescribed time stabilisation rather than the issue of finite time stabilisation, cf [20, 1, 15, 21]. In missile guidance [22] and other applications,

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the issue of prescribed time stabilisation has a long history and two main approaches for solving this problem have been considered: (a) *proportional navigation feedback* (PNF), which is a feedback linear in the state x and with time-varying gains blowing up to infinity towards the prescribed fixed time; (b) optimal control with a terminal constraint, where the dependency of the gains with respect to the state is implicit. Stemming from the PNF design for second order chains of integrators, a general approach is proposed in [18] for n-th order perturbed chains of integrators, i.e., in (1)). The feedback law has the form $u(t) = K^T P_1(\frac{1}{T-t})(x(t)) + P_2(\frac{1}{T-t})$, where P_1 and P_2 are polynomials (either matrix or real valued) and the vector K must be chosen in a rather involved way. The first term in the feedback is definitely of PNF type but the second one is only necessary for the convergence argument and does not appear for second order models for instance. Anyway, the controller in [18] does tend to zero as t tends to T even though the gains blow up and it exhibits excellent robustness properties in the case of matched uncertainties. However, the authors only suggest that it behaves poorly in case of measurement noise or unmatched uncertainties and also claim that all known techniques (including theirs) do not work in case of unmatched uncertainties. Finally, due to the rather complicated stability analysis as well as the involved construction of the feedback, it is not clear how to measure quantitatively the limitations of that feedback (see Section 3.2 in [18]) and to possibly improve its results.

The first part of the present paper aims at revisiting PNF design with a new perspective. We show that it can be naturally seen as a particular instance of weighted-homogeneous control systems (cf. [17] for instance) with the usual homogeneity coefficient not anymore constant but being equal to an appropriate time-varying function. Indeed, recall that a PNF has the form

$$u_{PNF}(t) = \sum_{i=1}^{n} \frac{k_i x_i(t)}{(T-t)^{n-i+1}}, \quad t \in [0,T), \tag{3}$$

where $K = (k_1, \cdots, k_n) \in \mathbb{R}^n$. Let $\mathbf{r} = (n-i+1)_{1\leq i\leq n}$ be the weight vector and set, for $\lambda > 0$, $D_\lambda^{\mathbf{r}}$ to be the diagonal matrix made of the λ^{n-i+1}'s. Rewriting $u_{PNF}(t) = K^T D_{\lambda(t)}^{\mathbf{r}} x(t)$ with $\lambda(t) = \frac{1}{T-t}$ suggests at once to consider the new state $y(t) = D_{\lambda(t)}^{\mathbf{r}} x(t)$ for which u_{PNF} simply reduces to $K^T y$. Now, the dynamics of y with respect to the time scale $s(t) = \ln(\frac{T}{T-t})$ turns out to be given by

$$\frac{dy}{ds} = (D_{\mathbf{r}} + J_n)y + (b(s)u(s) + d(s))e_n, \quad s \geq 0, \tag{4}$$

where $D_{\mathbf{r}}$ is the constant diagonal matrix made of the $n-i+1$'s. Hence, the original problem of prescribed time stabilisation of (1) *in time* $T > 0$ has been reduced to the *asymptotic* stabilisation of an n-th order perturbed chain of integrators with the extra term $D_{\mathbf{r}}y$ in (4) with respect to (1). Note that, to the best of our knowledge, it seems to be the first time that one considers a time-varying homogeneity coefficient in the context of stabilisation of weighted-homogeneous systems in that general manner. Usually, the homogeneity coefficient, when non constant, is state-dependent (cf. [16] as the pioneering reference for fixed-time stabilisation of linear systems, then [5] and [8] for instance, in the case of second order and n-th order perturbed chains of integrators respectively.)

With the previous viewpoint, it is immediate to see that PNF (and its variant given in [18]) is nothing more but the stabilisation of (4) with a linear feedback. As a consequence, we recover all the results of [18] with much simpler arguments and the limitations (as well as the advantages) of such a feedback appear in a transparent way. In particular, our convergence analysis easily reduces to the verification of a Linear Matrix Inequality (LMI), see Proposition 10 below, whose solution is essentially given in [4]. Moreover, the fact that measurement noise and unmatched uncertainties in (1) cannot be handled with that linear feedback is obvious since the corresponding disturbances become amplified by $D^{\mathbf{r}}_{\lambda(t)}$ in (4) and one can measure explicitly their destabilising effect.

One can then turn to other types of stabilisation for (4). If the settling time of the system associated to some feedback law $u = F(s, y)$ is infinite (i.e., the supremum over the initial conditions x_0 of the time needed to reach the origin for the trajectory of (4) fed by $u = F(s, y)$ and starting at x_0), then we will unavoidably face the numerical challenge of plotting $y(s(t)) = D^{\mathbf{r}}_{\lambda(t)}x(t))$ in (1), with $D^{\mathbf{r}}_{\lambda(t)}$ growing unbounded as t tends to T. Therefore, we should aim at feedback laws $u = F(s, y)$ providing fixed-time convergence for the y variable. On the other hand, recall that the n-th order perturbed chain of integrators is the basic model for sliding mode control, cf. [17], for which there exist plenty of efficient finite time stabilizers with eventually good robustness properties. At the heart of these stabilizers, lies the technique of weigthed-homogeneity with *constant* homogeneity coefficient. We will show that this technique easily extends to handle (4) and its extra linear term $D_{\mathbf{r}}y$ to produce fixed-time stabilizers for (4) under the assumption that bounds on b and d in (4) are known a priori. In particular, under that assumption, this resolves in a satisfactory manner one of the issues raised in [18], namely that of avoiding a gain growing unbounded without sacrifice on the regulation accuracy in x.

The second objective of the paper consists in addressing the difficult issues of robustness with respect to measurement noise and unmatched uncertainties for prescribed time stabilisation of (1). As mentioned earlier regarding time-varying homogeneity approach, the disturbances corresponding to these perturbations become, at the best, amplified by $D^{\mathbf{r}}_{\lambda(t)}$ in (4). It is not clear at all how to handle (4) with disturbances growing unbounded. This is why we present in the second part of the paper a feedback design that does not involve any time-varying function $\lambda(\cdot)$. This will allow us to provide partial robustness results in case of measurement noise and unmatched disturbances on the feedback. Here, robustness must be understood in the ISS setting of Definition 4 and not anymore according to Definitions 2 and 3. Our construction is based on fixed-time stabilisation with a control on the settling time in the case of an unperturbed chain of integrators and then on the use a simple trick to extend that solution to prescribed-time stabilisation. To perform that strategy, one must get an explicit hold on several parameters. To be more precise, the fixed-time stabilisation design relies on sliding mode feedbacks with state-dependent homogeneity degree. This idea was first considered in [8] and [9] with a completely explicit feedback law. The latter bears a serious drawback since it is discontinuous. This defect has been removed in a subsequent work in [13], relying on an appropriate perturbation argument. However that latter solution does not bear an explicit character, which is an issue in order to estimate the settling time, and hence it requires important extra work for practical implementations. Moreover, it can be adapted only to a restricted set of perturbations.

Our feedback design for fixed-time stabilisation relies on the sliding mode feedback laws proposed by [10] for finite-time stabilisation of an n-th order pure chain of integrators . Recall that, in that reference, it is proved that, for every homogeneity parameter $\kappa \in [-\frac{1}{n}, \frac{1}{n}]$, there exists a control law $u = \omega^H_\kappa(x)$ which stabilizes $\dot{x} = J_n x + u\ e_n$ and a Lyapunov function V_κ for the closed-loop system satisfying $\dot{V}_\kappa \leq -CV_\kappa^{\frac{2+2\kappa}{2+\kappa}}$, for some positive constant C, independent of κ. One of the main avantages of these feedbacks and Lyapunov functions is that they admit explicit closed forms formulas computable once the dimension n is given. In order to first obtain fixed-time stabilisation, we choose, as in [9], a feedback law of the type $u = \omega^H_{\kappa(x)}(x)$, where the homogeneity parameter is a state function and, by using the smart idea of [13], we can also make $x \mapsto \kappa(x)$ continuous. We finally use a standard homogeneity trick to pass from fixed-time to prescribed-time stabilisation.

The structure of the paper goes as follows. In Section 2, general stability notions and homogeneity properties are recalled. In Section 3, time-varying weighted homogeneity is considered for n-th order perturbed chains of integrators: Subsection 3.1 studies thoroughly linear time-varying homogeneous feedbacks while in Subsection 3.2, we provide sliding mode based feedbacks with uniformly bounded gains. We gather in Section 4 a new design of a sliding mode inspired feedback for which we characterise explicitly the parameters and we give some ISS properties in presence of measurement noise and unmatched disturbances. Finally we collect in an appendix the proofs of technical results used in the text.

2. Stability definitions

In this paper, we will consider various non autonomous differential equations $\dot{x} = f(x,t)$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a vector field. When it exists, the solution of $\dot{x} = f(x,t)$ for an initial condition $x_0 \in \mathbb{R}^n$ is denoted by $X(t,x_0)$. We recall the main stability notions needed in the paper, see [12].

Definition 1. *Let Ω be an open neighborhood of the origin assumed to be an an equilibrium point of f.*[1] *At 0, the system is said to be:*

(a) Lyapunov stable *if for any $x_0 \in \Omega$ the solution $X(t,x_0)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \delta$ then $\|X(t,x_0)\| \leq \epsilon$, $\forall t \geq 0$.;*
(b) asymptotically stable *if it is Lyapunov stable and for any $\kappa > 0$, $\epsilon > 0$ there exists $T(\kappa,\epsilon) \geq 0$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \kappa$ then $\|X(t,x_0)\| \leq \epsilon$, $\forall t \geq T(\kappa,\epsilon)$;*
(c) finite-time converging from Ω *if for any $x_0 \in \Omega$ there exists $0 \leq T < +\infty$ such that $X(t,x_0) = 0$ for all $t \geq T$. The function $\mathcal{T}(x_0) = \inf\{T \geq 0 \mid X(t,x_0) = 0, \forall t \geq T\}$ is called the* settling time for x_0 *of the system;*
(d) finite-time stable *if it is Lyapunov stable and finite-time converging from Ω.*
(e) fixed-time stable *if it is finite-time stable and $\sup_{x_0 \in \Omega} \mathcal{T}(x_0) < +\infty$ and the latter is referred as the* settling time *of the system.*

Furthermore, for prescribed-time stability and robustness issues, we consider disturbances $d : [0,\infty) \to \mathbb{R}^p$ which are measurable functions where $\|d\|_{[t_0,t_1)}$ denotes the essential supremum over any time interval $[t_0,t_1)$ in $[0,\infty)$. If $[t_0,t_1) = [0,\infty)$, we say that d is bounded if $\|d\|_\infty := \|d\|_{[0,\infty)}$ is finite. We have the following two definitions (cf. [18] and [13]).

Definition 2. *A system $\dot{x} = f(x,t,d)$ is* prescribed-time input-to-state stable in time T *(PT-ISS) if there exist functions $\beta \in \mathcal{KL}$*[2]*, $\gamma \in \mathcal{K}$ and $\lambda : [t_0, t_0+T) \to \mathbb{R}_+^*$ such that λ tends to infinity as t tends to t_0+T and, for all $t \in [t_0, t_0+T)$ and bounded d, $\|x(t)\| \leq \beta\left(\|x_0\|, \lambda(t)\right) + \gamma\left(\|d\|_{[t_0,t]}\right)$.*

Definition 3. *A system $\dot{x} = f(x,t,d)$ is* fixed-time input-to-state stable in time T and convergent to zero *(PT-ISS-C) if there exist functions β, $\beta_f \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $\lambda : [t_0, t_0+T) \to \mathbb{R}_+^*$ such that λ tends to infinity as t tends to t_0+T and, for all $t \in [t_0, t_0+T)$ and bounded d, $\|x(t,d,x_0)\| \leq \beta_f\left(\beta\left(\|x_0\|, t-t_0\right) + \gamma\left(\|d\|_{[t_0,t]}\right), \lambda(t)\right)$.*

Definition 4. *A system $\dot{x} = f(x,t,d)$ is* input-to-state practically stable *(ISpS) if, for any bounded disturbance d, there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $c > 0$ such that, for all $t \geq 0$ and bounded d, $\|x(t,d,x_0)\| \leq \beta\left(\|x_0\|, t\right) + \gamma\left(\|d\|_{[0,t]}\right) + c$. The system is* input-to-state stable *(ISS) if $c = 0$.*

[1]Meaning that for $f(t,0) = 0$ for all $t \geq 0$.

[2]A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to a class $\mathcal{K}$ if it is strictly increasing and continuous with $\gamma(0) = 0$. A function α is said to belong to a class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it increases to infinity. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to a class $\mathcal{KL}$ if for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot,t) \in \mathcal{K}_\infty$ and if for each fixed $s \in \mathbb{R}_+$, $\beta(s,t) \underset{t\to\infty}{\longrightarrow} 0$.

Note that $(PT - ISS)$ is a much stronger property than ISS.

Remark 5. *Definitions 2 and 3 have been given in* [18] *but with the explicit choice* $\lambda(t) = \frac{t-t_0}{T+t_0-t}$.

Next, basic definitions of homogeneity are collected.

Definition 6.

(i) *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be* homogeneous *of degree* $m \in \mathbb{R}$ *with respect to the weights* $\mathbf{r} = (r_1, ..., r_n) \in \mathbb{R}^n_{>0}$ *if for every* $x \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^*_+$, $f(D^{\mathbf{r}}_\lambda\ x) = \lambda^m f(x)$, *where* $D^{\mathbf{r}}_\lambda = \operatorname{diag}(\lambda^{r_i})^n_{i=1}$ *defines a family of dilations. We also say that* f *is* $\mathbf{r}$*-homogeneous of degree* m.

(b) *A vector field* $\Phi = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be* homogeneous *of degree* $m \in \mathbb{R}$ *if for all* $1 \leq k \leq n$, *for all* $x \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^*_+$, $f_k(D^{\mathbf{r}}_\lambda\ x) = \lambda^{m+r_k} f_k(x)$, *i.e., each coordinate function* f_k *is homogeneous of degree* $m + r_k$. *We also say that* F *is* $\mathbf{r}$*-homogeneous of degree* m.

(c) *Let* Φ *be a continuous vector field. If* Φ *is* $\mathbf{r}$*-homogeneous of degree* m, *then the system* $\dot{x} = \Phi(x),\ x \in \mathbb{R}^n$ *is* $\mathbf{r}$-homogeneous *of degree* m.

The next lemma is important in the proof of our results in Section 4 (see for instance [13]).

Lemma 7. [14] *Let* $\dot{x} = f(x,t)$ *be a* $\mathbf{r}$*-homogenous system of degree* κ *asymptotically stable at the origin. Then at the origin, the system is globally finite-time stable if* $\kappa < 0$, *globally exponentially stable if* $\kappa = 0$ *and globally fixed-time stable with respect to any open set containing the origin if* $\kappa > 0$.

3. Time-varying homogeneity

Let n be a positive integer, $(e_i)_{1 \leq i \leq n}$ the canonical basis of $\mathbb{R}^n$ and J_n the n-th Jordan block, i.e. $J_n e_i = e_{i-1}$, $1 \leq i \leq n$, with the convention that $e_0 = 0$. For $\lambda > 0$, using the notation above for $D^{\mathbf{r}}_\lambda$ (see also [4]), set, for $r_i := n - i + 1$, $1 \leq i \leq n$,

$$D^{\mathbf{r}}_\lambda = \operatorname{diag}(\lambda^{r_i})^n_{i=1},\ D^{\mathbf{r}}_\lambda J_n \left(D^{\mathbf{r}}_\lambda\right)^{-1} = \lambda J_n,\ D^{\mathbf{r}}_\lambda e_n = \lambda e_n. \tag{5}$$

(The second relation above simply says that the linear vector field induced by J_n on $\mathbb{R}^n$ is $\mathbf{r}$-homogeneous of degree -1.)

In the literature devoted to prescribed time stabilization (see [18] and references therein) and as clearly stated in Definitions 2 and 3, the quantity $\frac{t-t_0}{T+t_0-t}$ is a new time scale which tends to infinity as t tends to the prescribed convergence time T. This fact suggests to consider the homogeneity parameter λ depending on the time t in such a way that, if one sets the new time to be equal to

$$s : [0, T) \to \mathbb{R}^*_+, \quad s(t) = \int_0^t \lambda(\xi) d\xi, \tag{6}$$

then $s(t)$ tends to infinity as t tends to T. If $x(t)$ denotes a trajectory of (1), it is natural to consider the change of coordinates and time given by

$$y(s) = D^{\mathbf{r}}_{\lambda(t)} x(t),\ \forall\ t \in [0, T). \tag{7}$$

In order to analyse the dynamics of y, we use y' to denote the derivative of y with respect to the new time s. Using (1), (5) and (6), we obtain:

$$\lambda\ y' = \dot{y} = \dot{\lambda}\ \frac{\partial D^{\mathbf{r}}_\lambda}{\partial \lambda} (D^{\mathbf{r}})^{-1}_\lambda y + \lambda\ \left(J_n\ y + b\ u\ e_n + d\ e_n\right). \tag{8}$$

For every $\mu > 0$, we also have that

$$\frac{\partial D_\mu^{\mathbf{r}}}{\partial \mu}(D^{\mathbf{r}})_\mu^{-1} = \frac{1}{\mu} D_{\mathbf{r}}, \quad \text{with} \quad D_{\mathbf{r}} := \operatorname{diag}(r_i)_{1 \leq i \leq n}.$$

Then (8) becomes

$$y' = \left(\frac{\dot{\lambda}}{\lambda^2} D_{\mathbf{r}} + J_n \right) y + (b(s)\ u(s) + d(s))\, e_n. \tag{9}$$

Here we consider the control u and both b and f as functions of the new time s.

Let $a : [0, T] \to \mathbb{R}$ be a non negative continuous function so that the C^1 function $A : [0, T] \to \mathbb{R}$ defined by $A(t) = \int_t^T a(s)\ ds$ is positive on $[0, T)$. Setting

$$\lambda(t) := \frac{1}{A(t)}, \quad t \in [0, T), \tag{10}$$

one gets that

$$\frac{\dot{\lambda}(t)}{\lambda^2(t)} = a(t), \quad t \in [0, T).$$

It is then immediate to see that the function $\lambda : [0, T) \to \mathbb{R}_+^*$ is increasing, tends to infinity as t tends to T and the time s defined in (6) realizes an increasing C^1 bijection from $[0, T)$ to $[0, \infty)$.

We still use $a(s)$ to denote $a(t)$. With this choice, (9) becomes

$$y' = (a(s) D_{\mathbf{r}} + J_n)\, y + \big(b(s)\ u(s) + d(s)\big) e_n. \tag{11}$$

To solve the original problem of designing a feedback control u that renders the system FT-ISS-C in time $T > 0$, the idea consists in choosing

$$u = F(y(s)), \tag{12}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function to be chosen later.

Remark 8. *An important feature in stabilisation of control systems is the fact that one usually requires the feedback law to remain bounded and ideally, to tend to zero as the state x tends to zero, even if in presence of disturbance. In the context of prescribed time stabilisation of* (1)*, this feature is automatically guaranteed by our view point of time-varying homogeneity since the feedback law takes the form* (12)*: bounding $u(t)$ uniformly on $[0, T]$ simply reduces to bound the artificial state $y(s)$ uniformly on $\mathbb{R}_+$.*

Remark 9. *For the stabilisation of* (11)*, one can of course rely on linear feedback laws, as done in the next section (and already done in* [18]*) but also on sliding mode type of feedbacks which insure fixed time stabilisation (in the scale s) with robust properties, see Subsection 3.2 below.*

3.1. **Linear feedback.** We now revisit the results obtained in [18] at the light of the time-varying homogeneity introduced in the previous section. To establish the connection with that reference, one must compares our change of variable defined in (7) and the one considered in [18]. At once, one can see that the function $\mu(\cdot) = \frac{T^{m+n}}{(T-t)^{m+n}}$ in Eq. (31) of [18] corresponds, up to a positive constant, to the time-varying homogeneity parameter $\lambda(\cdot)$ where $a(t)$ is chosen as $a(t) = (T - t)^{m-1}$ (with $m \geq 2$ integer). In opposite to [18], in our approach one does not have to take time derivatives of $\lambda(\cdot)$ (or equivalently of $\mu(\cdot)$), and hence our computations are simpler (in particular no need of Lemmas 2 and 3 in [18]).

As for the feedback control in [18], it is given by $u = -\frac{1}{b}\left(d + L_0 + L_1 + kz\right)$ (cf. Eqs. (42) and (50) in [18]), where L_0 is a linear combination of successive derivatives of μ and the state components, L_1 contains a gain matrix K_{n-1}, k is a scalar gain and z is a change of variable of the n-th coordinate of the state. The above expression of the feedback u shows that this choice of feedback can be essentially reduced to a linear one (realized by the constant k and the $\mathbb{R}^{n-1}$ vector K_{n-1} in [18]). This is the reason why we take here $F(y) = -K^T y$ for some vector $K \in \mathbb{R}^n$ to be fixed later. In that case, after replacing u in (11), it follows

$$y' = \big(a(s)D_{\mathbf{r}} + J_n - b(s)\ e_n K^T\big)\, y + d(s)\ e_n, \tag{13}$$

that is an equation of the type $y' = M(s)y + f(s)\ e_n$ where $M(s) = a(s)D_{\mathbf{r}} + J_n - b(s)\ e_n K^T$ with $b(s)$ subject to (2). In [4], such systems were considered (without the term $a(s)D_{\mathbf{r}}$) and it was proven that there exists a positive constant $\mu > 0$, a real symmetric positive definite $n \times n$ matrix $S > 0$ and a vector $K \in \mathbb{R}^n$ such that

$$\Big(J_n - b\ e_n K^T\Big)^T S + S\Big(J_n - b\ e_n K^T\Big) \le -\mu\,\mathrm{Id}_n, \quad \forall\ \underline{b} \le b \le \bar{b}, \tag{14}$$

where Id_n denotes the $n \times n$ identity matrix and S, K and μ depend on $\underline{b}$ and $\bar{b}$.

A careful examination of the argument shows actually that one can remove the upper bound on the parameter b. We thus get the slightly stronger result, whose proof is given in Appendix, for sake of completeness.

Proposition 10. *Let $n \in \mathbb{N}$ and $\underline{b} > 0$. Then there exists a positive constant $\rho > 0$, a real symmetric positive definite $n \times n$ matrix S and $K \in \mathbb{R}^n$ so that*

$$\Big(J_n - b\ e_n K^T\Big)^T S + S\Big(J_n - b\ e_n K^T\Big) \le -\rho\,\mathrm{Id}_n, \quad \forall b \ge \underline{b}. \tag{15}$$

With an obvious perturbation argument, we immediately derive the following proposition:

Proposition 11. *Using the notation of Proposition 10, there exist $\rho_0, C_0 \in \mathbb{R}_+^*$, a real symmetric positive definite $n \times n$ matrix S and $K \in \mathbb{R}^n$ so that*

$$\Big(aD_{\mathbf{r}} + J_n - b\ e_n K^T\Big)^T S + S\Big(aD_{\mathbf{r}} + J_n - b\ e_n K^T\Big) \le -\rho_0 Id_n, \quad \forall a \in [-C_0, C_0], \quad b \ge \underline{b}. \tag{16}$$

We apply the above proposition to derive ISS properties of (13) and we prove the following proposition.

Proposition 12. *Consider the dynamics given in (11) and $D_\eta^{\mathbf{r}}$ defined in (5). Then there exists $K \in \mathbb{R}^n$ and $\eta_1 > 0$ such that, for $\eta \ge \eta_1$, the state feedback $u = -K^T D_\eta^{\mathbf{r}} y$ provides the following estimate:*

$$|y_i(s)| \le \frac{\max(1, \eta^{n-1})}{\eta^{n-i}} \exp(-C_S \rho_1 \eta\ s)\|y(0)\| + \frac{C_S}{\eta^{n-i+1}} \max_{r \in [0,s]} |d(r)|, \quad \forall s \ge 0, \quad 1 \le i \le n. \tag{17}$$

where C_S and ρ_1 are positive constants only depending on the lower bound $\underline{b}$.

Proof. Fix $\eta > 0$ and set $z_\eta = D_\eta^{\mathbf{r}} y$. From (13), with $u = -K_\eta y = -K z_\eta$, one gets that z_η verifies the following dynamics, after setting the time $\xi := \eta s$,

$$\frac{dz_\eta}{d\xi} = \frac{D_\eta^{\mathbf{r}} y'}{\eta} = \left(\frac{a(\xi/\eta)}{\eta} D_{\mathbf{r}} + J_n - b(\xi/\eta)\ e_n K^T\right) z + d(\xi/\eta)\ e_n. \tag{18}$$

Set $C_a = \sup_{s \ge 0} |a(s)| = \max_{t \in [0,T]} |a(t)|$.

One takes the Lyapunov function $V(z_\eta) = z_\eta^T S z_\eta$ and takes its time derivative along (18). Then, by taking $\eta \ge \frac{C_a}{C_0}$ and using Proposition 11, one gets

$$\begin{aligned} \frac{dV(z_\eta(\xi))}{d\xi} &\leq -\rho_0\|z_\eta(\xi)\|^2 + 2|z_\eta^T(\xi)Se_n| \max_{r\in[0,\xi/\eta]}|d(r)| \\ &\leq -\rho_0\|z_\eta(\xi)\|^2 + C_S\|z_\eta^T(\xi)\| \max_{r\in[0,\xi/\eta]}|d(r)|, \end{aligned} \tag{19}$$

where C_S stands for "any" constant only depends on S, i.e. on $\underline{b}$ and C_a.

One deduces that there exists constants C_S such that

$$\|z_\eta(\xi)\| \leq \exp(-C_S\mu\xi)\|z_\eta(0)\| + C_S \max_{r\in[0,\xi/\eta]}|d(r)|, \quad \forall\xi\geq 0.$$

We now write the previous inequality in terms of $y(s)$. After noticing that

$$\eta^{n-i+1}|y_i(s)| = |(z_\eta)_i(\xi)| \leq \|z_\eta(\xi)\|,\ 1\leq i\leq n, \quad \|z_\eta(0)\| \leq \eta\max(1,\eta^{n-1})\|y(0)\|,$$

one gets (17).

□

One can rewrite the previous argument using an LMI formulation. For that purpose, one needs a result similar to Proposition 11, which involves the extra parameter η. More precisely, one easily shows the following proposition.

Proposition 13. *Let $n\in\mathbb{N}$ and $\underline{b}\in\mathbb{R}_+^*$. Then there exist a positive constant ρ, a real symmetric positive definite $n\times n$-matrix S and a vector $K\in\mathbb{R}^n$ such that, for every $C>0$ there exists η_1 so that, the following holds true,*

(20)

$$\left(aD_{\mathbf{r}}+J_n-b\ e_nK_\eta^T\right)^T S_\eta+S_\eta\left(aD_{\mathbf{r}}+J_n-b\ e_nK_\eta^T\right) \leq -\mu_*\eta\ (D^{\mathbf{r}})_\eta^2, \quad a\in[-C,C],\ \eta\geq\eta_1\ b\geq\underline{b},$$

where $S_\eta = D_\eta^{\mathbf{r}}SD_\eta^{\mathbf{r}}$ and $K_\eta = D_\eta^{\mathbf{r}}K$.

To see that, simply take $K_\eta = D_\eta^{\mathbf{r}}K_1$ and multiply the LMI (16) on the left and on the right by $D_\eta^{\mathbf{r}}$. That yields (20).

Using Proposition 12 and the fact that

$$\lambda(t)^{n-i+1}|x_i(t)| \leq |y_i(s)|,\ 1\leq i\leq n,\ t\geq 0,$$

we deduce PT-ISS-C for x in any time $T>0$.

We gather in the following corollary our findings, which are similar to Theorem 1 in [18].

Corollary 14. *Consider the dynamics given in* (1)*. Let $a:[0,T]\to\mathbb{R}$ any non negative continuous function such that $\int_t^T a(\xi)d\ \xi>0$ for $t\in[0,T)$. Then set*

$$\lambda(t) = \frac{1}{\int_t^T a(\xi)d\xi}, \quad s(t) = \int_0^t \lambda(\xi)d\xi, \quad 0\leq t<T. \tag{21}$$

There exists $K\in\mathbb{R}^n$ such that, for every $\eta\geq 1$, the state feedback

$$u = -K^T D^{\mathbf{r}}_{\eta\lambda(t)}x(t), \quad t\geq 0, \tag{22}$$

where $D_\lambda^{\mathbf{r}}$ is defined in(5)*, provides the following estimate, for every $t\geq 0$ and $1\leq i\leq n$,*

$$|x_i(t)| \leq \frac{1}{(\eta\lambda(t))^{n-i+1}}\Big(\eta\max(1,\eta^{n-1})\exp\left(-C_S\mu\eta\ s(t)\right)\|x(0)\| + C_S\max_{r\in[0,t]}|d(t)|\Big). \tag{23}$$

where C_S and μ are positive constants only depending on the lower bound $\underline{b}$.

Remark 15. *The case where $a(t)=(T-t)^m$ with m positive integer corresponds to* [18] *and one can choose another a which goes faster to 0 as t tends to T, for instance $exp(-1/T-t)/(T-t)^2$, which yields to $\lambda(t)=exp(-1/(T-t))$ and then faster rates of convergence.*

One should now refer to Section 3.2 in [18] which provides the advantages and limits of such a feedback regulation. In the sequel, we only insist on what we believe are the advantages of our approach with respect to that of [18] as well as the inherent limitations in terms of robustness of feedback strategies based on time-varying homogeneity.

Remark 16. *The disturbance we consider here has a simpler expression than that of* [18]*, the latter being bounded by* $|d(t)|\psi(x)$*, with* d *any measurable function on* $[0,\infty)$ *and* $\psi \geq 0$ *a known scalar-valued continuous function. To lighten the presentation, we do not consider the function* ψ *since the analysis in this case is entirely similar to the above by using Eq.* (25) *in* [18].

Remark 17. *Let us compare our results with those obtained in* [18]*. First of all, we recover at once the main result of that reference (Theorem* 2 *and Inequality* (79)*) by choosing the function* $a(\cdot)$ *appearing in the theorem to be equal to* $C(T-t)^m$ *where* C *is a positive constant and* m *a positive integer. We have though slightly better results since we can prescribe the rate of exponential decrease as well as the estimate on the error term modeled by* $f(\cdot)$ *thanks to the occurence of the parameter* η *in our findings. Indeed the choice of the function* $\lambda(\cdot)$ *in* [18] *(called* $\mu(\cdot)$ *in Equation* (30) *of* [18]*) must be specific because it relies on the fact that time derivatives of* $\mu(\cdot)$ *must be expressed as polynomials in* $\lambda(\cdot)$*, cf. Lemmas* 2 *to* 4 *in the reference therein. Instead, using our presentation, it turns out there is more freedom in the choice of* λ*. More importantly, our presentation yields simpler proofs of convergence with a unique time scale for variables and everything boiling down to LMIs. Another advantage of the more transparent structure of the feedback is given in the subsequent remarks, where we are able to explain in a very explicit manner the limitations of the present feedback law, as they are suggested in the discussion* 3.2 *in* [18]*, as well as in the conclusion of that reference.*

Remark 18. *As noticed in* [18]*, the linear feedback defined in* (22) *is not suitable if it is subject to measurement noise on* $x(\cdot)$*. More precisely, this amounts to have instead of* (22) *a feedback* $\tilde{u}$ *given by*

$$\tilde{u}(t) = -K^T D^{\mathbf{r}}_{\eta\lambda(t)}\big(x(t)+d(t)\big) = u(t) - K^T D^{\mathbf{r}}_{\eta\lambda(t)} d(t), \quad t \geq 0,$$

i.e., with a disturbance d *in* (1) *of the form* $\eta\lambda(t)\max(1,(\eta\lambda)^{n-1}(t))\|d(t)\|$*. We can only derive from Corollary 14 the following estimate, for every* $t \geq 0$ *and* $1 \leq i \leq n$*,*

(24)
$$|x_i(t)| \leq \frac{\eta\max(1,\eta^{n-1})}{(\eta\lambda)^{n-i+1}(t)}\exp\left(-C_S\mu\eta\ s(t))\right)\|x(0)\| + C_S\frac{\max(1,(\eta\lambda)^{n-1}(t))\max_{r\in[0,t]}\|d(r)\|}{(\eta\lambda)^{n-i}(t)}.$$

The right-hand side blows up as t *tends to* T*, except for* $i=1$*, with a loss of regulation accuracy (we do not have anymore convergence to zero but to an arbitrary small neighborhood).*

On the other hand, by choosing η *of the amplitude of* $\lambda(t)$ *as* t *tends to* T*, we deduce at once from* (24) *the following corollary.*

Corollary 19. *With the notations of Corollary 14, assume that one feeds the dynamics given in* (1) *with the pertubed feedback* $\tilde{u}(t) = -K^T D^{\mathbf{r}}_{\eta\lambda(t)}\big(x(t)+d(t)\big)$*. Then for every time* $T' < T$*, there exists* $\eta > 0$ *such that, one gets that*

$$\max_{t\in[0,T']}|x_i(t)| \leq \eta\|x(0)\| + C_{T',T}\max_{t\in[0,T']}|d(t)|, \quad \forall t \geq 0, \quad 1 \leq i \leq n, \tag{25}$$

where $C_{T',T}$ *is a positive constant, tending to infinity as* T' *tends to* T*.*

The previous result of semi-global nature has been already suggested in [18] *and has been obtained in the present paper thanks to the extra parameter* η*. In particular, it follows the idea that in order to obtain estimates for prescribed time control in time* T'*, one can use the previous strategy of prescribed time control in a time* $T > T'$ *and then use* (25)*. This estimate is a* *direct result of the use of the time-varying function $\lambda(\cdot)$ but, as regards measurement noise it is definitely not satisfactory.*

Remark 20. *Looking back at* (12)*, the most natural choice is a linear feedback and it has been (essentially) first addressed in* [18] *and revisited here. One can also use other feedback laws, especially those providing finite-time stability (in the scale time s). If there is measurement noise, i.e., of the type $x+d$, the feedback implemented in* (11) *will be $u(s) = F(y(s) + D^{\mathbf{r}}_{\lambda(t(s))}d(s))$ and it is likely that one can find appropriate perturbations so that the last coordinate of x will become unbounded as t tends to T. Already, in the linear case, for a double integrator for instance, it is easy to choose bounded disturbances d in the case where $a(t) = 1$ such that $y_2(s)$ has the magnitude of $\lambda^2(s)$, and then $x_2(t)$ has the magnitude of $\lambda(t)$ as t tends to T. It is not difficult to extend that remark to any feedback law F which is differentiable at zero. Such a fact prevents to get any type of ISS results and it indicates that no property such as $(PT-ISS)$ can hold in presence of measurement noise. This is why time-varying homogeneity based feedbacks are not, in our opinion, well-suited for prescribed-time stabilization in presence of measurement noise. One must to follow another approach and this is the purpose of the next section.*

3.2. **Fixed-time feedbacks.** We now consider feedback laws in (12) which will provide fixed-time stabilisation for (11) under the assumption of a priori knowledge on the uncertainties bounds. More precisely, we will simply show that the feedback law provided in [9, Theorem 5] does the job and we have the following.

Proposition 21. *Set $\varepsilon = \pm$. Assume that there exists $\alpha \in (0, 1/n)$, $c > 0$, two continuous feedback laws $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ and two C^1 functions $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_+$, which are positive definite, $\mathbf{r}$-homogeneous of degree larger than one and such that the following holds true:*

- (a)**:** *u_ε stabilizes the n-th order pure chain of integrator $\dot{x} = J_n x + ue_n$ in finite-time and along the trajectories of the corresponding closed-loop system, one has*

$$\dot{V}_\varepsilon \leq -cV_\varepsilon^{1+\varepsilon\alpha}; \tag{26}$$

- (b)**:** *for every $x \in \mathbb{R}^n$, the following geometric condition holds true:*

$$\frac{\partial V_\varepsilon}{\partial x_n} u_\varepsilon(x) \leq 0 \text{ and } u_\varepsilon(x) = 0 \Rightarrow \frac{\partial V_\varepsilon}{\partial x_n} = 0. \tag{27}$$

Define the feedback law $\omega_0 : \mathbb{R}^n \to \mathbb{R}$ as

$$\omega_0(x) = \begin{cases} u_+(x), & \text{if} \quad V_-(x) > 1, \\ u_-(x), & \text{if} \quad V_-(x) \leq 1. \end{cases} \tag{28}$$

Consider the dynamics (11) *and assume that $b(\cdot)$ verifies* (2) *and $\|d\|_\infty \leq D$ for some positive constant D. Then the feedback law $\omega : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$\omega(x) = \frac{1}{\underline{b}}\Big(\omega_0(x) + D sgn(\omega_0(x))\Big), \tag{29}$$

where ω_0 is defined in (28) *and sgn stands for the set-valued function "sign", globally stabilises* (11) *in fixed-time.*

Here the *sgn* function makes the closed-loop system corresponding to (11) and $u = \omega$ a differential inclusion and its trajectories must be understood in the Filippov sense, cf. [6]. Note also that examples of feedbacks u_ε and the Lyapunov functions V_ε verifying Items (a) and (b) are also provided in Definition 23 and Proposition 24 given in the next section.

Proof. First of all notice that, by multiplying (11) by $D^{\mathbf{r}}_\mu$ with $\mu \geq 1$ and considering the new state $D^{\mathbf{r}}_\mu y(s/\mu)$, $C_a = \sup_{s\geq 0} |a(s)|$ becomes C_a/μ and hence arbitrary small.

Set $E := \min_{V_-(x)=1} V_+(x) > 0$ and consider the sets

$$S_1 = \{x \in \mathbb{R}^n \ : \ V_+(x) \leq E\}, \ S_2 = \{x \in \mathbb{R}^n \ : \ V_-(x) \leq 1\}.$$

By definition of E, we have that $S_1 \subset S_2$. We claim that the closed-loop system corresponding to (11) and $u = \omega$ is globally fixed-time stable with respect to S_2. For that purpose, we compute the time derivative of V_+ along the trajectories outside S_2 and get

$$\begin{aligned}\dot{V}_+ &= a\langle \nabla V_+(y), D_{\mathbf{r}} y\rangle + \sum_{i=1}^{n-1} \frac{\partial V_\varepsilon}{\partial x_i} x_{i+1} + \frac{\partial V_\varepsilon}{\partial x_n}(b\omega + d),\\ &\leq a\langle \nabla V_+(y), D_{\mathbf{r}} y\rangle - cV_+^{1+\alpha} + \frac{\partial V_\varepsilon}{\partial x_n}\Big((\frac{b}{\underline{b}} - 1)\omega_0 + sgn(\omega_0)(\frac{b}{\underline{b}} D - |d|)\Big),\\ &\leq \frac{c}{2} V_+ - cV_+^{1+\alpha} \leq -\frac{c}{2} V_+^{1+\alpha}.\end{aligned} \tag{30}$$

To get the above we have used Item (a), i.e.,

$$\sum_{i=1}^{n-1} \frac{\partial V_\varepsilon}{\partial x_i} x_{i+1} + + \frac{\partial V_\varepsilon}{\partial x_n}\omega_0(x) \leq -cV_+^{1+\alpha},$$

Item (b), and the fact that the function $\langle \nabla V_+(y), D_{\mathbf{r}} y\rangle$ having the same degree of $\mathbf{r}$-homogeneity as V is smaller than $\frac{c}{2} V_+^{1+\alpha}$ outside S_2 for C_a small enough. The claim is proved by using Lemma 7.

As soon as a trajectory x of the closed-loop system corresponding to (11) and $u = \omega$ reaches S_2, it verifies $V_-(x) = 1$. Morever for trajectories in S_2, a computation entirely similar to (30) yields the differential inequality $\dot{V}_- \leq -\frac{c}{2} V_-^{1-\alpha}$, which proves that any trajectory starting at $V_-(x) = 1$ enters in S_2, remains in it for all subsequent times and, again according to Lemma 7, converges to the origin in a uniform finite-time. That concludes the proof of Proposition 21.

□

Remark 22. *Note that the feedback ω defined in* (29) *exhibits a discontinuity at $V_- = 1$. By using the feedback law of Theorem 28, one can remove that discontinuity, if in addition, an upper bound for b is assumed to be known.*

4. Robust prescribed-time stabilisation

In the previous section, a linear feedback $u = K^T y$ was considered but this choice faces a pernicious problem as soon as there is some noise measurement on the state. We propose in this section an alternative feedback law for prescribed-time stabilisation with ISS properties in presence of measurement noise and unmatched uncertainties. The construction of this feedback runs in two steps, the first one deals with the fixed-time stabilisation in the unperturbed case and the second addresses the ISS issue in the perturbed case.

4.1. A special fixed-time stabilisation design.

The unperturbed case associated with (1), namely

$$\dot{x} = J_n x + u\ e_n, \tag{31}$$

which is referred in the sequel as the *n-th order pure chain of integrators.*

To proceed, we rely on the original idea of [9] and use the perturbation trick of [13] to provide an explicit and continuous feedback law.

We next provide the necessary material needed to describe the feedback design given in [9]. The following construction, which is based on a backstepping procedure, has been given first in [10] and we will modify it to handle the present situation.

Definition 23. *Let $\ell_j > 0$, $j = 1, \cdots, n$ be positive constants. For $\kappa \in \left[-\frac{1}{n}, \frac{1}{n}\right]$, define the weights $\mathbf{r}(\kappa) = (r_1, \cdots, r_n)$ by $r_j = 1 + (j-1)\kappa$, $j = 1, \cdots, n$. Define the feedback control law*

$$u = \omega_\kappa^H(x) := v_n, \tag{32}$$

where the $v_j = v_j(x)$ are defined inductively by:

$$v_0 = 0, \ v_j = -\ell_j \lceil \lceil x_j \rfloor^{\beta_{j-1}} - \lceil v_{j-1} \rfloor^{\beta_{j-1}} \rfloor^{\frac{r_j+\kappa}{r_j \beta_{j-1}}}, \tag{33}$$

and where the β_i's are defined by $\beta_0 = r_2$, $(\beta_j + 1) r_{j+1} = \beta_0 + 1 > 0$, $j = 1, ..., n-1$.

For $1 \le j \le n$, we also consider the union of the homogeneous unit spheres associated with $\mathbf{r}$, $\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$, i.e.,

$$S^j = \bigcup_{\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]} \left\{ x \in \mathbb{R}^j \ \mid \ |x_1|^{\frac{2}{r_1}} + \cdots + |x_n|^{\frac{2}{r_n}} = 1 \right\}. \tag{34}$$

Then S^j is clearly a compact subset of $\mathbb{R}^j$ and dealing with this set constitutes the main difference with [10].

We have then the following proposition.

Proposition 24. *There exist positive constants $\ell_j > 0$, $j = 1, \cdots, n$ such that for every $\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$, the feedback law $u = \omega_\kappa^H(x)$ defined in* (32) *stabilizes the system* (31). *Moreover, there exists a homogeneous C^1-function $V_\kappa : \mathbb{R}^n \to \mathbb{R}_+$ given by*

$$V_\kappa(x) = \sum_{j=1}^{n} \frac{\left(|x_j|^{\beta_{j-1}+1} - |v_{j-1}|^{\beta_{j-1}+1} \right)}{\beta_{j-1} + 1} - \lceil v_{j-1} \rfloor^{\beta_{j-1}} (x_j - v_{j-1}), \tag{35}$$

which is a Lyapunov function for the closed-loop system (31) *with the state feedback ω_κ^H, and it satisfies*

$$\dot{V}_\kappa \le -C V_\kappa^{1+\alpha(\kappa)}, \qquad \alpha(\kappa) := \frac{\kappa}{2+\kappa}, \tag{36}$$

for some positive constant C, independent of κ. Moreover, V_κ is $\mathbf{r}(\kappa)$-homogeneous of degree $(2+\kappa)$ with respect to the family of dilations $\left(D_\lambda^{\mathbf{r}(\kappa)}\right)_{\lambda > 0}$.

Remark 25. (i)**:** *The previous proposition is essentially Theorem* 3.1 *of* [10], *except that the gains ℓ_i are uniform with respect to $\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$. The choice of $\frac{1}{2n}$ has been made because the previous proposition actually holds true for $\kappa \in \left[-\frac{1}{n}, \frac{1}{n}\right]$ at the exception that $V_{\frac{1}{n}}$ is not C^1 on $\mathbb{R}^n$.*

(ii)**:** *The critical exponent $1+\alpha(\kappa)$ appearing in* (36) *is larger than one if $\kappa > 0$ and smaller than one if $\kappa < 0$.*

(iii)**:** *For $\kappa = 0$, one gets a linear feedback and V_0 is a positive definite quadratic form, hence there exists a real symmetric positive definite $n \times n$ matrix P such that $V_0(x) = x^T P x$ for every $x \in \mathbb{R}^n$. Finally, the time derivative of V_0 is associated with the $n \times n$ matrix $L^T P + PL$ where L is the companion matrix associated with the coefficients $-l_1, \cdots, -l_n$ on the last line. We deduce at once that L is Hurwitz since the differential inequality* (36) *for $\kappa = 0$ is equivalent to the LMI, $A^T P + PA \le -CP$. We set $Q := -(A^T P + PA)$, which is a real symmetric positive definite $n \times n$ matrix.*

Proof. The argument follows closely that of Theorem 3.1 of [10], but we will bring some technical changes to obtain the required uniformity with respect to $\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$. Moreover, in order to show in the next section the explicit character of our construction, we will provide quantitative estimates on the several constants involved in the construction, which are new with respect to [10].

Let $\kappa \in \left[-\frac{1}{2n}, \frac{1}{2n}\right]$ and set, for $1 \leq j \leq n$, $x^{(j)} = (x_1, \cdots, x_j)$,

$$W_{\kappa,j}(x^{(j)}) = \int_{v_{j-1}}^{x_j} \left(\lfloor s \rceil^{\beta_{j-1}} - \lfloor v_{j-1} \rceil^{\beta_{j-1}} \right) ds$$

$$= \frac{1}{\beta_{i-1}+1} \left(|x_j|^{\beta_{j-1}+1} - |v_{j-1}|^{\beta_{j-1}+1} \right) - \lfloor v_{j-1} \rceil^{\beta_{j-1}} (x_j - v_{j-1}), \tag{37}$$

and

$$V_{\kappa,0} := 0, \quad V_{\kappa,j} := W_{\kappa,j} + V_{\kappa,j-1}. \tag{38}$$

It holds that $V_\kappa(x) = V_{\kappa,n}(x^{(n)}) = \sum_{i=1}^n W_{\kappa,j}(x^{(j)})$. The choice of the ℓ_j is made recursively at each step $1 \leq j \leq n$ by considering, as in [10], the following expression

$$\frac{d}{dt} V_{\kappa,j} = \frac{d}{dt} V_{\kappa,j-1} + \frac{\partial V_{\kappa,j-1}}{\partial x_{j-1}}(x_j - v_{j-1}) + \sum_{i=1}^{j-1} \frac{\partial W_{\kappa,j}}{\partial x_i} x_{i+1} + \frac{\partial W_{\kappa,j}}{\partial x_j} v_j, \tag{39}$$

where $\frac{d}{dt} V_{\kappa,j-1}$ is used to denote the time derivative of $V_{\kappa,j-1}$ is taken along the $(j-1)$th pure chain of integrators. Note that the functions $\frac{\partial V_{\kappa,j-1}}{\partial x_{j-1}}$ and $\frac{\partial W_{\kappa,j}}{\partial x_i}$ are continuous.

We also get that, for $1 \leq j \leq n$, one has

$$\frac{\partial V_{\kappa,j-1}}{\partial x_{j-1}}(x_j - v_{j-1}) = \frac{\partial W_{\kappa,j-1}}{\partial x_{j-1}}(x_j - v_{j-1}) = \left(\lfloor x_{j-1} \rceil^{\beta_{j-2}} - \lfloor v_{j-2} \rceil^{\beta_{j-2}} \right)(x_j - v_{j-1}) \tag{40}$$

$$\frac{\partial W_{\kappa,j}}{\partial x_j} v_j = -\ell_j Z_{\kappa,j}, \quad Z_{\kappa,j} = \left| \lfloor x_j \rceil^{\beta_{j-1}} - \lfloor v_{j-1} \rceil^{\beta_{j-1}} \right|^{2(1+\kappa)/r_j \beta_{j-1}}. \tag{41}$$

We will need the following elementary fact: for every α in a compact set of $\mathbb{R}_+^*$ and $M > 0$, there exists two positive constants A, B such that, for every real numbers $|x|, |y| \leq M$,

$$A|x-y|^{\max(1,\alpha)} \leq \left| \lfloor x \rceil^\alpha - \lfloor y \rceil^\alpha \right| \leq B|x-y|^{\min(1,\alpha)}. \tag{42}$$

We next prove by induction on $1 \leq j \leq n$, that there exists positive real numbers $\ell_1, \cdots, \ell_n$ such that

$$\max\{\frac{d}{dt} V_{\kappa,j} \mid -\frac{1}{2n} \leq \kappa \leq \frac{1}{2n}, \; x^{(j)} \in S^j\} \leq -\frac{l_1}{2^{j-1}}. \tag{43}$$

By homogeneity and for $j = n$, one immediately gets (36) and the conclusion of the proposition.

In the rest of the argument, we use K_j, M_j, L_j to denote positive constants depending on S^j and $\ell_1, \cdots, \ell_{j-1}$ but independent of ℓ_j. For $j = 1$, (43) reduces to $\frac{d}{dt} V_{\kappa,1} = -l_1$ and any positive l_1 does the job. For the inductive step with $2 \leq j \leq n$, assume that $\ell_1, \cdots, \ell_{j-1}$ have been built with the required properties, in particular we have $\frac{d}{dt} V_{\kappa,j-1} \leq -\frac{l_1}{2^{j-2}}$ on S^{j-1}.

From (40), we get

$$\left| \frac{\partial V_{\kappa,j-1}}{\partial x_{j-1}}(x_j - v_{j-1}) \right| \leq K_j |x_j - v_{j-1}|. \tag{44}$$

For $1 \leq i \leq j-1$, the continuous function $\frac{\partial W_{\kappa,j}}{\partial x_i}$ is $\mathbf{r}(\kappa)$-homogeneous of degree $(2+\kappa)$ with respect to the family of dilations $\left(D_\varepsilon^{\mathbf{r}(\kappa)}\right)_{\varepsilon>0}$. (Actually, one restricts this homogeneity to $x^{(j)}$.) Moreover, it is equal to zero if $x_j = v_{j-1}$. Hence, by using repeatedly (42) and noticing that

$v_1, \cdots, v_{j-1}$ do not depend on ℓ_j, one deduces that there exists $L_j, M_j > 0$ such that, for every $x^{(j)} \in S^j$, if $\tilde{\beta}_j = \min(1, \beta_{j-1})$, then, for every $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$ and $x^{(j)} \in S^j$,

$$|\sum_{i=1}^{j-1} \frac{\partial W_{\kappa,j}}{\partial x_i} x_{i+1}| \leq L_j |x_j - v_{j-1}|^{\tilde{\beta}_j}, \tag{45}$$

$$|Z_{\kappa,j}| \geq M_j |x_j - v_{j-1}|^{2(1+\kappa)/r_j \tilde{\beta}_j}. \tag{46}$$

(Note that we used in the above that $|x_j| \leq 1$ for $1 \leq j \leq n$ and $x^{(j)} \in S^j$ as well as a bound on the v_j obtained with an immediate inductive argument based on (33).)

Inserting (44), (45) and (46) in (43), one deduces that,

$$\frac{d}{dt} V_{\kappa,j} \leq -\frac{l_1}{2^{j-2}} + (K_j + L_j)|x_j - v_{j-1}|^{\tilde{\beta}_j} - l_j M_j |x_j - v_{j-1}|^{2(1+\kappa)/r_j \tilde{\beta}_j}. \tag{47}$$

Set $\xi_j = \big(\frac{l_1}{(K_j+L_j)2^{j-1}}\big)^{1/\tilde{\beta}_j}$. By definition, one gets that $\frac{d}{dt} V_{\kappa,j} \leq -\frac{l_1}{2^{j-1}}$ if $|x_j - v_{j-1}| \leq \xi_j$. Now, if $|x_j - v_{j-1}| > \xi_j$, one chooses l_j such that

$$\ell_j \geq \frac{(K_j + L_j)}{M_j \xi_j^{2(1+\kappa)/r_j \tilde{\beta}_j - 1/\tilde{\beta}_j}}. \tag{48}$$

This is possible since the right-hand side of the above inequality does not depend on ℓ_j. In that case, $\frac{d}{dt} V_{\kappa,j} \leq -\frac{l_1}{2^{j-2}}$. This concludes the proof of the inductive step.

□

Remark 26. *One can notice in the above argument a difference with respect of that of* [10] *which consists in introducing the constants K_j, L_j and M_j. The latter provide an explicit choice in order to be as explicit as possible in view of numerical determination of the constants $\ell_1, \cdots, \ell_n$.*

We next consider a state varying homogeneity degree given next.

Definition 27. *For $m \in (0,1)$ and $\kappa_0 \in (0, \frac{1}{2n})$, define the following continuous function $\kappa : \mathbb{R}^n \to [-\kappa_0, \kappa_0]$ by*

$$\kappa(x) = \begin{cases} \kappa_0, & \text{if} \quad V_0(x) > 1 + m, \\ \kappa_0\big(1 + \frac{V_0(x) - (1+m)}{m}\big), & \text{if} \quad 1 - m \leq V_0(x) \leq 1 + m, \\ -\kappa_0, & \text{if} \quad V_0(x) < 1 - m. \end{cases} \tag{49}$$

We also need the following notation. For $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$ and a, b non negative real numbers, let $B^\kappa_{a,b}$, $B^\kappa_{\leq a}$ and $B^\kappa_{\geq b}$ respectively be the subsets of $\mathbb{R}^n$ defined respectively by

$$\begin{aligned} B^\kappa_{a,b} &:= \{x \in \mathbb{R}^n, \mid a \leq V_\kappa(x) \leq b\}, \\ B^\kappa_{<a} &:= \{x \in \mathbb{R}^n, \mid V_\kappa(x) < a\}, \\ B^\kappa_{>b} &:= \{x \in \mathbb{R}^n, \mid b < V_\kappa(x)\}, \\ B^\kappa_{a} &:= \{x \in \mathbb{R}^n, \mid V_\kappa(x) = a\}. \end{aligned}$$

The last set corresponds to the weighted spheres associated with the positive definite functions V_κ.

In the spirit of [13], we are now able to consider the feedbacks which will ultimately yield prescribed time stability, which is described next.

Theorem 28. *Assume that the uncertainty b is bounded, i.e.,*

$$\bar{b} \geq b(t) \geq \underline{b}, \quad t \geq 0, \tag{50}$$

for some positive constants $\overline{b}, \underline{b}$. Then, there exists $m \in (0,1)$ and $\kappa_0 \in (0, \frac{1}{2n})$ such that, the undisturbed n-th order chain of integrators defined by

$$\dot{x}(t) = J_n x(t) + b(t)u(t), \quad \overline{b} \geq b(t) \geq \underline{b}, \quad t \geq 0, \tag{51}$$

together with an adapted feedback law given by $\omega^H_{\kappa(x)}(x)$, with $\kappa(\cdot)$ defined in (49) is globally fixed-time stable at the origin in at most time $T(m, \kappa_0)$ upper bounded as

$$T(m,\kappa_0) \leq \frac{1}{C}\Big(\frac{r(m,\kappa_0)^{-\alpha(\kappa_0)}}{\alpha(\kappa_0)} - 2\ln(2m) + \frac{r(m,-\kappa_0)^{-\alpha(-\kappa_0)}}{-\alpha(-\kappa_0)}\Big), \tag{52}$$

where $r(m,\kappa_0) > 0$ (and $r(m,-\kappa_0) > 0$) is the largest (smallest) number $r > 0$ such that $B^{\kappa_0}_{<r}$ ($B^{-\kappa_0}_{<r}$) is contained in (contains) $B^0_{<1+m}$ ($B^0_{<1-m}$) and the constant C has been introduced in (36).

By adapted, we mean the following: strictly speaking, we must choose the feedback law $\omega^H_{\kappa(x)}(x)/\underline{b}$. However, we can replace ℓ_n by either $\ell_n/\underline{b}$ or by $\underline{b}\ell_n$ in order to satisfy (48). Hence, with no loss of generality, we assume $\underline{b} = 1$.

Proof. For this result, we follow the perturbative argument considered in the proof of Lemma 2 in [13]. For that purpose, the time derivative of V_0 along non trivial trajectories of System (51) closed by the feedback law given by $\omega^H_{\kappa(x)}(x)$ can be written as

$$\dot{V}_0 = 2x^T P(J_n x + b\omega^H_{\kappa(x)}(x)e_n) \leq -x^T Q x + 2|x^T P e_n|\delta(x), \;\; \delta(x) := \overline{b}|\omega^H_{\kappa(x)}(x) - \omega^H_0(x)|. \tag{53}$$

We have to first to show that trajectories of

$$\dot{x} = J_n x + \omega^H_{\kappa(x)}(x)\; e_n, \tag{54}$$

are well-defined and second that trajectories starting in $B^0_{>1+m}$ reach B^0_{1+m} in finite time, then "cross" it till reaching B^0_{1-m} in finite time and finally remain in $B^0_{<1-m}$ for all larger times, while converging to zero in finite time.

Since the right-hand side of (54) is continuous, there exist solutions from any initial condition defined at least on a non trivial interval. Clearly, there exists $R > 0$ such that trajectories starting at any $x_0 \in B^{\kappa_0}_{>R}$ stay in the compact set $B^{\kappa_0}_{<V_{\kappa_0}(x_0)}$ and hence are defined for all times.

Both the convergence parts of the claim follow from the arguments of [2] and Lemma 7, where one proves the following

- the closed-loop system (54) is $\mathbf{r}(\kappa_0)$-homogeneous of degree $2+\kappa_0$ in $B^0_{>1+m}$ and hence converges in finite-time to B^0_{1+m},
- the closed-loop system (54) is $\mathbf{r}(-\kappa_0)$-homogeneous of degree $2-\kappa_0$ in $B^0_{<1-m}$ and hence converges in finite-time to the origin.

For the remaining part of the argument, it amounts to show that, for $m \in (0,1)$ and $\kappa_0 \in (0, 1/n)$ small enough, the time derivative of V_0 along trajectories of (54) is negative in $B^0_{1-m,1+m}$. To see that, it is enough to notice that the function δ defined in (53) is continuous and tends to zero if either m or κ_0 tends to zero.

It remains to provide a first quantitative estimate of the "fixed-time" part of the theorem. The time needed for the closed-loop system (54) to converge to B^0_{1+m} is at most equal to the time $T_+(m,\kappa_0)$ needed to converge to $B^{\kappa_0}_{<r(m,\kappa_0)}$. By integrating (36), one derives that

$$T_+(m,\kappa_0) \leq \frac{1}{C\alpha(\kappa_0) r(m,\kappa_0)^{\alpha(\kappa_0)}}.$$

A similar reasoning yields that the time $T_-(m,\kappa_0)$ needed to converge from $B^{-\kappa_0}_{<r(m,-\kappa_0)}$ to the origin verifies the following

$$T_-(m,\kappa_0) \leq \frac{1}{-C\alpha(-\kappa_0)r(m,-\kappa_0)^{\alpha(-\kappa_0)}}.$$

(Recall that $\alpha(-\kappa_0) = \frac{-\kappa_0}{2-\kappa_0} < 0$.) It remains to upper bound the time $T_0(m,\kappa_0)$ needed to "cross" $B^0_{1-m,1+m}$. For that purpose, choose m and κ_0 small enough so that

$$M_\delta(m,\kappa_0) := \max_{x\in B^0_{1-m,1+m}} |\delta(x)| \leq \frac{C(1-m)}{2}. \tag{55}$$

In that case, (53) becomes $\dot{V} \leq -CV/2$ and one gets

$$T_0(m,\kappa_0) \leq \frac{-2\ln(2m)}{C}.$$

We conclude that the closed-loop system (54) is globally-fixed time stable with respect to the origin in settling time less than or equal to $T(m,\kappa_0)$ given by

$$T(m,\kappa_0) := T_+(m,\kappa_0) + T_0(m,\kappa_0) + T_-(m,\kappa_0).$$

One derives (52) and this concludes the proof of the theorem.

□

Remark 29. *The above result is the counterpart of Lemma 2 in* [13] *for our feedback law* ω^H_κ*. Note that in that reference, the statements of Lemma 2 and Theorem 4 as well as the argument of Lemma 2 consider the euclidean norm* $\|x\|$ *instead of* B^0_1 *in the definition of* $\kappa(\cdot)$*. As one can see from the above argument, using that norm cannot not provide the required results. However* [13] *does consider the correct controller in Lemma 3 and in the last section of the corresponding reference.*

It remains to use a standard time rescaling technique with homogeneity (cf. [11] and [9]) to pass from the result of fixed-time stability contained in Theorem 28 to a result about prescribed-time stability.

Theorem 30. *Let* $m \in (0,1)$*,* $\kappa_0 \in (0,1/n)$ *defined in Theorem 28 and the feedback law* $\omega^H_{\kappa(x)}(x)$ *defined in* (49) *which renders System* (31) *globally fixed-time stable at the origin in settling time less than or equal to* $T(m,\kappa_0)$ *defined in* (52)*. Then, given any* $T > 0$*, the the feedback law* $\omega^H_{\kappa(D_\lambda x)}(D^{\mathbf{r}}_\lambda x)$ *renders System* (31) *globally fixed-time stable at the origin in settling time less than or equal to* T *as soon as* $\lambda \geq T(m,\kappa_0)/T$*.*

Proof. For $\lambda > 0$, one sets $y(s) = D_\lambda x(t)$ with the time scale $s = \lambda t$. One deduces at once that y converges in finite time to the origin with a settling time upper bounded by $T(m,\kappa_0)$ as well as x, with a settling time upper bounded by $T(m,\kappa_0)/\lambda$. To guarantee that the latter is less than or equal to T, it is enough to choose λ as stated.

□

4.2. **Explicit determination of the main parameters.** In order to fully compare our controller $u = \omega^H_{\kappa(x)}$, with $x \mapsto \kappa(x)$ given in Definition 27 with the controller provided in [13], we must explain how to choose the parameters $m \in (0,1)$ and $\kappa_0 \in \left(0, \frac{1}{2n}\right)$ introduced in Definition 27. We also have to estimate the quantities $r(m,\kappa_0)$ and $r(m,-\kappa_0)$ in order to get a hold on the upper bound $T(m,\kappa_0)$ of the settling time to reach precise estimates of the rescaling factor λ appearing in Theorem 30.

For that purpose, we first need an explicit bound on the coordinates of $x \in B^\kappa_{1-m,1+m}$ with $-\frac{1}{2n} \leq \kappa \leq \frac{1}{2n}$. This is the content of the next lemma.

Lemma 31. *Let $m \in (0,1)$. Then, there exists an explicit positive constant X_n (depending on m and the ℓ_j's) such that, for $0 \le j \le n$, $x \in B^{\kappa}_{1-m,1+m}$ and $-\frac{1}{2n} \le \kappa \le \frac{1}{2n}$, $|x_j|, |v_j| \le X_n$.*

Proof. Fix $m \in (0,1)$, $x \in B^{\kappa}_{1-m,1+m}$ and $-\frac{1}{2n} \le \kappa \le \frac{1}{2n}$. The proof of the lemma goes by induction on j, where we prove the statement with a constant X_j explicitly depending on m and the ℓ_j's.

This is clearly true for $j = 1$ since $|v_1| = \ell_1 |x_1|^{1+\kappa}$ and $\frac{|x_1|^{1+\beta_0}}{1+\beta_0} \le V_\kappa(x) \le 1+m$. Assume that the thesis holds true for $j-1 \ge 1$. One then deduces from the definition of $W_{\kappa,j}$ in (37) and the induction hypothesis that

$$|x_j|^{1+\beta_{j-1}} \le (2+\beta_{j-1}) X_{j-1}^{1+\beta_{j-1}} + (1+\beta_{j-1}) X_{j-1}^{\beta_{j-1}} |x_j| + (1+\beta_{j-1})(1+m).$$

Since $\beta_{j-1} > 0$, one deduces at once a first explicit bound for x_j and then for v_j by using (33). □

The following lemma provides the required differences between useful quantities evaluated at any $\kappa \in [-\frac{1}{2n}, \frac{1}{2n}]$ and $\kappa = 0$. For $0 \le j \le n$, we introduce the notation $v^{\kappa}_j := v_j$, where the latter has been defined in (33).

Lemma 32. *Let $m \in (0,1)$. Then there exists explicit positive constants C^1_n, C^2_n (depending on m and the ℓ_j's) such that,*

$$\max\{|\omega^H_\kappa(x) - \omega^H_0(x) \ \mid \ -\frac{1}{2n} \le \kappa \le \frac{1}{2n},\ x \in B^{\kappa}_{1-m,1+m}\} \le C^1_n |\kappa|^{\min(1,r_n)}, \tag{56}$$

and

$$\max\{|V_\kappa(x) - V_0(x) \ \mid \ -\frac{1}{2n} \le \kappa \le \frac{1}{2n},\ x \in B^{\kappa}_{1-m,1+m}\} \le C^2_n |\kappa|^{\min(1,r_n)}. \tag{57}$$

Proof. Fix $m \in (0,1)$. We will actually prove by induction on $1 \le j \le n$, that there exists an explicit positive constant C^1_j (depending on m and $\ell_1, \cdots, \ell_j$) such that,

$$\max\{|v^{\kappa}_j(x) - v^0_j(x) \ \mid \ -\frac{1}{2n} \le \kappa \le \frac{1}{2n},\ x^{(j)} \in B^{\kappa}_{j,m}\} \le C^1_j |\kappa|^{\min(1,r_j)}, \tag{58}$$

where $B^{\kappa}_{j,m}$ is the set of $x^{(j)} \in \mathbb{R}^j$ for which $1-m \le V_{\kappa,j}(x^{(j)}) \le 1+m$.

The result is immediate for $j = 0$ and hence we turn to the inductive step for $1 \le j \le n$, assuming that the hypothesis holds for $j-1$.

Let $-\frac{1}{2n} \le \kappa \le \frac{1}{2n}$ and $x^{(j)} \in B^{\kappa}_{j,m}$. Then

$$v^{\kappa}_j(x) - v^0_j(x) = -\ell_j \lceil \lceil x_j \rfloor^{\beta_{j-1}} - \lceil v^{\kappa}_{j-1} \rfloor^{\beta_{j-1}} \rfloor^{\frac{r_{j+1}}{r_j \beta_{j-1}}} + \ell_j (x_j - v^0_{j-1}) = -\ell_j (F+G),$$

where

$$\begin{aligned} F &= \lceil \lceil x_j \rfloor^{\beta_{j-1}} - \lceil v^{\kappa}_{j-1} \rfloor^{\beta_{j-1}} \rfloor^{\frac{r_{j+1}a}{r_j \beta_{j-1}}} - \lceil \lceil x_j \rfloor^{\beta_{j-1}} - \lceil v^{0}_{j-1} \rfloor^{\beta_{j-1}} \rfloor^{\frac{r_{j+1}a}{r_j \beta_{j-1}}}, \\ G &= \lceil \lceil x_j \rfloor^{\beta_{j-1}} - \lceil v^{0}_{j-1} \rfloor^{\beta_{j-1}} \rfloor^{\frac{r_{j+1}}{r_j \beta_{j-1}}} - (x_j - v^0_{j-1}). \end{aligned}$$

By applying (42) with $\alpha = \frac{r_{j+1}}{r_j \beta_{j-1}}$, then with $\alpha = \beta_{j-1}$ and A, B depending on X_n obtained in Lemma 31, we get

$$|F| \le B \Big| \lceil v^{\kappa}_{j-1} \rfloor^{\beta_{j-1}} - \lceil v^{0}_{j-1} \rfloor^{\beta_{j-1}} \Big|^{\min(1, \frac{r_j+\kappa}{r_j \beta_{j-1}})} \le B^2 |v^{\kappa}_{j-1} - v^0_{j-1}|^{\nu_j}, \tag{59}$$

where $\nu_j := \min(1, \beta_{j-1}) \min(1, \frac{r_j+\kappa}{r_j \beta_{j-1}}) \le 1$.

To bound G, we consider

$$\begin{aligned} M &:= \max(|x_j|,|v^0_{j-1}|),\ \varepsilon := sign(x_j v^0_{j-1}) \in \{-1,1\}, \\ \tau &:= \frac{\max(|x_j|,|v^0_{j-1}|)}{M} \in [0,1],\ N := 1+\varepsilon t^{\beta j-1}, \end{aligned}$$

where we have assumed with no loss of generality that $M>0$.

An easy computation yields that

$$G = (M^{\frac{r_{j+1}}{r_j}} - M)N^{\frac{r_{j+1}}{r_j\beta_{j-1}}} + M\big(N^{\frac{r_{j+1}}{r_j\beta_{j-1}}} - N + \varepsilon(t^{\beta_{j-1}} - t)\big). \tag{60}$$

We now use the following elementary fact: for $x\geq 0$ and $\alpha>0$,

$$|x^\alpha - x| \leq |\alpha - 1|\ln(x)x^{\min(1,\alpha)}.$$

By applying that fact to (60), we deduce that there exists an explicit positive constant D_j (depending on m and $\ell_1,\cdots,\ell_{j-1}$) such that $|G|\leq |D_j|\kappa|$. From (59) and the previous inequality, we get that

$$|v^\kappa_j(x) - v^0_j(x)| \leq \ell_l\big(B^2|v^\kappa_{j-1} - v^0_{j-1}|^{\nu_j} + D_j|\kappa|\big).$$

By applying the induction hypothesis on $|v^\kappa_{j-1} - v^0_{j-1}|$, we prove the inductive step with $C^1_j := l_j(B^2 C^{\nu_{j-1}}_{j-1} + D_j)$. This concludes the proof of (58).

We now turn to the proof of (57). It is enough to prove the result for one single $W_{\kappa,j}$. Hence let $-\frac{1}{2n}\leq\kappa\leq\frac{1}{2n}$ and $x^{(j)}\in B^\kappa_{j,m}$. On gets

$$\begin{aligned} W_{\kappa,j}(x^{(j)}) - W_{0,j}(x^{(j)}) &= \frac{\beta_{j-1}-1}{\beta_{j-1}+1}\Big(|v^\kappa_{j-1}|^{\beta_{j-1}+1} - |v^0_{j-1}|^{\beta_{j-1}+1}\Big) \\ &+ \frac{1}{2}\Big(|x^\kappa_j|^{\beta_{j-1}+1} - x_j^2 - (|v^\kappa_{j-1}|^{\beta_{j-1}+1} - (v^0_{j-1})^2)\Big) \\ &- x_j\Big(\lceil v^\kappa_{j-1}\rfloor^{\beta_{j-1}} - \lceil v^0_{j-1}\rfloor^{\beta_{j-1}} + (v^\kappa_{j-1} - v^0_{j-1})\Big) \\ &+ |v^\kappa_{j-1}|^{\beta_{j-1}+1} - (v^0_{j-1})^2). \end{aligned}$$

Following the same type of estimates used to derive (58), one gets (57).

□

We can now provide explicit bounds on κ_0, for the results of the previous section to hold.

Proposition 33. *Let $m\in(0,1)$. Then there is an explicit $\kappa_0(m)\in[-\frac{1}{2n},-\frac{1}{2n}]$ such that, for every $\kappa_0\in(0,\kappa_0(m))$, the statements of Theorem 28 and Theorem 30 hold true.*

Proof. To determine $\kappa_0(m)$, we rewrite (53) as follows,

$$\dot V_0 \leq -CV_0 + 2\sqrt{V_0}\sqrt{V_0(e_n)}|\delta(x)|.$$

The constant C above has been characterized in (36).

Along trajectories of System (31) closed by the feedback law given by $\omega^H_{\kappa(x)}(x)$ inside $B^0_{1-m,1+m}$, one gets by using $1-m\leq V_0(x)\leq 1+m$ and (58) that

$$CV_0 \geq C(1-m), \quad 2\sqrt{V_0}\sqrt{V_0(e_n)}|\delta(x)| \leq 2\sqrt{1+m}\sqrt{V_0(e_n)}C^1_n\kappa_0^{1-(n-1)/2n}.$$

One chooses then $\kappa_0(m)>0$ so that $\dot V_0\leq -\frac{CV_0}{2}$ inside $B^0_{1-m,1+m}$, which yields that

$$\kappa_0(m) := \Big(\frac{C(1-m)}{4\sqrt{1+m}\sqrt{V_0(e_n)}C^1_n}\Big)^{\frac{2n}{n+1}}.$$

As for Theorem 30, the only task to complete for an explicit characterization of the parameter λ appearing in the statement consists in estimating explicitly lower bounds for $r(m,\kappa_0)$ and

$r(m,-\kappa_0)$. We provide indications for only $r(m,\kappa_0)$. By definition, every $x \in B^{\kappa_0}_{<r(m,\kappa_0)}$ belongs to $B^0_{<1+m}$. There exists $x \in B^{\kappa_0}_{r(m,\kappa_0)} \cap B^0_{\leq 1+m}$ and then $|r(m,\kappa_0)-(1+m)| \leq C_n^2 \kappa_0^{1-(n-1)/2n}$ according to (57). One deduces immediately an explicit lower bound for $r(m,\kappa_0)$

□

4.3. **ISS-type of result.** In this section, we provide the second step for our partial solution of the prescribed-time stabilization of the n-th order chain of integrators in presence of disturbances. More precisely, the aim consists in stabilizing (31) with a static feedback law $u = F(x)$, in a robust manner, i.e., with respect to measurement noise and external disturbances. The corresponding n-th order perturbed chain of integrators is given by

$$\dot{x} = J_n x + b(x)F(x+d_1)\ e_n + d_2, \tag{61}$$

where $d_1 \in \mathbb{R}^n$ is the measurement noise and $d_2 \in \mathbb{R}^n$ the external perturbation. We set $d = (d_1, d_2) \in \mathbb{R}^{2n}$ and we refer to it as the perturbation. Note that we are allowing unmatched uncertainties.

We now provide an ISS type of result regarding the robust properties of the perturbed system (61) stabilized with $k(x) = \omega^H_{\kappa(x)}(x)$ given by

$$\dot{x} = J_n x + b\omega^H_{\kappa(x+d_1)}(x)\ e_n + d_2, \quad x, d_1, d_2 \in \mathbb{R}^n, \tag{62}$$

where b verifies (50). As before, we can assume with no loss of generality that $\underline{b} = 1$. We have the following result.

Theorem 34. *With the assumptions of Theorem 30, System* (62) *is (ISS) for any bounded* $d = (d_1, d_2)$. *If* $d_1 = 0$ *and* d_2 *is parallel to* e_n *(matched uncertainty), then convergence occurs in fixed time. The same conclusion holds for any prescribed time* T *by using the feedback* $k_\mu(x) = \omega^H_{\kappa(D^{\mathbf{r}}_\mu x)}(D^{\mathbf{r}}_\mu x)$ *with* $\mu > 0$ *depending on* T.

Remark 35. *This result improves* [13, Corollary 1] *where only the property (ISpS) was obtained.*

Remark 36. *Using* k_μ *instead of* k_1 *will modify the gain functions in Definition 4 since the disturbance* $d = (d_1, d_2)$ *must be modified to* $(D^{\mathbf{r}}_\mu d_1, D^{\mathbf{r}}_\mu d_2/\mu)$.

To prove the theorem, we are not able to exhibit an ISS-Lyapunov function but, by taking into account Theorem 28 and using the characterization of (ISS) provided by [19, Theorem 2.1], it is enough to prove the following proposition.

Proposition 37. *There exists a function* F *of class* $\mathcal{KL}$ *such that, for every bounded disturbances* $d_1, d_2 : \mathbb{R}_+ \to \mathbb{R}^n$ *and any trajectory of* (62), *the following holds true*

$$\limsup_{t\to\infty} Z(x(t)) \leq F(\|d_1\|_\infty + \|d_2\|_\infty), \tag{63}$$

where

$$Z(x) = \min\Big(V_0(x), V^{1+\alpha(\kappa_0)}_{\kappa_0}(x), V^{1-\alpha(\kappa_0)}_{-\kappa_0}(x)\Big), \quad x \in \mathbb{R}^n. \tag{64}$$

Proof. The argument is similar to Item $(S-\infty)$ in [3, Proposition 2]. It is based on the following three inequalities, whose proofs are given in Appendix.

- **(i):** On the open set $B^0_{>1+m}$, the time derivative $\dot{V}_{\kappa_0}$ of V_{κ_0} along trajectories of (62) verifies almost everywhere

$$\dot{V}_{\kappa_0} \leq -\frac{C}{2} V^{1+\alpha(\kappa_0)}_{\kappa_0} + F_1(\|d_1\|_\infty + \|d_2\|_\infty), \tag{65}$$

  where F_1 is a function of class $\mathcal{KL}$.

(ii): On the set $B^0_{1-m,1+m}$, the time derivative $\dot{V}_0$ of V_0 along trajectories of (62) verifies almost everywhere

$$\dot{V}_0 \leq -\frac{C}{2}V_0 + F_2(\|d_1\|_\infty + \|d_2\|_\infty), \tag{66}$$

where F_2 is a function of class $\mathcal{KL}$.

(iii): On the open set $B^0_{<1-m}$, the time derivative $\dot{V}_{-\kappa_0}$ of $V_{-\kappa_0}$ along non trivial trajectories of (62) verifies almost everywhere

$$\dot{V}_{-\kappa_0} \leq -\frac{C}{2}V_{-\kappa_0}^{1+\alpha(-\kappa_0)} + F_3(\|d_1\|_\infty + \|d_2\|_\infty), \tag{67}$$

where F_3 is a function of class $\mathcal{KL}$.

Let $x(\cdot)$ be a non trivial trajectory of (62).

Assuming that we have at hand the above inequalities. Suppose first that there exists a time $t_0 \geq 0$ such that one of the following situations occurs:

(a): for every $t \geq t_0$, $x(t) \in B^0_{>1+m}$. By using (65), one gets that

$$\limsup_{t\to\infty} V_{\kappa_0}^{1+\alpha(\kappa_0)}(x(t)) \leq \frac{2F_1(\|d_1\|_\infty + \|d_2\|_\infty)}{C};$$

(b): for every $t \geq t_0$, $x(t) \in B^0_{1-m,1+m}$. By using (66), one gets that

$$\limsup_{t\to\infty} V_0(x(t)) \leq \frac{2F_2(\|d_1\|_\infty + \|d_2\|_\infty)}{C};$$

(c): for every $t \geq t_0$, $x(t) \in B^0_{<1-m}$. By using (67), one gets that

$$\limsup_{t\to\infty} V_{\kappa_0}^{1+\alpha(-\kappa_0)}(x(t)) \leq \frac{2F_3(\|d_1\|_\infty + \|d_2\|_\infty)}{C}.$$

Let I_+ (I_- respectively) be the set of times t such that $x(t) \in B^0_{>1+m}$ ($x(t) \in B^0_{<1-m}$ respectively). If such a t_0 does not exists, either I_+ or I_- is an infinite (countable) union of disjoint non trivial intervals (s_k, t_k), $k \geq 0$, where $\lim_{k\to\infty} s_k = \infty$. We analyse only the case where $I_+ = \cup_{k\geq 0}(s_k, t_k)$ since handling the other case is entirely similar.

Set $C_V := \max_{x\in B^0_{1+m}} V_{\kappa_0}$. For $k \geq 0$, consider the trajectory $x(\cdot)$ on $[t_k, s_{k+1}]$. Recall that $V_0(x(t_k)) = V_0(x(s_{k+1})) = 1+m$ by definition of t_k, s_{k+1}. Then, there exists $\tilde{t}_k \in [t_k, s_{k+1})$ such that $V_0(x(\tilde{t}_k)) \leq V_0(x(s_{k+1}))$ and $V_0(x(t_)) \geq 1-m$ on $[\tilde{t}_k, s_{k+1}]$. Integrating (66) from $\tilde{t}_k$ to s_{k+1} yields that $\frac{C(1-m)}{2} \leq F_2(\|d_1\|_\infty + \|d_2\|_\infty)$. Set now $L := \limsup_{t\to\infty} V_{\kappa_0}^{1+\alpha(\kappa_0)}(x(t))$. If $L \leq C_V^{1+\alpha(\kappa_0)}$, then

$$L \leq \frac{2C_V^{1+\alpha(\kappa_0)}}{C(1-m)}F_2(\|d_1\|_\infty + \|d_2\|_\infty).$$

Otherwise, assume that $L > C_V^{1+\alpha(\kappa_0)}$. Consider then the non empty set of $v > C_V^{1+\alpha(\kappa_0)}$ for which there exist two sequences $t_k \leq \tilde{t}_k < \tilde{s}_{k+1} < s_k$ such that

$$V_{\kappa_0}^{1+\alpha(\kappa_0)}(x(\tilde{t}_k)) = V_{\kappa_0}^{1+\alpha(\kappa_0)}(x(\tilde{s}_{k+1})) = v \text{ and } V_{\kappa_0}^{1+\alpha(\kappa_0)}(x(t)) \geq v,\ t \in [\tilde{t}_k, \tilde{s}_{k+1}].$$

Clearly L is the supremum of such v's. Integrating (65) between $\tilde{t}_k$ and $\tilde{s}_{k+1}$ yields at once that $v \leq \frac{2F_1(\|d_1\|_\infty+\|d_2\|_\infty)}{C}$. We deduce at once that the content of Item (b) above holds true. By collecting all the cases, we conclude the proof of Proposition 37.

□

5. Conclusion

In this paper, we have addressed the issue of prescribed-time stabilisation of an n-chain of integrators, $n \geq 1$, either pure or perturbed. We have first recasted the results obtained in [18] within the framework of time-varying homogeneity and hence provided simpler proofs. As noticed in [18], the feedback laws (linear or finite time) arising from this time-varying approach do not perform well when the n-chain of integrators is subject to perturbations (especially measurement noise), even if one stops before the prescribed settling time. We instead propose to rely on feedback laws handling fixed-time stabilisation and to apply a standard trick of time-scale reparametrisation and homogeneity to render the modified stabilisers fit for prescribed-time stabilisation of an n-th order chain perturbed of integrators. We perform that strategy in two steps. The first one consists in using feedbacks similar to those of [9] and then by relying on a nice deformation argument proposed in [13]. In a second step, we obtain an ISS type of result in the presence of measurement noise for prescribed-time stabilisation of an n-th perturbed chain of integrators. However, such an approach is meaningful if one can get an explicit hold on the various parameters involved in the above construction. This is why we devoted a section for such an objective.

6. Appendix

6.1. **Proof of Proposition 10.** We next prove the result for $\eta = 1$ and the argument is inspired from the proof of Lemma 4.0 of [7], and partly given in [4]. Given a vector $K = (k_1, \cdots, k_n)^T \in \mathbb{R}^n$ with positive entries, we consider the invertible $n \times n$ matrix M_K defined by

$$M_K = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 0 & k_1 & \cdots & k_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_1 \end{pmatrix}. \tag{68}$$

Note that

$$M_K e_n = K, \ M_K^T e_1 = K, \ M_K J_n M_K^{-1} = J_n.$$

The last equation comes from the fact that M_K is a polynomial function of J_n, namely $M_K = \sum_{i=1}^n k_i J_n^{i-1}$.

Multiplying the LMI (14) on the left and on the right by $(M_K^T)^{-1}$ and M_K^{-1} respectively yields the following LMI

$$\Big(J_n - b\ K e_1^T\Big)^T S_1 + S_1 \Big(J_n - bKe_1^T\Big) \leq -\rho M_K^T M_K, \quad \forall b \geq \underline{b}.$$

where $S_1 = (M_K^T)^{-1} S M_K^{-1}$. Let $\rho_* > 0$ such that $\rho M_K^T M_K \geq \rho_* Id_n$.

We are left to prove that there exists $\rho_* > 0$, S_1 symmetric positive definite and a vector $K \in \mathbb{R}^n$ so that the following LMI holds true,

$$\Big(J_n - b\ K e_1^T\Big)^T S_1 + S_1 \Big(J_n - b\ K e_1^T\Big) \leq -\rho_* Id_n, \quad \forall b \geq \underline{b}. \tag{69}$$

For $n = 1$, (69) reduces to $-2bkS \leq -\mu_*$. By taking $S = 1/2$ and $k = 1/\underline{b}$ we get the result with $\rho_* = 1$.

Let n be a positive integer larger than or equal to two. Set $\tilde{e}_1 = (1, \cdots, 0)^T \in \mathbb{R}^{n-1}$ and $K = (k_1, L^T)^T$ with $L \in \mathbb{R}^{n-1}$ to be determined. Notice that

$$J_n - b\ K e_1^T = \begin{pmatrix} -b\ k_1 & \tilde{e}_1^T \\ -b\ L & J_{n-1} \end{pmatrix}.$$

For $\Omega \in \mathbb{R}^{n-1}$, consider the $n \times n$ matrix A_Ω given by

$$A_\Omega = \begin{pmatrix} 1 & 0 \\ \Omega & Id_{n-1} \end{pmatrix},$$

We make the linear change of variable $y = A_\Omega x$ and we require the following condition on (k_1, L), i.e., $k_1\Omega + L = 0$. One gets that

$$A_\Omega(J_n - b\ Ke_1^T)A_\Omega^{-1} = \begin{pmatrix} -(b\ k_1 + \tilde{e}_1^T\Omega) & \tilde{e}_1^T \\ -(J_{n-1} + \Omega\tilde{e}_1^T)\Omega & J_{n-1} + \Omega\tilde{e}_1^T \end{pmatrix}.$$

This linear change of variable amounts to multiply (69) on the left by $(A_\Omega^T)^{-1}$ and on the right by A_Ω^{-1} and we still denote by S the matrix $(A_\Omega^T)^{-1}SA_\Omega^{-1}$. We now pick Ω so that $J_{n-1} + \Omega\tilde{e}_1^T$ is Hurwitz and there exists a positive constant $\mu > 0$ and a real symmetric positive definite $(n-1) \times (n-1)$ matrix $S_{n-1} > 0$ such that

$$(J_{n-1} + \Omega\tilde{e}_1^T)^T S_{n-1} + S_{n-1}(J_{n-1} + \Omega\tilde{e}_1^T) - \leq \rho_* Id_{n-1}.$$

After choosing $S = \begin{pmatrix} 1 & 0 \\ 0 & S_{n-1} \end{pmatrix}$, one simply finds $k_1 > 0$ large enough to get the result.

Remark 38. *One must notice the similarity of the argument which is essentially that of* [7] *and* [4], *with the corresponding one in* [18]. *The one given here is more transparent and also allows to use the extra degree of freedom given by* η.

6.2. **Proof of Equations** (65)**,** (66) **and** (67)**.** For $\kappa \in \{-\kappa_0, 0, \kappa_0\}$, taking the time derivative $\dot{V}_\kappa$ of V_{κ_0} along a trajectory of (62) yields the inequality

$$\dot{V}_\kappa \leq -CV_{\kappa_0}^{1+\alpha(\kappa)}(x) + b\langle \nabla V_\kappa(x), e_n\rangle \Big(\omega^H_{\kappa(x+d_1)}(x) - \omega^H_{\kappa(x)}(x)\Big) + \langle \nabla V_\kappa(x), d_2\rangle.$$

We will prove that in each region of interest, there exists $\mathcal{KL}$ functions F_1, F_2 such that

$$\bar{b}\Big|\langle \nabla V_\kappa(x), e_n\rangle \Big(\omega^H_{\kappa(x+d_1)}(x) - \omega^H_{\kappa(x)}(x)\Big)\Big| \leq \frac{C}{4}V_\kappa^{1+\alpha(\kappa)}(x) + F_1(\|d_1\|_\infty), \tag{70}$$

and

$$\Big|\langle \nabla V_\kappa(x), d_2\rangle\Big| \leq \frac{C}{4}V_{\kappa_0}^{1+\alpha(\kappa)}(x) + F_2(\|d_2\|_\infty). \tag{71}$$

Once this is established, one gets the conclusion by taking $F = F_1 + F_2$.

We start by proving (71). For $\kappa \in \{-\kappa_0, 0\}$, the region of interest is bounded. Hence one immediately concludes by applying Cauchy-Schwartz inequality and taking an upper bound for the continuous function $\|\nabla V_\kappa\|$ on the region of interest. For $\kappa = \kappa_0$, we recall that, for $1 \leq i \leq n$, $\langle \nabla V_\kappa(x), e_i\rangle$ is $\mathbf{r}(\kappa_0)$-homogeneous of degree $(2+\kappa_0) - r_i$ with respect to the family of dilations $\big(D_\lambda^{\mathbf{r}(\kappa_0)}\big)_{\lambda>0}$. It is therefore immediate to see that there exists a positive constant C_i such that $|\langle \nabla V_{\kappa_0}(x), e_i\rangle| \leq C_i V_{\kappa_0}^{1-\frac{r_i}{2+\kappa_0}}$ over $\mathbb{R}^n$. One deduces that

$$\Big|\langle \nabla V_{\kappa_0}(x), d_2\rangle\Big| \leq \sum_{i=1}^n C_i V_{\kappa_0}^{1-\frac{r_i}{2+\kappa_0}} |(d_2)_i|. \tag{72}$$

Since every r_i is positive and hence $1 - \frac{r_i}{2+\kappa_0} < 1 + \alpha(\kappa_0)$, one can apply an appropriately weighted Holder inequality to get (71).

We know turn to an argument for (70). We provide an argument only for $\kappa = \kappa_0$ since for the other cases it is similar. If $\kappa(x + d_1) \neq \kappa_0$, then $V_0(x + d_1) \leq 1 + m$ implying that $\|x\| \leq C_1\|d_1\|$ for some positive constant independent of x, d_1. Hence one can bound the left-hand side of (70) by $F_1(\|d_1\|)$ for some $\mathcal{KL}$ function F_1, and then conclude. We now treat the case where

$\kappa(x+d_1)=\kappa_0$. Recall that $\omega^H_{\kappa_0}$ is $\mathbf{r}(\kappa_0)$-homogeneous of degree $r_{n+1}:=1+n\kappa_0$ with respect to the family of dilations $\big(D^{\mathbf{r}(\kappa_0)}_\lambda\big)_{\lambda>0}$. For non zero $x\in\mathbb{R}^n$, we define the normalized vector

$$[x]^{\kappa_0}:=\frac{x}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}}\in B_1^{\kappa_0}.$$

Then one gets on $B^{\kappa_0}_{>1+m}$,

$$\begin{aligned}\langle\nabla V_{\kappa_0}(x),e_n\rangle\Big(\omega^H_{\kappa_0}(x+d_1)-\omega^H_{\kappa_0}(x)\Big)=&\\ V_{\kappa_0}^{1-\frac{r_n}{2+\kappa_0}}(x)\langle\nabla V_{\kappa_0}([x]^{\kappa_0}),e_n\rangle\Big(V_{\kappa_0}^{\frac{r_{n+1}}{2+\kappa_0}}(x+d_1)\omega^H_{\kappa_0}([x+d_1]^{\kappa_0})&-V_{\kappa_0}^{\frac{r_{n+1}}{2+\kappa_0}}(x)\omega^H_{\kappa_0}([x]^{\kappa_0})\Big)\\ =V_{\kappa_0}^{1+\alpha(\kappa_0)}(x)\langle\nabla V_{\kappa_0}([x]^{\kappa_0}),e_n\rangle M(x,d_1),&\end{aligned}\tag{73}$$

where

$$M(x,d_1):=\Big(\frac{V_{\kappa_0}(x+d_1)}{V_{\kappa_0}(x)}\Big)^{\frac{r_{n+1}}{2+\kappa_0}}\omega^H_{\kappa_0}([x+d_1]^{\kappa_0})-\omega^H_{\kappa_0}([x]^{\kappa_0}).$$

Moreover, we have the following result: there exists a positive constant B such that, for every $x,d\in\mathbb{R}^n$ with $\|x\|,\|d\|\le 1$, one has

$$|\omega^H_{\kappa_0}(x+d)-\omega^H_{\kappa_0}(x)|\le B\|d\|^{r_{n+1}},\tag{74}$$

which is an immediate consequence of (42).

Consider $\rho>0$ to be fixed small later. Assume first that

$$\frac{V_{\kappa_0}(d_1)}{V_{\kappa_0}(x)}=V_{\kappa_0}(\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}})\le\rho.$$

We rewrite $M(x,d_1)$ as

$$M(x,d_1)=\Big[\Big(\frac{V_{\kappa_0}(x+d_1)}{V_{\kappa_0}(x)}\Big)^{\frac{r_{n+1}}{2+\kappa_0}}-1\Big]\omega^H_{\kappa_0}([x+d_1]^{\kappa_0})+\Big(\omega^H_{\kappa_0}([x+d_1]^{\kappa_0})-\omega^H_{\kappa_0}([x]^{\kappa_0})\Big).\tag{75}$$

The term in brackets in (75) can be written as

$$\Big(\frac{V_{\kappa_0}(x+d_1)}{V_{\kappa_0}(x)}\Big)^{\frac{r_{n+1}}{2+\kappa_0}}-1=V_{\kappa_0}([x]^{\kappa_0}+\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}})-V_{\kappa_0}([x]^{\kappa_0}),$$

Notice that $[x]^{\kappa_0}$ and $\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}}$ belong to the compact set $B^{\kappa_0}_{\le 1+m}$ and hence, since V_{κ_0} is of class C^1, there exists a positive constant C_2 independent of x,d_1, such that

$$\Big|V_{\kappa_0}([x]^{\kappa_0}+\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}})-V_{\kappa_0}([x]^{\kappa_0})\Big|\le C_2V_{\kappa_0}(\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}})^{\frac{1}{2+\kappa_0}}\le C_2\rho^{\frac{1}{2+\kappa_0}}.$$

By using (74), we can bound the term in parentheses in (75) as follows,

$$|\omega^H_{\kappa_0}([x+d_1]^{\kappa_0})-\omega^H_{\kappa_0}([x]^{\kappa_0})|\le B\|[x+d_1]^{\kappa_0}-[x]^{\kappa_0}\|^{r_{n+1}}.$$

In turn, one deduces that

$$[x+d_1]^{\kappa_0}-[x]^{\kappa_0}=\Big[\Big(\frac{V_{\kappa_0}(x)}{V_{\kappa_0}(x+d_1)}\Big)^{\frac{r_{n+1}}{2+\kappa_0}}-1\Big][x]^{\kappa_0}+\Big(\frac{V_{\kappa_0}(x+d_1)}{V_{\kappa_0}(x)}\Big)^{\frac{r_{n+1}}{2+\kappa_0}}\frac{d_1}{V_{\kappa_0}^{\frac{1}{2+\kappa_0}}}.$$

Using the homogeneity property of V_{κ_0}, one gets that there exists a positive constant C_3 independent of x, d_1 such that $M(x,d_1) \leq C_3\rho^{\frac{1}{2+\kappa_0}}$. Since $\omega^H_{\kappa_0}$ is bounded on B^0_1, one gets

$$\Big|\langle \nabla V_\kappa(x), e_n\rangle \Big(\omega^H_{\kappa(x+d_1)}(x) - \omega^H_{\kappa(x)}(x)\Big)\Big| \leq \frac{C}{4} V_{\kappa_0}^{1+\alpha(\kappa)}(x),$$

for ρ small enough, and hence (70).

We now assume that

$$\frac{V_{\kappa_0}(d_1)}{V_{\kappa_0}(x)} > \rho. \tag{76}$$

In that case, the conclusion follows if one can prove that there exists $C_4 > 0$ independent of x, d_1 such that

$$V_{\kappa_0}(x+d_1) \leq C_4 V_{\kappa_0}(d_1) + F_1(\|d_1\|), \tag{77}$$

for some $\mathcal{KL}$ function F_1. Indeed, in (73), the term in parentheses becomes bounded by $F_2(\|d_1\|)$ for some $\mathcal{KL}$ function F_2 and then one gets (70) after using Holder's inequality with appropriate weights.

We are then left to prove (77). For that purpose set $f(s) := V_{\kappa_0}(x+sd_1)$ for $s \in [0,1]$ and let $s^* \in\in [0,1]$ such that $f(s^*) = \max_{s\in[0,1]} f(s)$. We will prove (77) with $f(s^*)$ on the left-hand side and hence the conclusion. We can therefore assume with no loss of generality that $s^* = 1$. One obtains that

$$\begin{aligned} f(1) - f(0) &\leq \int_0^1 |f'(s)|ds = \int_0^1 |\langle \nabla V_{\kappa_0}(x+sd_1), d_1\rangle| ds \\ &\leq \sum_{i=1}^n \int_0^1 \Big|\frac{\partial V_{\kappa_0}}{\partial x_i}(x+sd_1)(d_1)_i\Big| ds \leq C\sum_{i=1}^n \int_0^1 V_{\kappa_0}^{\frac{2+\kappa_0-r_i}{2+\kappa_0}}(x+sd_1)|(d_1)_i| ds \\ &\leq C\sum_{i=1}^n V_{\kappa_0}^{\frac{2+\kappa_0-r_i}{2+\kappa_0}}(x+d_1)|(d_1)_i| \leq \frac{f(1)}{2} + CF_3(\|d\|), \end{aligned}$$

for some $\mathcal{KL}$ function F_3. In the above, we have used (72), the definition of $s^* = 1$ and for the final inequality, Holder's inequality with appropriate weights. Combining the previous inequality with (76), one concludes the argument for (77) and hence that of (70).

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