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# Theory and analysis of acoustic-gravity waves in a free-surface compressible and stratified ocean

F. Auclair<sup>\*1</sup>, L. Debreu<sup>2</sup>, E. Duval<sup>2</sup>, P. Marchesiello<sup>3</sup>, E. Blayo<sup>2</sup>, and M. Hilt<sup>1</sup>

<sup>1</sup>Laboratoire d'Aérodynamique, Université de Toulouse, CNRS, UPS, France

<sup>2</sup>Univ. Grenoble Alpes, Inria, CNRS, Grenoble INP, LJK, 38000 Grenoble, France

<sup>3</sup>LEGOS, IRD, Université de Toulouse, CNRS, CNES, France

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*Abstract:* Waves propagate in a free-surface ocean due to compressibility and gravity (and surface tension at much smaller scale). Analytical solutions have long been derived independently for acoustic and gravity waves, i.e., acoustic waves or internal-gravity rays in an unbounded ocean, surface-gravity waves in a free-surface-ocean, and acoustic or internal modes in a bounded ocean. In the present study, capillarity waves and earth-rotation are neglected and a simple, unified model based on inner and boundary dispersion relations is derived for waves propagating in a compressible, stratified, free-surface ocean. Wave solutions are identified and visually analyzed in phase-space. Taylor developments are then carried out with respect to small parameters describing stratification and compressibility and are compared with numerical approximations of the intersection of inner and boundary dispersion surfaces. Finally, the model recovers the known approximations for swell, long-surface waves, internal-gravity rays, internal modes, acoustic waves or acoustic modes, and also provides modification of these solutions due to stratification and compressibility.

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\*Corresponding author: francis.auclair@aero.obs-mip.fr

## 21 1 Introduction

22 Many types of waves are known to propagate in the ocean, and textbooks (LeBlond and Mysak, 1981;  
23 Gill, 1982; Pedlosky, 1982) have detailed the derivation of their analytical solutions for decades. These  
24 waves can be classified in several categories depending on the type of mechanisms directly involved in  
25 their propagation. Neglecting Earth's rotation and associated planetary waves, two fundamental cate-  
26 gories are of particular interest in the present study: acoustic (sound) waves, which are a consequence of  
27 ocean compressibility, and gravity waves, which are sustained by the gravity force. Table 1 gives a short  
28 (and necessarily incomplete) list of such waves. A particular type of waves is most often characterized  
29 by a space-time dispersion relation, linking its time frequency (or period) with its space wavenumber  
30 (or wavelength). The phase and group velocities and wave dispersion capacity can be derived from the  
31 dispersion relations.

32 If ocean waves are short enough and generated far enough from the surface and bottom boundaries, they  
33 can propagate in the ocean as in any unbounded medium. However, when they are generated in the vicin-  
34 ity of these boundaries or when their wavelengths are large compared to the ocean depth, ocean waves  
35 are known to take specific forms, and the ocean basin is a *wave-guide* propagating *wave modes*. For  
36 example, the ocean stratification introduces internal gravity modes with "long" horizontal wavelengths.  
37 Another example are acoustic modes which are associated with compressibility effects and which have  
38 been recently re-discovered (Smith, 2015). Such internal and acoustic waves are qualified as "modes"  
39 since their vertical wavelength is constrained by the vertical extend of the domain. The ocean free-surface  
40 is permanently shaken by a myriad of horizontally propagating waves and it is not always clear whether  
41 these waves are modes or just vertically-evanescent edge-waves. Capillary waves, swells, tidal waves,  
42 tsunamis are well-known examples of such free surface waves.

43 Deriving a dispersion relation for acoustic waves or for internal wave rays in an unbounded ocean is  
44 rather straightforward. The method generally includes two steps: small amplitude is assumed; only spe-  
45 cific wave-restoring mechanisms and medium characteristics are retained in the simplest possible wave  
46 dispersion model (compressibility and pressure force for acoustic waves, gravity and vertical advection  
47 of isopycnal surfaces for internal waves). The linear nature of the resulting model has two main advan-  
48 tages: analytical solutions can be derived and waves can be superimposed without interaction (Lighthill,  
49 1967).

50 The introduction of free-surface brings more complexity. Small-amplitude is usually postulated in this  
51 case also and both gravity and free-surface motions are retained in the dispersion model. However,  
52 surface waves are "edge waves" propagating at the interface between the atmosphere and the ocean, and  
53 the surface kinematic relation (the free-surface general boundary condition) leads to a transcendental  
54 dispersion relation with trigonometric terms. As a result, deriving analytical solutions requires further  
55 simplifications. Specific analytical solutions can then be found in the literature depending, for example,  
56 on relative depth, .i.e., the product of horizontal wavenumber  $k_x$  and ocean depth  $H$  (Table 1). Long  
57 gravity wave solutions are a particular case of small  $k_x H$ , well-known to propagate horizontally with  
58  $\sqrt{gH}$  phase and group velocities (where  $g$  is the acceleration of gravity).

Waves	Assumptions	Frequency ( $\Omega$ )	Vertical wavenumber $k_z$
Acoustic waves	Compressible, unbounded	$\Omega_{aw}^2 = c_s^2(k_x^2 + k_z^2)$	
Internal gravity rays	Stratified, unbounded	$\Omega_{ivr}^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2}$	
Acoustic gravity modes	Compressible, bounded	$\Omega_{am}^2 = c_s^2(k_x^2 + k_z^2)$	$k_{z,am} = \frac{\pi}{2H} + \frac{m\pi}{H}$
Swell	Free-surface	$\Omega_{sw}^2 = gk_x \tanh(k_x H)$	$k_{z,sw} \approx k_x$
Long surface waves	Free-surface, shallow	$\Omega_{lsw}^2 = gH k_x^2$	$k_{z,lsw} \approx k_x$
Internal gravity modes	Stratified, bounded	$\Omega_{im}^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2}$	$k_{z,im} = \frac{n\pi}{H}$

Table 1: Simplified models of ocean waves and their dispersion relations in a vertical section, for an unbounded ocean (top) and for a bounded ocean (bottom).  $\Omega$  is the time frequency of the wave,  $k_x$  and  $k_z$  are wavenumbers,  $g$  is the acceleration of gravity,  $H$  a reference depth,  $N$  a reference Brunt-Väisälä frequency and  $c_s$  the speed of sound.  $n$  and  $m$  are two positive integer numbers.

59 The derivation of mixed acoustic-gravity waves in an ocean bounded by a free surface is even more  
60 challenging and is the central subject of this paper. The main objective is to specify the effect of com-  
61 pressibility and stratification on the preceding particular cases of wave solutions. Another target is to  
62 found analytical solutions to validate recent developments of the CROCO ocean model, resolving non-  
63 hydrostatic equations based on a pseudo-compressible approach (Auclair et al., 2018).

64 Eckart (1960) provides one of the most comprehensive view of oceanic waves. More recently Dukowicz  
65 (2013) also tackled this problem and proposed a review of "*Various approximations in atmosphere and*  
66 *ocean models based on an exact treatment of gravity wave dispersion*". In that paper, acoustic-gravity  
67 waves were shown to satisfy a system of two dispersion relations and the impact of several usual as-  
68 sumptions of ocean models was evaluated. The present study builds on Eckart's and Dukowicz's results  
69 and more specifically focuses on the impact of both stratification and compressibility on acoustic-gravity  
70 wave solutions: Taylor expansions of dispersion relations and resulting expressions for wavelength and  
71 frequency are derived in terms of compressibility and stratification. In addition:

- 72 • A systematic graphic analysis of wave solutions is proposed in 3D frequency/wavenumber phase-  
73 space, unfolding their dependency to the vertical wavenumber. The graphic presentation allows us  
74 to synthesize all possible connections between usual acoustic, internal and surface wave solutions.
- 75 • Surface waves are systematically studied together with internal and acoustic modes.
- 76 • Long-wave solutions are investigated in details in order to better understand from which solution  
77 branch they asymptotically derive; approximate parametric relations are derived for each type of  
78 wave.

79 In the section 2, a linear model of ocean wave propagation is proposed with bottom and surface bound-  
80 ary conditions, and a corresponding system of two dispersion relations (which will be called *inner* and  
81 *boundary* dispersion relations) is derived. The inner dispersion relation, which does not take into account  
82 the bottom and surface boundary conditions, is studied in details in Section 3, and the wave solutions  
83 propagating in an unbounded ocean are investigated. Waves propagating in a bounded ocean, which also  
84 have to satisfy the boundary relation dispersion, are then studied in Section 4. Conclusions are drawn in  
85 Section 5.

## 86 2 Linear model for surface and internal acoustic-gravity waves

### 87 2.1 General model for a compressible, viscous ocean

Ocean dynamics can be described with a small number of macroscopic variables: velocity ( $\mathbf{v}$ ), pressure and density ( $p$  and  $\rho$ ), temperature and salinity ( $T$  and  $S$ ). In a Cartesian framework, the general equations governing the motion of a compressible, viscous ocean are then:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \quad (1a)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla p + \nabla \cdot \underbrace{(\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T) + \mu_2(\nabla \cdot \mathbf{v}) \mathbf{I})}_{\boldsymbol{\tau}} + \rho \mathbf{g} \quad (1b)$$

$$\frac{\partial \rho T}{\partial t} = -\nabla \cdot (\rho T \mathbf{v}) + \rho \nabla \cdot \kappa_T \nabla T \quad (1c)$$

$$\frac{\partial \rho S}{\partial t} = -\nabla \cdot (\rho S \mathbf{v}) + \rho \nabla \cdot \kappa_S \nabla S \quad (1d)$$

$$\rho = \rho(T, S, p) \quad (1e)$$

88 where  $\mathbf{I}$  is the identity matrix, superscript T indicates transposition,  $\mu$  and  $\mu_2$  are the kinetic and bulk  
 89 (or second) viscosities,  $\kappa_T$  and  $\kappa_S$  are the heat and salt diffusivities. The first equations are written in a  
 90 conservative form. They specify basic conservation principles: conservation of mass for Equation (1a),  
 91 conservation of momentum for Equation (1b) and conservation of heat and salt for equations (1c) and  
 92 (1d). Equation (1e) is a functional relation describing the thermodynamic equation of state (EOS).

### 93 2.2 Surface and bottom boundary conditions

At the bottom ( $z = -H$ ) and surface ( $z = \zeta$ ) of the ocean, boundary conditions must be specified for each  
 variable (or for its derivatives). A simple condition of no penetration and no-slip at the ocean bottom can  
 be written:

$$\mathbf{v}(\mathbf{x}_H, z = -H, t) = \mathbf{0} \quad (2)$$

Neglecting surface-tension pressure drop, surface pressure is given by:

$$p(\mathbf{x}_H, z = \zeta, t) = p_{atm} \quad (3)$$

with  $p_{atm}$  the atmospheric pressure imposed at the surface of the ocean. Surface capillarity waves are  
 consequently filtered out and will be neglected in the remaining of this work. The surface kinematic  
 condition expresses the motion of the free-surface and relates the free-surface anomaly  $\zeta$  to the surface  
 vertical velocity  $w$ :

$$\frac{d\zeta(\mathbf{x}_H, t)}{dt} = w(\mathbf{x}_H, z = \zeta, t) \quad (4)$$

94 This kinematic boundary condition allows the propagation of surface gravity waves.

### 95 2.3 Pressure and density decomposition and EOS simplification

96 Waves are defined as small disturbances to a motionless thermodynamic equilibrium state, and both  
 97 pressure and density can be decomposed into an equilibrium component and a small increment. In  
 98 addition, as a first approximation, the impact of atmospheric pressure  $p_{atm}$  can be neglected — it can  
 99 take an active part in wave generation, but only plays a minor role during propagation.

The usual decomposition is now formalized for pressure (5a) and density (5b):

$$\begin{aligned}
p(\mathbf{x}, t) &= \underbrace{p_{\text{atm}}(\mathbf{x}_H, t)}_{\approx 0} + g \underbrace{\int_z^\zeta \rho_h(\mathbf{x}_H, z', t) dz'}_{p_h(\mathbf{x}, t)} + \delta p(\mathbf{x}, t) \\
&= g \underbrace{\int_z^\zeta \hat{\rho}_h(z') dz'}_{\hat{p}_h(z)} + g \underbrace{\int_z^\zeta (\rho_h(\mathbf{x}_H, z', t) - \hat{\rho}_h(z')) dz'}_{p'_h(\mathbf{x}, t)} + \delta p(\mathbf{x}, t)
\end{aligned} \tag{5a}$$

$$\begin{aligned}
\rho(\mathbf{x}, t) &= \underbrace{\hat{\rho}_{TS}(z) + \rho'_{TS}(\mathbf{x}, t)}_{\rho_{TS}(\mathbf{x}, t) = \rho(T, S, p=0)} + \frac{1}{c_s^2} \underbrace{(\hat{p}_h(z) + p'_h(\mathbf{x}, t) + \delta p(\mathbf{x}, t))}_{\partial \rho / \partial p|_{T, S} p(\mathbf{x}, t)} + O((\delta p)^2) \\
&\approx \underbrace{\hat{\rho}_{TS}(z)}_{\hat{\rho}_h(z)} + \underbrace{\rho'_{TS}(\mathbf{x}, t)}_{\rho'_h(\mathbf{x}, t)} + \frac{p'_h(\mathbf{x}, t)}{c_s^2} + \frac{\delta p(\mathbf{x}, t)}{c_s^2}
\end{aligned} \tag{5b}$$

100 with  $\partial \rho / \partial p|_{T, S} = c_s^2$  at constant entropy,  $\partial \hat{p}_h / \partial z = -\hat{\rho}_h(z)g$ ,  $\partial p'_h / \partial z = -\rho'_h(z)g$ .

101 The first decomposition  $p = p_h + \delta p$  is defined by an hydrostatic component  $p_h$  and a nonhydrostatic  
102 pressure increment  $\delta p$ . It is based on a division of the pressure field into a slow varying component  
103 in hydrostatic equilibrium and a fast varying nonhydrostatic component. Density can be decomposed  
104 in a similar way with  $\rho_h$  the hydrostatic component and  $\delta \rho = \delta p / c_s^2$  the non-hydrostatic increment.  
105 The hydrostatic component of the density can be further decomposed into a depth-dependent reference  
106 value  $\hat{\rho}_h(z)$  and an increment  $\rho'_h$ , leading to a similar decomposition for the hydrostatic component of the  
107 pressure into a barotropic component  $\hat{p}_h$  and a baroclinic increment  $p'_h$ .

The second decomposition focuses on compressibility: it can be viewed as a first-order approximation  
of the EOS with respect to the total pressure  $p$ . This Taylor development is carried out in the vicinity  
of the hydrostatic pressure and is first formulated for density. The decomposition  $\rho = \rho_{TS} + p/c_s^2$  is  
based on a first-order decomposition with respect to total pressure  $p$ . The component  $\rho_{TS}$  is a function  
of temperature and salinity and  $p/c_s^2$  is the first-order compressibility increment due to total pressure.  
The EOS approximation (1e) under this form  $\rho = \rho_{TS} + p/c_s^2$  along with tracer conservation equations,  
omitting the diffusion terms, allows us to write:

$$\frac{d\rho}{dt} = \frac{1}{c_s^2} \frac{dp}{dt} \tag{6}$$

or using mass conservation:

$$-\rho \nabla \cdot \mathbf{v} = \frac{1}{c_s^2} \frac{dp}{dt} \tag{7}$$

Without loss of generality, the present study can now be restricted to the  $(O, x, z)$  vertical plan to simplify  
notations. Equation (7) can then be expanded to:

$$-\rho \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = \frac{1}{c_s^2} \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + w \frac{\partial p}{\partial z} \right) \tag{8}$$

A Taylor expansion of model equations can now be carried out in the vicinity of the reference profiles  
( $\hat{p}_h(z)$ ,  $\hat{\rho}_h(z)$ ) and of a resting fluid ( $u = w = 0$ ). Small amplitude wave-induced increments are given by

$\delta V = (p'_h + \delta p, \rho'_h + \delta p/c_s^2, u, w)$ . At first order in  $\delta V$ , conservation of mass and vertical advection of pressure and density can be rewritten. The left-hand side of (8) becomes:

$$-\rho \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = - \left( \frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z} - w \frac{\partial \hat{\rho}_h}{\partial z} \right) + \mathcal{O}(\delta V^2)$$

while the right-hand side of (8) becomes:

$$\frac{1}{c_s^2} \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + w \frac{\partial p}{\partial z} \right) = \frac{1}{c_s^2} \left( \frac{\partial p}{\partial t} + w \frac{\partial \hat{p}_h}{\partial z} \right) + \mathcal{O}(\delta V^2) = \frac{1}{c_s^2} \left( \frac{\partial p}{\partial t} - \hat{\rho}_h g w \right) + \mathcal{O}(\delta V^2)$$

leading to:

$$\frac{\partial p}{\partial t} = -c_s^2 \left( \frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z} \right) + \underbrace{\left( g + \frac{c_s^2 \partial \hat{\rho}_h(z)/\partial z}{\hat{\rho}_h(z)} \right)}_{-c_s^2 N^2/g} \hat{\rho}_h w + \mathcal{O}(\delta V^2) \quad (9)$$

108 where we have introduced  $N^2(z) = -g \left( 1/\hat{\rho}_h(z) \partial \hat{\rho}_h(z)/\partial z + g/c_s^2 \right)$  the Brunt-Väisälä frequency for a  
 109 compressible ocean (Gill 1982, p169). By definition,  $N^2$  is zero for an homogeneous ocean for which  
 110  $\rho_{T,S}(\mathbf{x}, t) = \rho_0$ .

The free-surface variations are introduced through the surface boundary condition for pressure:

$$p(z=0) = \hat{\rho}_h(0) g \zeta + \mathcal{O}(\delta V^2)$$

and the kinematic surface boundary condition can be written for pressure:

$$\frac{dp}{dt}(z=0) = g \hat{\rho}_h(0) w(z=0) + \mathcal{O}(\delta V^2)$$

## 111 2.4 Linear inviscid wave model

Based on the pressure and density decomposition proposed in Section 2.3, a simpler, inviscid, linear, rotation-less  $p - \rho$  model can be used to model acoustic, internal and surface waves. At first order in wave-induced increment  $\delta V$ , the conservation of momentum and mass and the EOS read:

$$\frac{\partial \hat{\rho}_h u}{\partial t} = -\frac{\partial p}{\partial x} \quad (10a)$$

$$\frac{\partial \hat{\rho}_h w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g \quad (10b)$$

$$\frac{\partial \rho}{\partial t} = - \left( \frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z} \right) \quad (10c)$$

$$\frac{\partial p}{\partial t} = -c_s^2 \left( \frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z} \right) - (c_s^2 N^2/g) \hat{\rho}_h w \quad (10d)$$

with the (flat) bottom and surface conditions:

$$w(z=-H) = 0 \quad (11a)$$

$$\frac{\partial p}{\partial t}(z=0) = \hat{\rho}_h g w(z=0) \quad (11b)$$

112 As in Dukowicz (2013), in the following, we will express part of the results in terms of  $D(z)$ , a vertical  
 113 length scale associated with stratification, defined by  $\frac{1}{D(z)} = \frac{N^2(z)}{g} + \frac{g}{c_s^2}$ . The background stratification  
 114 satisfies  $\frac{\partial \hat{\rho}_h(z)}{\partial z} = -\frac{1}{D(z)} \hat{\rho}_h(z)$ .

## 115 2.5 General propagation equation and polarization relations

**Form of wave solutions** Dispersion relations can be derived by postulating and specifying the wave form. Horizontally-propagating surface waves, wave modes propagating in the ocean wave guide, internal wave rays and acoustic waves all satisfy the following "polarization" relations:

$$\begin{pmatrix} \hat{\rho}_{hu} \\ \hat{\rho}_{hw} \\ \rho \\ p \end{pmatrix} = \begin{pmatrix} \tilde{U}(z) \\ \tilde{W}(z) \\ \tilde{\rho}(z) \\ \tilde{p}(z) \end{pmatrix} e^{i(k_x x - \Omega t)} \quad (12a)$$

116 where  $k_x$  and  $\Omega$  are respectively the horizontal wavenumber and wave frequency.

**Inner equations** These relations can be introduced in the propagation model (10a)-(10d). After some developments, an ordinary differential equation can be obtained for  $\tilde{W}(z)$ :

$$\tilde{W}''(z) + \frac{1}{D(z)} \tilde{W}'(z) + \left( k_x^2 \frac{N^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{D'(z)}{D^2(z)} \right) \tilde{W}(z) = 0 \quad (13)$$

with the following polarization relations:

$$\tilde{U}(z) = -ik_x \frac{(c_s^2 - gD(z)) \tilde{W}(z) + c_s^2 D(z) \tilde{W}'(z)}{D(z) (\Omega^2 - c_s^2 k_x^2)} \quad (14a)$$

$$\tilde{\rho}(z) = -i \frac{k_x^2 (c_s^2 - gD(z)) \tilde{W}(z) + \Omega^2 D(z) \tilde{W}'(z)}{D(z) \Omega (\Omega^2 - c_s^2 k_x^2)} \quad (14b)$$

$$\tilde{p}(z) = -i\Omega \frac{(c_s^2 - gD(z)) \tilde{W}(z) + c_s^2 D(z) \tilde{W}'(z)}{D(z) (\Omega^2 - c_s^2 k_x^2)} \quad (14c)$$

To obtain this relation and in particular to eliminate its pressure dependence, one needs to exclude the possibility that:

$$\Omega^2 = c_s^2 k_x^2 \quad (15)$$

which is the dispersion relation for acoustic Lamb waves in the atmosphere (Apel, 1987). As for the atmospheric case, it can be shown that these waves are characterized by  $\tilde{W}(z) = 0$ ,  $\tilde{p}(z) = P_0 e^{-gz/c_s^2}$ . However, the free surface boundary condition (11b) constrains  $P_0$  to be zero and only the trivial null solution is obtained. Therefore, Lamb waves do not exist in a free surface ocean and, as pointed out by several authors, they are replaced by surface gravity waves, discussed later (see (4.4)).

*Stratification-induced change of variables*

To remove first-order terms and simplify future developments, the following change of variable can be made:

$$\tilde{W}(z) = \tilde{W}(0) F(z) \exp\left(\int_z^0 \frac{dz'}{2D(z')}\right), \quad (16)$$

with  $F(0) = 1$ . Following (13), the unknown function  $F(z)$  satisfies:

$$F''(z) + \left( k_x^2 \frac{N^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{1 + 2D'(z)}{4D(z)^2} \right) F(z) = 0 \quad (17)$$

117  $F(z)$  differs from the vertical momentum  $\tilde{W}(z)$  by the attenuation factor  $\exp\left(\int_z^0 \frac{dz'}{2D(z')}\right)$ . This factor re-

118 duces the vertical extent of wave anomalies based on the length scale  $D(z)$ . The stronger the stratification,  
119 the shorter  $D(z)$  and this vertical extent.

**Surface and bottom boundary conditions** The polarization relations must also be substituted in the surface boundary condition (11b), leading to:

$$-i\Omega \tilde{p}(0) = g\tilde{W}(0) \quad (18)$$

or, using (14),

$$\tilde{W}'(0) + \left( \frac{1}{D(0)} - \frac{gk_x^2}{\Omega^2} \right) \tilde{W}(0) = 0 \quad (19)$$

120 Note that (19) is identical to Eq. 71 of Dukowicz (2013) who suggests that the term  $1/D(0)$  can only  
 121 result from governing equations formulated in a Lagrangian vertical coordinate to properly account for  
 122 the surface boundary condition. Our result contradicts this statement since the term also appears in the  
 123 Eulerian coordinate system when no incompressibility nor Boussinesq approximation is made.

In terms of the unknown function  $F(z)$ , the surface boundary condition reads:

$$F'(0) + \left( \frac{1}{2D(0)} - \frac{gk_x^2}{\Omega^2} \right) F(0) = 0 \quad (20)$$

The bottom boundary condition (11a) simply leads to:

$$F(-H) = 0 \quad (21)$$

## 124 2.6 The constant Brunt-Väisälä frequency case

In the rest of the paper, the Brunt-Väisälä frequency case is assumed to be constant:  $N^2(z) = N_0^2$ , or equivalently  $D(z) = D(0) = D_0$ . Equation (16) can then be rewritten:

$$\tilde{W}(z) = \tilde{W}(0) \exp\left(-\frac{z}{2D_0}\right) F(z) \quad (22)$$

and  $\hat{\rho}_h(z)$  is given by  $\hat{\rho}_h(z) = \hat{\rho}_h(0) e^{-z/D_0}$ . The general expression of the vertical velocity perturbation profile for a constant scale height  $D_0$  is thus:

$$w(x, z, t) = \frac{1}{\hat{\rho}_h(z)} \tilde{W}(z) e^{i(k_x x - \Omega t)} = \frac{\tilde{W}(0)}{\hat{\rho}_h(0)} e^{z/2D_0} F(z) e^{i(k_x x - \Omega t)} \quad (23)$$

The vertical profile  $F(z)$  has to satisfy the following system of equations:

$$F''(z) + \overbrace{\left( k_x^2 \frac{N_0^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{1}{4D_0^2} \right)}^{\equiv k_z^2} F(z) = 0 \quad (24a)$$

$$F(0) = 1 \quad (24b)$$

$$F(-H) = 0 \quad (24c)$$

The general solution of (24a)- (24b)- (24c) is

$$F(z) = \frac{\sin(k_z(H+z))}{\sin(k_z H)} \quad (25)$$

where the vertical wavenumber  $k_z$  is defined in (24a) and is a function of  $k_x$  and  $\Omega$ .

Under the constant Brunt-Väisälä frequency assumption, the surface boundary condition (20) reads:

$$F'(0) + \left( \frac{1}{2D_0} - \frac{gk_x^2}{\Omega^2} \right) F(0) = 0 \quad (26)$$

125 **2.7 Inner and boundary dispersion relations**

The relation introduced in (24a) between the vertical wavenumber  $k_z$ , horizontal wavenumber  $k_x$  and wave frequency  $\Omega$  constitutes the first dispersion relation. It is rewritten as:

$$k_z^2 + k_x^2 \left( 1 - \frac{N_0^2}{\Omega^2} \right) - \frac{\Omega^2}{c_s^2} + \frac{1}{4D_0^2} = 0 \quad (27)$$

This relation does not account for surface or bottom boundary conditions and thus only deals with the propagation of waves in the inner ocean. It will now be referred to as the *inner* dispersion relation. The bottom boundary condition is accounted for in the general solution profile given by (25). Injecting this vertical profile  $F(z)$  into the surface boundary condition (26) then leads to the *boundary* dispersion relation:

$$\Omega^2 = \frac{gk_x^2 \tan(Hk_z)}{k_z + \frac{\tan(Hk_z)}{2D_0}} = \frac{gk_x^2}{\frac{1}{2D_0} + k_z \cotan(Hk_z)} \quad (28)$$

A wave propagating in a "bounded ocean" must satisfy both the *inner* and *boundary* (dimensional) dispersion relations (27) and (28). Note that the traditional inner and boundary dispersion relations for a Boussinesq, incompressible fluid (Gill, 1982) can be recovered from (27) and (28) by setting  $c_s \rightarrow +\infty$  (incompressibility) and then  $D_0 \rightarrow +\infty$  (incompressibility and Boussinesq approximations), leading to:

$$\Omega^2 = N^2 \frac{k_x^2}{k_x^2 + k_z^2} = gk_x^2 \frac{\tan(Hk_z)}{k_z}$$

**Dimensionless dispersion relations** As in Dukowicz (2013), two parameters are now defined to obtain dimensionless dispersion relations:

$$\epsilon_i^2 = \frac{N^2 H}{g}, \quad \epsilon_a^2 = \frac{gH}{c_s^2} \quad (29)$$

$\epsilon_i$  is thus a small parameter related to gravity, defined as the ratio of the order of magnitude of the first internal mode  $NH$  to the velocity of long surface waves  $\sqrt{gH}$ .  $\epsilon_a$  is a small parameter related to acoustics, defined as the ratio of the speed of long surface waves  $\sqrt{gH}$  to that of sound waves  $c_s$ . The depth scale can then be rewritten as:  $D_0 = H/(\epsilon_i^2 + \epsilon_a^2)$ . Three dimensionless variables are also further defined:

$$\omega = \Omega \sqrt{\frac{H}{g}}, \quad \delta_x = k_x H, \quad \delta_z = k_z H \quad (30)$$

The inner (27) and boundary (28) dispersion relations can be written in terms of the dimensionless parameters as:

$$\delta_x^2 + \delta_z^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \quad (31a)$$

$$\omega^2 = \frac{\delta_x^2 \tan(\delta_z)}{\delta_z + \frac{\epsilon_a^2 + \epsilon_i^2}{2} \tan(\delta_z)} \quad (31b)$$

126 In a free-surface ocean, wave solutions must satisfy simultaneously relations (31a) and (31b). This  
 127 means that only one parameter among the frequency  $\omega$  and horizontal and vertical wavenumbers ( $\delta_x$   
 128 and  $\delta_z$ ) can be imposed. The other two parameters must be adjusted for the wave to satisfy the two  
 129 dispersion relations. For short vertical wavenumbers and far from the bottom and surface boundaries,  
 130 wave solutions only need to satisfy the inner dispersion relation and to be dynamically consistent. Pure

131 acoustic waves or pure internal-gravity wave rays are known to propagate in the inner ocean as in an  
 132 unbounded ocean.

133 The resulting set of two equations (31a)-(31b) for the three variables  $(\delta_x, \delta_z, \omega)$  and the two parameters  
 134  $\epsilon_a, \epsilon_i$  is nonlinear, and simple general solutions cannot be found analytically. However much insight can  
 135 be gained on these waves by a geometrical investigation of surfaces (in phase space) that correspond  
 136 to the inner and boundary dispersion relations. These surfaces will respectively be named *inner and*  
 137 *boundary dispersion surfaces*.

138 In the following, we will use the fact that  $\delta_z$  is either real ( $\delta_z \in \mathbb{R}$ ) or purely imaginary ( $\delta_z \in i\mathbb{R}$ ) and thus  
 139 that  $\omega^2$  is always real and positive. This is proved in appendix A. It is important to note that  $\omega$  is forced  
 140 to be real by the boundary dispersion relation, and that other solutions are possible when considering the  
 141 inner dispersion relation only. These solutions will not be discussed in the following, even in Section 3  
 142 on waves in an unbounded ocean.

143 The waves associate with real  $\delta_z$  are *vertically propagating* while the ones associated with imaginary  $\delta_z$   
 144 are *vertically evanescent*.

145 The rest of the paper is organized as follows: the inner dispersion relation corresponding to waves in  
 146 an unbounded ocean is studied in Section 3. Additional constraints related to the boundary dispersion  
 147 relation, i.e., in a bounded ocean, are added in Section 4.

148 When not explicitly mentioned, the standard values of the parameters used in the rest of the paper are  
 listed in Table 2.

Gravity	$g$	$9.8 \text{ m.s}^{-2}$
Sound speed	$c_s$	$1500 \text{ m.s}^{-1}$
Depth	$H$	$4000 \text{ m}$
Brunt-Väisälä frequency	$N$	$10^{-3} \text{ s}^{-1}$
Acoustic small parameter	$\epsilon_a = \frac{\sqrt{gH}}{c_s}$	$\approx 0.132$
Internal small parameter	$\epsilon_i = \sqrt{\frac{N^2 H}{g}}$	$\approx 0.02020$
Depth scale	$D_0 = \frac{1}{\frac{N^2}{g} + \frac{g}{c_s^2}} = \frac{H}{\epsilon_a^2 + \epsilon_i^2}$	$\approx 224 \text{ km}$

Table 2: Main parameters used to plot dispersion relations.

149

### 150 3 Inner dispersion relation & waves in an unbounded ocean

151 The inner dispersion relation (31a) must be satisfied by any type of ocean waves whether or not the  
 152 ocean is considered as a locally unbounded medium (far from surface and bottom). We will show in the  
 153 present section that (i) in  $(\delta_x, \delta_z, \omega)$  phase-space, the inner dispersion relation leads to a three-branch  
 154 dispersion surface, (ii) two acoustic and stratification reference functions  $\omega_a(\delta_x, \delta_z)$  and  $\omega_i(\delta_x, \delta_z)$  are  
 155 good approximations of the branches, (iii) the upper and lower branches of the inner dispersion surface  
 156 (for respectively high and low frequencies) correspond to acoustic waves and internal rays propagating  
 157 in an unbounded ocean and (iv) the bounded central branch of the inner dispersion surface corresponds  
 158 to vertically vanishing waves, which are referred to as surface waves in the following.

159 As explained above, we consider here only regular, stable solutions, i.e., the case where frequency  $\omega$   
 160 is real. The stability of the solutions will generally be imposed by boundary conditions, as proved in  
 161 appendix A for the specific case of a free surface ocean (our concern here) and for physically meaningful  
 162 values of small parameters  $\epsilon_i, \epsilon_a$ .

### 163 3.1 Acoustic and stratification reference frequencies

To gain insight in the physics of wave solutions, the inner dispersion relation (31a) is rewritten below:

$$\delta_x^2 + \delta_z^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \quad (32)$$

Following Tolstoy (1963), it can be reformulated in the simpler form:

$$\frac{\omega^2}{\omega_a^2} + \frac{\omega_i^2}{\omega^2} = 1 \quad (33)$$

where  $\omega_i$  and  $\omega_a$  are functions of horizontal and vertical wavenumbers, defined by:

$$\omega_a^2(\delta_x, \delta_z) = \frac{1}{\epsilon_a^2} \left( \delta_x^2 + \delta_z^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \right) \quad (34a)$$

$$\omega_i^2(\delta_x, \delta_z) = \frac{\delta_x^2 \epsilon_i^2}{\delta_x^2 + \delta_z^2 + (\epsilon_a^2 + \epsilon_i^2)^2 / 4} \quad (34b)$$

164 These two functions are not roots of the inner dispersion equation (32) but they are useful in their ap-  
165 proximation and, more specifically, in their physical interpretation:

- 166 • If  $\epsilon_a^2 \omega^2 \gg \frac{\epsilon_i^2 \delta_x^2}{\omega^2}$  (high frequency) then (32) simplifies to  $\omega^2 \approx \omega_a^2$ , relevant to a compressible,  
167 homogeneous (unstratified) ocean.
- 168 • If  $\epsilon_a^2 \omega^2 \ll \frac{\epsilon_i^2 \delta_x^2}{\omega^2}$  (low frequency) then (32) simplifies to  $\omega^2 \approx \omega_i^2$ , relevant to an incompressible,  
169 stratified ocean.

170 Therefore,  $\omega_a$  can be interpreted as a *reference acoustic function* accounting for the compressibility con-  
171 tent of the inner dispersion relation, and is solution of the *inner* dispersion relation for an homogeneous  
172 ocean.  $\omega_i$  plays an equivalent role for ocean stratification: it can be interpreted as a *reference stratifica-*  
173 *tion function* and is solution of the *inner* dispersion relation for an incompressible ocean.

174

Recall that we consider only regular solutions for which the frequency  $\omega$  is real and  $\delta_z$  is either real or purely imaginary. Therefore (33) implies:

$$\omega_i^2(\delta_x, \delta_z) \leq \omega^2(\delta_x, \delta_z) \leq \omega_a^2(\delta_x, \delta_z) \quad \forall(\delta_x, \delta_z)$$

175 Therefore, frequency is always bounded by  $\omega_i$  and  $\omega_a$ .

176

Let us now define  $R(\delta_x, \delta_z)$  the ratio of stratification to acoustic frequencies:

$$R^2(\delta_x, \delta_z) = \frac{\omega_i^2}{\omega_a^2} = \frac{\epsilon_a^2 \epsilon_i^2 \delta_x^2}{\left( \delta_x^2 + \delta_z^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \right)^2} \quad (35)$$

$R(\delta_x, \delta_z)$  is an important criterion for locating the roots of the inner dispersion relation. The condition  $R^2 = \frac{\omega_i^2}{\omega_a^2} \leq \frac{1}{4}$  is required for Equation (33) to present two real roots, and thus for  $\omega^2$  to be real. These roots can be formulated for the squared frequency:

$$\omega_{\pm}^2 = \frac{\omega_a^2}{2} \left( 1 \pm \sqrt{1 - 4 \underbrace{\frac{\omega_i^2}{\omega_a^2}}_{\equiv R^2}} \right) \quad (36)$$

When  $R^2$  is small, the two roots are well separated and correspond to  $\omega_+ \approx \omega_a, \omega_- \approx \omega_i$ . It is also important to note that the product of the two roots  $\omega_-^2 \omega_+^2$  is always equal to  $\omega_a^2 \omega_i^2 = \frac{\epsilon_i^2}{\epsilon_a^2} \delta_x^2$  and thus we have:

$$\omega_i^2(\delta_x, \delta_z) \leq \omega_-^2(\delta_x, \delta_z) \leq \frac{\epsilon_i}{\epsilon_a} \delta_x \leq \omega_+^2(\delta_x, \delta_z) \leq \omega_a^2(\delta_x, \delta_z). \quad (37)$$

### 177 3.2 Three regions in $(\delta_x, \delta_z, \omega)$ phase-space

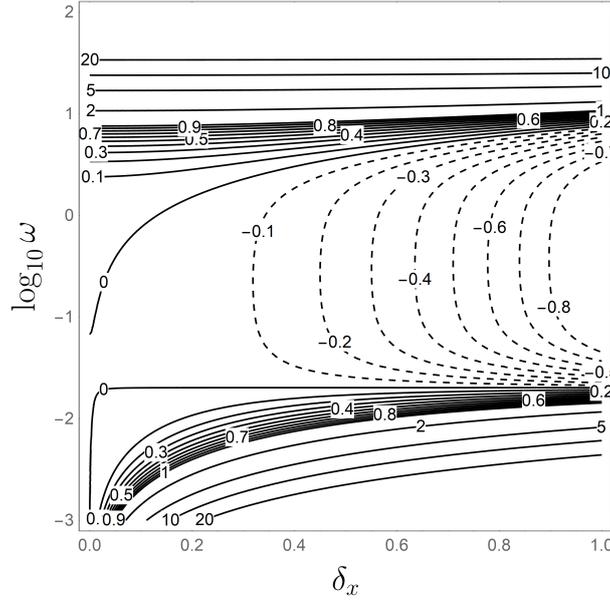


Figure 1: contours of  $\delta_z^2(\delta_x, \omega)$ . Plain lines represent positive values ( $\delta_z \in \mathbb{R}$ ), and dashed lines negative values ( $\delta_z \in i\mathbb{R}$ ).

178 Figure 1 shows variations of the squared vertical wavenumber  $\delta_z^2$  as a function of  $(\delta_x, \omega)$  for the values  
 179 of  $\epsilon_i, \epsilon_a$  given in table (2). Negative values are encountered for medium-range frequencies ( $10^{-1.7} < \omega <$   
 180  $10^{0.7}$ ) and large enough horizontal wavenumbers ( $\delta_x \geq 0.1 - 0.2$ ). This region is surrounded by high and  
 181 low frequency regions of positive  $\delta_z^2$ . The transition lines between these regions are given by  $\delta_z = 0$ :

- 182 • If  $\omega \gg 1$  and  $\epsilon_a \neq 0$  then  $\omega^2 \approx \omega_a^2$  and the transition line is given by  $\epsilon_a^2 \omega^2 \approx \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4}$ . This line  
 183 is a parabola and the frequency is not bounded when  $\delta_x$  increases.
- 184 • If  $\omega \ll 1$  and  $\epsilon_i \neq 0$  then  $\omega^2 \approx \omega_i^2$ , the equation of the transition line is  $\omega^2 \approx \delta_x^2 \epsilon_i^2 / (\delta_x^2 + (\epsilon_a^2 + \epsilon_i^2)^2 / 4)$ .  
 185 This line has an upper bound  $\omega_{c,i} = \epsilon_i$ . In dimensional form, this bound can be rewritten  $\Omega \leq N$   
 186 and is related to the well-known cut-off frequency for internal gravity waves.

187 Therefore, the inner dispersion relation (31a) authorizes three types of wave solutions: two with real  
 188 vertical wavenumbers ( $\delta_z^2 \geq 0$ ) and one with purely imaginary wavenumbers ( $\delta_z^2 < 0$ ).

#### 189 3.2.1 Real vertical wavenumber ( $\delta_z \in \mathbb{R}$ )

The ratio of reference functions  $R^2(\delta_x, \delta_z)$ , defined by (35), depends on two variables  $(\delta_x, \delta_z)$  and two parameters  $(\epsilon_i, \epsilon_a)$ . A study of its variations for  $\delta_x \in \mathbb{R}$  shows that, for non-vanishing  $(\epsilon_i, \epsilon_a)$ , it has an upper bound:

$$\max_{\delta_x} R^2(\delta_x, \delta_z) = \frac{1}{4} \frac{\epsilon_a^2 \epsilon_i^2}{\delta_z^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4}}. \quad (38)$$

This maximum value is attained for  $\delta_x^2 = \delta_z^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4}$ . We see that  $R^2$  approaches 1/4 (and thus the two roots  $\omega_+$  and  $\omega_-$  coincide) for both a near depth-independent vertical profile (i.e.  $\delta_z \approx 0^1$ ) and  $\epsilon_a \approx \epsilon_i$  (or in dimensional form  $N = g/c_s$ ). This last equality is true when the stratification and the compressibility effects have an identical contribution to the vertical variation of background stratification  $\hat{\rho}_h(z)$ . In all other cases, the vast majority, the two roots are well separated and given by:

$$\omega_-(\delta_x, \delta_z) \approx \omega_i(\delta_x, \delta_z), \quad \omega_+(\delta_x, \delta_z) \approx \omega_a(\delta_x, \delta_z) \quad (39)$$

190 These two roots are studied below.

**Modified internal waves (MIW)** The traditional dispersion relation for dispersive internal gravity wave rays in the context of a Boussinesq incompressible fluid (Gill 1982; see also Table 1 above) is:

$$\omega_{iwr}^2 = \epsilon_i^2 \frac{\delta_x^2}{\delta_x^2 + \delta_z^2} \quad (40)$$

A Taylor expansion of the gravity-wave root  $\omega_-^2$  given by (36) with respect to the small parameters  $\epsilon_a$  and  $\epsilon_i$  leads to:

$$\frac{\omega_-^2}{\omega_{iwr}^2} = 1 - \left( \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} - \frac{\epsilon_i^2 \epsilon_a^2 \delta_x^2}{(\delta_x^2 + \delta_z^2)^2} \right) + O((\epsilon_i^2 + \epsilon_a^2)^4) \quad (41)$$

$$= 1 - \underbrace{\frac{(\epsilon_i^2 + \epsilon_a^2)^2 \delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2 \delta_x^2}{4(\delta_x^2 + \delta_z^2)^2}}_{O((\epsilon_i^2 + \epsilon_a^2)^2)} + O((\epsilon_i^2 + \epsilon_a^2)^4), \quad (42)$$

while the development of  $\omega_i^2$  leads to

$$\frac{\omega_i^2}{\omega_{iwr}^2} = 1 - \underbrace{\frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)}}_{O((\epsilon_i^2 + \epsilon_a^2)^2)} + O((\epsilon_i^2 + \epsilon_a^2)^4).$$

191 Compared with  $\omega_i^2$ ,  $\omega_-^2$  includes corrective terms due to the two roots of the inner dispersion relation not  
 192 being fully separated. The corrective term  $\frac{\epsilon_i^2 \epsilon_a^2 \delta_x^2}{4(\delta_x^2 + \delta_z^2)^2}$  is naturally close to  $R^2$ . It is trivial to see from  
 193 formulation (42) that the combined effect of compressibility and stratification is always a reduction of  
 194 frequency, compared with the approximated value  $\omega_{iwr}$ :  $\omega_-^2 \leq \omega_{iwr}^2$ .

195

196 Ocean waves satisfying (41) will now be referred to as *Modified Internal Waves (MIW)*.

**Modified acoustic waves (MAW)** The well-known dispersion relation for acoustic waves in an homogeneous fluid is:

$$\omega_{aw}^2 = \frac{1}{\epsilon_a^2} (\delta_x^2 + \delta_z^2) \quad (43)$$

A Taylor development of the acoustic root ( $\omega_+$ ) with respect to  $\epsilon_a$  and  $\epsilon_i$  leads this time to:

$$\frac{\omega_+^2}{\omega_{aw}^2} = 1 + \left( \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} - \frac{\epsilon_i^2 \epsilon_a^2 \delta_x^2}{(\delta_x^2 + \delta_z^2)^2} \right) + O((\epsilon_i^2 + \epsilon_a^2)^4) \quad (44)$$

$$= 1 + \underbrace{\frac{(\epsilon_i^2 + \epsilon_a^2)^2 \delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2 \delta_x^2}{4(\delta_x^2 + \delta_z^2)^2}}_{O((\epsilon_i^2 + \epsilon_a^2)^2)} + O((\epsilon_i^2 + \epsilon_a^2)^4), \quad (45)$$

---

<sup>1</sup>We use the term *depth-independent* even if  $F(z)$  is a linear function of  $z$  when  $\delta_z$  is close to 0 (see (25)), so that the vertical profile of  $W(z)$  is approximately linear while the vertical profile of  $U(z)$  is approximately constant.

while the development of  $\omega_a^2$  leads to

$$\frac{\omega_a^2}{\omega_{aw}^2} = 1 + \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} + O((\epsilon_i^2 + \epsilon_a^2)^4).$$

$O((\epsilon_i^2 + \epsilon_a^2)^2)$

197 Compared with  $\omega_a^2$ ,  $\omega_+^2$  includes corrective terms due to the two roots of the inner dispersion relation  
 198 not being fully separated. Again, the corrective term is small and close to  $R^2$ . The combined effect of  
 199 compressibility and stratification is always an increase of frequency, compared with the approximated  
 200 value  $\omega_{aw}$ :  $\omega_+^2 \geq \omega_{aw}^2$ .

201

Ocean waves satisfying (44) will be called *Modified Acoustic Waves (MAW)* in the following. The mod-  
 ifications to usual internal and acoustic wave dispersion relations by compressibility and stratification  
 effects are expressed by:

$$\omega_+^2 \omega_-^2 = \frac{\epsilon_i^2}{\epsilon_a^2} \delta_x^2 = \omega_{aw}^2 \omega_{iwr}^2,$$

202 which explains the symmetry in the above developments for modified internal and acoustic waves.

### 203 3.2.2 Purely imaginary vertical wavenumber ( $\delta_z \in i \mathbb{R}$ )

If now  $\delta_z$  is a purely imaginary complex, it can be written  $\delta_z = i\delta_{z,i}$  with  $\delta_{z,i} \in \mathbb{R}$  and wave solutions are  
 vertically evanescent. The boundary dispersion relation now writes:

$$\omega^2 = \frac{\delta_x^2 \tanh(\delta_{z,i})}{\delta_{z,i} + \frac{\epsilon_a^2 + \epsilon_i^2}{2} \tanh(\delta_{z,i})} \quad (46)$$

and the inner dispersion relation:

$$\delta_x^2 - \delta_{z,i}^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \quad (47)$$

204 When horizontal and vertical wavenumbers are close together, the left-hand side and right-hand side of  
 205 (47) both vanish, i.e., the influence of stratification ( $\epsilon_i^2 \delta_x^2 / \omega^2$ ) and compressibility ( $\epsilon_a^2 \omega^2$ ) cancel out. In  
 206 other words, differences between horizontal and vertical wavenumbers are indication of the influence  
 207 of ocean stratification and/or compressibility. In an incompressible, homogeneous (unstratified) ocean,  
 208  $\epsilon_i = \epsilon_a = 0$  and vertical and horizontal wave-numbers are equal.

209 The developments of  $\omega_-^2, \omega_+^2$  for small  $\epsilon_i, \epsilon_a$  are identical to those for real vertical wavenumbers (41),  
 210 (44) just replacing  $\delta_z^2$  by  $-\delta_{z,i}^2$ .

The remaining question is that of root separation when  $\delta_z \in i \mathbb{R}$ . Unlike when  $\delta_z \in \mathbb{R}$  (previous sub-  
 section), the ratio  $R^2(\delta_x, i \delta_{z,i})$  can be equal to 1/4 even when  $\delta_z$  is not small. Relation (47) imposes  
 $0 \leq \delta_{z,i}^2 \leq \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4}$  and in this range of values,  $R^2$  is an increasing function of  $\delta_{z,i}^2$ . The value of  
 $R^2 = 1/4$  is attained for  $\delta_{z,i}^2 = \delta_{z,i,\star}^2$  given by:

$$\delta_{z,i,\star}^2 = \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} - 2\epsilon_a \epsilon_i \delta_x, \quad (48)$$

211 for which the inner dispersion relation has a double root  $\omega_+^2 = \omega_-^2 = \frac{\epsilon_i}{\epsilon_a} \delta_x$ . When  $\delta_{z,i}$  is less than  $\delta_{z,i,\star}$ ,  
 212 the two roots become separated. Again, since we have proved in appendix that only regular solutions  
 213 with real frequency can exist (when the domain is bounded), the case  $\delta_{z,i}^2 > \delta_{z,i,\star}^2$ , which would lead to  
 214  $R^2 > 1/4$  is not considered. Note that  $\omega_-, \omega_+$  as functions of  $\delta_{z,i}$  are not differentiable at the neutral point  
 215  $\delta_{z,i} = \delta_{z,i,\star}$ .

216 Even if (47) has two roots, the two corresponding branches are always connected, as shown above, and  
 217 thus form one family of ocean waves. Ocean waves satisfying (47) will be called *Modified Surface Waves*  
 218 (*MSW*) in the following.

219 **3.3 Summary: waves solutions in an unbounded ocean**

220 In the preceding analysis, three types of waves were identified. A synthesis is given by Figure 2 showing  
 221 the inner dispersion surfaces for a stratified compressible ocean, and Figure 3 showing limit cases for a  
 homogeneous incompressible ocean. The values of the parameters  $\epsilon_a, \epsilon_i$  are those of Table 2.

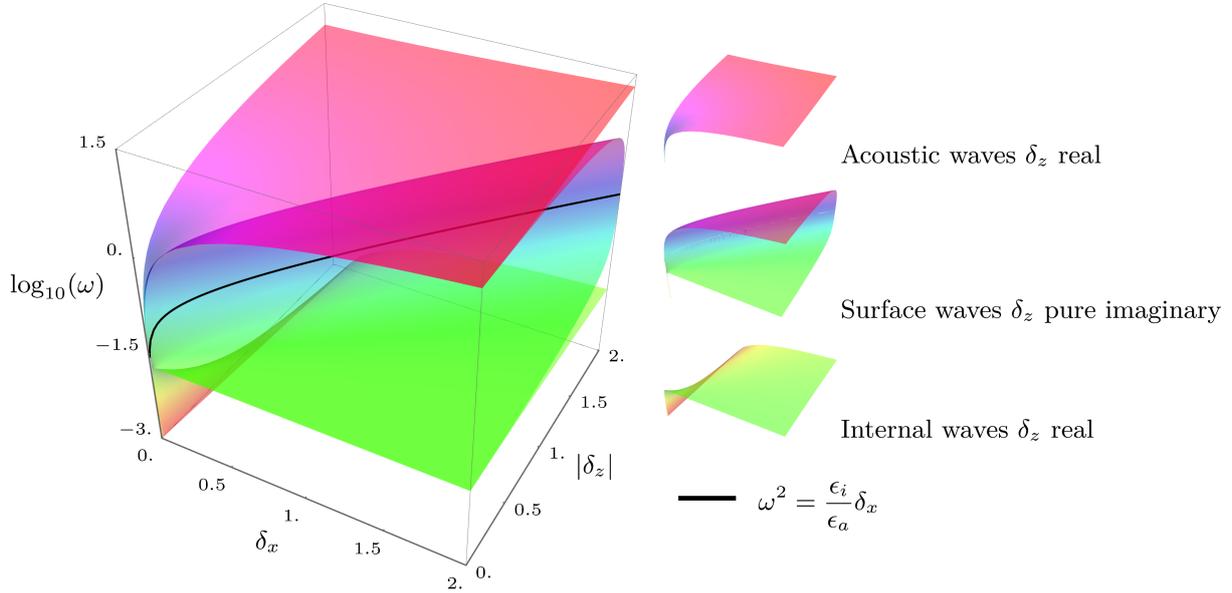


Figure 2: Inner dispersion surfaces in  $(\delta_x, \delta_z, \log_{10}(\omega))$  space for a stratified and compressible ocean. The surfaces are colored by the magnitude of frequency  $\omega$ .

222  
 223 Approximate frequency values for modified internal and acoustic waves are summarized in Table 3 and  
 224 Table 4 in dimensional form, for comparison with introductory Table 1. Since the practical existence and  
 225 characterization of modified surface waves is strongly dependent on boundary conditions, they are not  
 summarized here but will be detailed in the next section.

	Internal Waves	Acoustic Waves
a) $(\epsilon_i = \epsilon_a = 0)$	X	X
b) $(\epsilon_i = 0, \epsilon_a \neq 0)$	X	$\frac{\omega_+^2}{\omega_{aw}^2} \approx 1 + \frac{\epsilon_a^4}{4(\delta_x^2 + \delta_z^2)}$
c) $(\epsilon_i \neq 0, \epsilon_a = 0)$	$\frac{\omega_-^2}{\omega_{iwr}^2} \approx 1 - \frac{\epsilon_i^4}{4(\delta_x^2 + \delta_z^2)}$	X
d) $(\epsilon_i \neq 0, \epsilon_a \neq 0)$	$\frac{\omega_-^2}{\omega_{iwr}^2} \approx 1 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2 \delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2 \delta_x^2}{4(\delta_x^2 + \delta_z^2)^2}$	$\frac{\omega_+^2}{\omega_{aw}^2} \approx 1 + \frac{(\epsilon_i^2 + \epsilon_a^2)^2 \delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2 \delta_x^2}{4(\delta_x^2 + \delta_z^2)^2}$

Table 3: Modified Internal and Acoustic waves in an unbounded ocean.

226  
 227 In a more realistic bounded ocean, their existence is guaranteed only if their vertical scale is (much)  
 228 smaller than the ocean depth  $(|\delta_z| \gg 1)$  and if they do not interfere with the bottom or the surface of the  
 229 ocean. The next section will investigate the impact of adding the boundary dispersion relation (31b).

230 **4 Waves in a bounded ocean**

231 **4.1 Graphical investigation of Modified Surface Waves (MSW), Modified Acoustic Modes (MAM)**  
 232 **and Modified Internal Modes (MIM)**

233 The compressible and stratified ocean is now assumed to be bounded. Wave solutions must thus satisfy  
 234 both the inner (31a) and boundary (31b) dispersion relations. In phase space, they must lie at the in-

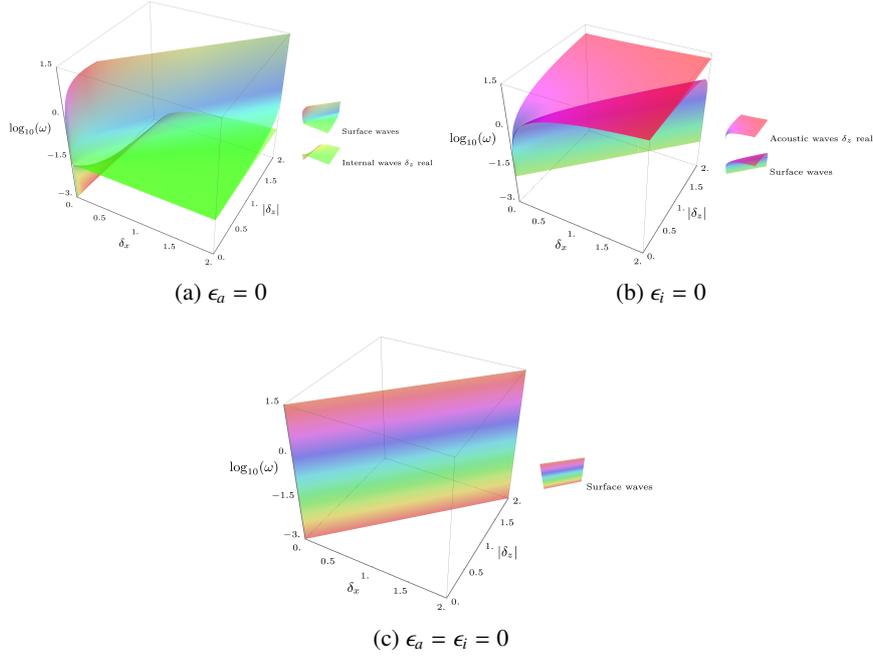


Figure 3: Inner dispersion surfaces for the limit cases of (a) an incompressible ocean (b) an homogeneous ocean and (c) an incompressible and homogeneous ocean (in this case, the wave frequency is free, i.e., not constrained by the inner dispersion relation).

Waves	Frequency ( $\Omega$ )
Modified Acoustic Waves (MAW)	$\Omega_{maw}^2 = c_s^2(k_x^2 + k_z^2) \left[ 1 + \frac{1}{4(k_x^2 + k_z^2)^2} \left( \frac{k_z^2}{D_0^2} + \left( \frac{g}{c_s^2} - \frac{N^2}{g} \right)^2 k_x^2 \right) \right]$
Modified Internal Waves (MIW)	$\Omega_{miw}^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2} \left[ 1 - \frac{1}{4(k_x^2 + k_z^2)^2} \left( \frac{k_z^2}{D_0^2} + \left( \frac{g}{c_s^2} - \frac{N^2}{g} \right)^2 k_x^2 \right) \right]$

Table 4: Compressibility and stratification induced modifications to the usual dispersion relations given in Table 1.  $\Omega$  is wave angular frequency,  $k_x$  and  $k_z$  are the wavenumbers,  $g$  is the acceleration of gravity,  $N$  a reference Brunt-Väisälä frequency and  $c_s$  the speed of sound.  $D_0$  is the background density vertical scale, given by  $1/D_0 = N^2/g + g/c_s^2$

235 intersections of the inner and boundary dispersion surfaces, which are plotted simultaneously on Figure  
236 4.

237 **Branches of the boundary dispersion surface** For real vertical wavenumbers ( $k_z \in \mathbb{R}$ , Figure 4a),  
238 the boundary dispersion surface is a piecewise surface. It includes several branches that are vertical in  
239  $(\delta_x, \delta_z, \log_{10}(\omega))$  space at  $\delta_z \approx n\pi$  (small  $\omega$ ) and at  $\delta_z \approx \pi/2 + m\pi$  (large  $\omega$ ) with  $n \in \mathbb{N}^*$  and  $m \in \mathbb{N}$ . The  
240 intersection of these branches with inner dispersion relation surfaces result in a number of constrained  
241 vertical wavenumbers (according to  $n$  and  $m$ ), the resulting wave solutions will thus be called *modes*.  
242 The two intersections correspond to *Modified Internal Modes (MIM)*,  $n \in \mathbb{N}^*$  and to *Modified Acoustic*  
243 *Modes (MAM)*,  $m \in \mathbb{N}$ .

244 For purely imaginary wavenumbers (Eq. (46)), the boundary surface (Figure 4b) looks like an horizontal  
245 hyperbolic surface which intersects the surface-wave surface. Far from the origin  $(\delta_x, \delta_z) = (0, 0)$ , at the  
246 intersection points,  $|\delta_z|$  is close to  $\delta_x$ .

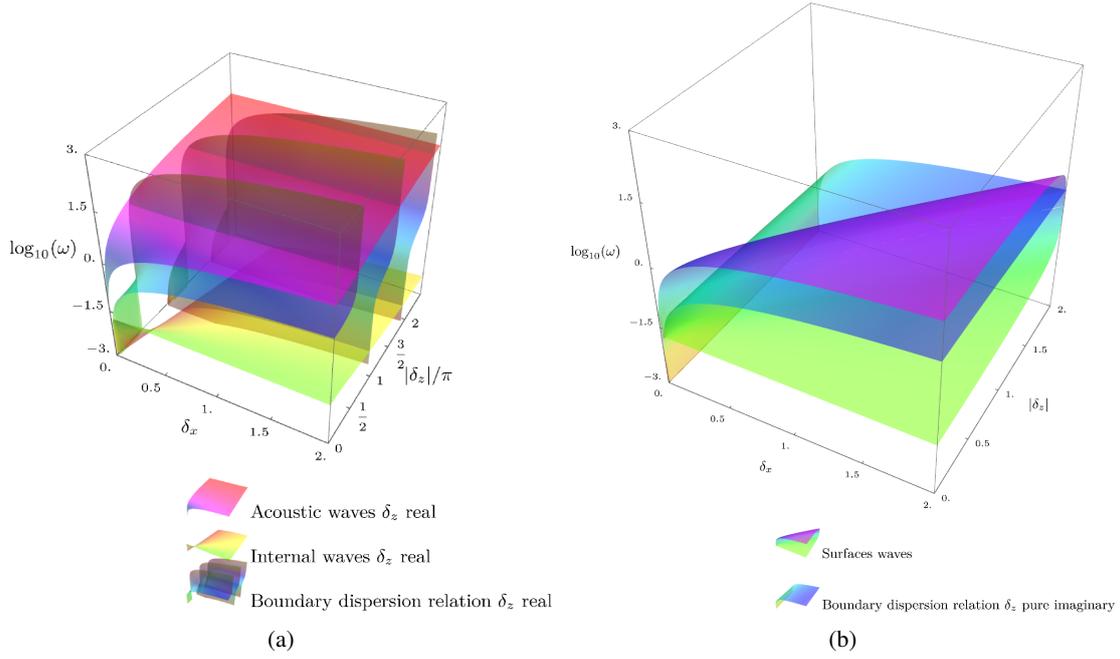


Figure 4: Dispersion surfaces in  $(\delta_x, \delta_z, \log_{10}(\omega))$  space and wave solutions. (a) Wave solutions with real  $\delta_z$ . Polychrome surfaces: inner dispersion surfaces (acoustic and internal branches) and boundary dispersion surface. (b) Wave solutions with purely imaginary  $\delta_z$ . Polychrome surfaces: inner dispersion surface (surface gravity-wave branch) and boundary dispersion surface.

247 **Long-wave solutions** For long waves ( $|\delta_z| \ll 1$ ), the boundary dispersion surfaces for real and purely  
 248 imaginary  $\delta_z$  coincide. Indeed, the development of the boundary relation is well approximated by  $\omega^2 \approx \delta_x^2$   
 249 in both cases (a better approximation is given in Eq. (59)). Figure 5 shows the different branches close  
 250 to the origin  $(\delta_x, \delta_z) = (0, 0)$ . The acoustic-wave surface is not shown since it does not intersect the  
 boundary dispersion surface near the origin (as proven in §4.4). While for very small  $\delta_x$ , the boundary

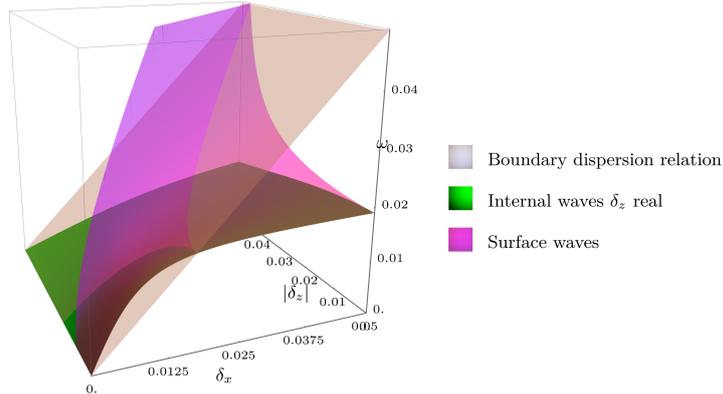


Figure 5: Dispersion surfaces in  $(\delta_x, \delta_z, \omega)$  space and wave solutions in the vicinity of the origin. Polychrome surfaces: inner dispersion surfaces (internal and surface branches) and Boundary dispersion surface.

251 dispersion surface intersects the internal-wave branch (with  $\delta_z$  real), for larger values, it intersects the  
 252 surface-wave branch (with  $\delta_z$  pure imaginary). As will be shown in section 4.4, the switch between  
 253 the two intersections is at  $\delta_x = \delta_{x,0} \approx \epsilon_i$  (and for a vanishing vertical wavenumber  $\delta_z = 0$ ). At these  
 254 wavenumbers, we also have  $\omega \approx \epsilon_i$  and thus, as mentioned in Dukowicz (2013), the barotropic mode  
 255 frequency saturates at the buoyancy frequency. However for larger horizontal wavenumbers ( $\delta_x \geq \epsilon_i$ ), a  
 256

257 vertically evanescent surface wave is found.

258 **Summary** In the context of a bounded ocean, three types of wave solutions are graphically identified,  
 259 spreading on the three branches of inner dispersion surface, while satisfying the boundary dispersion  
 260 relation: internal gravity (in a stratified ocean), acoustic (in a compressible ocean) and surface waves  
 261 (in a free-surface ocean). They are investigated in following using Taylor expansions of the general  
 262 roots  $\omega_{\pm}$ , with respect to small parameters  $(\epsilon_i, \epsilon_a)$ , leading to simple approximations of wave dispersion  
 263 relations. When needed, asymptotic relations are derived with respect to  $\delta_x, \delta_z$  or  $\omega$ . Taylor expansions  
 264 will give indications of how usual wave solutions can be modified by gravity and stratification  $(\epsilon_i)$ , and/or  
 265 by compressibility  $(\epsilon_a)$ .

## 266 4.2 Real $\delta_z$ : modified internal and acoustic waves

267 As shown in §3.2.1, as long as  $\delta_z$  is not close to zero, upper (acoustic) and lower (gravity) branches of the  
 268 inner dispersion surface for real  $\delta_z$  are well-separated and MAM and MIM solutions can thus be studied  
 269 independently. The case of long waves with  $\delta_z \approx 0$  is discussed in subsection 4.4.

### 270 4.2.1 Development of internal-gravity modes modified by compressibility (MIM)

Waves can propagate horizontally between the bottom and surface of the ocean as in a wave guide.  
 Internal gravity modes are well-known such examples (Gill, 1982). In the previous section, graphical  
 inspections of wave solutions confirmed that gravity waves with constrained vertical wavenumbers could  
 be found at the intersection of inner and boundary dispersion surfaces. We have also shown in §3.2.1 that  
 the root of the inner dispersion relation corresponding to internal gravity waves is well-approximated by  
 $\omega_i^2$ . Making  $\omega^2(\delta_x) \approx \omega_i^2$  in the boundary dispersion relation (31b) leads to

$$\frac{1}{\delta_z / \tan(\delta_z) + \frac{\epsilon_i^2 + \epsilon_a^2}{2}} \approx \frac{\epsilon_i^2}{\delta_x^2 + \delta_z^2 + \frac{1}{4}(\epsilon_i^2 + \epsilon_a^2)^2} \quad (49)$$

271 Since we assume here that  $\delta_z$  is not close to 0, the right-hand side of (49) is always small and thus  
 272  $\delta_z / \tan(\delta_z)$  has to be large, i.e., the vertical wavenumber  $\delta_z$  is close to  $n\pi$  with  $n$  a non-zero integer. This  
 273 agrees with the internal gravity wave solution found graphically in subsection 4.1.

In order to refine the approximation, we equate the squared frequency given by the surface dispersion  
 relation (31b) to  $\omega_i^2$ , given by the inner dispersion relation (36). This forms a non-linear equation for  
 $\delta_z$  whose solution can be approximated using two passes of a Newton algorithm starting from  $\delta_z = \delta_{z,n}$ .  
 Finally a Taylor development in  $\epsilon_i, \epsilon_a$  leads to:

$$\delta_z(\delta_x) = \delta_{z,mim}(\delta_x) = \delta_{z,n} \left( 1 + \frac{\epsilon_i^2}{\delta_x^2 + \delta_{z,n}^2} + \frac{(\delta_x^2 - \delta_{z,n}^2)}{(\delta_x^2 + \delta_{z,n}^2)^3} \epsilon_i^4 \right) + O((\epsilon_i^2 + \epsilon_a^2)^3) \quad (50)$$

A development for the frequency  $\omega^2$  can be obtained by injecting expression (50) of  $\delta_z$  in the general  
 expression (41) in the unbounded domain case. The usual dispersion relation given in Table 1 writes in  
 dimensional form:

$$\omega^2 = \omega_{iwr}^2 \Big|_{\delta_z = \delta_{z,n}} = \epsilon_i^2 \frac{\delta_x^2}{\delta_x^2 + \delta_{z,n}^2}$$

Considering only the first order correction to this relation, we can omit all terms of order 3 in  $(\epsilon_i^2 + \epsilon_a^2)$  in  
 (41) to get:

$$\omega^2(\delta_x) = \omega_{mim}^2(\delta_x) = \epsilon_i^2 \frac{\delta_x^2}{\delta_x^2 + \delta_{z,n}^2} \left( 1 - \frac{2\epsilon_i^2 \delta_{z,n}^2}{(\delta_x^2 + \delta_{z,n}^2)^2} \right) + O((\epsilon_i^2 + \epsilon_a^2)^3) \quad (51)$$

274 Therefore, the main correction to the usual dispersion relation comes from the fact that  $\delta_z$  is not exactly  
 275 equal to  $\delta_{z,n} = n\pi$ .

276 **4.2.2 Development of acoustic modes modified by gravity (MAM)**

Here, we use the fact that the root of the inner dispersion relation corresponding to internal gravity waves is well-approximated by  $\omega_a^2$ . Making  $\omega^2(\delta_x) \approx \omega_a^2$  in the boundary dispersion relation (31b) leads to:

$$\delta_z / \tan(\delta_z) + \frac{\epsilon_i^2 + \epsilon_a^2}{2} \approx \frac{\epsilon_a^2}{1 + \left( \delta_z^2 + \frac{1}{4} (\epsilon_i^2 + \epsilon_a^2)^2 \right) / \delta_x^2} \quad (52)$$

Since the right-hand side of (52) is small (bounded by  $\epsilon_a^2$ ),  $\delta_z / \tan(\delta_z)$  has to be small and thus the vertical wavenumber  $\delta_z$  is close to  $\pi/2 + m\pi$ , with  $m \in \mathbb{N}$ . Again, in order to get a more accurate expression of  $\delta_z$ , we equate the (inversed) squared frequency given by the surface dispersion relation (31b) to the (inversed)  $\omega_-^2$ , given by the inner dispersion relation (36), and perform the nonlinear equation's solution approximation followed by a Taylor development in  $\epsilon_i, \epsilon_a$  to obtain:

$$\begin{aligned} \delta_z(\delta_x) = \delta_{z,mam}(\delta_x) = \delta_{z,m} - \frac{(\delta_x^2 - \delta_{z,m}^2)}{2\delta_{z,m}(\delta_x^2 + \delta_{z,m}^2)} \epsilon_a^2 + \frac{\epsilon_i^2}{2\delta_{z,m}^2} \\ - \frac{\delta_x^6(\epsilon_a^2 - \epsilon_i^2)^2 + 3\delta_x^2(\epsilon_a^2 - \epsilon_i^2)^2\delta_{z,m}^2 - \delta_x^2(5\epsilon_a^2 - 3\epsilon_i^2)(\epsilon_a^2 + \epsilon_i^2)\delta_{z,m}^4 + (\epsilon_a^2 + \epsilon_i^2)^2\delta_{z,m}^6}{4(\delta_x^2 + \delta_{z,m}^2)^3\delta_{z,m}^3} \\ \underbrace{\hspace{15em}}_{O((\epsilon_i^2 + \epsilon_a^2)^2)} + O((\epsilon_i^2 + \epsilon_a^2)^3) \end{aligned} \quad (53)$$

An expansion for the frequency  $\omega^2$  can be obtained by injecting the expression (53) of  $\delta_z$  in the general expression (44) in the unbounded domain case. The main departure from to the usual acoustic wave frequency, given in Table 1 in dimensional form, is given by the second-order development:

$$\omega^2(\delta_x) = \omega_{mam}^2(\delta_x) = \frac{1}{\epsilon_a^2} \left[ \delta_x^2 + \delta_{z,m}^2 - \frac{(\delta_x^2 - \delta_{z,m}^2)}{(\delta_x^2 + \delta_{z,m}^2)} \epsilon_a^2 + \epsilon_i^2 \right] + O((\epsilon_i^2 + \epsilon_a^2)) \quad (54)$$

277 Here, stratification has a first-order (in  $\epsilon_i^2$ ) contribution to the modification of the homogeneous case fre-  
 278 quency. This first-order modification comes from the first-order modification on the vertical wavenumber  
 279 itself (53). However, it is clear that the associated impact is small since  $\epsilon_i^2$  is negligible compared with  
 280  $\delta_{z,m}^2$  in (53) and (54), because  $\delta_{z,m} \geq \pi/2$ . A similar conclusion was drawn in Smith (2015).

281 **4.3 Purely imaginary  $\delta_z$ : surface acoustic-gravity waves**

"Surface waves" generally refer to wave propagating horizontally as anomalies of the ocean free-surface (Gill, 1982). In the vertical direction, these surface wave anomalies are "evanescent", meaning that, with the notation chosen in the present study, the vertical wavenumber  $\delta_z$  is a purely imaginary complex number. A MSW defined by its triplet  $(\delta_x, \delta_z, \omega)$  must satisfy both the inner (31a) and boundary (31b) dispersion relations for  $\delta_z = i \delta_{z,i}$ :

$$\delta_x^2 - \delta_{z,i}^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \quad , \quad \omega^2 = \frac{\delta_x^2}{\frac{\delta_{z,i}}{\tanh(\delta_{z,i})} + \frac{\epsilon_a^2 + \epsilon_i^2}{2}} \quad (55)$$

282 In an homogeneous and incompressible ocean ( $\epsilon_i = \epsilon_a = 0$ ), we find  $\delta_{z,i} = \delta_x$ . This equality is of-  
 283 ten postulated in textbooks to reduce the number of variables. Vertical polarization relations are then  
 284 functions of  $\delta_x$  only (Gill, 1982) and, as a consequence, the only remaining dispersion relation is the  
 285 boundary dispersion relation (31b) for purely imaginary vertical wavenumber ( $\delta_z = i\delta_x$ ), or its approx-  
 286 imation  $\omega^2 = \delta_x \tanh \delta_x$ . In this case,  $\delta_{z,i}$  is reduced to a vertical length-scale of energy decay with  
 287 increasing distance from the surface; for very long waves ( $\delta_x \gg 1$ ), the surface wave is approximately  
 288 depth-independent.

**Surface waves existence** In the more general case (non homogeneous and compressible ocean), it is first necessary to study the existence of solutions to (55). Let us define  $f(\delta_x, \delta_{z,i})$  by:

$$f(\delta_x, \delta_{z,i}) = \delta_x^2 - \delta_{z,i}^2 - \left( \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \right), \quad \text{with } \omega^2 = \frac{\delta_x^2}{\frac{\delta_{z,i}}{\tanh(\delta_{z,i})} + \frac{\epsilon_a^2 + \epsilon_i^2}{2}}$$

289 The inner boundary relation translates into  $f(\delta_x, \delta_{z,i}) = 0$ . It is easy to check that, for a given  $\delta_{z,i}$ ,  
 290  $f(\delta_x, \delta_{z,i})$  is an increasing function of  $\delta_x^2$ . Let  $\delta_{x,0}$  be the value of  $\delta_x$  defined by  $f(\delta_{x,0}, \delta_z = 0) = 0$  ( $\delta_{x,0}$   
 291 will be given in (58), and is close to  $\epsilon_i$ ). In particular, we have  $f(\delta_x, 0) \leq f(\delta_{x,0}, 0) = 0$  for  $\delta_x \leq \delta_{x,0}$ . It is  
 292 also easy to verify that for a given  $\delta_x$  such that  $\delta_x \leq \delta_{x,0}$ , the maximum of  $f(\delta_x, \delta_z)$  is attained in  $\delta_{z,i} = 0$ .  
 293 This proves that if  $\delta_x < \delta_{x,0}$ ,  $f(\delta_x, \delta_{z,i}) < 0 \forall \delta_{z,i}$ . Therefore, modified surface waves can only exist under  
 294 the condition  $\delta_x \geq \delta_{x,0}$ .

**Dispersion relation** It can be shown that  $1 \leq \frac{\delta_{z,i}}{\tanh(\delta_{z,i})} \leq \delta_{z,i}^2 + 1$ ,  $\forall \delta_{z,i} \in \mathbb{R}$ . Injecting these two inequalities in the surface dispersion relation leads to:

$$\frac{\delta_x^2}{\omega^2} \leq \delta_{z,i}^2 + 1 + \frac{\epsilon_a^2 + \epsilon_i^2}{2}, \quad \text{and} \quad \omega^2 \leq \delta_x^2$$

Since the inner dispersion relation implies  $-\frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \leq \delta_x^2 - \delta_{z,i}^2 \leq \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2$ , we finally get:

$$-\frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \leq \delta_x^2 - \delta_{z,i}^2 \leq \epsilon_i^2 \delta_{z,i}^2 + \epsilon_a^2 \delta_x^2 + \epsilon_i^2 \left( 1 + \frac{\epsilon_a^2 + \epsilon_i^2}{2} \right)$$

Using the smallness of the parameters  $\epsilon_a$  and  $\epsilon_i$ , we can then conclude that  $\delta_{z,i}^2$  is indeed close to  $\delta_x^2$ . The crude assumption  $\delta_{z,i} \approx \delta_x$  is sufficiently accurate to recover usual swell-like approximations (Table 1), i.e., for sufficiently large  $\delta_x$  (or  $\delta_z$ ). However a more accurate expression is given by:

$$\delta_{z,i}^2 = \delta_x^2 - \delta_x \left( \epsilon_i^2 / \tanh(\delta_x) + \epsilon_a^2 \tanh(\delta_x) \right) + O((\epsilon_i^2 + \epsilon_a^2)^2). \quad (56)$$

In order to obtain (56), we introduce the frequency given by the boundary dispersion relation in the inner dispersion relation (see (55)) and solve the resulting nonlinear equation by performing one pass of a Newton algorithm. Note that the problem is formulated in terms of  $\delta_{z,i}^2(\delta_x)$  since it can be shown (by looking at the error estimate of the Newton algorithm) that a formulation in terms of  $\delta_{z,i}(\delta_x)$  is not accurate when  $\delta_x$  is relatively small. Note also that (56) requires  $\delta_x \geq \epsilon_i$  for  $\delta_{z,i}^2$  to be positive. This is consistent with the fact that  $\delta_x$  must be greater than  $\delta_{x,0} (\approx \epsilon_i)$ , as previously shown. However (56) does not allow us to recover the exact value  $\delta_{x,0}$  of  $\delta_x$  that cancels  $\delta_{z,i}$ , due to the first-order only approximation in terms of  $(\epsilon_i^2 + \epsilon_a^2)$ . The case of long surface waves with  $\delta_{z,i}$  close to zero will be treated separately in subsection 4.4.

The corresponding frequency is well approximated by:

$$\omega^2 = \delta_x \tanh(\delta_x) \left[ 1 - \frac{1}{2} \left( \epsilon_i^2 / \sinh(\delta_x)^2 + \epsilon_a^2 / \cosh(\delta_x)^2 - \epsilon_i^2 / (\delta_x \sinh(\delta_x) \cosh(\delta_x)) \right) \right] + O((\epsilon_i^2 + \epsilon_a^2)^2) \quad (57)$$

For very short waves ( $\delta_x \gg 1$ ), these relations simplify to:

$$\delta_{z,i} \approx \delta_x - \frac{1}{2} (\epsilon_i^2 + \epsilon_a^2), \quad \omega^2 \approx \delta_x$$

295 and we recover the frequency of short non hydrostatic waves  $\omega^2 = \delta_x$  (or  $\Omega = \sqrt{gk_x}$  in dimensional  
 296 form) with a slightly modified vertical wavenumber.

<sup>2</sup>This simple demonstration requires the assumption  $\epsilon_a^2 \leq 2 + \epsilon_i^2$ , i.e., the same used in appendix to prove that the imaginary part of  $\delta_z^2$  is zero.

297 **4.4 Long waves**

Here, we perform specific developments for long waves where the vertical profile is almost depth-independent  $\delta_z \approx 0$ ,  $\delta_z$  being either real or pure imaginary. Inserting the boundary dispersion relation (31b) into the inner dispersion relation (31a) and making a second-order Taylor expansion in  $\delta_z$  leads to:

$$\delta_z^2 = (\delta_{x,0}^2 - \delta_x^2) \frac{1 - \frac{\epsilon_a^2}{1+(\epsilon_i^2+\epsilon_a^2)/2}}{1 + \epsilon_i^2/3 - \frac{\delta_x^2}{3(1+(\epsilon_i^2+\epsilon_i^2)/2)^2} \epsilon_a^2} + O(\delta_z^4) \quad \text{with } \delta_{x,0}^2 = \frac{\epsilon_i^2 + (\epsilon_i^4 - \epsilon_a^4)/4}{1 - \frac{\epsilon_a^2}{1+(\epsilon_i^2+\epsilon_a^2)/2}} \approx \epsilon_i^2 \quad (58)$$

$\delta_{x,0}$  is the value of  $\delta_x$  for which  $\delta_z = 0$  is a solution of the inner and boundary dispersion relations. At first order in  $(\epsilon_i^2 + \epsilon_a^2)$ ,  $\delta_{x,0}$  is equal to  $\epsilon_i$ . The corresponding frequency can be obtained by inserting the approximation of the vertical wavenumber given by (58) into the boundary relation dispersion (31b). An approximation at second order in  $\delta_x^2$  and  $(\epsilon_i^2 + \epsilon_a^2)$  is:

$$\omega^2 = \delta_x^2 \left( 1 - \frac{1}{6} (\epsilon_i^2 + 3\epsilon_a^2) + \frac{\epsilon_a^2}{6} (\epsilon_a^2 + \epsilon_i^2) + \frac{1}{45} \epsilon_i^4 + O((\epsilon_i^2 + \epsilon_a^2)^3) \right) + O(\delta_x)^4. \quad (59)$$

298  **$\delta_x \leq \delta_{x,0}$ : barotropic mode** For  $\delta_x \leq \delta_{x,0} \approx \epsilon_i$ ,  $\delta_z$  given by (58) is real. This implies that  $\omega^2 \approx \delta_x^2 \leq$   
 299  $\epsilon_i \delta_x \leq \frac{\epsilon_i}{\epsilon_a} \delta_x$  (since  $\epsilon_a < 1$ ) and, thus, using the bounds on  $\omega_-^2, \omega_+^2$  given by (37), that these waves are  
 300 always issued from the internal-wave branch ( $\omega_-^2$ ), not from the acoustic-wave branch ( $\omega_+^2$ ). In other  
 301 words, there is no intersection between the boundary dispersion relation and the (real) acoustic-wave  
 302 branch for  $\delta_z$  close to zero. These waves can be seen as the barotropic mode ( $n = 0$ ) solution of the  
 303 internal gravity-wave branch. As noted by Dukowicz (2013), this mode, as higher internal modes, has its  
 304 frequency  $\omega^2$  approximately bounded by  $\epsilon_i^2$  (i.e.  $\Omega \leq N$ ), since  $\omega^2 \approx \delta_x^2 \leq \epsilon_i^2$ .

305  **$\delta_x \geq \delta_{x,0}$ : long surface waves** For  $\delta_x \geq \delta_{x,0}$  (but still small),  $\delta_z$  given by (58) is purely imaginary.  
 306 These long waves originate from the surface-wave branch. When  $\delta_x$  increases, (58) converges to (56)  
 307 (see Fig. (7)), which is a better approximation for medium and short surface waves. Figure 6 shows  
 308  $\delta_z^2$  as a function of  $\delta_x$  (for  $\delta_x \leq 2\epsilon_i \approx 0.04$ ). The two approximations given by (58), accurate for long  
 309 waves, and (56), accurate for medium/short waves, are plotted. On Figure 7, the modulus of the vertical  
 wavenumber is shown on a larger interval for  $\delta_x$  ( $\delta_x \leq 4\epsilon_i \approx 0.08$ ).

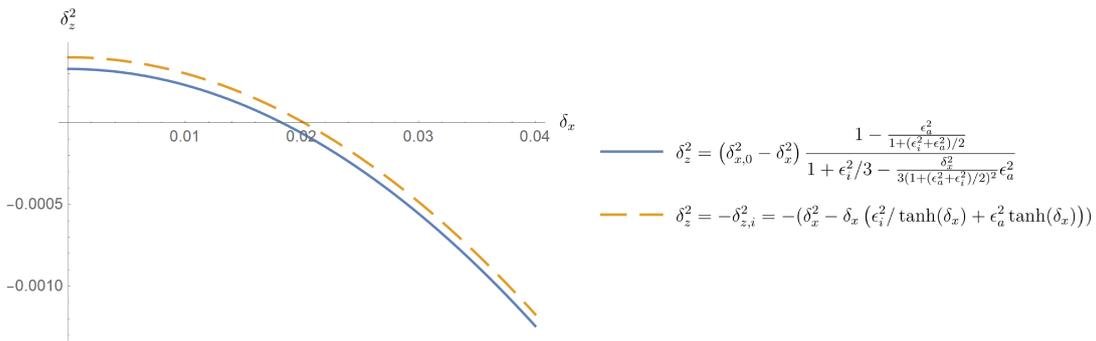


Figure 6: Long waves: squared vertical wavenumber  $\delta_z^2$  as a function of the horizontal wavenumber  $\delta_x$ . The interval where  $\delta_z^2$  is positive ( $\delta_z$  real), for  $\delta_x \leq \delta_{x,0}$ , corresponds to the barotropic mode while for  $\delta_x \geq \delta_{x,0}$ ,  $\delta_z^2$  is negative ( $\delta_z$  purely imaginary), corresponding to the surface-wave branch. The blue curve is the accurate approximation for long waves given by (58), while the dashed orange curve is the approximation of surface waves given by (56), which is more accurate for medium/short surface waves. An exact solution can be numerically computed (not shown) and is visually not distinguishable from the blue curve (accurate approximation for long waves).

310

311 Figure 8 presents the frequency  $\omega$  as a function of the horizontal wavenumber  $\delta_x$  according to different

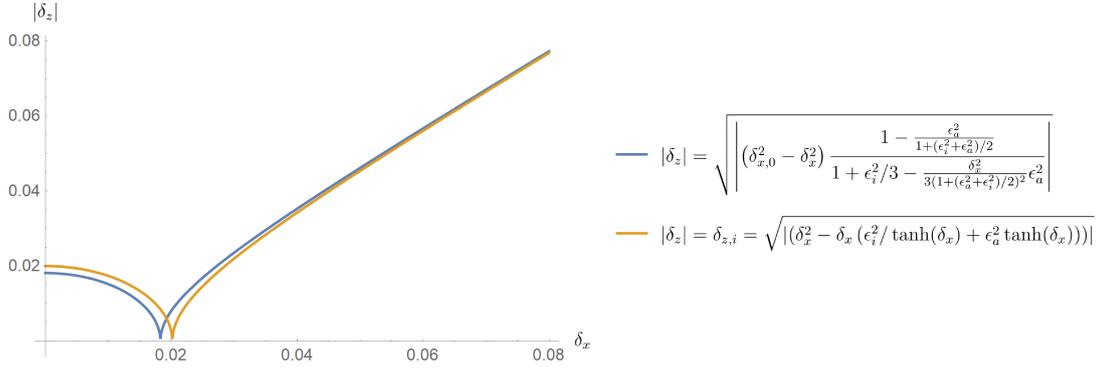


Figure 7: Long waves: modulus of the vertical wavenumber  $|\delta_z|$  as a function of the horizontal wavenumber  $\delta_x$ . The exact solution can be numerically computed (not shown) and is visually not distinguishable from the blue curve (accurate approximation). The vertical wavenumber of an incompressible and homogeneous ocean would follow the straight line  $\delta_z = \delta_x$ .

312 approximations.  $\omega_{\text{Long Waves}}$  and  $\omega_{\text{Medium Short Waves}}$  are computed by inserting the corresponding approxima-  
 313 tions (58) and (56) of  $\delta_z$  in the boundary dispersion relation (31b).  $\omega = \delta_x$  (resp.  $\omega = \sqrt{\delta_x}$ ) repre-  
 314 sents the classical long shallow-water waves (resp. short non-hydrostatic surface waves) approximation.  
 315  $\omega = \sqrt{\delta_x \tanh(\delta_x)}$  is the frequency of an homogeneous and incompressible (non hydrostatic) ocean. Un-  
 316 surprisingly, this last approximation is accurate over the full range of horizontal wavenumbers, even for  
 317 the set of compressible stratified equations. Indeed, even if for long waves,  $\delta_z$  is not directly linked to  $\delta_x$   
 318 (in particular  $\delta_z$  does not vanish for  $\delta_x \approx 0$ ),  $\delta_z$  remains small (less than  $\epsilon_i$ ) and thus  $\delta_z / \tanh(\delta_z)$  is close to  
 1.

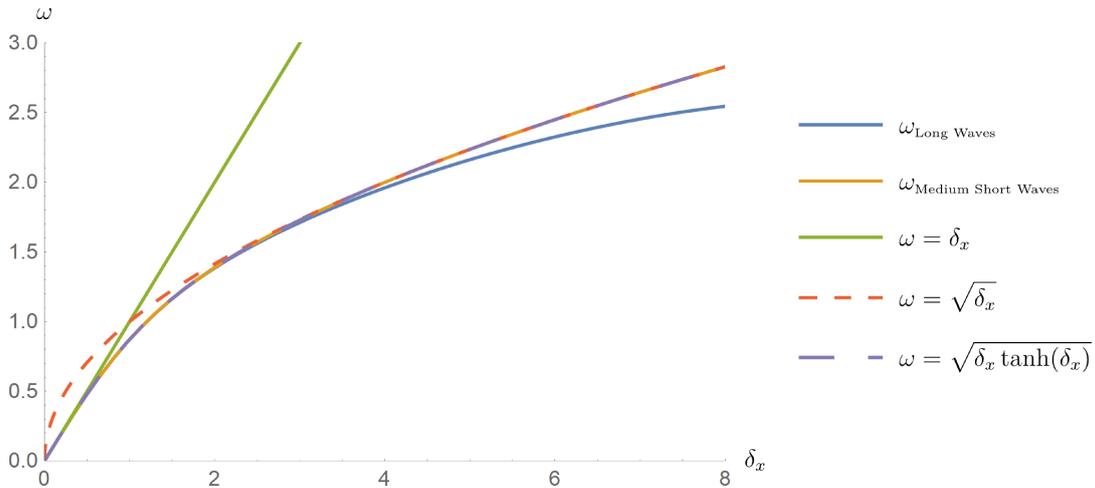


Figure 8: Frequency  $\omega$  as a function of  $\delta_x$  the horizontal wavenumber for different approximations.

319

320 **Remark on Lamb Waves and surface acoustic-gravity waves** In (2.5), horizontally propagating  
 321 acoustic waves (Lamb Waves) have been discarded due to the existence of the free surface boundary  
 322 condition which links the evolution of the surface pressure with the vertical velocity. If, instead of a free  
 323 surface, a rigid lid boundary condition is imposed, Lamb waves are recovered and replace the surface  
 324 waves studied above: while in an incompressible ocean, the rigid lid assumption modifies the barotropic  
 325 external wave in a wave with an infinite speed, in a compressible ocean, its transforms a surface wave in  
 326 a Lamb wave. Also note that in the rigid lid case, both the internal and acoustic modes have a vertical  
 327 wavenumbers given by  $\delta_{z,n} = n\pi, \delta_{z,m} = m\pi$ .

328 In the general case of a free surface boundary condition, there are similarities between surface waves and

329 Lamb waves: their frequencies lie between those of internal and acoustic modes and they are vertically  
 330 evanescent.

#### 331 4.5 Summary: waves solutions in a bounded ocean

332 In section (3) on waves in an unbounded ocean, we have seen that for the case of real vertical wavenum-  
 333 ber (*vertically propagating waves*) (3.2.1) internal and acoustic waves are well separated except for long  
 334 waves ( $\delta_z \approx 0$ ). We have here studied the consequences of the added surface dispersion relation.

335 At low frequency, modified internal gravity modes (MIM) have their frequencies bounded by above by  
 336 the buoyancy frequency  $\omega_- \leq \omega_{c,i} = \epsilon_i$  (i.e.  $\Omega_{c,i} = N$ ).  $\omega_{c,i}$  is the internal cut off frequency. Compress-  
 337 ibility effects on internal-gravity modes (MIM) are negligible (4.2.1). In comparison with the textbook  
 338 rigid lid approximated dispersion relation of MIM, the free surface boundary condition brings a small  
 339 modification to their vertical wavenumbers and associated frequencies.

340 At high frequency, modified acoustic modes (MAM) have their frequencies bounded by below by the  
 341 acoustic cut-off frequency  $\omega_{c,a}$ ,  $\omega_+ \geq \omega_{c,a}$  with is attained in the case of a bounded ocean for  $\delta_x = 0$ ,  $\delta_z =$   
 342  $\pi/2$  and writes  $\omega_{c,a}^2 = \frac{1}{\epsilon_a^2} \left( (\pi/2)^2 + \epsilon_a^2 + \epsilon_i^2 \right)$  (which is found by minimizing values of  $\omega$  satisfying Eq.

343 54) (or  $\Omega_{c,a}^2 = \left( \frac{c_s}{H} \right)^2 \left( \left( \frac{\pi}{2} \right)^2 + \epsilon_a^2 + \epsilon_i^2 \right)$ ). Again, stratification effects are small and the lowest order mod-  
 344 ification from the usual dispersion relation comes from the presence of a free surface boundary condition.

345 The free surface branch, which corresponds to pure imaginary wavenumber (*vertically evanescent waves*),  
 346 has been described in section (4.3). Here the acoustic and internal gravity branches are not well sepa-  
 347 rated. When frequency is increased, modified surface waves (MSW) can change from surface waves  
 348 modified by stratification to surfaces waves modified by compressibility. The crossing point is attained

349 when  $\omega^2 = \frac{\epsilon_i}{\epsilon_a} \delta_x$ . For the case of the bounded ocean here, only one triplet ( $\delta_{x,\star}, \delta_{z,\star}, \omega_\star$ ) can satisfy  
 350 this relation. Since for free surface waves,  $\omega^2$  is well approximated by  $\omega^2 = \delta_x \tanh(\delta_x)$ , this neutral  
 351 point exist only when  $\exists \delta_x$  s.t.  $\tanh(\delta_x) \approx \frac{\epsilon_i}{\epsilon_a}$  which requires  $\epsilon_i \leq \epsilon_a$ . When  $\epsilon_i \geq \epsilon_a$ , the free surface  
 352 waves branch always issues from the internal gravity wave branch. We also have noted that in a rigid lid  
 353 approximation where this family of surface waves cannot exist, they are replaced by Lamb waves with a  
 354 frequency of  $\Omega = c_s k_x$ .

355 The case of long waves ( $\delta_z \approx 0$ ) have been addressed in details separately (4.4). The interplay between  
 356 compressibility, stratification and free surface is more pronounced here. There is a transition between the  
 357 vertically propagating barotropic mode (LMIM) for very long horizontal waves to vertically evanescent  
 358 surface waves for higher horizontal wavenumbers (LMSW). This transition has been fully characterized  
 359 and occurs at a point where  $\delta_x = \delta_{x,0} \approx \epsilon_i$  given by (58) and where the frequency is closed to the  
 360 maximum value of internal waves  $\Omega \approx N$ .

361 *Orders of magnitudes:*

362 Orders of magnitudes for the characteristic horizontal/vertical length scales and frequencies introduced  
 363 above are given in table (5) for the reference ocean (parameters of table (2)), for a shallow ocean  $H = 10\text{m}$   
 364 and for a strongly stratified ocean  $N = 10^{-2}\text{s}^{-1}$ .

	<i>Notation</i>	<i>Reference</i> ( $H = 4000\text{m}$ , $N = 10^{-3}$ )	<i>10-m-deep</i>	$N = 10^{-2} \text{ s}^{-1}$
Parameters	$\epsilon_a$	0.132	$6.6 \cdot 10^{-3}$	0.13
	$\epsilon_i$	0.0202	$1.0 \cdot 10^{-3}$	0.20
Acoustic cut-off	$T_{c,a} = 2\pi \sqrt{H/g}/\omega_{c,a}$	10.6 s	0.026 s	10.5 s
Internal cut-off	$T_{c,i} = 2\pi \sqrt{H/g}/\omega_{c,i}$	1.7 h	1.7 h	10.5 mn
LMIM-LMSW cut-off	$\lambda_{x,0} = 2\pi H/\delta_{x,0}$	1367 km	62 km	123 km
	$T_{x,0} = 2\pi \sqrt{H/g}/\omega_{x,0}$	1.9 h	1.7 h	10.5 mn
MSW-neutral point	$\lambda_{x,*} = 2\pi H/\delta_{x,*}$	161 km	411 m	X
	$\lambda_{z,*} = 2\pi H/\delta_{z,*}$	164 km	411 m	X
	$T_* = 2\pi \sqrt{H/g}/\omega_*$	14 mn	41.7 s	X

Table 5: Orders of magnitude of various scales in term of horizontal/vertical length scales and wave periods. Parameters for the "Reference" ocean are given in Table (2). "10-m-deep" ocean is a 10-m-deep Reference ocean and " $N = 10^{-2} \text{ s}^{-1}$ " refers to a "Reference" ocean with  $N = 10^{-2} \text{ s}^{-1}$ .

365 Table 6 summarizes the main results of this paper. Approximations for the different type of waves (acous-  
366 tic, gravity and surface waves/modes) are indicated in their dimensional form. For sake of readability,  
367 some of the relations given in Table 6 are lower order versions of our derivations. They generally stem  
368 from first-order correction terms, compared with usual dispersion relations introduced in Table 1. In that  
369 case, a red link to the higher approximation is given.  
370 Corresponding analytical solutions for the vertical profiles of  $u, w, p, \rho$  are given in Appendix (B).

Waves	Frequency ( $\Omega$ )	Vertical wavenumber $k_z$
Modified Acoustic Modes (MAM) $k_{z,m} = \frac{1}{H}(\pi/2 + m\pi), m \geq 0$ (4.2.2)	$\Omega_{mam}^2 = c_s^2(k_x^2 + k_z^2) \left[ 1 - \frac{g}{Hc_s^2} \frac{k_x^2 - k_{z,m}^2}{(k_x^2 + k_{z,m}^2)^2} + \frac{N^2}{gH} \frac{1}{k_x^2 + k_{z,m}^2} \right]$	$k_{z,m} \left[ 1 - \frac{g}{Hc_s^2} \frac{k_x^2 - k_{z,m}^2}{2(k_x^2 + k_{z,m}^2)k_{z,m}^2} + \frac{N^2}{2gH^2} \frac{1}{k_{z,m}^2} \right]$ <span style="color: red;">(53)</span>
Modified Internal Modes (MIM) $k_{z,n} = \frac{1}{H}(n\pi), m \geq 1$ (4.2.1)	$\Omega_{mim}^2 = \frac{N^2 k_x^2}{k_x^2 + k_{z,n}^2} \left[ 1 - 2 \frac{N^2}{gH} \frac{k_{z,n}^2}{(k_x^2 + k_{z,n}^2)^2} \right]$	$k_{z,n} \left[ 1 + \frac{N^2}{gH} \frac{1}{k_x^2 + k_{z,n}^2} + \left( \frac{N^2}{gH} \right)^3 \frac{(k_x^2 - k_{z,n}^2)}{(k_x^2 + k_{z,n}^2)^3} \right]$
Barotropic mode $k_x \leq k_{x,0} \approx \frac{N^2}{g}$ (4.4)	$\Omega^2 = gHk_x^2 \left[ 1 - \frac{1}{6}(\epsilon_t^2 + 3\epsilon_a^2) \right]$ <span style="color: red;">(59)</span>	$\sqrt{k_{x,0}^2 - k_x^2}$ <span style="color: red;">(58)</span>
Long surface waves $k_x \geq k_{x,0} \approx \frac{N^2}{g}$ (4.4)	$\Omega^2 = gHk_x^2 \left[ 1 - \frac{1}{6}(\epsilon_t^2 + 3\epsilon_a^2) \right]$ <span style="color: red;">(59)</span>	$i \sqrt{k_x^2 - k_{x,0}^2}$ <span style="color: red;">(58)</span>
Medium and short surface waves $k_x \geq k_{x,0} \approx \frac{N^2}{g}$ (4.3)	$\Omega^2 = gk_x \tanh(Hk_x)$ <span style="color: red;">(57)</span>	$ik_x \sqrt{1 - \frac{1}{k_x} \left( \frac{N^2}{g \tanh(Hk_x)} + \frac{g}{c_s^2} \tanh(Hk_x) \right)}$

Table 6: Compressibility and stratification induced modifications to the usual dispersion relations given in table (1).  $\Omega$  is wave angular frequency,  $k_x$  and  $k_z$  are wavenumbers,  $g$  is the acceleration of gravity,  $H$  the water depth,  $N$  a reference Brunt-Väisälä frequency and  $c_s$  the speed of sound.  $\epsilon_t^2 = \frac{N^2 H}{g}$ ,  $\epsilon_a^2 = \frac{gH}{c_s^2}$ .

## 371 5 Conclusion

372 Several authors have made comprehensive studies on waves in the ocean. The primary purpose of this  
373 paper was to extend these studies by going at an higher order in the study of dispersion relations. This  
374 allows to look at the combined effect of compressibility and stratification on acoustic, internal and free  
375 surface waves.

376 We first proved in Appendix (A) that, for physically sounded values of the compressibility  $\epsilon_a$  and strat-  
377 ification  $\epsilon_i$  parameters, the solutions are either vertically propagating ( $\delta_z$  real) or vertical evanescent  $\delta_z$   
378 pure imaginary. The acoustic and gravity waves, satisfying the inner dispersion relation, have then been  
379 described in an unbounded ocean. The addition of the surface dispersion relation leads to the charac-  
380 terization of internal modes (MIM) and acoustic modes (MAM) along with surface waves (MSW). The  
381 analysis has been first performed in light of a graphical study of the inner and boundary dispersion rela-  
382 tions and their intersections. This has allowed us to characterized triplets  $(\delta_x, \delta_z, \omega)$  of solutions. Then  
383 Taylor developments have been performed at an order higher than the usual dispersion relations where  
384 combined effects of compressibility, stratification and freed surface boundary conditions are not taken  
385 into account.

386 The main results have been summarized qualitatively and quantitatively in (4.5) and corresponding ana-  
387 lytic solutions in Appendix (B).

388 These developments are particularly useful for the validation of recent free surface non-hydrostatic and  
389 potentially compressible numerical ocean models.

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## 394 References

- 395 Apel, J. R. (1987). *Principles of Ocean Physics*. Number v. 38 in International Geophysics Series.  
396 Academic Press, London ; Orlando.
- 397 Auclair, F., Bordois, L., Dossmann, Y., Duhaut, T., Paci, A., Ulses, C., and Nguyen, C. (2018). A non-  
398 hydrostatic non-boussinesq algorithm for free-surface ocean modelling. *Ocean Modelling*, 132:12 –  
399 29.
- 400 Dukowicz, J. K. (2013). Evaluation of various approximations in atmosphere and ocean modeling based  
401 on an exact treatment of gravity wave dispersion. *Monthly Weather Review*, pages 4487–4506.
- 402 Eckart, C. (1960). Chapter x - oceans with constant coefficients. In ECKART, C., editor, *Hydrodynamics*  
403 *of Oceans and Atmospheres*, pages 136 – 153. Pergamon.
- 404 Gill, A. E. (1982). *Atmosphere-Ocean Dynamics*. William L. Donn.
- 405 LeBlond, P. H. and Mysak, L. A. (1981). *Waves in the Ocean*. Elsevier.
- 406 Lighthill, M. J. (1967). Waves in fluids. *Communications on pure and applied mathematics*, pages  
407 267–293.
- 408 Pedlosky, J. (1982). *Geophysical Fluid Dynamics*. Springer.
- 409 Smith, J. A. (2015). Revisiting oceanic acoustic gravity surface waves. *Journal of Physical Oceanogra-*  
410 *phy*, pages 2953–2958.
- 411 Tolstoy, I. (1963). The theory of waves in stratified fluids including the effects of gravity and rotation.  
412 *Rev. Mod. Phys.*, 35:207–230.

413 **A  $\delta_z$  real or purely imaginary**

In this section, we prove that under the condition of smallness of parameters  $\epsilon_i$  and  $\epsilon_a$ ,  $\delta_z$  is either real or pure imaginary and thus that the frequency  $\omega$  is real.

The eigenvalue problem (24) and the surface boundary condition (26) are rewritten in non-dimensional form as:

$$G''(s) + \delta_z^2 G(s) = 0 \quad (60)$$

$$G(1) = 1 \quad (61)$$

$$G(0) = 0 \quad (62)$$

$$G'(1) + \left( \frac{\epsilon_i^2 + \epsilon_a^2}{2} - \frac{\delta_x^2}{\omega^2} \right) G(1) = 0 \quad (63)$$

where  $s = \frac{z+H}{H}$ ,  $G(s(z)) = F(z)$  and  $\epsilon_i^2 = \frac{N^2 H}{g}$ ,  $\epsilon_a^2 = \frac{gH}{c_s^2}$ ,  $\delta_x = k_x H$ ,  $\delta_z = k_z H$ ,  $\omega = \Omega \sqrt{\frac{H}{g}}$ .

$\delta_z$  is linked to  $\delta_x$  and  $\omega$  by the inner dispersion relation

$$\delta_z^2 = \left( \delta_x^2 \frac{\epsilon_i^2 - \omega^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} \right) \quad (64)$$

Multiplying (60) by  $\overline{G(s)}$  and integrating over  $[0, 1]$  we get:

$$\int_0^1 G''(s) \overline{G(s)} ds + \delta_z^2 \int_0^1 |G(s)|^2 ds = 0$$

Integration by parts leads to:

$$- \int_0^1 |G'(s)|^2 ds + \delta_z^2 \int_0^1 |G(s)|^2 ds + G'(1) \overline{G(1)} - G'(0) \overline{G(0)} = 0$$

and using (61), (62), (63)

$$\delta_z^2 \int_0^1 |G(s)|^2 ds + \left( \frac{\delta_x^2}{\omega^2} - \frac{\epsilon_i^2 + \epsilon_a^2}{2} \right) = \int_0^1 |G'(s)|^2 ds$$

Using the Poincaré inequality  $\int_0^1 |G(s)|^2 ds \leq \int_0^1 |G'(s)|^2 ds$  and  $1 = |G(1)|^2 \leq \int_0^1 |G'(s)|^2 ds$ , we obtain:

$$\delta_z^2 \mu + \left( \frac{\delta_x^2}{\omega^2} - \frac{\epsilon_i^2 + \epsilon_a^2}{2} \right) \nu = 1 \quad (65)$$

with  $0 \leq \mu \leq 1$  and  $0 \leq \nu \leq 1$ .

Taking the imaginary part of (65) and using (64)

$$\mu \left( \epsilon_a^2 \Im[\omega^2] + \delta_x^2 \epsilon_i^2 \Im[1/\omega^2] \right) + \nu \delta_x^2 \Im[1/\omega^2] = 0$$

or, using  $\Im[1/\omega^2] = -\Im[\omega^2]/|\omega|^4$ ,

$$\Im[\omega^2] \left( \mu \left( \epsilon_a^2 - \epsilon_i^2 \frac{\delta_x^2}{|\omega|^4} \right) - \nu \frac{\delta_x^2}{|\omega|^4} \right) = 0 \quad (66)$$

Let us assume that  $\omega$  is neither real or pure imaginary. This implies  $\Im[\omega^2] \neq 0$  and (66) leads to:

$$\mu \left( \epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2 \right) - \nu = 0 \quad (67)$$

We will show below that, in this case, solutions can exist only for non physical values of  $\epsilon_i, \epsilon_a$  satisfying  $\max(\epsilon_i, \epsilon_a) > \sqrt{2}$ .

Taking the real part of (65)

$$\mu \left( -\delta_x^2 + \delta_x^2 \epsilon_i^2 \Re[1/\omega^2] - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + \epsilon_a^2 \Re[\omega^2] \right) + \nu \left( \delta_x^2 \Re\left[\frac{1}{\omega^2}\right] - \frac{\epsilon_i^2 + \epsilon_a^2}{2} \right) = 1$$

or, using  $\Re[1/\omega^2] = \Re[\omega^2]/|\omega|^4$ ,

$$\mu \left( -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + \left( \frac{\delta_x^2 \epsilon_i^2}{|\omega|^4} + \epsilon_a^2 \right) \Re[\omega^2] \right) + \nu \left( \frac{\delta_x^2}{|\omega|^4} \Re[\omega^2] - \frac{\epsilon_i^2 + \epsilon_a^2}{2} \right) = 1.$$

(67) allows to simplify in:

$$\mu \left( -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 \Re[\omega^2] \right) - \nu \left( \frac{\epsilon_i^2 + \epsilon_a^2}{2} \right) = 1 \quad (68)$$

Eqs (67) and (68) can be summarized in

$$\begin{aligned} \alpha\mu - \beta\nu &= 1 \\ \gamma\mu - \nu &= 0 \end{aligned} \quad (69)$$

with

$$\alpha = -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 \Re[\omega^2], \beta = \frac{\epsilon_i^2 + \epsilon_a^2}{2}, \gamma = \epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2$$

Using  $\beta > 0$ , it is easy to check that (69) has solutions  $(\mu, \nu)$  positive and with magnitude less than one only if

$$\gamma > 1 \text{ and } \alpha > (1 + \beta)\gamma$$

or

$$0 \leq \gamma \leq 1 \text{ and } \alpha \geq 1 + \beta\gamma$$

414 The conditions above immediately exclude the cases  $\epsilon_a = 0$  (which leads to  $\gamma < 0$ ) and  $\Re[\omega^2] \leq 0$   
415 (which leads to  $\alpha < 0$ ). They will not be considered below.

- First case:  $0 \leq \gamma \leq 1$

This implies:

$$|\omega|^4 \leq \frac{\delta_x^2}{\epsilon_a^2} (1 + \epsilon_i^2) \quad (70)$$

$\alpha \geq 1 + \beta\gamma$  writes:

$$\Re[\omega^2] \geq \frac{1}{2\epsilon_a^2} \left[ 1 + \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} + \frac{\epsilon_i^2 + \epsilon_a^2}{2} \left( \epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2 \right) \right]$$

Now using the inequalities  $|\Re[\omega^2]|^2 \leq |\omega|^4$  and (70), we get

$$\frac{\delta_x^2}{\epsilon_a^2} (1 + \epsilon_i^2) \geq |\omega|^4 \geq \left( \frac{1}{2\epsilon_a^2} \left[ 1 + \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} + \frac{\epsilon_i^2 + \epsilon_a^2}{2} \left( \epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2 \right) \right] \right)^2 \quad (71)$$

Using a computing algebra software to simplify the technical exercise, we can prove that (71) has solutions if and only if

$$\epsilon_i^2 > \sqrt{4 + \epsilon_a^4}$$

416

which requires  $\epsilon_i > \sqrt{2}$ .

- Second case:  $\gamma > 1$

This implies:

$$|\omega|^4 > \frac{\delta_x^2}{\epsilon_a^2}(1 + \epsilon_i^2) \quad (72)$$

$\alpha > (1 + \beta)\gamma$  can be written as:

$$\gamma < \frac{\alpha}{1 + \beta}$$

or

$$|\omega|^4 < \frac{\delta_x^2}{\epsilon_a^2} \left[ \epsilon_i^2 + \frac{1}{1 + \frac{\epsilon_a^2 + \epsilon_i^2}{2}} \left( -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 \Re[\omega^2] \right) \right]$$

Now using  $0 \leq \Re[\omega^2] \leq |\omega|^2$  and adding (72), we get:

$$\frac{\delta_x^2}{\epsilon_a^2}(1 + \epsilon_i^2) < |\omega|^4 < \frac{\delta_x^2}{\epsilon_a^2} \left[ \epsilon_i^2 + \frac{1}{1 + \frac{\epsilon_a^2 + \epsilon_i^2}{2}} \left( -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 |\omega|^2 \right) \right]$$

which has non trivial solutions if and only if

$$\epsilon_a^2 > 2 + \epsilon_i^2$$

417

which requires  $\epsilon_a > \sqrt{2}$ .

418

This concludes the proof. If  $\epsilon_a = 0$  or  $\max(\epsilon_i, \epsilon_a) \leq \sqrt{2}$ , then  $\Im[\omega^2]$  is zero. This also leads to  $\Im[\delta_z^2] = 0$

419

and we conclude that, under these conditions,  $\delta_z$  is either real or purely imaginary.

420

## B Analytical solutions in the constant Brunt-Väisälä case

Let us define a vertical velocity profile  $W(z)$

$$W(z) = e^{-z/(2D_0)} F(z) = e^{-z/2D_0} \frac{\sin(k_z(H + z))}{\sin(k_z H)}$$

and a vertical pressure profile  $P(z)$

$$P(z) = \frac{\Omega^2}{g^2 k_x^2 (1 - (\Omega/(c_s k_x))^2)} (N_0^2 W(z) + g W'(z)).$$

with  $\frac{1}{D_0} = \frac{N_0^2}{g} + \frac{g}{c_s^2}$ , and where, for a given horizontal wavenumber  $k_x$ ,  $k_z$  and  $\Omega$  are solutions of the inner and boundary dispersion relations. Highly accurate approximations of them can be found in table (6).

Note that by construction (and using the boundary dispersion relation), we have  $W(0) = P(0) = 1$ .

The real analytical solutions of Eqs. (10a, 10b, 10c, 10d) which satisfy the bottom and surface conditions (11a, 11b) are given by:

$$\eta(x, t) = \eta_0 \cos(k_x x) \cos(\Omega t) \quad (73)$$

$$u(x, z, t) = \eta_0 \frac{gk_x}{\Omega} \sin(k_x x) \sin(\Omega t) \frac{\hat{\rho}_h(0)}{\hat{\rho}_h(z)} P(z) \quad (74)$$

$$w(x, z, t) = -\eta_0 \Omega \cos(k_x x) \sin(\Omega t) \frac{\hat{\rho}_h(0)}{\hat{\rho}_h(z)} W(z) \quad (75)$$

$$p(x, z, t) = \eta_0 g \hat{\rho}_h(0) \cos(k_x x) \cos(\Omega t) P(z) \quad (76)$$

$$\rho(x, z, t) = \eta_0 g \hat{\rho}_h(0) \cos(k_x x) \cos(\Omega t) \left( \frac{N_0^2}{g^2} W(z) + \frac{1}{c_s^2} P(z) \right) \quad (77)$$

with  $\hat{\rho}_h(z) = \hat{\rho}_h(0) e^{-z/D_0}$  and  $\eta_0$  a constant.

For an homogeneous  $N_0^2 \rightarrow 0$  and incompressible ( $c_s \rightarrow \infty$ , and thus  $D_0 \rightarrow \infty$ ) non hydrostatic ocean, the Airy's solutions are recovered (see e.g. (Gill 1982, section 5.3)). Indeed  $k_z = ik_x$ ,  $\Omega^2 = gk_x \tanh(Hk_x)$  and

$$W_{\text{Airy}}(z) = F_{\text{Airy}}(z) = \frac{\sinh(k_x(H+z))}{\sinh(k_x H)}, \quad P_{\text{Airy}}(z) = \frac{\Omega^2}{gk_x^2} W'_{\text{Airy}}(z) = \frac{\Omega^2}{gk_x} \frac{\cosh(k_x(H+z))}{\sinh(k_x H)},$$

leading to

$$\eta_{\text{Airy}}(x, t) = \eta_0 \cos(k_x x) \cos(\Omega t) \quad (78)$$

$$u_{\text{Airy}}(x, z, t) = \eta_0 \Omega \sin(k_x x) \sin(\Omega t) \frac{\cosh(k_x(H+z))}{\sinh(k_x H)} \quad (79)$$

$$w_{\text{Airy}}(x, z, t) = -\eta_0 \Omega \cos(k_x x) \sin(\Omega t) \frac{\sinh(k_x(H+z))}{\sinh(k_x H)} \quad (80)$$

$$p_{\text{Airy}}(x, z, t) = \hat{\rho}_h(0) g \eta_0 \cos(k_x x) \cos(\Omega t) \frac{\cosh(k_x(H+z))}{\cosh(k_x H)} \quad (81)$$

$$\rho_{\text{Airy}}(x, z, t) = 0 \quad (82)$$